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Error estimate for classical solutions to the heat equation in a moving thin domain and its limit equation

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Abstract. We consider the Neumann-type problem of the heat equation in a moving thin domain around a given closed moving hypersurface. The main result of this paper is an error estimate in the sup-norm for classical solutions to the thin domain problem and a limit equation on the moving hypersurface which appears in the thin-film limit of the heat equation. To prove the error estimate, we show a uniform a priori estimate for a classical solution to the thin domain problem based on the maximum principle. Moreover, we construct a suitable approximate solution to the thin domain problem from a classical solution to the limit equation based on an asymptotic expansion of the thin domain problem and apply the uniform a priori estimate to the difference of the approximate solution and a classical solution to the thin domain problem.

1. Introduction

1.1. Problem settings and main results

For $t \in [0, T]$, $T > 0$, let $\Gamma(t)$ be a given closed moving hypersurface in $\mathbb{R}^n$, $n \geq 2$ with unit outward normal vector field $\nu(\cdot, t)$. Also, let $g_0(\cdot, t)$ and $g_1(\cdot, t)$ be functions on $\Gamma(t)$ such that $g(y, t) = g_1(y, t) - g_0(y, t) \geq c$ for all $y \in \Gamma(t)$ with some constant $c > 0$ independent of t. For a sufficiently small $\varepsilon > 0$, we define a moving thin domain

$$\Omega_\varepsilon(t) = \{ y + r\nu(y, t) \mid y \in \Gamma(t),\ \varepsilon g_0(y, t) < r < \varepsilon g_1(y, t) \}, \quad t \in [0, T]$$

and consider the Neumann-type problem of the heat equation (a linear diffusion equation)

$$\begin{cases} \partial_t \rho^\varepsilon - k_d \Delta \rho^\varepsilon = f^\varepsilon & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon} \rho^\varepsilon + k_d^{-1} V_\varepsilon \rho^\varepsilon = 0 & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0). \end{cases} \tag{1.1}$$

Here $Q_{\varepsilon,T}$ and $\partial_\ell Q_{\varepsilon,T}$ are a space-time domain and its lateral boundary, given by

$$Q_{\varepsilon,T} = \bigcup_{t\in(0,T]} \Omega_\varepsilon(t) \times \{t\}, \quad \partial_\ell Q_{\varepsilon,T} = \bigcup_{t\in(0,T]} \partial\Omega_\varepsilon(t) \times \{t\},$$

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respectively. Also, ν_ε and V_ε are the unit outward normal vector field and the (scalar) outer normal velocity of $\partial\Omega_\varepsilon(t)$ (see Section 2 for details of the notations), respectively. Moreover, $k_d > 0$ is a diffusion coefficient independent of ε; and f^ε and ρ_0^ε are a given source term and initial data which may depend on ε, respectively. Note that the term $k_d^{-1} V_\varepsilon \rho^\varepsilon$ is added in the boundary condition so that the conservation law

$$\frac{d}{dt}\int_{\Omega_\varepsilon(t)} \rho^\varepsilon\,dx = \int_{\Omega_\varepsilon(t)} f^\varepsilon\,dx, \quad t \in (0,T)$$

holds, by the Reynolds transport theorem and integration by parts. It can be also be seen that the heat (or a substance) moves along the boundary so that it does not go out of or come into the moving thin domain through the boundary.

The purpose of this paper is to compare a solution to (1.1) with a solution to

$$\begin{cases} \partial^\circ(g\eta) - gV_\Gamma H\eta - k_d \operatorname{div}_\Gamma(g\nabla_\Gamma\eta) = gf & \text{on} \quad S_T, \\ \eta|_{t=0} = \eta_0 & \text{on} \quad \Gamma(0), \end{cases} \tag{1.2}$$

which we call the limit equation of (1.1). Here

$$S_T = \bigcup_{t\in(0,T]} \Gamma(t)\times\{t\}$$

is a space-time hypersurface; $\partial^\circ = \partial_t + V_\Gamma\nu\cdot\nabla$ is the normal time derivative; V_Γ is the (scalar) outer normal velocity of $\Gamma(t)$; H is the mean curvature of $\Gamma(t)$, which is just the sum of the principal curvatures of $\Gamma(t)$; and $\operatorname{div}_\Gamma$ and ∇_Γ are the surface divergence and the tangential gradient on $\Gamma(t)$, respectively (see Section 2 for details). Also, f and η_0 are a given source term and initial data, respectively.

In our previous work [35], we studied the thin-film limit problem for (1.1) with $k_d = 1$ and $f^\varepsilon \equiv 0$ in the L^2-framework and derived (1.2) from (1.1) rigorously by means of the convergence of a solution and the characterization of the limit. For an L^2-weak solution ρ^ε to (1.1), we proved that the average of ρ^ε in the thin direction converges weakly to a function η on S_T in an appropriate function space on S_T as $\varepsilon \to 0$. Moreover, we derived a weak form on S_T satisfied by η from the average of a weak form of (1.1) and obtained the limit equation in (1.2) as an equation to which the weak limit η is a unique L^2-weak solution. We also estimated the difference of η and the average of ρ^ε on S_T by a standard energy method and used it to get an L^2-error estimate in $Q_{\varepsilon,T}$ for ρ^ε and the constant extension of η in the normal direction of $\Gamma(t)$.

The above result shows that a solution to limit equation (1.2) approximates a solution to the thin domain problem given by (1.1) in the L^2-framework. However, the error estimate in the $L^2(Q_{\varepsilon,T})$-norm may have some ambiguity, since the volume of $Q_{\varepsilon,T}$ is of order ε. One approach to avoid such an ambiguity is to divide the $L^2(Q_{\varepsilon,T})$-norm by $\varepsilon^{1/2}$. In fact, the error estimate obtained in [35, Theorem 6.12] shows that

$$\varepsilon^{-1/2}\|\rho^\varepsilon - \overline{\eta}\|_{L^2(Q_{\varepsilon,T})} \le c(\varepsilon^{-1/2}\|\rho_0^\varepsilon - \overline{\eta}_0\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon\|\eta_0\|_{L^2(\Gamma(0))}) \tag{1.3}$$

for weak solutions ρ^ε to (1.1) and η to (1.2), respectively, when $k_d = 1$, $f^\varepsilon \equiv 0$, and $f \equiv 0$, where $\overline{\eta}(\cdot, t)$ is the constant extension of $\eta(\cdot, t)$ in the normal direction of $\Gamma(t)$ for each $t \in [0, T]$. Hence, we can say that η approximates ρ_ε of order ε in the L^2-framework.

The purpose of this paper is to give another approach to avoid the ambiguity due to the volume of $Q_{\varepsilon,T}$: an error estimate in the sup-norm. We estimate the difference of classical solutions to (1.1) and (1.2) in the sup-norm on $Q_{\varepsilon,T}$. It seems that such an attempt for a curved thin domain around a hypersurface is first done in this paper even if a curved thin domain does not move in time, although a similar idea has already appeared in the case of a stationary flat thin domain around a lower-dimensional domain (see, e.g., [34]).

To state our main results, we give some definitions and notations. For given data ρ_0^ε and f^ε, we say that ρ^ε is a classical solution to (1.1) if

$$\rho^\varepsilon \in C(\overline{Q_{\varepsilon,T}}), \quad \partial_i \rho^\varepsilon \in C(Q_{\varepsilon,T} \cup \partial_\ell Q_{\varepsilon,T}), \quad \partial_t \rho^\varepsilon, \partial_i \partial_j \rho^\varepsilon \in C(Q_{\varepsilon,T})$$

for all $i, j = 1, \dots, n$ and ρ^ε satisfies (1.1) at each point of $\overline{Q_{\varepsilon,T}}$. Also, for given data η_0 and f, we say that η is a classical solution to (1.2) if $\eta \in C(\overline{S_T}) \cap C^{2,1}(S_T)$ and η satisfies (1.2) at each point of $\overline{S_T}$, where

$$C^{2,1}(S_T) = \left\{\zeta \in C(S_T) \mid \partial^\circ \zeta, \underline{D}_i \zeta, \underline{D}_i \underline{D}_j \zeta \in C(S_T) \ \text{ for all } i, j = 1, \dots, n\right\}$$

and $\underline{D}_i$ is the i-th component of ∇_Γ (see Section 2.1). Here we do not touch the problem of the existence of classical solutions. It is known (see, e.g., [31, 33]) that there exists a classical solution to (1.1) if the given data have a sufficient Hölder regularity; for example, $\rho_0^\varepsilon \in C^{2+\alpha}(\overline{\Omega_\varepsilon(0)})$ and $f^\varepsilon \in C^{\alpha,\alpha/2}(\overline{Q_{\varepsilon,T}})$ (α in space and $\alpha/2$ in time) with some $\alpha \in (0, 1)$. Also, when the given data for (1.2) have a sufficient Hölder regularity—for example, with $\eta_0 \in C^{\alpha+2}(\Gamma(0))$ and $f \in C^{\alpha,\alpha/2}(\overline{S_T})$ with some $\alpha \in (0, 1)$—the existence of a classical solution to (1.2) can be shown by a standard localization argument and application of the existence result in the case of a flat domain, although there seems to be no literature giving the procedure explicitly in the case of a moving surface. An alternative approach is to relate a solution to (1.2) with a solution to the Neumann-type problem of a suitable parabolic equation on a tubular neighborhood of $\Gamma(t)$ and then use the existence result in the case of a flat domain, which was carried out in the study of a time-periodic solution to an advection-diffusion equation on a moving surface [16] (after transforming an original problem on a moving surface into a problem on a fixed-in-time surface). We also note that classical solutions to (1.1) and (1.2) are unique, by the maximum principle.

Let us fix some more notations. Let Ω be a spatial set in $\mathbb{R}^n$ or a space-time set in $\mathbb{R}^{n+1}$. For a bounded function ρ on Ω, we write

$$\|\rho\|_{\mathcal{B}(\Omega)} = \sup_{\Omega} |\rho|.$$

Note that, if $\Omega = \overline{Q_{\varepsilon,T}}$ is the closure of $Q_{\varepsilon,T}$ in $\mathbb{R}^{n+1}$, then

$$\|\rho\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} = \max\left\{\|\rho\|_{\mathcal{B}(Q_{\varepsilon,T})}, \|\rho(\cdot, 0)\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})}, \|\rho\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\right\},$$

since $Q_{\varepsilon,T}$ does not contain the initial set $\overline{\Omega_\varepsilon(0)} \times \{0\}$ and the lateral boundary $\partial_\ell Q_{\varepsilon,T}$. Similarly, when $\Omega = \overline{S_T}$ is the closure of S_T in $\mathbb{R}^{n+1}$, then

$$\|\rho\|_{\mathcal{B}(\overline{S_T})} = \max\bigl\{\|\rho\|_{\mathcal{B}(S_T)}, \|\rho(\cdot,0)\|_{\mathcal{B}(\Gamma(0))}\bigr\}.$$

In what follows, we always assume that a function ρ is bounded on Ω if we use the notation $\|\rho\|_{\mathcal{B}(\Omega)}$. When $\eta \in C^{2,1}(S_T)$ and η, $\partial^\circ\eta$, $\underline{D}_i\eta$, and $\underline{D}_i\underline{D}_j\eta$ are bounded on S_T, we write

$$\begin{aligned} \|\eta\|_{\mathcal{B}^{2,1}(S_T)} = \|\eta\|_{\mathcal{B}(S_T)} + \|\partial^\circ\eta\|_{\mathcal{B}(S_T)} \\ + \sum_{i=1}^n \|\underline{D}_i\eta\|_{\mathcal{B}(S_T)} + \sum_{i,j=1}^n \|\underline{D}_i\underline{D}_j\eta\|_{\mathcal{B}(S_T)}. \end{aligned} \tag{1.4}$$

We take a constant $\delta > 0$ such that $\Gamma(t)$ has a tubular neighborhood of radius δ in $\mathbb{R}^n$ for all $t \in [0,T]$ (see Section 2.2), and fix $\varepsilon_0 \in (0,1)$ such that $\varepsilon|g_i| \le \delta$ on $\overline{S_T}$ for all $\varepsilon \in (0,\varepsilon_0)$ and $i = 0,1$. Moreover, for a function ζ on $\overline{S_T}$ and for each $t \in [0,T]$, we denote by $\bar{\zeta}(\cdot,t)$ the constant extension of $\zeta(\cdot,t)$ in the normal direction of $\Gamma(t)$.

Now let us state our main results. First, we give an error estimate when $\Omega_\varepsilon(t)$ is a thin tubular neighborhood for each $t \in [0,T]$.

Theorem 1.1. *Let $\varepsilon \in (0,\varepsilon_0)$ and the given data*

$$\rho_0^\varepsilon \in C(\overline{\Omega_\varepsilon(0)}), \quad f^\varepsilon \in C(Q_{\varepsilon,T}), \quad \eta_0 \in C(\Gamma(0)), \quad f \in C(S_T)$$

be bounded. Also, let ρ^ε and η be classical solutions to (1.1) *and* (1.2)*, respectively. Suppose that*

(a) *$\partial^\circ\eta$, $\underline{D}_i\eta$, and $\underline{D}_i\underline{D}_j\eta$ with $i,j = 1,\dots,n$ are bounded on S_T,*

(b) *$g_0 = g_0(t)$ and $g_1 = g_1(t)$ depend only on $t \in [0,T]$.*

Then, there exists a constant $c_T > 0$ depending on T but independent of ε such that

$$\begin{aligned} \|\rho^\varepsilon - \bar{\eta}\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \le c_T(\|\rho_0^\varepsilon - \bar{\eta}_0\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f^\varepsilon - \bar{f}\|_{\mathcal{B}(Q_{\varepsilon,T})}) \\ + \varepsilon c_T(\|\eta_0\|_{\mathcal{B}(\Gamma(0))} + \|\eta\|_{\mathcal{B}^{2,1}(S_T)}). \end{aligned} \tag{1.5}$$

Note that the classical solutions ρ^ε and η to (1.1) and (1.2) are bounded on $\overline{Q_{\varepsilon,T}}$ and $\overline{S_T}$, respectively, since they are continuous on the compact sets $\overline{Q_{\varepsilon,T}}$ and $\overline{S_T}$, by the definitions. We also note that we leave the $\mathcal{B}^{2,1}(S_T)$-norm of η in the right-hand side of (1.5), which cannot be estimated just by the sup-norms of η_0 and f. It can be estimated by the Hölder norms of η_0 and f (more precisely, the $C^{2+\alpha}(\Gamma(0))$-norm of η_0 and the $C^{\alpha,\alpha/2}(\overline{S_T})$-norm of f with some $\alpha \in (0,1)$) if one uses a localization argument and a regularity estimate for a classical solution in the case of a flat domain (see, e.g., [31,33]). We also refer to [16, Theorem 3.1] for a regularity estimate for a classical solution to a parabolic equation on a fixed surface transformed from an advection-diffusion equation on a moving surface.

When $\Omega_\varepsilon(t)$ is not a tubular neighborhood, that is, g_0 and g_1 are not constant in the variable $y \in \Gamma(t)$, we require an additional regularity assumption on η.

Theorem 1.2. *Under the settings of Theorem 1.1, suppose that condition* (a) *is satisfied. Moreover, suppose instead of condition* (b) *that*

(b') *for $i = 0, 1$, the function $h_i = \nabla_\Gamma g_i \cdot \nabla_\Gamma \eta$ belongs to $C(\overline{S_T}) \cap C^{2,1}(S_T)$ and its derivatives $\partial^\circ h_i$, $\underline{D}_j h_i$, and $\underline{D}_j \underline{D}_k h_i$ with $j, k = 1, \dots, n$ are bounded on S_T.*

Then, there exists a constant $c_T > 0$ depending on T but independent of ε such that

$$\begin{aligned}\|\rho^\varepsilon - \overline{\eta}\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \le c_T(\|\rho_0^\varepsilon - \overline{\eta}_0\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} &+ \|f^\varepsilon - \overline{f}\|_{\mathcal{B}(Q_{\varepsilon,T})}) \\ &+ \varepsilon c_T(\|\eta_0\|_{\mathcal{B}(\Gamma(0))} + \|\eta\|_{\mathcal{B}^{2,1}(S_T)}) \\ &+ \varepsilon c_T \sum_{i=0,1}(\|h_i(\cdot,0)\|_{\mathcal{B}(\Gamma(0))} + \|h_i\|_{\mathcal{B}^{2,1}(S_T)}).\end{aligned} \tag{1.6}$$

By using a localization argument and a parabolic regularity theory in the case of a flat domain (see, e.g., [31]), one may show that condition (b') is satisfied if $\eta_0 \in C^{3+\alpha}(\Gamma(0))$ and $f \in C^{1+\alpha,(1+\alpha)/2}(\overline{S_T})$ with some $\alpha \in (0,1)$. Also, the last line of (1.6) can be estimated by the Hölder norms of η_0 and f of the above order.

The proofs of Theorem 1.1 and 1.2 are given in Section 4. We explain the idea of the proofs in the next subsection.

In (1.5) and (1.6), the volume of $Q_{\varepsilon,T}$ and $\Omega_\varepsilon(0)$ is irrelevant to the sup-norm. Thus, by Theorems 1.1 and 1.2, we can say that a classical solution to (1.2) approximates a classical solution to (1.1) of order ε in the sup-norm as in the L^2-framework. We should note that, however, unlike Theorem 1.2, an additional regularity assumption on an L^2-weak solution to (1.2) is not required for L^2-error estimate (1.3) even if $\nabla_\Gamma g_i \not\equiv 0$ for $i = 0, 1$ (see [35]). This is due to the fact that we use the strong form of (1.1) to show (1.6), while the weak form of (1.1) is used in the proof of (1.3). As we explain below, we need to construct a suitable approximate solution to the strong form of (1.1) involving the function $\nabla_\Gamma g_i \cdot \nabla_\Gamma \eta$, $i = 0, 1$ from a classical solution η to (1.2) in the proof of (1.6), since η itself does not satisfy the strong form of (1.1) even approximately. Hence, condition (b') of Theorem 1.2 is required in order to make the approximate solution well-defined at $t = 0$ and sufficiently regular in $Q_{\varepsilon,T}$. In the proof of (1.3), however, we take the average in the thin direction of the weak form of (1.1) and use the average of a weak solution to (1.1) as an approximate weak solution to (1.2). This method generates the $L^2(Q_{\varepsilon,T})$-norms of a weak solution to (1.1) and its gradient as additional terms in an L^2-error estimate, but they can be bounded by the $L^2(\Omega_\varepsilon(0))$-norm of an initial data (and an appropriate norm of a source term) by a standard energy estimate. Hence, we can avoid additional regularity assumptions on weak solutions to (1.1) and (1.2) even if $\nabla_\Gamma g_i \not\equiv 0$ for $i = 0, 1$ in the proof of (1.3).

