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Quantum Topology

Mutation invariance of Khovanov homology over $\mathbb{F}_2$

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Abstract. We prove that Khovanov homology and Lee homology with coefficients in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ are invariant under component-preserving link mutations.

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1. Introduction

Khovanov homology is a refinement of the Jones polynomial [Jon85] which was discovered by Mikhail Khovanov [Kho00] in the year 1999, and which was subsequently generalized through the work Eun Soo Lee [Lee05] and Dror Bar-Natan [BN05a]. In 2003, the author [Weh03] discovered a series of examples of mutant links [Con70] with different (integer coefficient) Khovanov homology. Despite this discovery, the question whether there are mutant knots with different Khovanov homology remained open. In this paper, we partially answer this question. We prove:

Theorem 1.1. *The graded homotopy type of* $\mathrm{Kh}(L)$ *is invariant under component-preserving link mutation.*

In this theorem, $\mathrm{Kh}(L)$ stands for a variant of Bar-Natan's formal Khovanov bracket [BN05a], which generalizes both Khovanov homology with $\mathbb{F}_2$ coefficients and Lee homology [Lee05] with $\mathbb{F}_2$ coefficients. To prove the theorem, we will employ an argument that was outlined by Bar-Natan in 2005 [BN05b]. While Bar-Natan's argument had some gaps, the author realized that these gaps can be filled if one works over $\mathbb{F}_2$ coefficients. In 2007, the author presented a complete proof of Theorem 1.1 at the 'Knots in Washington XXIV' conference in Washington D.C., and at the 'Link homology and categorification' conference in Kyoto [Weh07].

More recently, Jonathan Bloom [Blo10] discovered an alternative and completely independent proof of mutation invariance. Bloom's proof has the advantage that it works not only over $\mathbb{F}_2$ coefficients, but rather extends to a proof showing that the odd version of the integer coefficient Khovanov homology (defined as in [ORS07]) is invariant under arbitrary link mutations. On the other hand, the proof given in this paper has the advantage that it also implies that *Lee homology* with $\mathbb{F}_2$ coefficients is invariant under component-preserving link mutation.

The paper is organized as follows. In Section 2, we show that every *component-preserving link mutation* can be realized by a finite sequence of *crossed z-mutations* and *isotopies*. In Section 3, we introduce the variant of the formal Khovanov bracket that we will use throughout this paper. This variant takes values in a category whose morphisms are formal $\mathbb{F}_2$-linear combinations of properly embedded 2-cobordisms, decorated by finitely many distinct dots, and considered up to some relations. In Section 4, we discuss algebraic operations for manipulating the dots that appear in a decorated cobordism, and in Section 5, we use these operations to prove that our variant of the formal Khovanov bracket is invariant under crossed z-mutation.

2. Conway mutation

Let $U \subset \mathbb{R}^2$ be the closure of a domain in $\mathbb{R}^2$, and let $P \subset \partial U$ a finite subset of ∂U. A *tangle* above (U, P) is a properly embedded compact 1-manifold $\mathcal{T} \subset U \times \mathbb{R}$ with $\partial\mathcal{T} = P \times \{0\}$. To represent a tangle above (U, P), we use a plane diagram $T \subset U$ with $\partial T = P$. In the case where T is a plane diagram of a tangle $\mathcal{T}$ above the unit disk $U = \mathcal{D} := \{z \in \mathbb{C} = \mathbb{R}^2 \colon |z| \leq 1\}$, and P is the set $P := \{a, b, c, d\} \subset \partial\mathcal{D}$ where a, b, c, d are the points $\exp(i\pi n/4) \in \mathbb{C} = \mathbb{R}^2$ for $n = 1, 3, 5, 7$ (in this order), then we denote by $R_x(T)$, $R_y(T)$, $R_z(T)$ the plane diagrams of the tangles obtained by rotating $\mathcal{T} \subset \mathcal{D} \times \mathbb{R} \subset \mathbb{R}^3$ by 180° around the x-, y- and z-axis, respectively.

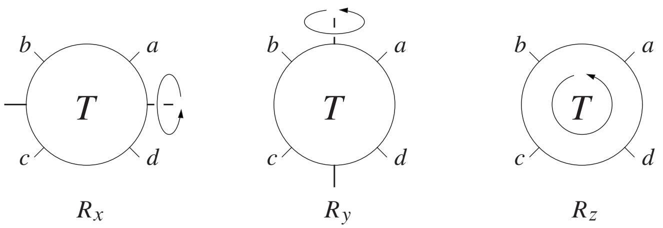

Figure 1. Rotations R_x, R_y, R_z.

Let $\mathcal{D}^c := \mathbb{R}^2 \backslash \mathrm{Int}(\mathcal{D})$ and $P := \{a, b, c, d\}$. If $\mathcal{T}$ is a tangle over $(\mathcal{D}, P)$, and $\mathcal{T}'$ is a tangle over $(\mathcal{D}^c, P)$, then the union $\mathcal{T} \cup \mathcal{T}'$ is a link $\mathcal{L} = \mathcal{T} \cup \mathcal{T}' \subset \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$.

Definition 2.1. Two links $\mathcal{L}$ and $\mathcal{L}'$ are called *elementary Conway mutants of each other* [Con70] if there is a rotation $R \in \{R_x, R_y, R_z\}$ and two tangle diagrams $T \subset \mathcal{D}$ and $T' \subset \mathcal{D}^c$ with $\partial T = \partial T' = P$ and such that $T \cup T'$ is a diagram for $\mathcal{L}$ and $R(T) \cup T'$ is a diagram for $\mathcal{L}'$. Depending on whether $R = R_x$, R_y or R_z, we say that the diagrams $T \cup T'$ and $R(T) \cup T'$ are related by *x-, y- or z-mutation*.

Remark 2.2. If $\mathcal{L}$ and $\mathcal{L}'$ are oriented, then we require that $T \cup T'$ is a diagram for $\mathcal{L}$, and $R(T) \cup T'$ or $R(-T) \cup T'$ (whichever of the two is oriented consistently) is a diagram for $\mathcal{L}'$.

Definition 2.3. We say that $T \cup T'$ and $R(T) \cup T'$ are related by a *crossed mutation* if the tangle corresponding to $T' \subset \mathcal{D}^c$ has *crossed connectivity*, i.e., if one of its arcs has endpoints at $\{a\} \times \{0\}$ and $\{c\} \times \{0\}$, and the other arc has endpoints at $\{b\} \times \{0\}$ and $\{d\} \times \{0\}$.

Definition 2.4. We say that $\mathcal{L} = \mathcal{T} \cup \mathcal{T}'$ and $\mathcal{L}' = R(\mathcal{T}) \cup \mathcal{T}'$ are related by a *component-preserving mutation* if the union $R(\alpha) \cup \alpha'$ is a connected component of $\mathcal{L}'$ if and only if the union $\alpha \cup \alpha'$ is a connected component of $\mathcal{L}$, for any two arc components $\alpha \subset \mathcal{T}$ and $\alpha' \subset \mathcal{T}'$.

The following lemma allows us to reduce Theorem 1.1 to Proposition 2.6 below.

