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# Instantaneous convexity breaking for the quasi-static droplet model

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**Abstract.** We consider a well-known quasi-static model for the shape of a liquid droplet. The solution can be described in terms of time-evolving domains in $\mathbb{R}^n$. We give an example to show that convexity of the domain can be instantaneously broken.

## 1. Introduction

We consider the following system of equations for a function $u(x,t)$ and domains $\Omega_t \subset \mathbb{R}^n$, for $t \geq 0$; this system is used to model the quasi-static shape evolution of a liquid droplet of height $u(x,t)$ occupying the region $\Omega_t$:

$$-\Delta u = \lambda_t \quad \text{on } \Omega_t, \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega_t, \tag{1.1b}$$

$$V = F(|Du|) \quad \text{on } \partial\Omega_t, \tag{1.1c}$$

$$\int_{\Omega_t} u\, dx = 1. \tag{1.1d}$$

In the above, $V$ is the velocity of the free boundary $\partial\Omega_t$ in the direction of the outward unit normal and $F : (0,\infty) \to \mathbb{R}$ is an analytic function with $F'(r) > 0$ for $r > 0$. The constant $\lambda_t > 0$ is determined by the integral condition on $u$.

The initial data is given by a domain $\Omega_0$, which we assume is bounded with smooth boundary $\partial\Omega_0$. Note that the domains $\Omega_t$ (assuming they are bounded with sufficiently regular boundary $\partial\Omega_t$) determine uniquely the solution $x \mapsto u(x,t)$. Thus, we may denote a solution of (1.1) by a family of evolving domains $\Omega_t$. In Section 2 we will explain what is meant by a *classical solution* to this problem.

The system of equations given in (1.1) has long been accepted as a model for droplet evolution in the physical literature [1,6,7,10,11]. There have been results on weak formulations of this equation by Glasner–Kim [5] and Grunewald–Kim [8]. Feldman–Kim [3] gave some conditions for global existence and convergence to an equilibrium. Escher–Guidotti [2] proved a short time existence result for classical solutions, which we describe in Section 2 below.

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In this note we address the following natural question:

**Question 1.1.** Is the convexity of $\Omega_t$ preserved by system (1.1)?

This question is implicit in the work of Glasner–Kim [5]. It was raised explicitly by Feldman–Kim [3, p. 822]: "Let us point out that, in particular, it is unknown whether the convexity of the drop is preserved in the system [(1.1)]."

In this note, we answer Question 1.1 by showing that convexity is *not* generally preserved. We make an assumption on $F$, namely that

$$\lim_{r\to 0^+} \frac{F''(r)}{F'(r)} \geq \gamma \quad \text{for some } \gamma > 0. \tag{1.2}$$

This includes the important cases $F(r) = r^3 - 1$ and $F(r) = r^2 - 1$ considered in [5] and [3, 8], respectively.

We construct an example where $\Omega_t$ is convex for $t = 0$, but not convex for $t \in (0, \delta]$ for some $\delta > 0$.

**Theorem 1.2.** *Assume that $F$ satisfies Assumption 1.2. There exist $\delta > 0$ and a bounded convex domain $\Omega_0 \subset \mathbb{R}^2$ with smooth boundary such that the solution $\Omega_t$ to (1.1) with this initial data is not convex for any $t \in (0, \delta]$.*

Escher–Guidotti [2] showed that as long as $\Omega_0$ is a bounded domain with sufficiently smooth boundary, there always exists a unique classical solution for a short time, and this is what is meant by "the solution $\Omega_t$" in the statement of Theorem 1.2. In Section 2, we describe more precisely the results of [2].

In Section 3 we give the proof of Theorem 1.2. The starting point is an explicit solution of the equation $-\Delta u = \lambda_0$ on an equilateral triangle [9]. We smooth out the corners to obtain our convex domain $\Omega_0$ and show that it immediately breaks convexity.

## 2. Short time existence

In this section, we recall the short time existence result of Escher–Guidotti [2].

We first give a definition of a solution of (1.1), following [2]. Note that the domains $\Omega_t$ determine uniquely the functions $u$, so we will describe the solution of (1.1) in terms of varying domains—given as graphs over the original boundary.