By the above explanations, one may guess that condition (b') of Theorem 1.2 can be removed by the use of the average method. It may be possible to do that, but the average method will generate the $\mathcal{B}(Q_{\varepsilon,T})$-norms of the derivatives of a classical solution ρ^ε

to (1.1) up to the second order (or more) as additional terms in an error estimate instead of the last line of (1.6). Then, unlike the norms in the last line of (1.6), the $\mathcal{B}(Q_{\varepsilon,T})$-norms of the derivatives of ρ^ε depend on ε, and we do not a priori know how these norms depend on ε. Hence, we need to estimate the derivatives of ρ^ε in $\mathcal{B}(Q_{\varepsilon,T})$ explicitly in terms of ε by the given data ρ_0^ε and f^ε, but such a task will be very tough and also require an additional regularity assumption on ρ_0^ε and f^ε (see, e.g., [31, 33] in the case of a general domain). We would like to avoid such tough work and an assumption on ρ_0^ε and f^ε, so we do not use the average method in this paper.

1.2. Idea of the proofs

The proofs of Theorems 1.1 and 1.2 are based on a uniform a priori estimate for a classical solution to (1.1) and a construction of a suitable approximate solution to (1.1) from a classical solution to (1.2). For classical solutions ρ^ε and η to (1.1) and (1.2), we give an approximate solution ρ_η^ε to (1.1) close to η so that the difference of ρ^ε and ρ_η^ε satisfies

$$\begin{cases} \partial_t(\rho^\varepsilon-\rho_\eta^\varepsilon)-k_d\Delta(\rho^\varepsilon-\rho_\eta^\varepsilon)=(f^\varepsilon-\bar{f})-f_\eta^\varepsilon & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon}(\rho^\varepsilon-\rho_\eta^\varepsilon)+k_d^{-1}V_\varepsilon(\rho^\varepsilon-\rho_\eta^\varepsilon)=-\psi_\eta^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ (\rho^\varepsilon-\rho_\eta^\varepsilon)|_{t=0}=\rho_0^\varepsilon-\rho_\eta^\varepsilon(\cdot,0) & \text{in} \quad \Omega_\varepsilon(0), \end{cases} \tag{1.7}$$

where f_η^ε and ψ_η^ε are error terms due to ρ_η^ε. Then, we prove the uniform a priori estimate

$$\begin{aligned} \|\rho^\varepsilon-\rho_\eta^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} &\leq c(\|\rho_0^\varepsilon-\rho_\eta^\varepsilon(\cdot,0)\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})}+\|f^\varepsilon-\bar{f}\|_{\mathcal{B}(Q_{\varepsilon,T})}) \\ &\quad +c\Big(\|f_\eta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})}+\frac{1}{\varepsilon}\|\psi_\eta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big) \end{aligned} \tag{1.8}$$

for the classical solution $\rho^\varepsilon-\rho_\eta^\varepsilon$ to (1.7), where $c>0$ is a constant independent of ε (see Theorem 3.4). Hence, we can get (1.5) and (1.6) by choosing ρ_η^ε so that $\rho_\eta^\varepsilon-\overline{\eta}$ and f_η^ε are of order ε in $\overline{Q_{\varepsilon,T}}$ and $Q_{\varepsilon,T}$, respectively, and ψ_η^ε is of order ε^2 on $\partial_\ell Q_{\varepsilon,T}$.

To prove (1.8), we consider a general parabolic equation

$$\begin{cases} \partial_t\chi^\varepsilon-\sum_{i,j=1}^n a_{ij}^\varepsilon\partial_i\partial_j\chi^\varepsilon+\sum_{i=1}^n b_i^\varepsilon\partial_i\chi^\varepsilon+c^\varepsilon\chi^\varepsilon=f^\varepsilon & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon}\chi^\varepsilon+\beta^\varepsilon\chi^\varepsilon=\psi^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ \chi^\varepsilon|_{t=0}=\chi_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0). \end{cases} \tag{1.9}$$

By a standard argument based on the maximum principle (see, e.g., [31,33]) with a careful analysis of the dependence of coefficients and constants on ε, we show that a uniform a priori estimate similar to (1.8) holds for a classical solution χ^ε to (1.9) provided that a_{ij}^ε, b_i^ε, and c^ε are uniformly bounded in $Q_{\varepsilon,T}$ with respect to ε and $\beta^\varepsilon\geq -c\varepsilon$ on $\partial_\ell Q_{\varepsilon,T}$ with some constant $c>0$ independent of ε (see Theorem 3.1). Here we note that the condition on β^ε cannot be replaced by the uniform lower bound $\beta^\varepsilon\geq -c$ due to the thickness of $\Omega_\varepsilon(t)$ and the fact that the direction of ν_ε on the inner boundary of $\Omega_\varepsilon(t)$ is opposite to that of ν_ε on the outer boundary of $\Omega_\varepsilon(t)$ (see Remark 3.3). Moreover, the factor ε^{-1} appears in (1.8) because of the condition on β^ε, and we cannot remove it since we have a contradiction if (1.8) holds without ε^{-1} (see Remark 4.1). In order to apply the result of Theorem 3.1 to (1.7), we also need to deal with the outer normal velocity V_ε of $\partial\Omega_\varepsilon(t)$. When ε is sufficiently small, V_ε is close to $-V_\Gamma$ on the inner boundary of $\Omega_\varepsilon(t)$ and to V_Γ on the outer boundary of $\Omega_\varepsilon(t)$, where V_Γ is the outer normal velocity of $\Gamma(t)$ (see Lemma 2.9). Hence, V_ε is of order one with respect to ε and can take both the positive and negative values on the whole boundary $\partial\Omega_\varepsilon(t)$, and thus V_ε does not satisfy the condition on β^ε in general. To overcome this difficulty, we eliminate the zeroth-order term $\pm V_\Gamma$ of V_ε by multiplying $\rho^\varepsilon - \rho^\varepsilon_\eta$ by the exponential of a suitable function involving V_Γ and the signed distance from $\Gamma(t)$. Let us also mention that we do not scale the thickness of $\Omega_\varepsilon(t)$ and use local coordinates of $\Gamma(t)$ in the proofs of the uniform a priori estimates. Our proofs avoid complicated calculations associated with the change of variables for the curved thin domain $\Omega_\varepsilon(t)$ with complicated geometry, so they seem to be readable and easy to understand.

Another task is to find a suitable approximate solution ρ^ε_η to (1.1) close to η. In order to derive (1.5) and (1.6) from (1.8), we need to choose ρ^ε_η so that the error terms f^ε_η and ψ^ε_η are of order ε and ε^2, respectively, that is, ρ^ε_η should satisfy (1.1) approximately of order ε in $Q_{\varepsilon,T}$ and ε^2 on $\partial_\ell Q_{\varepsilon,T}$. It turns out that η, the classical solution to (1.2), does not satisfy (1.1) approximately itself, due to the geometry and motion of $\partial\Omega_\varepsilon(t)$. Hence, we seek a suitable approximate solution by taking a formal asymptotic expansion

$$\rho^\varepsilon(y+\varepsilon z\nu(y,t),t)=\sum_{k=0}^{\infty}\varepsilon^k\eta_k(y,t,z),\quad (y,t)\in\overline{S_T},\ z\in[g_0(y,t),g_1(y,t)],$$

or equivalently,

$$\rho^\varepsilon(x,t)=\sum_{k=0}^{\infty}\varepsilon^k\eta_k(\pi(x,t),t,\varepsilon^{-1}d(x,t)),\quad (x,t)\in\overline{Q_{\varepsilon,T}},$$

where $y=\pi(x,t)$ is the closest point of x on $\Gamma(t)$, $d(x,t)$ is the signed distance of x from $\Gamma(t)$ so that $z=\varepsilon^{-1}d(x,t)$ is the scaled signed distance from $\Gamma(t)$, and the functions

$$\eta_k(y,t,z),\quad (y,t)\in\overline{S_T},\ z\in[g_0(y,t),g_1(y,t)],\quad k=0,1,\dots$$

are independent of ε. In Section 5, we substitute the right-hand side of the above expansion for (1.1) and determine η_k for which (1.1) is satisfied of order ε in $Q_{\varepsilon,T}$ and ε^2 on $\partial_\ell Q_{\varepsilon,T}$. Then, we find that $\eta_0=\eta$ should be a solution to (1.2) and it is enough to determine η_1 and η_2 for our purpose, but η_2 involves $\nabla_\Gamma g_i\cdot\nabla_\Gamma\eta_0$, $i=0,1$ in order to make the boundary condition of (1.1) satisfied of order ε^2. This is the reason why we require additional regularity assumption (b') on η in Theorem 1.2. We also note that limit equation (1.2) appears as a necessary condition on η_0 for the existence of η_2.

The actual proofs of Theorems 1.1 and 1.2 are given in Section 4 before carrying out formal calculations. In that section, we use the functions η_1 and η_2 to define the approximate solution ρ_η^ε by

$$\rho_\eta^\varepsilon(x,t) = \bar{\eta}(x,t) + \sum_{k=1,2} \varepsilon^k \eta_k(\pi(x,t), t, \varepsilon^{-1} d(x,t)), \quad (x,t) \in \overline{Q_{\varepsilon,T}}.$$

This ρ_η^ε is expressed as the sum of functions of the form

$$(d^k \bar{\zeta})(x,t) = d(x,t)^k \bar{\zeta}(x,t), \quad (x,t) \in \overline{Q_{\varepsilon,T}},\ k = 0,1,2,$$

where ζ is a function on $\overline{S_T}$ and $\bar{\zeta}$ is its constant extension in the normal direction of $\Gamma(t)$. Then, we express the derivatives of $d^k\bar{\zeta}$ approximately in terms of functions on $\overline{S_T}$ by using lemmas in Section 2 and apply the resulting expressions, the explicit forms of η_1 and η_2, and the fact that η satisfies (1.2) to show that ρ_η^ε indeed satisfies (1.1) approximately of order ε in $Q_{\varepsilon,T}$ and ε^2 on $\partial_\ell Q_{\varepsilon,T}$. Here we again note that the proofs of Theorems 1.1 and 1.2 avoid the scaling of the thickness of $\Omega_\varepsilon(t)$ and the use of local coordinates of $\Gamma(t)$.

1.3. Literature overview

Partial differential equations (PDEs) in thin domains appear in various fields like engineering, biology, and fluid mechanics. Many authors have studied PDEs (especially reaction-diffusion and the Navier–Stokes equations) in flat thin domains around lower-dimensional domains since the pioneering works by Hale and Raugel [22,23]. Moreover, there are several works on a reaction-diffusion equation and its stationary problem in a thin L-shaped domain [24], in a flat thin domain with holes [49], in thin tubes around curves or networks [28, 29, 57], and in curved thin domains around lower-dimensional manifolds [48, 50]. Curved thin domains around (hyper)surfaces also appears in the study of the Navier–Stokes equations [37–39, 54] and of the asymptotic behavior of the eigenvalues of the Laplacian [26,30,53,56]. We also refer to [51] for various examples of thin domains.

In the above cited papers, thin domains and their limit sets are stationary in time. Pereira and Silva [47] first studied a reaction-diffusion equation in a moving thin domain which has a moving boundary but whose limit set is a stationary domain. Elliott and Stinner [19] considered a moving thin domain around a moving surface in order to approximate a given advection-diffusion equation on a moving surface by a diffuse interface model (see also [20] for a numerical computation of the diffuse interface model). In [35], the present author first rigorously derived an unknown limit equation on a moving surface from a given equation in a moving thin domain by the average method in the case of the heat equation. Also, fluid and non-linear diffusion equations on moving surfaces are formally derived by the thin-film limit in [36,40], but the justification has not been done yet because of difficulties coming from the non-linearity of the equations.

PDEs on moving surfaces have also attracted interest of many researchers recently in view of applications. There are many works on the mathematical and numerical analysis of linear diffusion equations on moving surfaces like (1.2) (see, e.g., [7, 13–16, 44, 55] and the references cited therein). Other equations on moving surfaces were also studied, such as a Stefan problem [4], a porous medium equation [5], the Cahn–Hilliard equation [9, 18, 43], and the Hamilton–Jacobi equation [11]. The authors of [58] formulated equations of non-linear elasticity in an evolving ambient space like a moving surface. Additionally, the Stokes and Navier–Stokes equations on moving surfaces were proposed in [8] and [25, 27], respectively. These fluid equations are coupled systems of the motion of a surface described by the normal velocity and the evolution of a tangential fluid velocity. In [45, 52], numerical methods for the Navier–Stokes equations on a moving surface were proposed. Also, when the motion of a surface is given, the wellposedness of the tangential Navier–Stokes equations on a moving surface was shown in [45]. Some models of liquid crystals on moving surfaces were proposed and numerically computed in [41, 42]. In [1, 2, 12, 17], a coupled system of a mean curvature flow for a surface and a diffusion equation on the surface was studied. We also refer to [3, 6] for abstract frameworks for PDEs in evolving function spaces.

1.4. Organization of this paper

The rest of this paper is organized as follows: We fix notations and give basic results on surfaces and thin domains in Section 2. In Section 3 we show a uniform a priori estimate for a classical solution to the heat equation in the moving thin domain. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2. In Section 5 we explain a formal derivation of limit equation (1.2) and a suitable approximate solution used in the proofs of Theorems 1.1 and 1.2 based on an asymptotic expansion of the thin domain problem (see (1.1)). A short conclusion is given in Section 6.

2. Preliminaries

In this section we fix notations and give basic results on surfaces and thin domains. We assume that functions and surfaces appearing below are sufficiently smooth. Also, we consider a vector in $\mathbb{R}^n$, $n \geq 2$ as a column vector, and for a square matrix A we denote by A^T the transpose of A and by $|A| = \sqrt{\mathrm{tr}[A^T A]}$ the Frobenius norm of A.

2.1. Fixed surface

Let Γ be a closed (i.e., compact and without boundary), connected, and oriented smooth hypersurface in $\mathbb{R}^n$, $n \geq 2$. We assume that Γ is the boundary of a bounded domain Ω in $\mathbb{R}^n$ and denote by ν the unit outward normal vector field of Γ which points from Ω into $\mathbb{R}^n \setminus \Omega$. Let d be the signed distance function from Γ increasing in the direction of ν. Also, we write $\kappa_1, \dots, \kappa_{n-1}$ for the principal curvatures of Γ. Then, ν and $\kappa_1, \dots, \kappa_{n-1}$

are smooth and thus bounded on Γ by the smoothness of Γ. Hence, we may take a tubular neighborhood $\overline{N} = \{x \in \mathbb{R}^n \mid -\delta \leq d(x) \leq \delta\}$ of Γ with $\delta > 0$ such that for each $x \in \overline{N}$, there exists a unique $\pi(x) \in \Gamma$ satisfying

$$x = \pi(x) + d(x)\nu(\pi(x)), \quad \nabla d(x) = \nabla d(\pi(x)) = \nu(\pi(x)) \tag{2.1}$$

and that d and π are smooth on $\overline{N}$ (see, e.g., [21, Section 14.6]). Moreover, taking $\delta > 0$ sufficiently small, we may assume that

$$c_0^{-1} \leq 1 - r\kappa_\alpha(y) \leq c_0, \quad y \in \Gamma,\ r \in [-\delta, \delta],\ \alpha = 1, \dots, n-1 \tag{2.2}$$

with some constant $c_0 > 0$.