Lemma 2.5. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two links that are related by component-preserving mutation, and let D be a planar diagram of $\mathcal{L}$ and D' a planar diagram of $\mathcal{L}'$. Then D can be transformed into D' by a sequence of Reidemeister moves and crossed z-mutations.*

Proof. It is easy to see that the three different types of mutation (x-, y- and z-mutation) are topologically equivalent. Indeed, Figure 2 shows how a y-mutation can be obtained by performing a Reidemeister move of type II, followed by a z-mutation, followed by an isotopy in $\mathbb{R}^3$, and analogously, an x-mutation can be reduced to a

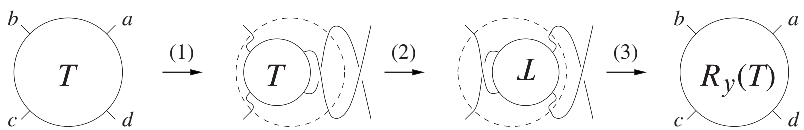

Figure 2. Decomposing a y-mutation into three steps: (1) a Reidemeister move of type II; (2) a z-mutation along the dashed circle; (3) an isotopy in $\mathbb{R}^3$ that rotates $\mathcal{T}$ around the x-axis and thus untwists the crossings on either side of $\mathcal{T}$.

z-mutation. Thus, we can assume without loss of generality that D and D' are related by a z-mutation, i.e., $D = T \cup T'$ and $D' = R_z(T) \cup T'$ for suitable tangle diagrams $T \subset \mathcal{D}$ and $T' \subset \mathcal{D}^c$. If T' has crossed connectivity, then there is nothing to prove, and if T has crossed connectivity, then we can interchange the roles of T and T' by applying a planar isotopy which moves T' into $\mathcal{D}$ and T out of $\mathcal{D}$. Thus, we only need to care about the case where neither T nor T' has crossed connectivity. In this case, either T or T' must have horizontal connectivity (i.e., represent a tangle that contains an arc with endpoints at $\{a\} \times \{0\}$ and $\{b\} \times \{0\}$), for otherwise the mutation would not be component-preserving. After interchanging the roles of T and T' if necessary, we can assume that T' has horizontal connectivity. But then the z-mutation in Step (2) of Figure is a crossed z-mutation, and hence Figure shows that $D = T \cup T'$ can be transformed into $R_y(T) \cup T'$ by Reidemeister moves and a crossed z-mutation. A similar argument shows $R_y(T) \cup T'$ can be transformed into $R_y(T) \cup R_x(T')$ by Reidemeister moves and a crossed z-mutation, and since $R_z = R_x \circ R_y$, the latter diagram is isotopic to $R_x\left(R_y(T) \cup R_x(T')\right) = R_z(T) \cup T' = D'$, whence the proof is complete. □

The following proposition is the main result of this paper. Its proof will be given in Section 5.

Proposition 2.6. *If two link diagrams are related by a crossed z-mutation, then their formal Khovanov brackets are isomorphic.*

3. Bar-Natan's formal Khovanov bracket

In this section, we briefly review the definition of Bar-Natan's formal Khovanov bracket. For more details, we refer the reader to [BN05a].

3.1. Chain complexes and chain maps in pre-additive categories. Let $\mathcal{C}$ be a pre-additive category. To $\mathcal{C}$, one can associate an additive category $\mathrm{Mat}(\mathcal{C})$, called the *matrix extension* or *additive closure* of $\mathcal{C}$ and defined as follows. An object of $\mathrm{Mat}(\mathcal{C})$ is a finite tuple $(O_1, \ldots, O_m)$ of objects $O_i \in \mathcal{C}$ (where m can be any non-negative integer). A morphism $F\colon (O_1, \ldots, O_n) \to (O'_1, \ldots, O'_m)$ is a matrix $F = (F_{ij})$ of morphisms $F_{ij} \in \mathrm{Hom}_{\mathcal{C}}(O_j, O'_i)$. The composition of two morphisms $F = (F_{ik})$ and $G = (G_{kl})$ is modelled on ordinary matrix multiplication: $(F \circ G)_{ij} := \sum_k F_{ik} \circ G_{kj}$. Direct sums are defined by concatenation: $(O_1, \ldots, O_n) \oplus (O'_1, \ldots, O'_m) := (O_1, \ldots, O_n, O'_1, \ldots, O'_m)$. By identifying an object $O \in \mathcal{C}$ with the 1-tuple $(O) \in \mathrm{Mat}(\mathcal{C})$, one can embed $\mathcal{C}$ into $\mathrm{Mat}(\mathcal{C})$ as a full subcategory. In particular, one can write every object $(O_1, \ldots, O_m) \in \mathrm{Mat}(\mathcal{C})$ as a direct sum $(O_1, \ldots, O_m) = \bigoplus_{i=1}^m O_i$.

Definition 3.1. A *bounded chain complex* in $\mathcal{C}$ is a pair $C = (C^*, d^*)$, where $C^* = \{C^i\}_{i \in \mathbb{Z}}$ is a sequence of objects $C^i \in \mathrm{Mat}(\mathcal{C})$, such that $C^i = 0$ for $|i| \gg 0$, and $d^* = \{d^i\}_{i \in \mathbb{Z}}$ is sequence of morphisms $d^i\colon C^i \to C^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all $i \in \mathbb{Z}$.

Definition 3.2. A *chain map* $F\colon (C_1^*, d_1^*) \to (C_2^*, d_2*)$ is a sequence of morphisms $F^i\colon C_1^i \to C_2^i$ such that $F^{i+1} \circ d_1^i = d_2^i \circ F^i$ for all $i \in \mathbb{Z}$.

We denote by $\mathrm{Kom}(\mathcal{C})$ the category whose objects are bounded chain complexes in $\mathcal{C}$ and whose morphisms are chain maps.

Remark 3.3. If $\mathsf{F}\colon \mathcal{C}_1 \to \mathcal{C}_2$ is an additive functor between two pre-additive categories $\mathcal{C}_1$ and $\mathcal{C}_2$, then F can be extended to an additive functor $\mathsf{F}\colon \mathrm{Mat}(\mathcal{C}_1) \to \mathrm{Mat}(\mathcal{C}_2)$ by setting $\mathsf{F}((O_1, \dots, O_m)) := (\mathsf{F}(O_1), \dots, \mathsf{F}(O_m))$ and $\mathsf{F}(F) := (\mathsf{F}(F_{ij}))$ for every object $(O_1, \dots, O_m) \in \mathrm{Mat}(\mathcal{C}_1)$ and every morphism $F = (F_{ij})$. Similarly, F can be extended to an additive functor $\mathsf{F}\colon \mathrm{Kom}(\mathcal{C}_1) \to \mathrm{Kom}(\mathcal{C}_2)$ by setting $\mathsf{F}((C^*, d^*))^i := (\mathsf{F}(C^i), \mathsf{F}(d^i))$ and $\mathsf{F}(F^*)^i := \mathsf{F}(F^i)$. In this paper, we make no distinction between the notation for the functor $\mathsf{F}\colon \mathcal{C}_1 \to \mathcal{C}_2$ itself, and the notation for the extensions of F.

3.2. Decorated cobordisms. In the following, U is the closure of a domain in $\mathbb{R}^2$, and P is a finite subset of ∂U.

Let $O_1, O_2 \subset U$ be two properly embedded unoriented compact 1-submanifolds in U with $\partial O_1 = \partial O_2 = P$. A *cobordims* between O_1 and O_2 is a compact properly embedded unoriented surface $S \subset U \times [0, 1]$ whose bottom boundary is O_1 and whose top boundary is O_2, and whose intersection with $(\partial U) \times [0, 1]$ consists of the vertical segments $P \times [0, 1]$. A *decorated cobordism* is a cobordism decorated by finitely many (possibly zero) distinct points or *dots*, which lie in the interior of S. Let $\mathrm{DC}(O_1, O_2)_\bullet$ be the set of isotopy classes of decorated cobordisms between O_1 and O_2. Moreover, let $\mathrm{DC}(O_1, O_2)_{\bullet/\ell}$ be the quotient of the $\mathbb{F}_2$-vector space spanned the elements of $\mathrm{DC}(O_1, O_2)_\bullet$ modulo the following *local relations*, called respectively the *sphere relation*, the *dot relation* and the *neck-cutting relation:*

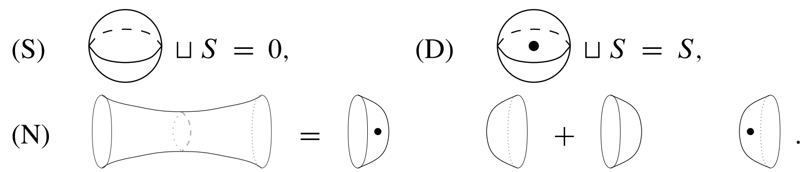

Figure 3. Local relations in $\mathrm{DC}(O_1, O_2)_{\bullet/\ell}$.

