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Quantum Topology

When the theories meet: Khovanov homology as Hochschild homology of links

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Abstract. We show that Khovanov homology and Hochschild homology theories share a common structure. In fact they overlap: Khovanov homology of the $(2,n)$ torus link can be interpreted as a Hochschild homology of the algebra underlining the Khovanov homology. In the classical case of Khovanov homology we prove the concrete connection. In the general case of Khovanov–Rozansky $\mathrm{sl}(n)$ homology and their deformations we conjecture the connection. The best framework to explore our ideas is to use a comultiplication-free version of Khovanov homology for graphs developed by L. Helme-Guizon and Y. Rong and extended here to the $\mathbb{M}$-reduced case, and in the case of a polygon extended to noncommutative algebras. In this framework we prove that for any unital algebra $\mathcal{A}$ the Hochschild homology of $\mathcal{A}$ is isomorphic to graph cohomology over $\mathcal{A}$ of a polygon. We expect that this paper will encourage a flow of ideas in both directions between Hochschild/cyclic homology and Khovanov homology theories.

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1. Hochschild homology and cyclic homology

We recall in this section definitions of Hochschild homology and cyclic homology and sketch two classical calculations for tensor algebras and symmetric tensor algebras. More calculations are reviewed in Section 4 in which we use our main result, Theorem 1.3, to obtain new results in Khovanov homology, in particular solving some conjectures from [8]. We follow [21] in our exposition of Hochschild homology.

Let k be a commutative ring and $\mathcal{A}$ a k-algebra (not necessarily commutative). Let $\mathbb{M}$ be a bimodule over $\mathcal{A}$ that is a k-module on which $\mathcal{A}$ operates linearly on the left and on the right in such a way that $(am)a' = a(ma')$ for $a, a' \in \mathcal{A}$ and $m \in \mathbb{M}$. The actions of $\mathcal{A}$ and k are always compatible (e.g., $m(\lambda a) = (m\lambda)a = \lambda(ma)$).

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When $\mathcal{A}$ has a unit element 1 we always assume that $1m = m1 = m$ for all $m \in \mathbb{M}$. Under this unital hypothesis, the bimodule $\mathbb{M}$ is equivalent to a right $\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}$-module via $m(a' \otimes a) = ama'$. Here $\mathcal{A}^{\mathrm{op}}$ denotes the opposite algebra of $\mathcal{A}$ that is $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are the same as sets but the product $a \cdot b$ in $\mathcal{A}^{\mathrm{op}}$ is the product ba in $\mathcal{A}$. The product map of $\mathcal{A}$ is usually denoted $\mu\colon \mathcal{A} \otimes \mathcal{A} \to A$, $\mu(a,b) = ab$.

In this paper we work only with unital algebras. We also assume, unless otherwise stated, that $\mathcal{A}$ is a free k-module, however in most cases, it suffices to assume that $\mathcal{A}$ is k-projective, or less restrictively, that $\mathcal{A}$ is flat over k. Throughout the paper the tensor product $\mathcal{A} \otimes \mathcal{B}$ denotes the tensor product over k, that is, $\mathcal{A} \otimes_k \mathcal{B}$.

Definition 1.1 ([11], [21]). The Hochschild chain complex $C_*(\mathcal{A}, \mathbb{M})$ is defined as

$$\cdots \xrightarrow{b} \mathbb{M} \otimes \mathcal{A}^{\otimes n} \xrightarrow{b} \mathbb{M} \otimes \mathcal{A}^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} \mathbb{M} \otimes \mathcal{A} \xrightarrow{b} \mathbb{M},$$

where $C_n(\mathcal{A}, \mathbb{M}) = \mathbb{M} \otimes \mathcal{A}^{\otimes n}$ and the Hochschild boundary is the k-linear map $b\colon \mathbb{M} \otimes \mathcal{A}^{\otimes n} \to \mathbb{M} \otimes \mathcal{A}^{\otimes n-1}$ given by the formula $b = \sum_{i=0}^{n}(-1)^i d_i$, where the face maps d_i are given by

$$\begin{aligned} d_0(m, a_1, \ldots, a_n) &= (ma_1, a_2, \ldots, a_n),\\ d_i(m, a_1, \ldots, a_n) &= (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \quad \text{for } 1 \le i \le n-1,\\ d_n(m, a_1, \ldots, a_n) &= (a_n m, a_1, \ldots, a_{n-1}). \end{aligned}$$

In the case when $\mathbb{M} = \mathcal{A}$ the Hochschild complex is called the *cyclic bar complex*. By definition, the n-th Hochschild homology group of the unital k-algebra $\mathcal{A}$ with coefficients in the $\mathcal{A}$-bimodule $\mathbb{M}$ is the n-th homology group of the Hochschild chain complex, denoted by $\mathrm{H}_n(\mathcal{A}, \mathbb{M})$. In the particular case $\mathbb{M} = \mathcal{A}$ we write $C_*(\mathcal{A})$ instead of $C_*(\mathcal{A}, \mathcal{A})$ and $\mathrm{HH}_*(\mathcal{A})$ instead of $\mathrm{H}_*(\mathcal{A}, \mathcal{A})$.

The algebra $\mathcal{A}$ acts on $C_n(\mathcal{A}, \mathbb{M})$ by $a \cdot (m, a_1, \ldots, a_n) = (am, a_1, \ldots, a_n)$. If $\mathcal{A}$ is a commutative algebra then the action commutes with boundary map b, therefore $\mathrm{H}_n(\mathcal{A}, \mathbb{M})$ (in particular, $\mathrm{HH}_*(\mathcal{A})$) is an $\mathcal{A}$-module.

If $\mathcal{A}$ is a graded algebra and $\mathbb{M}$ a coherently graded $\mathcal{A}$-bimodule, and the boundary maps are grading preserving, then the Hochschild chain complex is a bigraded chain complex with $b\colon C_{i,j}(\mathcal{A}, \mathbb{M}) \to C_{i-1,j}(\mathcal{A}, \mathbb{M})$, and $\mathrm{H}_{**}(\mathcal{A}, \mathbb{M})$ is a bigraded k-module. In the case of commutative $\mathcal{A}$ and $\mathcal{A}$-symmetric $\mathbb{M}$ (i.e., $am = ma$), $\mathrm{H}_{**}(\mathcal{A}, \mathbb{M})$ is bigraded $\mathcal{A}$-module. The main examples coming from the knot theory are $\mathcal{A}_m = \mathbb{Z}[x]/(x^m)$ (truncated polynomials over integers) and $\mathbb{M}$ the ideal in $\mathcal{A}_m$ generated by x^{m-1}.

We complete this survey by describing, after [21], two classical results in Hochschild homology – the computation of Hochschild homology for a tensor algebra and for a symmetric tensor algebra.

Theorem 1.2. *Let V be any k-module and let $\mathcal{A} = T(V) = k \oplus V \oplus V^{\otimes 2} \oplus \cdots$ be its tensor algebra. We denote by $\tau_n\colon V^{\otimes n} \to V^{\otimes n}$ the cyclic permutation,*

$\tau_n(v_1, \ldots, v_{n-1}, v_n) = (v_n, v_1, \ldots, v_{n-1})$. *Then the Hochschild homology of* $\mathcal{A} = T(V)$ *is:*

$\mathrm{HH}_0(\mathcal{A}) = \bigoplus_{i \geq 0} V^{\otimes i}/(1 - \tau_i)$;

$\mathrm{HH}_1(\mathcal{A}) = \bigoplus_{i \geq 1} (V^{\otimes i})^{\tau_i}$, *where* $(V^{\otimes i})^{\tau_i}$ *is the space of invariants, that is, the kernel of* $1 - \tau_i$;

$\mathrm{HH}_n(\mathcal{A}) = 0$ *for* $n \geq 2$.

