Minimal generating sets of Reidemeister moves

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Abstract. It is well known that any two diagrams representing the same oriented link are related by a finite sequence of Reidemeister moves $\Omega 1$, $\Omega 2$ and $\Omega 3$. Depending on orientations of fragments involved in the moves, one may distinguish 4 different versions of each of the $\Omega 1$ and $\Omega 2$ moves, and 8 versions of the $\Omega 3$ move. We introduce a minimal generating set of 4 oriented Reidemeister moves, which includes two $\Omega 1$ moves, one $\Omega 2$ move, and one $\Omega 3$ move. We then study which other sets of up to 5 oriented moves generate all moves, and show that only few of them do. Some commonly considered sets are shown not to be generating. An unexpected non-equivalence of different $\Omega 3$ moves is discussed.

1. Introduction

A standard way to describe a knot or a link in $\mathbb{R}^3$ is via its *diagram*, i.e., a generic plane projection of the link such that the only singularities are transversal double points, endowed with the over/undercrossing information at each double point. Two diagrams are equivalent if there is an orientation-preserving diffeomorphism of the plane that carries one diagram to the other diagram. A classical result of Reidemeister [6] states that any two diagrams of isotopic links are related by a finite sequence of simple moves[1] $\Omega 1$, $\Omega 2$, and $\Omega 3$, shown in Figure 1.
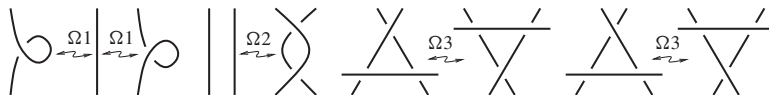
Figure 1. Reidemeister moves.

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[1]Our notation makes no distinction between a move and the inverse move.

Here we assume that two diagrams D, D' related by a move coincide outside an oriented embedded disk $C \subset \mathbb{R}^2$ (with an orientation of C induced by the standard orientation of $\mathbb{R}^2$), called the *changing disk*, and look as a corresponding pair R, R' of arc diagrams in Figure 1 inside C. In other words, there are two orientation-preserving diffeomorphisms $f, f' : C \to B^2$ of C to the standard oriented 2-disk B^2, such that $f_{\partial C} = f'_{\partial C}$ and $f(C \cap D) = R$, $f'(C \cap D') = R'$.

To deal with oriented links we consider oriented diagrams. Depending on orientations of fragments involved in the moves, one may distinguish four different versions of each of the $\Omega 1$ and $\Omega 2$ moves, and eight versions of the $\Omega 3$ move; see Figures 2, 3, and 4, respectively.

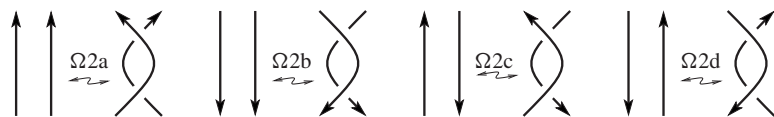Figure 2. Oriented Reidemeister moves of type 1.

Figure 3. Oriented Reidemeister moves of type 2.

When one checks that a certain function of knot or link diagrams defines a link invariant, it is important to minimize the number of moves. We will call a collection S of oriented Reidemeister moves a *generating set* if any oriented Reidemeister move Ω may be obtained by a finite sequence of isotopies and moves from the set S inside the changing disk of Ω.