In the first two relations, S stands for an arbitrary decorated cobordism, and in the third relation, the three pictures stand for three decorated cobordisms, which are identical everywhere except in a small ball $B^3 \subset U \times [0, 1]$ where they differ as shown. Using the above relations, one can deduce the important *double dot relation:*

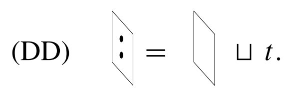

Figure 4. The double dot relation.

In the (DD) relation, t stands for a 2-sphere decorated by exactly three dots. Thus, the (DD) relation says that we can remove any pair of dots lying on the same component of a decorated cobordism, at the expense of adding a 2-sphere decorated by exactly three dots. We can endow $\mathrm{DC}(O_1, O_2)_{\bullet/\ell}$ with the structure an $\mathbb{F}_2[t]$-module by defining $t^n S$ to be the disjoint union of S with n disjoint copies of t.

Definition 3.4. Let $\mathcal{C}ob(U, P)_{\bullet/\ell}$ be the pre-additve category whose objects are unoriented properly embedded compact 1-manifolds $O \subset U$ with $\partial O = P$, and whose morphism sets are the $\mathbb{F}_2$-vector spaces $\mathrm{DC}(O_1, O_2)_{\bullet/\ell}$. Composition of morphisms $S_1\colon O_1 \to O_2$ and $S_2\colon O_2 \to O_3$ is given by stacking S_2 on top of S_1.

Let $\mathrm{Mat}(U, P) := \mathrm{Mat}(\mathcal{C}ob(U, P)_{\bullet/\ell})$ and $\mathrm{Kom}(U, P) := \mathrm{Kom}(\mathcal{C}ob(U, P)_{\bullet/\ell})$.

3.3. Quantum grading. To incorporate the *quantum grading* (or *j-grading*) of Khovanov homology, one has to redefine the objects of $\mathcal{C}ob(U, P)_{\bullet/\ell}$ as being pairs (O, n) where $O \subset U$ is a properly embedded compact 1-manifold with $\partial O = P$ as before, and n is an integer. A morphism $S\colon (O_1, n_1) \to (O_2, n_2)$ is given by a morphism $S\colon O_1 \to O_2$, i.e., by an element $S \in \mathrm{DC}(O_1, O_2)_{\bullet/\ell}$. The *quantum degree* of a morphism is defined by:

$$\deg(S) := e(S) - 2d(S) + n_2 - n_1,$$

where $e(S) := \chi(S) - |P|/2$ is the Euler measure of S, and $d(S)$ is the number of dots on S. Let $\mathcal{C}ob(U, P)^0_{\bullet/\ell}$ denote the category which has the same objects as $\mathcal{C}ob(U, P)_{\bullet/\ell}$, but whose morphisms $S\colon (O_1, n_1) \to (O_2, n_2)$ are required to satisfy $\deg(S) = 0$. Let $\mathrm{Mat}(U, P)^0 := \mathrm{Mat}(\mathcal{C}ob(U, P)^0_{\bullet/\ell})$ and $\mathrm{Kom}(U, P)^0 := \mathrm{Kom}(\mathcal{C}ob(U, P)^0_{\bullet/\ell})$. For each integer m, let $\{m\}$ denote the degree shift functor given by $(O, n)\{m\} := (O, m + n)$. Identifying $(O, 0)$ with O, we will henceforth write $O\{n\}$ instead of (O, n).

3.4. Formal Khovanov bracket. Now let $T \subset U$ be a tangle diagram with $\partial T = P$. Let χ be the set of crossings of T and $\{0, 1\}^\chi$ the set of all maps $\epsilon\colon \chi \to \{0, 1\}$. A crossing $c \in \chi$ (looking like: $\times$) can be resolved in two possible ways, $)($ and $\asymp$ called its 0-*resolution* and its 1-*resolution*, respectively. Given $\epsilon \in \{0, 1\}^\chi$, denote by T_ϵ the crossingless tangle diagram obtained from T by replacing every $c \in \epsilon^{-1}(0)$ by its 0-resolution, and every $c \in \epsilon^{-1}(1)$ by its 1-resolution. For $\epsilon, \epsilon' \in \{0, 1\}^\chi$ and $c \in \chi$, we will write $\epsilon <_c \epsilon'$ iff ϵ and ϵ' satisfy $\epsilon(c) = 0$ and $\epsilon'(c) = 1$, and $\epsilon(c') = \epsilon'(c')$ for all $c' \in \chi$ with $c' \neq c$. For such ϵ, ϵ', there is a preferred cobordism $S_{\epsilon'\epsilon}\colon T_\epsilon \to T_{\epsilon'}$ containing no dots, such that $S_{\epsilon',\epsilon} \cap (\mathrm{Nbd}(c) \times [0, 1])$ is a saddle cobordism between $)($ and $\asymp$, and $S_{\epsilon',\epsilon} \setminus (\mathrm{Nbd}(c) \times [0, 1])$ is the identity cobordism. For $\epsilon, \epsilon' \in \{0, 1\}^\chi$ and $c \in \chi$, let $(d_c)_{\epsilon'\epsilon}\colon T_\epsilon \to T_{\epsilon'}$ be the morphism defined by $(d_c)_{\epsilon'\epsilon} := S_{\epsilon',\epsilon}$ if $\epsilon <_c \epsilon'$, and $(d_c)_{\epsilon'\epsilon} := 0$ otherwise. Let $d_{\epsilon'\epsilon} := \sum_{c\in\chi}(d_c)_{\epsilon'\epsilon}$ and $|\epsilon| := |\epsilon^{-1}(1)| = \sum_{c\in\chi}\epsilon(c)$. Suppose T is oriented and let n_+ (n_-) be the number of positive (negative) crossings in T. If ϵ and ϵ' satisfy $|\epsilon| = i + n_-$ and $|\epsilon'| = i + 1 + n_-$ for an $i \in \mathbb{Z}$, then we set $d^i_{\epsilon'\epsilon} := d_{\epsilon'\epsilon}$.

Definition 3.5. The *formal Khovanov bracket* of T is the chain complex $\mathrm{Kh}(T) := (\mathrm{Kh}(T)^*, d^*) \in \mathrm{Kom}(U, P)^0$ defined by $\mathrm{Kh}(T)^i := \bigoplus_{|\epsilon|=i+n_-} T_\epsilon\{i + n_+ - 2n_-\}$ and $d^i := (d^i_{\epsilon'\epsilon})$.

Definition 3.5 is justified by the following lemma:

Lemma 3.6. $d^{i+1} \circ d^i = 0$ *for all* $i \in \mathbb{Z}$.

Proof. Ignoring differentials and gradings for a moment, we can identify $\mathrm{Kh}(T)$ with the object $\mathrm{Kh}(T) = \bigoplus_{\epsilon\in\{0,1\}^\chi} T_\epsilon \in \mathrm{Mat}(U, P)$. We can then identify the differential in $\mathrm{Kh}(T)$ with the endomorphism $d := (d_{\epsilon'\epsilon})$ of $\mathrm{Kh}(T) \in \mathrm{Mat}(U, P)$ (with $d_{\epsilon'\epsilon}$ defined as above). For $c \in \chi$, let d_c be the endomorphism of $\mathrm{Kh}(T) \in \mathrm{Mat}(U, P)$ defined by $d_c := ((d_c)_{\epsilon'\epsilon})$. We have $d_c \circ d_c = 0$ because for any three elements $\epsilon, \epsilon', \epsilon'' \in \{0, 1\}^\chi$, at least one of the two matrix entries $(d_c)_{\epsilon''\epsilon'}$ and $(d_c)_{\epsilon'\epsilon}$ is equal to zero. We also have $d_c \circ d_{c'} = d_{c'} \circ d_c$ for all $c, c' \in \chi$ because distant saddles can be time-reordered by isotopy. Since $d = \sum_{c\in\chi} d_c$, this implies $d \circ d = 0$, and thus the lemma follows. □

The following theorem was proved by Bar-Natan [BN05a].

