

Fusion categories and homotopy theory

Pavel Etingof, Dmitri Nikshych and Victor Ostrik

With an Appendix by Ehud Meir

Abstract. We apply the yoga of classical homotopy theory to classification problems of G -extensions of fusion and braided fusion categories, where G is a finite group. Namely, we reduce such problems to classification (up to homotopy) of maps from BG to classifying spaces of certain higher groupoids. In particular, to every fusion category \mathcal{C} we attach the 3-groupoid $\underline{\text{BrPic}}(\mathcal{C})$ of invertible \mathcal{C} -bimodule categories, called the Brauer–Picard groupoid of \mathcal{C} , such that equivalence classes of G -extensions of \mathcal{C} are in bijection with homotopy classes of maps from BG to the classifying space of $\underline{\text{BrPic}}(\mathcal{C})$. This gives rise to an explicit description of both the obstructions to existence of extensions and the data parametrizing them; we work these out both topologically and algebraically.

One of the central results of the article is that the 2-truncation of $\underline{\text{BrPic}}(\mathcal{C})$ is canonically equivalent to the 2-groupoid of braided auto-equivalences of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of \mathcal{C} . In particular, this implies that the Brauer–Picard group $\text{BrPic}(\mathcal{C})$ (i.e., the group of equivalence classes of invertible \mathcal{C} -bimodule categories) is naturally isomorphic to the group of braided auto-equivalences of $\mathcal{Z}(\mathcal{C})$. Thus, if $\mathcal{C} = \text{Vec}_A$, where A is a finite abelian group, then $\text{BrPic}(\mathcal{C})$ is the orthogonal group $O(A \oplus A^*)$. This allows one to obtain a rather explicit classification of extensions in this case; in particular, in the case $G = \mathbb{Z}_2$, we re-derive (without computations) the classical result of Tambara and Yamagami. Moreover, we explicitly describe the category of all $(\text{Vec}_{A_1}, \text{Vec}_{A_2})$ -bimodule categories (not necessarily invertible ones) by showing that it is equivalent to the hyperbolic part of the category of Lagrangian correspondences.

Mathematics Subject Classification (2010). 18D10, 55S35.

Keywords. Fusion categories, module categories, homotopy theory, higher groupoids.

Contents

1	Introduction	210
2	Preliminaries	213
3	Tensor product of module categories	220
4	Higher groupoids attached to fusion categories	228
5	Proof of Theorem 1.1	233
6	Invertibility of components of graded fusion categories	238

7	Classification of extensions (topological version)	239
8	Classification of extensions (algebraic version)	246
9	Examples of extensions	256
10	Lagrangian subgroups in metric groups and bimodule categories over Vec_A	260
11	Appendix by Ehud Meir: Group extensions as G -graded fusion categories	265
	References	271

1. Introduction

Fusion categories (introduced in [12]) form a class of relatively simple tensor categories. It would be very interesting to give a classification of fusion categories but this seems to be out of reach at the moment. A more feasible task is to come up with some new examples and constructions of such categories. In this article we are making a step in this direction. Namely, for a finite group G there is a natural notion of G -graded fusion category; see § 2.3 below.¹ The *trivial component* of a G -graded fusion category is itself a smaller fusion category and we say that a G -graded fusion category is G -extension of its trivial component. The goal of this article is to apply classical homotopy theory to classify G -extensions of a given fusion category.

To do so, we introduce the *Brauer–Picard groupoid of fusion categories* $\underline{\text{BrPic}}$. By definition, this is a 3-groupoid, whose objects are fusion categories, 1-morphisms from \mathcal{C} to \mathcal{D} are invertible $(\mathcal{C}, \mathcal{D})$ -bimodule categories, 2-morphisms are equivalences of such bimodule categories, and 3-morphisms are isomorphisms of such equivalences. This 3-groupoid can be truncated in the usual way to a 2-groupoid $\underline{\text{BrPic}}$ and further to a 1-groupoid (i.e., an ordinary groupoid) BrPic ; the group of automorphisms of \mathcal{C} in this groupoid is the *Brauer–Picard group* $\text{BrPic}(\mathcal{C})$ of \mathcal{C} , which is the group of equivalence classes of invertible \mathcal{C} -bimodule categories.

We also define the 2-groupoid $\underline{\text{EqBr}}$, whose objects are braided fusion categories, 1-morphisms are braided equivalences, and 2-morphisms are isomorphisms of such equivalences. It can be truncated in the usual way to an ordinary groupoid EqBr ; the group of automorphisms of a braided fusion category \mathcal{B} in this groupoid is the group $\text{EqBr}(\mathcal{B})$ of isomorphism classes of braided auto-equivalences of \mathcal{B} .

Let \mathcal{C} and \mathcal{D} be fusion categories. Any invertible $(\mathcal{C}, \mathcal{D})$ -bimodule category \mathcal{M} naturally gives rise to a Morita equivalence between \mathcal{C} and \mathcal{D} . Hence, by the result of Müger [28] it defines a braided equivalence of the Drinfeld centers $\mathcal{Z}(\mathcal{M}): \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{Z}(\mathcal{D})$. This implies that the operation \mathcal{Z} of taking the Drinfeld center is a 2-functor $\underline{\text{BrPic}} \rightarrow \underline{\text{EqBr}}$.

Our first main result, which is a strengthening of [13], Theorem 3.1, is

¹We note that one can find in the literature a different (but related) notion of *graded monoidal category*, see [15], [4].

Theorem 1.1. *The 2-functor \mathcal{Z} is a fully faithful embedding $\underline{\text{BrPic}} \rightarrow \underline{\text{EqBr}}$. In particular, for every fusion category \mathcal{C} we have a natural group isomorphism $\text{BrPic}(\mathcal{C}) \cong \text{EqBr}(\mathcal{Z}(\mathcal{C}))$.*

This result allows one to calculate the group $\text{BrPic}(\mathcal{C})$ in the case $\mathcal{C} = \text{Vec}_A$, the category of vector spaces graded by a group A . In particular, we immediately get the following corollary of Theorem 1.1:

Corollary 1.2. *If A is an abelian group and $\mathcal{C} = \text{Vec}_A$ then $\text{BrPic}(\mathcal{C}) = \text{O}(A \oplus A^*)$, the split orthogonal group of $A \oplus A^*$ (i.e., the group of automorphisms of $A \oplus A^*$ preserving the hyperbolic quadratic form $q(a, f) = f(a)$).*

To apply the above to classifying extensions, we recall that to the 3-groupoid $\underline{\text{BrPic}}$ one can attach its classifying space $B\underline{\text{BrPic}}$, defined up to homotopy equivalence. This space falls into connected components, labeled by Morita equivalence classes of fusion categories. Each connected component $B\underline{\text{BrPic}}(\mathcal{C})$ corresponding to a fusion category \mathcal{C} is a 3-type, i.e., it has three nontrivial homotopy groups: its fundamental group π_1 is $\text{BrPic}(\mathcal{C})$, π_2 is the group of isomorphism classes of invertible objects of $\mathcal{Z}(\mathcal{C})$, and $\pi_3 = \mathbf{k}^\times$ (the multiplicative group of the ground field).

It then follows from general abstract nonsense that extensions of \mathcal{C} by a group G are parametrized by maps of classifying spaces $BG \rightarrow B\underline{\text{BrPic}}(\mathcal{C})$. Thus, to classify extensions, one needs to classify the homotopy classes of such maps, which we proceed to do using the classical obstruction theory. This leads us to our second main result, which is the following explicit description of extensions of fusion categories and which is similar to the classical description of group extensions [9] (and is made more explicit in the body of the article).

Theorem 1.3. *Graded extensions of a fusion category \mathcal{C} by a finite group G are parametrized by triples (c, M, α) , where $c: G \rightarrow \text{BrPic}(\mathcal{C})$ is a group homomorphism, M belongs to a certain torsor T_c^2 over $H^2(G, \pi_2)$ (where G acts on π_2 via c), and α belongs to a certain torsor $T_{c,M}^3$ over $H^3(G, \mathbf{k}^\times)$. Here the data c, M must satisfy the conditions that certain obstructions $O_3(c) \in H^3(G, \pi_2)$ and $O_4(c, M) \in H^4(G, \mathbf{k}^\times)$ vanish.*

We also give a purely algebraic proof of Theorem 1.3, which does not rely on homotopy theory. (This proof spells out the computations that on the topological side are hidden in the machinery of homotopy theory.) After this, we proceed to examples and applications. In particular, we give a conceptual proof of the classification of categorifications of Tambara–Yamagami fusion rings [37] (the original proof is by a direct computation).²

²We note that the quest for a computation-free derivation of the remarkable result of Tambara and Yamagami was one of the motivations for this work.

At the end of the article we discuss a number of related topics. In particular, we describe explicitly the monoidal 2-category of all bimodule categories over $\mathcal{C} = \text{Vec}_A$ (not necessarily invertible ones). It turns out to be equivalent to a full subcategory of the category of Lagrangian correspondences for metric groups (abelian groups with a non-degenerate quadratic form).

1.1. Organization. Section 2 contains background material from the theory of fusion categories and their module categories. There is new material in Section 2.6, where we give a definition (due to V. Drinfeld) of the special orthogonal group of a metric group.

The notion of a tensor product of module categories over a fusion category \mathcal{C} plays a central role in this work. It extends categorically the notion of tensor product of modules over a ring. In Section 3 we define, following [36], the tensor product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ of a right \mathcal{C} -module category \mathcal{M} and a left \mathcal{C} -module category \mathcal{N} by a certain universal property. We prove its existence and give several equivalent characterizations of it useful for practical purposes. We also introduce a monoidal 2-category $\mathbf{Bimodc}(\mathcal{C})$ of \mathcal{C} -bimodule categories and explicitly describe the product of bimodule categories over the categories of vector spaces graded by abelian groups.

In Section 4 we study bimodule categories invertible under the above tensor product. We introduce for a fusion category \mathcal{C} its categorical *Brauer–Picard* 2-group $\underline{\text{BrPic}}(\mathcal{C})$ consisting of invertible \mathcal{C} -bimodule categories and for a braided fusion category \mathcal{B} its categorical *Picard* 2-group $\underline{\text{Pic}}(\mathcal{B})$ consisting of invertible \mathcal{B} -module categories.

Section 5 contains the proof of Theorem 1.1 and its generalization Theorem 5.2.

In Section 6 we prove that homogeneous components of a fusion category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ graded by a finite group G (i.e., a G -extension) are invertible bimodule categories over the trivial component \mathcal{C}_e .

In Section 7 we show that morphisms from a group G to various categorical groups attached to a (braided) fusion category \mathcal{C} (or, equivalently, maps between the corresponding classifying spaces) are in bijection with fundamental tensor category constructions involving G and \mathcal{C} : extensions, actions, braided G -crossed extensions, etc. Here we also give a topological version of the proof of Theorem 1.3.

In Section 8 we give a detailed algebraic version of the proof of Theorem 1.3. We give a formula for the associativity constraint obstruction $O_4(c, M)$ in terms of the Pontryagin–Whitehead quadratic function and prove a divisibility result, Theorem 8.16, for the order of $O_4(c, M)$ in $H^4(G, \mathbf{k}^\times)$.

In Section 9 we apply our classification of extensions to recover Tambara–Yamagami categories [37] as $\mathbb{Z}/2\mathbb{Z}$ -extensions of the category Vec_A of A -graded vector spaces, where A is an abelian group.

In Section 10 we explicitly describe tensor products of $(\text{Vec}_A - \text{Vec}_B)$ -bimodule categories, where A, B are finite abelian groups. This description is given in terms of elementary linear algebra and uses the language of Lagrangian correspondences.

Finally, in the appendix, written by Ehud Meir, it is explicitly shown using the Lyndon–Hochschild–Serre spectral sequence that in the case of pointed extensions, our classification of extensions reproduces the usual theory of extensions of groups with 3-cocycles.

Remark 1.4. 1. We emphasize that the homotopy-theoretic approach to monoidal categories in the style of this article is not new, and by now is largely a part of folklore. The principal goal of this article is to use this approach to obtain concrete results about classification of fusion categories.

2. We expect that the results of this article extend, with appropriate changes, to the case of not necessarily semisimple finite tensor categories, using the methods of [14]. One of the new features will be that in the non-semisimple case the groups $\text{Pic}(\mathcal{C})$, $\text{EqBr}(\mathcal{B})$ need not be finite groups – they may be affine algebraic groups of positive dimension.

Acknowledgments. We are deeply grateful to V. Drinfeld for many inspiring conversations. Without his influence, this article would not have been written. In particular, he suggested the main idea – to use homotopy theory of classifying spaces to describe extensions of fusion categories. We also thank Jacob Lurie for useful discussions (in particular, for explanations regarding Proposition 7.6), and Fernando Muro for explanations and references. The work of P.E. was partially supported by the NSF grant DMS-0504847. The work of D.N. was partially supported by the NSA grant H98230-07-1-0081 and the NSF grant DMS-0800545. The work of V.O. was partially supported by the NSF grant DMS-0602263.

2. Preliminaries

2.1. General conventions. In this article, we will freely use the basic theory of fusion categories and module categories over them. For basics on these topics, we refer the reader to [1], [31], [12], [7]. All fusion categories in this article will be over an algebraically closed field \mathbf{k} of characteristic zero, and all module categories will be semisimple left module categories (unless noted otherwise). We will also use the theory of higher categories and especially higher groupoids, for which we refer the reader to [25]. However, for the reader’s convenience, we recall some of the most important definitions and facts that are used below.

2.2. Categorical n -groups. For an integer $n \geq 1$, a *categorical n -group* is a monoidal n -groupoid whose objects are invertible. In particular, a categorical 0-group is an ordinary group, and a categorical 1-group (or simply a categorical group) is also called a *gr-category* (if the corresponding group of objects is abelian, such a structure is often called a Picard groupoid). Any categorical n -group can be viewed as an $(n + 1)$ -groupoid with one object, and vice versa.

Note that any categorical n -group can be truncated to a categorical $(n - 1)$ -group by forgetting the n -morphisms and identifying isomorphic $(n - 1)$ -morphisms. Conversely, any categorical $(n - 1)$ -group can be regarded as a categorical n -group by adding the identity n -morphism from every $(n - 1)$ -morphism to itself.

2.3. Graded tensor categories and extensions. Let G be a finite group. Recall that a G -grading on a tensor category \mathcal{C} is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

into a direct sum of full abelian subcategories such that the tensor product \otimes maps $\mathcal{C}_g \times \mathcal{C}_h$ to \mathcal{C}_{gh} for all $g, h \in G$. In this case, the trivial component \mathcal{C}_e is a full tensor subcategory of \mathcal{C} , and each \mathcal{C}_g is a \mathcal{C}_e -bimodule category. We will always assume that the grading is faithful, i.e., $\mathcal{C}_g \neq 0$ for all $g \in G$.

Definition 2.1. A G -extension of a fusion category \mathcal{D} is a G -graded fusion category \mathcal{C} whose trivial component is equivalent to \mathcal{D} .

2.4. Quadratic forms, bicharacters, metric groups, and Lagrangian subgroups.

Let E be a finite abelian group. A *bicharacter* on E with values in \mathbf{k}^\times is a biadditive map $b: E \times E \rightarrow \mathbf{k}^\times$. A *symmetric bicharacter* on E (also called an inner product or a symmetric bilinear form) is a bicharacter b such that $b(x, y) = b(y, x)$. A *skew-symmetric bicharacter* on E (also called a skew-symmetric bilinear form) is a bicharacter b such that $b(x, x) = 1$.

Let $E^* = \text{Hom}(E, \mathbf{k}^\times)$ be the character group of E . By acting on its first argument, any bicharacter b on E defines a group homomorphism $\hat{b}: E \rightarrow E^*$. We say that b is *non-degenerate* if \hat{b} is an isomorphism. Note that if E admits a non-degenerate skew-symmetric bicharacter, then $|E|$ is a square.

A *quadratic form* on E is a function $q: E \rightarrow \mathbf{k}^\times$ such that $q(x) = q(x^{-1})$, and $b_q(x, y) := q(x + y)/q(x)q(y)$ is a symmetric bilinear form. If the order of the group E is odd, the assignment $q \rightarrow b_q$ defines a bijection between symmetric bilinear forms and quadratic forms, but in general, it is not a bijection.

We will say that a quadratic form q is *non-degenerate* if the bilinear form b_q is non-degenerate. In this case we say that (E, q) is a *metric group*. To every metric group (E, q) , one can attach its *orthogonal group* $O(E, q)$, which is the group of automorphisms of E preserving q . For example, if A is any finite abelian group then $A \oplus A^*$ is a metric group, with hyperbolic quadratic form $q(a, f) := f(a)$. To simplify notation, we will denote the corresponding orthogonal group by $O(A \oplus A^*)$.

If E is a finite abelian group with a bicharacter b , and $N \subset E$ is a subgroup, then the *orthogonal complement* N^\perp is the set of $a \in E$ such that $b(x, a) = 1$ for any $x \in N$. If b is non-degenerate, then N^\perp is identified with E/N , so $|N| \cdot |N^\perp| = |E|$.

Let (E, q) be a metric group. We say that a subgroup L of E is *isotropic* if $q(a) = 1$ for any $a \in L$. This implies that $b_q(L) \subset (E/L)^*$, which implies that $|L|^2 \leq |E|$. We say that an isotropic subgroup L of E is *Lagrangian* if $|L|^2 = |E|$.

2.5. Frobenius–Perron dimensions in module categories. Let \mathcal{C} be a fusion category and let \mathcal{M} be a \mathcal{C} -module category. Recall that for a pair of objects $M, N \in \mathcal{M}$ their *internal Hom* is the object of \mathcal{C} , denoted by $\underline{\text{Hom}}(M, N)$, determined by the natural isomorphism

$$\text{Hom}_{\mathcal{C}}(X, \underline{\text{Hom}}(M, N)) \cong \text{Hom}_{\mathcal{M}}(X \otimes M, N), \quad X \in \mathcal{C}.$$

We use this notion to define *canonical* Frobenius–Perron dimensions of objects of \mathcal{M} . Let $K_0(\mathcal{C}), K_0(\mathcal{M})$ be the Grothendieck ring of \mathcal{C} and the Grothendieck group of \mathcal{M} . It follows from [12] that there is a unique $K_0(\mathcal{C})$ -module map

$$\text{FPdim}: K_0(\mathcal{M}) \rightarrow \mathbb{R}$$

determined by

$$\text{FPdim}(\underline{\text{Hom}}(M, N)) = \text{FPdim}(M) \text{FPdim}(N) \tag{1}$$

for all objects $M, N \in \mathcal{M}$.

Let \mathcal{M} be an indecomposable left \mathcal{C} -module category. Let $\mathcal{O}(\mathcal{C})$ and $\mathcal{O}(\mathcal{M})$ denote the sets of isomorphism classes of simple objects in \mathcal{C} and \mathcal{M} .

Proposition 2.2. $\sum_{M \in \mathcal{O}(\mathcal{M})} \text{FPdim}(M)^2 = \text{FPdim}(\mathcal{C})$.

Proof. Let $R_{\mathcal{C}} := \sum_{X \in \mathcal{O}(\mathcal{C})} \text{FPdim}(X)X \in K_0(\mathcal{C})$ be the virtual regular object of \mathcal{C} . We choose a Frobenius–Perron dimension function $d: K_0(\mathcal{M}) \rightarrow \mathbb{R}$ as in [12], Proposition 8.7, normalized by

$$\sum_{M \in \mathcal{O}(\mathcal{M})} d(M)^2 = \text{FPdim}(\mathcal{C})$$

and let $R_{\mathcal{M}} := \sum_{M \in \mathcal{O}(\mathcal{M})} d(M)M$. We compute

$$\begin{aligned} \sum_{M \in \mathcal{O}(\mathcal{M})} \text{FPdim}(M)^2 &= \text{FPdim}(\bigoplus_{M \in \mathcal{O}(\mathcal{M})} \underline{\text{Hom}}(M, M)) \\ &= \sum_{M \in \mathcal{O}(\mathcal{M})} [R_{\mathcal{C}} \otimes M : M] \\ &= \sum_{M \in \mathcal{O}(\mathcal{M})} d(M)[R_{\mathcal{M}} : M] \\ &= \sum_{M \in \mathcal{O}(\mathcal{M})} d(M)^2 = \text{FPdim}(\mathcal{C}), \end{aligned}$$

as required. □

Remark 2.3. The Frobenius–Perron dimensions in \mathcal{M} defined in (1) are completely determined by the following properties:

- (i) $\text{FPdim}(M) > 0$ for all $M \in \mathcal{O}(\mathcal{M})$,
- (ii) $\text{FPdim}(X \otimes M) = \text{FPdim}(X) \text{FPdim}(M)$ for all $X \in \mathcal{C}, M \in \mathcal{M}$,
- (iii) $\sum_{M \in \mathcal{O}(\mathcal{M})} \text{FPdim}(M)^2 = \text{FPdim}(\mathcal{C})$.

2.6. The special orthogonal group. Let (M, q) be a metric group. If $L_1, L_2 \subset M$ are Lagrangian subgroups, define $d(L_1, L_2) \in \mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$ to be the image of the number $|L_1|/|L_1 \cap L_2| = |L_2|/|L_1 \cap L_2| = |M|^{1/2}/|L_1 \cap L_2| \in \mathbb{N}$. Clearly $d(L_2, L_1) = d(L_1, L_2) = d(L_1, L_2)^{-1}$ and $d(L, L) = 1$.

The following proposition and its proof were provided to us by V. Drinfeld.

Proposition 2.4. $d(L_1, L_2)d(L_2, L_3) = d(L_1, L_3)$ for any Lagrangian subgroups $L_1, L_2, L_3 \subset M$.

The proposition follows from Lemmas 2.5–2.6 below.

Lemma 2.5. $d(L_1, L_2)d(L_2, L_3)/d(L_1, L_3) \in \mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$ is the image of $|A/B| \in \mathbb{N}$, where $A := (L_1 + L_2) \cap L_3$, $B := (L_1 \cap L_3) + (L_2 \cap L_3)$.

Proof. By definition, $d(L_1, L_2)d(L_2, L_3)/d(L_1, L_3) \in \mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$ is the image of

$$|L_1| \cdot |L_2| \cdot |L_3| \cdot |L_1 \cap L_2|^{-1} \cdot |L_1 \cap L_3|^{-1} \cdot |L_2 \cap L_3|^{-1} \in \mathbb{N}.$$

On the other hand,

$$\begin{aligned} |B| &= |L_1 \cap L_3| \cdot |L_2 \cap L_3| / |L_1 \cap L_2 \cap L_3|, \\ |A| &= |L_1 + L_2| \cdot |L_3| / |L_1 + L_2 + L_3| \\ &= |L_1| \cdot |L_2| \cdot |L_3| \cdot |L_1 \cap L_2|^{-1} \cdot |L_1 + L_2 + L_3|^{-1}. \end{aligned}$$

Finally, $L_1 \cap L_2 \cap L_3 = (L_1 + L_2 + L_3)^\perp$, so $|L_1 \cap L_2 \cap L_3| \cdot |L_1 + L_2 + L_3| = |M| = |L_i|^2$ is a square. \square

By Lemma 2.5, proving Proposition 2.4 amounts to showing that $|A/B|$ is a square. To this end, it suffices to construct a non-degenerate skew-symmetric bicharacter $c: (A/B) \times (A/B) \rightarrow \mathbf{k}^\times$.

Here is the construction. Let $x, y \in A := (L_1 + L_2) \cap L_3$. Represent x and y as

$$x = x_1 + x_2, \quad y = y_1 + y_2, \quad x_i, y_i \in L_i,$$

and set $c(x, y) := b(x_1, y_2) = b(x, y_2) = b(x_1, y_2)$, where $b: M \times M \rightarrow \mathbf{k}^\times$ is the symmetric bicharacter associated to q . It is easy to see that $c: A \times A \rightarrow \mathbf{k}^\times$ is a well-defined bicharacter.

Lemma 2.6. (i) $c(x, x) = 1$.

(ii) The kernel of $c: A \times A \rightarrow \mathbf{k}^\times$ equals B .

Proof. (i) $c(x, x) = b(x_1, x_2) = q(x)q(x_1)^{-1}q(x_2)^{-1} = 1$ because $x \in L_3$, $x_1 \in L_1$, and $x_2 \in L_2$.

(ii) An element $x \in A$ belongs to the kernel of c if and only if $b(x, y) = 1$ for all $y \in L_2 \cap (L_1 + L_3)$. The orthogonal complement of $L_2 \cap (L_1 + L_3)$ with respect

to $b: M \times M \rightarrow \mathbf{k}^\times$ equals $L_2 + (L_1 \cap L_3)$, so $\text{Ker } c = A \cap (L_2 + (L_1 \cap L_3))$. Since $A \subset L_3$ we see that $\text{Ker } c \subset (L_2 \cap L_3) + (L_1 \cap L_3) = B$. On the other hand, $B \subset A$ and $B \subset L_2 + (L_1 \cap L_3)$, so $B \subset \text{Ker } c$. \square

Now for a metric group E and $g \in \text{O}(E, q)$ define $\det(g) \in \mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$ to be the image of $|(g - 1)E| \in \mathbb{N}$.

Proposition 2.7. *The map $\det: \text{O}(E, q) \rightarrow \mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$ is a homomorphism.*

Proof. Let $g, h \in \text{O}(E, q)$, and let $M = E \oplus E$ with quadratic form $Q(x, y) = q(y)/q(x)$, $x, y \in E$. Let $L_1, L_2, L_3 \subset M$ be the graphs of Id , g^{-1} , and h . They are Lagrangian, and $L_1 \cap L_2 = \text{Ker}(g - 1)$, so $d(L_1, L_2) = \det(g)$. Similarly, $d(L_1, L_3) = \det(h)$, and $d(L_2, L_3) = \det(gh)$. Thus, by Proposition 2.4, $\det(gh) = \det(g)\det(h)$. \square

Proposition 2.8. *If L is a Lagrangian subgroup of E then $\det(g) = d(L, g(L))$.*

Proof. First, note that by Proposition 2.4, $d(L, g(L))$ is independent on the choice of L . So let us call this function $\delta(g)$. Next, note that $\delta(g) = \delta(g, 1)$, where $(g, 1) \in \text{O}(E \oplus E, q^{-1} \oplus q)$. Finally, note that

$$\delta(g, 1) = d(E_{\text{diag}}, (g, 1)(E_{\text{diag}})) = \det(g),$$

where E_{diag} is the diagonal copy of E . \square

Definition 2.9. The kernel of the homomorphism

$$\det: \text{O}(E, q) \rightarrow \mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$$

is called the *special orthogonal group* and denoted by $\text{SO}(E, q)$.

Remark 2.10. If E is a vector space over \mathbb{F}_p with $p > 2$, then it is easy to see that \det is the usual determinant (so Definition 2.9 agrees with the familiar one from linear algebra). Indeed, in this case any orthogonal transformation is the composition of reflections, and it is clear that on reflections the two definitions of the determinant coincide. On the other hand, if E is a vector space over \mathbb{F}_2 and q takes values ± 1 (i.e., in \mathbb{F}_2), then $\det(g)$ coincides with the Dickson invariant of g [8] (which is also known as Dickson’s pseudodeterminant [20]), while the usual determinant is trivial.

2.7. Module categories over Vec_G . Let G be a finite group and let $\mathcal{C} := \text{Vec}_G$ be the fusion category of G -graded vector spaces. We will denote simple objects of Vec_G simply by $g \in G$.

Recall that equivalence classes of indecomposable left Vec_G -module categories correspond to pairs (H, ψ) where $H \subset G$ is a subgroup and $\psi \in Z^2(H, \mathbf{k}^\times)$ is a 2-cocycle (modulo cohomological equivalence). Namely, let \mathcal{M} be an indecomposable

left Vec_G -module category, and let Y be a simple object of \mathcal{M} . Set $H := \{x \in G \mid x \otimes Y \cong Y\}$ and choose an isomorphism $u_x: x \otimes Y \xrightarrow{\sim} Y$ for any $x \in H$. Let $\psi(x_1, x_2), x_1, x_2 \in H$, be the scalar such that the map

$$Y \xrightarrow{u_{x_1 x_2}^{-1}} (x_1 \otimes x_2) \otimes Y \xrightarrow{\sim} x_1 \otimes (x_2 \otimes Y) \xrightarrow{\text{id}_{x_1} \otimes u_{x_2}} x_1 \otimes Y \xrightarrow{u_{x_1}} Y$$

is given by $\psi(x_1, x_2)\text{id}_Y$. Then $\psi \in Z^2(H, \mathbf{k}^\times)$ and we constructed a pair (H, ψ) corresponding to \mathcal{M} (observe that a different choice of isomorphisms u_x would produce ψ' cohomologous to ψ). Note that the set of isomorphism classes of simple objects of $\mathcal{M} = \mathcal{M}(H, \psi)$ is in bijection with the set G/H of right cosets of H in G .