Fix $\alpha \in (0, 1)$. Assume that $\Omega_0$ is a bounded domain in $\mathbb{R}^n$ whose boundary $\Gamma_0 := \partial\Omega_0$ is a smooth hypersurface. Let $\nu(x)$ denote the unit outward normal to $\Gamma_0$ at $x$. Then, there exists a maximal constant $\sigma(\Omega_0) > 0$ such that for any given function $\rho \in C^{2+\alpha}(\Gamma_0)$ with $\|\rho\|_{C^1(\Gamma_0)} \leq \sigma$, the set

$$\Gamma_\rho = \left\{x + \rho(x)\nu(x) \mid x \in \Gamma_0\right\}$$

is a $C^{2+\alpha}$ hypersurface in $\mathbb{R}^n$, which is the boundary of a bounded domain $\Omega = \Omega(\rho)$.

We can now describe a solution of (1.1) in terms of a time-varying family $\rho(x,t)$, that is, given

$$\rho \in C([0,T], C^{2+\alpha}(\Gamma_0)) \cap C^1([0,T], C^{1+\alpha}(\Gamma_0))$$

with $\sup_{t\in[0,T]} \|\rho(\cdot,t)\|_{C^1(\Gamma_0)} < \sigma(\Omega_0)$, write $\Omega_t$, $t \in [0,T]$, for the corresponding family of domains, with boundaries $\Gamma_t := \Gamma_{\rho(t)}$. The velocity $V$ of the boundary in the direction of the outward normal at a point $y = x + \rho(x,t)\nu(x) \in \Gamma_t$ is given by

$$V = \frac{\partial \rho}{\partial t}(x,t)\nu(x) \cdot n(y,t),$$

where $n(y,t)$ is the outward unit normal to $\Gamma_t$ at the point $y$.

Since the domains $\Omega_t$ have $C^{2+\alpha}$ boundaries, there exists for each $t$ a unique solution $u(\cdot,t) \in C^{2+\alpha}(\overline{\Omega}_t)$ and $\lambda_t \in \mathbb{R}$ of

$$-\Delta u = \lambda_t \quad \text{on } \Omega_t, \qquad u|_{\Gamma_t} = 0, \qquad \int_{\Omega_t} u\,dx = 1;$$

see, for example, [4, Theorem 6.14].

Then, we say that such a $\rho$ is a *classical solution* of (1.1) with initial domain $\Omega_0$ if the velocity $V(y)$ at each $y \in \Gamma_t$, for $t \in [0,T]$, satisfies

$$V = F(|Du|).$$

The main theorem of Escher–Guidotti [2] implies, in particular, the following:

**Theorem 2.1.** *There exist a $T > 0$ and a unique classical solution*

$$\rho \in C([0,T], C^{2+\alpha}(\Gamma_0)) \cap C^1([0,T], C^{1+\alpha}(\Gamma_0))$$

*of the quasi-static droplet model* (1.1) *with initial domain $\Omega_0$ whose boundary $\Gamma_0$ is smooth.*

In fact, they prove more: they also allow their initial domain to have its boundary in $C^{2+\alpha}$. Note that this result does not require Assumption 1.2.

## 3. Proof of Theorem 1.2

In this section we give a proof of Theorem 1.2. We work in $\mathbb{R}^2$, using $x$ and $y$ as coordinates. The heart of the proof is the following lemma, which makes use of Assumption 1.2:

**Lemma 3.1.** *There exist a bounded convex domain $\Omega_0$ with smooth boundary $\Gamma_0$ and real numbers $0 < x_0 < x_1$ with the following properties:*

(i) *$\Omega_0$ is contained in $\{y \geq 0\}$.*

(ii) *$(x,0) \in \partial\Omega_0$ for $x_0 \leq x \leq x_1$.*

(iii) *Let* $u(x, y)$ *solve*

$$-\Delta u = \lambda_0 \quad \text{on } \Omega_0, \qquad u|_{\Gamma_0} = 0, \qquad \int_{\Omega_0} u \, dxdy = 1$$