Theorem 3.7. *The graded homotopy type of* $\mathrm{Kh}(T)$ *is a tangle invariant.*

3.5. Relation with Khovanov homology and Lee homology. If T is a link diagram (i.e., $\partial T = \emptyset$), then the formal Khovanov bracket of T refines both the $\mathbb{F}_2$-coefficient *Khovanov homology* [Kho00] and the $\mathbb{F}_2$-coefficient *Lee homology* [Lee05] of T. Indeed, let $\mathrm{Hom}(\emptyset, -)$ be the functor which maps an object $O \in \mathcal{C}ob(U, \emptyset)_{\bullet/\ell}$ to the graded morphism set $\mathrm{Hom}(\emptyset, O)$, regarded as a graded $\mathbb{F}_2[t]$-module via the (DD) relation. Then the $\mathbb{F}_2$-coefficient Khovanov homology of T is the homology of the chain complex $\mathsf{F}_{\mathrm{Kh}}(\mathrm{Kh}(T))$, where $\mathsf{F}_{\mathrm{Kh}}(-) := \mathrm{Hom}(\emptyset, -) \otimes_{t=0} \mathbb{F}_2$, and the $\mathbb{F}_2$-coefficient Lee homology of T is the homology of the chain complex of $\mathsf{F}_{\mathrm{Lee}}(\mathrm{Kh}(T))$, where $\mathsf{F}_{\mathrm{Lee}}(-) := \mathrm{Hom}(\emptyset, -) \otimes_{t=1} \mathbb{F}_2$.

3.6. Tensor products. In this subsection, we describe a special case of the 'categorified planar algebra' structure of $\mathrm{Kh}(T)$ that was introduced in [BN05a, Section 5]. Assume that we have the following situation:

- U' and U'' are the closures of two disjoint domains in $\mathbb{R}^2$ and $U := U' \cup U''$.
- P_1 and P_2 are finite subsets of $(\partial U') \setminus U''$ and $(\partial U'') \setminus U'$, respectively.
- P_0 is a finite subset of $U' \cap U''$.
- $P' := P_0 \cup P_1$ and $P'' := P_0 \cup P_2$ and $P := P_1 \cup P_2$.

In this situation, there is a natural functor

$$\mathcal{C}ob(U', P')_{\bullet/\ell} \times \mathcal{C}ob(U'', P'')_{\bullet/\ell} \longrightarrow \mathcal{C}ob(U, P)_{\bullet/\ell}$$

which takes a pair of objects (O', O'') (or morphisms (S', S'')) to the union $O' \cup O''$ (or $S' \cup S''$). We write this functor as a tensor product, and we extend it to a functor $\mathrm{Mat}(U', P') \times \mathrm{Mat}(U'', P'') \to \mathrm{Mat}(U, P)$ by declaring that the tensor product distributes over direct sums, i.e., $(O'_1 \oplus O'_2) \otimes (O''_1 \oplus O''_2) := (O'_1 \otimes O''_2) \oplus (O'_1 \otimes O''_2) \oplus (O'_2 \otimes O''_2) \oplus (O'_2 \otimes O''_2)$ and $(F' \otimes F'')_{i\otimes k, j\otimes l} = F'_{ij} \otimes F''_{kl}$. Given two chain complexes $C' \in \mathrm{Kom}(U', P')$ and $C'' \in \mathrm{Kom}(U'', P'')$, we define $C' \otimes C'' \in \mathrm{Kom}(U, P)$ to be the chain complex whose underlying object is the tensor product $C' \otimes C'' \in \mathrm{Mat}(U, P)$, and whose differential is the endomorphism (in $\mathrm{Mat}(U, P)$) given by

$$d_{C'\otimes C''} := d_{C'} \otimes 1_{C'} + 1_{C''} \otimes d_{C''},$$

where $d_{C'}, d_{C''}, 1_{C'}, 1_{C''}$ are the differentials and the identity morphisms of C' and C'', respectively. As for the gradings, it is understood that both the homological grading and the quantum grading are additive under tensor products. The following theorem was shown (in greater generality) in [BN05a, Section 5].

Theorem 3.8. *Let $T' \subset U'$ and $T'' \subset U''$ be tangle diagrams with $\partial T' = P'$ and $\partial T'' = P''$. Then* $\mathrm{Kh}(T' \cup T'')$ *is canonically isomorphic to* $\mathrm{Kh}(T') \otimes \mathrm{Kh}(T'')$.

3.7. Delooping. Let '$\bigcirc$' denote the connected 1-manifold consisting of a single circle. More generally, let '$\bigcirc^n$' denote the 1-manifold consisting of n disjoint circles, and let $\emptyset\{1\}$ and $\emptyset\{-1\}$ denote degree-shifted copies of the empty 1-manifold. The following lemma is well-known (see e.g. [BN07, Lemma 4.1]).

Lemma 3.9. *The objects $\bigcirc$ and $\emptyset\{1\} \oplus \emptyset\{-1\}$ are isomorphic in* $\mathrm{Mat}(U, \emptyset)^0$.

Proof. Let $V := \emptyset\{1\} \oplus \emptyset\{-1\}$, and let $G\colon \bigcirc \to V$ and $H\colon V \to \bigcirc$ be the morphisms given by the matrices $(G_{11}, G_{21})^t$ and (H_{11}, H_{12}), where $G_{11}, G_{21}, H_{11}, H_{12}$ are cobordisms homeomorphic to disks, with G_{21} and H_{11} containing no dots, and G_{11} and H_{12} containing a single dot each. Using the local relations shown in Figure 3, one can easily check that $G \circ H$ and $H \circ G$ are the identity morphism of V and $\bigcirc$, respectively. □

Let $\mathcal{C} \subset \mathcal{C}ob(U, P)_{\bullet/\ell}$ be the full subcategory containing of all objects of the form $O\{n\}$, where O is a 1-manifold without closed components, and $n \in \mathbb{Z}$ is an arbitrary integer (in fact, we will henceforth drop the $\{n\}$ from the notation). Note that every object $O \in \mathcal{C}ob(U, P)_{\bullet/\ell}$ can be written in the form $O = O' \otimes \bigcirc^n$, where $O' \in \mathcal{C}$ and $n \geq 0$, and the tensor product '$\otimes$' denotes a disjoint union. (This notation is consistent with the one used in the previous subsection for $P_0 = \emptyset$). By applying the isomorphism $G\colon \bigcirc \to V$ defined in the proof of Lemma 3.9 repeatedly to each circle in $O = O' \otimes \bigcirc^n$, we can define a functor which sends the object $O \in \mathrm{Mat}(U, P)$ to an isomorphic object in $\mathrm{Mat}(\mathcal{C})$. Formally, this functor is defined as follows.

Definition 3.10. The *delooping functor* $\mathsf{D}\colon \mathrm{Mat}(U, P) \to \mathrm{Mat}(\mathcal{C})$ sends an object $O = O' \otimes \bigcirc^n$ (with $O' \in \mathcal{C}$) to the object $\mathsf{D}(O) := O' \otimes V^{\otimes n}$, and a morphism $S\colon O'_1 \otimes \bigcirc^{n_1} \to O'_2 \otimes \bigcirc^{n_2}$ to the morphism $\mathsf{D}(S) := (1 \otimes G^{\otimes n_2}) \circ S \circ (1 \otimes H^{\otimes n_1})$ where V and G, H are as in the proof of Lemma 3.9, and 1 stands for the identity morphism of either O'_1 or O'_2.

4. Operations involving dots

In this section, we define algebraic operations for manipulating the dots that decorate a decorated cobordism.

4.1. Dot multiplication. Let U be the closure of a domain in $\mathbb{R}^2$ and P be a finite subset of ∂U. Let $O \subset U$ be an object of the pre-additive category $\mathcal{C}ob(U, P)_{\bullet/\ell}$ defined in Section 3.2, and let $p \in O$ be an arbitrary point on O.