For any $x \in G$ set $H^x := xHx^{-1}$ and define $\psi^x \in Z^2(H^x, \mathbf{k}^\times)$ by

$$\psi^x(xy_1x^{-1}, xy_2x^{-1}) := \psi(y_1, y_2), \quad y_1, y_2 \in H.$$

Two Vec_G -module categories $\mathcal{M}(H, \psi)$ and $\mathcal{M}(H', \psi')$ are equivalent if and only if there is $x \in G$ such that $H' = xHx^{-1}$ and ψ' is cohomologous to ψ^x .

If H is abelian, then $H^2(H, \mathbf{k}^\times)$ is the group of skew-symmetric bicharacters of H . Thus, if A is a finite abelian group, then the indecomposable left module categories over Vec_A are $\mathcal{M}(H, \psi)$, where $H \subset A$ is a subgroup and ψ is a skew-symmetric bicharacter of H .

Let A, B be abelian groups, $\phi: B \rightarrow A$ be a group homomorphism (not necessarily injective), and ξ be a skew-symmetric bicharacter of B with coefficients in \mathbf{k}^\times . Let $K = \text{Ker } \phi$, K^\perp be the orthogonal complement of K in B under ξ , and $H = \phi(K^\perp)$. It is easy to show that ξ descends to a skew-symmetric bicharacter of H , which we will denote by ψ .

Proposition 2.11. *Let \mathcal{N} be the category of A -graded vector spaces which are right-equivariant under the action of B (via ϕ) with 2-cocycle ξ . Then, as a left Vec_A module, $\mathcal{N} \cong m \cdot \mathcal{M}(H, \psi)$, where*

$$m = \frac{|K| \cdot |K^\perp|}{|B|} = |K \cap \text{Rad}(\xi)|.$$

Proof. The simple objects of \mathcal{N} are obviously parameterized by pairs (z, ρ) , where $z \in A/\phi(B)$, and ρ is an irreducible projective representation of K with cohomology class $\xi|_K$, which implies that the number of simple objects of \mathcal{N} and $m \cdot \mathcal{M}(H, \psi)$ is the same.

Now consider the stabilizer S of a pair (z, ρ) in A . Obviously, S is contained in $\phi(B)$ since an element of S must preserve z . Further, if $g \in B$, the action of $\phi(g)$ on ρ is by tensoring with the character $\xi(g, \cdot)$. So the condition that $\phi(g)$ fixes ρ is that $\xi(g, k) = 1$ for any $k \in \text{Rad}(\xi|_K) = K^\perp \cap K$, i.e., $g \in K + K^\perp$ (indeed, we have the equality $(K \cap K^\perp)^\perp = K^{\perp\perp} + K^\perp = K + K^\perp$ since $K^{\perp\perp} = K + \text{Rad}(\xi)$ and K^\perp contains $\text{Rad}(\xi)$). Thus $S = H$. It is straightforward to check that the corresponding second cohomology class on S is exactly ψ . The proposition is proved. \square

We also have the following proposition, the proof of which is easy and omitted.

Proposition 2.12. *Let A be a finite abelian group, $H, B \subset A$ subgroups, and let $\psi \in H^2(H, \mathbf{k}^\times)$ be a skew-symmetric bicharacter. Then one has an equivalence of left Vec_B -module categories*

$$\mathcal{M}(H, \psi)|_{\text{Vec}_B} \cong m \cdot \mathcal{M}(H \cap B, \psi|_{H \cap B}),$$

where m is the index of $B + H$ in A .

2.8. The center of a bimodule category. Let \mathcal{C} be a fusion category with unit object $\mathbf{1}$ and associativity constraint $\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$, and let \mathcal{M} be a \mathcal{C} -bimodule category. The following definition was given in [18].

Definition 2.13. The *center* of \mathcal{M} is the category $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ of \mathcal{C} -bimodule functors from \mathcal{C} to \mathcal{M} .

Explicitly, the objects of $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ are pairs (M, γ) , where M is an object of \mathcal{M} and

$$\gamma = \{\gamma_X: X \otimes M \xrightarrow{\sim} M \otimes X\}_{X \in \mathcal{C}}$$

is a natural family of isomorphisms making the following diagram commutative:

$$\begin{array}{ccc} X \otimes (M \otimes Y) & \xrightarrow{\alpha_{X,M,Y}^{-1}} & (X \otimes M) \otimes Y \\ \text{id}_X \otimes \gamma_Y \uparrow & & \downarrow \gamma_X \otimes \text{id}_Y \\ X \otimes (Y \otimes M) & & (M \otimes X) \otimes Y \\ \alpha_{X,Y,M}^{-1} \downarrow & & \uparrow \alpha_{M,X,Y}^{-1} \\ (X \otimes Y) \otimes M & \xrightarrow{\gamma_{X \otimes Y}} & M \otimes (X \otimes Y), \end{array}$$

where the α 's denote the associativity constraints in \mathcal{M} .

Indeed, a \mathcal{C} -bimodule functor $F: \mathcal{C} \rightarrow \mathcal{M}$ is completely determined by the pair $(F(\mathbf{1}), \{\gamma_X\}_{X \in \mathcal{C}})$, where $\gamma = \{\gamma_X\}_{X \in \mathcal{C}}$ is the collection of isomorphisms

$$\gamma_X: X \otimes F(\mathbf{1}) \xrightarrow{\sim} F(X) \xrightarrow{\sim} F(\mathbf{1}) \otimes X$$

coming from the \mathcal{C} -bimodule structure on F .

Remark 2.14. $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ is a semisimple abelian category. It has a natural structure of a $\mathcal{Z}(\mathcal{C})$ -module category. Also it is clear that $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$.

2.9. The opposite module category. Let \mathcal{C} be a fusion category and \mathcal{M} a right \mathcal{C} -module category. Let \mathcal{M}^{op} be the category opposite to \mathcal{M} . Then \mathcal{M}^{op} is a left \mathcal{C} -module category with the \mathcal{C} -action $\odot_{\mathcal{M}^{\text{op}}}$ given by $X \odot_{\mathcal{M}^{\text{op}}} M := M \otimes^* X$ and the associativity constraint given by

$$(X \otimes Y) \odot_{\mathcal{M}^{\text{op}}} M = M \otimes^*(X \otimes Y) = (M \otimes^* Y) \otimes^* X = X \odot_{\mathcal{M}^{\text{op}}} (Y \odot_{\mathcal{M}^{\text{op}}} M).$$

Similarly, if \mathcal{N} is a left \mathcal{C} -module category, then \mathcal{N}^{op} is a right \mathcal{C} -module category, with the \mathcal{C} -action $\odot_{\mathcal{N}^{\text{op}}}$ given by $N \odot_{\mathcal{N}^{\text{op}}} X := X^* \otimes N$. Note that $(\mathcal{M}^{\text{op}})^{\text{op}}$ is canonically equivalent to \mathcal{M} as a \mathcal{C} -module category. Indeed, the identity functor $\mathcal{M} \rightarrow (\mathcal{M}^{\text{op}})^{\text{op}}$ has an obvious structure of module functor since

$$M \odot_{(\mathcal{M}^{\text{op}})^{\text{op}}} X = X^* \odot_{\mathcal{M}^{\text{op}}} M = M \otimes^*(X^*) = M \otimes X.$$

More generally, given a $(\mathcal{C}, \mathcal{D})$ -bimodule category \mathcal{M} , the above definitions make \mathcal{M}^{op} a $(\mathcal{D}, \mathcal{C})$ -bimodule category.

3. Tensor product of module categories

3.1. Definition of the tensor product of module categories over a fusion category.

Let \mathcal{C}, \mathcal{D} be fusion categories. By definition, a $(\mathcal{C}, \mathcal{D})$ -bimodule category is a module category over $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$, where \mathcal{D}^{rev} is the category \mathcal{D} with reversed tensor product.

Let $\mathcal{M} = (\mathcal{M}, m)$ be a right \mathcal{C} -module category and let $\mathcal{N} = (\mathcal{N}, n)$ be a left \mathcal{C} -module category. Here m and n are the associativity constraints:

$$\begin{aligned} m_{M,X,Y} &: M \otimes (X \otimes Y) \rightarrow (M \otimes X) \otimes Y, \\ n_{X,Y,N} &: (X \otimes Y) \otimes N \rightarrow X \otimes (Y \otimes N), \end{aligned}$$

where $X, Y \in \mathcal{C}, M \in \mathcal{M}, N \in \mathcal{N}$.

Let \mathcal{A} be a semisimple abelian category.

Definition 3.1. Let $F : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{A}$ be a bifunctor additive in every argument. We say that F is \mathcal{C} -balanced if there is a natural family of isomorphisms

$$b_{M,X,N} : F(M \otimes X, N) \cong F(M, X \otimes N)$$

satisfying the commutative diagram

$$\begin{array}{ccc} F(M \otimes (X \otimes Y), N) & \xrightarrow{m_{M,X,Y}} & F((M \otimes X) \otimes Y, N) \\ b_{M,X \otimes Y, N} \downarrow & & \downarrow b_{M \otimes X, Y, N} \\ F(M, (X \otimes Y) \otimes N) & & F(M \otimes X, Y \otimes N) \\ & \swarrow n_{X,Y,N}^{-1} & \swarrow b_{M,X,Y \otimes N} \\ & F(M, X \otimes (Y \otimes N)) & \end{array} \quad (2)$$

for all $M \in \mathcal{M}, N \in \mathcal{N}, X, Y \in \mathcal{C}$.

Remark 3.2. A bifunctor $\mathcal{M} \times \mathcal{N} \rightarrow \mathcal{A}$ as above canonically extends to a functor $\mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{A}$, where $\mathcal{M} \boxtimes \mathcal{N}$ is the Deligne product of abelian categories [6]. Clearly, one can formulate the balancing property in terms of functors $\mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{A}$.

We define tensor product of \mathcal{C} -module categories by “categorifying” the definition of a tensor product of modules over a ring. This extends the notion of Deligne’s tensor product of abelian categories (i.e., module categories over Vec) to the context of module categories over tensor categories. In the setting of additive \mathbf{k} -linear (not necessarily abelian) categories the notion of tensor product of module categories was given by D. Tambara in [36].

Definition 3.3. A *tensor product* of a right \mathcal{C} -module category \mathcal{M} and a left \mathcal{C} -module category \mathcal{N} is an abelian category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ together with a \mathcal{C} -balanced functor

$$B_{\mathcal{M}, \mathcal{N}}: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \tag{3}$$

inducing, for every abelian category \mathcal{A} , an equivalence between the category of \mathcal{C} -balanced functors from $\mathcal{M} \times \mathcal{N}$ to \mathcal{A} and the category of functors from $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ to \mathcal{A} :

$$\text{Fun}_{\text{bal}}(\mathcal{M} \times \mathcal{N}, \mathcal{A}) \cong \text{Fun}(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \mathcal{A}).$$

Remark 3.4. Equivalently, the bifunctor (3) is universal for all \mathcal{C} -balanced bifunctors from $\mathcal{M} \times \mathcal{N}$ to abelian categories. In other words, for any \mathcal{C} -balanced functor $F: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{A}$ there exists a unique additive functor $F': \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \rightarrow \mathcal{A}$ making the following diagram commutative

$$\begin{array}{ccc}
 \mathcal{M} \times \mathcal{N} & & \\
 \downarrow B_{\mathcal{M}, \mathcal{N}} & \searrow F & \\
 \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} & \xrightarrow{F'} & \mathcal{A}.
 \end{array} \tag{4}$$

If \mathcal{M} and \mathcal{N} are \mathcal{C} -bimodule categories then so is $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$.

3.2. Tensor product as a category of module functors. Let us show that the tensor product of bimodule categories introduced in Definition 3.3 does exist.

Let \mathcal{C} be a fusion category, let \mathcal{M} be a right \mathcal{C} -module category and \mathcal{N} be a left \mathcal{C} -module category. There is an obvious equivalence

$$\mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\sim} \text{Fun}(\mathcal{M}^{\text{op}}, \mathcal{N}): M \boxtimes N \mapsto \text{Hom}_{\mathcal{M}}(?, M) \otimes N. \tag{5}$$

Observe that the equivalence (5) sends $(M \otimes X) \boxtimes N$ and $M \boxtimes (X \otimes N)$ to $\text{Hom}_{\mathcal{M}}(?, M \otimes X) \otimes N = \text{Hom}_{\mathcal{M}}(X \circlearrowleft_{\mathcal{M}^{\text{op}}}, M) \otimes N$

and

$$\text{Hom}_{\mathcal{M}}(?, M) \otimes (X \otimes N) = X \otimes (\text{Hom}_{\mathcal{M}}(?, M) \otimes N),$$

respectively. Thus under the equivalence (5) \mathcal{C} -balanced functors $\mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{A}$ correspond to functors $F: \text{Fun}(\mathcal{M}^{\text{op}}, \mathcal{N}) \rightarrow \mathcal{A}$ endowed with a natural isomorphism

$$F(T(X \otimes ?)) \cong F(X \otimes T(?)), \quad \text{where } T: \mathcal{M}^{\text{op}} \rightarrow \mathcal{N}, X \in \mathcal{C}, \quad (6)$$

satisfying a coherence condition similar to diagram (2) (by abuse of notation we use here \otimes instead of $\odot_{\mathcal{M}^{\text{op}}}$).

Proposition 3.5. *There is an equivalence of abelian categories*

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \text{Fun}_{\mathcal{C}}(\mathcal{M}^{\text{op}}, \mathcal{N}). \quad (7)$$

Proof. Let $F: \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{A}$ be the extension of some \mathcal{C} -balanced bifunctor as in Remark 3.2 and let $G: \mathcal{A} \rightarrow \mathcal{M} \boxtimes \mathcal{N}$ be its right adjoint. Using the equivalence (5) and coherence (6) one can check that for every $A \in \mathcal{A}$ the functor $G(A)$ in $\text{Fun}(\mathcal{M}^{\text{op}}, \mathcal{N})$ has a canonical structure of a \mathcal{C} -module functor. Thus, G factors through the obvious forgetful functor $U: \text{Fun}_{\mathcal{C}}(\mathcal{M}^{\text{op}}, \mathcal{N}) \rightarrow \text{Fun}(\mathcal{M}^{\text{op}}, \mathcal{N})$:

$$\begin{array}{ccc} & \text{Fun}(\mathcal{M}^{\text{op}}, \mathcal{N}) & \\ & \uparrow U & \swarrow G \\ \text{Fun}_{\mathcal{C}}(\mathcal{M}^{\text{op}}, \mathcal{N}) & \leftarrow & \mathcal{A} \end{array}$$

Taking left adjoints we recover diagram (4). □

Remark 3.6. (i) It is easy to see that if \mathcal{M} is a $(\mathcal{D}, \mathcal{C})$ -bimodule category and \mathcal{N} is a $(\mathcal{C}, \mathcal{E})$ -bimodule category, then (7) is an equivalence of $(\mathcal{D}, \mathcal{E})$ -bimodule categories.

(ii) Let \mathcal{M} be a right \mathcal{C} -module category, \mathcal{N} a $(\mathcal{C}, \mathcal{D})$ -bimodule category, and \mathcal{K} a left \mathcal{D} -module category. Then there is a canonical equivalence $(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}) \boxtimes_{\mathcal{D}} \mathcal{K} \cong \mathcal{M} \boxtimes_{\mathcal{C}} (\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{K})$ of categories. Hence the notation $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{K}$ will yield no ambiguity.

We refer the reader to the work of J. Greenough [19] for an alternative proof of Proposition 3.5. It is shown in [19] that for any fusion category \mathcal{C} its bimodule categories equipped with the tensor product $\boxtimes_{\mathcal{C}}$ form a (non-semi-strict) monoidal 2-category in the sense of Kapranov and Voevodsky [23]. We denote this monoidal 2-category by **Bimodc**(\mathcal{C}).

More generally, one can define the tricategory **Bimodc** of bimodule categories over fusion categories, in which 1-morphisms from \mathcal{C} to \mathcal{D} are $(\mathcal{C}, \mathcal{D})$ -bimodule categories (with composition being the tensor product of bimodule categories as defined above), 2-morphisms are bimodule functors between such bimodule categories, and 3-morphisms are morphisms of such bimodule functors. Then **Bimodc**(\mathcal{C}) consists of 1-morphisms from \mathcal{C} to \mathcal{C} in **Bimodc**, and the corresponding 2-morphisms and 3-morphisms.

Remark 3.7. The tricategory **Bimodc** is a categorification of the 2-category **Bimod**, whose objects are rings, 1-morphisms are bimodules, and 2-morphisms are homomorphisms of bimodules.

3.3. Tensor product as the center of a bimodule category. Let \mathcal{C} be a fusion category. Below we describe the tensor product of \mathcal{C} -module categories in a way convenient for computations. Recall that the center of a \mathcal{C} -bimodule category was defined in Section 2.8.

As before, let \mathcal{M} be a right \mathcal{C} -module category and let \mathcal{N} be a left \mathcal{C} -module category. The category $\mathcal{M} \boxtimes \mathcal{N}$ has a natural structure of a \mathcal{C} -bimodule category. It turns out that its center $\mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N})$ can be identified with $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$.

Let $F: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{A}$ be a \mathcal{C} -balanced functor. Let $\bar{F}: \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{A}$ be the extension of F and let $G: \mathcal{A} \rightarrow \mathcal{M} \boxtimes \mathcal{N}$ be the functor right adjoint to \bar{F} . Let

$$i: \text{Hom}_{\mathcal{A}}(\bar{F}(V), W) \cong \text{Hom}_{\mathcal{M} \boxtimes \mathcal{N}}(V, G(W))$$

be the adjunction isomorphism. Let

$$c_{X,G(A)}: G(A) \otimes (X \boxtimes \mathbf{1}) \cong (\mathbf{1} \boxtimes X) \otimes G(A), \quad A \in \mathcal{A},$$

be the image under i of the isomorphism

$$b_{V,*X}: F(V \otimes (*X \boxtimes \mathbf{1})) \cong F((\mathbf{1} \boxtimes *X) \otimes V), \quad V \in \mathcal{M}.$$

Then $G(A)$ is an object of $\mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N})$ and the functor $G': \mathcal{A} \rightarrow \mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N})$: $A \mapsto G(A)$ satisfies $UG' = G$, where

$$U: \mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N}) \rightarrow \mathcal{M} \boxtimes \mathcal{N}$$

is the obvious forgetful functor. Let

$$I_{\mathcal{M},\mathcal{N}}: \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N})$$

be the right adjoint of U .

Proposition 3.8. *There is a canonical equivalence*

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N})$$

such that $I_{\mathcal{M},\mathcal{N}}: \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N})$ is identified with the extension of the universal bifunctor $\mathcal{B}_{\mathcal{M},\mathcal{N}}: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$.

Proof. From the above discussion we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M} \boxtimes \mathcal{N} & & \\ \uparrow U & \swarrow G & \\ \mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}) & \leftarrow & \mathcal{A} \end{array}$$

Taking the adjoint diagram gives the result. □

Remark 3.9. There is yet one more description of $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$. Namely, let $A \in \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ be the object representing the functor $\otimes: \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \rightarrow \mathcal{C}$. Then $A = \bigoplus_X X^* \boxtimes X$ (summation taken over simple objects of \mathcal{C}) is an algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$, and $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is equivalent to the category of left A -modules in $\mathcal{M} \boxtimes \mathcal{N}$. The canonical functor

$$\mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$$

is identified with the middle multiplication by A .

3.4. Tensor product of module categories over a braided category. Let \mathcal{B} be a braided fusion category. Since every left (or right) \mathcal{B} -module category is automatically a \mathcal{B} -bimodule category (using the braiding in \mathcal{B}), one can tensor any two such categories to get a third one. Let $\mathbf{Mod}(\mathcal{B})$ denote the monoidal 2-category of (left) \mathcal{B} -module categories. Clearly, $\mathbf{Mod}(\mathcal{B})$ is a full subcategory of the monoidal 2-category $\mathbf{Bimod}(\mathcal{B})$ of all \mathcal{B} -bimodule categories.

Remark 3.10. The monoidal 2-category $\mathbf{Mod}(\mathcal{B})$ is a categorification of the monoidal category $\mathbf{Mod}(A)$ of modules over a commutative ring A . Note that unlike $\mathbf{Mod}(A)$ the monoidal 2-category $\mathbf{Mod}(\mathcal{B})$ is, in general, not symmetric or braided; in fact, in this category, $X \otimes Y$ may be non-isomorphic to $Y \otimes X$.

Let \mathcal{C} be a fusion category. Recall [14] that there is a 2-equivalence

$$\mathcal{Z}_{\mathcal{C}}: \mathbf{Bimod}(\mathcal{C}) \xrightarrow{\sim} \mathbf{Mod}(\mathcal{Z}(\mathcal{C})): \mathcal{M} \mapsto \mathcal{Z}_{\mathcal{C}}(\mathcal{M}), \quad (8)$$

where the center $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ is defined in Section 2.8.

The next proposition is proved in [19]. We include its proof for the reader's convenience.

Proposition 3.11. *The 2-equivalence $\mathcal{Z}_{\mathcal{C}}$ is monoidal. That is, for any pair \mathcal{M}, \mathcal{N} of \mathcal{C} -bimodule categories we have a natural equivalence*

$$\mathcal{Z}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}) \cong \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \boxtimes_{\mathcal{Z}(\mathcal{C})} \mathcal{Z}_{\mathcal{C}}(\mathcal{N}) \quad (9)$$

which satisfies appropriate compatibility conditions.

Proof. By Proposition 3.8 the left hand side of (9) is identified as a $\mathcal{Z}(\mathcal{C})$ -module category with $\mathcal{Z}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}}(\mathcal{M} \boxtimes \mathcal{N})$ where the left and right actions of the object $X \boxtimes Y \in \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ on $M \boxtimes N \in \mathcal{M} \boxtimes \mathcal{N}$ are given by

$$\begin{aligned} (X \boxtimes Y) \otimes (M \boxtimes N) &:= (X \otimes M) \boxtimes (Y \otimes N), \\ (M \boxtimes N) \otimes (X \boxtimes Y) &:= (M \otimes Y) \boxtimes (N \otimes X). \end{aligned}$$

On the other hand, combining Proposition 3.5 and the 2-equivalence (8) we obtain a sequence of $\mathcal{Z}(\mathcal{C})$ -module category equivalences

$$\begin{aligned} \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \boxtimes_{\mathcal{Z}(\mathcal{C})} \mathcal{Z}_{\mathcal{C}}(\mathcal{N}) &\cong \text{Fun}_{\mathcal{Z}(\mathcal{C})}(\mathcal{Z}_{\mathcal{C}}(\mathcal{M})^{\text{op}}, \mathcal{Z}_{\mathcal{C}}(\mathcal{N})) \\ &\cong \text{Fun}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}}(\mathcal{M}^{\text{op}}, \mathcal{N}) \\ &\cong \mathcal{Z}_{\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}}(\mathcal{M} \boxtimes \mathcal{N}), \end{aligned}$$

where the bimodule action of $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ on $\mathcal{M} \boxtimes \mathcal{N}$ is the same as the one described above. \square

Remark 3.12. It is possible to show that (8) is an equivalence of monoidal 2-categories.

Corollary 3.13. *Any Morita equivalence between fusion categories \mathcal{C}_1 and \mathcal{C}_2 canonically gives rise to an equivalence of monoidal 2-categories $\mathbf{Bimodc}(\mathcal{C}_1)$ and $\mathbf{Bimodc}(\mathcal{C}_2)$.*

Proof. By [28], the Morita equivalence between \mathcal{C}_1 and \mathcal{C}_2 gives rise to an equivalence $\mathcal{Z}(\mathcal{C}_1) \cong \mathcal{Z}(\mathcal{C}_2)$ as braided fusion categories, and so the result follows from Proposition 3.11. \square

The next statement was formulated in [7], Section 4.3.

Corollary 3.14. *Let \mathcal{C}_1 and \mathcal{C}_2 be Morita equivalent fusion categories. Let \mathcal{K}_i , $i = 1, 2$, be the 2-category of fusion categories \mathcal{C} equipped with a tensor functor $\mathcal{C}_i \rightarrow \mathcal{C}$. Then \mathcal{K}_1 and \mathcal{K}_2 are 2-equivalent.*

Proof. This follows from the observation that \mathcal{K}_i can be interpreted as the 2-category of algebras in \mathcal{C}_i , $i = 1, 2$ (cf. [7], Remarks 4.38 (i)). \square

3.5. Tensor product of module categories over \mathbf{Vec}_A , where A is a finite abelian group. Let A be an abelian group and $\mathcal{C} = \mathbf{Vec}_A$. Since \mathcal{C} is a symmetric category, any left \mathcal{C} -module category can be viewed as a right \mathcal{C} -module category, and thus the dual \mathcal{M}^{op} of a left \mathcal{C} -module category \mathcal{M} is again a left \mathcal{C} -module category. Also, the tensor product over \mathcal{C} of two left \mathcal{C} -module categories is again a left \mathcal{C} -module category.

For any subgroup $H \subset A$ and a skew-symmetric bicharacter ψ on H let $\mathcal{M}(H, \psi)$ be the \mathcal{C} -module category constructed in Section 2.7. The following Lemma is easy, and its proof is omitted.

Lemma 3.15. *One has $\mathcal{M}(H, \psi)^{\text{op}} = \mathcal{M}(H, \psi^{-1})$.*

Now let us give an explicit description of the tensor product of \mathcal{C} -module categories.

We repeat the construction preceding Proposition 2.11. Let $H_1, H_2 \subset A$ be subgroups of a finite abelian group A and let ψ_1, ψ_2 be skew-symmetric bicharacters on them. Consider the group $H_1 \cap H_2$ embedded antidiagonally (i.e., by $h \mapsto (-h, h)$) into $H_1 \oplus H_2$. Let $(H_1 \cap H_2)^\perp$ be the orthogonal complement of this group under the bicharacter $\psi_1 \times \psi_2$ on $H_1 \oplus H_2$. Let H be the image of $(H_1 \cap H_2)^\perp$ in $H_1 + H_2 \subset A$, under the map $(h_1, h_2) \mapsto h_1 + h_2$. We have an exact sequence

$$0 \rightarrow \text{Rad}((\psi_1 \times \psi_2)|_{H_1 \cap H_2}) \rightarrow (H_1 \cap H_2)^\perp \rightarrow H \rightarrow 0.$$

Therefore, the bicharacter $(\psi_1 \times \psi_2)|_{(H_1 \cap H_2)^\perp}$ descends to a bicharacter on H , which we will denote by ψ .

Proposition 3.16. *One has*

$$\mathcal{M}(H_1, \psi_1) \boxtimes_{\mathcal{C}} \mathcal{M}(H_2, \psi_2) = m \cdot \mathcal{M}(H, \psi),$$

where

$$m = \frac{|(H_1 \cap H_2)^\perp| \cdot |H_1 \cap H_2|}{|H_1| \cdot |H_2|} = |H_1 \cap H_2 \cap \text{Rad}(\psi_1 \times \psi_2)|.$$

Proof. Using Lemma 3.15, we get

$$\mathcal{M}(H_1, \psi_1) \boxtimes_{\mathcal{C}} \mathcal{M}(H_2, \psi_2) = \text{Fun}_{\mathcal{C}}(\mathcal{M}(H_1, \psi_1^{-1}), \mathcal{M}(H_2, \psi_2)).$$

According to [31], this category can be described as the category of A -graded vector spaces which are left equivariant under the action of H_1 with 2-cocycle ψ_1 and right equivariant under the action of H_2 with 2-cocycle ψ_2 . Since A is abelian, this is the same as considering A -graded vector spaces which are right-equivariant under the action of $H_1 \oplus H_2$ with cocycle $\psi_1 \times \psi_2$. So the result follows immediately from Proposition 2.11. \square

Corollary 3.17. (i) *The \mathcal{C} -module category $\mathcal{M}(H, \psi)$ is invertible (in the sense of § 4.4 below) if and only if ψ is non-degenerate.*

(ii) *The group of equivalence classes of invertible \mathcal{C} -module categories is naturally isomorphic to the group $H^2(A^*, \mathbf{k}^\times)$ of skew-symmetric bicharacters of A^* via $\mathcal{M}(H, \psi) \mapsto \psi^\vee|_{A^*}$, where ψ^\vee is the bicharacter on H^* dual to ψ (i.e., $\widehat{\psi^\vee} = \widehat{\psi}^{-1}$).*

Proof. This follows from Proposition 3.16 via a direct calculation. \square

Example 3.18. Assume that $H_1 = H_2 = A$, and ψ_1, ψ_2 are such that $\psi_1\psi_2$ is a non-degenerate bicharacter. In this case, Proposition 3.16 implies that $H = A$ and

$$\widehat{\psi} = \widehat{\psi}_1 \circ (\widehat{\psi_1\psi_2})^{-1} \circ \widehat{\psi}_2 = \widehat{\psi}_2 \circ (\widehat{\psi_1\psi_2})^{-1} \circ \widehat{\psi}_1.$$

Note that if ψ_1, ψ_2 are themselves non-degenerate, this is a special case of Corollary 3.17.