The main idea of the proof is to show that there is a quasi-isomorphism[1] from the Hochschild chain complex of $T(V)$ to the "small" chain complex

$$C^{\text{small}}(T(V)) \colon \cdots \to 0 \to \mathcal{A} \otimes V \xrightarrow{\hat{b}} \mathcal{A},$$

where the module $\mathcal{A}$ is in degree 0 and where the map $\hat{b}$ is given by $\hat{b}(a \otimes v) = av - va$. Therefore $\hat{b}$ restricted to $V^{\otimes n-1} \otimes V$ is precisely $(1 - \tau_n) \colon V^{\otimes n} \to V^{\otimes n}$ and Theorem 1.2 follows. See Proposition 3.1.2 and Theorem 3.1.4 of [21].

Theorem 1.3. *Let* V *be a module over* k *and let* $S(V)$ *be the symmetric tensor algebra over* V; $S(V) = k \oplus V \oplus S^2(V) \oplus \cdots$. *If* V *is free of dimension* n *generated by* $x_1, \ldots, x_n$ *then* $S(V)$ *is the polynomial algebra* $k[x_1, \ldots, x_n]$. *Assume that* V *is a flat* k*-module (e.g., a free module). Then there is an isomorphism*

$$S(V) \otimes \Lambda^n V \cong \mathrm{HH}_n(S(V)),$$

where $\Lambda^* V$ *is the exterior algebra of* V. *If* V *is free of dimension* n *generated by* $x_1, \ldots, x_n$ *then* $\Lambda^m V$ *is a free* $\binom{n}{m}$*-dimensional* k*-module with a basis* $\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_m} \mid i_1 < i_2 < \cdots < i_m\}$.

The above theorem is a special case of Hochschild–Konstant–Rosenberg theorem about Hochschild homology of smooth algebras [12], which we discuss in Section 4. Here we stress, after Loday, that the isomorphism $\varepsilon_* \colon S(V) \otimes \Lambda^* V \to \mathrm{HH}_*(S(V))$ is induced by a chain map, which is not true in general for smooth algebras. $S(V) \otimes \Lambda^* V$ is a chain complex with the zero boundary maps. The chain map $\varepsilon \colon S(V) \otimes \Lambda^n V \to C_n(S(V), S(V))$ is given by $\varepsilon(a_0 \otimes a_1 \wedge \cdots \wedge a_n) = \varepsilon_n(a_0, a_1, \ldots, a_n)$, where ε_n is the antisymmetrization map given as the sum $\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma((a_0, a_1, \ldots, a_n))$ and the permutation $\sigma \in S_n$ acts by $\sigma((a_0, a_1, \ldots, a_n)) = (a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$.

In Section 2 we describe Khovanov homology of links and its ("comultiplication-free") version for graphs introduced in [10]. In order to compare them with Hochschild homology we offer various generalizations, relaxing the condition that underlying algebra needs to be commutative (in the case of a polygon or a line graph), and allowing an $\mathbb{M}$-reduced case. Another innovation in our presentation is that we start with cohomology of a functor from the category of subsets of a fixed set to the category

[1] A chain map $f \colon C \to C'$ is called a quasi-isomorphism or homologism if the induced map on homology, $f_* \colon \mathrm{H}_n(C) \to \mathrm{H}_n(C')$, is an isomorphism for all n.

of modules. In this setting we describe graph cohomology introduced in [10] and its generalizations to $\mathbb{M}$-reduced cohomology and to cohomology of a polygon or a line graphs with a noncommutative underlying algebra. A set in this case is the set of edges of a graph. Finally, we describe a generalization to "supersets" which allows homology for signed graphs, link diagrams and, in some cases, links (with the classical Khovanov homology as the main example). In examples of functors from "supersets" we utilize comultiplication in the underlying algebra $\mathcal{A}$. Following Khovanov [14], we assume that $\mathcal{A}$ is a Frobenius algebra [1], [18].

In Section 3 we prove our main result relating Hochschild homology $\mathrm{HH}_*(\mathcal{A})$ to graph cohomology and Khovanov homology of links, and the homology $\mathrm{H}_n(\mathcal{A}, \mathbb{M})$ to reduced ($\mathbb{M}$-reduced, more precisely) cohomology of graphs and links.

In Section 4 we use our main result to describe graph cohomology of polygons for various algebras solving, in particular, several problems from [8].

We envision the connection between Connes cyclic homology and Khovanov-type homology, possibly when analyzing symmetry of graphs and links. For the idea of cyclic homology it is the best to quote from J-L. Loday [21]:

> "… in his search for a non-commutative analogue of de Rham homology theory, A. Connes discovered in 1981 the following striking phenomenon:
>
> the Hochschild boundary map b is still well defined when one factors out the module $A \otimes A^{\otimes n} = A^{\otimes n+1}$ by the action of the (signed) cyclic permutation of order $n + 1$. Hence a new complex was born whose homology is now called (at least in characteristic zero) *cyclic homology*."

2. Khovanov homology

We start this section with a very abstract definition based on Khovanov construction but, initially, devoid of topological or geometric context. In this setting we recall and generalize the concept of graph cohomology [10] and of classical Khovanov homology for unoriented framed links. We follow, in part, the exposition in [9], [10] and [26]. We review, after [8], the connection between Khovanov homology of links and graph cohomology of associated Tait graphs. We define homology of link diagrams related to graph cohomology for any commutative algebra[2].

2.1. Cohomology of a functor on sets. Let k be a commutative ring and E a finite set.

Definition 2.1. Let Φ be a functor from the category of subsets of E (i.e., subsets of E are objects and inclusions are morphisms) to the category of k-modules. We define the "Khovanov cohomology" of the functor, $\mathrm{H}^i(\Phi)$, as follows. We start from the graded k-module $\{C^i(\Phi)\}$, where $C^i(\Phi)$ is the direct sum of $\Phi(s)$ over all

[2] We can extend the definition to noncommutative algebras in the case of $(2, n)$-torus link diagrams.

$s \subset E$ of i elements ($|s| = i$). To define $d: C^i(\Phi) \to C^{i+1}(\Phi)$ we first define face maps $d_e(s) = \Phi(s \subset s \cup e)(s)$ for $e \in E - s$ (notice that $s \subset s \cup e$ is the unique morphism in $\mathrm{Mor}(s, s \cup e)$). Now, as usual,[3] $d(s) = \sum_{e \notin s}(-1)^{t(s,e)} d_e(s)$, where $t(s,e)$ requires ordering of elements of E and is equal the number of elements of s smaller then e. Because Φ is a functor, we have $d_{e_2} d_{e_1}(s) = d_{e_1} d_{e_2}(s)$ for any $e_1, e_2 \notin s$. The sign convention guarantees that $d^2 = 0$ and $(\{C^i(\Phi)\}, d)$ is a cochain complex. Now we define, in a standard way, cohomology as $\mathrm{H}^i(\Phi) = \ker(d(C^i(\Phi) \to C^{i+1}(\Phi))/d(C^{i-1}(\Phi))$. The standard argument shows that $\mathrm{H}^i(\Phi)$ is independent on ordering of E.