Definition 4.1. The *dot multiplication map* is the endomorphism $X_p\colon O \to O$ given by the cobordism $O \times [0, 1]$, decorated by a single dot lying in the interior of the segment $\{p\} \times [0, 1] \subset O \times [0, 1]$. If p is a point of $\partial O = P$, then we move the dot slightly into the interior of $O \times [0, 1]$, so that the result is a decorated cobordism in the sense of Section 3.2.

If $O_1, O_2 \subset U$ are two objects of $\mathcal{C}ob(U, P)_{\bullet/\ell}$ containing a point $p \in O_1 \cap O_2$, and $S\colon O_1 \to O_2$ is a decorated cobordism commuting with X_p, then we define

$$x_p S := X_p \circ S = S \circ X_p.$$

The above definitions extend to $\mathrm{Mat}(U, P)$ as follows. Let $O = (O_1, \ldots, O_m)$ be an object in $\mathrm{Mat}(U, P)$ and $p \in \bigcap O_i$. Then the dot multiplication map $X_p\colon O \to O$ is the endomorphism whose off-diagonal entries are zero and whose diagonal entry $(X_p)_{ii}$ is the decorated cobordism $x_p(O_i \times [0, 1])$. Similarly, if $F\colon O \to O'$ is a morphism commuting with X_p for a point $p \in \bigcap O_i \cap \bigcap O'_j$, then we define $x_p F := X_p \circ F = F \circ X_p$.

Definition 4.2. The *endpoint ring* $\mathbb{F}_2[P]$ is the commutative polynomial ring with coefficients in $\mathbb{F}_2$ in formal variables x_p, one for each $p \in P$.

Since every morphism in $\mathcal{C}ob(U, P)_{\bullet/\ell}$ contains the segment $\{p\} \times [0, 1]$ and hence commutes with X_p for all $p \in P$, the endpoint ring $\mathbb{F}_2[P]$ acts on morphism sets of $\mathcal{C}ob(U, P)_{\bullet/\ell}$ (or Mat $U, P)$ by $x_p \cdot S := x_p S = X_p \circ S = S \circ X_p$.

4.2. Dot derivation. Let $O_1, O_2 \subset U$ be two compact embedded 1-manifolds with $\partial O_1 = \partial O_2 = P$, and let $S \in \mathrm{DC}(O_1, O_2)_\bullet$ be a decorated cobordism containing $m \geq 0$ dots.

Definition 4.3. The *derivative of S with respect to the dot* is the sum

$$\partial_\bullet S := S_1 + \cdots + S_m \in \mathrm{DC}(O_1, O_2)_{\bullet/\ell},$$

where S_i is the decorated cobordism obtained from S by removing the ith dot.

Lemma 4.4. *The map $\partial_\bullet \colon S \mapsto \partial_\bullet S$ descends to a linear endomorphism of* $\mathrm{DC}(O_1, O_2)_{\bullet/\ell}$.

Proof. We have to check that $\partial_\bullet$ is compatible with the local relation shown in Figure 3. Applying $\partial_\bullet$ to the two sides of the (S) relation yields zero on both sides, and so there is nothing to prove in this case. Applying $\partial_\bullet$ to the (D) relation yields zero on the right-hand side and an undecorated sphere on the left-hand side. But an undecorated sphere is equivalent to zero by the (S) relation, whence $\partial_\bullet$ is also compatible with the (D) relation. Compatibility with the (N) relation follows because $\partial_\bullet$ applied to the left-hand side of (N) gives zero, and $\partial_\bullet$ applied to the right-hand side of (N) yields a sum of two identical term, which is zero because we are working with $\mathbb{F}_2$ coefficients. □

The above lemma implies that $\partial_\bullet$ acts on the morphism sets of $\mathcal{C}ob(U, P)_{\bullet/\ell}$, and the following lemma says that $\partial_\bullet$ satisfies Leibniz' rule with respect to composition of morphisms.

Lemma 4.5. *We have* $\partial_\bullet(S \circ S') = (\partial_\bullet S) \circ S' + S \circ \partial_\bullet S'$.

Proof. Obvious from the definition of $\partial_\bullet$. □

Corollary 4.6. *If S satisfies $S \circ S = 0$, then S commutes with $\partial_\bullet S$.*

Proof. Since coefficients are in $\mathbb{F}_2$ and since $\partial_\bullet$ satisfies Leibniz' rule by Lemma 4.5, we can write the commutator of S with $\partial_\bullet S$ as $[S, \partial_\bullet S] = S \circ \partial_\bullet S + (\partial_\bullet S) \circ S = \partial_\bullet(S \circ S)$, and thus the corollary follows. □

We extend $\partial_\bullet$ to morphisms of $\mathrm{Mat}(U, P)$ (or $\mathrm{Kom}(U, P)$) by setting $\partial_\bullet(F_{ij}) := (\partial_\bullet F_{ij})$. It is easy to see that Lemma 4.5 and Corollary 4.6 remain true for this extended version of $\partial_\bullet$.

Remark 4.7. Note that $\partial_\bullet$ raises the quantum degree by 2 and satisfies $\partial_\bullet \circ \partial_\bullet = 0$ (again we are using that coefficients are in $\mathbb{F}_2$). Thus, the subcategory $\mathcal{C}ob(U, P)^{ev}_{\bullet/\ell} \subset \mathcal{C}ob(U, P)_{\bullet/\ell}$, which has the same objects as $\mathcal{C}ob(U, P)_{\bullet/\ell}$ but whose morphisms are required to have even quantum degree (i.e., $\deg(S) \in 2\mathbb{Z}$), becomes a differential graded category when equipped with the derivation $\partial_\bullet$.

4.3. Dot rotation. In this subsection, we assume that $U = \mathcal{D}$ is the closed unit disk in $\mathbb{R}^2$ and $P \subset \partial U$ is the set $P = \{a, b, c, d\}$ defined in Section 2. As in Section 2, we denote by R_z the self map of $\mathcal{D} \times [0, 1] \subset \mathbb{R}^3$ given by 180° rotation around the z-axis. Since $R_z(P) = P$, the rotation R_z acts on objects and morphisms of $\mathcal{C}ob(\mathcal{D}, P)_{\bullet/\ell}$ by sending an object $O \subset \mathcal{D}$ to the rotated object $R_z(O)$, and a morphism $S \subset \mathcal{D} \times [0, 1]$ to the rotated morphism $R_z(S)$. Since this action is compatible with the composition of morphisms, it defines a functor

$$R_z\colon \mathcal{C}ob(\mathcal{D}, P)_{\bullet/\ell} \longrightarrow \mathcal{C}ob(\mathcal{D}, P)_{\bullet/\ell}.$$

The goal of this subsection is to re-express this functor in terms of the algebraic operations introduced in the previous two subsections. To do this, we first define

$$r_z\colon \mathbb{F}_2[P] \longrightarrow \mathbb{F}_2[P]$$

to be the ring automorphism induced by mapping $x_p \in \mathbb{F}_2[P] := \mathbb{F}_2[x_a, x_b, x_c, x_d]$ to $r_z(x_p) := x_{R_z(p)} \in \mathbb{F}_2[P]$ for all $p \in P$. Explicitly, r_z exchanges x_a with x_c and x_b with x_d. The following lemma is obvious.