Let I_n and $\nu \otimes \nu$ be the $n \times n$ identity matrix and the tensor product of ν with itself. We set $P = (P_{ij})_{i,j} = I_n - \nu \otimes \nu$ on Γ, which is the orthogonal projection onto the tangent plane of Γ. For a function η on Γ, we define the tangential gradient and the tangential derivatives of η by

$$\nabla_\Gamma \eta(y) = P(y)\nabla\tilde{\eta}(y), \quad \underline{D}_i \eta(y) = \sum_{j=1}^{n} P_{ij}(y)\partial_j\tilde{\eta}(y), \quad y \in \Gamma,\ i = 1, \dots, n,$$

so that $\nabla_\Gamma \eta = (\underline{D}_1\eta, \dots, \underline{D}_n\eta)^T$, where $\tilde{\eta}$ is an extension of η to $\overline{N}$. Then,

$$\nu \cdot \nabla_\Gamma \eta = 0, \quad P\nabla_\Gamma \eta = \nabla_\Gamma \eta \quad \text{on} \quad \Gamma \tag{2.3}$$

and the values of $\nabla_\Gamma \eta$ and $\underline{D}_i\eta$ are independent of the choice of $\tilde{\eta}$. In particular, if we take the constant extension $\overline{\eta} = \eta \circ \pi$ of η in the normal direction of Γ, then we have

$$\nabla\overline{\eta}(y) = \nabla_\Gamma \eta(y), \quad \partial_i\overline{\eta}(y) = \underline{D}_i\eta(y), \quad y \in \Gamma,\ i = 1, \dots, n \tag{2.4}$$

by (2.1) and $d(y) = 0$ for $y \in \Gamma$. In what follows, a function with an "overline" (e.g., $\overline{f}$) always stands for the constant extension of a function on Γ in the normal direction of Γ. We set

$$\Delta_\Gamma \eta = \sum_{i=1}^{n} \underline{D}_i\underline{D}_i\eta, \quad |\nabla_\Gamma^2 \eta| = \left(\sum_{i,j=1}^{n} |\underline{D}_i\underline{D}_j\eta|^2\right)^{1/2} \quad \text{on} \quad \Gamma$$

and call Δ_Γ the Laplace–Beltrami operator on Γ. For a (not necessarily tangential) vector field u on Γ, we define the surface divergence of u by

$$\mathrm{div}_\Gamma u = \sum_{i=1}^{n} \underline{D}_i u_i \quad \text{on} \quad \Gamma,$$

where $u = (u_1, \dots, u_n)^T$. Also, we define

$$W_{ij} = -\underline{D}_i\nu_j, \quad H = -\mathrm{div}_\Gamma \nu \quad \text{on} \quad \Gamma, \quad i, j = 1, \dots, n$$

and call $W = (W_{ij})_{i,j}$ and H the Weingarten map (or the shape operator) and the mean curvature of Γ, respectively. The matrix W is symmetric since $W = -\nabla^2 d$ on Γ, by (2.1) and (2.4). Moreover, $W\nu = -\nabla_\Gamma(|\nu|^2/2) = 0$ by $|\nu| = 1$ on Γ, and thus W has the eigenvalue zero and $WP = PW = W$ on Γ. It is also known (see, e.g., [32]) that the other eigenvalues of W are the principal curvatures $\kappa_1, \dots, \kappa_{n-1}$. Hence,

$$H = \mathrm{tr}[W] = \sum_{\alpha=1}^{n-1} \kappa_\alpha, \quad |W|^2 = \mathrm{tr}[W^2] = \sum_{\alpha=1}^{n-1} \kappa_\alpha^2 \quad \text{on} \quad \Gamma.$$

In particular, we have

$$\begin{aligned} |H(y)| &\le (n-1) \max_{\alpha=1,\dots,n-1} \max_{z\in\Gamma} |\kappa_\alpha(z)|, \\ |W(y)| &\le (n-1)^{1/2} \max_{\alpha=1,\dots,n-1} \max_{z\in\Gamma} |\kappa_\alpha(z)| \end{aligned} \tag{2.5}$$

for all $y \in \Gamma$. Also, it follows from (2.2) that the matrix

$$I_n - d(x)\overline{W}(x) = I_n - rW(y)$$

is invertible for all $x = y + r\nu(y) \in \overline{N}$ with $y \in \Gamma$ and $r \in [-\delta, \delta]$. We set

$$R(x) = (R_{ij}(x))_{i,j} = \left\{I_n - d(x)\overline{W}(x)\right\}^{-1}, \quad x \in \overline{N}. \tag{2.6}$$

Let us give a few lemmas related to the derivatives of the constant extension of a function on Γ. In Lemmas 2.1–2.4 below, we denote by c a general positive constant depending only on n, δ, the constant c_0 appearing in (2.2), and the quantities

$$\max_{\alpha=1,\dots,n-1} \max_{y\in\Gamma} |\kappa_\alpha(y)|, \qquad \max_{i,j,k=1,\dots,n} \max_{y\in\Gamma} |\underline{D}_i W_{jk}(y)|.$$

Lemma 2.1. *For all $x \in \overline{N}$, we have*

$$|R(x)| \le c, \quad |I_n - R(x)| \le c|d(x)|. \tag{2.7}$$

Proof. For $x \in \overline{N}$, let $y = \pi(x) \in \Gamma$ and $r = d(x) \in [-\delta, \delta]$ so that

$$R(x) = \left\{I_n - rW(y)\right\}^{-1}.$$

Since $W(y)$ is symmetric and has the eigenvalues $\kappa_1(y), \dots, \kappa_{n-1}(y)$ and zero, we can take an orthonormal basis of $\mathbb{R}^n$ consisting of the corresponding eigenvectors of $W(y)$. Using this, we easily find that

$$|R(x)|^2 = \sum_{\alpha=1}^{n-1} \left\{1 - r\kappa_\alpha(y)\right\}^{-2} + 1, \quad |I_n - R(x)|^2 = \sum_{\alpha=1}^{n-1} \left(\frac{r\kappa_\alpha(y)}{1 - r\kappa_\alpha(y)}\right)^2.$$

Hence, we obtain (2.7) by (2.2) and $r = d(x)$. ∎

Lemma 2.2. *Let η be a function on Γ and $\overline{\eta} = \eta \circ \pi$ be its constant extension. Then,*

$$\nabla\overline{\eta}(x) = R(x)\overline{\nabla_\Gamma \eta}(x), \quad \overline{\nu}(x) \cdot \nabla\overline{\eta}(x) = 0 \tag{2.8}$$

for all $x \in \overline{N}$. Moreover,

$$|\nabla\overline{\eta}(x)| \le c|\overline{\nabla_\Gamma \eta}(x)|, \tag{2.9}$$

$$|\nabla\overline{\eta}(x) - \overline{\nabla_\Gamma \eta}(x)| \le c|d(x)||\overline{\nabla_\Gamma \eta}(x)|. \tag{2.10}$$

Proof. For the mapping $\pi = (\pi_1, \dots, \pi_n)$ given in (2.1), we write

$$\nabla\pi = \begin{pmatrix} \partial_1\pi_1 & \cdots & \partial_1\pi_n \\ \vdots & \ddots & \vdots \\ \partial_n\pi_1 & \cdots & \partial_n\pi_n \end{pmatrix}.$$

Since $\pi(x) = x - d(x)\overline{\nu}(\pi(x))$ by (2.1), we see by (2.4) with $y = \pi(x)$ that

$$\nabla\pi(x)\{I_n - d(x)\overline{W}(x)\} = \overline{P}(x), \quad \nabla\pi(x) = \overline{P}(x)R(x) = R(x)\overline{P}(x),$$

where the last equality is due to $PW = WP$ on Γ. We differentiate $\overline{\eta}(x) = \overline{\eta}(\pi(x))$ and use the above equality, (2.3), and (2.4) to get the first equality of (2.8). Also, the second equality holds since $\overline{\eta}$ is the constant extension of η in the direction of $\overline{\nu}$. Inequalities (2.9) and (2.10) follow from (2.7) and (2.8). ■

Lemma 2.3. *For all $x \in \overline{N}$ and $i = 1, \dots, n$, we have*

$$|\partial_i R(x) - \overline{\nu}_i(x)\overline{W}(x)| \le c|d(x)|. \tag{2.11}$$

Also, let η be a function on Γ and $\overline{\eta} = \eta \circ \pi$ be its constant extension. Then,

$$|\nabla^2\overline{\eta}(x)| \le c(|\overline{\nabla_\Gamma \eta}(x)| + |\overline{\nabla^2_\Gamma \eta}(x)|) \tag{2.12}$$

for all $x \in \overline{N}$, where $\nabla^2\overline{\eta} = (\partial_i\partial_j\overline{\eta})_{i,j}$, and

$$\left|\partial_i\partial_j\overline{\eta}(x) - \overline{\underline{D}_i\underline{D}_j\eta}(x) - \overline{\nu}_i(x)\sum_{k=1}^n \overline{W}_{jk}(x)\overline{\underline{D}_k\eta}(x)\right| \le c|d(x)|(|\overline{\nabla_\Gamma \eta}(x)| + |\overline{\nabla^2_\Gamma \eta}(x)|) \tag{2.13}$$

for all $x \in \overline{N}$ and $i, j = 1, \dots, n$.

Proof. We apply ∂_i to

$$R(x)\{I_n - d(x)\overline{W}(x)\} = I_n, \quad x \in \overline{N}$$

and use $\nabla d = \overline{\nu}$ in $\overline{N}$ to find that

$$\partial_i R(x) = \overline{\nu}_i(x)R(x)\overline{W}(x)R(x) + d(x)R(x)\partial_i\overline{W}(x)R(x).$$

By this equality, (2.5), (2.7), (2.9), and the boundedness of $\underline{D}_i W_{jk}$ on Γ, we obtain (2.11). Also, for a function η on Γ, it follows from (2.8) that

$$\partial_j \overline{\eta}(x) = \sum_{k=1}^{n} R_{jk}(x) \overline{\underline{D}_k \eta}(x), \quad x \in \overline{N}.$$

We apply ∂_i to both sides and use (2.8) with η replaced by $\underline{D}_k \eta$ to get

$$\partial_i \partial_j \overline{\eta}(x) = \sum_{k=1}^{n} \partial_i R_{jk}(x) \overline{\underline{D}_k \eta}(x) + \sum_{k,l=1}^{n} R_{jk}(x) R_{il}(x) \overline{\underline{D}_l \underline{D}_k \eta}(x).$$

Hence, we have (2.12) and (2.13) by this equality, (2.7), and (2.11). ∎

The next lemma plays a fundamental role in the proofs of Theorems 1.1 and 1.2.

Lemma 2.4. *Let η be a function on Γ and $\overline{\eta} = \eta \circ \pi$ be its constant extension. Then,*

$$|\Delta \overline{\eta}(x) - \overline{\Delta_\Gamma \eta}(x)| \le c|d(x)|(|\overline{\nabla_\Gamma \eta}(x)| + |\overline{\nabla_\Gamma^2 \eta}(x)|), \tag{2.14}$$

$$|\Delta(d\overline{\eta})(x) + \overline{H}(x)\overline{\eta}(x)| \le c|d(x)|(|\overline{\eta}(x)| + |\overline{\nabla_\Gamma \eta}(x)| + |\overline{\nabla_\Gamma^2 \eta}(x)|), \tag{2.15}$$

$$|\Delta(d^2\overline{\eta})(x) - 2\overline{\eta}(x)| \le c|d(x)|(|\overline{\eta}(x)| + |\overline{\nabla_\Gamma \eta}(x)| + |\overline{\nabla_\Gamma^2 \eta}(x)|) \tag{2.16}$$

for all $x \in \overline{N}$. Here we write $(d^k \overline{\eta})(x) = d(x)^k \overline{\eta}(x)$ for $x \in \overline{N}$ and $k = 1, 2$.

Proof. Inequality (2.14) follows from (2.13) and

$$\sum_{i,k=1}^{n} \nu_i W_{ik} \underline{D}_k \eta = (W\nu) \cdot \nabla_\Gamma \eta = 0 \quad \text{on} \quad \Gamma$$

by the symmetry of W and $W\nu = 0$ on Γ. Next we see that

$$\Delta(d\overline{\eta}) = (\operatorname{div} \overline{\nu})\overline{\eta} + 2\overline{\nu} \cdot \nabla \overline{\eta} + d\Delta \overline{\eta} = (\operatorname{div} \overline{\nu})\overline{\eta} + d\Delta \overline{\eta} \quad \text{in} \quad \overline{N}$$

by $\nabla d = \overline{\nu}$ in $\overline{N}$ and (2.8). Moreover,

$$|\operatorname{div} \overline{\nu} + \overline{H}| = |\operatorname{div} \overline{\nu} - \overline{\operatorname{div}_\Gamma \nu}| \le c|d||\overline{W}| \le c|d| \quad \text{in} \quad \overline{N} \tag{2.17}$$

by $H = -\operatorname{div}_\Gamma \nu$ on Γ, (2.10), $\underline{D}_i \nu_j = -W_{ij}$ on Γ, and (2.5). By these relations and (2.12), we obtain (2.15). We also observe that

$$\Delta(d^2\overline{\eta}) = 2|\overline{\nu}|^2\overline{\eta} + 2d\{(\operatorname{div} \overline{\nu})\overline{\eta} + 2\overline{\nu} \cdot \nabla \overline{\eta}\} + d^2 \Delta \overline{\eta} = 2\overline{\eta} + 2d(\operatorname{div} \overline{\nu})\overline{\eta} + d^2 \Delta \overline{\eta}$$

in $\overline{N}$ by $\nabla d = \overline{\nu}$ in $\overline{N}$, $|\nu| = 1$ on Γ, and (2.8). We apply (2.17) to the above equality and then use (2.5), (2.12), and $|d| \le \delta$ in $\overline{N}$ to get (2.16). ∎

2.2. Moving surface

For each $t \in [0,T]$, $T>0$, let $\Gamma(t)$ be a given closed, connected, and oriented smooth hypersurface in $\mathbb{R}^n$. As in Section 2.1, we assume that $\Gamma(t)$ is the boundary of a bounded domain $\Omega(t)$ in $\mathbb{R}^n$ and denote by $\nu(\cdot,t)$ the unit outward normal vector field of $\Gamma(t)$ which points from $\Omega(t)$ into $\mathbb{R}^n \setminus \Omega(t)$. Moreover, we set $\Gamma_0 = \Gamma(0)$ and

$$S_T = \bigcup_{t\in(0,T]} \Gamma(t)\times\{t\}, \quad \overline{S_T} = (\Gamma_0\times\{0\})\cup S_T,$$

and use the same notations as in Section 2.1.

We assume that there exists a smooth mapping $\Phi\colon \Gamma_0\times[0,T]\to\mathbb{R}^n$ such that $\Phi(\cdot,t)$ is a diffeomorphism from Γ_0 onto $\Gamma(t)$ for each $t\in[0,T]$ with $\Phi(\cdot,0)=\mathrm{Id}$. Then, the unit outward normal vector field ν and the principal curvatures $\kappa_1,\dots,\kappa_{n-1}$ of $\Gamma(t)$ are smooth and thus bounded as functions on $\overline{S_T}$. Hence, we can take a constant $\delta>0$ independent of t and a tubular neighborhood

$$\overline{N(t)} = \left\{x\in\mathbb{R}^n \mid -\delta \le d(x,t) \le \delta\right\} \tag{2.18}$$

of $\Gamma(t)$ such that for each $x\in\overline{N(t)}$, there exists a unique $\pi(x,t)\in\Gamma(t)$ satisfying (2.1) and that d and π are smooth on $\overline{N_T}$, where

$$\overline{N_T} = \bigcup_{t\in[0,T]} \overline{N(t)}\times\{t\}.$$

Moreover, (2.2) holds for $(y,t)\in\overline{S_T}$ instead of $y\in\Gamma$ with a constant c_0 independent of t, and the functions $\underline{D}_i W_{jk}$ are bounded on $\overline{S_T}$. Therefore, we can apply Lemmas 2.1–2.4 on $\Gamma(t)$ for all $t\in[0,T]$ with a constant c independent of t.

We define the (scalar) outer normal velocity of $\Gamma(t)$ by

$$V_\Gamma(\Phi(Y,t),t) = \partial_t\Phi(Y,t)\cdot\nu(\Phi(Y,t),t), \quad (Y,t)\in\Gamma_0\times[0,T].$$

Then, V_Γ is smooth on $\overline{S_T}$. Note that the evolution of $\Gamma(t)$ (as a subset of $\mathbb{R}^n$) is determined by the normal velocity field $V_\Gamma\nu$; in other words, if we define a mapping $\Phi_\nu\colon\Gamma_0\times[0,T]\to\mathbb{R}^n$ by

$$\Phi_\nu(Y,0)=Y, \quad \partial_t\Phi_\nu(Y,t) = (V_\Gamma\nu)(\Phi_\nu(Y,t),t), \quad (Y,t)\in\Gamma_0\times[0,T], \tag{2.19}$$

then $\Phi_\nu(\cdot,t)$ is a diffeomorphism from Γ_0 onto $\Gamma(t)$ for each $t\in[0,T]$. Moreover,

$$\partial_t d(y,t) = -V_\Gamma(y,t), \quad (y,t)\in\overline{S_T}, \tag{2.20}$$

since the signed distance d from $\Gamma(t)$ increases in the direction of ν. We can also compute the time derivatives of d and π in $\overline{N_T}$ as follows:

Lemma 2.5. *For all* $(x,t) \in \overline{N_T}$*, we have*

$$\partial_t d(x,t) = -\overline{V_\Gamma}(x,t), \tag{2.21}$$

$$\partial_t \pi(x,t) = \overline{V_\Gamma}(x,t)\overline{\nu}(x,t) + d(x,t)R(x,t)\overline{\nabla_\Gamma V_\Gamma}(x,t), \tag{2.22}$$

where R *is the matrix given by* (2.6)*.*

Proof. Since $d(\pi(x,t),t) = 0$ by $\pi(x,t) \in \Gamma(t)$, we have

$$\partial_t d(\pi(x,t),t) + \partial_t \pi(x,t) \cdot \nabla d(\pi(x,t),t) = 0.$$

Also, since $\pi(x,t) = x - d(x,t)\nabla d(x,t)$ by (2.1), it follows that

$$\partial_t \pi(x,t) = -\partial_t d(x,t)\nabla d(x,t) - d(x,t)\partial_t \nabla d(x,t).$$

Then, we see by $\nabla d(\pi(x,t),t) = \nabla d(x,t)$ and $|\nabla d(x,t)|^2 = 1$ that

$$\begin{aligned}\partial_t \pi(x,t) \cdot \nabla d(\pi(x,t),t) &= -\partial_t d(x,t)|\nabla d(x,t)|^2 - \frac{1}{2}d(x,t)\partial_t(|\nabla d(x,t)|^2) \\ &= -\partial_t d(x,t).\end{aligned}$$

We deduce from the above equalities and (2.20) with $y = \pi(x,t) \in \Gamma(t)$ that

$$\partial_t d(x,t) = \partial_t d(\pi(x,t),t) = -V_\Gamma(\pi(x,t),t) = -\overline{V}_\Gamma(x,t).$$

Hence, (2.21) holds. Moreover, we see by this equality and (2.8) that

$$\partial_t \nabla d(x,t) = \nabla \partial_t d(x,t) = -\nabla(\overline{V}_\Gamma(x,t)) = -R(x,t)\overline{\nabla_\Gamma V_\Gamma}(x,t).$$

Hence, (2.22) follows from the above equalities and (2.1). ∎

Let η be a function on $\overline{S_T}$. We define the normal time derivative of η by

$$\partial^\circ \eta(\Phi_\nu(Y,t),t) = \frac{\partial}{\partial t}(\eta(\Phi_\nu(Y,t),t)), \quad (Y,t) \in \Gamma_0 \times [0,T], \tag{2.23}$$

where Φ_ν is the mapping given by (2.19). Note that

$$\partial^\circ \eta(y,t) = \partial_t \tilde{\eta}(y,t) + (V_\Gamma \nu)(y,t) \cdot \nabla \tilde{\eta}(y,t), \quad (y,t) \in \overline{S_T}$$

for any extension $\tilde{\eta}$ of η to $\overline{N_T}$. In particular, setting $\overline{\eta}(x,t) = \eta(\pi(x,t),t)$ for $(x,t) \in \overline{N_T}$, which is the constant extension of η in the normal direction of $\Gamma(t)$, we have

$$\partial_t \overline{\eta}(y,t) = \partial^\circ \eta(y,t), \quad (y,t) \in \overline{S_T}, \tag{2.24}$$

by $\nu \cdot \nabla \overline{\eta} = \nu \cdot \nabla_\Gamma \eta = 0$ on $\overline{S_T}$ (see (2.3) and (2.4)).