3.6. Tensor product of bimodule categories. Let us now compute the tensor product of bimodule categories over the categories of vector spaces graded by finite abelian groups.

Let A_1, A_2, A_3 be finite abelian groups. Let $H \subset A_1 \oplus A_2, H' \subset A_2 \oplus A_3$ be subgroups, and let ψ, ψ' be skew-symmetric bicharacters of H, H' , respectively. Let us repeat, with some modifications, the construction preceding Proposition 2.11.

Namely, let $H \circ H'$ be the subgroup of elements $(a_1, -a_2, a_2, a_3)$ in $H \oplus H'$, and let $H \cap H' \subset A_2$ be the intersection of H and H' with A_2 . We regard $H \cap H'$ as a subgroup of $H \circ H'$ via the antidiagonal embedding $h \rightarrow (-h, h)$ and let $(H \cap H')^\perp$ denote the orthogonal complement of $H \cap H'$ in $H \circ H'$ with respect to the bicharacter $(\psi \times \psi')|_{H \circ H'}$. Finally, let H'' be the image of $(H \cap H')^\perp$ in $A_1 \oplus A_3$. Obviously, the bicharacter $(\psi \times \psi')|_{H \circ H'}$ descends to a skew-symmetric bicharacter of H'' , which we denote by ψ'' .

Let $\mathcal{M}(H, \psi)$ (respectively $\mathcal{M}(H', \psi')$ and $\mathcal{M}(H'', \psi'')$) be the module category over $\text{Vec}_{A_1 \oplus A_2} = \text{Vec}_{A_1} \boxtimes \text{Vec}_{A_2}$ (respectively over $\text{Vec}_{A_2 \oplus A_3}$ and $\text{Vec}_{A_1 \oplus A_3}$) as described in § 2.7. Since $\text{Vec}_{A_i}^{\text{rev}} = \text{Vec}_{A_i}$ for $i = 2, 3$, we can consider $\mathcal{M}(H, \psi)$ (respectively $\mathcal{M}(H', \psi')$ and $\mathcal{M}(H'', \psi'')$) as $(\text{Vec}_{A_1}, \text{Vec}_{A_2})$ -bimodule (respectively as $(\text{Vec}_{A_2}, \text{Vec}_{A_3})$ -bimodule and $(\text{Vec}_{A_1}, \text{Vec}_{A_3})$ -bimodule) category. Then we have the following proposition, whose proof is parallel to the proof of Proposition 3.16.

For a subgroup $B \subset A$ of a finite abelian group A write B_\perp for the annihilator of B in A^* (to avoid confusion with the orthogonal complement with respect to a bicharacter, we use a subscript rather than a superscript).

Proposition 3.19. $\mathcal{M}(H, \psi) \boxtimes_{\text{Vec}_{A_2}} \mathcal{M}(H', \psi') = m \cdot \mathcal{M}(H'', \psi'')$, where

$$m = \frac{|H \cap H'| \cdot |(H \cap H')^\perp| \cdot |A_2|}{|H| \cdot |H'|} = \frac{|H \cap H'| \cdot |(H \cap H')^\perp| \cdot |H_\perp \cap H'_\perp|}{|H \circ H'|}.$$

Proof. Let $B := H \oplus H' \oplus A_2$, and $\phi: B \rightarrow A_1 \oplus A_2 \oplus A_2 \oplus A_3$ be the homomorphism given by the formula $\phi(h, h', a) = (h, h') + (0, a, -a, 0)$. Let ξ denote the bicharacter $\psi \times \psi' \times 1$ of B .

Using Lemma 3.15, we get

$$\mathcal{N} := \mathcal{M}(H, \psi) \boxtimes_{\text{Vec}_{A_2}} \mathcal{M}(H', \psi') = \text{Fun}_{\text{Vec}_{A_2}}(\mathcal{M}(H, \psi^{-1}), \mathcal{M}(H', \psi')).$$

Thus, according to [31], \mathcal{N} can be described as the category of $A_1 \oplus A_2 \oplus A_2 \oplus A_3$ -graded vector spaces which are right equivariant under the action B with cocycle ξ . By Proposition 2.11, this means that as a left $\text{Vec}_{A_1 \oplus A_2 \oplus A_2 \oplus A_3}$ -module category, \mathcal{N} is equivalent to $r \cdot \mathcal{M}(E, \theta)$, with $E = \phi(K_B^\perp)$, $\theta = \xi|_E$ (the pushforward of ξ to E , which is obviously well defined), and

$$r = \frac{|K| \cdot |K_B^\perp|}{|H| \cdot |H'| \cdot |A_2|},$$

where $K = \text{Ker}(\phi) = H \cap H'$ embedded into $H \oplus H' \oplus A_2$ via $a \mapsto (a, -a, a)$. (Here K_B^\perp stands for the orthogonal complement of K in B .) Thus, by Proposition 2.12, as a left $\text{Vec}_{A_1 \oplus A_3}$ -module category, \mathcal{N} is indeed a multiple of $\mathcal{M}(H'', \psi'')$.

It remains to prove the formulas for the coefficient m . From the above we get

$$\begin{aligned} m &= \frac{|H \cap H'| \cdot |K_B^\perp|}{|H| \cdot |H'| \cdot |A_2|} |\text{Coker}(K_B^\perp \rightarrow A_2 \oplus A_2)| \\ &= \frac{|H \cap H'| \cdot |A_2|}{|H| \cdot |H'|} |\text{Ker}(K_B^\perp \rightarrow A_2 \oplus A_2)| \\ &= \frac{|H \cap H'| \cdot |A_2|}{|H| \cdot |H'|} |(H \cap H')^\perp|, \end{aligned}$$

which is the first formula for m . To get the second formula, note that we have an exact sequence

$$0 \rightarrow H \circ H' \rightarrow H \oplus H' \rightarrow A_2 \rightarrow (H_\perp \cap H'_\perp)^* \rightarrow 0,$$

so

$$\frac{|A_2|}{|H| \cdot |H'|} = \frac{|H_\perp \cap H'_\perp|}{|H \circ H'|}.$$

Substituting this into the first formula, we get

$$m = \frac{|H \cap H'| \cdot |H_\perp \cap H'_\perp|}{|H \circ H'|} |(H \cap H')^\perp|,$$

which is the second formula for m . □

4. Higher groupoids attached to fusion categories

4.1. Invertible bimodule categories and the Brauer–Picard 3-groupoid. Let \mathcal{C} , \mathcal{D} be fusion categories. Recall from Section 2.9 that given a $(\mathcal{C}, \mathcal{D})$ -bimodule category \mathcal{M} its opposite \mathcal{M}^{op} is a $(\mathcal{D}, \mathcal{C})$ -bimodule category.

Definition 4.1. We will say that a $(\mathcal{C}, \mathcal{D})$ -bimodule category \mathcal{M} is *invertible* if there exist bimodule equivalences

$$\mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \cong \mathcal{D} \quad \text{and} \quad \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{op}} \cong \mathcal{C}.$$

Let \mathcal{M} be a $(\mathcal{C}, \mathcal{D})$ -bimodule category. We will denote $\text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ (respectively $\text{Fun}(\mathcal{M}, \mathcal{M})_{\mathcal{D}}$) the category of left (respectively right) module endofunctors of \mathcal{M} . Note that these categories $\text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ and $\text{Fun}(\mathcal{M}, \mathcal{M})_{\mathcal{D}}$ are at the same time multifusion categories and bimodule categories (over \mathcal{D} and \mathcal{C} , respectively).

Note that for any object X in \mathcal{D} (respectively \mathcal{C}) the right (respectively left) multiplication by X gives rise to a left (respectively right) \mathcal{C} -module (respectively

\mathcal{D} -module) endofunctor of \mathcal{M} denoted $R(X)$ (respectively $L(X)$). Thus, we have tensor functors

$$R: X \mapsto R(X): \mathcal{D}^{\text{rev}} \rightarrow \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \tag{10}$$

and

$$L: X \mapsto L(X): \mathcal{C} \rightarrow \text{Fun}(\mathcal{M}, \mathcal{M})_{\mathcal{D}}. \tag{11}$$

Proposition 4.2. *Let \mathcal{M} be a $(\mathcal{C}, \mathcal{D})$ -bimodule category. The following conditions are equivalent:*

- (i) \mathcal{M} is invertible.
- (ii) There exists a \mathcal{D} -bimodule equivalence $\mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \cong \mathcal{D}$.
- (iii) There exists a \mathcal{C} -bimodule equivalence $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{op}} \cong \mathcal{C}$.
- (iv) The functor (10) is an equivalence.
- (v) The functor (11) is an equivalence.

Proof. By definition, (i) is equivalent to (ii) and (iii) combined.

Recall that, by Proposition 3.5, $\mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \cong \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ as \mathcal{D} -bimodule categories. We also have $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{op}} \cong \text{Fun}_{\mathcal{D}}(\mathcal{M}^{\text{op}}, \mathcal{M}^{\text{op}}) = \text{Fun}(\mathcal{M}, \mathcal{M})_{\mathcal{D}}$ as \mathcal{C} -bimodule categories. So (iv) implies (ii) and (v) implies (iii).

Next suppose that \mathcal{M} is invertible and let $\phi: \mathcal{D} \cong \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ be a \mathcal{D} -bimodule equivalence. Let $F = \phi(\mathbf{1})$. Then any functor in $\text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is isomorphic to $F \circ R(X)$ for some $X \in \mathcal{D}$. This is only possible when R is an equivalence. Thus, (ii) implies (iv). The proof that (iii) implies (v) is completely similar.

It remains to show that (iv) is equivalent to (v). If (10) is an equivalence then \mathcal{D}^{rev} is the dual $\mathcal{C}_{\mathcal{M}}^*$ of \mathcal{C} with respect to \mathcal{M} and the functor (11) identifies with the canonical tensor functor $\mathcal{C} \rightarrow (\mathcal{C}_{\mathcal{M}}^*)_{\mathcal{M}}^*$. By [14], Theorem 3.27, this functor is an equivalence. So (iv) implies (v). The opposite implication is completely similar. \square

Remark 4.3. In view of Proposition 4.2 invertible $(\mathcal{C}, \mathcal{D})$ -bimodule categories can be thought of as *Morita equivalences* $\mathcal{C} \rightarrow \mathcal{D}$.

Corollary 4.4. *An invertible $(\mathcal{C}, \mathcal{D})$ -bimodule category is indecomposable as both a left \mathcal{C} -module category and a right \mathcal{D} -module category.*

Definition 4.5. The *Brauer–Picard groupoid of fusion categories* $\underline{\text{BrPic}}$ is a 3-groupoid, whose objects are fusion categories, 1-morphisms from \mathcal{C} to \mathcal{D} are invertible $(\mathcal{C}, \mathcal{D})$ -bimodule categories, 2-morphisms are equivalences of such bimodule categories, and 3-morphisms are isomorphisms of such equivalences.

In other words, $\underline{\text{BrPic}}$ is the subcategory of **Bimodc** obtained by extracting the invertible morphisms at all levels.

The 3-groupoid $\underline{\text{BrPic}}$ can be truncated (by forgetting 3-morphisms and identifying isomorphic 2-morphisms) to a 2-groupoid BrPic , and further truncated (by

forgetting 2-morphisms and identifying isomorphic 1-morphisms) to a 1-groupoid (i.e., an ordinary groupoid) BrPic .

In particular, for every fusion category \mathcal{C} we have the following hierarchy of objects:

- the categorical 2-group $\underline{\text{BrPic}}(\mathcal{C})$ of automorphisms of \mathcal{C} in $\underline{\text{BrPic}}$ (which is obtained by extracting invertible objects and morphisms at all levels from $\text{Bimodc}(\mathcal{C})$);
- the categorical group $\text{BrPic}(\mathcal{C})$ of automorphisms of \mathcal{C} in BrPic ;
- the group $\text{BrPic}(\mathcal{C})$ of automorphisms of \mathcal{C} in BrPic , which we will call the *Brauer–Picard group* of \mathcal{C} .

Remark 4.6. For any pair of isomorphic objects in $\underline{\text{BrPic}}(\mathcal{C})$ the set of morphisms between them is a torsor over the group $\text{Inv}(\mathcal{Z}(\mathcal{C}))$ of invertible objects of $\mathcal{Z}(\mathcal{C})$.

Remark 4.7. The 3-groupoid $\underline{\text{BrPic}}$ is a categorification of the 2-groupoid $\underline{\text{Pic}}$, whose objects are rings, 1-morphisms from A to B are invertible (A, B) -bimodules, and 2-morphisms are isomorphisms of such bimodules. In particular, the Brauer–Picard group $\text{BrPic}(\mathcal{C})$ is a categorical analog of the classical Picard group $\text{Pic}(A)$ of a ring A .

Remark 4.8. The terminology “Brauer–Picard group” is justified by the following observation.

For a moment, let \mathbf{k} be any field (not necessarily algebraically closed).

Proposition 4.9. $\text{BrPic}(\text{Vec}_{\mathbf{k}})$ is isomorphic to the classical Brauer group $\text{Br}(\mathbf{k})$.

Proof. First of all, note that bimodule categories over $\mathcal{C} = \text{Vec}_{\mathbf{k}}$ is the same thing as module categories.

Let R be a finite dimensional simple algebra over \mathbf{k} . Then $\mathcal{M}(R) := R\text{-mod}$ is an indecomposable module category over \mathcal{C} . It is easy to see that $\text{Mat}_N(R)\text{-mod}$ is naturally equivalent to $R\text{-mod}$ as a module category, and that

$$\mathcal{M}(R) \boxtimes_{\mathcal{C}} \mathcal{M}(S) = \mathcal{M}(R \otimes_{\mathbf{k}} S).$$

Also, any indecomposable semisimple module category over \mathcal{C} is of the form $\mathcal{M}(R)$ for a finite dimensional simple \mathbf{k} -algebra R , determined uniquely up to a Morita equivalence. This implies that $\mathcal{M}(R)$ is invertible if and only if R is central simple, which implies the claim. \square

Proposition 4.10. Let $\mathcal{C}_1, \mathcal{C}_2$ be two fusion categories of relatively prime Frobenius–Perron dimensions. Then

$$\text{BrPic}(\mathcal{C}_1 \boxtimes \mathcal{C}_2) = \text{BrPic}(\mathcal{C}_1) \times \text{BrPic}(\mathcal{C}_2).$$

Proof. This follows from [12], Proposition 8.55. \square

4.2. Integral bimodule categories. Let \mathcal{C} be an integral fusion category, i.e., a category such that the Frobenius–Perron dimension of any simple object of \mathcal{C} is an integer. Recall that the Frobenius–Perron dimensions in module categories were defined in Section 2.5. It is clear that the Frobenius–Perron dimension of any object in a \mathcal{C} -module category is a square root of an integer.

Definition 4.11. We say that a \mathcal{C} -module category \mathcal{M} is *integral* if $\text{FPdim}(M) \in \mathbb{Z}$ for every object $M \in \mathcal{M}$.

Equivalently, \mathcal{M} is integral if for every simple object $M \in \mathcal{M}$ the number $\text{FPdim}(\underline{\text{Hom}}(M, M))$ is the square of an integer.

Now let \mathcal{M} be an invertible \mathcal{C} -bimodule category. To avoid possible confusion let us agree that we compute Frobenius–Perron dimensions in \mathcal{M} by regarding it as a one-sided (left or right) \mathcal{C} -module category. In particular,

$$\sum_{M \in \mathcal{O}(\mathcal{M})} \text{FPdim}(M)^2 = \text{FPdim}(\mathcal{C}).$$

Definition 4.12. We will say that an invertible \mathcal{C} -bimodule category \mathcal{M} is integral if it is integral as one-sided (left or right) module category.

It is clear that here the choice of left or right \mathcal{C} -module structure is not important since the Frobenius–Perron dimensions in \mathcal{M} defined using these structures coincide.

Proposition 4.13. *Let \mathcal{C} be an integral fusion category. If \mathcal{M}, \mathcal{N} are invertible integral \mathcal{C} -bimodule categories then $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is integral.*

Proof. It is easy to see that according to our conventions the canonical \mathcal{C} -bimodule functor $F: \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ satisfies

$$\text{FPdim}(F(M \boxtimes N)) = \text{FPdim}(M) \text{FPdim}(N) \quad \text{for all } M, N \in \mathcal{M}.$$

Since this functor F is surjective, we conclude that $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is integral. Indeed, no integer can be equal to a sum of non-integer square roots. □

It is easy to show that equivalence classes of invertible integral \mathcal{C} -bimodule categories form a normal subgroup of $\text{BrPic}(\mathcal{C})$, denoted by $\text{BrPic}_+(\mathcal{C})$, such that $\text{BrPic}(\mathcal{C})/\text{BrPic}_+(\mathcal{C})$ is an elementary abelian 2-group. It gives rise to a full categorical 2-subgroup $\underline{\text{BrPic}}_+(\mathcal{C})$ of the categorical 2-group $\underline{\text{BrPic}}(\mathcal{C})$.

4.3. The categorical 2-group of outer auto-equivalences a fusion category. Let \mathcal{C} be a fusion category. Let us say that an invertible \mathcal{C} -bimodule category \mathcal{M} is *quasi-trivial* if it is equivalent to \mathcal{C} as a left module category. It is easy to see that if \mathcal{M} is quasi-trivial, then there exists a tensor auto-equivalence $\phi: \mathcal{C} \rightarrow \mathcal{C}$ such that $\mathcal{M} = \mathcal{C}$ with the left action of \mathcal{C} by left multiplication, and the right action

of \mathcal{C} by right multiplication twisted by ϕ . Moreover, ϕ is uniquely determined up to composing with conjugation by an invertible object of \mathcal{C} . In other words, it is uniquely determined as an *outer auto-equivalence*.

Now define the categorical 2-group $\underline{\text{Out}}(\mathcal{C})$ to be the 2-subgroup of $\underline{\text{BrPic}}(\mathcal{C})$ which includes only the quasi-trivial invertible bimodule categories (and all the corresponding equivalences and isomorphisms). This 2-group can be truncated to a 1-group $\underline{\text{Out}}(\mathcal{C})$ and further to the usual group $\text{Out}(\mathcal{C})$ of isomorphism classes of outer tensor auto-equivalences of \mathcal{C} (i.e., auto-equivalences modulo conjugations by invertible objects).

4.4. The Picard 2-groupoid of a braided fusion category. Let \mathcal{B} be a braided fusion category. The monoidal 2-category $\mathbf{Modc}(\mathcal{B})$ contains a categorical 2-group $\underline{\text{Pic}}(\mathcal{B})$, obtained by extracting invertible objects and morphisms at all levels, which we will call *the Picard 2-group* of \mathcal{B} . This categorical 2-group is a categorical analog of the categorical 1-group of invertible modules over a commutative ring A (or, more generally, of the Picard 1-group, or groupoid, of a scheme). By truncating it one obtains a categorical 1-group $\underline{\text{Pic}}(\mathcal{B})$ and an ordinary group $\text{Pic}(\mathcal{B})$, called *the Picard group* of the braided category \mathcal{B} .

Remark 4.14. If \mathcal{B} is a braided fusion category then $\underline{\text{BrPic}}(\mathcal{B})$ contains $\underline{\text{Pic}}(\mathcal{B})$ as a full categorical 2-subgroup (of bimodule categories in which the left and right action are related via the braiding).

4.5. The 2-groupoid of equivalences. Following [16], we define the 2-groupoid $\underline{\text{Eq}}$, whose objects are fusion categories, 1-morphisms are tensor equivalences, and 2-morphisms are isomorphisms of such equivalences. It can be truncated to an ordinary groupoid Eq . So for every fusion category \mathcal{C} , we obtain the groupoid $\underline{\text{Eq}}(\mathcal{C})$ of tensor auto-equivalences of \mathcal{C} and the corresponding group $\text{Eq}(\mathcal{C})$ of isomorphism classes of tensor auto-equivalences of \mathcal{C} .

4.6. The 2-groupoid of braided equivalences. Here is the braided version of the construction of the previous subsection. We define the 2-groupoid $\underline{\text{EqBr}}$, whose objects are braided fusion categories, 1-morphisms are braided equivalences, and 2-morphisms are isomorphisms of such equivalences. It can be truncated to an ordinary groupoid EqBr . So for every braided fusion category \mathcal{B} we obtain the groupoid $\underline{\text{EqBr}}(\mathcal{B})$ of braided auto-equivalences of \mathcal{B} and the corresponding group $\text{EqBr}(\mathcal{B})$ of isomorphism classes of braided auto-equivalences of \mathcal{B} .

4.7. The finiteness theorem

Theorem 4.15. *The groups $\text{BrPic}(\mathcal{C})$, $\text{Out}(\mathcal{C})$, $\text{Eq}(\mathcal{C})$, $\text{EqBr}(\mathcal{B})$, $\text{Pic}(\mathcal{B})$ are finite.*

Proof. This follows from the finiteness results from [12] (Theorem 2.31, Corollary 2.35). \square

5. Proof of Theorem 1.1

It is sufficient to prove for every fusion category \mathcal{C} the functor $\mathcal{Z}: \underline{\mathbf{BrPic}}(\mathcal{C}) \rightarrow \underline{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$ is an equivalence.

5.1. A monoidal functor $\Phi: \underline{\mathbf{BrPic}}(\mathcal{C}) \rightarrow \underline{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$. Let \mathcal{M} be an indecomposable right \mathcal{C} -module category. Let $\mathcal{C}_{\mathcal{M}}^*$ denote the dual of \mathcal{C} with respect to \mathcal{M} , i.e., the category of right \mathcal{C} -module endofunctors of \mathcal{M} . By [31] $\mathcal{C}_{\mathcal{M}}^*$ is a fusion category. We can regard \mathcal{M} as a $\mathcal{C}_{\mathcal{M}}^* \boxtimes \mathcal{C}^{\text{rev}}$ -module category. Its $\mathcal{C}_{\mathcal{M}}^* \boxtimes \mathcal{C}^{\text{rev}}$ -module endofunctors can be identified, on the one hand, with functors of left multiplication by objects of $\mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*)$ and, on the other hand, with functors of right multiplication by objects of $\mathcal{Z}(\mathcal{C})$. Combined, these identifications yield a canonical equivalence of braided categories

$$\mathcal{Z}(\mathcal{C}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*). \tag{12}$$

This result is due to Schauenburg; see [35].

Now suppose that \mathcal{M} is an invertible \mathcal{C} -bimodule category. Let us view it as a right \mathcal{C} -module category. By Proposition 4.2 and Remark 4.3 we have an equivalence of tensor categories

$$\mathcal{C}_{\mathcal{M}}^* \cong \mathcal{C} \tag{13}$$

obtained by identifying right \mathcal{C} -module endofunctors of \mathcal{M} with the functors of left multiplication by objects of \mathcal{C} .

Thus, we have a braided tensor equivalence

$$\Phi(\mathcal{M}): \mathcal{Z}(\mathcal{C}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}_{\mathcal{M}}^*) \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}), \tag{14}$$

where the first equivalence is (12) and the second one is induced from (13).

Clearly, a \mathcal{C} -bimodule equivalence between $\mathcal{M}, \mathcal{N} \in \underline{\mathbf{BrPic}}(\mathcal{C})$ gives rise to an isomorphism of tensor functors $\Phi(\mathcal{M})$ and $\Phi(\mathcal{N})$.

To see that the functor (14) is monoidal, observe that the \mathcal{C} -bimodule functor of right multiplication by an object $Z \in \mathcal{Z}(\mathcal{C})$ on $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is isomorphic to the well-defined functor of “middle” multiplication by $(\Phi(\mathcal{N}))(Z)$, which, in turn, is isomorphic to the functor of left multiplication by $(\Phi(\mathcal{M}) \circ \Phi(\mathcal{N}))(Z)$. This gives a natural isomorphism of tensor functors $\Phi(\mathcal{M}) \circ \Phi(\mathcal{N}) \cong \Phi(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N})$, i.e., a monoidal structure on Φ .

5.2. A functor $\Psi: \underline{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \rightarrow \underline{\mathbf{BrPic}}(\mathcal{C})$. Let α be a braided tensor auto-equivalence of $\mathcal{Z}(\mathcal{C})$. Below we recall a construction of an invertible \mathcal{C} -bimodule category from α given in [13].

Let $F: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ and $I: \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ denote the forgetful functor and its adjoint. Given an algebra A in \mathcal{C} let $A\text{-mod}_{\mathcal{C}}$ and $A\text{-bimod}_{\mathcal{C}}$ denote the categories of left A -modules and A -bimodules in \mathcal{C} , respectively.

The object $I(\mathbf{1})$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$ and so is

$$L := \alpha^{-1}(I(\mathbf{1})). \tag{15}$$

Furthermore, there is a tensor equivalence

$$\mathcal{C} \xrightarrow{\sim} L\text{-mod}_{\mathcal{Z}(\mathcal{C})}: X \mapsto I(X). \tag{16}$$

Note that L is indecomposable in $\mathcal{Z}(\mathcal{C})$ but might be decomposable as an algebra in \mathcal{C} , i.e.,

$$L = \bigoplus_{i \in J} L_i,$$

where $L_i, i \in J$, are indecomposable algebras in \mathcal{C} such that the multiplication of L is zero on $L_i \otimes L_j, i \neq j$. Here and below we abuse notation and write L for an object of $\mathcal{Z}(\mathcal{C})$ and its forgetful image in \mathcal{C} .

For any $i \in J$ let

$$\Psi_i(\alpha) := L_i\text{-mod}_{\mathcal{C}}. \tag{17}$$

Clearly, it is a right \mathcal{C} -module category. We would like to show that $\Psi_i(\alpha)$ is, in fact, an invertible \mathcal{C} -bimodule category.

Consider the following commutative diagram of tensor functors:

$$\begin{array}{ccc} \mathcal{Z}(\mathcal{C}) & \xrightarrow{Z \mapsto L_i \otimes Z} & \mathcal{Z}(L_i\text{-bimod}_{\mathcal{C}}) \\ \downarrow Z \mapsto L \otimes Z & & \downarrow F_i \\ L\text{-mod}_{\mathcal{Z}(\mathcal{C})} & \xrightarrow{F} \bigoplus L_i\text{-bimod}_{\mathcal{C}} \subset L\text{-bimod}_{\mathcal{C}} \xrightarrow{\pi_i} & L_i\text{-bimod}_{\mathcal{C}} \end{array} \tag{18}$$

Here $F_i: \mathcal{Z}(L_i\text{-bimod}_{\mathcal{C}}) \rightarrow L_i\text{-bimod}_{\mathcal{C}}$ is the forgetful functor and π_i is a projection from $L\text{-bimod}_{\mathcal{C}} = \bigoplus_{i,j} (L_i - L_j)\text{-bimod}_{\mathcal{C}}$ to its (i, i) component. We have $\pi_i(L \otimes X) = L_i \otimes X$ for all $X \in \mathcal{C}$. The top arrow is an equivalence and the forgetful functor $\mathcal{Z}(L_i\text{-bimod}_{\mathcal{C}}) \rightarrow L_i\text{-bimod}_{\mathcal{C}}$ (the right down arrow) is surjective. Hence, the composition $G_i := \pi_i F$ of the functors in the bottom row is surjective. But G_i is a tensor functor between fusion categories of equal Frobenius–Perron dimension and hence it is an equivalence by [14], Proposition 2.20.

In view of (16) this gives a tensor equivalence between \mathcal{C} and $\mathcal{C}_{\Psi_i(\alpha)}^*$. Hence, $\Psi_i(\alpha)$ is a \mathcal{C} -bimodule category. It is easy to see that the above functor G_i identifies with (11) when $\mathcal{M} = \Psi_i(\alpha)$, therefore $\Psi_i(\alpha)$ is invertible by Proposition 4.2.