Lemma 4.8. *$R_z(fS) = r_z(f)R_z(S)$ for every morphism S in $\mathcal{C}$ and every $f \in \mathbb{F}_2[P]$.*

Now let $\mathcal{C}$ be the full subcategory of $\mathcal{C}ob(\mathcal{D}, P)_{\bullet/\ell}$ containing all objects without closed components, and let $\mathsf{D}\colon \operatorname{Mat}(\mathcal{D}, P) \to \operatorname{Mat}(\mathcal{C})$ be the delooping functor defined as in Section 3.7. The subcategory $\mathcal{C}$ contains two preferred objects: $O_0 := [a, d] \cup [b, c]$ and $O_1 := [a, b] \cup [c, d]$, where $[p, q] \subset \mathcal{D}$ denotes the straight line segment connecting the points $p, q \in P$. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ over the objects O_0 and O_1. (More precisely, $\mathcal{C}'$ contains all objects that are of the form $O\{n\}$ where $O \in \{O_0, O_1\}$ and $\{n\}$ is a grading shift by an arbitrary $n \in \mathbb{Z}$). Since every object in $\mathcal{C}$ is isotopic relative to the boundary (and hence isomorphic in $\mathcal{C}$) to exactly one of the two objects O_0 and O_1, we can define a natural functor $\mathsf{S}\colon \mathcal{C} \to \mathcal{C}'$ by sending $O \in \mathcal{C}$ to O_0 or O_1, whichever of the two is isomorphic to O. Of course, this functor extends to $\operatorname{Mat}(\mathcal{C})$ (or $\operatorname{Kom}(\mathcal{C})$), and we will also write S for this extended functor.

Definition 4.9. The *enhanced delooping functor* is the composition $\mathsf{D}' := \mathsf{S} \circ \mathsf{D}$.

Lemma 4.10. *$\mathsf{D}'(O)$ is isomorphic to O for every $O \in \operatorname{Mat}(\mathcal{D}, P)$ (or $\operatorname{Kom}(\mathcal{D}, P)$).*

Proof. Clear from the definitions of D and S. □

Since O_0 and O_1 are invariant under rotation by 180°, the functor R_z acts as the identity on the set $\operatorname{Ob}(\mathcal{C}') = \{O_0, O_1\}$.

Definition 4.11. The *dot rotation functor* is the endofunctor $R_\bullet\colon \mathcal{C}' \to \mathcal{C}'$ which acts as the identity on the set $\mathrm{Ob}(\mathcal{C}') = \{O_0, O_1\}$ and which takes a morphism S to the morphism

$$R_\bullet(S) := S + (x_a + x_c)\partial_\bullet S\,.$$

Lemma 4.12. *$R_z(S) = R_\bullet(S)$ for every morphism S in $\mathcal{C}'$.*

Proof. Let $S \subset \mathcal{D} \times [0, 1]$ be a decorated cobordism representing a morphism in $\mathcal{C}'$. Using the local relations shown in Figures 3 and 4, we can write as $S = S' \sqcup t^n =: t^n S'$, where t^n is a disjoint union of $n \geq 0$ two-spheres, each or them decorated by exactly three dots, and S' is a decorated cobordism whose every component is homeomorphic to a disk and decorated by at most one dot. Let S'' be the undecorated cobordism underlying S'. Then S'' has to be either a saddle cobordism or one of the two identity cobordisms $O_0 \times [0, 1]$ or $O_1 \times [0, 1]$ (as these are the only undecorated cobordisms in $\mathcal{C}'$ that have the property that all of their connected components are homeomorphic to disks). In particular, S'' is invariant under R_z and has at most two connected components. Moreover, every connected component of S'' contains at least one of the two segments $\{a\} \times [0, 1]$ or $\{c\} \times [0, 1]$, and this means that we can write S' as $S' = x_a^{n_a} x_c^{n_c} S''$ for appropriate $n_a, n_c \in \{0, 1\}$ (where e.g. $x_a x_c S''$ denotes the decorated cobordism $X_a \circ X_c \circ S''$ as in Section 4.1). Writing f for the monomial $t^n x_a^{n_a} x_c^{n_c} \in \mathbb{F}_2[t, x_a, x_c]$ and using Lemma 4.8, we obtain:

$$R_z(S) = r_z(f) R_z(S'') = r_z(f) S''.$$

One can easily check that $\partial_\bullet t = 0$, and since S'' contains no dots, we also have $\partial_\bullet S'' = 0$. Using Lemma 4.5 we therefore obtain $\partial_\bullet S = \partial_\bullet(f S'') = (\partial f)S''$, where $\partial\colon \mathbb{F}_2[t, x_a, x_c] \to \mathbb{F}_2[t, x_a, x_c]$ is the $\mathbb{F}_2[t]$-linear map defined by $\partial := \partial/\partial x_a + \partial/\partial x_c$. Thus:

$$R_\bullet(S) = [f + (x_a + x_c)(\partial f)]S''.$$

Comparing the above expressions for $R_z(S)$ and $R_\bullet(S)$, we see that it suffices to prove the equivalence $r_z(f) \equiv f + (x_a + x_c)(\partial f)$ modulo local relations. We do this by case by case analysis: if $f = t^n$, then $r_z(f) = f$ and $\partial f = 0$, so the result follows. If $f = t^n x_a$, then $r_z(f) = t^n x_c$ and $\partial f = t^n$, so $r_z(f) = t^n x_c = 2t^n x_a + t^n x_c = f + (x_a + x_c)(\partial f)$; the case $f = t^n x_c$ is analogous. Finally, if $f = t^n x_a x_c$, then $r_z(f) = f$ and

$$(x_a + x_c)(\partial f)S'' = t^n (x_a + x_c)^2 S'' = t^n (x_a^2 + x_c^2) S'' = 2t^{n+1} S'' = 0,$$

where we have used the (DD) relation and the fact that coefficients are in $\mathbb{F}_2$. □

Corollary 4.13. *$R_\bullet(\mathsf{D}'(O))$ is isomorphic to $R_z(O)$ for all $O \in \mathrm{Mat}(\mathcal{D}, P)$ (or $\mathrm{Kom}(\mathcal{D}, P)$).*

Proof. The functors D and S are clearly equivariant under the rotation R_z, and hence $\mathsf{D}' = \mathsf{S} \circ \mathsf{D}$ commutes with R_z. Using Lemmas 4.10 and 4.12, we thus obtain $R_z(O) \cong \mathsf{D}'(R_z(O)) = R_z(\mathsf{D}'(O)) = R_\bullet(\mathsf{D}'(O))$. □

4.4. Dot migration. Let T' be a tangle diagram in $\mathcal{D}^c := \{z \in \mathbb{C} = \mathbb{R}^2 \colon |z| \geq 1\}$ with $\partial T' = P = \{a, b, c, d\}$. Assume that T' has crossed connectivity as in Proposition 2.6, i.e., that it represents a a tangle $\mathcal{T}' \subset \mathcal{D}^c \times \mathbb{R}$ which contains an arc connecting the endpoints $\{a\} \times \{0\}$ and $\{c\} \times \{0\}$. Let $\alpha \subset T'$ be the projection of this arc, and let $c_1, \ldots, c_m \subset \alpha$ be the crossings of T' along α, enumerated in the order shown in Figure 5.

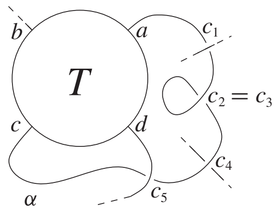

Figure 5. Crossings $c_1, \ldots, c_m$ along the arc $\alpha \subset T'$.

For $k = 2, \ldots, m$, let $e_k \subset \alpha$ be the connected component of $\alpha \setminus \bigcup_k c_k$ which lies between c_{k-1} and c_k, and let $p_k \in e_k$ denote the midpoint of e_k. Put $p_1 := a$ and $p_{m+1} := c$.

Definition 4.14. Let $X_1, \ldots, X_{m+1}$ be the endomorphisms of $\bigoplus_{i \in \mathbb{Z}} \mathrm{Kh}(T')^i \in \mathrm{Mat}(\mathcal{D}^c, P)$ defined by $X_k := X_{p_k}$, where X_{p_k} is the dot multiplication map defined in Section 4.1.