Let us give auxiliary results related to the normal time derivative.

Lemma 2.6. *For all* $(y,t) \in \overline{S_T}$*, we have*

$$\partial^\circ \nu(y,t) = -\nabla_\Gamma V_\Gamma(y,t). \tag{2.25}$$

Proof. Since $\partial_t \overline{\nu} = \partial_t \nabla d = \nabla \partial_t d$ in $\overline{N_T}$ by (2.1), we apply (2.21) to this equality, set $x = y \in \Gamma(t)$, and use (2.4) and (2.24) to get (2.25). ∎

Lemma 2.7. *Let* η *be a function on* $\overline{S_T}$ *and* $\overline{\eta}$ *be its constant extension. Then,*

$$\partial_t \overline{\eta}(x,t) = \overline{\partial^\circ \eta}(x,t) + d(x,t) R(x,t) \overline{\nabla_\Gamma V_\Gamma}(x,t) \cdot \overline{\nabla_\Gamma \eta}(x,t) \tag{2.26}$$

for all $(x,t) \in \overline{N_T}$*. Moreover,*

$$|\partial_t \overline{\eta}(x,t)| \leq c(|\overline{\partial^\circ \eta}(x,t)| + |\overline{\nabla_\Gamma \eta}(x,t)|), \tag{2.27}$$

$$|\partial_t \overline{\eta}(x,t) - \overline{\partial^\circ \eta}(x,t)| \leq c|d(x)||\overline{\nabla_\Gamma \eta}(x)|. \tag{2.28}$$

Proof. We differentiate $\overline{\eta}(x,t) = \overline{\eta}(\pi(x,t),t)$ with respect to t and use (2.4) and (2.24) with $y = \pi(x,t) \in \Gamma(t)$ to find that

$$\begin{aligned}
\partial_t \overline{\eta}(x,t) &= \partial_t \overline{\eta}(\pi(x,t),t) + \partial_t \pi(x,t) \cdot \nabla \overline{\eta}(\pi(x,t),t) \\
&= \partial^\circ \eta(\pi(x,t),t) + \partial_t \pi(x,t) \cdot \nabla_\Gamma \eta(\pi(x,t),t) \\
&= \overline{\partial^\circ \eta}(x,t) + \partial_t \pi(x,t) \cdot \overline{\nabla_\Gamma \eta}(x,t).
\end{aligned}$$

Then, we apply (2.22) and use (2.3) to get (2.26). Also, (2.27) and (2.28) follow from inequalities (2.7) and (2.26) and the smoothness of V_Γ on $\overline{S_T}$. ∎

2.3. Moving thin domain

Let g_0 and g_1 be smooth functions on $\overline{S_T}$. We set $g = g_1 - g_0$ on $\overline{S_T}$ and assume that there exists a constant $c > 0$ such that

$$g(y,t) \geq c \quad \text{for all} \quad (y,t) \in \overline{S_T}. \tag{2.29}$$

For $\varepsilon > 0$, we define a moving thin domain $\Omega_\varepsilon(t)$ by

$$\Omega_\varepsilon(t) = \{y + r\nu(y,t) \mid y \in \Gamma(t),\ \varepsilon g_0(y,t) < r < \varepsilon g_1(y,t)\}, \quad t \in [0,T]$$

and its inner and outer boundaries $\Gamma_\varepsilon^0(t)$ and $\Gamma_\varepsilon^1(t)$ by

$$\Gamma_\varepsilon^i(t) = \{y + \varepsilon g_i(y,t)\nu(y,t) \mid y \in \Gamma(t)\}, \quad t \in [0,T],\ i = 0,1. \tag{2.30}$$

We denote by $\partial\Omega_\varepsilon(t) = \Gamma_\varepsilon^0(t) \cup \Gamma_\varepsilon^1(t)$ the whole boundary of $\Omega_\varepsilon(t)$ and set

$$Q_{\varepsilon,T} = \bigcup_{t \in (0,T]} \Omega_\varepsilon(t) \times \{t\}, \quad \partial_\ell Q_{\varepsilon,T} = \bigcup_{t \in (0,T]} \partial\Omega_\varepsilon(t) \times \{t\},$$

$$\overline{Q_{\varepsilon,T}} = \bigcup_{t \in [0,T]} \overline{\Omega_\varepsilon(t)} \times \{t\} = (\overline{\Omega_\varepsilon(0)} \times \{0\}) \cup Q_{\varepsilon,T} \cup \partial_\ell Q_{\varepsilon,T}.$$

Since g_0 and g_1 are smooth and thus bounded on $\overline{S_T}$, we may take a sufficiently small $\varepsilon_0 \in (0,1)$ so that $\varepsilon|g_i| \leq \delta$ on $\overline{S_T}$ for all $\varepsilon \in (0,\varepsilon_0)$ and $i = 0,1$, where $\delta > 0$ is the time-independent constant appearing in (2.18). Hence, for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T]$, we have $\overline{\Omega_\varepsilon(t)} \subset \overline{N(t)}$ and we can use the lemmas in the previous subsections on $\overline{\Omega_\varepsilon(t)}$ with constants independent of t and ε. In what follows, we assume $\varepsilon \in (0,\varepsilon_0)$ and denote by c a general positive constant independent of t and ε (but possibly depending on T). Also, we frequently use the inequality $0 < \varepsilon < 1$ in what follows without mention.

Let ν_ε and V_ε be the unit outward normal vector field and the (scalar) outer normal velocity of $\partial\Omega_\varepsilon(t)$, respectively. We express them in terms of functions on $\Gamma(t)$ below. Let

$$\tau_\varepsilon^i(y,t) = \left\{I_n - \varepsilon g_i(y,t)W(y,t)\right\}^{-1}\nabla_\Gamma g_i(y,t), \quad (y,t) \in \overline{S_T},\ i = 0,1. \tag{2.31}$$

Note that, since W is symmetric and $W\nu = 0$ on $\overline{S_T}$,

$$\tau_\varepsilon^i \cdot \nu = \nabla_\Gamma g_i \cdot \left\{(I_n - \varepsilon g_i W)^{-1}\nu\right\} = \nabla_\Gamma g_i \cdot \nu = 0 \quad \text{on} \quad \overline{S_T}. \tag{2.32}$$

Also, by (2.7) with $x = y + \varepsilon g_i(y,t)\nu(y,t) \in \Gamma_\varepsilon^i(t)$ and the smoothness of g_i on $\overline{S_T}$,

$$|\tau_\varepsilon^i(y,t)| \leq c, \quad |\tau_\varepsilon^i(y,t) - \nabla_\Gamma g_i(y,t)| \leq c\varepsilon, \quad (y,t) \in \overline{S_T},\ i = 0,1 \tag{2.33}$$

with a constant $c > 0$ independent of ε. As in the previous subsections, we denote by $\overline{\eta}$ the constant extension of a function η on $\overline{S_T}$ in the normal direction of $\Gamma(t)$.

Lemma 2.8. *For $i = 0,1$, $t \in [0,T]$, and $x \in \Gamma_\varepsilon^i(t)$, we have*

$$\nu_\varepsilon(x,t) = \frac{(-1)^{i+1}}{\sqrt{1 + \varepsilon^2|\overline{\tau}_\varepsilon^i(x,t)|^2}}\left\{\overline{\nu}(x,t) - \varepsilon\overline{\tau}_\varepsilon^i(x,t)\right\}. \tag{2.34}$$

Proof. We refer to [39, Lemma 3.9] for the proof. Note that the paper [39] deals with a fixed surface in $\mathbb{R}^3$, but the proof given there is applicable to our case. ∎

Lemma 2.9. *For $i = 0,1$, $t \in [0,T]$, and $x \in \Gamma_\varepsilon^i(t)$, we have*

$$V_\varepsilon(x,t) = \frac{(-1)^{i+1}}{\sqrt{1 + \varepsilon^2|\overline{\tau}_\varepsilon^i(x,t)|^2}}\left(\overline{V_\Gamma} + \varepsilon\,\overline{\partial^\circ g_i} + \varepsilon^2\overline{g}_i\overline{\tau}_\varepsilon^i \cdot \overline{\nabla_\Gamma V_\Gamma}\right)(x,t). \tag{2.35}$$

Proof. Fix $i = 0,1$. For $X \in \Gamma_\varepsilon^i(0)$ and $t \in [0,T]$, we set $Y = \pi(X,0) \in \Gamma(0)$ and

$$\Phi_\varepsilon^i(X,t) = \Phi_\nu(Y,t) + \varepsilon g_i(\Phi_\nu(Y,t),t)\nu(\Phi_\nu(Y,t),t), \tag{2.36}$$

where Φ_ν is the mapping given by (2.19). Then, since $\Phi_\nu(\cdot,t)$ is a diffeomorphism from Γ_0 onto $\Gamma(t)$ for each $t \in [0,T]$, and since relation (2.1) holds for all $(x,t) \in \overline{N_T}$ and $\Gamma_\varepsilon^i(t)$ is given by (2.30), we observe that $\Phi_\varepsilon^i(\cdot,t)$ is a diffeomorphism from $\Gamma_\varepsilon^i(0)$ onto $\Gamma_\varepsilon^i(t)$ for each $t \in [0,T]$. Hence, the outer normal velocity of $\Gamma_\varepsilon^i(t)$ is given by

$$V_\varepsilon(\Phi_\varepsilon^i(X,t),t) = \nu_\varepsilon(\Phi_\varepsilon^i(X,t),t) \cdot \partial_t\Phi_\varepsilon^i(X,t), \quad (X,t) \in \Gamma_\varepsilon(0) \times [0,T].$$

Now let $x = \Phi_\varepsilon^i(X,t) \in \Gamma_\varepsilon^i(t)$ and $y = \pi(x,t) = \Phi_\nu(Y,t) \in \Gamma(t)$. We differentiate (2.36) with respect to t and use (2.19), (2.23), and (2.25) to get

$$\partial_t \Phi_\varepsilon^i(X,t) = (V_\Gamma \nu)(y,t) + \varepsilon\bigl\{\partial^\circ g_i(y,t)\nu(y,t) - g_i(y,t)\nabla_\Gamma V_\Gamma(y,t)\bigr\}. \tag{2.37}$$

Hence, noting that $\overline{\eta}(x,t) = \eta(y,t)$ for a function η on $\overline{S_T}$, we take the inner product of (2.34) and (2.37) and use $|\nu| = 1$ and $\tau_\varepsilon^i \cdot \nu = \nabla_\Gamma V_\Gamma \cdot \nu = 0$ on $\overline{S_T}$ to obtain (2.35). ■

3. Uniform a priori estimate for a solution to the thin domain problem

The purpose of this section is to show a uniform a priori estimate for a classical solution to the heat equation in the moving thin domain. To this end, we first consider an initial-boundary value problem of a parabolic equation

$$\begin{cases} \partial_t \chi^\varepsilon - \displaystyle\sum_{i,j=1}^n a_{ij}^\varepsilon \partial_i \partial_j \chi^\varepsilon + \sum_{i=1}^n b_i^\varepsilon \partial_i \chi^\varepsilon + c^\varepsilon \chi^\varepsilon = f^\varepsilon & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon} \chi^\varepsilon + \beta^\varepsilon \chi^\varepsilon = \psi^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ \chi^\varepsilon|_{t=0} = \chi_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0). \end{cases} \tag{3.1}$$

Here a_{ij}^ε, b_i^ε, c^ε, and f^ε are given functions defined on $Q_{\varepsilon,T}$, and β^ε and ψ^ε are given functions defined on $\partial_\ell Q_{\varepsilon,T}$. In addition, χ_0^ε is a given initial data defined on $\overline{\Omega_\varepsilon(0)}$. Also, we write $\partial_{\nu_\varepsilon} = \nu_\varepsilon \cdot \nabla$ for the outer normal derivative on $\partial_\ell Q_{\varepsilon,T}$. We assume that

$$\begin{aligned} a_{ij}^\varepsilon(x,t) = a_{ji}^\varepsilon(x,t) \quad &\text{for all} \quad (x,t) \in Q_{\varepsilon,T},\ i,j = 1,\dots,n, \\ \sum_{i,j=1}^n a_{ij}^\varepsilon(x,t)\xi_i\xi_j \geq 0 \quad &\text{for all} \quad (x,t) \in Q_{\varepsilon,T},\ \xi = (\xi_1,\dots,\xi_n)^T \in \mathbb{R}^n \end{aligned} \tag{3.2}$$

and say that χ^ε is a classical solution to (3.1) if

$$\chi^\varepsilon \in C(\overline{Q_{\varepsilon,T}}), \quad \partial_i \chi^\varepsilon \in C(Q_{\varepsilon,T} \cup \partial_\ell Q_{\varepsilon,T}), \quad \partial_t \chi^\varepsilon, \partial_i \partial_j \chi^\varepsilon \in C(Q_{\varepsilon,T})$$

for all $i,j = 1,\dots,n$ and χ^ε satisfies (3.1) at each point of $\overline{Q_{\varepsilon,T}}$.

Theorem 3.1. *Let $\varepsilon \in (0,\varepsilon_0)$ and χ^ε be a classical solution to* (3.1). *Suppose that the given data χ_0^ε, f^ε, and ψ^ε are bounded on their domains. Moreover, suppose that* (3.2) *holds and there exist constants $c_1, c_2 > 0$ independent of ε such that*

$$\max_{i,j=1,\dots,n} |a_{ij}^\varepsilon(x,t)| \leq c_1, \quad \max_{i=1,\dots,n} |b_i^\varepsilon(x,t)| \leq c_1, \quad |c^\varepsilon(x,t)| \leq c_1 \tag{3.3}$$

for all $(x,t) \in Q_{\varepsilon,T}$ and

$$\beta^\varepsilon(x,t) \geq -c_2\varepsilon \quad \textit{for all} \quad (x,t) \in \partial_\ell Q_{\varepsilon,T}. \tag{3.4}$$

Then, there exists a constant $c_T > 0$ depending on T but independent of ε, χ_0^ε, f^ε, ψ^ε, and χ^ε such that

$$\|\chi^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \leq c_T\Big(\|\chi_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \frac{1}{\varepsilon}\|\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big). \tag{3.5}$$

Note that we do not assume that the coefficients and given data in (3.1) are continuous on their domains (but $\chi_0^\varepsilon = \chi^\varepsilon|_{t=0}$ must be continuous on $\overline{\Omega_\varepsilon(0)}$, by the continuity of χ^ε). This is not important in this paper, but it may be useful for other problems.

To prove Theorem 3.1, we follow a standard argument of the proof of an a priori estimate based on the maximum principle (see, e.g., [31,33]), but it is necessary to analyze carefully the dependence of coefficients and constants on ε. We first introduce an auxiliary function and then give the proof of Theorem 3.1. Recall that we write $\overline{\eta}$ for the constant extension of a function η on $\overline{S_T}$ in the normal direction of $\Gamma(t)$.