We claim that definition (17) does not depend on a choice of $i \in J$.

Lemma 5.1. *For all $i, j \in J$ there is an equivalence of \mathcal{C} -bimodule categories $\Psi_i(\alpha)$ and $\Psi_j(\alpha)$.*

Proof. Let us consider the category $\mathcal{D} := L\text{-mod}_{\mathcal{C}}$. It is a *multifusion* category in the sense of [12], Section 2.4, i.e., it has a decomposition

$$\mathcal{D} = \bigoplus_{ij \in J} \mathcal{D}_{ij}$$

such that \mathcal{D}_{ii} is a fusion category and \mathcal{D}_{ij} is a $(\mathcal{D}_{ii}, \mathcal{D}_{jj})$ -bimodule category for all $i, j \in J$. Furthermore for $X \in \mathcal{D}_{ij}$ and $Y \in \mathcal{D}_{kl}$ we have $X \otimes Y \in \mathcal{D}_{il}$ if $j = k$ and $X \otimes Y = 0$ if $j \neq k$.

It follows from the result of Schauenburg [35], Corollary 4.5, that $\mathcal{Z}(\mathcal{D}) \cong \text{Vec}$ as a tensor category. Therefore, $\mathcal{D}_{ij} \cong \text{Vec}$ for all $i, j \in J$, i.e., simple objects of \mathcal{D} can be labeled E_{ij} in such a way that the tensor product \otimes_L satisfies the usual matrix multiplication rules:

$$E_{ij} \otimes_L E_{kl} = \delta_{jk} E_{il}, \quad i, j, k, l \in J.$$

It follows that $L_i = E_{ii}$ and $L_i\text{-mod}_{\mathcal{C}}$ is spanned by $E_{ik}, k \in J$. Thus, the functor

$$X \mapsto E_{ji} \otimes_L X : L_i\text{-mod}_{\mathcal{C}} \rightarrow L_j\text{-mod}_{\mathcal{C}}, \quad i, j \in J,$$

is an equivalence of \mathcal{C} -bimodule categories. □

Let us choose a \mathcal{C} -bimodule category $\Psi(\alpha) \in \text{BrPic}(\mathcal{C})$ in the equivalence class of \mathcal{C} -bimodule categories $\Psi_i(\alpha), i \in J$.

Let $f : \alpha \xrightarrow{\sim} \alpha'$ be an isomorphism in $\text{EqBr}(\mathcal{Z}(\mathcal{C}))$. It gives rise to an equivalence of the corresponding algebras L, L' in $\overline{\mathcal{Z}(\mathcal{C})}$ and, consequently, to a \mathcal{C} -bimodule equivalence $\Psi_i(f) : \Psi_i(\alpha) \xrightarrow{\sim} \Psi_i(\alpha')$. By Lemma 5.1 we obtain a \mathcal{C} -bimodule equivalence $\Psi(f) : \Psi(\alpha) \xrightarrow{\sim} \Psi(\alpha')$.

Thus, we have a functor

$$\Psi : \text{EqBr}(\mathcal{Z}(\mathcal{C})) \rightarrow \text{BrPic}(\mathcal{C}).$$

It remains to check that Ψ is an inverse of the monoidal functor Φ introduced in Section 5.1.

5.3. Equivalences $\Phi \circ \Psi \cong \text{Id}_{\text{EqBr}(\mathcal{Z}(\mathcal{C}))}$ and $\Psi \circ \Phi \cong \text{Id}_{\text{BrPic}(\mathcal{C})}$. First we prove an equivalence $\Phi \circ \Psi \cong \text{Id}_{\text{EqBr}(\mathcal{Z}(\mathcal{C}))}$. Given $\alpha \in \text{EqBr}(\mathcal{Z}(\mathcal{C}))$ let $\mathcal{M} = \Psi(\alpha) \cong L_i\text{-mod}_{\mathcal{C}}$, where the algebra L_i is defined as in Section 5.2. From (14) we see that $\Phi(\mathcal{M})$ is defined by

$$\Phi(\mathcal{M}) : \mathcal{Z}(\mathcal{C}) \xrightarrow{Z \mapsto L_i \otimes Z} \mathcal{Z}(L_i\text{-bimod}_{\mathcal{C}}) \xrightarrow{\iota} \mathcal{Z}(\mathcal{C}),$$

where the second equivalence ι is induced from the inverse of the equivalence in the bottom row of (18). Since $\mathcal{C} \cong I(\mathbf{1})\text{-mod}_{\mathcal{Z}(\mathcal{C})}$ we have

$$\iota^{-1}(Z) = \pi_i F \alpha^{-1} I(Z) = \pi_i F \alpha^{-1} (I(\mathbf{1}) \otimes Z) = L_i \otimes \alpha^{-1}(Z)$$

for all $Z \in \mathcal{Z}(\mathcal{C})$. Therefore, $\Phi \circ \Psi(\alpha) \cong \alpha$.

Next, we prove that $\Psi \circ \Phi \cong \text{Id}_{\text{BrPic}(\mathcal{C})}$. Take $\mathcal{M} \in \text{BrPic}(\mathcal{C})$. Let $A \in \mathcal{C}$ be an algebra such that $\mathcal{M} \cong A\text{-mod}_{\mathcal{C}}$ as a right \mathcal{C} -module category. Since \mathcal{M} is invertible, we have an equivalence $\mathcal{C} \cong A\text{-bimod}_{\mathcal{C}}$ by Proposition 4.2.

Construct a braided auto-equivalence $\alpha := \Phi(\mathcal{M}) \in \underline{\text{EqBr}}(\mathcal{Z}(\mathcal{C}))$ as in (14). Upon the identification $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(A\text{-bimod}_{\mathcal{C}})$ we have

$$\alpha(Z) = A \otimes Z, \quad Z \in \mathcal{Z}(\mathcal{C}),$$

where $A \otimes Z$ has an obvious structure of a central object in the category of A -bimodules. So the algebra L in $\mathcal{Z}(\mathcal{C})$ defined by (15) is identified with the algebra $A \otimes I(\mathbf{1})$ in $\mathcal{Z}(A\text{-bimod}_{\mathcal{C}})$. Hence, the category of $L\text{-mod}_{\mathcal{C}}$ is identified with the category $\tilde{\mathcal{M}}$ of $A - I(\mathbf{1})$ -bimodules in \mathcal{C} (recall that $I(\mathbf{1})$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$). Indecomposable components of $\tilde{\mathcal{M}}$ are equivalent to $A\text{-mod}_{\mathcal{C}}$ and so they are identified with \mathcal{M} as \mathcal{C} -bimodule categories, i.e., $\Psi \circ \Phi(\mathcal{M}) \cong \mathcal{M}$, as required.

It is easy to check that Φ and Ψ are bijective on morphisms (cf. Remark 4.6).

This completes the proof of Theorem 1.1.

5.4. Generalization. Let \mathcal{B} be a non-degenerate braided fusion category (see [7], Definition 2.28). By [7], Proposition 3.7, this means that the braiding on \mathcal{B} induces an equivalence $\mathcal{B} \boxtimes \mathcal{B}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{B})$. Now let \mathcal{M} be an invertible module category over \mathcal{B} (see Section 4.4) and let $\mathcal{B}_{\mathcal{M}}^* = \text{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{M})$. Combining the equivalence above with (12) we get an equivalence $\mathcal{B} \boxtimes \mathcal{B}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{B}_{\mathcal{M}}^*)$. The compositions

$$\alpha_+ : \mathcal{B} = \mathcal{B} \boxtimes \mathbf{1} \subset \mathcal{B} \boxtimes \mathcal{B}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{B}_{\mathcal{M}}^*) \rightarrow \mathcal{B}_{\mathcal{M}}^*,$$

and

$$\alpha_- : \mathcal{B} = \mathbf{1} \boxtimes \mathcal{B}^{\text{rev}} \subset \mathcal{B} \boxtimes \mathcal{B}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{B}_{\mathcal{M}}^*) \rightarrow \mathcal{B}_{\mathcal{M}}^*$$

are called *alpha-induction functors*; see e.g. [31]. Proposition 4.2 says that invertibility of \mathcal{M} is equivalent to α_+ and α_- being tensor equivalences. Thus

$$\alpha_+ = \alpha_- \circ \theta_{\mathcal{M}}$$

where $\theta_{\mathcal{M}} : \mathcal{B} \rightarrow \mathcal{B}$ is an auto-equivalence. One verifies directly that $\theta_{\mathcal{M}}$ is actually a *braided* auto-equivalence of \mathcal{B} . Furthermore, the same argument as the one in the end of Section 5.1 shows that $\theta_{\mathcal{M}}$ naturally extends to a functor $\underline{\text{Pic}}(\mathcal{B}) \rightarrow \underline{\text{EqBr}}(\mathcal{B})$.

Conversely, let $\gamma \in \underline{\text{EqBr}}(\mathcal{B})$. Then $\text{id} \boxtimes \gamma \in \underline{\text{EqBr}}(\mathcal{B} \boxtimes \mathcal{B}^{\text{rev}}) = \underline{\text{EqBr}}(\mathcal{Z}(\mathcal{B}))$. Thus Theorem 1.1 assigns to γ an invertible \mathcal{B} -bimodule category \mathcal{M}_{γ} . It follows immediately from definitions that right and left actions of \mathcal{B} on \mathcal{M}_{γ} are related by the braiding, so \mathcal{M}_{γ} is an invertible module category over \mathcal{B} . It is clear that this assignment $\gamma \mapsto \mathcal{M}_{\gamma}$ extends naturally to a functor $\underline{\text{EqBr}}(\mathcal{B}) \rightarrow \underline{\text{Pic}}(\mathcal{B})$. A careful examination of the constructions involved shows the following result:

Theorem 5.2. *For a non-degenerate braided fusion category \mathcal{B} the functors above are mutually inverse equivalences of $\underline{\text{Pic}}(\mathcal{B})$ and $\underline{\text{EqBr}}(\mathcal{B})$.*

Details of the proof of Theorem 5.2 will be given in a subsequent article.

Remark 5.3. (i) We notice that the construction of $\theta_{\mathcal{M}}$ above makes sense for arbitrary braided fusion category \mathcal{B} ; see [31]. Thus, we have a monoidal functor

$$\Theta: \underline{\text{Pic}}(\mathcal{B}) \rightarrow \underline{\text{EqBr}}(\mathcal{B}): \mathcal{M} \mapsto \theta_{\mathcal{M}}. \tag{19}$$

However it is clear that (19) does not produce an equivalence as in Theorem 5.2. For example it is clear that for a symmetric braided fusion category $\alpha_+ = \alpha_-$ for any \mathcal{M} , so $\theta_{\mathcal{M}} = \text{id}_{\mathcal{B}}$ for any \mathcal{M} in this case.

(ii) For a fusion category \mathcal{C} the braided category $\mathcal{Z}(\mathcal{C})$ is non-degenerate; see [7], Corollary 3.9. Thus combining Theorem 1.1 and Proposition 3.11 we get an equivalence $\underline{\text{Pic}}(\mathcal{Z}(\mathcal{C})) \simeq \underline{\text{EqBr}}(\mathcal{Z}(\mathcal{C}))$ in this case. One verifies that this equivalence and equivalence from Theorem 5.2 are canonically identified.

Remark 5.4. Given a braided category \mathcal{B} we have a monoidal functor $\Theta: \underline{\text{Pic}}(\mathcal{B}) \rightarrow \underline{\text{EqBr}}(\mathcal{B})$ given by (19). Recall that in Section 5.1 we constructed a monoidal equivalence $\Phi: \underline{\text{BrPic}}(\mathcal{C}) \rightarrow \underline{\text{EqBr}}(\mathcal{Z}(\mathcal{C}))$ for any fusion category \mathcal{C} . The following conceptual explanation of these functors were suggested to us by V. Drinfeld.

Namely, let \mathcal{A} be a monoidal 2-category (see [23]). Then the monoidal category $\underline{\text{End}}(\mathbf{1}_{\mathcal{A}})$ of endofunctors of the unit object of \mathcal{A} has a canonical structure of a braided category (this is a higher categorical version of the well-known fact that endomorphisms of the unit object in a monoidal category form a commutative monoid). The categorical group \mathcal{A}^\times of invertible objects of \mathcal{A} acts on $\underline{\text{End}}(\mathbf{1}_{\mathcal{A}})$ by tensor conjugation. Hence, we have a monoidal functor

$$\mathcal{A}^\times \rightarrow \underline{\text{EqBr}}(\underline{\text{End}}(\mathbf{1}_{\mathcal{A}})). \tag{20}$$

For $\mathcal{A} = \mathbf{Bimodc}(\mathcal{C})$, the monoidal 2-category of \mathcal{C} -bimodule categories over a fusion category \mathcal{C} , one has $\underline{\text{End}}(\mathbf{1}_{\mathcal{A}}) = \mathcal{Z}(\mathcal{C})$ and the above functor (20) is precisely the functor $\Phi: \underline{\text{BrPic}}(\mathcal{C}) \rightarrow \underline{\text{EqBr}}(\mathcal{Z}(\mathcal{C}))$ from Section 5.1. For $\mathcal{A} = \mathbf{Modc}(\mathcal{B})$, the monoidal 2-category of module categories over a braided fusion category \mathcal{B} , it gives the functor (19).

5.5. The truncation of the categorical 2-group of outer auto-equivalences of a fusion category. For any fusion category \mathcal{C} , we have a natural homomorphism of categorical groups $\xi: \underline{\text{Eq}}(\mathcal{C}) \rightarrow \underline{\text{Out}}(\mathcal{C})$, attaching to every tensor auto-equivalence its class of outer auto-equivalences.

Proposition 5.5. *If \mathcal{C} has no nontrivial invertible objects, then ξ is an isomorphism of $\underline{\text{Eq}}(\mathcal{C})$ onto the truncation $\underline{\text{Out}}(\mathcal{C})$.*

Proof. Let \mathcal{M} be a quasi-trivial invertible bimodule category over \mathcal{C} . Then there exists a unique, up to an isomorphism, equivalence of left module categories $\mathcal{C} \rightarrow \mathcal{M}$, so we may assume that $\mathcal{M} = \mathcal{C}$ as a left module category. Then the right action of \mathcal{C} is given by some uniquely determined auto-equivalence ϕ . Then we can define $\xi^{-1}(\mathcal{M}) = \phi$. □

6. Invertibility of components of graded fusion categories

Let G be a finite group and let

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

be a graded fusion category, cf. Section 2.3. The trivial component \mathcal{C}_e is a tensor subcategory of \mathcal{C} , and each \mathcal{C}_g is a \mathcal{C}_e -bimodule category. It follows that for all $g, h \in G$ the tensor product of \mathcal{C} restricts to a \mathcal{C}_e -balanced bifunctor

$$\otimes : \mathcal{C}_g \times \mathcal{C}_h \rightarrow \mathcal{C}_{gh},$$

which gives rise to a functor

$$M_{g,h} : \mathcal{C}_g \boxtimes_{\mathcal{C}_e} \mathcal{C}_h \rightarrow \mathcal{C}_{gh}. \tag{21}$$

Theorem 6.1. *Let $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a G -extension. Then:*

- (i) *each \mathcal{C}_g , $g \in G$, is an invertible \mathcal{C}_e -bimodule category;*
- (ii) *the functor $M_{g,h} : \mathcal{C}_g \boxtimes_{\mathcal{C}_e} \mathcal{C}_h \rightarrow \mathcal{C}_{gh}$, $g, h \in G$, is an equivalence of \mathcal{C}_e -bimodule categories.*

Proof. For each $g \in G$ let us pick a non-zero object Y_g in \mathcal{C}_g . Then $A_g = Y_g \otimes Y_g^*$ is an algebra in \mathcal{C}_e (and, therefore, in \mathcal{C}). By [14], [31] the regular left \mathcal{C} -module category \mathcal{C} is equivalent to the category of right A_g -modules in \mathcal{C} , and the left \mathcal{C}_e -module category \mathcal{C}_g is equivalent to the category of right A_g -modules in \mathcal{C}_e . Furthermore, there are tensor equivalences

$$F_g : \mathcal{C} \xrightarrow{\sim} A_g\text{-bimodules in } \mathcal{C} : X \mapsto Y_g \otimes X \otimes Y_g^*, \quad g \in G.$$

Let R_g , $g \in G$, denote the restriction of F_g to \mathcal{C}_e . It establishes a tensor equivalence

$$R_g : \mathcal{C}_e \xrightarrow{\sim} A_g\text{-bimodules in } \mathcal{C}_e \cong \text{Fun}_{\mathcal{C}_e}(\mathcal{C}_g, \mathcal{C}_g).$$

It is straightforward to see that R_g coincides with the functor defined in (10). Passing from right to left A_g -modules, one gets an equivalence $L_g : \mathcal{C}_e^{\text{rev}} \xrightarrow{\sim} \text{Fun}(\mathcal{C}_g, \mathcal{C}_g)_{\mathcal{C}_e}$. By Proposition 4.2, \mathcal{C}_g is an invertible \mathcal{C}_e -bimodule category. This proves (i).

To prove (ii), note that tensor equivalences F_g , $g \in G$, make the category of (A_g, A_h) -bimodules in \mathcal{C} into a \mathcal{C} -bimodule category, with the left (respectively right) action of an object X in \mathcal{C} by multiplication by $F_g(X)$ (respectively by $F_h(X)$). Thus we have \mathcal{C} -bimodule equivalences

$$F_{g,h} : \mathcal{C} \cong (A_g - A_h)\text{-bimodules in } \mathcal{C} : X \mapsto Y_g \otimes X \otimes Y_h^*, \quad g, h \in G.$$

Therefore, the restriction of $F_{g^{-1},h}$ to \mathcal{C}_{gh} establishes a \mathcal{C}_e -bimodule equivalence between \mathcal{C}_{gh} and the category of $(A_{g^{-1}}, A_h)$ -bimodules in \mathcal{C}_e . The latter category is equivalent to $\text{Fun}_{\mathcal{C}_e}(\mathcal{C}_{g^{-1}}, \mathcal{C}_h) \cong \mathcal{C}_g \boxtimes_{\mathcal{C}_e} \mathcal{C}_h$.

Thus, we have constructed a \mathcal{C}_e -bimodule equivalence

$$\mathcal{C}_g \boxtimes_{\mathcal{C}_e} \mathcal{C}_h \rightarrow \mathcal{C}_{gh}, \quad g, h \in G.$$

It is easy to see that it coincides with the functor (21) induced by the \mathcal{C}_e -balanced bifunctor $\otimes: \mathcal{C}_g \times \mathcal{C}_h \rightarrow \mathcal{C}_{gh}$. Indeed, both functors are identified with

$$\mathcal{C}_{gh} \rightarrow \text{Fun}_{\mathcal{C}_e}(\mathcal{C}_{g^{-1}}, \mathcal{C}_h): X \mapsto ? \otimes X,$$

so the proof is complete. □

Corollary 6.2. *The dual category of $\mathcal{C}_e \boxtimes \mathcal{C}_e^{\text{rev}}$ with respect to each \mathcal{C}_e -bimodule category \mathcal{C}_g , $g \in G$, is equivalent to the center $\mathcal{Z}(\mathcal{C}_e)$ of \mathcal{C}_e :*

$$(\mathcal{C}_e \boxtimes \mathcal{C}_e^{\text{rev}})_{\mathcal{C}_g}^* \cong \mathcal{Z}(\mathcal{C}_e).$$

Proof. This follows by [14], Theorem 3.34, since $(\mathcal{C}_e)_{\mathcal{C}_g}^* \cong \mathcal{C}_e^{\text{rev}}$ by Theorem 6.1. □

Thus, a G -extension \mathcal{C} defines a group homomorphism

$$c: G \rightarrow \text{BrPic}(\mathcal{C}_e).$$

The tensor product and associator of \mathcal{C} give rise to an additional data which we will investigate next.

7. Classification of extensions (topological version)

7.1. The classifying space of a categorical n -group. It is well known that any categorical n -group \mathcal{G} gives rise to a (connected) classifying space $B\mathcal{G}$ (well defined up to homotopy), which determines the equivalence class of \mathcal{G} uniquely (so that $B\mathcal{G}$ carries the same information as \mathcal{G}). Moreover, the homotopy groups of $B\mathcal{G}$ are as follows: $\pi_i(B\mathcal{G}) = \text{Mor}_{i+1}(X_i, X_i)$ for any i -morphism X_i for $i = 1, \dots, n + 1$, and $\pi_i(B\mathcal{G}) = 0$ if $i \geq n + 2$.

A convenient model for the space $B\mathcal{G}$ is the simplicial complex given by the well-known “nerve” construction. For the convenience of the readers, we recall this construction in the case of $n = 2$ (which is the highest value of n we will need). For brevity we omit associativity isomorphisms.

Step 0. We start with one 0-simplex.

Step 1. For every isomorphism class x of objects of \mathcal{G} , we pick an object representing x (which we also call x , abusing the notation) and add a 1-simplex s_x .

Step 2. For every isomorphism classes of objects x_1, x_2 and an isomorphism class of 1-morphisms $f: x_1 \otimes x_2 \rightarrow x_1 x_2$, where $x_1 x_2$ is the representative of $x_1 \otimes x_2$ chosen in the previous step, we pick a 1-morphism representing f (which we also call f , abusing the notation), and add a 2-simplex s_f such that $\partial s_f = s_{x_1} + s_{x_2} - s_{x_1 x_2}$.

Step 3. For each isomorphism classes of objects x_1, x_2, x_3 , isomorphism classes of 1-morphisms $f_{1,2}: x_1 \otimes x_2 \rightarrow x_1x_2$, $f_{2,3}: x_2 \otimes x_3 \rightarrow x_2x_3$, $f_{12,3}: x_1x_2 \otimes x_3 \rightarrow x_1x_2x_3$, $f_{12,3}: x_1 \otimes x_2x_3 \rightarrow x_1x_2x_3$, where $x_1x_2, x_1x_2x_3$, etc. are representatives of tensor products chosen in Step 1, and a 2-morphism

$$g: f_{12,3} \circ (f_{1,2} \otimes \text{id}_3) \rightarrow f_{1,23} \circ (\text{id}_1 \otimes f_{2,3})$$

we add a 3-simplex s_g such that $\partial s_g = s_{f_{1,2}} - s_{f_{1,23}} + s_{f_{12,3}} - s_{f_{2,3}}$.

Step 4. Given isomorphism classes of objects x_1, x_2, x_3, x_4 , isomorphism classes of 1-morphisms $f_{1,2}: x_1 \otimes x_2 \rightarrow x_1x_2$, $f_{2,3}: x_2 \otimes x_3 \rightarrow x_2x_3$, $f_{3,4}: x_3 \otimes x_4 \rightarrow x_3x_4$, $f_{12,3}: x_1x_2 \otimes x_3 \rightarrow x_1x_2x_3$, $f_{1,23}: x_1 \otimes x_2x_3 \rightarrow x_1x_2x_3$, $f_{23,4}: x_2x_3 \otimes x_4 \rightarrow x_2x_3x_4$, $f_{2,34}: x_2 \otimes x_3x_4 \rightarrow x_2x_3x_4$, $f_{1,234}: x_1 \otimes x_2x_3x_4 \rightarrow x_1x_2x_3x_4$, $f_{123,4}: x_1x_2x_3 \otimes x_4 \rightarrow x_1x_2x_3x_4$, $f_{12,34}: x_1x_2 \otimes x_3x_4 \rightarrow x_1x_2x_3x_4$, and 2-morphisms

$$\begin{aligned} g_{1,2,3}: f_{12,3} \circ (f_{1,2} \otimes \text{id}_3) &\rightarrow f_{1,23} \circ (\text{id}_1 \otimes f_{2,3}), \\ g_{2,3,4}: f_{23,4} \circ (f_{2,3} \otimes \text{id}_4) &\rightarrow f_{2,34} \circ (\text{id}_2 \otimes f_{3,4}), \\ g_{1,23,4}: f_{123,4} \circ (f_{1,23} \otimes \text{id}_4) &\rightarrow f_{1,234} \circ (\text{id}_1 \otimes f_{23,4}), \\ g_{12,3,4}: f_{123,4} \circ (f_{12,3} \otimes \text{id}_4) &\rightarrow f_{12,34} \circ (\text{id}_{12} \otimes f_{3,4}), \\ g_{1,2,34}: f_{12,34} \circ (f_{1,2} \otimes \text{id}_{34}) &\rightarrow f_{1,234} \circ (\text{id}_1 \otimes f_{2,34}) \end{aligned}$$

such that

$$(1_1 \otimes g_{2,3,4}) \circ g_{1,23,4} \circ (g_{1,2,3} \otimes 1_4) = g_{1,2,34} \circ g_{12,3,4},$$

we add a single 4-simplex s whose boundary is

$$\partial s = s_{g_{1,2,3}} - s_{g_{1,2,34}} + s_{g_{1,23,4}} - s_{g_{12,3,4}} + s_{g_{2,3,4}}.$$

Step k , $k \geq 5$. Any boundary of a k -simplex, $k \geq 5$, is filled in with a k -simplex. Note that the obtained model is a Kan complex.

7.2. Homotopy groups of classifying spaces of higher groupoids attached to fusion categories

Proposition 7.1. *Let \mathcal{C} be a fusion category and let $\underline{\text{BrPic}}(\mathcal{C})$ be its Brauer–Picard 2-group introduced in Section 4.1. We have:*

- (i) $\pi_1(\underline{\text{BBrPic}}(\mathcal{C})) = \text{BrPic}(\mathcal{C})$;
- (ii) $\pi_2(\underline{\text{BBrPic}}(\mathcal{C})) = \text{Inv}(\mathcal{Z}(\mathcal{C}))$, the group of isomorphism classes of invertible objects in the Drinfeld center of \mathcal{C} ;
- (iii) $\pi_3(\underline{\text{BBrPic}}(\mathcal{C})) = \mathbf{k}^\times$;
- (iv) $\pi_i(\underline{\text{BBrPic}}(\mathcal{C})) = 0$ for all $i \geq 4$.

Proof. (i) is clear. To prove (ii), we need to calculate the group of equivalence classes of automorphisms of any object. Take the object \mathcal{C} regarded as a \mathcal{C} -bimodule. Its endomorphisms as a \mathcal{C} -bimodule is the dual category to \mathcal{C} with respect to $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$, so it is $\mathcal{Z}(\mathcal{C})$ [14], Corollary 3.37. Thus the automorphisms are the invertible objects in $\mathcal{Z}(\mathcal{C})$. To prove (iii), we need to compute the group of automorphisms of any 1-morphism. Take this 1-morphism to be the neutral object in $\mathcal{Z}(\mathcal{C})$. Then the group of automorphisms is \mathbf{k}^\times . (iv) is clear since by construction we have killed all the homotopy groups of degree ≥ 4 . \square

Proposition 7.2. *Let \mathcal{C} be a fusion category, and let $\underline{\text{Out}}(\mathcal{C})$ its categorical 2-group of outer auto-equivalences introduced in Section 4.3. We have:*

- (i) $\pi_1(B\underline{\text{Out}}(\mathcal{C})) = \text{Out}(\mathcal{C})$;
- (ii) $\pi_2(B\underline{\text{Out}}(\mathcal{C})) = \text{Inv}(\mathcal{Z}(\mathcal{C}))$;
- (iii) $\pi_3(B\underline{\text{Out}}(\mathcal{C})) = \mathbf{k}^\times$;
- (iv) $\pi_i(B\underline{\text{Pic}}(\mathcal{C})) = 0$ for $i \geq 4$.

Proposition 7.3. *Let \mathcal{B} be a braided fusion category, and let $\underline{\text{Pic}}(\mathcal{B})$ its Picard 2-group introduced in Section 4.4. We have:*

- (i) $\pi_1(B\underline{\text{Pic}}(\mathcal{B})) = \text{Pic}(\mathcal{B})$;
- (ii) $\pi_2(B\underline{\text{Pic}}(\mathcal{B})) = \text{Inv}(\mathcal{B})$, the group of isomorphism classes of invertible objects of \mathcal{B} ;
- (iii) $\pi_3(B\underline{\text{Pic}}(\mathcal{B})) = \mathbf{k}^\times$;
- (iv) $\pi_i(B\underline{\text{Pic}}(\mathcal{B})) = 0$ for $i \geq 4$.