As explained in the proof of Lemma 3.6, the differential in $\mathrm{Kh}(T')$ can be regarded as an endomorphism d of the object $\bigoplus_{i \in \mathbb{Z}} \mathrm{Kh}(T')^i \in \mathrm{Mat}(\mathcal{D}^c, P)$, and this endomorphism can be written as a sum $d = \sum_{c \in \chi} d_c$. Recall that the matrix entries $d_{\epsilon'\epsilon}$ and $(d_c)_{\epsilon'\epsilon}$ are either zero or given by a saddle cobordism $S_{\epsilon'\epsilon} \subset \mathcal{D}^c \times [0, 1]$. Let $r \colon \mathcal{D}^c \times [0, 1] \to \mathcal{D}^c \times [0, 1]$ be the reflection along $\mathcal{D}^c \times \{1/2\}$, and let $d_k := d_{c_k}$.

Definition 4.15. The *dot migration homotopies* $h_1, \ldots, h_m$ are the endomorphisms of $\bigoplus_{i \in \mathbb{Z}} \mathrm{Kh}(T')^i = \bigoplus_{\epsilon \in \{0,1\}^\chi} T'_\epsilon \in \mathrm{Mat}(\mathcal{D}^c, P)$ defined by $h_k := d_k^\dagger$ where $(d_k^\dagger)_{\epsilon'\epsilon} := r((d_k)_{\epsilon\epsilon'})$.

Arguing as in the proof of Lemma 3.6, one can easily show:

Lemma 4.16. *We have*

(1) $h_k \circ h_k = 0$,
(2) $h_k \circ h_l = h_l \circ h_k$,
(3) $h_k \circ d_c = d_c \circ h_k$,

for all $k, l = 1, \ldots, m$ *and all crossings* $c \neq c_k$.

The next lemma says that h_k is a homotopy between X_k and X_{k+1}.

Lemma 4.17. $d \circ h_k + h_k \circ d = X_k + X_{k+1}$.

Proof. Since $d = \sum_{c \in \chi} d_c$ and since h_k commutes with d_c for all $c \in \chi$ with $c \neq c_k$, we have $d \circ h_k + h_k \circ d = d_k \circ h_k + h_k \circ d_k$, and so it is enough to prove $d_k \circ h_k + h_k \circ d_k = X_k + X_{k+1}$. Since this is a purely local equation, we can restrict ourselves to the case where $k = 1$ and $\chi = \{c_1\}$, i.e., where T' has only one crossing. Then $\mathrm{Kh}(T') = T'_0 \oplus T'_1$ (here we ignore the homological grading and the quantum grading), where T'_0 and T'_1 are the crossingless diagrams obtained by replacing the crossing c_1 ($= \times$) by its 0-resolution ()() and its 1-resolution ($\asymp$), respectively. We can regard the differential $d = d_1$ in $\mathrm{Kh}(T')$ as an endomorphism of the object $T'_0 \oplus T'_1 \in \mathrm{Mat}(\mathcal{D}^c, P)$. As such, it is given by a 2×2 matrix, whose only non-zero entry is $d_{10} = S_{10}$, where S_{10} is a saddle cobordism (as in Section 3.4). Similarly, the homotopy $h := h_1$ is given by a 2×2-matrix whose only non-zero entry is the saddle cobordism $h_{01} = r(S_{10})$. Thus, $(h \circ d)_{00} = r(S_{10}) \circ S_{10}$ and $(d \circ h)_{11} = S_{10} \circ r(S_{10})$, and all other matrix entries in $h \circ d$ and $d \circ h$ are zero. The cobordism $r(S_{10}) \circ S_{10}$ is a composition of two 'opposite' saddle cobordisms, and it is easy to see that such a composition results in a cobordism looking like the identity cobordism $T'_0 \times [0, 1]$, except that the two components of)($\times [0, 1]$ are connected by a tube. Applying the (N) relation to this tube, we obtain

$$(h \circ d)_{00} = r(S_{10}) \circ S_{10} = (x_1 + x_2)(T'_0 \times [0, 1]), = (x_1 + x_2)1_{00}$$

where 1_{00} is the identity morphism of T'_0. Similarly, we obtain $(d \circ h)_{11} = (x_1 + x_2)1_{11}$ where 1_{11} is the identity morphism of T'_1. Thus, $d \circ h + h \circ d = (x_1 + x_2)1 = X_1 + X_2$, as desired. □

Lemma 4.18. $h_k \circ d \circ h_k = 0$.

Proof. By the previous lemma, we have $d \circ h_k = h_k \circ d + X_k + X_{k+1}$, and inserting this into $h_k \circ d \circ h_k$, we obtain $h_k \circ d \circ h_k = h_k \circ h_k \circ d + h_k \circ (X_k + X_{k+1})$. The first term on the right-hand side vanishes because $h_k \circ h_k = 0$, and to see that the second term vanishes, we can assume that T' consists of a single crossing, i.e., $k = 1$ and $\chi = \{c_1\}$ as in the proof of the previous lemma. Then $(h_1)_{01} = r(S_{10})$ (as in the proof of the previous lemma), and since the cobordism $r(S_{10})$ has only one connected component, we have $x_1 r(S_{10}) = x_2 r(S_{10})$, whence $h_1 \circ X_1 = h_1 \circ X_2$. Using that coefficients are in $\mathbb{F}_2$, we get $h_1 \circ (X_1 + X_2) = 2h_1 \circ X_1 = 0$. □

5. Proof of Proposition 2.6

In this section, we use the notations of Section 2, and we assume that the hypotheses of Proposition 2.6 are satisfied.

In particular, T denotes a tangle diagram in the unit disk $\mathcal{D} \subset \mathbb{R}^2$, and T' a tangle diagram in $\mathcal{D}^c := \mathbb{R}^2 \setminus \mathrm{Int}(\mathcal{D})$. The endpoints of T and T' lie in the set $\partial T = \partial T' = P = \{a, b, c, d\} \subset \partial\mathcal{D}$. As in Proposition 2.6, we assume that T' represents a tangle $\mathcal{T}' \subset \mathcal{D}^c \times \mathbb{R} \subset \mathbb{R}^3$ which has crossed connectivity, i.e., contains an arc connecting the endpoints $\{a\} \times \{0\}$ and $\{c\} \times \{0\}$. We also assume that the mutation is a z-mutation, i.e., that it consists in replacing T by $R_z(T)$. Let $L := T \cup T'$ and $L' := R_z(T) \cup T'$ denote the link diagrams before and after mutation. Using the tensor product theorem (Theorem 3.8), we can write the formal Khovanov brackets of L and L' as

$$\mathrm{Kh}(L) = \mathrm{Kh}(T) \otimes \mathrm{Kh}(T') \quad \text{and} \quad \mathrm{Kh}(L') = \mathrm{Kh}(R_z(T)) \otimes \mathrm{Kh}(T').$$

Let $\mathcal{C}' \subset \mathcal{C}ob(\mathcal{D}, P)_{\bullet/\ell}$ be the full subcategory generated by the two objects $O_0 := [a, d] \cup [b, c]$ and $O_1 := [a, b] \cup [c, d]$ where $[p, q] \subset \mathcal{D}$ denotes the straight line segment connecting the points $p, q \in P$ as in Section 4.3. Let $\mathsf{D}' \colon \mathrm{Mat}(\mathcal{D}, P) \to \mathrm{Mat}(\mathcal{C}')$ denote the enhanced delooping functor (Definition 4.9) and $R_\bullet \colon \mathrm{Mat}(\mathcal{C}') \to \mathrm{Mat}(\mathcal{C}')$ the dot rotation functor (Definition 4.11). By Lemma 4.10, $\mathrm{Kh}(T)$ is isomorphic to $\mathsf{D}'(\mathrm{Kh}(T))$, and hence $\mathrm{Kh}(L)$ is isomorphic to the complex

$$A := \mathsf{D}'(\mathrm{Kh}(T)) \otimes \mathrm{Kh}(T').$$

Since the construction of $\mathrm{Kh}(T)$ is equivariant with respect to the rotation R_z, we have $\mathrm{Kh}(R_z(T)) = R_z(\mathrm{Kh}(T))$. Moreover, Corollary 4.13 implies that $R_z(\mathrm{Kh}(T))$ is isomorphic to $R_\bullet(\mathsf{D}'(\mathrm{Kh}(T)))$, and hence $\mathrm{Kh}(L')$ is isomorphic to the complex