Lemma 3.2. *For $(x,t) \in \overline{Q_{\varepsilon,T}}$, we define*

$$\sigma_\varepsilon(x,t) = \big\{d(x,t) - \varepsilon\overline{g}_0(x,t)\big\}\big\{d(x,t) - \varepsilon\overline{g}_1(x,t)\big\}. \tag{3.6}$$

Then, there exists a constant $c_3 > 0$ independent of ε such that

$$|\sigma_\varepsilon(x,t)| \leq c_3\varepsilon^2, \quad |\partial_t\sigma_\varepsilon(x,t)| \leq c_3\varepsilon, \quad |\nabla\sigma_\varepsilon(x,t)| \leq c_3\varepsilon, \quad |\nabla^2\sigma_\varepsilon(x,t)| \leq c_3 \tag{3.7}$$

for all $(x,t) \in \overline{Q_{\varepsilon,T}}$, and

$$\nu_\varepsilon(x,t)\cdot\nabla\sigma_\varepsilon(x,t) \geq c_3\varepsilon \quad \text{for all} \quad (x,t) \in \partial_\ell Q_{\varepsilon,T}. \tag{3.8}$$

Proof. We see that (3.7) holds by (2.9), (2.12), (2.27), the smoothness of d on $\overline{N_T}$ and of g_0 and g_1 on $\overline{S_T}$, and $|d| \leq c\varepsilon$ in $\overline{Q_{\varepsilon,T}}$. Also, since

$$\nabla\sigma_\varepsilon = (d - \varepsilon\overline{g}_1)(\overline{\nu} - \varepsilon R\,\overline{\nabla_\Gamma g_0}) + (d - \varepsilon\overline{g}_0)(\overline{\nu} - \varepsilon R\,\overline{\nabla_\Gamma g_1}) \quad \text{in} \quad \overline{N_T}$$

by $\nabla d = \overline{\nu}$ in $\overline{N_T}$ and (2.8), where R is given by (2.6); and since $d(x,t) = \varepsilon\overline{g}_i(x,t)$ for each $x \in \Gamma_\varepsilon^i(t)$, $t \in [0,T]$, and $i = 0, 1$ by (2.30), we have

$$\nabla\sigma_\varepsilon(x,t) = (-1)^{i+1}\varepsilon\overline{g}(x,t)\big\{\overline{\nu}(x,t) - \varepsilon\overline{\tau}_\varepsilon^i(x,t)\big\}, \quad x \in \Gamma_\varepsilon^i(t),\ t \in [0,T],\ i = 0, 1,$$

where $g = g_1 - g_0$ and τ_ε^i is given by (2.31) on $\overline{S_T}$. Using this equality, (2.34), and $\tau_\varepsilon^i \cdot \nu = 0$ on $\overline{S_T}$ (see (2.32)), and then applying (2.29), we find that

$$\nu_\varepsilon(x,t)\cdot\nabla\sigma_\varepsilon(x,t) = \varepsilon\overline{g}(x,t)\sqrt{1 + \varepsilon^2|\overline{\tau}_\varepsilon^i(x,t)|^2} \geq \varepsilon\overline{g}(x,t) \geq c\varepsilon$$

for all $x \in \Gamma_\varepsilon^i(t)$, $t \in [0,T]$, and $i = 0, 1$. Hence, (3.8) is valid. ■

Proof of Theorem 3.1. Let χ^ε be a classical solution to (3.1) and σ_ε be the function given by (3.6). Also, let c_1 and c_2 be the constants appearing in (3.3) and (3.4), and c_3 be the constant given in Lemma 3.2. We define

$$c_4 = \frac{c_2 + 1}{c_3}, \quad \zeta^\varepsilon(x,t) = e^{-c_4\sigma_\varepsilon(x,t)}\chi^\varepsilon(x,t), \quad (x,t) \in \overline{Q_{\varepsilon,T}}.$$

Then, noting that $a_{ij}^\varepsilon = a_{ji}^\varepsilon$ and that $e^{-c_4\sigma_\varepsilon} = 1$ on $\partial_\ell Q_{\varepsilon,T}$ since $\sigma_\varepsilon = 0$ on $\partial_\ell Q_{\varepsilon,T}$, we observe that ζ^ε satisfies

$$\begin{cases} \partial_t \zeta^\varepsilon - \sum_{i,j=1}^n a_{ij}^\varepsilon \partial_i \partial_j \zeta^\varepsilon + \sum_{i=1}^n B_i^\varepsilon \partial_i \zeta^\varepsilon + C^\varepsilon \zeta^\varepsilon = e^{-c_4\sigma_\varepsilon} f^\varepsilon & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon} \zeta^\varepsilon + \gamma^\varepsilon \zeta^\varepsilon = \psi^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ \zeta^\varepsilon|_{t=0} = e^{-c_4\sigma_\varepsilon(\cdot,0)} \chi_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0), \end{cases} \tag{3.9}$$

where the functions B_i^ε and C^ε on $Q_{\varepsilon,T}$ and γ^ε on $\partial_\ell Q_{\varepsilon,T}$ are given by

$$B_i^\varepsilon = -2c_4 \sum_{j=1}^n a_{ij}^\varepsilon \partial_j \sigma_\varepsilon + b_i^\varepsilon, \quad i = 1, \dots, n,$$

$$C^\varepsilon = c_4 \partial_t \sigma_\varepsilon - c_4 \sum_{i,j=1}^n a_{ij}^\varepsilon \partial_i \partial_j \sigma_\varepsilon - c_4^2 \sum_{i,j=1}^n a_{ij}^\varepsilon (\partial_i \sigma_\varepsilon)(\partial_j \sigma_\varepsilon) + c_4 \sum_{i=1}^n b_i^\varepsilon \partial_i \sigma_\varepsilon + c^\varepsilon$$

on $Q_{\varepsilon,T}$, and $\gamma^\varepsilon = c_4(\nu^\varepsilon \cdot \nabla \sigma_\varepsilon) + \beta^\varepsilon$ on $\partial_\ell Q_{\varepsilon,T}$. Moreover, $|C^\varepsilon| \leq c_5$ in $Q_{\varepsilon,T}$ by (3.3) and (3.7), where $c_5 > 0$ is a constant depending only on c_1, c_2, and c_3 and thus independent of ε. Also, it follows from (3.4), (3.8), and $c_4 = (c_2 + 1)/c_3$ that

$$\gamma^\varepsilon \geq c_4 \cdot c_3 \varepsilon - c_2 \varepsilon = \varepsilon > 0 \quad \text{on} \quad \partial_\ell Q_{\varepsilon,T}. \tag{3.10}$$

Thus, setting

$$\widehat{C}^\varepsilon(x,t) = C^\varepsilon(x,t) + c_5 + 1 \geq -|C^\varepsilon(x,t)| + c_5 + 1 \geq 1 \tag{3.11}$$

for $(x,t) \in Q_{\varepsilon,T}$ and

$$\begin{aligned} Z_\pm^\varepsilon(x,t) &= \pm e^{-(c_5+1)t} \zeta^\varepsilon(x,t) \\ &\quad - \left(\|e^{-c_4\sigma_\varepsilon(\cdot,0)} \chi_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|e^{-c_4\sigma_\varepsilon} f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \frac{1}{\varepsilon} \|\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})} \right) \end{aligned}$$

for $(x,t) \in \overline{Q_{\varepsilon,T}}$, we see that $Z_\pm^\varepsilon$ satisfies

$$\begin{cases} \partial_t Z_\pm^\varepsilon - \sum_{i,j=1}^n a_{ij}^\varepsilon \partial_i \partial_j Z_\pm^\varepsilon + \sum_{i=1}^n B_i^\varepsilon \partial_i Z_\pm^\varepsilon + \widehat{C}^\varepsilon Z_\pm^\varepsilon \leq 0 & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon} Z_\pm^\varepsilon + \gamma^\varepsilon Z_\pm^\varepsilon \leq 0 & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ Z_\pm^\varepsilon|_{t=0} \leq 0 & \text{in} \quad \Omega_\varepsilon(0). \end{cases} \tag{3.12}$$

Note that, due to (3.10), we need to multiply the supremum of $|\psi^\varepsilon|$ by ε^{-1} in the definition of $Z_\pm^\varepsilon$ in order to get the second inequality of (3.12). Then, as in the proof of the maximum principle (see, e.g., [31,33]), we can show that the maximum of $Z_\pm^\varepsilon$ on $\overline{Q_{\varepsilon,T}}$ must be non-positive by using (3.2) and (3.10)–(3.12) and by noting that $\partial_{\nu_\varepsilon} Z_\pm^\varepsilon(x,t) \geq 0$ if $Z_\pm^\varepsilon$ attains its maximum at $(x,t) \in \partial_\ell Q_{\varepsilon,T}$. Hence, $Z_\pm^\varepsilon \le 0$, that is,

$$|\zeta^\varepsilon| \le e^{(c_5+1)t}\Big(\|e^{-c_4\sigma_\varepsilon(\cdot,0)}\chi_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|e^{-c_4\sigma_\varepsilon} f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \frac{1}{\varepsilon}\|\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big)$$

in $\overline{Q_{\varepsilon,T}}$. By this inequality, $\chi^\varepsilon = e^{c_4\sigma_\varepsilon}\zeta^\varepsilon$ in $\overline{Q_{\varepsilon,T}}$, and (3.7), we conclude that (3.5) holds with $c_T = c_6 e^{(c_5+1)T}$ with a constant $c_6 > 0$ independent of ε. ∎

Remark 3.3. By the above proof, one may expect to remove the factor ε^{-1} in (3.5) if (3.4) is replaced by $\beta^\varepsilon \ge -c$ on $\partial_\ell Q_{\varepsilon,T}$ with a constant $c > 0$ independent of ε, but we must have another constant growing as $\varepsilon \to 0$ in that case. Indeed, when $\beta^\varepsilon \ge -c$ on $\partial_\ell Q_{\varepsilon,T}$, we need to employ $\varepsilon^{-1}\sigma_\varepsilon$ instead of σ_ε in the proof of Theorem 3.1 in order to get $\gamma^\varepsilon \ge 1$ on $\partial_\ell Q_{\varepsilon,T}$ instead of (3.10) for the coefficient γ^ε in (3.9). However, by taking $\varepsilon^{-1}\sigma_\varepsilon$ we must have $C^\varepsilon \ge -c\varepsilon^{-1}$ in $Q_{\varepsilon,T}$ for the coefficient C^ε in (3.9) even if we carefully compute the derivatives of σ_ε. Hence, if we intend to remove ε^{-1} in (3.5), then we need to replace $c_T = c_6 e^{(c_5+1)T}$ by $c_6 e^{(c_5+1)T/\varepsilon}$, which grows as $\varepsilon \to 0$ much faster than ε^{-1}.

Now we consider the heat equation with source terms

$$\begin{cases} \partial_t \rho^\varepsilon - k_d \Delta \rho^\varepsilon = f^\varepsilon & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon}\rho^\varepsilon + k_d^{-1} V_\varepsilon \rho^\varepsilon = \psi^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0). \end{cases} \tag{3.13}$$

In this case, the outer normal velocity V_ε of $\partial\Omega_\varepsilon(t)$ is of order one with respect to ε by (2.35). Thus, in order to apply Theorem 3.1, we need to introduce an auxiliary function to eliminate the zeroth-order term of V_ε.

Theorem 3.4. *Let $\varepsilon \in (0,\varepsilon_0)$ and the given data*

$$\rho_0^\varepsilon \in C(\overline{\Omega_\varepsilon(0)}), \quad f^\varepsilon \in C(Q_{\varepsilon,T}), \quad \psi^\varepsilon \in C(\partial_\ell Q_{\varepsilon,T})$$

be bounded. Also, let ρ^ε be a classical solution to (3.13). Then, there exists a constant $c_T > 0$ depending on T but independent of ε, ρ_0^ε, f^ε, ψ^ε, and ρ^ε such that

$$\|\rho^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \le c_T\Big(\|\rho_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \frac{1}{\varepsilon}\|\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big). \tag{3.14}$$

Proof. Let ρ^ε be a classical solution to (3.13). We define

$$\lambda(x,t) = -k_d^{-1} d(x,t)\overline{V_\Gamma}(x,t), \quad \chi^\varepsilon(x,t) = e^{-\lambda(x,t)}\rho^\varepsilon(x,t)$$

for $(x,t) \in \overline{Q_{\varepsilon,T}}$. Then, χ^ε satisfies (3.1) with

$$a_{ij}^\varepsilon = k_d\delta_{ij}, \quad b_i^\varepsilon = -2k_d\partial_i\lambda, \quad c^\varepsilon = \partial_t\lambda - k_d(\Delta\lambda + |\nabla\lambda|^2) \quad \text{in} \quad Q_{\varepsilon,T},$$
$$\beta^\varepsilon = \nu_\varepsilon \cdot \nabla\lambda + k_d^{-1}V_\varepsilon \quad \text{on} \quad \partial_\ell Q_{\varepsilon,T},$$

and f^ε, ψ^ε, and χ_0^ε replaced by $e^{-\lambda} f^\varepsilon$, $e^{-\lambda}\psi^\varepsilon$, and $e^{-\lambda(\cdot,0)}\rho_0^\varepsilon$. Here δ_{ij} is the Kronecker delta. Then, since d and V_Γ are smooth on $\overline{N_T}$ and $\overline{S_T}$, we see by (2.9), (2.12), and (2.27) that (3.3) holds with a constant $c_1 > 0$ independent of ε. Also, since

$$\nabla\lambda = -k_d^{-1}\overline{V_\Gamma}\,\overline{\nu} - k_d^{-1} d\,\nabla\overline{V_\Gamma} \quad \text{in} \quad \overline{Q_{\varepsilon,T}}$$

by $\nabla d = \overline{\nu}$ in $\overline{N_T}$, and since ν_ε is of the form in (2.34), we have

$$\nu_\varepsilon \cdot \nabla\lambda = \frac{(-1)^{i+1}k_d^{-1}}{\sqrt{1+\varepsilon^2|\overline{\tau_\varepsilon^i}|^2}}(-\overline{V_\Gamma} + \varepsilon^2\overline{g}_i\overline{\tau_\varepsilon^i}\cdot\nabla\overline{V_\Gamma}) \quad \text{on} \quad \Gamma_\varepsilon^i(t),$$

for $t \in (0,T]$ and $i = 0,1$. Here τ_ε^i is given by (2.31) and we also used

$$|\nu| = 1,\ \tau_\varepsilon^i \cdot \nu = 0 \quad \text{on} \quad \overline{S_T}, \qquad \overline{\nu}\cdot\nabla\overline{V_\Gamma} = 0 \quad \text{in} \quad \overline{N_T}, \qquad d = \varepsilon\overline{g}_i \quad \text{on} \quad \Gamma_\varepsilon^i(t);$$

see (2.32) for the second equality. By the above equality and (2.35), we get

$$\beta_\varepsilon = \frac{(-1)^{i+1}k_d^{-1}}{\sqrt{1+\varepsilon^2|\overline{\tau_\varepsilon^i}|^2}}\{\varepsilon\,\overline{\partial^\circ g_i} + \varepsilon^2\overline{g}_i\overline{\tau_\varepsilon^i}\cdot(\nabla\overline{V_\Gamma} + \overline{\nabla_\Gamma V_\Gamma})\} \quad \text{on} \quad \Gamma_\varepsilon^i(t)$$

for $t \in (0,T]$ and $i = 0,1$. Thus, by (2.9), (2.33), and the smoothness of g_0, g_1, and V_Γ on $\overline{S_T}$, we find that $|\beta_\varepsilon| \leq c\varepsilon$ on $\partial_\ell Q_{\varepsilon,T}$ with a constant $c > 0$ which does not depend on ε. Hence, (3.4) is valid and we can apply Theorem 3.1 to obtain

$$\|\chi^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \leq c_T\Big(\|e^{-\lambda(\cdot,0)}\rho_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|e^{-\lambda}f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \frac{1}{\varepsilon}\|e^{-\lambda}\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big).$$

Recall that χ^ε satisfies equation (3.1) with f^ε, ψ^ε, and χ_0^ε replaced by $e^{-\lambda}f^\varepsilon$, $e^{-\lambda}\psi^\varepsilon$, and $e^{-\lambda(\cdot,0)}\rho_0^\varepsilon$. Now we observe that $\rho^\varepsilon = e^\lambda\chi^\varepsilon$ in $\overline{Q_{\varepsilon,T}}$ and that $\lambda = -k_d^{-1}d\,\overline{V_\Gamma}$ is bounded on $\overline{N_T}$ independently of ε. Therefore, we obtain (3.14), by the above estimate for χ^ε. ■

Remark 3.5. The idea of the proof of Theorem 3.4 also applies to the problem in (3.1) under conditions (3.3) and $\beta^\varepsilon \geq -c$ on $\partial_\ell Q_{\varepsilon,T}$ instead of (3.4), provided that there exists a function ω^ε on $\overline{Q_{\varepsilon,T}}$ such that ω^ε, $\partial_t\omega^\varepsilon$, $\nabla\omega^\varepsilon$, and $\nabla^2\omega^\varepsilon$ are uniformly bounded on $\overline{Q_{\varepsilon,T}}$ with respect to ε and that $\beta^\varepsilon - (-1)^{i+1}\omega^\varepsilon \geq -c\varepsilon$ on $\Gamma_\varepsilon^i(t)$ for all $t \in (0,T]$ and $i = 0,1$. In this case, for a classical solution χ^ε to (3.1), we see that $e^{d\omega^\varepsilon}\chi^\varepsilon$ satisfies a problem similar to (3.1) with coefficients in $Q_{\varepsilon,T}$ satisfying (3.3) and with the coefficient on $\partial_\ell Q_{\varepsilon,T}$ given by

$$\widehat{\beta}^\varepsilon = \beta^\varepsilon - (\nu^\varepsilon\cdot\overline{\nu})\omega^\varepsilon - \varepsilon\overline{g}_i(\nu^\varepsilon\cdot\nabla\omega^\varepsilon) \quad \text{on} \quad \Gamma_\varepsilon^i(t)$$

for $t \in (0,T]$ and $i = 0,1$. Then, it follows from (2.32)–(2.34), the mean value theorem for $(1+s)^{-1/2}$ with $s \in \mathbb{R}$, and the assumption on ω^ε that

$$\widehat{\beta}^\varepsilon = \beta^\varepsilon - (-1)^{i+1}\omega^\varepsilon + (-1)^{i+1}\omega^\varepsilon\Big(1 - \frac{1}{\sqrt{1+\varepsilon^2|\overline{\tau_\varepsilon^i}|^2}}\Big) - \varepsilon\overline{g}_i(\nu^\varepsilon\cdot\nabla\omega^\varepsilon)$$

$$\geq \beta^\varepsilon - (-1)^{i+1}\omega^\varepsilon - |\omega^\varepsilon|\left|1 - \frac{1}{\sqrt{1+\varepsilon^2|\overline{\tau}_\varepsilon^i|^2}}\right| - c\varepsilon|\nabla\omega^\varepsilon| \geq -c\varepsilon$$

on $\Gamma_\varepsilon^i(t)$ for all $t \in (0,T]$ and $i = 0,1$. Hence, we can apply Theorem 3.1 to $e^{d\omega^\varepsilon}\chi^\varepsilon$ and then obtain (3.5) for χ^ε by noting that $d\omega^\varepsilon$ is uniformly bounded on $\overline{Q_{\varepsilon,T}}$.

4. Proofs of the main results

In this section we prove Theorems 1.1 and 1.2. We give the proof of Theorem 1.2 below, and Theorem 1.1 can be obtained by setting $\nabla_\Gamma g_0 = \nabla_\Gamma g_1 = 0$ on $\overline{S_T}$ in this proof. As in the previous sections, we denote by $\overline{\eta}$ the constant extension of a function η on $\overline{S_T}$ in the normal direction of $\Gamma(t)$.