Proposition 7.4. *Let \mathcal{C} be a fusion category, and let $\underline{\text{Eq}}(\mathcal{C})$ be the categorical group of auto-equivalences of \mathcal{C} introduced in Section 4.5. We have:*

- (i) $\pi_1(B\underline{\text{Eq}}(\mathcal{C})) = \text{Eq}(\mathcal{C})$;
- (ii) $\pi_2(B\underline{\text{Eq}}(\mathcal{C})) = \text{Aut}_\otimes(\text{Id}_\mathcal{C})$, the group of tensor isomorphisms of the identity functor of \mathcal{C} ;
- (iii) $\pi_i(B\underline{\text{Eq}}(\mathcal{C})) = 0$ for $i \geq 3$.

Proposition 7.5. *Let \mathcal{B} be a braided fusion category, and let $\underline{\text{EqBr}}(\mathcal{B})$ be the categorical group of braided auto-equivalences of \mathcal{B} introduced in Section 4.6. We have:*

- (i) $\pi_1(B\underline{\text{EqBr}}(\mathcal{B})) = \text{EqBr}(\mathcal{B})$;
- (ii) $\pi_2(B\underline{\text{EqBr}}(\mathcal{B})) = \text{Aut}_\otimes(\text{Id}_\mathcal{B})$;
- (iii) $\pi_i(B\underline{\text{EqBr}}(\mathcal{B})) = 0$ for $i \geq 3$.

The proofs of Propositions 7.2, 7.3, 7.4 and 7.5 are analogous to that of Proposition 7.1.

7.3. The Whitehead half-square and the braiding. Recall that for any $i, j > 1$ we have the Whitehead bracket $[\cdot, \cdot]: \pi_i \times \pi_j \rightarrow \pi_{i+j-1}$ on homotopy groups of any topological space. Also, since there is a map $S^3 \rightarrow S^2$ of Hopf invariant 1, we have the Whitehead half-square map $W: \pi_2 \rightarrow \pi_3$ such that $W(x+y) - W(x) - W(y) = [x, y]$.

The following proposition was pointed out to us by V. Drinfeld.

Proposition 7.6. *Let \mathcal{B} be a braided fusion category. For the space $B\text{BrPic}(\mathcal{B})$, the Whitehead half-square map $W: \pi_2 \rightarrow \pi_3$ is given by the braiding c_{ZZ} on invertible objects $Z \in \mathcal{B}$. Therefore, the Whitehead bracket $[\cdot, \cdot]: \pi_2 \times \pi_2 \rightarrow \pi_3$ coincides with the squared braiding $c_{ZY}c_{YZ}$ on invertible objects $Y, Z \in \mathcal{B}$.*

Proof. For any pointed space X , the fundamental groupoid of the double loop space $\Omega^2(X)$ is a braided monoidal category (see [33], Section 2.3). Note that $\pi_2(X) = \pi_0(\Omega^2(X))$ and $\pi_3(X) = \pi_1(\Omega^2(X))$. So, the map $Z \mapsto c_{ZZ}$, where c denotes the braiding of the above category, defines a map $\pi_2(X) \rightarrow \pi_3(X)$. We claim that this map is the Whitehead half-square map. To prove this, it suffices to treat the universal example $X = S^2$. That is, one needs to show that for $X = S^2$ the map in question is the map $\mathbb{Z} = \pi_2(S^2) \rightarrow \mathbb{Z} = \pi_3(S^2)$ given by $n \mapsto n^2$. This is done by a straightforward verification. \square

In particular, taking $\mathcal{B} = \mathcal{Z}(\mathcal{C})$, we find that the Whitehead half-square map and the Whitehead bracket for $B\text{BrPic}(\mathcal{C})$ are given by the braiding on invertible objects of $\mathcal{Z}(\mathcal{C})$.

Remark. Proposition 7.6 can also be derived from [2], Chapter IV.

7.4. Classification of extensions. Now we would like to classify G -extensions \mathcal{C} of a given fusion category \mathcal{D} . As we have seen in Theorem 6.1, such a category necessarily defines a group homomorphism $c: G \rightarrow \text{BrPic}(\mathcal{D})$. We would like to study additional data and conditions on them that define a category \mathcal{C} given a homomorphism c .

Theorem 7.7. *Equivalence classes of G -extensions \mathcal{C} of \mathcal{D} are in bijection with morphisms of categorical 2-groups $G \rightarrow \text{BrPic}(\mathcal{D})$, or, equivalently, with homotopy classes of maps between their classifying spaces: $BG \rightarrow B\text{BrPic}(\mathcal{D})$.*

Proof. Let us consider what it takes to define a continuous map $\xi: BG \rightarrow B\text{BrPic}(\mathcal{D})$, using the simplicial model of $\text{BrPic}(\mathcal{D})$ described above. Note that since our model of $B\text{BrPic}(\mathcal{D})$ is a Kan complex, any map ξ is homotopic to a simplicial map, so it suffices to restrict our attention to simplicial maps (which we will do from now on).

Step 1. Defining the map ξ at the level of 1-skeletons (up to homotopy) obviously amounts to a choice of a set-theoretical map of fundamental groups $c: G \rightarrow \text{BrPic}(\mathcal{D})$. On the categorical side, this is just a choice of an assignment

$g \mapsto c(g) = \mathcal{C}_g, g \in G$, where \mathcal{C}_g is an invertible bimodule category over \mathcal{D} . They can be combined into a single \mathcal{D} -bimodule category $\mathcal{C} = \bigoplus_g \mathcal{C}_g$.

Step 2. Extendability of this ξ to the level of 2-skeletons amounts to the condition that c is a group homomorphism. On the categorical side, this means that one has equivalences $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}$ and in particular $\mathcal{C}_e \cong \mathcal{D}$.

Next, any choice of an extension of ξ to the level of 2-skeletons amounts to picking the equivalences $M_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \rightarrow \mathcal{C}_{gh}$, which defines a functor \otimes of tensor multiplication on \mathcal{C} .

Step 3. Further, extendability of such a ξ to the level of 3-skeletons amounts, on the categorical side, to the condition that there exists a functorial isomorphism

$$\alpha: (\bullet \otimes \bullet) \otimes \bullet \rightarrow \bullet \otimes (\bullet \otimes \bullet)$$

(respecting the \mathcal{D} -bimodule structure but not necessarily satisfying the pentagon relation), and once a good M (for which α exists) has been fixed, the freedom of choosing an extension of ξ to the level of 3-skeletons is a choice of α .

Step 4. Finally, once ξ has been extended to 3-skeletons, its extendability to the level of 4-skeletons amounts to the condition that α satisfies the pentagon relation. Once such an α has been fixed, there is a unique extension of ξ to the level of 4-skeletons.

Step 5. Once ξ has been extended to a map of 4-skeletons, it canonically extends to a map of skeletons of all dimensions.

The theorem is proved. □

7.5. Proof of Theorem 1.3. Theorem 1.3 follows from Theorem 7.7 and classical obstruction theory in algebraic topology. Let us describe this derivation in more detail. For brevity we denote the homotopy groups $\underline{BBrPic}(\mathcal{D})$ just by π_i , without specifying the space.

Let us go back to the proof of Theorem 7.7. At Step 3, it may be necessary to modify $M = (M_{g,h})$ to secure the existence of α . But even if we allow modifications of M , there is an obstruction $O_3(c)$ to the existence of α . Let us discuss the nature of this obstruction.

If we have a map ξ of 2-skeletons, then the condition for this map to be extendable to 3-skeletons is that for every 3-simplex $\sigma \subset BG, \xi(\partial\sigma)$ represents the trivial element in the group π_2 . Thus we get an obstruction which is a 3-cochain of G with values in π_2 . It is easy to see that this 3-cochain is actually a cocycle (where G acts on π_2 via the homomorphism c), i.e., we get an obstruction $\psi \in Z^3(G, \pi_2)$. Now M can be modified by adding a 2-cochain χ on G with coefficients in π_2 , and this modification replaces ψ with $\psi + d\chi$. This implies that the actual obstruction to extending ξ to 3-skeletons (allowing the modifications of M) is the cohomology class $[\psi] = O_3(c) \in H^3(G, \pi_2)$.

If the obstruction $O_3(c)$ vanishes, then, as we see from the above, the freedom of choosing M so that ξ is extendable to 3-skeletons is in $H^2(G, \pi_2)$. That is, we can

modify M by adding a cocycle $\chi \in Z^2(G, \pi_2)$, but if χ is a coboundary, then the homotopy class of the extension does not change.

Further, at Step 4, there is an obstruction $O_4(c, M)$ to choosing α . Let us discuss its nature.

If we have a map ξ of 3-skeletons, then the condition for this map to be extendable to 4-skeletons is that for every 4-simplex $\sigma \subset BG$, $\xi(\partial\sigma)$ represents the trivial element in the group π_3 . Thus we get an obstruction which is a 4-cochain of G with values in π_3 . It is easy to see that this 4-cochain is actually a cocycle, i.e., we get an obstruction $\eta \in Z^4(G, \pi_3)$. Now α can be modified by adding a 3-cochain θ on G with coefficients in π_3 , and this modification replaces η with $\eta + d\theta$. This implies that the actual obstruction to extending ξ to 4-skeletons (allowing the modifications of α) is the cohomology class $[\eta] = O_4(c, M) \in H^4(G, \pi_3)$.

If the obstruction $O_4(c, M)$ vanishes, then, as we see from the above, the freedom of choosing α so that ξ is extendable to 4-skeletons is in $H^3(G, \pi_3)$. That is, we can modify α by adding a cocycle $\theta \in Z^3(G, \pi_3)$, but if θ is a coboundary, then the homotopy class of the extension does not change.

This proves Theorem 1.3.

7.6. Classification of group actions on fusion categories

Proposition 7.8. (i) *Actions of a group G by auto-equivalences of a fusion category \mathcal{C} , up to an isomorphism, are in natural bijection with homotopy classes of mappings $BG \rightarrow \text{BEq}(\mathcal{C})$.*

(ii) *Actions of a group G by braided auto-equivalences of a braided fusion category \mathcal{B} , up to an isomorphism, are in natural bijection with homotopy classes of mappings $BG \rightarrow \text{BEqBr}(\mathcal{B})$.*

Proof. (i) We argue as in the previous subsection. Namely, a map between 2-skeletons of the spaces in question is the same thing as an assignment $g \rightarrow F_g$ that attaches to every $g \in G$ a tensor equivalence $F_g: \mathcal{C} \rightarrow \mathcal{C}$ and a collection of functorial isomorphisms $\eta_{g,h}: F_g \circ F_h \rightarrow F_{gh}$, $g, h \in G$. This map is extendable to 3-skeletons if and only if $\eta_{g,h}$ satisfies the 2-cocycle condition, i.e., if the data $(F_g, \eta_{g,h})$ is an action of G on \mathcal{C} by tensor auto-equivalences. Note that the extension to 3-skeletons is unique if it exists, and extends uniquely to skeletons of higher dimensions. So (i) is proved.

(ii) is proved similarly. □

Corollary 7.9 ([16], Theorem 5.5). (i) *Actions of a group G by tensor auto-equivalences of a fusion category \mathcal{C} , up to an isomorphism, are parametrized by pairs (c, η) , where $c: G \rightarrow \text{Eq}(\mathcal{B})$ is a homomorphism such that the corresponding first obstruction $O_3(c) \in H^3(G, \text{Aut}_{\otimes}(\text{Id}_{\mathcal{C}}))$ vanishes and $\eta = (\eta_{g,h})$ is the equivalence class of the identification $F_g \circ F_h \rightarrow F_{gh}$ belonging to a torsor over $H^2(G, \text{Aut}_{\otimes}(\text{Id}_{\mathcal{C}}))$.*

(ii) *Actions of a group G by braided auto-equivalences of a braided fusion category \mathcal{B} , up to an isomorphism, are parametrized by pairs (c, η) , where $c: G \rightarrow \text{EqBr}(\mathcal{B})$ is a homomorphism such that the corresponding first obstruction $O_3(c) \in H^3(G, \text{Aut}_\otimes(\text{Id}_\mathcal{B}))$ vanishes and $\eta = (\eta_{g,h})$ is the equivalence class of the identification $F_g \circ F_h \rightarrow F_{gh}$ belonging to a torsor over $H^2(G, \text{Aut}_\otimes(\text{Id}_\mathcal{B}))$.*

7.7. Classification of quasi-trivial extensions. Let G be a finite group, and \mathcal{D} a fusion category. We call a G -extension \mathcal{C} of \mathcal{D} *quasi-trivial* if every category \mathcal{C}_g is a quasi-trivial bimodule category over \mathcal{D} . This condition is equivalent to the condition that \mathcal{C} is strongly G -graded in the sense of [16], i.e., every category \mathcal{C}_g contains an invertible object.

The following proposition is a corollary of Theorem 7.7.

Proposition 7.10. *Quasi-trivial G -extensions of \mathcal{D} , up to graded equivalence, are in natural bijection with homotopy classes of mappings $BG \rightarrow \underline{B\text{Out}}(\mathcal{D})$.*

Remark 7.11. Note that a mapping $\tau: BG \rightarrow \underline{B\text{Out}}(\mathcal{D})$ is representable as $\tau = B\xi \circ \zeta$, where $\xi: \underline{\text{Eq}}(\mathcal{D}) \rightarrow \underline{\text{Out}}(\mathcal{D})$ is defined in Section 5.5 and $\zeta: BG \rightarrow \underline{\text{Eq}}(\mathcal{D})$ if and only if the corresponding quasi-trivial extension is trivial, i.e., \mathcal{C} is the semidirect product category $\text{Vec}_G \rtimes \mathcal{D}$ for the G -action on \mathcal{D} corresponding to ζ .

7.8. Classification of faithfully graded braided G -crossed fusion categories. Let G be a finite group. The notion of a braided G -crossed fusion category is due to Turaev; see [39], [38]. By definition, it is a G -graded category

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g, \tag{22}$$

equipped with an action of G such that $g(\mathcal{C}_h) = \mathcal{C}_{ghg^{-1}}$ and a natural family of isomorphisms

$$c_{X,Y}: X \otimes Y \xrightarrow{\sim} g(Y) \otimes X, \quad g \in G, X \in \mathcal{C}_g, Y \in \mathcal{C},$$

called the G -braiding. The above action and G -braiding are required to satisfy certain natural compatibility conditions. In particular, the trivial component $\mathcal{B} := \mathcal{C}_e$ is a braided fusion category. We refer the reader to [39], [38] for the precise definition and to [7], § 4.4.3, for a detailed discussion of braided G -crossed categories.

Below we only consider braided G -crossed fusion categories with a *faithful* grading (22). The general case will be treated elsewhere.

Theorem 7.12. *Let \mathcal{B} be a braided fusion category. Equivalence classes of braided G -crossed categories \mathcal{C} having a faithful G -grading with the trivial component \mathcal{B} are in bijection with morphisms of categorical 2-groups $G \rightarrow \underline{\text{Pic}}(\mathcal{B})$, or, equivalently, with homotopy classes of maps between their classifying spaces $BG \rightarrow \underline{B\text{Pic}}(\mathcal{B})$.*

Proof. Since $\underline{\text{Pic}}(\mathcal{B}) \subseteq \underline{\text{BrPic}}(\mathcal{B})$, it follows from Theorem 7.7 that a morphism $G \rightarrow \underline{\text{Pic}}(\mathcal{B})$ determines a G -extension \mathcal{C} of \mathcal{B} . It remains to check that the additional condition that each $\mathcal{C}_g, g \in G$, is an invertible \mathcal{B} -module category is equivalent to the existence of an action of G and a G -braiding on \mathcal{C} .

Indeed, if the image of G is inside $\underline{\text{Pic}}(\mathcal{B})$ then for all $g, h \in G$ the category $\text{Fun}_{\mathcal{B}}(\mathcal{C}_g, \mathcal{C}_{gh})$ of \mathcal{B} -module functors from \mathcal{C}_g to \mathcal{C}_{gh} is identified, on the one hand, with functors of right tensor multiplication by objects of \mathcal{C}_h and, on the other hand, with functors of left tensor multiplication by objects of $\mathcal{C}_{ghg^{-1}}$. So there is an equivalence $g: \mathcal{C}_h \rightarrow \mathcal{C}_{ghg^{-1}}$ defined by the isomorphism of \mathcal{B} -module functors

$$? \otimes Y \cong g(Y) \otimes ? : \mathcal{C}_g \rightarrow \mathcal{C}_{gh}, \quad Y \in \mathcal{C}_h. \tag{23}$$

Extending it to \mathcal{C} by linearity we obtain an action of G by tensor auto-equivalences of \mathcal{C} . Furthermore, evaluating (23) on $X \in \mathcal{C}_g$ we obtain a natural family of isomorphisms

$$X \otimes Y \xrightarrow{\sim} g(Y) \otimes X, \quad g \in G, X \in \mathcal{C}_g, Y \in \mathcal{C},$$

which gives a G -braiding on \mathcal{C} .

To prove the converse, one can follow the proof of Theorem 6.1 to verify that components of a braided G -crossed category are invertible module categories over its trivial component. □

Remark 7.13. The somewhat similar problem of classifying G -extensions of braided 2-groups is discussed in by E. Jenkins in [21] (the notion of a G -extension of a braided 2-group is defined in [7], Appendix E, Definition E.8).

8. Classification of extensions (algebraic version)

Now we would like to retell the contents of the previous section in a purely algebraic language, without using homotopy theory, and thus give an algebraic proof of Theorem 1.3.

8.1. Decategorification. We start by recalling a well-known decategorified version of Theorem 1.3. Let G be a group and let $R = \bigoplus_{g \in G} R_g$ be a G -graded ring. Recall that R is called *strongly graded* if the multiplication map $R_g \otimes_{\mathbb{Z}} R_h \rightarrow R_{gh}$ is surjective; in this situation we say that R is strongly G -graded extension of R_e . By [5] for a strongly G -graded ring R the induced maps $R_g \otimes_{R_e} R_h \rightarrow R_{gh}$ are isomorphisms; in particular R_g is an invertible R_e -bimodule for any $g \in G$. Thus any strongly G -graded ring R defines a homomorphism $g \mapsto R_g$ of 2-groupoids $G \rightarrow \underline{\text{Pic}}(R_e)$, where $\underline{\text{Pic}}(R_e)$ is the 2-groupoid of invertible R_e -bimodules. Conversely, it is clear that any homomorphism of 2-groupoids $G \rightarrow \underline{\text{Pic}}(S)$ determines a strongly G -graded extension of S .

By definition, the group of 1-morphisms in $\underline{\text{Pic}}(S)$ is the group of isomorphism classes of invertible S -bimodules $\text{Pic}(S)$ and the group of 2-endomorphisms of the unit 1-morphism (which is S considered as an S -bimodule) is the group $\mathcal{Z}(S)^\times$ of invertible elements of the center $\mathcal{Z}(S)$ of S . Thus the group $\text{Pic}(S)$ acts on the abelian group $\mathcal{Z}(S)^\times$ and the equivalence class of 2-groupoid $\underline{\text{Pic}}(S)$ is completely determined by a class $\omega \in H^3(\text{Pic}(S), \mathcal{Z}(S)^\times)$. Thus we obtain the following result:

Theorem 8.1 (see [3]). *There exists a class $\omega \in H^3(\text{Pic}(S), \mathcal{Z}(S)^\times)$ such that strongly G -graded extensions of a ring S corresponding to a homomorphism $\phi: G \rightarrow \text{Pic}(S)$ form an $H^2(G, \mathcal{Z}(S)^\times)$ -torsor which is nonempty if and only if $\phi^*(\omega) = 0 \in H^3(G, \mathcal{Z}(S)^\times)$.*

8.2. Cohomological data determined by a G -extension. Let

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

be a G -graded fusion category and let $\mathcal{D} := \mathcal{C}_e$ (i.e., \mathcal{C} is a G -extension of \mathcal{D}). By Theorem 6.1 for each pair $g, h \in G$ there is an equivalence of \mathcal{D} -bimodule categories

$$M_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}, \tag{24}$$

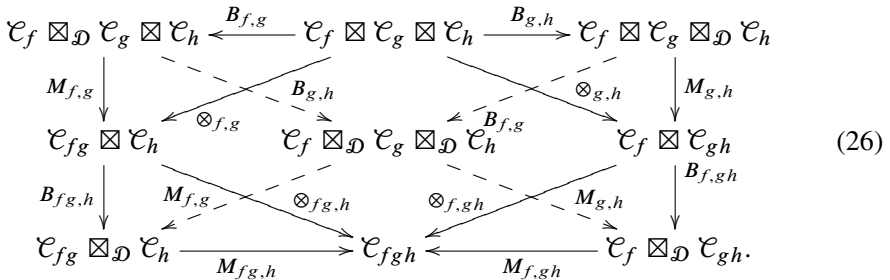
which comes from the restriction of the tensor product of $\otimes: \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$ to

$$\otimes_{g,h}: \mathcal{C}_g \boxtimes \mathcal{C}_h \rightarrow \mathcal{C}_{gh}. \tag{25}$$

For all $g, h \in G$ let

$$B_{g,h} := B_{\mathcal{C}_g, \mathcal{C}_h}: \mathcal{C}_g \boxtimes \mathcal{C}_h \rightarrow \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h$$

be the canonical functor coming from Definition 3.3. For all $f, g, h \in G$ consider the following diagram of \mathcal{D} -bimodule categories and functors:



In this diagram we will refer to polygons formed by solid lines as those in the “front” and to polygons formed by dotted lines as the “rear”. The four triangles in the front commute by the universal property of tensor product of module categories. The square

in the front commutes up to an associativity constraint, which is an isomorphism of \mathcal{D} -bimodule functors. Hence, the perimeter of the diagram commutes up to a natural isomorphism of \mathcal{D} -bimodule functors. The upper rear quadrangle commutes by Remark 3.6 (ii) and the left and right rear quadrangles commute since functors $B_{g,h}$, $g, h \in G$, come from \mathcal{D} -balanced functors. Therefore, the lower quadrangle in the rear commutes up to a natural isomorphism of \mathcal{D} -bimodule functors:

$$\alpha_{f,g,h} : M_{f,gh}(\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M_{g,h}) \cong M_{f,g,h}(M_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h}). \tag{27}$$

Equivalently, the \mathcal{D} -bimodule functor

$$T_{f,g,h} := M_{f,g,h}(M_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h})(\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M_{g,h}^{-1})M_{f,g,h}^{-1} : \mathcal{C}_{fgh} \rightarrow \mathcal{C}_{fgh} \tag{28}$$

is isomorphic (as a \mathcal{D} -bimodule functor) to the identity.

The pentagon axiom for the tensor product in \mathcal{C} implies the equality of natural transformations

$$\begin{aligned} & M_{f,ghk}(\text{id}_f \boxtimes_{\mathcal{D}} \alpha_{g,h,k}) \circ \alpha_{f,gh,k}(\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M_{g,h} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_k}) \\ & \quad \circ M_{fgh,k}(\alpha_{f,g,h} \boxtimes_{\mathcal{D}} \text{id}_k) \\ & = \alpha_{f,g,hk}(\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_g} \boxtimes_{\mathcal{D}} M_{h,k}) \circ \alpha_{f,g,h,k}(M_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_k}) \end{aligned} \tag{29}$$

for all $f, g, h, k \in G$ (note that we use the notation Id for the identity functor and id for the identity morphism).

8.3. An action determined by a G -extension. Let $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a G -extension of $\mathcal{C}_e =: \mathcal{D}$. We continue to use the notation introduced in Section 8.3, see (24) and (25).

For a \mathcal{D} -bimodule category \mathcal{M} , any \mathcal{D} -bimodule functor $\mathcal{M} \rightarrow \mathcal{M}$ is by definition an object of the dual category $(\mathcal{D} \boxtimes \mathcal{D}^{\text{rev}})_{\mathcal{M}}^*$. Hence by Corollary 6.2 the group $Z_g := \text{Aut}_{\mathcal{D}}(\mathcal{C}_g)$ of \mathcal{D} -bimodule auto-equivalences of \mathcal{C}_g is abelian and is isomorphic to the group $Z := Z_e$ of isomorphism classes of invertible objects in $\mathcal{Z}(\mathcal{D})$.

Observe that for all $g, f \in G$ there are group isomorphisms $i_{f,g} : Z_g \cong Z_{gf}$ and $j_{f,g} : Z_g \cong Z_{fg}$ defined by

$$i_{f,g}(b) := M_{g,f} \circ (b \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_f}) \circ M_{g,f}^{-1}, \tag{30}$$

$$j_{f,g}(b) := M_{f,g} \circ (\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} b) \circ M_{f,g}^{-1} \tag{31}$$

for all $b \in Z_g$. One can easily check that for all $f, h, g \in G$ there are equalities

$$i_{fh,g} = i_{h,gf}i_{f,g} \quad \text{and} \quad j_{fh,g} = j_{f,hg}j_{h,g}. \tag{32}$$

It follows from (27) that the isomorphisms i and j commute with each other, i.e., for all $f, g, h \in G$ there is an equality

$$i_{f,hg}j_{h,g} = j_{h,gf}i_{f,g}. \tag{33}$$

Define an action of G on Z by $\rho: G \rightarrow \text{Aut}(Z): g \mapsto \rho_g$ where

$$\rho_g := i_{g^{-1},g} j_{g,1}, \quad g \in G. \tag{34}$$

Proposition 8.2. *The action $\rho: G \rightarrow \text{Aut}(Z)$ depends only on the homomorphism $c: G \rightarrow \text{BrPic}(\mathcal{C}_e)$; it does not depend on the choice of $M_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}$.*

Proof. It suffices to show that isomorphisms $i_{f,g}, j_{f,g}$ defined by equations (30) and (31) do not depend on the choice of equivalences $M_{g,h}$.

Indeed, if $M'_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}$ is another \mathcal{D} -bimodule equivalence then $M'_{g,h} = L_{g,h} \circ M_{g,h}$ where $L_{g,h} \in \text{Aut}_{\mathcal{D}}(\mathcal{C}_{gh})$. It follows that isomorphisms $\text{Aut}_{\mathcal{D}}(\mathcal{C}_g) \cong \text{Aut}_{\mathcal{D}}(\mathcal{C}_{gh})$ determined by $M_{g,h}$ and $M'_{g,h}$ differ by a conjugation by $L_{g,h}$. But since $\text{Aut}_{\mathcal{D}}(\mathcal{C}_{gh})$ is abelian, this conjugation is trivial. \square

The following summarizes Sections 8.2 and 8.3. A G -extension \mathcal{C} determines the following data:

- (1) a fusion category \mathcal{D} , a collection of invertible \mathcal{D} -bimodule categories $\mathcal{C}_g, g \in G$ such that $\mathcal{C}_e \cong \mathcal{D}$, and an action ρ of G by automorphisms of the group Z of invertible objects of $\mathcal{Z}(\mathcal{D})$;
- (2) a collection of \mathcal{D} -bimodule isomorphisms $M_{gh}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}$ such that each $T_{f,g,h}$ defined by (28) is isomorphic to the identity as a \mathcal{D} -bimodule functor;
- (3) natural isomorphisms $\alpha_{f,g,h}$ (27) satisfying identity (29).

In the next few subsections we will investigate when a set of data with the above properties gives rise to a G -extension.

8.4. Obstruction to the existence of tensor product. Let us consider a situation opposite to the one studied in Section 8.2. Let G be a finite group. Suppose that we are given a fusion category \mathcal{D} , a group homomorphism $c: G \rightarrow \text{BrPic}(\mathcal{D}), g \mapsto \mathcal{C}_g$, and there are \mathcal{D} -bimodule equivalences

$$M_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}$$

for all $g, h \in G$ such that isomorphisms (27) exist. Let $\rho: G \rightarrow \text{Aut}(Z)$, where $Z := \text{Aut}_{\mathcal{D} \boxtimes_{\mathcal{D} \text{op}}(\mathcal{D})}$ be the action of G defined in Section 8.3. By Proposition 8.2, ρ depends only on c and not on the choice $M_{g,h}, g, h \in G$. We would like to parameterize fusion category structures on $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ which give rise to this data.

First, let us investigate the existence of a G -graded quasi-tensor category structure on \mathcal{C} . By a *quasi-tensor category* we mean a category \mathcal{C} with a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ such that $\otimes \circ (\otimes \times \text{Id}_{\mathcal{C}}) \cong \otimes \circ (\text{Id}_{\mathcal{C}} \times \otimes)$ (so we do not yet require existence of an associativity constraint for \otimes).