$$B := R_\bullet(\mathsf{D}'(\mathrm{Kh}(T))) \otimes \mathrm{Kh}(T').$$

To prove Proposition 2.6, it is now enough to show A is isomorphic to B. By definition, $R_\bullet$ acts as the identity on the set $\mathrm{Ob}(\mathcal{C}') = \{O_0, O_1\}$, and so we have $A = B$ if we ignore the differentials in A and B (i.e., if we just consider the objects $\bigoplus_{i\in\mathbb{Z}} A^i$ and $\bigoplus_{i\in\mathbb{Z}} B^i$ of $\mathrm{Mat}(\mathbb{R}^2, \emptyset)$ instead of the actual complexes $A = (A^*, d_A^*)$ and $B = (B^*, d_B^*)$). The differentials in A and B are given by

$$d_A = \delta \otimes 1 + 1 \otimes d \quad \text{and} \quad d_B = R_\bullet(\delta) \otimes 1 + 1 \otimes d,$$

where δ is the differential in $\mathsf{D}'(\mathrm{Kh}(T))$ and d is the differential in $\mathrm{Kh}(T')$, and 1 stands for an identity morphism. To prove that the complexes A and B are isomorphic, we must therefore construct an automorphism φ of the object $A = B \in \mathrm{Mat}(\mathbb{R}^2, \emptyset)$ which satisfies $\varphi \circ d_A = d_B \circ \varphi$.

Let $\mathcal{T}' \subset \mathcal{D}^c \times \mathbb{R}$ be the tangle represented by $T' \subset \mathcal{D}^c$. Let $\alpha \subset T'$ the projection of the arc of $\mathcal{T}'$ connecting $\{a\} \times \{0\}$ to $\{c\} \times \{0\}$, and let $c_1, \ldots, c_m$ be the sequence of crossings along α, as in Figure 5. As in Section 4.4, we denote $h_1, \ldots, h_m$ the dot migration homotopies (Definition 4.15) and by $X_1, \ldots, X_{m+1}$ the maps $X_k := X_{p_k}$ (Definition 4.14). For $k = 1, \ldots, m$, we define φ_k to be the endomorphism of $A = B \in \mathrm{Mat}(\mathbb{R}^2, \emptyset)$ given by

$$\varphi_k := 1 \otimes 1 + (\partial_\bullet \delta) \otimes h_k,$$

where $\partial_\bullet$ is the derivative with respect to the dot (Definition 4.3).

Definition 5.1. Let φ be the composition $\varphi := \varphi_1 \circ \cdots \circ \varphi_m \in \mathrm{End}_{\mathrm{Mat}(\mathbb{R}^2,\emptyset)}(A = B)$.

Using Lemma 4.16 and the fact that coefficients are in $\mathbb{F}_2$, it is easy to check that $\varphi_k \circ \varphi_k = 1 \otimes 1$ and $\varphi_k \circ \varphi_l = \varphi_l \circ \varphi_k$ for all k, l, and hence also $\varphi \circ \varphi = 1 \otimes 1$. In particular, φ is invertible.

Remark 5.2. Since every self-crossing of α appears twice in the list $c_1, \ldots, c_m$, every endomorphism φ_k corresponding to a self-crossing of α appears twice in φ. Since φ_k squares to the identity, we can thus ignore all self-crossings of α, and define φ as the product over all φ_k for which c_k is not a self-crossing of α.

To see that φ satisfies $\varphi \circ d_A = d_B \circ \varphi$ as desired, we need several technical lemmas.

Lemma 5.3. *φ commutes with $\delta \otimes 1$.*

Proof. Corollary 4.6 tells us that $\partial_\bullet \delta$ commutes with δ, and this immediately implies that each φ_k (and hence also φ) commutes with $\delta \otimes 1$. □

Lemma 5.4. $\varphi_k \circ (1 \otimes d) \circ \varphi_k^{-1} = 1 \otimes d + (\partial_\bullet \delta) \otimes (X_k + X_{k+1})$.

Proof. Direct calculation using $\varphi_k = \varphi_k^{-1} = 1 \otimes 1 + (\partial_\bullet d) \otimes h_k$ yields

$$\varphi_k \circ (1 \otimes d) \circ \varphi_k^{-1} = 1 \otimes d + (\partial_\bullet \delta) \otimes (d \circ h_k + h_k \circ d) + (\partial_\bullet \delta)^2 \otimes (h_k \circ d \circ h_k)$$

and now the claim follows from Lemmas 4.17 and 4.18. □

Corollary 5.5. $\varphi \circ (1 \otimes d) \circ \varphi^{-1} = 1 \otimes d + (\partial_\bullet \delta) \otimes (X_a + X_c)$.

Proof. Recall that $X_l = X_{p_l} = x_{p_l} 1$ and from this it easily follows that $X_l \circ h_k = x_{p_l} h_k = h_k \circ X$ for all k, l. Thus φ_k commutes with $(\partial_\bullet \delta) \otimes X_l$ for all k, l. Recalling that $\varphi = \varphi^{-1} = \varphi_1 \circ \cdots \circ \varphi_m$ and using Lemma 5.4 repeatedly, one can now conclude

$$\varphi \circ (1 \otimes d) \circ \varphi^{-1} = d \otimes 1 + (\partial_\bullet \delta) \otimes [(X_1 + X_2) + (X_2 + X_3) + \cdots + (X_m + X_{m+1})],$$

and the telescope sum in the square brackets collapses to $X_1 + X_{m+1}$ because all intermediate terms appear twice and hence cancel. Since $p_1 = a$ and $p_{m+1} = c$ (see Section 4.4), we have $X_1 = X_a$ and $X_{m+1} = X_c$, whence $X_1 + X_{m+1} = X_a + X_c$. □

We are now ready to prove Proposition 2.6.

Proof of Proposition 2.6. We have to show that $\varphi \circ d_A \circ \varphi^{-1} = d_B$. This is now a direct calculation:

$$\begin{aligned} \varphi \circ d_A \circ \varphi^{-1} &\overset{(1)}{=} \varphi \circ (\delta \otimes 1 + 1 \otimes d) \circ \varphi^{-1} \\ &\overset{(2)}{=} \delta \otimes 1 + \varphi \circ (1 \otimes \delta) \circ \varphi^{-1} \\ &\overset{(3)}{=} \delta \otimes 1 + 1 \otimes d + (\partial_\bullet \delta) \otimes (X_a + X_c) \\ &\overset{(4)}{=} \delta \otimes 1 + 1 \otimes d + \big((x_a + x_c)(\partial_\bullet \delta)\big) \otimes 1 \\ &\overset{(5)}{=} R_\bullet(\delta) \otimes 1 + 1 \otimes d \\ &\overset{(6)}{=} d_B. \end{aligned}$$

Equalities (1) and (6) are the definitions of d_A and d_B, respectively. Equality (5) is the definition of $R_\bullet$. Equality (2) follows because φ commutes with $\delta \otimes 1$ by Lemma 5.3. Equality (3) is Corollary 5.5. To see (4), observe that $1 \otimes X_a = X_a \otimes 1$ because $1 \otimes X_a$ and $X_a \otimes 1$ are both obtained from the identity morphism $1 \otimes 1$ by inserting a dot into the line segment $\{a\} \times [0, 1]$ (cf. Definitions 4.1 and 4.14). Therefore

$$(\partial_\bullet \delta) \otimes X_a = (1 \otimes X_a) \circ [(\partial_\bullet \delta) \otimes 1] = (X_a \otimes 1) \circ [(\partial_\bullet \delta) \otimes 1] = (x_a \partial_\bullet \delta) \otimes 1,$$

and similarly $(\partial_\bullet \delta) \otimes X_c = (x_c \partial_\bullet \delta) \otimes 1$. □

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