*for a constant* $\lambda_0$. *Then,* $V(x) := F(|Du(x, 0)|)$ *satisfies*

$$\frac{V(x_0) + V(x_1)}{2} > V\Big(\frac{x_0 + x_1}{2}\Big).$$

*Proof.* We begin with the following explicit solution of the "torsion problem," $-\Delta v =$ const., on the equilateral triangle [9]. Let $D$ be the equilateral triangle of side length $2a$ given by

$$0 < y < \sqrt{3}(a - |x|).$$

The function

$$v = cy((y - a\sqrt{3})^2 - 3x^2) \quad \text{for } c := \frac{5}{3a^5}$$

satisfies

$$-\Delta v = 4ac\sqrt{3},$$

vanishes on the boundary of $D$, and satisfies

$$\int_D v \, dxdy = 1.$$

On the bottom edge of the triangle

$$E_1 = \big\{(x, 0) \in \mathbb{R}^2 \mid -a \le x \le a\big\},$$

we have

$$v_y(x, 0) = 3c(a^2 - x^2).$$

Hence,

$$V(x) = F(3c(a^2 - x^2)),$$

and

$$V''(x) = 36c^2x^2F''(3c(a^2 - x^2)) - 6cF'(3c(a^2 - x^2)). \tag{3.1}$$

Recalling that $c = 5/(3a^5)$, we may then choose $a > 0$ sufficiently small so that

$$36c^2x^2 \ge 2\frac{6c}{\gamma} \quad \text{for } |x| \ge a/2, \tag{3.2}$$

where $\gamma > 0$ is given by Assumption 1.2. From now on, we fix this $a$ (and hence, $c$).

It follows from (3.1), (3.2), and Assumption 1.2 that $V''(x) > 0$ for $|x|$ sufficiently close to $a$. In particular, there exists $0 < x_0 < x_1 < a$ with

$$\frac{V(x_0) + V(x_1)}{2} > V\Big(\frac{x_0 + x_1}{2}\Big). \tag{3.3}$$

The above example readily implies the existence of a smooth domain $\Omega_0$ satisfying the conditions in the lemma. Indeed, we only have to "smooth the corners" of the triangle domain $D$.

Denote the vertices of $D$ by $p_1, p_2, p_3$. Let $\{D_k\}_{k=1}^{\infty}$ be a sequence of bounded convex domains with smooth boundaries such that for each $k \geq 1$:

(1) $D_k \subset D_{k+1} \subset D$ (the sequence is nested and increasing).

(2) $D \setminus D_k \subset \bigcup_{i=1}^{3} B_{k^{-1}}(p_i)$, where $B_r(p)$ denotes the ball of radius $r$ centered at $p$.

Such a sequence $\{D_k\}$ can be constructed by "rounding out the corners" of the triangle $D$ in a ball of radius $k^{-1}$ centered at each corner.

For each $k \geq 1$, let $u_k$ on $D_k$ be the solutions of

$$-\Delta u_k = 4ac\sqrt{3} \quad \text{on } D_k, \qquad u|_{\partial D_k} = 0,$$

where we recall that $a$ and $c$ are fixed constants.

It follows from property (1) above and the maximum principle that for each $k \geq 1$

$$0 < u_k \leq u_{k+1} \leq v \quad \text{on } D_k, \tag{3.4}$$

from which we conclude a pointwise limit on the triangle $D$

$$0 \leq u_\infty(x) := \lim_{k\to\infty} u_k(x) \leq v(x) \quad \text{for } x \in D, \tag{3.5}$$

and define $u_\infty(x)$ to be zero on $\partial D$.

By standard elliptic estimates (see, for example, [4, Theorem 6.19] and the remark after it), the convergence above will hold in $C^\ell(K)$ for any compact set $K \subset\subset (\overline{D} \setminus \{p_1, p_2, p_3\})$ and any $\ell \geq 0$. Hence, $u_\infty \in C^\infty(\overline{D} \setminus \{p_1, p_2, p_3\})$ and $-\Delta u_\infty = 4ac\sqrt{3}$ on $D$. Moreover, by (3.4) and the continuity of $v$, it is easily verified that $u_\infty$ is also continuous at the corners $p_1, p_2, p_3$ and thus on all of $\overline{D}$. By the maximum principle, $u_\infty = v$. Note also that

$$\int_{D_k} u_k \, dx dy \to 1 \quad \text{as } k \to \infty.$$

Then, for sufficiently large $k$, the domain $\Omega_0 := D_k$ will satisfy conditions (i), (ii), and (iii), with

$$u := \frac{u_k}{\int_{D_k} u_k \, dx dy}, \qquad \lambda_0 := \frac{4ac\sqrt{3}}{\int_{D_k} u_k \, dx dy}.$$