As before, let $Z_g = \text{Aut}_{\mathcal{D}}(\mathcal{C}_g)$. Then $Z := Z_e$ is the group of invertible objects in $\mathcal{Z}(\mathcal{D})$. We have isomorphisms $i_{g,1}: Z \cong Z_g$ for all $g \in G$.

For all $f, g, h \in G$ let

$$T_{f,g,h} = M_{fg,h}(M_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h})(\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M_{g,h}^{-1})M_{f,gh}^{-1}.$$

Then $\bar{T}_{f,g,h} = i_{fgh,1}^{-1}(T_{f,g,h})$ defines a function on G^3 with values in the abelian group Z . One can directly check that $\bar{T}_{f,g,h}$ is an element of $Z^3(G, Z)$, i.e., a 3-cocycle on G with values in Z (the latter is a G -module via ρ). Let us find how this function depends on the choice of equivalences $M_{g,h}$.

Suppose that each $M_{g,h}$ is replaced by $M'_{g,h} = L_{g,h} \circ M_{g,h}$ where $L_{g,h} \in Z_{gh}$ as above. Then the corresponding function on G^3 with values in Z_{fgh} is

$$\begin{aligned} T'_{f,g,h} &= M'_{fg,h}(M'_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h})(\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M'_{g,h}{}^{-1})M'_{f,gh}{}^{-1} \\ &= L_{fg,h}M_{fg,h}(L_{f,g}M_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h})(\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M_{g,h}^{-1}L_{g,h}^{-1})M_{f,gh}^{-1}L_{f,gh}^{-1} \\ &= L_{fg,h}i_{h,fg}(L_{f,g})j_{f,gh}(L_{g,h}^{-1})L_{f,gh}^{-1}T_{f,g,h}. \end{aligned}$$

Let $\bar{L}_{g,h} := i_{gh,1}^{-1}(L_{g,h}) \in Z$. We compute, using equations (32), (33) and definition (34) of the action ρ :

$$\begin{aligned} \bar{T}'_{f,g,h} &= i_{fgh,1}^{-1}(T'_{f,g,h}) \\ &= \bar{L}_{fg,h}i_{fgh,1}^{-1}i_{h,fg}i_{fg,1}(\bar{L}_{f,g})i_{fgh,1}^{-1}j_{f,gh}i_{gh,1}(\bar{L}_{g,h}^{-1})\bar{L}_{f,gh}^{-1}\bar{T}_{f,g,h} \\ &= \bar{L}_{fg,h}\bar{L}_{f,g}\rho_f(\bar{L}_{g,h}^{-1})\bar{L}_{f,gh}^{-1}\bar{T}_{f,g,h}. \end{aligned}$$

Thus, the function \bar{T}' differs from \bar{T} by a coboundary. This yields a cohomology class in $H^3(G, Z)$ independent on the choice of the equivalences $M_{g,h}$.

Definition 8.3. We will call the cohomology class of \bar{T} in $H^3(G, Z)$ the *tensor product obstruction class* and denote it $O_3(c)$.

Theorem 8.4. *Let G be a finite group and let \mathcal{D} be a fusion category. Let*

$$c : G \rightarrow \text{BrPic}(\mathcal{D}) : g \mapsto \mathcal{C}_g$$

be a group homomorphism. Then there exist \mathcal{D} -bimodule category equivalences $\mathcal{C}_e \cong \mathcal{D}$ and $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}$, $g, h \in G$, defining a \mathcal{D} -bimodule tensor product \otimes on $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ such that $\otimes \circ (\otimes \boxtimes \text{Id}_{\mathcal{C}}) \cong \otimes \circ (\text{Id}_{\mathcal{C}} \boxtimes \otimes)$ if and only if the obstruction class $O_3(c)$ is the trivial element of $H^3(G, Z)$.

Proof. Consider the diagram of functors (26). It was explained above that its natural \mathcal{D} -bimodule commutativity is equivalent to the natural \mathcal{D} -bimodule isomorphism $T_{f,g,h} \cong \text{Id}_{\mathcal{C}_{fgh}}$, $f, g, h \in G$. The latter is equivalent to \bar{T} being cohomologous to 1 in $Z^3(G, Z)$. □

8.5. Construction of a quasi-tensor product

Theorem 8.5. *Suppose that the obstruction class $O_3(c)$ vanishes. Then isomorphism classes of \mathcal{D} -bimodule tensor products on $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ form a torsor over the second cohomology group $H^2(G, Z)$.*

Proof. Choose \mathcal{D} -bimodule equivalences $M_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}$ for all $g, h \in G$ and natural isomorphisms of \mathcal{D} -bimodule functors

$$\alpha_{f,g,h}: M_{f,g,h}(M_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h}) \xrightarrow{\sim} M_{f,g,h}(\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M_{g,h}), \quad (35)$$

which give rise to a natural isomorphism

$$\alpha: \otimes \circ (\otimes \times \text{Id}_{\mathcal{C}}) \xrightarrow{\sim} \otimes \circ (\text{Id}_{\mathcal{C}} \times \otimes).$$

The computations done in the previous subsection show that replacing each $M_{g,h}$ by $L_{g,h} \circ M_{g,h}$ where $L_{g,h} \in Z_{gh}$ makes the two sides of (35) differ by

$$\bar{L}_{f,g,h} \bar{L}_{f,g} \rho_f(\bar{L}_{g,h}^{-1}) \bar{L}_{f,g,h}^{-1} \in Z.$$

Thus, substituting $L_{g,h} \circ M_{g,h}$ for $M_{g,h}$ does not affect the existence of an isomorphism α as in (35) if and only if $\bar{L} \in Z^2(G, Z)$ is a 2-cocycle. Clearly, two 2-cocycles define isomorphic tensor products if and only if they are cohomologous. \square

Thus, in the case when $O_3(c)$ is cohomologically trivial, one defines a tensor product on $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ as follows. Choose \mathcal{D} -bimodule equivalences

$$M_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}$$

and natural isomorphisms $\alpha_{f,g,h}$ as in (35). Then each $M_{g,h}$ gives rise to a product $\otimes_{g,h}: \mathcal{C}_g \boxtimes \mathcal{C}_h \rightarrow \mathcal{C}_{gh}$ and $\alpha_{f,g,h}$ gives rise to an isomorphism

$$\otimes_{f,g,h} \circ (\text{Id}_{\mathcal{C}_f} \boxtimes \otimes_{g,h}) \cong \otimes_{f,g,h} \circ (\otimes_{f,g} \boxtimes \text{Id}_{\mathcal{C}_h}).$$

8.6. Obstruction to the existence of an associativity constraint. We continue to assume that the tensor product obstruction $O_3(c)$ vanishes, i.e., that c gives rise to a \mathcal{D} -bimodule quasi-tensor product on $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$.

Let us determine when this quasi-tensor product is in fact a tensor product, i.e., when it admits an associativity constraint satisfying the pentagon equation. Choose a collection of \mathcal{D} -bimodule category equivalences

$$M = \{M_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}\}$$

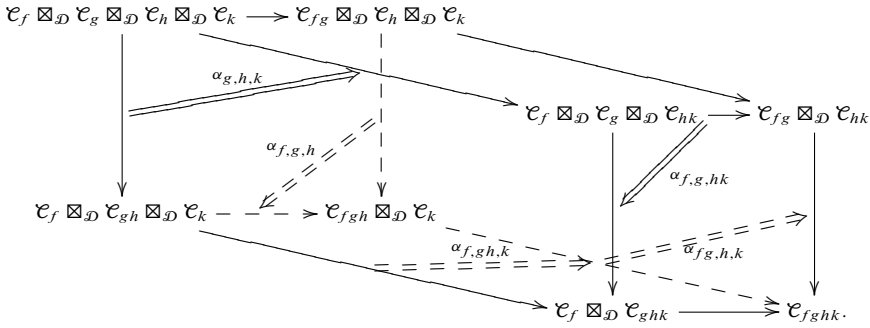
and natural isomorphisms $\alpha_{f,g,h}$ as in (35).

Definition 8.6. We will call $M = \{M_{g,h}: \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \cong \mathcal{C}_{gh}\}_{g,h \in G}$ a *system of products*.

By Theorem 8.5, M is an element of an $H^2(G, Z)$ -torsor.

Note that each $\alpha_{f,g,h}$ is determined up to an automorphism of a simple object in $Z(\mathcal{C}_\ell)$, i.e., up to a non-zero scalar.

For all $f, g, h, k \in G$ let us consider the following cube whose vertices are \mathcal{D} -bimodule categories, edges are \mathcal{D} -bimodule equivalences $M_{a,b}$, and faces are natural isomorphisms $\alpha_{a,b,c}$, $a, b, c \in G$, see (35) (to keep the diagram readable, only the faces are labeled):



The composition of the natural transformations corresponding to faces of this cube is a \mathcal{D} -bimodule automorphism of the functor

$$M_{fgh,k} \circ (M_{fg,h} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_k}) \circ (M_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_k}),$$

i.e., the scalar

$$\begin{aligned} v_{f,g,h,k} & := (M_{f,g} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_h} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_k}) \alpha_{f,g,h,k}^{-1} \circ (\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_g} \boxtimes_{\mathcal{D}} M_{h,k}) \alpha_{f,g,hk}^{-1} \quad (36) \\ & \circ M_{f,ghk} (\text{id}_f \boxtimes_{\mathcal{D}} \alpha_{g,h,k}) \circ \alpha_{f,gh,k} (\text{Id}_{\mathcal{C}_f} \boxtimes_{\mathcal{D}} M_{g,h} \boxtimes_{\mathcal{D}} \text{Id}_{\mathcal{C}_k}) \\ & \circ M_{fgh,k} (\alpha_{f,g,h} \boxtimes_{\mathcal{D}} \text{id}_k). \end{aligned}$$

The commutativity of the above cube is equivalent to the existence of an associativity constraint for \mathcal{C} satisfying the pentagon axiom. The cube commutes if and only if $\{\alpha_{f,g,h}\}_{f,g,h \in G}$ can be chosen in such a way that $v_{f,g,h,k} = 1$. See [23] regarding the notions of composition of faces and commutativity of a polytope.

It is easy to check that the above v is a 4-cocycle on G with values in \mathbf{k}^\times . Let us determine how v changes when we change the choice of α . Let

$$\alpha'_{f,g,h} = \alpha_{f,g,h} \lambda_{f,g,h}, \quad \text{where } \lambda_{f,g,h} \in \mathbf{k}^\times. \quad (37)$$

Then the corresponding scalar is

$$v'_{f,g,h,k} = v_{f,g,h,k} \lambda_{f,g,hk}^{-1} \lambda_{fg,h,k}^{-1} \lambda_{g,h,k} \lambda_{f,gh,k} \lambda_{f,g,h}. \quad (38)$$

Therefore, there is a canonical element in $H^4(G, \mathbf{k}^\times)$ (the class of ν) which depends only on c and the choice of M .

Definition 8.7. We will call this element the *associativity constraint obstruction class* and denote it $O_4(c, M)$.

Theorem 8.8. *Suppose that a homomorphism $c : G \rightarrow \text{BrPic}(\mathcal{D})$ is such that $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ (with $\mathcal{C}_e = \mathcal{D}$) admits a \mathcal{D} -bimodule quasi-tensor product via a choice of a system of products M . Then this product admits an associativity constraint satisfying the pentagon equation if and only if $O_4(c, M)$ is the trivial element of $H^4(G, \mathbf{k}^\times)$.*

Proof. Clear from the discussion above. □

Let $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a G -extension and let $\alpha_{X,Y,Z}$ be the associativity constraint for the tensor product of \mathcal{C} , where X, Y, Z are objects in \mathcal{C} . Given a 3-cocycle $\omega \in Z^3(G, \mathbf{k}^\times)$ one can define a new associator

$$\alpha'_{X,Y,Z} := \omega(f, g, h)\alpha_{X,Y,Z}$$

for all $X \in \mathcal{C}_f, Y \in \mathcal{C}_g, Z \in \mathcal{C}_h$.

Let $\alpha'_{X,Y,Z}$ be another associativity constraint for the tensor product of \mathcal{C} . We will say that α' is *equivalent* to α if $\alpha' = \alpha^\omega$ for some coboundary ω . Clearly, equivalent associators determine equivalent tensor categories.

Theorem 8.9. *Suppose that the obstruction classes $O_3(c)$ and $O_4(c, M)$ vanish. Then the equivalence classes of associativity constraints for the tensor product of $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ coming from the system of products M form a torsor T_c^2 over $H^3(G, \mathbf{k}^\times)$.*

Proof. The proof is similar to the proof of Theorem 8.5. We need to establish a bijection between the choices of $\alpha = \{\alpha_{f,g,h}\}_{f,g,h \in G}$ leading to associativity constraints on \mathcal{C} and elements of $H^3(G, \mathbf{k}^\times)$. If one such α is chosen, any other choice has a form (37). From equation (38) we see that the corresponding coboundary $\nu'_{f,g,h,k}$ is equal to 1 precisely when

$$\lambda_{f,g,h}\lambda_{f,g,h,k}\lambda_{g,h,k} = \lambda_{f,g,h,k}\lambda_{f,g,hk}, \quad f, g, h, k \in G,$$

i.e., when λ is a 3-cocycle. Moreover, two 3-cocycles are cohomologous if and only if the corresponding associators on \mathcal{C} are equivalent. □

8.7. The associativity constraint and the Pontryagin–Whitehead quadratic function. Let \mathcal{E} be a pointed braided fusion category, and let A be the group of isomorphism classes of invertible objects of \mathcal{E} . Let G be a finite group acting on

\mathcal{E} by braided auto-equivalences. In this situation we can define a quadratic map $\text{PW}: H^2(G, A) \rightarrow H^4(G, \mathbf{k}^\times)$, which we call the *Pontryagin–Whitehead quadratic function*, as follows.

Let

$$L: G \times G \rightarrow A: (f, g) \mapsto L_{f,g}$$

be a 2-cocycle of G with coefficients in A , i.e., a collection of simple objects of \mathcal{E} such that there exist isomorphisms

$$\zeta_{f,g,h}: L_{f,g,h} \otimes L_{f,g} \cong L_{f,gh} \otimes f(L_{g,h}), \quad f, g, h \in G. \quad (39)$$

Then we can consider the automorphism of $L_{fgh,k} \otimes L_{f,g,h} \otimes L_{f,g}$ (identified with a scalar) $\nu_{f,g,h,k}$, given by the composition

$$\begin{aligned} L_{fgh,k} \otimes L_{f,g,h} \otimes L_{f,g} &\rightarrow L_{fgh,k} \otimes L_{f,gh} \otimes f(L_{g,h}) \\ &\rightarrow L_{f,g,hk} \otimes f(L_{gh,k}) \otimes f(L_{g,h}) \rightarrow L_{f,g,hk} \otimes f(L_{g,hk}) \otimes fg(L_{h,k}) \\ &\rightarrow L_{f,g,hk} \otimes L_{f,g} \otimes fg(L_{h,k}) \rightarrow L_{f,g,hk} \otimes fg(L_{h,k}) \otimes L_{f,g} \\ &\rightarrow L_{fgh,k} \otimes L_{f,g,h} \otimes L_{f,g}, \end{aligned}$$

where we suppress the associativity isomorphisms, and all the maps except the fifth map are given by the isomorphisms $\zeta_{x,y,z}$ from (39) for appropriate x, y, z , while the fifth map is given by the braiding acting on $L_{f,g} \otimes fg(L_{h,k})$.

Proposition 8.10. (i) ν is a 4-cocycle of G with coefficients in \mathbf{k}^\times .

(ii) If ζ is changed by a cochain $\xi_{x,y,z}$, then ν is multiplied by $d\xi$ (so the cohomology class of ν does not change).

(iii) If $L_{f,g}$ is changed by a coboundary, i.e., replaced by

$$L'_{f,g} = X_{fg} \otimes L_{f,g} \otimes f(X_g^{-1}) \otimes X_f^{-1},$$

where (X_f) is a collection of simple objects, then ν is changed by a coboundary.

Proof. Straightforward verification. □

Definition 8.11. The map $\text{PW}: H^2(G, A) \rightarrow H^4(G, \mathbf{k}^\times)$ is defined by

$$\text{PW}(L) = \nu.$$

Note that if the action of G on \mathcal{E} is altered by an element $\theta \in H^2(G, A^*)$ (by changing the isomorphisms $\eta_{g,h}: F_g \circ F_h \xrightarrow{\sim} F_{gh}$), then the map PW is modified according to the rule

$$\text{PW}'(L) = \text{PW}(L)(L, \theta),$$

where $(,) : H^2(G, A) \times H^2(G, A^*) \rightarrow H^4(G, \mathbf{k}^\times)$ is the evaluation map combined with the cup product in the cohomology of G .

Let q be the quadratic form on A defined by the braiding on \mathcal{E} ($q(Z) = c_{ZZ}$), and b_q be the corresponding symmetric bilinear form ($b_q(Y, Z) = c_{YZ}c_{ZY}$). The following proposition shows that PW is indeed a quadratic function.

Proposition 8.12. $\text{PW}(L_1 L_2) = \text{PW}(L_1) \text{PW}(L_2) b_q(L_1, L_2)$.

Proof. This is verified by a direct computation from the definition, by using the hexagon relations for the braiding. \square

Now let us assume that $|A|$ is odd. Then $\mathcal{E} = \text{Vec}_A$, and the braiding on \mathcal{E} is canonically defined by the quadratic form q on A , which is, in turn, determined by the corresponding symmetric bilinear form b_q . Thus, every homomorphism $\phi: G \rightarrow O(A, q)$ canonically defines an action of G on \mathcal{E} . In this case, we can pick the associativity morphisms and the maps $\zeta_{x,y,z}$ to be the identities, and one gets

$$\forall f,g,h,k = c_{L_{f,g},fg}(L_{h,k}). \tag{40}$$

Proposition 8.13. *For the canonical action of G on \mathcal{E} , one has*

$$\text{PW}(L) = b_q(L^{1/2}, L)$$

(i.e., $\text{PW}(L)$ is b_q applied to the cup product of $L^{1/2}$ with L). Thus, for the canonical action shifted by $\theta \in H^2(G, A^*)$, one has

$$\text{PW}(L) = b_q(L^{1/2}, L)(L, \theta).$$

Proof. This follows from formula (40). \square

Remark 8.14. The map PW can be alternatively characterized as follows. Since G acts on \mathcal{E} by braided auto-equivalences, it acts canonically on the Drinfeld center $\mathcal{Z}(\mathcal{E})$. Note that $\mathcal{Z}(\mathcal{E})$ is a pointed category, and its group of simple objects is $A \oplus A^*$. Thus, an element $L \in H^2(G, A)$ is nothing but a way to alter the canonical action of G on $\mathcal{Z}(\mathcal{E})$ (keeping its action on isomorphism classes of objects fixed), so that its action on $\mathcal{E} \subset \mathcal{Z}(\mathcal{E})$ remains the same. By Theorem 1.1 and Theorem 1.3, having fixed L , we fix a collection of \mathcal{E} -bimodule categories $\mathcal{E}_g, g \in G$, with a tensor product functor on them. Then $\text{PW}(L)$ is nothing but the obstruction $O_4(c, L)$ to the existence of the associativity constraint for this tensor product functor.

Now let \mathcal{D} be any fusion category. Let $c: G \rightarrow \text{BrPic}(\mathcal{D})$ be a group homomorphism, $c(g) = \mathcal{C}_g$, and let $M = (M_{g,h})$ be a choice of isomorphisms $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \rightarrow \mathcal{C}_{gh}$ defining a tensor product functor on $\mathcal{C} = \bigoplus_g \mathcal{C}_g$. Then G acts on the braided category $\mathcal{Z}(\mathcal{D})$, in particular, on the subcategory of its invertible objects. Thus, we have the Pontryagin–Whitehead quadratic function

$$\text{PW}_M: H^2(G, \pi_2) \rightarrow H^4(G, \pi_3),$$

where $\pi_2 = \text{Inv}(\mathcal{Z}(\mathcal{D})), \pi_3 = \mathbf{k}^\times$.

Proposition 8.15. *For any $L \in H^2(G, \pi_2)$, one has*

$$O_4(c, LM)/O_4(c, M) = \text{PW}_M(L).$$

Thus, for $L_1, L_2 \in H^2(G, \pi_2)$, one has

$$\frac{O_4(c, L_1 L_2 M) O_4(c, M)}{O_4(c, L_1 M) O_4(c, L_2 M)} = [L_1, L_2],$$

where $[,]$ is the Whitehead bracket combined with the cup product in the cohomology of G .

Proof. The first statement follows by replacing M by LM in the commutative cube in Section 8.6 and applying the definition of PW_M . The second statement follows from the first one and Proposition 8.12. □

Remark. A version of the map PW is discussed in [2], Chapter V, and under additional assumptions the above results can be derived from the statements in [2].

8.8. A divisibility theorem. The following theorem is somewhat analogous to the Anderson–Moore–Vafa theorem for tensor categories (see [10]).

Theorem 8.16. *Let D be the Frobenius–Perron dimension of \mathcal{D} . Then the order of $O_4(c, M)$ in $H^4(G, \mathbf{k}^\times)$ divides D^4 .*

Proof. For $a \in G$, let $R_a = \bigoplus_{X \in \text{Irr} \mathcal{C}_a} \text{FPdim}(X) X$ be the regular (virtual) object of \mathcal{C}_a (where the Frobenius–Perron dimensions in \mathcal{C}_a are normalized in such a way that $\text{FPdim}(R_a) = D$). Let us apply equation (36) to the product $R_f \otimes R_g \otimes R_h \otimes R_k$ and compute the determinants of both sides (where the determinant is understood in the sense of [10], Section 2). Since ν is a scalar, on the left-hand side we get $\nu_{f,g,h,k}^{D^4}$. To compute the right-hand side, we use that $R_a \otimes R_b = DR_{ab}$. Then the right-hand side takes the form

$$\det(\alpha_{f,g,h,k})^{-D} \det(\alpha_{f,g,hk})^{-D} \det(\alpha_{g,h,k})^D \det(\alpha_{f,gh,k})^D \det(\alpha_{f,g,h})^D.$$

Thus we see that

$$\nu^{D^4} = d(\det(\alpha)^D),$$

where d is the differential in the standard complex of G with coefficients in \mathbf{k}^\times . This implies the statement. □

9. Examples of extensions

Throughout this section we freely use the notation and terminology from the previous sections.

9.1. Extensions of finite groups. Let G be a finite group. The problem of finding all G -extensions of a fusion category \mathcal{D} (i.e., G -graded fusion categories $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ with a prescribed identity component $\mathcal{C}_e = \mathcal{D}$) includes, as a special case, the classical theory of group extensions [9].

Indeed, let H be a finite group and let $\mathcal{D} = \text{Vec}_H$ be the fusion category of H -graded vector spaces with the trivial associator. For any automorphism $\alpha \in \text{Aut}(H)$ let \mathcal{M}_α be the \mathcal{D} -bimodule category which is \mathcal{D} as an abelian category, with the actions given by

$$\mathbf{k}_h \otimes \mathbf{k}_x = \mathbf{k}_{\alpha(h)x} \quad \text{and} \quad \mathbf{k}_x \otimes \mathbf{k}_h = \mathbf{k}_{xh}, \quad h, x \in H,$$

where $\mathbf{k}_h, h \in H$ are the simple objects of Vec_H and with the usual vector space associator. Note that \mathcal{M}_α is a typical example of an indecomposable \mathcal{D} -bimodule category which is equivalent to \mathcal{D} as a right \mathcal{D} -module category.

It is easy to check that \mathcal{M}_α is isomorphic to the regular \mathcal{D} -bimodule category if and only if α is an inner automorphism and that $\mathcal{M}_\alpha \boxtimes_{\mathcal{D}} \mathcal{M}_\beta \cong \mathcal{M}_{\alpha\beta}$ for all $\alpha, \beta \in \text{Aut}(H)$. In particular, each \mathcal{M}_α is an invertible \mathcal{D} -bimodule category.

Thus, in this case a homomorphism $c: G \rightarrow \text{BrPic}(\mathcal{D})$ with the property that each \mathcal{C}_g is equivalent to \mathcal{D} as a right \mathcal{D} -module category is the same thing as a homomorphism $c: G \rightarrow \text{Out}(H)$ to the quotient of $\text{Aut}(H)$ by the subgroup of inner automorphisms. For such a homomorphism choose a representative $\gamma_g \in \text{Aut}(H)$ from each coset $c(g)$ and let $\mathcal{C}_g = \mathcal{M}_{\gamma_g}$.

If there is a fusion category structure on $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ then this category is pointed and hence is equivalent to a category of K -graded vector spaces for some group K , possibly with a 3-cocycle ω . It is clear that this K is an extension of G by H , i.e., there is a short exact sequence of finite groups

$$1 \rightarrow H \rightarrow K \rightarrow G \rightarrow 1. \tag{41}$$

In this case the group Z is isomorphic to $Z(H) \oplus \text{Hom}(H, \mathbf{k}^\times)$, where $Z(H)$ is the center of H and $\text{Hom}(H, \mathbf{k}^\times)$ is the group of homomorphisms from H to \mathbf{k}^\times . Indeed, as we observed earlier, Z is isomorphic to the group of invertible objects of $\mathcal{Z}(\mathcal{D}) (= \mathcal{Z}(\text{Vec}_H))$.

One can easily check that the obstruction class $O_3(c)$ belongs to $H^3(G, Z(H)) \subset H^3(G, Z(H)) \oplus \text{Hom}(H, \mathbf{k}^\times)$ and coincides with the Eilenberg–Mac Lane obstruction to the existence of extension (41), with a given action $G \rightarrow \text{Out}(H)$; see [9]. When this obstruction vanishes, we have a choice of $M = (M_1, M_2)$ where M_1 belongs to a torsor T_1 over $H^2(G, Z(H))$, and M_2 belongs to a torsor T_2 over $H^2(G, \text{Hom}(H, \mathbf{k}^\times))$. One can check that the torsor T_1 is exactly the one classifying group extensions; see [9]. Furthermore, the torsor T_2 is canonically trivial, since every group extension canonically determines a categorical extension. Finally, it is easy to check that the obstruction $O_4(c, M_1, M_2)$ is linear in M_2 , and it follows from Proposition 8.15 that

$$\frac{O_4(c, LM_1, M_2)}{O_4(c, M_1, M_2)} = (L, M_2) \in H^4(G, \mathbf{k}^\times)$$

for any $L \in H^2(G, Z(H))$. Thus, our theory of categorical extensions reproduces the classical theory of group extensions.

Remark 9.1. If H is an abelian group, then it is clear that $O_3(c)$ vanishes and the torsor T_1 is canonically trivial ($T_1 = H^2(G, H)$). In this case, we have

$$O_4(c, M_1, M_2) = (M_1, M_2) \in H^4(G, \mathbf{k}^\times).$$

9.2. Invertible fiber functors and Tambara–Yamagami categories. Recall that a fiber functor on a tensor category \mathcal{C} is the same thing as a \mathcal{C} -module category structure on Vec .

Let G be a finite group and let $\mathcal{C} = \text{Vec}_G$ be the tensor category of G -graded vector spaces. We will describe all invertible \mathcal{C} -bimodule category structures on Vec . Let ϕ be a 2-cocycle on $G \times G^{\text{op}}$ and let \mathcal{M}_ϕ denote the Vec_G -bimodule category based on Vec with the action $(a, b) \otimes \mathbf{k} = \mathbf{k}$, where \mathbf{k} is a one-dimensional vector space, and an associativity constraint

$$\phi((a_1, a_2), (b_1, b_2))\text{id}_{\mathbf{k}} : ((a_1, a_2) \otimes (b_1, b_2)) \otimes \mathbf{k} \xrightarrow{\sim} (a_1, a_2) \otimes ((b_1, b_2) \otimes \mathbf{k}).$$

Every \mathcal{C} -bimodule category structure on Vec is equivalent to \mathcal{M}_ϕ .

Recall the Schur isomorphism, see [24], 2.2.10,

$$s : H^2(G \times G, \mathbf{k}^\times) \xrightarrow{\sim} H^2(G, \mathbf{k}^\times) \times H^2(G, \mathbf{k}^\times) \times (G_{\text{ab}} \otimes_{\mathbb{Z}} G_{\text{ab}})^*,$$

where $G_{\text{ab}} = G/G'$ is the abelianization of G . Note that $(G_{\text{ab}} \otimes_{\mathbb{Z}} G_{\text{ab}})^*$ is isomorphic to the group of bicharacters on G (or, equivalently, on G_{ab}).