Let ρ^ε and η be classical solutions to (1.1) and (1.2), respectively. Based on a formal asymptotic expansion of (1.1) carried out in Section 5 (see also the explanations in Section 1.2), we define an approximate solution ρ_η^ε to (1.1) by using η as follows: let

$$\zeta_i = \frac{1}{g}\{\nabla_\Gamma g_i \cdot \nabla_\Gamma \eta - k_d^{-1}(\partial^\circ g_i)\eta + k_d^{-2} g_i V_\Gamma^2 \eta\} \tag{4.1}$$

on $\overline{S_T}$ and $i = 0,1$. For $(x,t) \in \overline{Q_{\varepsilon,T}}$, we define

$$\begin{aligned} \eta_2^\varepsilon(x,t) &= \varepsilon d(x,t)(\overline{g}_1\overline{\zeta}_0 - \overline{g}_0\overline{\zeta}_1)(x,t) + \frac{1}{2}d(x,t)^2(\overline{\zeta}_1 - \overline{\zeta}_0)(x,t), \\ \rho_\eta^\varepsilon(x,t) &= \overline{\eta}(x,t) - k_d^{-1} d(x,t)(\overline{V_\Gamma \eta})(x,t) + \eta_2^\varepsilon(x,t). \end{aligned} \tag{4.2}$$

We also define error terms due to ρ_η^ε by

$$\begin{aligned} f_\eta^\varepsilon &= \partial_t \rho_\eta^\varepsilon - k_d \Delta \rho_\eta^\varepsilon - \overline{f} \quad &&\text{on} \quad Q_{\varepsilon,T}, \\ \psi_\eta^\varepsilon &= \partial_{\nu_\varepsilon}\rho_\eta^\varepsilon + k_d^{-1} V_\varepsilon \rho_\eta^\varepsilon \quad &&\text{on} \quad \partial_\ell Q_{\varepsilon,T}, \end{aligned}$$

so that $\rho^\varepsilon - \rho_\eta^\varepsilon$ satisfies

$$\begin{cases} \partial_t(\rho^\varepsilon - \rho_\eta^\varepsilon) - k_d\Delta(\rho^\varepsilon - \rho_\eta^\varepsilon) = (f^\varepsilon - \overline{f}) - f_\eta^\varepsilon & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon}(\rho^\varepsilon - \rho_\eta^\varepsilon) + k_d^{-1}V_\varepsilon(\rho^\varepsilon - \rho_\eta^\varepsilon) = -\psi_\eta^\varepsilon & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ (\rho^\varepsilon - \rho_\eta^\varepsilon)|_{t=0} = \rho_0^\varepsilon - \rho_\eta^\varepsilon(\cdot,0) & \text{in} \quad \Omega_\varepsilon(0). \end{cases}$$

Hence, it follows from Theorem 3.4 that

$$\begin{aligned} \|\rho^\varepsilon - \rho_\eta^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} &\leq c_T(\|\rho_0^\varepsilon - \rho_\eta^\varepsilon(\cdot,0)\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f^\varepsilon - \overline{f}\|_{\mathcal{B}(Q_{\varepsilon,T})}) \\ &\quad + c_T\left(\|f_\eta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \frac{1}{\varepsilon}\|\psi_\eta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\right) \end{aligned}$$

with a constant $c_T > 0$ depending on T but independent of ε, ρ^ε, η, and the given data. By this inequality, (4.2), and $\eta|_{t=0} = \eta_0$ on $\Gamma(0)$, we obtain

$$\begin{aligned}\|\rho^\varepsilon - \bar{\eta}\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} &\leq c_T(\|\rho_0^\varepsilon - \bar{\eta}_0\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})} + \|f^\varepsilon - \bar{f}\|_{\mathcal{B}(Q_{\varepsilon,T})}) \\ &\quad + c_T(\|d(\overline{V_\Gamma \eta})\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} + \|\eta_2^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})}) \\ &\quad + c_T\Big(\|f_\eta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \frac{1}{\varepsilon}\|\psi_\eta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}\Big). \end{aligned} \tag{4.3}$$

Let us further estimate the right-hand side. In what follows, we use the notation in (1.4) and denote by c a general positive constant independent of ε, ρ^ε, η, and the given data. We also frequently use the facts that g_0, g_1, and V_Γ are smooth and thus bounded on $\overline{S_T}$ along with their derivatives, and that g satisfies (2.29), without mention.

First, we observe by (4.2) and $|d| \leq c\varepsilon$ in $\overline{Q_{\varepsilon,T}}$ that

$$\|d(\overline{V_\Gamma \eta})\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} + \|\eta_2^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \leq c\varepsilon(\|\eta\|_{\mathcal{B}(\overline{S_T})} + \|\zeta_0\|_{\mathcal{B}(\overline{S_T})} + \|\zeta_1\|_{\mathcal{B}(\overline{S_T})}). \tag{4.4}$$

Note that $\overline{S_T}$ includes $\Gamma(0) \times \{0\}$. Next we consider f_η^ε. We see that

$$|\partial_t \rho_\eta^\varepsilon - \overline{\partial^\circ \eta} - k_d^{-1}(\overline{V_\Gamma^2 \eta})| \leq c\varepsilon \sum_{\xi=\eta,\zeta_0,\zeta_1} (|\bar{\xi}| + |\overline{\partial^\circ \xi}| + |\overline{\nabla_\Gamma \xi}|) \tag{4.5}$$

in $Q_{\varepsilon,T}$, by (2.21), (2.27), (2.28), and $|d| \leq c\varepsilon$ in $Q_{\varepsilon,T}$. Also,

$$|\Delta \rho_\eta^\varepsilon - \overline{\Delta_\Gamma \eta} - k_d^{-1}(\overline{V_\Gamma H \eta}) - (\bar{\zeta}_1 - \bar{\zeta}_0)| \leq c\varepsilon \sum_{\xi=\eta,\zeta_0,\zeta_1} (|\bar{\xi}| + |\overline{\nabla_\Gamma \xi}| + |\overline{\nabla_\Gamma^2 \xi}|) \tag{4.6}$$

in $Q_{\varepsilon,T}$, by (2.14)–(2.16) and $|d| \leq c\varepsilon$ in $Q_{\varepsilon,T}$. Moreover, since

$$\zeta_1 - \zeta_0 = \frac{1}{g}\{\nabla_\Gamma g \cdot \nabla_\Gamma \eta - k_d^{-1}(\partial^\circ g)\eta + k_d^{-2} g V_\Gamma^2 \eta\} \quad \text{on} \quad S_T$$

by (4.1) and $g_1 - g_0 = g$ on S_T, and since η satisfies (1.2), we have

$$\begin{aligned} &\partial^\circ \eta + k_d^{-1} V_\Gamma^2 \eta - k_d\{\Delta_\Gamma \eta + k_d^{-1} V_\Gamma H \eta + (\zeta_1 - \zeta_0)\} \\ &= \frac{1}{g}\{g\partial^\circ \eta + (\partial^\circ g)\eta - g V_\Gamma H \eta - k_d\, g\Delta_\Gamma \eta - k_d \nabla_\Gamma g \cdot \nabla_\Gamma \eta\} \\ &= \frac{1}{g}\{\partial^\circ(g\eta) - g V_\Gamma H \eta - k_d \operatorname{div}_\Gamma(g\nabla_\Gamma \eta)\} = \frac{gf}{g} = f \quad \text{on} \quad S_T. \end{aligned} \tag{4.7}$$

Thus, by (4.5)–(4.7) and $f_\eta^\varepsilon = \partial_t \rho_\eta^\varepsilon - k_d \Delta \rho_\eta^\varepsilon - \bar{f}$ in $Q_{\varepsilon,T}$, we find that

$$|f_\eta^\varepsilon| \leq c\varepsilon \sum_{\xi=\eta,\zeta_0,\zeta_1} (|\bar{\xi}| + |\overline{\partial^\circ \xi}| + |\overline{\nabla_\Gamma \xi}| + |\overline{\nabla_\Gamma^2 \xi}|)$$

in $Q_{\varepsilon,T}$, which implies that

$$\|f_\eta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} \leq c\varepsilon(\|\eta\|_{\mathcal{B}^{2,1}(S_T)} + \|\zeta_0\|_{\mathcal{B}^{2,1}(S_T)} + \|\zeta_1\|_{\mathcal{B}^{2,1}(S_T)}). \tag{4.8}$$

Note that here $\Gamma(0) \times \{0\}$ is not included. Now let us estimate ψ_η^ε on $\partial_\ell Q_{\varepsilon,T}$. Let $i = 0, 1$ and $t \in (0, T]$. Since ν_ε and V_ε are of the form in (2.34) and (2.35), we have

$$\begin{aligned} \psi_\eta^\varepsilon &= \nu_\varepsilon \cdot \nabla \rho_\eta^\varepsilon + k_d^{-1} V_\varepsilon \rho_\eta^\varepsilon \\ &= \frac{(-1)^{i+1}}{\sqrt{1+\varepsilon^2|\overline{\tau}_\varepsilon^i|^2}} \left\{ (\overline{\nu} - \varepsilon \overline{\tau}_\varepsilon^i) \cdot \nabla \rho_\eta^\varepsilon + k_d^{-1}(\overline{V_\Gamma} + \varepsilon\, \overline{\partial^\circ g_i} + \varepsilon^2 \overline{g}_i \overline{\tau}_\varepsilon^i \cdot \overline{\nabla_\Gamma V_\Gamma})\rho_\eta^\varepsilon \right\} \end{aligned} \tag{4.9}$$

on $\Gamma_\varepsilon^i(t)$, where τ_ε^i is given by (2.31). To estimate the second line, we first note that

$$\overline{\nu} \cdot \nabla d = |\overline{\nu}|^2 = 1, \quad \overline{\nu} \cdot \nabla \overline{\xi} = 0 \quad \text{in} \quad \overline{N_T}$$

for a function ξ on $\overline{S_T}$. We apply these equalities to ρ_η^ε of the form in (4.2) to get

$$\overline{\nu} \cdot \nabla \rho_\eta^\varepsilon = -k_d^{-1}(\overline{V_\Gamma \eta}) + \varepsilon(\overline{g}_1\overline{\zeta}_0 - \overline{g}_0\overline{\zeta}_1) + d(\overline{\zeta}_1 - \overline{\zeta}_0) \quad \text{in} \quad \overline{Q_{\varepsilon,T}}.$$

Then, we see by $d = \varepsilon \overline{g}_i$ on $\Gamma_\varepsilon^i(t)$, $g_1 - g_0 = g$ on S_T, and (4.1) that

$$\begin{aligned} \overline{\nu} \cdot \nabla \rho_\eta^\varepsilon &= -k_d^{-1}(\overline{V_\Gamma \eta}) + \varepsilon(\overline{g\zeta_i}) \\ &= -k_d^{-1}(\overline{V_\Gamma \eta}) + \varepsilon\left\{ \overline{\nabla_\Gamma g_i} \cdot \overline{\nabla_\Gamma \eta} - k_d^{-1}(\overline{\partial^\circ g_i})\overline{\eta} + k_d^{-2}(\overline{g_i V_\Gamma^2 \eta}) \right\} \end{aligned}$$

on $\Gamma_\varepsilon^i(t)$ and thus, by noting that $\varepsilon k_d^{-2}(\overline{g_i V_\Gamma^2 \eta}) = k_d^{-1}\overline{V_\Gamma} \cdot k_d^{-1}(\varepsilon \overline{g}_i)(\overline{V_\Gamma \eta})$,

$$\begin{aligned} &(\overline{\nu} - \varepsilon \overline{\tau}_\varepsilon^i) \cdot \nabla \rho_\eta^\varepsilon + k_d^{-1}(\overline{V_\Gamma} + \varepsilon\, \overline{\partial^\circ g_i} + \varepsilon^2 \overline{g}_i \overline{\tau}_\varepsilon^i \cdot \overline{\nabla_\Gamma V_\Gamma})\rho_\eta^\varepsilon \\ &\quad = k_d^{-1}\overline{V_\Gamma}\left\{ \rho_\eta^\varepsilon - \overline{\eta} + k_d^{-1}(\varepsilon \overline{g}_i)(\overline{V_\Gamma \eta}) \right\} + \varepsilon(\overline{\nabla_\Gamma g_i} \cdot \overline{\nabla_\Gamma \eta} - \overline{\tau}_\varepsilon^i \cdot \nabla \rho_\eta^\varepsilon) \\ &\qquad + \varepsilon k_d^{-1}(\overline{\partial^\circ g_i})(\rho_\eta^\varepsilon - \overline{\eta}) + \varepsilon^2 k_d^{-1} \overline{g}_i (\overline{\tau}_\varepsilon^i \cdot \overline{\nabla_\Gamma V_\Gamma})\rho_\eta^\varepsilon \end{aligned} \tag{4.10}$$

on $\Gamma_\varepsilon^i(t)$. Moreover, since ρ_η^ε is given by (4.2) and $d = \varepsilon \overline{g}_i$ on $\Gamma_\varepsilon^i(t)$, we have

$$|\rho_\eta^\varepsilon - \overline{\eta} + k_d^{-1}(\varepsilon \overline{g}_i)(\overline{V_\Gamma \eta})| = |\eta_2^\varepsilon| \le c\varepsilon^2(|\overline{\zeta}_0| + |\overline{\zeta}_1|) \tag{4.11}$$

on $\Gamma_\varepsilon^i(t)$ and thus,

$$|\rho_\eta^\varepsilon - \overline{\eta}| \le c\varepsilon(|\overline{\eta}| + |\overline{\zeta}_0| + |\overline{\zeta}_1|), \quad |\rho_\eta^\varepsilon| \le c(|\overline{\eta}| + |\overline{\zeta}_0| + |\overline{\zeta}_1|) \tag{4.12}$$

on $\Gamma_\varepsilon^i(t)$. Also, since ρ_η^ε is given by (4.2) and $\overline{\tau}_\varepsilon^i \cdot \nabla d = \overline{\tau}_\varepsilon^i \cdot \overline{\nu} = 0$ in $\overline{N_T}$ by (2.32),

$$[\overline{\tau}_\varepsilon^i \cdot \nabla \rho_\eta^\varepsilon](x,t) = [\overline{\tau}_\varepsilon^i \cdot \nabla \overline{\eta}](x,t) - k_d^{-1} d(x,t)[\overline{\tau}_\varepsilon^i \cdot \nabla(\overline{V_\Gamma \eta})](x,t) + [\overline{\tau}_\varepsilon^i \cdot \nabla \eta_2^\varepsilon](x,t)$$

for $(x,t) \in \overline{Q_{\varepsilon,T}}$. Noting that η_2^ε is given by (4.2), we deduce from this equality, (2.9) and (2.10) with $d = \varepsilon \overline{g}_i$ on $\Gamma_\varepsilon^i(t)$, and (2.33) that

$$|\overline{\tau}_\varepsilon^i \cdot \nabla \rho_\eta^\varepsilon - \overline{\nabla_\Gamma g_i} \cdot \overline{\nabla_\Gamma \eta}| \le c\varepsilon \sum_{\xi = \eta, \zeta_0, \zeta_1} (|\overline{\xi}| + |\overline{\nabla_\Gamma \xi}|) \tag{4.13}$$

on $\Gamma_\varepsilon^i(t)$. Hence, we apply (2.33) and (4.10)–(4.13) to (4.9) to get

$$\begin{aligned}|\psi_\eta^\varepsilon| &\le \left|(\overline{\nu}-\varepsilon\overline{\tau}_\varepsilon^i)\cdot\nabla\rho_\eta^\varepsilon + k_d^{-1}(\overline{V_\Gamma}+\varepsilon\,\overline{\partial^\circ g_i}+\varepsilon^2\overline{g}_i\overline{\tau}_\varepsilon^i\cdot\overline{\nabla_\Gamma V_\Gamma})\rho_\eta^\varepsilon\right| \\ &\le c\varepsilon^2\sum_{\xi=\eta,\zeta_0,\zeta_1}(|\overline{\xi}|+|\overline{\nabla_\Gamma\xi}|)\end{aligned}$$

on $\Gamma_\varepsilon^i(t)$ with $i=0,1$ and $t\in(0,T]$. Therefore,

$$\|\psi_\eta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})} \le c\varepsilon^2(\|\eta\|_{\mathcal{B}^{2,1}(S_T)}+\|\zeta_0\|_{\mathcal{B}^{2,1}(S_T)}+\|\zeta_1\|_{\mathcal{B}^{2,1}(S_T)}). \tag{4.14}$$

Finally, we obtain (1.6) by applying (4.4), (4.8), and (4.14) to (4.3) and noting that

$$\|\eta\|_{\mathcal{B}(\overline{S_T})} \le \|\eta_0\|_{\mathcal{B}(\Gamma(0))}+\|\eta\|_{\mathcal{B}^{2,1}(S_T)}$$

by $\eta|_{t=0}=\eta_0$ on $\Gamma(0)$, and that

$$\begin{aligned}\|\zeta_i\|_{\mathcal{B}(\overline{S_T})}+\|\zeta_i\|_{\mathcal{B}^{2,1}(S_T)} &\le \|\zeta_i(\cdot,0)\|_{\mathcal{B}(\Gamma(0))}+\|\zeta_i\|_{\mathcal{B}(S_T)}+\|\zeta_i\|_{\mathcal{B}^{2,1}(S_T)} \\ &\le c(\|\eta_0\|_{\mathcal{B}(\Gamma(0))}+\|\eta\|_{\mathcal{B}^{2,1}(S_T)}) \\ &\quad + c(\|h_i(\cdot,0)\|_{\mathcal{B}(\Gamma(0))}+\|h_i\|_{\mathcal{B}^{2,1}(S_T)})\end{aligned}$$

for $i=0,1$ with $h_i=\nabla_\Gamma g_i\cdot\nabla_\Gamma\eta$ on $\overline{S_T}$ by (4.1). The proof of Theorem 1.2 is complete.