Here we are using (3.3) and the fact that $x \mapsto F(|Du_k(x,0)|)$ will converge uniformly to $x \mapsto F(|Dv(x,0)|)$ on $[x_0, x_1]$ as $k \to \infty$. This completes the proof of the lemma. ■

*Proof of Theorem* 1.2. Let $\Omega_0$ and $u$ be given as in Lemma 3.1. By Theorem 2.1, there exists a unique classical solution of (1.1) for a short time interval $[0, T]$ with $T > 0$.

The boundary $\Gamma_t$ of $\Omega_t$ can be written as a graph over $\Gamma_0 := \partial\Omega_0$. In particular, using $x$ as a coordinate, part of $\Gamma_t$ is given by a graph $y = g(x,t)$ for $x_0 \le x \le x_1$, with $g(x,0) = 0$ for $x_0 \le x \le x_1$ and with the unit normal to $\Omega_0$ being in the negative $y$ direction.

We may assume that

$$g \in C([0,T], C^{2+\alpha}([x_0,x_1])) \cap C^1([0,T], C^{1+\alpha}([x_0,x_1])).$$

Moreover, $(\partial g/\partial t)(x,0)$ represents the *negative* of the velocity in the normal direction at time $t = 0$. Hence, by condition (iii) of Lemma 3.1,

$$\frac{1}{2}\Big(\frac{\partial g}{\partial t}(x_0,0) + \frac{\partial g}{\partial t}(x_1,0)\Big) < \frac{\partial g}{\partial t}\Big(\frac{x_0+x_1}{2}, 0\Big).$$

Then, for $t \in (0,\delta]$ for $\delta > 0$ sufficiently small, we have

$$\frac{1}{2}\Big(g(x_0,t) + g(x_1,t)\Big) < g\Big(\frac{x_0+x_1}{2}, t\Big).$$

In particular, $x \mapsto g(x,t)$ is not convex for $(x,t) \in [x_0,x_1] \times (0,\delta]$. Hence, $\Omega_t$ is not a convex domain for $t \in (0,\delta]$. ■

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## References

[1] R. G. Cox, The spreading of a liquid on a rough solid surface. *J. Fluid Mech.* **131** (1983), 1–26 Zbl 0597.76102 MR 718031

[2] J. Escher and P. Guidotti, Local well-posedness for a quasi-stationary droplet model. *Calc. Var. Partial Differential Equations* **54** (2015), no. 1, 1147–1160 Zbl 1329.35363 MR 3385195

[3] W. M. Feldman and I. C. Kim, Dynamic stability of equilibrium capillary drops. *Arch. Ration. Mech. Anal.* **211** (2014), no. 3, 819–878 Zbl 1293.35240 MR 3158808

[4] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. Class. Math., Springer, Berlin, 2001 Zbl 1042.35002 MR 1814364

[5] K. Glasner and I. C. Kim, Viscosity solutions for a model of contact line motion. *Interfaces Free Bound.* **11** (2009), no. 1, 37–60 Zbl 1166.35386 MR 2487023

[6] K. B. Glasner, A boundary integral formulation of quasi-steady fluid wetting. *J. Comput. Phys.* **207** (2005), no. 2, 529–541 Zbl 1213.76068 MR 2144627

[7] H. P. Greenspan, On the motion of a small viscous droplet that wets a surface. *J. Fluid Mech.*, **84** (1978), no. 1, 125–143 Zbl 0373.76040

[8] N. Grunewald and I. Kim, A variational approach to a quasi-static droplet model. *Calc. Var. Partial Differential Equations* **41** (2011), no. 1–2, 1–19 Zbl 1228.35087 MR 2782795

[9] G. Keady and A. McNabb, The elastic torsion problem: solutions in convex domains. *New Zealand J. Math.* **22** (1993), no. 2, 43–64 Zbl 0814.35133 MR 1244022

[10] L. Tanner, The spreading of silicone oil drops on horizontal surfaces. *J. Phys. D* **12** (1979), 1473–1484

[11] O. V. Voinov, Hydrodynamics of wetting. *Fluid Dyn.* **11** (1976), 714–721

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