Below we will abuse notation and identify cocycles with their cohomology classes. Let us write

$$s(\phi) = (\phi_1, \phi_2, \phi_{12}). \tag{42}$$

Here $\phi_1, \phi_2 \in H^2(G, \mathbf{k}^\times)$ define the left and right \mathcal{C} -module structures on \mathcal{M}_ϕ and $\phi_{12} \in (G_{\text{ab}} \otimes_{\mathbb{Z}} G_{\text{ab}})^*$ defines its \mathcal{C} -bimodule structure.

Remark 9.2. The category $(\mathcal{M}_\phi)^{\text{op}}$ is also a \mathcal{C} -bimodule category based on Vec . It is easy to check that $(\mathcal{M}_\phi)^{\text{op}} \cong \mathcal{M}_{\tilde{\phi}}$ where

$$\tilde{\phi}((a, a'), (b, b')) := \phi((a', a), (b', b))^{-1}.$$

Thus, $\tilde{\phi}_1 = \phi_2^{-1}$, $\tilde{\phi}_2 = \phi_1^{-1}$ and $\tilde{\phi}_{12}(a, b) = \phi_{12}(b, a)$, $a, b \in G_{\text{ab}}$.

Given two 2-cocycles ϕ, ϕ' on $G \times G$, the category $\text{Fun}_{\mathcal{C}}(\mathcal{M}_\phi, \mathcal{M}_{\phi'})$ is equivalent, as an abelian category, to the category $\text{Rep}_\mu(G)$ of projective representations of G with the Schur multiplier $\mu = \phi'_1/\phi_1$. This category is acted upon by $\text{Vec}_{G \times G^{\text{op}}}$ via

$$((a, b) \otimes \pi)(x) = \phi_{12}(x, a)\pi(x)\phi'_{12}(x, b),$$

where $a, b, x \in G, \pi \in \text{Rep}_\mu(G)$. The associativity constraint isomorphism between $((a, a') \otimes (b, b')) \otimes \pi$ and $(a, a') \otimes ((b, b') \otimes \pi)$ is given by $\phi_2(a, b)\phi'_2(b', a')$.

Proposition 9.3. (i) Let G be a finite group and let $\omega \in H^3(G, \mathbf{k}^\times)$. Let $\text{Vec}_{G,\omega}$ be the corresponding pointed fusion category. If Vec has a structure of an invertible $\text{Vec}_{G,\omega}$ -bimodule category then G is abelian and ω is cohomologically trivial.

(ii) Let G be abelian. Then \mathcal{M}_ϕ is an invertible Vec_G -bimodule category if and only if ϕ_{12} is a non-degenerate bicharacter on G .

(iii) The category \mathcal{M}_ϕ has order 2 in $\text{BrPic}(\text{Vec}_G)$ if and only if $\phi_1 = \phi_2^{-1}$ and ϕ_{12} is a symmetric non-degenerate bicharacter.

Proof. (i) Since $\text{Vec}_{G,\omega}$ has a fiber functor, ω must be trivial. By Proposition 4.2 the dual of Vec_G with respect to its module category Vec must be pointed, which forces G to be abelian.

(ii) The computations done before this Proposition show that $\text{Fun}_{\mathcal{C}}(\mathcal{M}_\phi, \mathcal{M}_\phi) \cong \mathcal{C}$ as a \mathcal{C} -bimodule category if and only if ϕ_{12} is non-degenerate (there are no conditions on ϕ_1, ϕ_2).

(iii) This is equivalent to existence of a \mathcal{C} -bimodule equivalence $\mathcal{M}_\phi^{\text{op}} \cong \mathcal{M}_\phi$, so we can apply Remark 9.2. □

Example 9.4. In [37] D. Tambara and S. Yamagami classified all $\mathbb{Z}/2\mathbb{Z}$ -graded fusion categories $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$ in which \mathcal{C}_+ is a pointed category and \mathcal{C}_- has a unique simple object. They showed that any such category is determined, up to a tensor equivalence, by a finite abelian group A , an isomorphism class of a non-degenerate symmetric bilinear form $\chi: A \times A \rightarrow \mathbf{k}^\times$, and a square root of $|A|$ in \mathbf{k} . The classification of [37] uses direct calculations of associativity constraints as solutions of a system of pentagon equations.

Let us derive this classification from our description of graded categories in Section 8 and Proposition 9.3. Let $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ be a fusion category with $\mathbb{Z}/2\mathbb{Z}$ -grading satisfying the above properties. Its trivial component \mathcal{C}_0 is a pointed fusion category. By Proposition 9.3 (i), $\mathcal{C}_0 \cong \text{Vec}_A$, for some finite abelian group A . The invertible Vec_A -bimodule category \mathcal{C}_1 has order 2 in $\text{BrPic}(\text{Vec}_A)$. By Proposition 9.3, $\mathcal{C}_1 \cong \mathcal{M}_\phi$ where ϕ is such that $\phi_1 = \phi_2^{-1}$ and ϕ_{12} is a non-degenerate symmetric bicharacter of A , cf. (42).

We have $Z := \text{Inv}(\mathcal{Z}(\text{Vec}_A)) = A \oplus A^*$. Let us identify A with A^* using the bicharacter ϕ_{12} ; then we have $Z = A \oplus A$, and as a $\mathbb{Z}/2\mathbb{Z}$ -module, $Z = \text{Fun}(\mathbb{Z}/2\mathbb{Z}, A)$. Therefore, by the Shapiro lemma, $H^i(\mathbb{Z}/2\mathbb{Z}, Z) = 0$ for $i > 0$. Thus, $O_3(c) = 0$, and there is no freedom in choosing M .

Observe that $H^4(\mathbb{Z}/2\mathbb{Z}, \mathbf{k}^\times) = 0$. Thus the associativity constraint obstruction O_4 vanishes and hence there are precisely two non-equivalent tensor category structures on \mathcal{C} corresponding to two elements of the group $H^3(\mathbb{Z}/2\mathbb{Z}, \mathbf{k}^\times) \cong \mathbb{Z}/2\mathbb{Z}$.

Let τ be a tensor auto-equivalence of Vec_A . Let \mathcal{C}_1^τ denote the Vec_A -bimodule category obtained from \mathcal{C}_1 by twisting the action of Vec_A by means of τ , i.e., by letting the result of action of $X \boxtimes Y \in \text{Vec}_A \boxtimes \text{Vec}_A^{\text{ev}}$ on $M \in \mathcal{M}$ to be $(\tau(X) \boxtimes \tau(Y)) \otimes M$. Clearly, we can replace \mathcal{C}_1 by \mathcal{C}_1^τ without changing the corresponding extension.

The group of tensor auto-equivalences of Vec_A is isomorphic to the semidirect product $H^2(A, \mathbf{k}^\times) \rtimes \text{Aut}(A)$. Choosing τ to be the element corresponding to (ϕ_1^{-1}, α) , where α is any automorphism of A , we see that ϕ can be chosen in such a way that $\phi_1 = 1$ and the choice of ϕ_{12} matters only up to an automorphism of A .

Thus, we obtain the same parameterization as in [37].

9.3. Categories \mathcal{C} graded by a group G of order coprime to $\text{FPdim}(\mathcal{C}_e)$. If $|G|$ and $D := \text{FPdim}(\mathcal{D})$ are coprime, the classification of extensions of \mathcal{D} by G simplifies, as the cohomological obstructions O_3 and O_4 automatically vanish. Namely, we have the following result.

Theorem 9.5. *Let \mathcal{D} be a fusion category of Frobenius–Perron dimension D relatively prime to $|G|$. Then any homomorphism $c : G \rightarrow \text{BrPic}(\mathcal{D})$ can be upgraded to a G -graded fusion category with trivial component $\mathcal{C}_e = \mathcal{D}$, and such categories are parametrized by a torsor $T_{c,M}^3$ over $H^3(G, \mathbf{k}^\times)$ (up to a grading-preserving equivalence).*

Proof. This follows from Theorem 1.3 and Theorem 8.16. Indeed, the order of the group $\pi_2 = \text{Inv}(\mathcal{Z}(\mathcal{D}))$ divides D^2 ([12], Proposition 8.15), so it is relatively prime to $|G|$. Thus, $H^i(G, \pi_2) = 0$, $i \geq 1$. So $O_3(c)$ vanishes, and there is no freedom in choosing M . Also, by Theorem 8.16, the obstruction $O_4(c, M)$ vanishes. So the graded category \mathcal{C} exists and the freedom in its construction is just the freedom of choosing α , which lies in a torsor over $H^3(G, \mathbf{k}^\times)$, as desired. \square

For applications of this theorem, see [22].

10. Lagrangian subgroups in metric groups and bimodule categories over Vec_A

Let Bimod_{ab} be the category whose objects are categories Vec_A where A is a finite abelian group, and morphisms from Vec_A to Vec_B are equivalence classes of (not necessarily invertible) $(\text{Vec}_B, \text{Vec}_A)$ -bimodule categories, with composition of morphisms being the tensor product of bimodule categories. The goal of this section is to describe this category explicitly.

First we need to set up some linear algebra, which is well known, but we work out the details for the reader's convenience.

10.1. The category of Lagrangian correspondences. Let us define the category Lag of Lagrangian correspondences. We define the objects of this category to be metric groups (E, q) . Morphisms from (E_1, q_1) to (E_2, q_2) are, by definition, formal \mathbb{Z}_+ -linear combinations of Lagrangian subgroups in $(E_1 \oplus E_2, q_1^{-1} \oplus q_2)$.

The composition of morphisms is defined as follows. For Lagrangian subgroups $L \in \text{Mor}((E_1, q_1), (E_2, q_2))$, $M \in \text{Mor}((E_2, q_2), (E_3, q_3))$ we define $M \circ L$ to be the set of all pairs $(a_1, a_3) \in E_1 \oplus E_3$ such that there exists $a_2 \in E_2$ for which

$(a_1, a_2) \in L$ and $(a_2, a_3) \in M$. Also, let $m(M, L)$ be the number of such a_2 . Then the composition of morphisms is defined by the condition that it is biadditive and

$$M \bullet L = m(M, L)M \circ L.$$

To validate this definition, we must prove the following lemma.

Lemma 10.1. (i) *The set $M \circ L$ is a Lagrangian subgroup of the metric group $(E_1 \oplus E_3, q_1^{-1} \oplus q_3)$.*

(ii) *The function m satisfies the 2-cocycle condition,*

$$m(N, M \circ L)m(M, L) = m(N \circ M, L)m(N, M),$$

so that the operation \bullet is associative.

Proof. (i) First of all, it is easy to check that $M \circ L$ is an isotropic subgroup. Next, $M \circ L$ is the quotient of the intersection of the subgroup $L \oplus M$ with the diagonal copy of $E_1 \oplus E_2 \oplus E_3$ in $E_1 \oplus E_2 \oplus E_2 \oplus E_3$ by the group $N = M \cap L \cap E_2$. It is easy to see that the image of $L \oplus M$ in $E_2 \oplus E_2/E_2^{\text{diag}} = E_2$ is the orthogonal complement N^\perp of N . Thus, the order of the intersection of $L \oplus M$ with $E_1 \oplus E_2 \oplus E_3$ is $|M| \cdot |L|/|N^\perp|$, and hence the order of $M \circ L$ is $|M| \cdot |L|/|E_2| = (|E_1| \cdot |E_3|)^{1/2}$, i.e., $M \circ L$ is Lagrangian.

(ii) This is a straightforward computation. □

Thus, we have defined the category Lag. Note that the identity morphism of (E, q) in this category is the diagonal subgroup of $E \oplus E$.

Proposition 10.2. *The groupoid of isomorphisms in Lag is naturally isomorphic to the groupoid of isometries of metric groups. In particular, the group of automorphisms of (E, q) in Lag is naturally isomorphic to $O(E, q)$.*

Proof. Let $(E, q), (E', q') \in \text{Lag}$. Let $L \subset E \oplus E'$ be Lagrangian under the form $q^{-1} \oplus q'$. If L defines an isomorphism then $L \circ M = \text{id}$ for some Lagrangian $M \subset E' \oplus E$, which implies that the intersection L with E' is zero. Similarly, the intersection of L with E is zero (because $M \circ L = \text{id}$). This means that L is the graph of some isomorphism of groups $g: E \rightarrow E'$, and since L is Lagrangian, this isomorphism is an isometry. Conversely, if $g: E \rightarrow E'$ is an isometry then the graph of g is Lagrangian in $E \oplus E'$. It is easy to see that the composition of Lagrangian subgroups goes under this identification to the composition of isometries. The proposition is proved. □

10.2. Subgroups with a skew-symmetric bicharacter in an abelian group. Let A be a finite abelian group. Denote by $C(A)$ the set of pairs (H, ψ) , where $H \subset A$ is a subgroup and ψ is a skew-symmetric bicharacter of H . Also, for a metric group (E, q) , let $\mathcal{L}(E, q)$ be the set of Lagrangian subgroups of E .

The following proposition is a special case of a more general result proved in [30].

Proposition 10.3. *There is a natural bijection*

$$\tau: C(A) \rightarrow \mathcal{L}(A \oplus A^*, q), \tag{43}$$

where q is the standard hyperbolic quadratic form of $A \oplus A^*$ given by $q(a, f) = f(a)$. This bijection is given by the formula $\tau(H, \psi) = \{(h, z) \mid z \in \psi(h)\}$, where $\psi(h) \in H^* = A^*/H_\perp$ is regarded as a coset of H_\perp in A^* .

Proof. It is clear that the subgroup $L = \{(h, z) \mid z \in \psi(h)\} \subset A \oplus A^*$ is isotropic. Also, $|L| = |H| \cdot |H_\perp| = |A|$, so L is Lagrangian. Thus the map τ is well defined. Now we prove that τ is invertible by constructing the inverse map. Namely, given a Lagrangian subgroup $L \subset A \oplus A^*$, set $\sigma(L) = (H, \psi)$, where H is the image of L in A and

$$\psi(h_1, h_2) := (h'_1, h_2),$$

with h'_1 any lifting of h_1 into L .

To prove that σ is well defined, we need to show that $\psi(h_1, h_2)$ is independent on the choice of the lifting h'_1 . In other words, we must show that if v is an element of $L \cap H^*$ then for any $h \in H$ we have $(v, h) = 1$. But this holds because $(v, h) = (v, h')$ for any lifting h' of h to L , and $(v, h') = 1$ since $v, h' \in L$ and b_q is the standard inner product on $A \oplus A^*$.

Now we should prove that $(\psi(h), h) = 1$, i.e., that $(h', h) = 1$ if $h \in H$ and h' is a lift of h in L . We have

$$(h', h) = q(h)q(h')/q(h' - h).$$

Now we see that all three factors on the right-hand side are equal to 1: the first one because $h \in A$, the third one because $h' - h \in A^*$, and the second one because $h' \in L$ and L is Lagrangian.

Finally, we should check that σ is indeed inverse to τ . We have $(\sigma \circ \tau)(H, \psi) = (H, \psi')$, where $\psi'(h_1, h_2) = (z_1, h_2)$ with $z_1 \in \psi(h_1)$. Thus $\psi' = \psi$, and we are done (since τ is a map of finite sets). \square

10.3. The structure of the category Bimod_{ab} . Now we will define a functor T from the category Bimod_{ab} to the full subcategory Lag_{hyp} of Lag , whose objects are groups of the form $A \oplus A^*$ with the hyperbolic quadratic form q . Namely, recall that if G is an abelian group, then equivalence classes of indecomposable left module categories over Vec_G are parametrized by the set $C(G)$ defined in the previous subsection. Now, for any indecomposable $(\text{Vec}_A, \text{Vec}_B)$ -bimodule category \mathcal{M} , regard \mathcal{M} as a $\text{Vec}_{A \oplus B}$ -module category via $(a, b) \otimes M = a \otimes M \otimes b^{-1}$ and consider its equivalence class $[\mathcal{M}] \in C(A \oplus B)$. Put

$$T(\mathcal{M}) := \gamma\tau([\mathcal{M}]) \in \mathcal{L}(A \oplus A^* \oplus B \oplus B^*, q_A \oplus q_B^{-1}),$$

where $\gamma \in \text{Aut}(A \oplus A^* \oplus B \oplus B^*)$ is defined by the formula $\gamma(a, a^*, b, b^*) = (a, a^*, -b, b^*)$ and τ is defined in (43). Extend T to decomposable module categories by additivity.

Theorem 10.4. *The assignment T is a functor, i.e., for any $(\text{Vec}_{A_1}, \text{Vec}_{A_2})$ -bimodule category \mathcal{N} and $(\text{Vec}_{A_2}, \text{Vec}_{A_3})$ -bimodule category \mathcal{N}' one has*

$$T(\mathcal{N} \boxtimes_{\text{Vec}_{A_2}} \mathcal{N}') = T(\mathcal{N}) \circ T(\mathcal{N}').$$

Proof. Let A_1, A_2, A_3 be abelian groups and let $(H, \psi) \in C(A_1 \oplus A_2), (H', \psi') \in C(A_2 \oplus A_3)$. We would like to find (H'', ψ'') such that

$$\gamma\tau(H, \psi) \bullet \gamma\tau(H', \psi') = m \cdot \gamma\tau(H'', \psi'')$$

and compute the value of m .

By the definition of τ , the subgroup $L := \gamma\tau(H, \psi) \subset A_1 \oplus A_1^* \oplus A_2 \oplus A_2^*$ is the set of all (a_1, f_1, a_2, f_2) such that $(a_1, -a_2) \in H$ and $(f_1, f_2) - \widehat{\psi}(a_1, -a_2) \in H_\perp$. Similarly, the subgroup $L' := \gamma\tau(H', \psi') \subset A_2 \oplus A_2^* \oplus A_3 \oplus A_3^*$ is the set of all (a_2, f_2, a_3, f_3) such that $(a_2, -a_3) \in H'$ and $(f_2, f_3) - \widehat{\psi}'(a_2, -a_3) \in H'_\perp$.

Now, $L \circ L' = m \cdot L''$, where L'' is the set of all (a_1, f_1, a_3, f_3) such that there exist a_2, f_2 with $(a_1, -a_2) \in H, (f_1, f_2) - \widehat{\psi}(a_1, -a_2) \in H_\perp, (a_2, -a_3) \in H'$ and $(f_2, f_3) - \widehat{\psi}'(a_2, -a_3) \in H'_\perp$. Moreover, m is the number of pairs (a_2, f_2) satisfying these conditions.

Let $L'' = \gamma\tau(H'', \psi'')$. It can be checked directly from the above conditions that H'', ψ'' are the same as in Proposition 3.19. Moreover, the number m is the number of pairs (a_2, f_2) , so we have

$$m = |\text{Ker}((H \cap H')^\perp \rightarrow A_1 \oplus A_3)| \cdot |H_\perp \cap H'_\perp|$$

(the first factor represents the number of choices of a_2 and the second one stands for the number of choices of f_2). Thus,

$$m = \frac{|(H \cap H')^\perp|}{|H''|} \cdot |H_\perp \cap H'_\perp|.$$

But $H'' = H \circ H' / (H \cap H')$, so we get

$$m = \frac{|(H \cap H')^\perp| \cdot |H \cap H'|}{|H \circ H'|} \cdot |H_\perp \cap H'_\perp|,$$

which coincides with the second formula for m in Proposition 3.19. The theorem is proved. □

Corollary 10.5. *T is an equivalence of categories $\text{Bimod}_{\text{ab}} \rightarrow \text{Lag}_{\text{hyp}}$.*

Proof. This follows from Theorem 10.4 and Proposition 10.3. □

Remark 10.6. Note that we have obtained another (direct) proof of Corollary 1.2, which does not use Theorem 1.1. (Namely, Corollary 1.2 follows from Theorem 10.4 and Proposition 10.2.) One can check that the two proofs provide the same isomorphism

$$\text{BrPic}(\text{Vec}_A) \cong \text{O}(A \oplus A^*).$$

Remark 10.7. The isomorphism of Corollary 1.2 can be understood in topological terms as follows. Recall that

$$\pi_2(\underline{BBrPic}(\text{Vec}_A)) = A \oplus A^*, \quad \pi_3(\underline{BBrPic}(\text{Vec}_A)) = \mathbf{k}^\times,$$

and by Proposition 7.6 the Whitehead half-square $\pi_2 \rightarrow \pi_3$ is the hyperbolic quadratic form q on $A \oplus A^*$. Thus, the action of π_1 on π_2 must preserve this form, i.e., we have a homomorphism

$$\eta: \text{BrPic}(\text{Vec}_A) \rightarrow \text{O}(A \oplus A^*).$$

One can show that this η coincides with the isomorphism of Corollary 1.2, i.e., with the restriction of T to invertible Vec_A -bimodule categories.

10.4. The number of simple objects in an invertible bimodule category over Vec_A

Proposition 10.8. *Let $g \in \text{O}(A \oplus A^*)$ and let \mathcal{C}_g be the corresponding invertible bimodule category. Let P be the projection $A \oplus A^* \rightarrow A$ and K be the kernel of $P \circ g|_{A^*}$. Then the number of isomorphism classes of simple objects of \mathcal{C}_g equals $|K|$.*

Proof. The Lagrangian subspace L in $A \oplus A^* \oplus A \oplus A^*$ corresponding to g is the set of $(a, f, g(a, f))$, where $a \in A$, $f \in A^*$. The corresponding subgroup H in $A \oplus A$ (such that $\mathcal{C}_g = \mathcal{M}(H, \psi)$ for some ψ) is the projection of L to $A \oplus A$. Thus, H projects onto A (via the first coordinate), and the kernel is the set of possible first coordinates of $g(a, f)$, $f \in A^*$, i.e., the image of $P \circ g|_{A^*}$. Thus, $|H| = |A|/|K|$, and we are done. \square

10.5. Integral Vec_A -bimodule categories. Recall that for an integral fusion category \mathcal{C} we defined in Section 4.2 the categorical 2-subgroup $\underline{\text{BrPic}}_+(\mathcal{C}) \subset \underline{\text{BrPic}}(\mathcal{C})$ consisting of integral invertible \mathcal{C} -bimodule categories.

Proposition 10.9. *If A is an abelian group then $\text{BrPic}_+(\text{Vec}_A) = \text{SO}(A \oplus A^*)$.*

Proof. This follows easily from Corollary 1.2. Namely, by Proposition 4.10, we may assume without loss of generality that A is a p -group for some prime p . In this case, the dimensions of simple objects in a bimodule category are either an integer or half-integer powers of p .

Let $\mathcal{C} = \text{Vec}_A$. For $g \in \text{BrPic}(\mathcal{C}) = \text{O}(A \oplus A^*)$, let $P: A \oplus A^* \rightarrow A$ be the projection, K be the kernel of $P \circ g|_{A^*}$, and I be the image of $P \circ g|_{A^*}$. Then by Proposition 10.8, the dimensions of simple objects of \mathcal{C}_g are $(|A|/|K|)^{1/2} = |I|^{1/2}$ (as $I = A^*/K$). This is an integer if and only if $|I| = p^n$, where n is even, i.e., if and only if $d(A^*, g(A^*)) = 1$, which implies the statement by Proposition 2.8. \square

11. Appendix by Ehud Meir: Group extensions as G -graded fusion categories

11.1. Introduction. In this appendix we will discuss a special class of extensions of a fusion category by a finite group. Let Γ be a finite group which fits into a short exact sequence of groups $1 \rightarrow N \rightarrow \Gamma \rightarrow G \rightarrow 1$. Suppose that we have a 3-cocycle $\omega \in H^3(\Gamma, \mathbf{k}^\times)$ and the corresponding fusion category $\mathcal{C} = \text{Vec}_{\Gamma, \omega}$. This category has a subcategory $\mathcal{D} = \text{Vec}_{N, \omega}$ (where by ω we also mean the restriction of ω to N), and \mathcal{C} is a G -extension of \mathcal{D} . It is possible to classify directly extensions of \mathcal{D} by G which are also pointed; one needs to give an extension Γ of G by N , and to give an extension of the cocycle ω on N to a cocycle on Γ . We will explain here why this solution and the solution given by the theory of G -extensions developed in the article are equivalent. We will do so in the following way: we take a parameterization (c, M, α) of a pointed G -graded extension of \mathcal{D} , as in Theorem 1.3, and we explain why this parameterization is equivalent to giving an extension Γ of G by N and an extension of ω to a cocycle on Γ . In order to do so we first study the groups $\text{Aut}_{\otimes}(\mathcal{D})$ and $\text{Out}_{\otimes}(\mathcal{D})$ of tensor auto-equivalences and outer tensor auto-equivalences of \mathcal{D} , respectively, since these two groups will play a decisive role in understanding the triple (c, M, α) . We then describe the group $T = \text{Inv}(\mathcal{Z}(\mathcal{D}))$ of invertible objects of the center in order to understand the obstruction $O_3(c)$ which lies in $H^3(G, T)$. Using this, we explain how to “translate” a triple (c, M, α) to an extension Γ of G by N together with a 3-cocycle on Γ which is an extension of ω . If H is any finite group and $\omega \in H^3(H, \mathbf{k}^\times)$, we denote the simple objects of $\text{Vec}_{H, \omega}$ by $\{V_h\}_{h \in H}$.

11.2. The groups $\text{Aut}_{\otimes}(\mathcal{D})$ and $\text{Out}_{\otimes}(\mathcal{D})$. Let $\Phi \in \text{Aut}_{\otimes}(\mathcal{D})$. By considering the way in which Φ acts on simple objects of \mathcal{D} (which correspond to elements of N) we get an automorphism ϕ of N . The additional data which we need in order to turn Φ into a tensor auto-equivalence of \mathcal{D} is an isomorphism, for every $a, b \in N$,

$$\Phi(V_{\phi^{-1}(a)}) \otimes \Phi(V_{\phi^{-1}(b)}) \rightarrow \Phi(V_{\phi^{-1}(a)} \otimes V_{\phi^{-1}(b)}).$$

This isomorphism is given by a scalar which we denote $\gamma_{\Phi}(a, b)$. It is easy to see that the equation that γ_{Φ} should satisfy is

$$\partial \gamma_{\Phi}(a, b, c) = \omega(\phi^{-1}(a), \phi^{-1}(b), \phi^{-1}(c)) \omega^{-1}(a, b, c) = \phi \cdot \omega / \omega.$$

In other words, in order for ϕ to furnish a tensor auto-equivalence, it is necessary and sufficient that $\phi \cdot \omega = \omega$ in $H^3(N, \mathbf{k}^\times)$. We denote the subgroup of all such automorphisms by $\text{Aut}(N, \omega)$. We thus have an onto map $\pi: \text{Aut}_{\otimes}(\mathcal{D}) \twoheadrightarrow \text{Aut}(N, \omega)$. A direct calculation shows that the kernel of this map is $H^2(N, \mathbf{k}^\times)$. We thus have a short exact sequence

$$1 \rightarrow H^2(N, \mathbf{k}^\times) \rightarrow \text{Aut}_{\otimes}(\mathcal{D}) \rightarrow \text{Aut}(N, \omega) \rightarrow 1.$$

Notice that in the case $\omega \neq 1$ this sequence does not necessarily split.

For every $n \in N$ we have an auto-equivalence C_n of conjugation by V_n . This is the auto-equivalence which sends the object V_a to $(V_n \otimes V_a) \otimes V_{n^{-1}}$, and the tensor structure is defined in the obvious way. Notice that in particular this gives us a canonical 2-cochain t_n such that $\partial t_n = \omega(n^{-1} ? n) / \omega(?)$. As expected, this defines a homomorphism of groups $\text{Con}: N \rightarrow \text{Aut}_{\otimes}(\mathcal{D})$. The image of Con is a normal subgroup, and we denote the quotient of $\text{Aut}_{\otimes}(\mathcal{D})$ by $\text{im}(\text{Con})$ by $\text{Out}_{\otimes}(\mathcal{D})$.

11.3. The group $\text{Inv}(\mathcal{Z}(\mathcal{D}))$. We now describe the group $T = \text{Inv}(\mathcal{Z}(\mathcal{D}))$. This is a special case of Theorem 5.2 of [17], where the group of invertible objects of a general group-theoretical category was described. An invertible object of $\mathcal{Z}(\mathcal{D})$ would be an invertible object of \mathcal{D} (that is V_z for some $z \in N$) such that for every $a \in N$ we have an isomorphism $V_z \otimes V_a \rightarrow V_a \otimes V_z$ (and thus, $z \in Z(N)$, the center of N). The element z should satisfy however another condition. The map $V_z \otimes V_a \rightarrow V_a \otimes V_z$ (if it exists) is just multiplication by a scalar. Denote this scalar by $r(a)$. Then a direct calculation shows that the set of scalars $r(a)$ define on V_z a structure of a central object if and only if the equation

$$r(a)r(b)r(ab)^{-1} = \omega(z, a, b)\omega(a, b, z)\omega^{-1}(a, z, b)$$

holds. We have the following fact, which can be easily proved directly.