Remark 4.1. By the idea of the above proof, we can also show that the factor ε^{-1} in the uniform a priori estimate given by (3.14) cannot be removed. Indeed, assume that the inequality

$$\|\rho^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \le c(\|\rho_0^\varepsilon\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})}+\|f^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})}+\|\psi^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}) \tag{4.15}$$

holds for a classical solution ρ^ε to (3.13) with a constant $c>0$ independent of ε. For an arbitrary smooth function ζ on $\overline{S_T}$, we define

$$\rho_\zeta^\varepsilon(x,t)=\overline{\zeta}(x,t)-k_d^{-1}d(x,t)(\overline{V_\Gamma\zeta})(x,t)+\frac{1}{2}d(x,t)^2\overline{\zeta}_2(x,t), \quad (x,t)\in\overline{Q_{\varepsilon,T}},$$

where ζ_2 is a function on $\overline{S_T}$ given later (and ρ_ζ^ε is in fact independent of ε), and set

$$f_\zeta^\varepsilon=\partial_t\rho_\zeta^\varepsilon-k_d\Delta\rho_\zeta^\varepsilon \quad \text{on} \quad Q_{\varepsilon,T}, \qquad \psi_\zeta^\varepsilon=\partial_{\nu_\varepsilon}\rho_\zeta^\varepsilon+k_d^{-1}V_\varepsilon\rho_\zeta^\varepsilon \quad \text{on} \quad \partial_\ell Q_{\varepsilon,T}.$$

Then, since ρ_ζ^ε satisfies (3.13) with the above f_ζ^ε and ψ_ζ^ε, we can use (4.15) to get

$$\|\rho_\zeta^\varepsilon\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})} \le c(\|\rho_\zeta^\varepsilon(\cdot,0)\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})}+\|f_\zeta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})}+\|\psi_\zeta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})})$$

and thus, by the definition of ρ_ζ^ε, $|d|\le c\varepsilon$ in $\overline{Q_{\varepsilon,T}}$ and

$$\|\overline{\zeta}\|_{\mathcal{B}(\overline{Q_{\varepsilon,T}})}=\|\zeta\|_{\mathcal{B}(\overline{S_T})}, \quad \|\overline{\zeta}(\cdot,0)\|_{\mathcal{B}(\overline{\Omega_\varepsilon(0)})}=\|\zeta(\cdot,0)\|_{\mathcal{B}(\Gamma(0))}\le\|\zeta\|_{\mathcal{B}(\overline{S_T})},$$

we find that

$$\|\zeta\|_{\mathcal{B}(\overline{S_T})} \le c(\|\zeta(\cdot,0)\|_{\mathcal{B}(\Gamma(0))} + \|f_\zeta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \|\psi_\zeta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})}) + c\varepsilon(\|\zeta\|_{\mathcal{B}(\overline{S_T})} + \|\zeta_2\|_{\mathcal{B}(\overline{S_T})}). \tag{4.16}$$

Moreover, since

$$\bar{\nu}(x,t)\cdot\nabla\rho_\zeta^\varepsilon(x,t) = -k_d^{-1}(\overline{V_\Gamma\zeta})(x,t) + d(x,t)\bar{\zeta}_2(x,t), \quad (x,t)\in\overline{Q_{\varepsilon,T}}$$

by $\nabla d = \bar{\nu}$, $|\bar{\nu}|^2 = 1$, and $\bar{\nu}\cdot\nabla\bar{\xi} = 0$ in $\overline{N_T}$ for a function ξ on $\overline{S_T}$, we observe by (4.9) with η replaced by ζ and by (2.33) and $d = \varepsilon\bar{g}_i$ on $\Gamma_\varepsilon^i(t)$ that

$$|\psi_\zeta^\varepsilon| \le c\varepsilon \sum_{\xi=\zeta,\zeta_2}(|\bar{\xi}| + |\overline{\nabla_\Gamma\xi}|) \quad \text{on} \quad \partial_\ell Q_{\varepsilon,T}. \tag{4.17}$$

Also, as in (4.5) and (4.6), we have

$$|f_\zeta^\varepsilon - \bar{f}_\zeta| \le c\varepsilon \sum_{\xi=\zeta,\zeta_2}(|\bar{\xi}| + |\overline{\partial^\circ\xi}| + |\overline{\nabla_\Gamma\xi}| + |\overline{\nabla_\Gamma^2\xi}|) \quad \text{in} \quad Q_{\varepsilon,T} \tag{4.18}$$

by (2.14)–(2.16), (2.21), (2.27), (2.28), and $|d| \le c\varepsilon$ in $\overline{Q_{\varepsilon,T}}$, where

$$f_\zeta = \partial^\circ\zeta + k_d^{-1}V_\Gamma^2\zeta - k_d\Delta_\Gamma\zeta - V_\Gamma H\zeta - k_d\zeta_2 \quad \text{on} \quad S_T.$$

Thus, setting

$$\zeta_2 = k_d^{-1}(\partial^\circ\zeta + k_d^{-1}V_\Gamma^2\zeta - k_d\Delta_\Gamma\zeta - V_\Gamma H\zeta) \quad \text{on} \quad \overline{S_T},$$

we get $f_\zeta = 0$ on S_T, and it follows from (4.17), (4.18), and $f_\zeta = 0$ on S_T that

$$\|f_\zeta^\varepsilon\|_{\mathcal{B}(Q_{\varepsilon,T})} + \|\psi_\zeta^\varepsilon\|_{\mathcal{B}(\partial_\ell Q_{\varepsilon,T})} \le c\varepsilon(\|\zeta\|_{\mathcal{B}^{2,1}(S_T)} + \|\zeta_2\|_{\mathcal{B}^{2,1}(S_T)}).$$

Now we apply this inequality to (4.16) to find that

$$\|\zeta\|_{\mathcal{B}(\overline{S_T})} \le c\|\zeta(\cdot,0)\|_{\mathcal{B}(\Gamma(0))} + c\varepsilon\sum_{\xi=\zeta,\zeta_2}(\|\xi\|_{\mathcal{B}(\overline{S_T})} + \|\xi\|_{\mathcal{B}^{2,1}(S_T)})$$

with a constant $c > 0$ independent of ε. Hence, we send $\varepsilon \to 0$ to get

$$\|\zeta\|_{\mathcal{B}(\overline{S_T})} \le c\|\zeta(\cdot,0)\|_{\mathcal{B}(\Gamma(0))}$$

for any smooth function ζ on $\overline{S_T}$, but this is a contradiction, since we can take ζ such that $\zeta(\cdot,0) = 0$ on $\Gamma(0)$ but $\zeta(\cdot,t)$ does not vanish on $\Gamma(t)$ for some $t \in (0,T]$. Therefore, inequality (4.15) does not hold, that is, we cannot remove ε^{-1} in (3.14). In the same way, we can show that any smooth function on $\overline{S_T}$ approximates a classical solution to (1.1) of order ε as in (1.5) and (1.6) if (4.15) holds, which is again absurd. Hence, we can also consider that the factor ε^{-1} in (3.14) is crucial in order that η should be a solution to limit equation (1.2) in error estimates (1.5) and (1.6).

5. Formal derivation of a limit equation and an approximate solution

We explain how to derive formally limit equation (1.2) and the approximate solution ρ_η^ε of the form in (4.2) from the thin domain problem given by (1.1). Throughout this section, we write $O(\varepsilon^k)$ with $k \geq 1$ for any quantity of order at least ε^k.

5.1. Asymptotic expansions of the time derivative, gradient, and Laplacian

Let ρ be a function on $\overline{Q_{\varepsilon,T}}$. We make the change of variables to write

$$\eta(y,t,z) = \rho(y + \varepsilon z \nu(y,t), t), \quad (y,t) \in \overline{S_T},\ z \in [g_0(y,t), g_1(y,t)],$$

where z is the scaled signed distance from $y \in \Gamma(t)$. Note that the domain of the variables (y,t,z) is independent of ε. Let us express $\partial_t \rho(x,t)$, $\nabla \rho(x,t)$, and $\Delta \rho(x,t)$ by asymptotic expansions with respect to ε in terms of $\eta(y,t,z)$ and its derivatives.

Since $y = \pi(x,t)$ and $z = \varepsilon^{-1} d(x,t)$ for $(x,t) \in \overline{Q_{\varepsilon,T}}$ by (2.1), we can write

$$\rho(x,t) = \eta(\pi(x,t), t, \varepsilon^{-1} d(x,t)), \quad (x,t) \in \overline{Q_{\varepsilon,T}}.$$

By a slight abuse of the notation, we write

$$\overline{\eta}(x,t,z) = \eta(\pi(x,t), t, z), \quad (x,t) \in \overline{N_T},\ z \in [\overline{g}_0(x,t), \overline{g}_1(x,t)]$$

so that

$$\rho(x,t) = \overline{\eta}(x,t,\varepsilon^{-1} d(x,t)), \quad (x,t) \in \overline{Q_{\varepsilon,T}}. \tag{5.1}$$

In what follows, we sometimes suppress the arguments x and t of functions which do not have the third argument z. For example, we write (5.1) as $\rho = \overline{\eta}(x,t,\varepsilon^{-1} d)$. Also, we consider that the tangential derivatives apply to the argument y of $\eta(y,t,z)$.

Let $(x,t) \in \overline{Q_{\varepsilon,T}}$ and R be given by (2.6). We differentiate (5.1) with respect to time and use (2.21) and (2.26) to find that

$$\begin{aligned}\partial_t \rho &= \partial_t \overline{\eta}(x,t,\varepsilon^{-1} d) + \varepsilon^{-1} (\partial_t d) \partial_z \overline{\eta}(x,t,\varepsilon^{-1} d) \\ &= \overline{\partial^\circ \eta}(x,t,\varepsilon^{-1} d) + d(R\, \overline{\nabla_\Gamma V_\Gamma}) \cdot \overline{\nabla_\Gamma \eta}(x,t,\varepsilon^{-1} d) - \varepsilon^{-1} \overline{V_\Gamma}\, \partial_z \overline{\eta}(x,t,\varepsilon^{-1} d).\end{aligned}$$

Then, setting $y = \pi(x,t)$ and $z = \varepsilon^{-1} d(x,t)$ and noting that

$$R(x,t) = \left\{ I_n - \varepsilon z W(y,t) \right\}^{-1} = I_n + \varepsilon z W(y,t) + O(\varepsilon^2) \tag{5.2}$$

by the Neumann series expansion, we have

$$\partial_t \rho(x,t) = -\varepsilon^{-1} V_\Gamma(y,t) \partial_z \eta(y,t,z) + \partial^\circ \eta(y,t,z) + O(\varepsilon). \tag{5.3}$$

Next we differentiate (5.1) with respect to x_i and apply (2.8) and $\nabla d = \overline{\nu}$ in $\overline{N_T}$ to get

$$\partial_i \rho = \sum_{j=1}^n R_{ij} \overline{\underline{D}_j \eta}(x,t,\varepsilon^{-1} d) + \varepsilon^{-1} \overline{\nu}_i \partial_z \overline{\eta}(x,t,\varepsilon^{-1} d) \tag{5.4}$$

for $i = 1, \dots, n$. Then, we use (5.2) with $y = \pi(x,t)$ and $z = \varepsilon^{-1}d(x,t)$ to find that

$$\nabla\rho(x,t) = \varepsilon^{-1}\partial_z\eta(y,t,z)\nu(y,t) + \nabla_\Gamma\eta(y,t,z) + \varepsilon z W(y,t)\nabla_\Gamma\eta(y,t,z) + O(\varepsilon^2). \tag{5.5}$$

Also, we apply ∂_i to (5.4) and use (2.8) and $\nabla d = \overline{\nu}$ in $\overline{N_T}$ to get

$$\begin{aligned}\partial_i\partial_i\rho = {}& \sum_{j=1}^n(\partial_i R_{ij})\overline{\underline{D}_j\eta}(x,t,\varepsilon^{-1}d) + \sum_{j,k=1}^n R_{ij}R_{ik}\overline{\underline{D}_k\underline{D}_j\eta}(x,t,\varepsilon^{-1}d)\\ &+ \sum_{j=1}^n R_{ij}\left\{\varepsilon^{-1}\overline{\nu}_i\partial_z(\overline{\underline{D}_j\eta})(x,t,\varepsilon^{-1}d)\right\}\\ &+ \varepsilon^{-1}\sum_{j=1}^n R_{ij}(\overline{\underline{D}_j\nu_i})\partial_z\overline{\eta}(x,t,\varepsilon^{-1}d)\\ &+ \varepsilon^{-1}\overline{\nu}_i\sum_{j=1}^n R_{ij}\overline{\underline{D}_j(\partial_z\eta)}(x,t,\varepsilon^{-1}d) + \varepsilon^{-2}\overline{\nu}_i^2\partial_z^2\overline{\eta}(x,t,\varepsilon^{-1}d).\end{aligned}$$

We take the sum of both sides for $i = 1, \dots, n$. Then, since

$$R^T\overline{\nu} = R\overline{\nu} = (I_n - d\overline{W})^{-1}\overline{\nu} = \overline{\nu}, \quad \text{i.e.,} \quad \sum_{i=1}^n R_{ij}\overline{\nu}_i = \overline{\nu}_j \quad \text{in} \quad \overline{N_T}$$

for $j = 1, \dots, n$, by the symmetry of W and $W\nu = 0$ on $\overline{S_T}$; and since

$$R_{ij}(x,t) = \delta_{ij} + \varepsilon z W_{ij}(y,t) + O(\varepsilon^2), \quad \partial_i R_{ij}(x,t) = (\nu_i W_{ij})(y,t) + O(\varepsilon)$$

for $(x,t) \in \overline{Q_{\varepsilon,T}}$ and $i, j = 1, \dots, n$ by (2.11) with $|d| \le c\varepsilon$ in $\overline{Q_{\varepsilon,T}}$ and by (5.2), where δ_{ij} is the Kronecker delta and $y = \pi(x,t)$ and $z = \varepsilon^{-1}d(x,t)$; and since

$$|\nu|^2 = 1, \quad \underline{D}_j\nu_i = -W_{ji}, \quad \sum_{i=1}^n \underline{D}_i\nu_i = \mathrm{div}_\Gamma\nu = -H \quad \text{on} \quad \overline{S_T}, \tag{5.6}$$

we have

$$\begin{aligned}\Delta\rho(x,t) = {}& \sum_{i,j=1}^n(\nu_i W_{ij})(y,t)\underline{D}_j\eta(y,t,z) + \Delta_\Gamma\eta(y,t,z)\\ &+ \varepsilon^{-1}\sum_{j=1}^n \nu_j(y,t)\partial_z(\underline{D}_j\eta)(y,t,z)\\ &- \varepsilon^{-1}H(y,t)\partial_z\eta(y,t,z) - z\sum_{i,j=1}^n(W_{ij}W_{ji})(y,t)\partial_z\eta(y,t,z)\\ &+ \varepsilon^{-1}\sum_{j=1}^n \nu_j(y,t)\underline{D}_j(\partial_z\eta)(y,t,z) + \varepsilon^{-2}\partial_z^2\eta(y,t,z) + O(\varepsilon)\end{aligned}$$

with $y = \pi(y,t)$ and $z = \varepsilon^{-1}d(x,t)$. Moreover, we see that

$$\sum_{i,j=1}^{n}(\nu_i W_{ij})(y,t)\underline{D}_j\eta(y,t,z) = (W^T\nu)(y,t)\cdot\nabla_\Gamma\eta(y,t,z) = 0,$$
$$\sum_{i,j=1}^{n}(W_{ij}W_{ji})(y,t) = \sum_{i,j=1}^{n}W_{ij}(y,t)^2 = |W(y,t)|^2 \tag{5.7}$$

by $W^T = W$ and $W\nu = 0$ on $\overline{S_T}$, and that

$$\sum_{j=1}^{n}\nu_j(y,t)\partial_z(\underline{D}_j\eta)(y,t,z) = \partial_z(\nu(y,t)\cdot\nabla_\Gamma\eta(y,t,z)) = 0,$$
$$\sum_{j=1}^{n}\nu_j(y,t)\underline{D}_j(\partial_z\eta)(y,t,z) = \nu(y,t)\cdot[\nabla_\Gamma(\partial_z\eta)](y,t,z) = 0,$$

since ν is independent of the variable z and $\nu\cdot\nabla_\Gamma = 0$. Therefore,

$$\begin{aligned}\Delta\rho(x,t) &= \varepsilon^{-2}\partial_z^2\eta(y,t,z) - \varepsilon^{-1}H(y,t)\partial_z\eta(y,t,z)\\ &\quad - z|W(y,t)|^2\partial_z\eta(y,t,z) + \Delta_\Gamma\eta(y,t,z) + O(\varepsilon).\end{aligned} \tag{5.8}$$

Recall that we take $(x,t)\in\overline{Q_{\varepsilon,T}}$ and set $y = \pi(x,t)$ and $z = \varepsilon^{-1}d(x,t)$.

Remark 5.1. Asymptotic expansions (5.3), (5.5), and (5.8) were already given in the study of an asymptotic expansion of a solution to the Cahn–Hilliard equation [10,43,46]. The expressions given there look slightly different from (5.3), (5.5), and (5.8), but one may observe that they are the same if one uses (2.8), (5.2), and

$$\Delta d(x,t) = \operatorname{div}(\overline{\nu}(x,t)) = -H(y,t) - \varepsilon z|W(y,t)|^2 + O(\varepsilon^2)$$

for $(x,t)\in\overline{Q_{\varepsilon,T}}$ with $y = \pi(x,t)$ and $z = \varepsilon^{-1}d(x,t)$, which follows from $\nabla d = \overline{\nu}$ in $\overline{N_T}$, (2.8), (5.2), (5.6), and (5.7).