In the case when E are edges of a graph G we can define specific functors in various ways taking into account a structure of G. We construct below our main example: a generalization of a graph cohomology, defined in [10], to $\mathbb{M}$-reduced case and its translation to homology of alternating diagrams. In the case of the algebra $\mathcal{A}_2 = \mathbb{Z}[x]/(x^2)$, this homology agrees partially with the classical Khovanov homology (see Theorem 2.7). To deal with all link diagrams we will later expand Definition 2.1 to "supersets" (Definition 2.4) in a construction which can incorporate multiplication and comultiplication in $\mathcal{A}$ (Example 2.5).

Definition 2.2 (Cohomology introduced in [10] and extended to $\mathbb{M}$-reduced case).

(1) We define here $\mathbb{M}$-reduced cohomology denoted by $\mathrm{H}^*_{\mathcal{A},\mathbb{M}}(G, v_1)$. If we assume that $\mathbb{M} = \mathcal{A}$ we obtain (comultiplication-free) cohomology of graphs, $\mathrm{H}^*_{\mathcal{A}}(G)$, defined in [10].

Let G be a graph with an edge set $E = E(G)$, and a chosen base vertex v_1. Fix a commutative k-algebra $\mathcal{A}$ and an $\mathcal{A}$-module $\mathbb{M}$. We define a functor Φ on a category of subsets of E as follows:

Objects: To define the functor Φ on objects $s \subset E$, we define it more generally on any subgraph $H \subset G$, starting from a connected H, to be $\Phi(H) = \mathbb{M}$ if $v_1 \in H$, and $\Phi(H) = \mathcal{A}$ if $v_1 \notin H$. If H has connected components $H_1, \ldots, H_k$ then we define $\Phi(H) = \Phi(H_1) \otimes \cdots \otimes \Phi(H_k)$. Finally, $\Phi(s) = \Phi([G:s])$, where $[G:s]$ is a subgraph of G containing all vertices of G and edges s.

In what follows $k(s)$ is the number of components of $[G:s]$.

Morphisms: It suffices to define $\Phi(s \subset s \cup e)$ where $e \notin s$. The definition depends now on the position of e with respect to $[G:s]$, as follows:

(i) Assume that e connects different components of $[G:s]$:

(i′) If e connects components u_i and u_{i+1} not containing v_1 then we define

$$\Phi(s \subset s \cup e)(m, a_1, \ldots, a_i, a_{i+1}, \ldots, a_{k(s)-1}) = (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_{k(s)-1}).$$

(i″) If e connects a component of $[G:s]$ containing v_1 with another component of $[G:s]$, say u_1, then we put $\Phi(s \subset s \cup e)(m, a_1, \ldots, a_{k(s)-1}) = (ma_1, \ldots, a_{k(s)-1})$.

[3] We build a cochain complex from a presimplicial category.

(ii) Assume that e connects vertices of the same component of $[G:s]$, then $\Phi(s \subset s \cup e)$ is the identity map on $\Phi(s) = \Phi(s \cup e) = \mathbb{M} \otimes \mathcal{A}^{\otimes k(s)-1}$

In the proof that Φ is a functor commutativity of $\mathcal{A}$ is important (compare (3)).

(2) We can modify the functor Φ from (1) to a new functor, $\widehat{\Phi}$, which differs from Φ only in the rule (1) (ii). That is, in the case in which e connects vertices of the same component of $[G:s]$ we put $\widehat{\Phi}(s \subset s \cup e)$ equal to zero. We denote by $\widehat{\mathrm{H}}^*_{\mathcal{A},\mathbb{M}}(G, v_1)$ the cohomology yielded by the functor $\widehat{\Phi}$. For $\mathbb{M} = \mathcal{A}$ this cohomology, $\widehat{\mathrm{H}}^*_{\mathcal{A}}(G)$, is introduced in [10].

(3) For a polygon or a line graph the cohomology $\widehat{\mathrm{H}}^*_{\mathcal{A},\mathbb{M}}(G, v_1)$ is also defined for a noncommutative algebra $\mathcal{A}$ and any $\mathcal{A}$-bimodule $\mathbb{M}$. We consider a polygon or a line graph as a directed graph: from left to right (in the case of a line graph, Figure 3.1) and in the anti-clockwise orientation (in the case of a polygon). In the formula for the morphism, $\hat{\Phi}(s \subset s \cup e)$, of Definition 2.2 (2) we use the product xy if x is the weight of the initial point of the directed edge e connecting different components of $[G:s]$. We use this graph cohomology of the directed polygon when comparing graph cohomology with Hochschild homology.

Notice that cohomology described in (1) and (2) coincide to certain degree. Namely $\mathrm{H}^i_{\mathcal{A},\mathbb{M}}(G, v_1) = \widehat{\mathrm{H}}^i_{\mathcal{A},\mathbb{M}}(G, v_1)$ for all $i < \ell - 1$, where ℓ is the girth of G, that is, the length of the shortest cycle in G.

Furthermore, if k is a principal ideal domain (e.g., $k = \mathbb{Z}$) and $\mathcal{A}$ and $\mathbb{M}$ are free k-modules then $\mathrm{Tor}(\mathrm{H}^i_{\mathcal{A},\mathbb{M}}(G, v_1)) = \mathrm{Tor}(\widehat{\mathrm{H}}^i_{\mathcal{A},\mathbb{M}}(G, v_1))$ for $i = \ell - 1$ (compare Theorem 2.7).

Remark 2.3. (i) One can generalize[4] construction in Definition 2.2 (1) and (2) by choosing the sequence of elements $f_1, f_2, f_3, \ldots, f_{|E|}$ in $\mathcal{A}$ and modifying functors Φ and $\hat{\Phi}$ on morphisms to get the functors Φ' and $\hat{\Phi}'$. We put $\Phi'(s \subset s \cup e) = f_{|s|+1}\Phi(s \subset s \cup e)$ and, similarly, $\hat{\Phi}'(s \subset s \cup e) = f_{|s|+1}\hat{\Phi}(s \subset s \cup e)$.

(ii) If $\mathcal{A}$ is not commutative and we work with cohomology of a line graph or a polygon (as in Definition 2.2 (3)) we have to assume, in order to have $d^2 = 0$, that f_i's are in the center of $\mathcal{A}$. We define $f \cdot (m, a_1, \ldots, a_{k(s)-1})(s) = (fm, a_1, \ldots, a_{k(s)-1})(s)$.

We can define Khovanov cohomology on an alternating link diagram, D, by considering associated plane graph $G(D)$ (Tait graph; compare Figure 2.1) and its cohomology described in Definition 2.2 and Remark 2.3.

In order to define Khovanov cohomology on any link diagram (and take both multiplication and comultiplication into account) we have to define cohomology on any signed planar graph. We can start, as in Definition 2.1, from the very general setting

[4] We are motivated here by [25] and to catch more of the spirit of [25] we can consider two sets of constants: $f_1, f_2, \ldots, f_{|E|}$ and $f'_1, f'_2, \ldots, f'_{|E|}$ so that $f_i f'_{i+1} = f'_i f_{i+1}$. We use the constant f_i in the case the set s has i elements and e is connecting different components of $[G:s]$. We use the constant f'_i in the case the set s has i elements and e has endpoints on the same component of $[G:s]$.

(again cohomology of a functor) and produce specific examples of a cohomology of signed planar graphs using a coherent algebra and coalgebra structures (Frobenius algebra).