Fact 11.1. *For every $z \in Z(N)$, the function*

$$c_z(a, b) = \omega(z, a, b)\omega(a, b, z)\omega^{-1}(a, z, b)$$

is a 2-cocycle of N with values in \mathbf{k}^{\times} . The conjugation map $\text{Con}: N \rightarrow \text{Aut}_{\otimes}(\mathcal{D})$ maps $Z(N)$ to $H^2(N, \mathbf{k}^{\times})$ via $z \mapsto c_z$.

So conjugation by V_z , where $z \in Z(N)$, is not necessarily the trivial auto-equivalence of \mathcal{D} . It is the auto-equivalence given by the 2-cocycle c_z . An object V_z , $z \in Z(N)$, has a structure of a central object if and only if conjugation by V_z is trivial, that is, if and only if c_z is the trivial cocycle. We denote the kernel of $z \mapsto c_z$ by $Z(N, \omega)$ (so this is also the kernel of $N \rightarrow \text{Aut}_{\otimes}(\mathcal{D})$). Thus, we have an onto map $T \twoheadrightarrow Z(N, \omega)$. What is its kernel? To give V_1 a structure of an object of $\mathcal{Z}(\mathcal{D})$ is the same thing as to give a function $r: N \rightarrow \mathbf{k}^{\times}$ which satisfies $r(a)r(b) = r(ab)$, i.e., a 1-cocycle. Since 1-coboundaries are trivial, we can describe T as an extension of the form

$$1 \rightarrow H^1(N, \mathbf{k}^{\times}) \rightarrow T \rightarrow Z(N, \omega) \rightarrow 1.$$

In case $\omega \neq 1$, this sequence does not necessarily split.

The group $\text{Aut}_{\otimes}(\mathcal{D})$ acts naturally on T . As objects of T are central in \mathcal{D} , it is easy to see that inner automorphisms would act trivially on T . We therefore have an induced action of $\text{Out}_{\otimes}(\mathcal{D})$ on T .

11.4. The homomorphism c . If \mathcal{C} is a pointed extension of \mathcal{D} , it is easy to see that, for every $g \in G$, the bimodule category \mathcal{D}_g is a quasi-trivial bimodule (as defined in Section 4.3). It follows that there are auto-equivalences $\Phi(g) \in \text{Aut}_{\otimes}(\mathcal{D})$ for $g \in G$ such that $\mathcal{D}_g \cong \mathcal{D}^{\Phi(g)}$, that is, \mathcal{D}_g is the same category as \mathcal{D} , but the action of $\mathcal{D} \boxtimes \mathcal{D}^{op}$ is given by

$$(V_a \boxtimes V_c) \otimes V_b = (V_a \otimes V_b) \otimes \Phi(g)(V_c).$$

It can easily be seen that the bimodule category \mathcal{D}_g defines the auto-equivalence $\Phi(g)$ only up to conjugation by an invertible object of \mathcal{D} . So $\text{Out}_{\otimes}(\mathcal{D})$ is a subgroup of $\text{BrPic}(\mathcal{D})$, and the image of $c: G \rightarrow \text{BrPic}(\mathcal{D})$ lies inside $\text{Out}_{\otimes}(\mathcal{D})$. For each $g \in G$, choose an auto-equivalence $\Phi(g)$ of \mathcal{D} whose image in $\text{Out}_{\otimes}(\mathcal{D})$ is $c(g)$. We thus have an isomorphism of functors

$$p_{g,h}: \Phi(g)\Phi(h) \xrightarrow{\sim} C_{n(g,h)}\Phi(gh), \tag{44}$$

where $n(g, h) \in N$. Notice that we need to make a choice here as $n(g, h)$ is defined only up to a coset of $Z(N, \omega)$ in N . We can think of the morphism $p_{g,h}$ as a 1-cochain which satisfies a certain boundary condition. We also make a choice in choosing the $p_{g,h}$'s. As explained above, we think of $\Phi(g)$ as an automorphism $\phi(g)$ of N together with a 2-cochain γ_g on N which satisfies

$$\partial\gamma_g = \phi(g) \cdot \omega / \omega. \tag{45}$$

Equation (44) simply means that we have the equality $\phi(g)\phi(h) = c_{n(g,h)}\phi(gh)$ of automorphisms of N , where c_n means the automorphism of conjugation by n , and also that the 2-cocycle

$$U_{g,h} = \gamma_g(\phi(g) \cdot \gamma_h)t_{n(g,h)}^{-1}(c_n \cdot \gamma_{gh}^{-1})$$

where t_n was described above, is trivial and is equal to $\partial p_{g,h}$ (this is the boundary condition that $p_{g,h}$ should satisfy in order to be an isomorphism between the functors described above).

11.5. The first obstruction. We now explain what the first obstruction $O_3(c)$ looks like in our context. Recall that $O_3(c)$ is an element of $H^3(G, T)$. Assume that g, h, k are elements of G . Let us describe $O_3(c)(g, h, k) \in T$. In order to do so we need to choose equivalences of \mathcal{D} -bimodule categories $\mathcal{D}_g \boxtimes_{\mathcal{D}} \mathcal{D}_h \cong \mathcal{D}_{gh}$ for every $g, h \in G$. By the universal property of tensor product of bimodule categories, this is the same as to give a balanced \mathcal{D} -bimodule functor $F_{g,h}: \mathcal{D}_g \boxtimes \mathcal{D}_h \rightarrow \mathcal{D}_{gh}$ for every $g, h \in G$ such that the universal property from Definition 3.3 holds. We choose

$$F_{g,h}(V_a \boxtimes V_b) = (V_a \otimes \Phi(g)(V_b)) \otimes V_{n(g,h)}.$$

The isomorphism $p_{g,h}$ of auto-equivalences of \mathcal{D} given by equation (44) equips the functor $F_{g,h}$ with a structure of a balanced \mathcal{D} -bimodule functor. The idea now is

that we have two functors from $\mathcal{D}_g \boxtimes \mathcal{D}_h \boxtimes \mathcal{D}_k$ into \mathcal{D}_{ghk} , namely $F_{g,hk}F_{h,k}$ and $F_{gh,k}F_{g,h}$. As both functors can be used to identify \mathcal{D}_{ghk} with $\mathcal{D}_g \boxtimes_{\mathcal{D}} \mathcal{D}_h \boxtimes_{\mathcal{D}} \mathcal{D}_k$, there is an equivalence of \mathcal{D} -bimodule categories $y_{g,h,k}: \mathcal{D}_{ghk} \rightarrow \mathcal{D}_{ghk}$ such that $F_{g,hk}F_{h,k} \cong y_{g,h,k}F_{gh,k}F_{g,h}$. Since \mathcal{D} -bimodule equivalences of \mathcal{D}_{ghk} correspond to elements of T as explained in Section 8.4, this $y_{g,h,k}$ corresponds to $O_3(c)(g, h, k)$. Using these considerations, a more explicit description of $O_3(c)$ is obtained in the following way: for $g, h, k \in G$, the isomorphisms of functors given in equation (44) provide us the following isomorphism of functors:

$$\begin{aligned} & C_{n(g,h)n(gh,k)n(g,hk)^{-1}\phi(g)(n(h,k)^{-1})} \\ & \cong C_{n(g,h)}C_{n(gh,k)}C_{n(g,hk)}^{-1}\Phi(g)C_{n(h,k)}^{-1}\Phi(g)^{-1} \\ & \cong \Phi(g)\Phi(h)\Phi(gh)^{-1}\Phi(gh)\Phi(k)\Phi(ghk)^{-1}\Phi(ghk)\Phi(hk)^{-1}\Phi(g)^{-1} \\ & \quad \Phi(g)\Phi(hk)\Phi(k)^{-1}\Phi(h)^{-1}\Phi(g)^{-1} \\ & \cong \text{Id}. \end{aligned}$$

To give an isomorphism of functors $C_n \cong \text{Id}$ is the same as to give a structure of a central object on V_n . This (invertible) central object $V_{n(g,h)n(gh,k)n(g,hk)^{-1}\phi(g)(n(h,k)^{-1})}$ would be $O_3(c)(g, h, k)$. Notice that choosing different isomorphisms $p_{g,h}$ or different coset representatives $n(g, h)$ changes $O_3(c)$ only by a coboundary and thus will give an equivalent cocycle. We will be interested in the case in which $O_3(c)$ vanishes in $H^3(G, T)$. If it vanishes, we call an element $\rho \in C^2(G, T)$ which satisfies $\partial\rho = O_3(c)$ a *solution* for $O_3(c)$ (and use similar terminology for other obstructions). The choice of a solution in this case therefore corresponds to the choice of a system of products. In our case this is equivalent to choosing the elements $n(g, h)$ and the morphisms $p_{g,h}$ in such a way that the obstruction we get is the trivial 3-cocycle. This means that the equation $n(g, h)n(gh, k) = \phi(g)(n(h, k))n(g, hk)$ holds in N (and not only up to a coset of $Z(N, \omega)$), and also that the functions $p_{g,h}$ satisfy a certain boundary condition which we will consider later.

11.6. Vanishing of the first obstruction, the Eilenberg–Mac Lane obstruction, and the choice of a solution. The description of T as an extension of $Z(N, \omega)$ by $H^1(N, \mathbf{k}^\times)$ will help us understand the vanishing of $O_3(c)$ in two steps. Assume first that we know that the image of $O_3(c)$ in $H^3(G, Z(N, \omega))$ (which is denoted by $\overline{O_3(c)}$) vanishes. This means that we can change the elements $n(g, h)$ by elements of $Z(N, \omega)$ (that is, to take different coset representatives) in such a way that the equation

$$n(g, h)n(gh, k) = \phi(g)(n(h, k))n(g, hk) \tag{46}$$

holds in N . A solution to $\overline{O_3(c)}$ in $H^3(G, Z(N, \omega))$ would therefore be a choice of coset representatives $n(g, h)$ which satisfy equation (46). This would give us a group extension

$$1 \rightarrow N \rightarrow \Gamma \rightarrow G \rightarrow 1$$

We think of elements of Γ as products of the form $n\bar{g}$ where $n \in N$ and $g \in G$. The product of two such elements would be $n\bar{g}m\bar{h} = n\phi(g)(m)n(g, h)\bar{g}\bar{h}$. Equation (46) is thus equivalent to the associativity of Γ . It can be checked that the image of $O_3(c)$ in $H^3(G, Z(N))$ coincides with the Eilenberg–Mac Lane obstruction for the existence of a group extension of G by N with the given “outer” action $\bar{c}: G \rightarrow \text{Out}_{\otimes}(\mathcal{D}) \rightarrow \text{Out}(N)$. See [26] for a description of this obstruction. Suppose that we have chosen a solution ρ for $\widehat{O_3(c)}$ (and therefore we get a group extension Γ of G by N). Lift ρ to a 2-cochain $\tilde{\rho}$ of G with values in T . The cocycle $O_3(c)\partial\tilde{\rho}^{-1}$ has all its values in the subgroup $H^1(N, \mathbf{k}^\times)$. It is easy to see that the class of this cocycle in $H^3(G, H^1(N, \mathbf{k}^\times))$ is well defined and does not depend on the choice of the particular lifting but only on the choice of the solution ρ . We denote this cocycle by $\widehat{O_3(c)}_\rho$. It is easy to see that the vanishing of $O_3(c)$ is equivalent to the fact that $\widehat{O_3(c)}$ vanishes, and that we can find for it a solution ρ such that $\widehat{O_3(c)}_\rho$ vanishes as well. A solution for $O_3(c)$ will then be of the form $\tilde{\rho}\mu$, where μ is a solution for $\widehat{O_3(c)}_\rho$. So the situation we will consider from now on is the following: we have a group extension Γ of G by N , and we also have $\widehat{O_3(c)}_\rho$, the “remainder” of the first obstruction $O_3(c)$. We next describe the second obstruction $O_4(c, M)$ and see how all this data corresponds to data from the spectral sequence of the group extension.

11.7. The second obstruction. Let us now describe the second obstruction $O_4(c, M)$ (we assume that we have a solution μ for $\widehat{O_3(c)}_\rho$). We “almost have” the extension $\bar{\omega}$ of ω to Γ in the following sense: if we knew $\bar{\omega}(\bar{g}, \bar{h}, \bar{k})$ for every $g, h, k \in G$, we would have known $\bar{\omega}(a, b, c)$ for any $a, b, c \in \Gamma$. This is because the system of products enables us to express any value of $\bar{\omega}$ solely in terms of the values $\bar{\omega}(\bar{g}, \bar{h}, \bar{k})$ for $g, h, k \in G$. So choose such values arbitrarily, for example, $\bar{\omega}(\bar{g}, \bar{h}, \bar{k}) = 1$ for every $g, h, k \in G$. Now consider the hexagon diagram for $\bar{g}, \bar{h}, \bar{k}, \bar{l}$, where $g, h, k, l \in G$. It will be commutative up to a scalar, which will be $O_4(c, M)(g, h, k, l)$. A choice of different arbitrary values would give us a cohomologous cocycle. A solution for $O_4(c, M)$ means a collection of values $\bar{\omega}(\bar{g}, \bar{h}, \bar{k})$ which will make $\bar{\omega}$ a 3-cocycle on Γ . By choosing a different solution, we will get another extension of ω to Γ , which differs by a pullback to Γ of a class $\alpha \in H^3(G, \mathbf{k}^\times)$. In the context of the data (c, M, α) , we assume that we have one fixed solution η for $O_4(c, M)$, and that we take the solution $\eta\alpha$, where $\alpha \in H^3(G, \mathbf{k}^\times)$.

11.8. The Lyndon–Hochschild–Serre spectral sequence. We have already seen that the data (c, M, α) yields an extension Γ of G by N . We now explain how it determines the extension $\bar{\omega}$ of ω from N to Γ . In order to do so we will use the Lyndon–Hochschild–Serre (abbreviated LHS) spectral sequence

$$E_2^{p,q} = H^p(G, H^q(N, \mathbf{k}^\times)) \Rightarrow E_\infty^{p,q} = H^{p+q}(\Gamma, \mathbf{k}^\times)$$

A general discussion of this spectral sequence can be found in [34] and in [26]. We will use this spectral sequence to understand how the vanishing of the obstructions and

the mere existence of $c : G \rightarrow \text{Out}_{\otimes}(\mathcal{D})$ imply that ω can be extended to a 3-cocycle on Γ , and how the choices of c , M and α give us a specific extension of ω to Γ . The idea is the following: we consider ω as an element of $E_2^{0,3} = H^0(G, H^3(N, \mathbf{k}^\times))$. Using the theory of spectral sequences, we know that ω is extendable to Γ if and only if $d_2(\omega) = 0$ in $E_2^{2,2}$, $d_3(\omega) = 0$ in $E_3^{3,1}$ and $d_4(\omega) = 0$ in $E_4^{4,0}$. If this is true, the theory of spectral sequences also gives us all possible extensions of ω to Γ . They are parameterized in the following way: if $d_2(\omega) = 0$, this means that a certain equation has a “solution” (we will soon see this explicitly). We need to choose a solution γ , and then ω and γ together will define an element $i^{\gamma,\omega} \in E_2^{3,1} = H^3(G, H^1(N, \mathbf{k}^\times))$. This element $i^{\gamma,\omega}$ is in the kernel of d_2 , and the image of $i^{\gamma,\omega}$ in $E_3^{3,1}$ is $d_3(\omega)$ (recall that E_3 is the cohomology of (E_2, d_2)). The fact that $d_3(\omega) = 0$ means that the cocycle $i^{\gamma,\omega}$ is trivial for some of the solutions γ . We need to choose only such γ 's. Again, the fact that $i^{\gamma,\omega} = 0$ means that some equation has a solution, and we need once again to choose such a solution, which we will denote by p . Exactly like at the previous step, ω , γ and p define an element $j^{\omega,\gamma,p} \in E_2^{4,0} = H^4(G, H^0(N, \mathbf{k}^\times)) = H^4(G, \mathbf{k}^\times)$. The cohomology class $j^{\omega,\gamma,p}$ is obviously in the kernel of d_2 and d_3 , as they are trivial on $E_2^{4,0}$ and $E_3^{4,0}$, respectively. The image of $j^{\omega,\gamma,p}$ in $E_4^{4,0}$ is exactly $d_4(\omega)$. The fact that $d_4(\omega) = 0$ is equivalent to the fact that we can choose γ and p such that $j^{\omega,\gamma,p} = 0$ in $H^4(G, \mathbf{k}^\times)$, and we will choose only such γ 's and p 's. The construction of $j^{\omega,\gamma,p}$ gives it as a cocycle rather than just as a cohomology class. Therefore we also need a 3-cochain $\beta \in C^3(G, \mathbf{k}^\times)$ which satisfies $\partial\beta = j^{\omega,\gamma,p}$, that is, we need a solution to this equation. The tuple (γ, p, β) will give us, by the theory of spectral sequences, the desired extension of ω to Γ . We will now explain the connection between the tuple (γ, p, β) and the data (c, M, α) . We do so by considering the different pages of the spectral sequence.

11.8.1. The first differential in page E_2 . Let us describe $d_2(\omega)$. The cocycle ω is G -invariant, and therefore for every $g \in G$ we can find a 2-cochain γ_g such that $\partial\gamma_g = \phi(g) \cdot \omega/\omega$. Let $g, h \in G$. Since $\phi(g)\phi(h) = c_{n(g,h)}\phi(gh)$, we have a two cocycle on N with values in \mathbf{k}^\times ,

$$U_{g,h}^\gamma = \frac{\gamma_g g \cdot \gamma_h}{t_{n(g,h)} c_n \cdot \gamma_{gh}}.$$

The function which takes (g, h) , for $g, h \in G$, to the cocycle $U_{g,h}^\gamma$ is a 2-cocycle of G with values in $H^2(N, \mathbf{k}^\times)$. Different choices of γ_g 's give us cohomologous cocycles. The cocycle $U_{g,h}^\gamma$ is trivial if and only if there is a choice of γ_g 's for which $U_{g,h}^\gamma$ is a coboundary for every $g, h \in G$. A direct calculation shows that $d_2(\omega) = U_{g,h}^\gamma$. We claim that the existence of Φ implies that $d_2(\omega)$ is trivial. This is because we can choose the γ_g 's we have in the definition of Φ , in equation (45), and for this choice we know that $U_{g,h}^\gamma = \partial p_{g,h}$. So the “equation” we have here is $U_{g,h}^\gamma = 1$ in $H^2(N, \mathbf{k}^\times)$, and the solution γ is given by Φ , which comes from the homomorphism c .

This γ is the first part of the data needed in order to define the extension of ω .

11.8.2. The second differential in page E_3 . We consider now the cocycle $i^{\gamma,\omega} \in H^3(G, H^1(N, \mathbf{k}^\times))$. In order to construct $i^{\gamma,\omega}$ we need to choose 1-cochains $p_{g,h}$ for every $g, h \in G$ in such a way that the equation $\partial p_{g,h} = U_{g,h}^\gamma$ holds. We have such 1-cochains given in equation (44). If we take the 1-cochains from equation (44) and compute $i^{\gamma,\omega}$, we get $i^{\gamma,\omega} = \widehat{O_3(c)}_\rho$. So the vanishing of the first obstruction implies also that $d_3(\omega) = 0$, and the second part we need in order to define the extension of ω is the collection of isomorphisms $p_{g,h}$ (which comes from the system of products M). Again, $p = \{p_{g,h}\}$ is a solution to an equation which says that $\widehat{O_3(c)}_\rho$ is trivial (recall that $O_3(c)$ was defined using the $p_{g,h}$'s).

11.8.3. The third differential in page E_4 . Finally, consider the cocycle $j^{\omega,\gamma,p}$. A direct calculation shows that this is exactly $O_4(c, M)$. So the vanishing of the second obstruction implies that $d_4(\omega) = 0$. The last choice we need to make is to choose a 3-cochain $\beta \in C^3(G, \mathbf{k}^\times)$ such that $\partial\beta = j^{\omega,\gamma,p}$. But the data (c, M, α) determines such a solution. The solution is $\beta = \eta\alpha$, where η is the fixed solution for $O_4(c, M)$ we assumed to exist.

References

- [1] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*. Univ. Lecture Ser. 21, Amer. Math. Soc., Providence, RI, 2001. [MR 1797619](#) [Zbl 0965.18002](#)
- [2] H. J. Baues, *Combinatorial homotopy and 4-dimensional complexes*. de Gruyter Exp. Math. 2, Walter de Gruyter & Co., Berlin 1991. [MR 1096295](#) [Zbl 0716.55001](#)
- [3] A. M. Cegarra and A. R. Garzón, Obstructions to Clifford system extensions of algebras. *Proc. Indian Acad. Sci. Math. Sci.* **111** (2001), 151–161. [MR 1836363](#) [Zbl 0985.16030](#)
- [4] A. M. Cegarra, A. R. Garzón, and J. A. Ortega, Graded extensions of monoidal categories. *J. Algebra* **241** (2001), 620–657. [MR 1843316](#) [Zbl 0988.18004](#)
- [5] E. C. Dade, Group-graded rings and modules. *Math. Z.* **174** (1980), 241–262. [MR 593823](#) [Zbl 0424.16001](#)
- [6] P. Deligne, Catégories tannakiennes. In *The Grothendieck Festschrift*, Vol. II, Progr. Math. 87, Birkhäuser, Boston 1990, 111–195. [MR 1106898](#) [Zbl 0727.14010](#)
- [7] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, On braided fusion categories I. *Selecta Math. (N.S)* **16** (2010), 1–119. [MR ???](#) [Zbl 05716902](#)
- [8] L. E. Dickson, *Linear groups*. B. G. Teubner, Leipzig 1901. [JFM 32.0128.01](#)
- [9] S. Eilenberg and S. Mac Lane, Cohomology theory in abstract groups II: Group extensions with a non-Abelian kernel. *Ann. of Math. (2)* **48** (1947), 326–341. [MR 0020996](#) [Zbl 0029.34101](#)
- [10] P. Etingof, On Vafa's theorem for tensor categories. *Math. Res. Lett.* **9** (2002), 651–657. [MR 1906068](#) [Zbl 1035.18004](#)
- [11] P. Etingof, D. Nikshych, and V. Ostrik, An analogue of Radford's S^4 formula for finite tensor categories. *Internat. Math. Res. Notices* **2004** (2004), 2915–2933. [MR 2097289](#) [Zbl 1079.16024](#)

- [12] P. Etingof, D. Nikshych, and V. Ostrik, On fusion categories. *Ann. of Math. (2)* **162** (2005), 581–642. [MR 2183279](#) [Zbl 1125.16025](#)
- [13] P. Etingof, D. Nikshych, and V. Ostrik, Weakly group-theoretical and solvable fusion categories. *Adv. Math.*, to appear; Preprint 2008. [arXiv:0809.3031](#) [math.QA]
- [14] P. Etingof and V. Ostrik, Finite tensor categories. *Moscow Math. J.* **4** (2004), 627–654. [MR 2119143](#) [Zbl 1077.18005](#)
- [15] A. Fröhlich and C. T. C. Wall, Graded monoidal categories. *Compositio Math.* **28** (1974), 229–285. [MR 0349804](#) [Zbl 0327.18007](#)
- [16] C. Galindo, Clifford theory for tensor categories. Preprint 2009. [arXiv:0902.1088](#)
- [17] S. Gelaki and D. Naidu, Some properties of group-theoretical categories. *J. Algebra* **322** (2009), 2631–2641. [MR 2559854](#) [Zbl 05655713](#)
- [18] S. Gelaki, D. Naidu, and D. Nikshych, Centers of graded fusion categories. *Algebra Number Theory* **3** (2009), 959–990. [MR 2587410](#) [Zbl 05704459](#)
- [19] J. Greenough, Monoidal 2-structure of bimodule categories. *J. Algebra*, to appear; Preprint 2009. [arXiv:0911.4979v4](#) [math.QA]
- [20] L. C. Grove, *Classical groups and geometric algebra*. Grad. Stud. Math. 39, Amer. Math. Soc., Providence, RI, 2002. [MR 1859189](#) [Zbl 0990.20001](#)
- [21] E. Jenkins, Extensions of groups by braided 2-groups. In preparation.
- [22] D. Jordan and E. Larson, On the classification of certain fusion categories. *J. Noncommut. Geom.* **3** (2009), 481–499. [MR 2511638](#) [Zbl 05578954](#)
- [23] M. Kapranov, V. Voevodsky, 2-categories and Zamolodchikov tetrahedra equations. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods* (University Park, PA, 1991), Proc. Sympos. Pure Math. 56, Amer. Math. Soc., Providence, RI, 1994, 177–259. [MR 1278735](#) [Zbl 0809.18006](#)
- [24] G. Karpilovsky, *The Schur multiplier*. London Math. Soc. Monogr. (N.S.) 2, Oxford University Press, Oxford 1987. [MR 1200015](#) [Zbl 0619.20001](#)
- [25] J. Lurie, *Higher topos theory*. Ann. of Math. Stud. 170, Princeton University Press, Princeton, NJ, 2009. [MR 2522659](#) [Zbl 1175.18001](#)
- [26] S. Mac Lane, *Homology*. Grundlehren Math. Wiss. 114, Springer-Verlag, Berlin 1963. [MR 0156879](#) [Zbl 0133.26502](#)
- [27] S. Mac Lane, *Categories for the working mathematician*. 2nd ed., Grad. Texts in Math. 5, Springer-Verlag, New York 1998. [MR 1712872](#) [Zbl 0906.18001](#)
- [28] M. Müger, From subfactors to categories and topology I: Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Algebra* **180** (2003), 81–157. [MR 1966524](#) [Zbl 1033.18002](#)
- [29] D. Naidu, Categorical Morita equivalence for group-theoretical categories. *Comm. Algebra* **35** (2007), 3544–3565. [MR 2362670](#) [Zbl 1143.18009](#)
- [30] D. Naidu and D. Nikshych, Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups. *Comm. Math. Phys.* **279** (2008), 845–872. [MR 2386730](#) [Zbl 1139.16028](#)
- [31] V. Ostrik, Module categories, weak Hopf algebras and modular invariants. *Transform. Groups* **8** (2003), 177–206. [MR 1976459](#) [Zbl 1044.18004](#)

- [32] V. Ostrik, Module categories over the Drinfeld double of a finite group. *Internat. Math. Res. Notices* **2003** (2003), 1507–1520. [MR 1976233](#) [Zbl 1044.18005](#)
- [33] F. Quinn, Group categories and their field theories. In *Proceedings of the Kirbyfest* (Berkeley, 1998), *Geom. Topol. Monogr.* 2, *Geom. Topol. Publ.*, Coventry 1999, 407–453. [MR 1734419](#) [Zbl 0964.18003](#)
- [34] J. J. Rotman, *An introduction to homological algebra*. Pure Appl. Math. 85, Academic Press, New York 1979. [MR 0538169](#) [Zbl 0441.18018](#)
- [35] P. Schauenburg, The monoidal center construction and bimodules. *J. Pure Appl. Algebra* **158** (2001), 325–346. [MR 1822847](#) [Zbl 0984.18006](#)
- [36] D. Tambara, A duality for modules over monoidal categories of representations of semisimple Hopf algebras. *J. Algebra* **241** (2001), 515–547. [MR 1843311](#) [Zbl 0987.16034](#)
- [37] D. Tambara and S. Yamagami, Tensor categories with fusion rules of self-duality for finite abelian groups. *J. Algebra* **209** (1998), 692–707. [MR 1659954](#) [Zbl 0923.46052](#)
- [38] V. Turaev, Crossed group-categories. *Arab. J. Sci. Eng. Sect. C Theme Issues* **33** (2008), 483–503. [MR 2500054](#) [Zbl 1185.18009](#)
- [39] V. Turaev, *Homotopy quantum field theory*. EMS Tracts Math. 10, European Math. Soc. Publ. House, Zurich 2010. [Zbl 05716448](#)

Received September 17, 2009

P. Etingof, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

E-mail: etingof@math.mit.edu

D. Nikshych, Department of Mathematics and Statistics, University of New Hampshire, Durham, NH 03824, U.S.A.

E-mail: nikshych@math.unh.edu

V. Ostrik, Department of Mathematics, University of Oregon, Eugene, OR 97403, U.S.A.

E-mail: vostrik@math.uoregon.edu