5.2. Asymptotic expansion of the heat equation in the moving thin domain

Now let ρ^ε be a solution to the thin domain problem given by (1.1). To derive limit equation (1.2) and the approximate solution ρ_η^ε of the form in (4.2), we consider an asymptotic expansion of ρ^ε as in the case of a flat stationary thin domain (see, e.g., [34, Section 15.1]) but in a slightly simplified form: we assume that ρ^ε and f^ε are of the form

$$\rho^\varepsilon(y+\varepsilon z\nu(y,t),t) = \sum_{k=0}^{\infty}\varepsilon^k\eta_k(y,t,z), \quad (y,t)\in\overline{S_T},\ z\in[g_0(y,t),g_1(y,t)],$$
$$f^\varepsilon(y+\varepsilon z\nu(y,t),t) = f(y,t) + O(\varepsilon), \quad (y,t)\in S_T,\ z\in(g_0(y,t),g_1(y,t)),$$

or equivalently,

$$\begin{aligned} \rho^\varepsilon(x,t) &= \sum_{k=0}^{\infty} \varepsilon^k \eta_k(\pi(x,t),t,\varepsilon^{-1}d(x,t)), \quad (x,t) \in \overline{Q_{\varepsilon,T}}, \\ f^\varepsilon(x,t) &= f(\pi(x,t),t) + O(\varepsilon), \quad (x,t) \in Q_{\varepsilon,T}, \end{aligned} \tag{5.9}$$

where the functions $\eta_k(y,t,z)$ are independent of ε and $f(y,t)$ is a function of $(y,t) \in S_T$ independent of ε. Our aim is to give η_0, η_1, and η_2 for which the right-hand side of (5.9) satisfies (1.1) approximately of order ε in $Q_{\varepsilon,T}$ and ε^2 on $\partial_\ell Q_{\varepsilon,T}$. In what follows, we set $y = \pi(x,t)$ for $(x,t) \in \overline{Q_{\varepsilon,T}}$ and suppress the arguments y and t of $\eta_k(y,t,z)$ and functions on $\overline{S_T}$.

First we consider the heat equation in $Q_{\varepsilon,T}$. We see by (5.3) that

$$\partial_t \rho^\varepsilon(x,t) = -\varepsilon^{-1} V_\Gamma \partial_z \eta_0(z) + \partial^\circ \eta_0(z) - V_\Gamma \partial_z \eta_1(z) + O(\varepsilon)$$

for $(x,t) \in Q_{\varepsilon,T}$ and $z = \varepsilon^{-1} d(x,t) \in (g_0, g_1)$. Also, by (5.8),

$$\begin{aligned} \Delta \rho^\varepsilon(x,t) = \varepsilon^{-2} \partial_z^2 \eta_0(z) + \varepsilon^{-1} \bigl\{ -H \partial_z \eta_0(z) + \partial_z^2 \eta_1(z) \bigr\} \\ - z|W|^2 \partial_z \eta_0(z) + \Delta_\Gamma \eta_0(z) - H \partial_z \eta_1(z) + \partial_z^2 \eta_2(z) + O(\varepsilon). \end{aligned}$$

We substitute these expressions for

$$\partial_t \rho^\varepsilon(x,t) - k_d \Delta \rho^\varepsilon(x,t) = f^\varepsilon(x,t) = f + O(\varepsilon), \quad (x,t) \in Q_{\varepsilon,T}.$$

Then, for $z \in (g_0, g_1)$, we have

$$\begin{aligned} \varepsilon^{-2} \bigl\{ -k_d \partial_z^2 \eta_0(z) \bigr\} + \varepsilon^{-1} \bigl\{ -V_\Gamma \partial_z \eta_0(z) + k_d H \partial_z \eta_0(z) - k_d \partial_z^2 \eta_1(z) \bigr\} \\ + \partial^\circ \eta_0(z) - V_\Gamma \partial_z \eta_1(z) + k_d z |W|^2 \partial_z \eta_0(z) \\ - k_d \Delta_\Gamma \eta_0(z) + k_d H \partial_z \eta_1(z) - k_d \partial_z^2 \eta_2(z) = f + O(\varepsilon). \end{aligned}$$

Since each function in this equation is independent of ε, it follows that

$$-k_d \partial_z^2 \eta_0(z) = 0, \tag{5.10}$$

$$-V_\Gamma \partial_z \eta_0(z) + k_d H \partial_z \eta_0(z) - k_d \partial_z^2 \eta_1(z) = 0, \tag{5.11}$$

and

$$\begin{aligned} \partial^\circ \eta_0(z) - V_\Gamma \partial_z \eta_1(z) + k_d z |W|^2 \partial_z \eta_0(z) \\ - k_d \Delta_\Gamma \eta_0(z) + k_d H \partial_z \eta_1(z) - k_d \partial_z^2 \eta_2(z) = f \end{aligned} \tag{5.12}$$

for $z \in (g_0, g_1)$.

Next we deal with the boundary condition on $\partial_\ell Q_{\varepsilon,T}$. For $t \in (0,T]$ and $i = 0, 1$, let $x \in \Gamma_\varepsilon^i(t)$ and $z = \varepsilon^{-1} d(x,t) = g_i$. Since ν_ε and V_ε are of the form in (2.34) and (2.35), respectively, the boundary condition is equivalent to

$$[(\bar{\nu} - \varepsilon \bar{\tau}_\varepsilon^i) \cdot \nabla \rho^\varepsilon](x,t) + k_d^{-1} \bigl[(\overline{V_\Gamma} + \varepsilon \overline{\partial^\circ g_i} + \varepsilon^2 \bar{g}_i \bar{\tau}_\varepsilon^i \cdot \overline{\nabla_\Gamma V_\Gamma}) \rho^\varepsilon \bigr](x,t) = 0. \tag{5.13}$$

Here τ_ε^i is given by (2.31) and thus,

$$\overline{\tau}_\varepsilon^i(x,t) = (I_n - \varepsilon g_i W)^{-1}\nabla_\Gamma g_i = \nabla_\Gamma g_i + \varepsilon g_i W \nabla_\Gamma g_i + O(\varepsilon^2),$$

by the Neumann series expansion. Also, we see by (5.5) that (note that here $z = g_i$)

$$\begin{aligned}\nabla\rho^\varepsilon(x,t) = \varepsilon^{-1}\partial_z\eta_0(z)\nu + \nabla_\Gamma\eta_0(z) + \partial_z\eta_1(z)\nu \\ + \varepsilon\{zW\nabla_\Gamma\eta_0(z) + \nabla_\Gamma\eta_1(z) + \partial_z\eta_2(z)\nu\} + O(\varepsilon^2).\end{aligned}$$

By the above expressions, $|\nu|^2 = 1$, $W^T\nu = W\nu = 0$, and $\nu\cdot\nabla_\Gamma = 0$, we get

$$\begin{aligned}[(\overline{\nu} - \varepsilon\overline{\tau}_\varepsilon^i)\cdot\nabla\rho^\varepsilon](x,t) = \varepsilon^{-1}\partial_z\eta_0(z) + \partial_z\eta_1(z) \\ + \varepsilon\big\{\partial_z\eta_2(z) - \nabla_\Gamma g_i\cdot\nabla_\Gamma\eta_0(z)\big\} + O(\varepsilon^2).\end{aligned}$$

Note that $\overline{\tau}_\varepsilon^i$ in the left-hand side is multiplied by ε. We also have

$$\begin{aligned}&\big[(\overline{V_\Gamma} + \varepsilon\,\overline{\partial^\circ g_i} + \varepsilon^2\overline{g}_i\overline{\tau}_\varepsilon^i\cdot\overline{\nabla_\Gamma V_\Gamma})\rho^\varepsilon\big](x,t) \\ &\quad = V_\Gamma\eta_0(z) + \varepsilon\big\{(\partial^\circ g_i)\eta_0(z) + V_\Gamma\eta_1(z)\big\} + O(\varepsilon^2),\end{aligned}$$

and we substitute the above expressions for (5.13) to obtain

$$\begin{aligned}&\varepsilon^{-1}\partial_z\eta_0(z) + \partial_z\eta_1(z) + k_d^{-1}V_\Gamma\eta_0(z) \\ &\quad + \varepsilon\big\{\partial_z\eta_2(z) - \nabla_\Gamma g_i\cdot\nabla_\Gamma\eta_0(z) + k_d^{-1}(\partial^\circ g_i)\eta_0(z) + k_d^{-1}V_\Gamma\eta_1(z)\big\} = O(\varepsilon^2).\end{aligned}$$

Since each function in the left-hand side is independent of ε, and since the above equality holds at $z = g_i$ (recall that we suppress the arguments y and t), we find that

$$\partial_z\eta_0(g_i) = 0, \tag{5.14}$$

$$\partial_z\eta_1(g_i) + k_d^{-1}V_\Gamma\eta_0(g_i) = 0, \tag{5.15}$$

and

$$\partial_z\eta_2(g_i) - \nabla_\Gamma g_i\cdot\nabla_\Gamma\eta_0(g_i) + k_d^{-1}(\partial^\circ g_i)\eta_0(g_i) + k_d^{-1}V_\Gamma\eta_1(g_i) = 0 \tag{5.16}$$

for $i = 0, 1$.

Now let us determine η_0, η_1, and η_2. By (5.10) and (5.14), we see that

$$\partial_z^2\eta_0(z) = 0, \quad z\in(g_0,g_1), \qquad \partial_z\eta_0(g_i) = 0, \quad i = 0,1. \tag{5.17}$$

Hence, $\partial_z\eta_0(z) = 0$ for $z\in[g_0,g_1]$ and $\eta_0(z) = \eta_0$ is independent of z. Note that at this point we do not know what equation η_0 should satisfy on S_T. Next we have

$$\partial_z^2\eta_1(z) = 0, \quad z\in(g_0,g_1), \quad \partial_z\eta_1(g_i) = -k_d^{-1}V_\Gamma\eta_0, \quad i = 0,1$$

by (5.11), (5.15), and $\eta_0(z) = \eta_0$. Hence,

$$\partial_z\eta_1(z) = -k_d^{-1}V_\Gamma\eta_0, \quad \eta_1(z) = \eta_1(g_0) - k_d^{-1}(z-g_0)V_\Gamma\eta_0, \quad z\in[g_0,g_1].$$

Here we may choose $\eta_1(g_0)$ arbitrarily, since we only consider the approximation of (1.1) of order ε in $Q_{\varepsilon,T}$ and ε^2 on $\partial_\ell Q_{\varepsilon,T}$. We take $\eta_1(g_0) = -k_d^{-1} g_0 V_\Gamma \eta_0$ so that

$$\eta_1(z) = -k_d^{-1} z V_\Gamma \eta_0, \quad z \in [g_0, g_1]. \tag{5.18}$$

However, if we require a higher-order approximation of (1.1), then $\eta_1(g_0)$ should be determined by a higher-order asymptotic expansion of (1.1). For η_2, we have

$$\begin{cases} \partial_z^2 \eta_2(z) = k_d^{-1} \partial^\circ \eta_0 + k_d^{-2} V_\Gamma^2 \eta_0 - \Delta_\Gamma \eta_0 \\ \qquad\qquad - k_d^{-1} V_\Gamma H \eta_0 - k_d^{-1} f, & z \in (g_0, g_1), \\ \partial_z \eta_2(g_i) = \nabla_\Gamma g_i \cdot \nabla_\Gamma \eta_0 - k_d^{-1} (\partial^\circ g_i) \eta_0 + k_d^{-2} g_i V_\Gamma^2 \eta_0, & i = 0, 1 \end{cases} \tag{5.19}$$

by (5.12), (5.16), $\eta_0(z) = \eta_0$, and (5.18). In order that η_2 exists, it is necessary that

$$\int_{g_0}^{g_1} \partial_z^2 \eta_2(z)\, dz = \partial_z \eta_2(g_1) - \partial_z \eta_2(g_0).$$

Then, noting that the right-hand side of the first equation of (5.19) is independent of z, we substitute (5.19) for the above equality and use $g_1 - g_0 = g$ to get

$$\begin{aligned} &g\{k_d^{-1} \partial^\circ \eta_0 + k_d^{-2} V_\Gamma^2 \eta_0 - \Delta_\Gamma \eta_0 - k_d^{-1} V_\Gamma H \eta_0 - k_d^{-1} f\} \\ &\quad = \nabla_\Gamma g \cdot \nabla_\Gamma \eta_0 - k_d^{-1} (\partial^\circ g) \eta_0 + k_d^{-2} g V_\Gamma^2 \eta_0, \end{aligned} \tag{5.20}$$

and this equation can be rewritten as

$$\partial^\circ (g \eta_0) - g V_\Gamma H \eta_0 - k_d \operatorname{div}_\Gamma (g \nabla_\Gamma \eta_0) = g f. \tag{5.21}$$

Hence, we obtain limit equation (1.2) as the necessary condition on the zeroth-order term η_0 in order that the second-order term η_2 exists. Moreover, when (5.20) is satisfied, we can rewrite (5.19) as

$$\begin{cases} \partial_z^2 \eta_2(z) = \dfrac{1}{g} \{\nabla_\Gamma g \cdot \nabla_\Gamma \eta_0 - k_d^{-1} (\partial^\circ g) \eta_0 + k_d^{-2} g V_\Gamma^2 \eta_0\}, & z \in (g_0, g_1), \\ \partial_z \eta_2(g_i) = \nabla_\Gamma g_i \cdot \nabla_\Gamma \eta_0 - k_d^{-1} (\partial^\circ g_i) \eta_0 + k_d^{-2} g_i V_\Gamma^2 \eta_0, & i = 0, 1. \end{cases}$$

Then, setting

$$\zeta_i = \frac{1}{g} \{\nabla_\Gamma g_i \cdot \nabla_\Gamma \eta_0 - k_d^{-1} (\partial^\circ g_i) \eta_0 + k_d^{-2} g_i V_\Gamma^2 \eta_0\}, \quad i = 0, 1, \tag{5.22}$$

we see that the above problem is of the form

$$\partial_z^2 \eta_2(z) = \zeta_1 - \zeta_0, \quad z \in (g_0, g_1), \qquad \partial_z \eta_2(g_i) = g \zeta_i, \quad i = 0, 1.$$

Hence, $\partial_z \eta_2(z) = g \zeta_0 + (z - g_0)(\zeta_1 - \zeta_0)$ for $z \in [g_0, g_1]$ and

$$\begin{aligned} \eta_2(z) &= \eta_2(g_0) + (z - g_0) g \zeta_0 + \frac{1}{2} (z - g_0)^2 (\zeta_1 - \zeta_0) \\ &= \left\{\eta_2(g_0) - g_0 g \zeta_0 + \frac{1}{2} g_0^2 (\zeta_1 - \zeta_0)\right\} + z\{g \zeta_0 - g_0(\zeta_1 - \zeta_0)\} + \frac{1}{2} z^2 (\zeta_1 - \zeta_0) \end{aligned}$$

for $z \in [g_0, g_1]$. Moreover, as in the case of $\eta_1(g_0)$, we may take

$$\eta_2(g_0) = g_0 g \zeta_0 - \frac{1}{2} g_0^2 (\zeta_1 - \zeta_0).$$

Hence, noting that $g\zeta_0 - g_0(\zeta_1 - \zeta_0) = g_1\zeta_0 - g_0\zeta_1$, we obtain

$$\eta_2(z) = z(g_1\zeta_0 - g_0\zeta_1) + \frac{1}{2} z^2 (\zeta_1 - \zeta_0), \quad z \in [g_0, g_1]. \tag{5.23}$$

Finally, we conclude by (5.9), (5.17), (5.18), and (5.21)–(5.23) that if we set

$$\rho_\eta^\varepsilon(x,t) = \sum_{k=0}^{2} \varepsilon^k \eta_k(\pi(x,t), t, \varepsilon^{-1} d(x,t)), \quad (x,t) \in \overline{Q_{\varepsilon,T}},$$

where $\eta_0(y,t,z) = \eta_0(y,t)$ is independent of z and satisfies (1.2), then ρ_η^ε is of the form in (4.2) and satisfies (1.1) approximately of order ε in $Q_{\varepsilon,T}$ and ε^2 on $\partial_\ell Q_{\varepsilon,T}$.

6. Conclusion

We compare a classical solution ρ^ε to heat equation (1.1) in the moving thin domain $\Omega_\varepsilon(t)$ of thickness of order ε and a classical solution η to limit equation (1.2) on the moving surface $\Gamma(t)$ which appears in the thin-film limit of (1.1) as $\varepsilon \to 0$. Our main result consists in error estimates (1.5) and (1.6) of order ε in the sup-norm for the difference of ρ^ε and η, which avoids the ambiguity due to the volume of $\Omega_\varepsilon(t)$. The proof is based on the uniform a priori estimate (see (3.14)) for a classical solution to the thin domain problem (see (3.13)) with non-zero boundary data and a construction of a suitable approximate solution to (1.1) from a classical solution to (1.2). In order to prove the uniform a priori estimate given by (3.14), we carefully re-examine the maximum principle for parabolic equations in $\Omega_\varepsilon(t)$ in view of the dependence of coefficients and constants on ε. Also, we carry out a formal asymptotic expansion of (1.1) with respect to ε to find a suitable approximate solution to (1.1).

As a concluding remark, we mention that it may be possible to extend our result to the general parabolic equation given by (3.1) in $\Omega_\varepsilon(t)$ under suitable assumptions on the coefficients and the given data. To this end, one first finds a limit equation on $\Gamma(t)$ and an approximate solution to (3.1) by carrying out a formal asymptotic expansion of (3.1), as in Section 5, and then estimates the difference of classical and approximate solutions to (3.1) by using the uniform a priori estimate in (3.5) and the results in Section 2. In this case, however, a limit equation will be more complicated than (1.2) and, in particular, it will involve not only the limits (in some sense) as $\varepsilon \to 0$ of a_{ij}^ε, b_i^ε, c^ε, and f^ε in the first equation of (3.1), but also the limits of β^ε and ψ^ε appearing in the boundary condition of (3.1).

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