Definition 2.4. Let k be a commutative ring and $E = E_+ \cup E_-$ a finite set divided into two disjoint subsets (positive and negative sets). We consider the category of subsets of E ($E \supset s = s_+ \cup s_-$ where $s_\pm = s \cap E_\pm$). The set $\mathrm{Mor}(s, s')$ is either empty or has one element if $s_- \subset s'_-$ and $s_+ \supset s'_+$. Objects are graded by $\sigma(s) = |s_-| - |s_+|$. Let us call this category the *superset category* (as the set E is initially $\mathbb{Z}_2$-graded). We define "Khovanov cohomology" for every functor, Φ, from the superset category to the category of k-modules. We define cohomology of Φ in the similar way as for a functor from the category of sets (which corresponds to the case $E = E_-$). The cochain complex corresponding to Φ is defined to be $\{C^i(\Phi)\}$ where $C^i(\Phi)$ is the direct sum of $\Phi(s)$ over all $s \subset E$ with $\sigma(s) = i$. To define $d : C^i(\Phi) \to C^{i+1}(\Phi)$ we first define face maps $d_e(s)$ where $e = e_- \notin s_-$ ($e_- \in E_-$) or $e = e_+ \in s_+$. In such a case $d_{e_-}(s) = \Phi(s \subset s \cup e_-)(s)$ and $d_{e_+}(s) = \Phi(s \supset s - e_+)$. We define $d(s) = \sum_{e \notin s} (-1)^{t(s,e)} d_e(s)$, where $t(s, e)$ requires ordering of elements of E and is equal the number of elements of s_- smaller then e plus the number of elements of s_+ bigger than e. We obtain the cochain complex whose cohomology does not depend on ordering of E.

Example 2.5. Let G be a signed plane graph with an edge set $E = E_+ \cup E_-$, where E_+ is the set of positive edges and E_- is the set of negative edges. We define the functor from the superset category $\mathbb{E}$ using the fact that G is the (signed) Tait graph of a link diagram $D(G)$ with the infinite white region; see Figure 2.1 for conventions.

Figure 2.1

To define the functor Φ we fix a Frobenius algebra $\mathcal{A}$ with multiplication μ and comultiplication Δ (the main example used by Khovanov is the algebra of truncated polynomials $\mathcal{A}_m = \mathbb{Z}[x]/(x^m)$ with a coproduct $\Delta(x^k) = \sum_{i+j=m-1+k} x^i \otimes x^j$). To get our functor on objects, $\Phi(s)$, we consider the Kauffman state defined by s (so also denoted by s) and we associate $\mathcal{A}$ to every circle of D_s obtained from $D(G)$ by smoothing every crossing according to s and then taking tensor product of these copies of $\mathcal{A}$ (compare [26]). Define Φ on generating morphisms via product or coproduct depending on whether D_s has more or less circles than $D_{s'}$. For A_2 we obtain the classical Khovanov homology.

Remark 2.6. One can also extend Example 2.5 to include the concept of M-reduced cohomology. We can, for example, consider M to be an ideal in $\mathcal{A}$ with $\Delta(\mathbb{M}) \subset$

$M \otimes M$. An example, considered by Khovanov, is A_m with M generated by x^{m-1} (in which case $\Delta(x^{m-1}) = x^{m-1} \otimes x^{m-1}$).

One can build more delicate (co)homology theory for ribbon graphs (flat vertex graph) using the fact that they embed uniquely into the closed surface. For $A = A_2$ it can be achieved using the approach presented in [3], while for more general (Frobenius) algebras it is not yet done (most likely one should not use Frobenius algebra alone but its proper enhancement like in A_2 case).

In [8] we proved the following relation between graph cohomology and classical Khovanov homology of alternating links.

Theorem 2.7. *Let D be the diagram of an unoriented framed alternating link and let G be its Tait graph. Let ℓ be the length of the shortest cycle in G. For all $i < \ell - 1$, we have*

$$\mathrm{H}^{i,j}_{\mathcal{A}_2}(G) \cong \mathrm{H}_{a,b}(D) \quad \text{with} \begin{cases} a = E(G) - 2i, \\ b = E(G) - 2V(G) + 4j, \end{cases}$$

where $\mathrm{H}_{a,b}(D)$ are the Khovanov homology groups of the unoriented framed link defined by D, as explained in [26]. Furthermore, $\mathrm{Tor}(\mathrm{H}^{i,j}_{\mathcal{A}_2}(G)) = \mathrm{Tor}(\mathrm{H}_{a,b}(D))$ *for $i = \ell - 1$.*

We also speculate that for general $\mathrm{sl}(m)$ Khovanov–Rozansky homology [16], [17] the graph cohomology of an n-gon with $\mathcal{A} = \mathcal{A}_m = \mathbb{Z}[x]/(x^m)$ keeps essential information on $\mathrm{sl}(m)$-homology of torus links of type $(2, n)$.

3. Relation between Hochschild homology and Khovanov homology

The main goal of our paper is to demonstrate a relation between Khovanov homology and Hochschild homology. Initially I observed this connection by showing that for every unital commutative algebra $\mathcal{A}$ the graph cohomology of $(n+1)$-gon, $\mathrm{H}^i_{\mathcal{A}}(P_{n+1})$, is isomorphic to Hochschild homology of $\mathcal{A}$, $\mathrm{H}_{n-i}(\mathcal{A})$; $0 < i < n$. From this, via Theorem 2.7, relation between classical Khovanov homology of the $(2, n+1)$ torus link and Hochschild homology of $\mathcal{A}_2 = \mathbb{Z}[x]/(x^2)$, follows. This relation was also observed independently by Magnus Jacobsson [13].

In this paper we prove a more general result. In order to formulate it we use an extended version of (Khovanov-type) graph cohomology (working with noncommutative algebras and $\mathbb{M}$-reduced cohomology):

(i) We can work with a noncommutative algebra, because, as mentioned in Section 2 (Definition 2.2 (3)), for a polygon the graph cohomology is also defined for noncommutative algebras.

(ii) We can fix an $\mathcal{A}$-bialgebra $\mathbb{M}$ and compare $\mathbb{M}$-reduced Khovanov-type graph cohomology with the Hochschild homology of $\mathcal{A}$ with coefficients in $\mathcal{A}$-bimodule $\mathbb{M}$.

One can observe that we generalize notion of graph (co)homology while we keep the original definition of Hochschild homology. Our point is that the graph (co)homology is the proper generalization of Hochschild homology: from a polygon to any graph. We have this interpretation only for a commutative $\mathcal{A}$. It seems to be, that if one works with general graphs and not necessary commutative algebras then these algebras should satisfy some "multiface" properties. Very likely that planar algebras or operads provide the proper framework.[5]

The interpretation of Hochschild homology as a homology of $\mathcal{A}$ treated as an algebra over $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ allows us to use the standard tool of homological algebra, that is we find appropriate (partial) free resolution of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ module $\mathcal{A}$ using graph cochain complex of a line graph (Figure 3.1). The graph cohomology of the polygon is the cohomology obtained from this resolution. In Theorem 3.1 we use cohomology $\widehat{\mathrm{H}}^{i}_{\mathcal{A},\mathbb{M}}(P_{n+1})$ because we do not assume that $\mathcal{A}$ is commutative.

Theorem 3.1. *Let $\mathcal{A}$ be a unital algebra,[6] let $\mathbb{M}$ be an $\mathcal{A}$-bimodule and P_{n+1}, the $(n+1)$-gon. Then for $0 < i \leq n$ we have*

$$\widehat{\mathrm{H}}^{i}_{\mathcal{A},\mathbb{M}}(P_{n+1}) = \mathrm{H}_{n-i}(\mathcal{A}, \mathbb{M}).$$

Furthermore, if $\mathcal{A}$ is a graded algebra and $\mathbb{M}$ a coherently graded module then $\widehat{\mathrm{H}}^{i,j}_{\mathcal{A},\mathbb{M}}(P_{n+1}) = \mathrm{H}_{n-i,j}(\mathcal{A}, \mathbb{M})$ for $0 < i \leq n$ and every j.

Corollary 3.2. $\widehat{\mathrm{H}}^{i,j}_{\mathcal{A}}(P_{n+1}) = \mathrm{HH}_{n-i,j}(\mathcal{A})$ *for* $0 < i \leq n$ *and every* j. *Furthermore, for a commutative $\mathcal{A}$, $\mathrm{H}^{i,j}_{\mathcal{A}}(P_{n+1}) = \widehat{\mathrm{H}}^{i,j}_{\mathcal{A}}(P_{n+1})$ for $0 < i < n$, and $\mathrm{H}^{n,*}_{\mathcal{A}}(P_{n+1}) = 0$, $\widehat{\mathrm{H}}^{n,j}_{\mathcal{A}}(P_{n+1}) = \mathrm{HH}_{0,*}(\mathcal{A}) = \mathcal{A}$. For a general $\mathcal{A}$, $\widehat{\mathrm{H}}^{n,*}_{\mathcal{A}}(P_{n+1}) = \mathrm{HH}_{0,*}(\mathcal{A}) = \mathcal{A}/(ab - ba)$.*

Poof of Theorem 3.1. We consider graph cohomology for a unital, possibly noncommutative algebra $\mathcal{A}$ and any $\mathcal{A}$-bimodule $\mathbb{M}$. There is no difference in the proof between commutative and noncommutative case except that we have to prove some property of cohomology given in [10] for a commutative $\mathcal{A}$ (Lemma 3.3).

[5] The author's idea of working with directed graphs (quivers) seems to apply, as observed by Y. Rong, only for line graphs and polygons. However, P. Turner demonstrated at Knots in Washington XXVII Conference how to work with general graphs (http://atlas-conferences.com/cgi-bin/abstract/caxq-22).

[6] We assume in this paper that $\mathcal{A}$ is a free k-module, but one could relax the condition to have $\mathcal{A}$ to be projective or, more generally, flat over a commutative ring with identity k; compare [21]. We require $\mathcal{A}$ to be a unital algebra in order to have an isomorphism $\mathbb{M} \otimes_{\mathcal{A}^{\epsilon}} \mathcal{A}^{\otimes n+2} = \mathbb{M} \otimes \mathcal{A}^{\otimes n}$; the isomorphism is given by $\mathbb{M} \otimes_{\mathcal{A}^{\epsilon}} \mathcal{A}^{\otimes n+2} \ni (m, a_0, a_1, \ldots, a_n, a_{n+1}) \to (a_{n+1} m a_0, 1, a_1, \ldots, a_n, 1)$, which one can write succinctly as $(a_{n+1} m a_0, a_1, \ldots, a_n) \in \mathbb{M} \otimes \mathcal{A}^{\otimes n}$. We should stress that in $\mathbb{M} \otimes_{\mathcal{A}^{\epsilon}} \mathcal{A}^{\otimes n+2}$ the tensor product is taken over $\mathcal{A}^{\epsilon} = \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$, while in $\mathbb{M} \otimes \mathcal{A}^{\otimes n}$ the tensor product is taken over k.

The main idea of our proof is to interpret the graph cochain complex of a line graph as a (partial) resolution of $\mathcal{A}$. It was proved in [10] that $\mathrm{H}^i_{\mathcal{A}}$(line graph)$= 0$, $i > 0$, for a commutative algebra $\mathcal{A}$. We give here the proof for any unital algebra $\mathcal{A}$. Let L_n be the (directed) line graph of $n+1$ vertices $(v_0, \dots, v_n)$ and n edges $(e_1, \dots, e_n)$, see Figure 3.1.

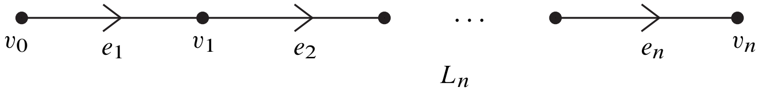

Figure 3.1

Lemma 3.3. *The graph cochain complex of L_n, $C^*_{\mathcal{A}}(L_n)\colon C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \cdots \to C^{n-1} \xrightarrow{d^{(n-1)}} C^n$, is acyclic, except for the first term. That is, $\widehat{\mathrm{H}}^i_{\mathcal{A}}(L_n) = 0$ for $i > 0$ and $\widehat{\mathrm{H}}^0_{\mathcal{A}}(L_n)$ is usually nontrivial.*[7]

For a line graph $\widehat{\mathrm{H}}_{\mathcal{A}} = \mathrm{H}_{\mathcal{A}}$ so we will use $\mathrm{H}_{\mathcal{A}}$ to simplify notation. We prove Lemma 3.3 by induction on n. For $n = 0$, L_0 is the one vertex graph, thus $\mathrm{H}^*_{\mathcal{A}} = \mathrm{H}^0_{\mathcal{A}} = \mathcal{A}$ and the Lemma 3.3 holds. Assume that the lemma holds for L_k with $k < n$. In order to perform inductive step we construct the long exact sequence of cohomology of line graphs. For a commutative $\mathcal{A}$ it is a special case[8] of the exact sequence of graph cohomology in [10], which in turn resemble the skein exact sequence of Khovanov homology [26]:

$$\begin{gathered} 0 \to \mathrm{H}^0_{\mathcal{A}}(L_n) \to \mathrm{H}^0_{\mathcal{A}}(L_{n-1}) \otimes \mathcal{A} \xrightarrow{\partial} \mathrm{H}^0_{\mathcal{A}}(L_{n-1}) \to \mathrm{H}^1_{\mathcal{A}}(L_n) \to \cdots \\ \cdots \to \mathrm{H}^{i-1}_{\mathcal{A}}(L_{n-1}) \to \mathrm{H}^i_{\mathcal{A}}(L_n) \to \mathrm{H}^i_{\mathcal{A}}(L_{n-1}) \otimes \mathcal{A} \to \cdots \end{gathered}$$

such that $\partial\colon \mathrm{H}^0_{\mathcal{A}}(L_n) \otimes \mathcal{A} \to \mathrm{H}^0_{\mathcal{A}}(L_{n-1})$ is an epimorphism. From this exact sequence the inductive step follows.

To construct the above exact sequence we consider the short exact sequence of chain complexes (for the notation see Figure 3.1):

$$0 \to C^{i-1}(L_n/e_n) \xrightarrow{\alpha} C^i(L_n) \xrightarrow{\beta} C^i(L_n - e_n) \to 0.$$

[7] From the fact that the chromatic polynomial of L_n is equal to $\lambda(\lambda-1)^n$ it follows that $\mathrm{rank}(\widehat{\mathrm{H}}^0_{\mathcal{A}}(L_n)) = \mathrm{rank}(\mathcal{A})(\mathrm{rank}(\mathcal{A})-1)^n$; we assume here that k is a principal ideal domain. It was proven in [10] that for a commutative $\mathcal{A}$ decomposable into $k1 \oplus \mathcal{A}/k$ one has that $\mathrm{H}^*_{\mathcal{A}}(L_n) = \mathrm{H}^0_{\mathcal{A}}(L_n) = \mathcal{A} \otimes (\mathcal{A}/k)^{\otimes n}$.

[8] One can construct an exact sequence of functor cohomology imitating deleting–contracting exact sequence. One have to define properly two functors on $E \cup e$, one “covariant” and one “contravariant” and the exact sequence will be based on a functor on subsets of E and these two additional functors. We will discuss this idea in a sequel paper.

This exact sequence is constructed in the same way as in the case of commutative $\mathcal{A}$, that is $\alpha(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-i})(e_{j_1}, \ldots, e_{j_{i-1}}) = (a_0 \otimes a_1 \otimes \cdots \otimes a_{n-i})(e_{j_1}, \ldots, e_{j_{i-1}}, e_n)$. Further, β is defined in such a way that if $e_n \in s$ then $\beta(S) = 0$, and if e is not in s then β is the identity map (up to $(-1)^{|s|}$).

Exactness of the sequence follows from the definition. This exact sequence leads to the long exact sequence of cohomology:

$$
\begin{aligned}
0 \to \mathrm{H}^0_{\mathcal{A}}(L_n) \to \mathrm{H}^0_{\mathcal{A}}(L_n - e_n) \xrightarrow{\partial} \mathrm{H}^0_{\mathcal{A}}(L_n/e_n) \to \cdots \\
\cdots \to \mathrm{H}^{i-1}_{\mathcal{A}}(L_n/e_n) \to \mathrm{H}^i_{\mathcal{A}}(L_n) \to \mathrm{H}^i_{\mathcal{A}}(L_n - e_n) \to \cdots .
\end{aligned}
$$

Now $L_{n-1} = L_n/e_n$ and $L_n - e_n$ is L_{n-1} with an additional isolated vertex, therefore by a Künneth formula (see for Example 1.0.16 of [21]) we have $\mathrm{H}^i_{\mathcal{A}}(L_n - e_n) = \mathrm{H}^i_{\mathcal{A}}(L_{n-1}) \otimes \mathcal{A}$. From this we get the exact sequence used in the proof of Lemma 3.3. To see surjectivity of ∂ notice that the map $\mathrm{H}^0_{\mathcal{A}}(L_n - e_n) \to \mathrm{H}^0_{\mathcal{A}}(L_{n-1})$ is a surjectivity almost by the definition (we decorate the last vertex of L_n by 1, to see the surjectivity).

More formally, there is a chain map epimorphism $\partial_c \colon C^i_{\mathcal{A}}(L_n - e_n) \to C^i_{\mathcal{A}}(L_{n-1})$ which is obtained by multiplying the weight of component containing v_{n-1} by the weight of v_n which descends to ∂. This map possesses a chain map section $\partial_c^{-1} \colon C^i_{\mathcal{A}}(L_{n-1}) \to C^i_{\mathcal{A}}(L_n - e_n)$ such that in the image v_n has always weight 1. Because $\partial_c \partial_c^{-1} = \mathrm{Id}$, on the cohomology level ∂ is a surjectivity.

We can continue now with the proof of Theorem 3.1. The (partially) acyclic chain complex of Lemma 3.3 is the chain complex of $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ modules. It is a (partial) free resolution of the $\mathcal{A}^e$-module $\mathcal{A}$. Upon tensoring this resolution with $\mathbb{M}$ considered as a right module over $\mathcal{A}^e$ we obtain the cochain complex

$$
\begin{aligned}
&\{\mathbb{M} \otimes_{\mathcal{A}^e} C^i\}_{i=0}^{n-1} \colon \\
&\mathbb{M} \otimes_{\mathcal{A}^e} C^0 \xrightarrow{\partial^0} \mathbb{M} \otimes_{\mathcal{A}^e} C^1 \xrightarrow{\partial^1} \cdots \to \mathbb{M} \otimes_{\mathcal{A}^e} C^{n-2} \xrightarrow{\partial^{n-1}} \mathbb{M} \otimes_{\mathcal{A}^e} C^{n-1} \to 0,
\end{aligned}
$$

whose cohomology (except possibly H^0) are the Hochschild homology of $\mathcal{A}$ with coefficients in $\mathbb{M}$ (compare for example [28]). Having in mind relation between indexing we get that $\mathrm{H}^i = \mathrm{H}_{n-i}(\mathcal{A}, \mathbb{M})$ for $i > 0$. To get exactly the chain complex of the $\mathbb{M}$-reduced (directed) graph cohomology of P_n, $\widehat{\mathrm{H}}^*_{\mathcal{A},\mathbb{M}}(P_n)$, we extend this chain complex to

$$
\begin{aligned}
&\{\mathbb{M} \otimes_{\mathcal{A}^e} C^i\}_{i=0}^{n} \colon \\
&\mathbb{M} \otimes_{\mathcal{A}^e} C^0 \xrightarrow{\partial^0} \mathbb{M} \otimes_{\mathcal{A}^e} C^1 \xrightarrow{\partial^1} \cdots \to \mathbb{M} \otimes_{\mathcal{A}^e} C^{n-1} \xrightarrow{\partial^{n-1}} \mathbb{M} \otimes_{\mathcal{A}^e} C^n = \mathbb{M} \to 0,
\end{aligned}
$$

where the homomorphism ∂^{n-1} is the zero map.

To complete the proof of Theorem 3.1 we show that this complex is exactly the same as the cochain complex of the $\mathbb{M}$-reduced (directed) graph cohomology of P_n. We consider carefully the map $\mathbb{M} \otimes_{\mathcal{A}^e} C^j \xrightarrow{\partial^j} \mathbb{M} \otimes_{\mathcal{A}^e} C^{j+1}$. In the calculation we follow the proof of Proposition 1.1.13 of [21]. The idea is to "bend" the line graph L_n to the polygon P_n and to show that this corresponds to tensoring over $\mathcal{A}^e$ with $\mathbb{M}$; see Figure 3.2.

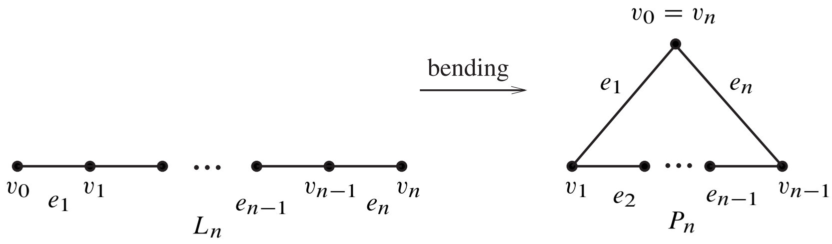

Figure 3.2

Let us order components of $[G : s]$ (G is equal to P_n or L_n) in the anti-clockwise orientation, starting from the component containing v_0 (decorated by an element of $\mathbb{M}$ if $G = P_n$). We denote the elements of $C^j_{\mathcal{A}}(L_n)$ and $C^j_{\mathcal{A},\mathbb{M}}(P_n)$ by $(a_{i_0}, a_{i_1}, \ldots, a_{i_{n-j-1}}, a_{i_{n-j}})(s)$ and $(m, a_{i_1}, \ldots, a_{i_{n-j-1}})(s)$, respectively. The element $(m, a_{i_0}, a_{i_1}, \ldots, a_{i_{n-j-1}}, a_{i_{n-j}})(s)$ is sent to $(a_{i_{n-j}} m a_{i_0}, a_{i_1}, \ldots, a_{i_{n-j-1}})(s)$ under the isomorphism $\mathbb{M} \otimes_{\mathcal{A}^e} C^j_{\mathcal{A}}(L_n) \to C^j_{\mathcal{A},\mathbb{M}}(P_n)$ ($j < n$). It is easily checked that this yields a cochain map (cf. [21]), so it induces the isomorphism on cohomology. Note that $(m, a_{i_0}, a_{i_1}, \ldots, a_{i_{n-j-1}}, a_{i_{n-j}}) = (a_{i_{n-j}} m a_{i_0}, 1, a_{i_1}, \ldots, a_{i_{n-j-1}}, 1)$ in $\mathbb{M} \otimes_{\mathcal{A}^e} C^j_{\mathcal{A}}(L_n)$. The proof of Theorem 3.1 is completed. □

4. Calculations and speculations

There is an extensive literature on Hochschild homology and a lot of ingenious methods of computing it (e.g., [21], [28], [23], [20]). Our main result, Theorem 3.1, allows us to use these methods to compute graph cohomology for polygons and, to some extent, for other graphs (using for example an observation that some properties of cohomology of a polygon propagate to graphs containing it (cf. [2], [8]). Properties of Hochschild homology (and, equally well, cyclic homology) should eventually shed light on Khovanov-type homology of links.

We start from adapting Theorem 1.3 about Hochschild homology of symmetric tensor algebra. The simplest case of one variable polynomials $\mathcal{A} = \mathcal{A}_\infty = \mathbb{Z}[x]$ allows us to extend Theorem 27 of [8] from the triangle to any polygon. $\mathcal{A}_\infty$ is a graded algebra with x^i being of degree i. Consequently Hochschild homology of $\mathcal{A}_\infty$ is a bigraded module. We treat $\mathcal{A}_\infty$ as a $\mathbb{Z}$-module (an abelian group) and to simplify description of homology we use the Poincaré polynomial of $\mathrm{HH}_{**}(\mathcal{A}_\infty)$ to describe the free part of homology. Recall that the Poincaré polynomial (or series) of bigraded finitely generated $\mathbb{Z}$-modules H_{**} is $\mathrm{PP}(t,q) = \mathrm{PP}(\mathrm{H}_{**})(t,p) = \sum_{i,j} a_{i,j} t^i q^j$, where $a_{i,j}$ is the rank of the group $\mathrm{H}_{i,j}$.

Corollary 4.1. *For an n-gon P_n the graph cohomology groups $\mathrm{H}^{i,j}_{\mathcal{A}_\infty}(P_n)$ are free abelian with Poincaré polynomial $(q+q^2+q^3+\dots)^3 + t^{n-2}(q+q^2+q^3+\dots) = (\frac{q}{1-q})^3 + t^{n-2}\frac{q}{1-q}$.*

Proof. From Theorem 3.1 we obtain $\mathrm{H}^{i,j}(P_n)$ for $0 < i < n-1$. It was observed in [10] that $\mathrm{H}^{i,j}(P_n) = 0$ for $i \geq n-1$. To find $\mathrm{H}^{0,*}(P_n)$ we use the fact that the chromatic polynomial of P_n is equal to $(\lambda-1)^n + (-1)^n(\lambda-1)$ and the graph cohomology categorify the chromatic polynomial. That is, if we substitute $t = -1$ and $1+q+q^2+q^3+\cdots = \lambda$ in the Poincaré polynomial we obtain the chromatic polynomial [10]. □

Another illustration of the power of our connection is for the algebra $\mathcal{A} = \mathcal{A}_{p(x)} = \mathbb{Z}[x]/(p(x))$, where $p(x)$ is a polynomial in $\mathbb{Z}[x]$. We discuss the general case later, here let us notice that the two special cases of $p(x) = x^m$ and $p(x) = x^m - 1$ are of great interest in knot theory (in Khovanov–Rozansky homology [16] and its deformations [6]). Let us apply first the knowledge of Hochschild homology for $\mathcal{A}_m = \mathbb{Z}[x]/(x^m)$ (cf. [21]) to solving Conjectures 30 and 31 of [8].

Theorem 4.2. (*Free*) *The Poincaré polynomial of* $\mathrm{HH}_{**}(\mathcal{A}_m)$ *is equal to*

$$\begin{aligned}(1+q+\cdots+q^{m-1}) &+ t(q+q^2+\cdots+q^{m-1})\\ &+ (t^2+t^3)(q+q^2+\cdots+q^{m-1})q^m\\ &+ (t^4+t^5)(q+q^2+\cdots+q^{m-1})q^{2m}\\ &+\cdots+(t^{2i}+t^{2i+1})(q+q^2+\cdots+q^{m-1})q^{im}+\cdots.\end{aligned}$$

(*Torsion*) $\mathrm{Tor}(\mathrm{H}_{**}(\mathcal{A}_m)) = \bigoplus_{i=1}^{\infty} \mathrm{H}_{2i-1,im}(\mathcal{A}_m)$*, where each summand is isomorphic to* $\mathbb{Z}_m$.

We solve Conjectures 30 and 31 of [8] by applying Theorems 4.2 and 3.1.

Corollary 4.3. (*Odd*) *For* $n = 2g+1$ *we have*

$$\mathrm{Tor}(\mathrm{H}^{*,*}_{\mathcal{A}_m}(P_{2g+1})) = \mathrm{H}^{v-2,m}_{\mathcal{A}_m}(P_{2g+1}) \oplus \mathrm{H}^{v-4,2m}_{\mathcal{A}_m}(P_{2g+1}) \oplus \cdots \oplus \mathrm{H}^{1,gm}_{\mathcal{A}_m}(P_{2g+1}),$$

with each summand isomorphic to $\mathbb{Z}_m$.

The Poincaré polynomial of $\mathrm{H}^{*,*}_{\mathcal{A}_m}(P_{2q+1})$ *is equal to*

$$\begin{aligned}&(q+\cdots+q^{m-1})^v\\ &\quad + (q+\cdots+q^{m-1})(t^{v-2} + (t^{v-3}+t^{v-4}q^m+\cdots+(t^2+t)q^{m(g-1)}).\end{aligned}$$

(*Even*) *For* $n = 2g + 2$ *we have*

$$\mathrm{Tor}(\mathrm{H}^{*,*}_{\mathcal{A}_m}(P_{2g+2})) = \mathrm{H}^{v-2,m}_{\mathcal{A}_m}(P_{2g+1}) \oplus \mathrm{H}^{v-4,2m}_{\mathcal{A}_m}(P_{2g+1}) \oplus \cdots \oplus \mathrm{H}^{2,gm}_{\mathcal{A}_m}(P_{2g+2}),$$

with each summand isomorphic to $\mathbb{Z}_m$.

The Poincaré polynomial of $\mathrm{H}^{*,*}_{\mathcal{A}_m}(P_{2q+2})$ *is equal to*

$$(q + \cdots + q^{m-1})^n + q^{m(n/2)-1}(q + \ldots q^{m-1}) \\ + (q + \cdots + q^{m-1})(t^{n-2} + (t^{n-3} + t^{n-4})q^m + \cdots + (t^3 + t^2)q^{m(g-1)} + tq^{mg}).$$

Assume that $m = 2$ in Corollary 4.3. Then, using Theorem 2.7, we can recover Khovanov computation of homology of the torus link $T_{2,n}$ [14], [15]. In particular we get:

Corollary 4.4 ([15]). *Let* $T_{2,-n}$ *be a left-handed torus link of type* $(2,-n)$, $n > 2$. *Then the torsion part of the Khovanov homology of* $T_{2,-n}$ *is given by* (*in the description of homology we use notation of* [26] *treating* $T_{2,-n}$ *as a framed link*):

(*Odd*) *For n odd, all the torsion of* $\mathrm{H}_{**}(T_{2,-n})$ *is supported by*

$$\mathrm{H}_{n-2,3n-4}(T_{2,-n}) = \mathrm{H}_{n-4,3n-8}(T_{2,-n}) = \cdots = \mathrm{H}_{-n+4,-n+8}(T_{2,-n}) = \mathbb{Z}_2.$$

(*Even*) *For n even, all the torsion of* $\mathrm{H}_{**}(T_{2,-n})$ *is supported by*

$$\mathrm{H}_{n-4,3n-8}(T_{2,-n}) = \mathrm{H}_{n-6,3n-12}(T_{2,-n}) = \cdots = \mathrm{H}_{-n+4,-n+8}(T_{2,-n}) = \mathbb{Z}_2.$$

For a right-handed torus link of type $(2,n)$, $n > 2$, *we can use the formula for the mirror image* (*Khovanov duality theorem; see for example* [2], [3]): $\mathrm{H}_{-i,-j}(\bar{D}) = (\mathrm{H}_{ij}(D)/\mathrm{Tor}(\mathrm{H}_{ij}(D)) \oplus \mathrm{Tor}(\mathrm{H}_{i-2,j}(D))$.

The result on Hochschild homology of symmetric algebras has a major generalization to the large class of algebras called *smooth algebras*.

Theorem 4.5 ([21], [12]). *For any smooth algebra* $\mathcal{A}$ *over k, the antisymmetrization map* $\varepsilon_*\colon \Omega^*_{\mathcal{A}|k} \to \mathrm{HH}_*(\mathcal{A})$ *is an isomorphism of graded algebras. Here* $\Omega^n_{\mathcal{A}|k} = \Lambda^n\Omega^1_{\mathcal{A}|k}$ *is an* $\mathcal{A}$*-module of differential n-forms.*

We refer to [21] for a precise definition of a smooth algebra, here we only recall that the following are examples of smooth algebras:

(i) any finite extension of a perfect field k (e.g., a field of characteristic zero);

(ii) the ring of algebraic functions on a nonsingular variety over an algebraically closed field k, e.g., $k[x]$, $k[x_1,\ldots,x_n]$, $k[x,y,z,t]/(xt - yz - 1)$ [21].

Not every quotient of a polynomial algebra is a smooth algebra. For example, $C[x,y])(x^2y^3)$ or $\mathbb{Z}[x]/(x^m)$ are not smooth. The broadest, to my knowledge, treatment of Hochschild homology of algebras $C[x_1,\ldots,x_n]/(\mathrm{Ideal})$ is given by

Kontsevich in [20]. For us the motivation came from one variable polynomials, Theorem 40 of [8]. In particular we generalize Theorem 40 (i) from a triangle to any polygon, that is, we compute the graph cohomology of a polygon for truncated polynomial algebras and their deformations. Thus, possibly, we can approximate Khovanov–Rozansky $\mathrm{sl}(n)$ homology and their deformations.

Theorem 4.6. (i) $\mathrm{HH}_i(\mathcal{A}_{p(x)}) = \mathbb{Z}[x]/(p(x), p'(x))$ *for i odd and* $\mathrm{HH}_i(\mathcal{A}_{p(x)}) = \{[q(x)] \in \mathbb{Z}[x]/(p(x)) \mid q(x)p'(x)$ *is divisible by* $p(x)\}$ *for i even $i \geq 2$. In both cases the $\mathbb{Z}$ rank of the group is equal to the degree of* $\gcd(p(x), p'(x))$.

(ii) *In particular, for $p(x) = x^{m+1}$ we obtain homology of the ring of truncated polynomials, $\mathcal{A}_{m+1} = \mathbb{Z}[x]/(x^{m+1})$, for which*

$$\begin{aligned} \mathrm{HH}_i(\mathcal{A}_{m+1}) &= \mathbb{Z}_{m+1} \oplus \mathbb{Z}^m && \textit{for } i \textit{ odd},\\ \mathrm{HH}_i(\mathcal{A}_{m+1}) &= \mathbb{Z}^m && \textit{for } i \textit{ even}, i \geq 2, \end{aligned}$$

and

$$\mathrm{HH}_0(\mathcal{A}_{m+1}) = \mathcal{A} = \mathbb{Z}^{m+1}.$$

(iii) *The graph cohomology $\mathrm{H}^i_{\mathcal{A}_{p(x)}}(P_n)$ of a polygon P_n is given by*

$$\mathrm{H}^{n-2i}_{\mathcal{A}_{p(x)}}(P_n) = \mathcal{A}_{p(x)}/(p'(x)) \quad \textit{for } 1 \leq i \leq \tfrac{v-1}{2},$$

and

$$\mathrm{H}^{n-2i-1}_{\mathcal{A}_{p(x)}}(P_n) = \ker(\mathcal{A}_{p(x)} \xrightarrow{p'(x)} \mathcal{A}_{p(x)}) \quad \textit{for } 1 \leq i \leq \tfrac{v-2}{2}.$$

Furthermore, $\mathrm{H}^k_{\mathcal{A}_{p(x)}}(P_n) = 0$ for $k \geq n-1$ and $\mathrm{H}^0_{\mathcal{A}_{p(x)}}(P_n)$ is a free abelian group of rank $(d-1)^n + (-1)^n(d-1)$ for n even (d denotes the degree of $p(x)$) and it is of $\mathrm{rank}(d-1)^n + (-1)^n(d-1) - \mathrm{rank}(\mathrm{H}^1_{\mathcal{A}_{p(x)}}(P_n))$ *if n is odd (notice that $(d-1)^n + (-1)^n(d-1)$ is the Euler characteristic of $\{\mathrm{H}^i_{\mathcal{A}_{p(x)}}(P_n)\}$).*

Proof. Theorem 4.6 (i) is proven by considering a resolution of $\mathcal{A}_{p(x)}$ as an $\mathcal{A}^e_{p(x)} = \mathcal{A}_{p(x)} \otimes \mathcal{A}^{\mathrm{op}}_{p(x)}$ module,

$$\cdots \to \mathcal{A}_{p(x)} \otimes \mathcal{A}_{p(x)} \xrightarrow{u} \mathcal{A}_{p(x)} \otimes \mathcal{A}_{p(x)} \xrightarrow{v} \mathcal{A}_{p(x)} \otimes \mathcal{A}_{p(x)} \xrightarrow{u} \cdots \to \mathcal{A}_{p(x)},$$

where $u = x \otimes 1 - 1 \otimes x$ and $v = \Delta_{p(x)}(1)$ is a coproduct given by $\Delta_{p(x)}(1) = \sum_{i=0}^n a_i \Delta_i(1)$, where $p(x) = \sum_{i=0}^n a_i x^i$, and $\Delta_i(1) = x^{i-1} \otimes 1 + x^{i-2} \otimes x + \cdots + x \otimes x^{i-2} + 1 \otimes x^{i-1}$. □

A curious but not accidental observation is that by choosing coproduct $\Delta(1) = v$ we define a Frobenius algebra structure on $\mathcal{A}$. In a Frobenius algebra $(x \otimes 1)\Delta(1) = (1 \otimes x)\Delta(1)$, which makes $uv = vu = 0$ in our resolution. Furthermore, the distinguished element of the Frobenius algebra is $\mu\Delta(1) = p'(x)$.

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