While some dependencies between oriented Reidemeister moves are well known, the standard generating sets of moves usually include six different $\Omega 3$ moves, see e.g. Kauffman [3]. For sets with a smaller number of $\Omega 3$ moves there seems to be a number of different, often contradictory, results. In particular, Turaev [7], proof of Theorem 5.4, introduces a set of five oriented Reidemeister moves with only one $\Omega 3$ move. There is no proof (and in fact we will see in Section 3 that this particular set is not generating), with the only comment being a reference to a figure where, unfortunately, a move $\Omega 2$ which does not belong to the set is used. Wu [9] uses the same set of moves citing [7] but additionally incorrectly puts the total number of oriented $\Omega 3$ moves at 12 (instead of 8). Kauffman [3], p. 90, includes as an exercise a set of all $\Omega 1$ and $\Omega 2$ moves together with two $\Omega 3$ moves. Meyer [4] uses a set with four $\Omega 1$, two $\Omega 2$, and two $\Omega 3$ moves and states (again without a proof) that the minimal number of needed $\Omega 3$ moves is two. The number of $\Omega 3$ moves used by
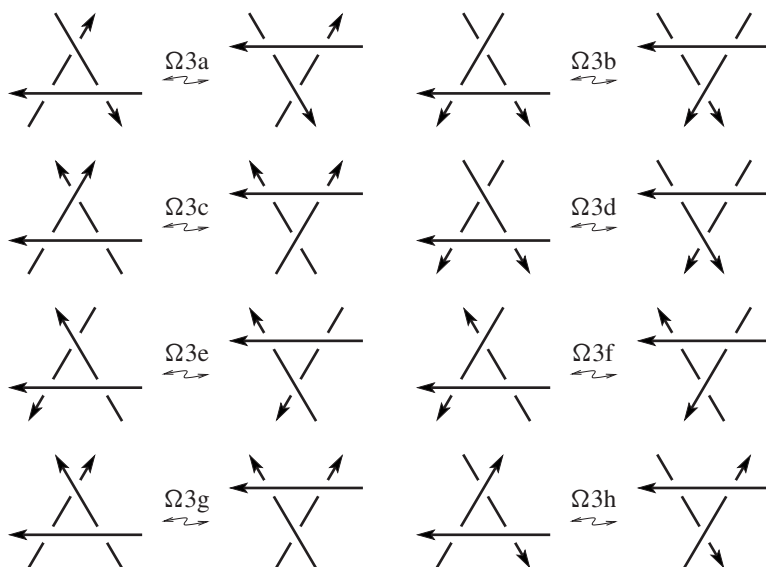
Figure 4. Oriented Reidemeister moves of type 3.

Östlund [5] is also two, but his classification of $\Omega 3$ moves works only for knots and is non-local (depending on the cyclic order of the fragments along the knot). Series of exercises in Chmutov et al. [1] (unfortunately without proofs) suggest that only one $\Omega 3$ suffices, but this involves all $\Omega 2$ moves. These discrepancies are most probably caused by the fact that while many people needed some statement of this kind, it was only an auxiliary technical statement, a proof of which would be too long and would take the reader away from the main subject. So only a brief comment was usually made. We believe that it is time for a careful treatment. In this note we introduce a simple generating set of four Reidemeister moves, which includes two $\Omega 1$ moves, one $\Omega 2$ move and one $\Omega 3$ move.

Theorem 1.1. *Let D and D' be two diagrams in $\mathbb{R}^2$ representing the same oriented link. Then one may pass from D to D' by isotopy and a finite sequence of four oriented Reidemeister moves $\Omega 1a$, $\Omega 1b$, $\Omega 2a$, and $\Omega 3a$, shown in Figure 5.*
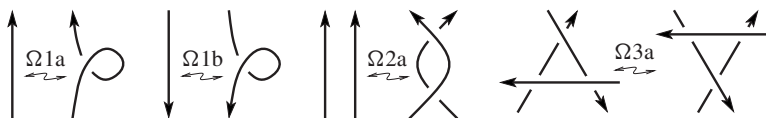

Figure 5. A generating set of Reidemeister moves.

This generating set of Reidemeister moves has the minimal number of generators.

Indeed, it is easy to show that any generating set should contain at least one move of each of the types two and three; Lemma 2.2 in Section 3 implies that there should be at least two moves of type one. Thus any generating set of Reidemeister moves should contain at least four moves.

Our choice of the move Ω3a as a generator may look unusual since this move (called a cyclic Ω3 move, see e.g. [3]) is rarely included in the list of generators, contrary to a more common move Ω3b which is the standard choice motivated by braid theory.[2] The reason is that, unexpectedly, these moves have different properties, as we discuss in detail in Section 3. Indeed, Theorem 1.2 below implies that any generating set of Reidemeister moves which includes Ω3b has at least five moves. If we consider sets of five Reidemeister moves which contain Ω3b, then it turns out that out of all combinations of Ω1 and Ω2 moves only 4 sets generate all Reidemeister moves. The only freedom is in the choice of Ω1 moves, while Ω2 moves are uniquely determined:

Theorem 1.2. *Let S be a set of at most five Reidemeister moves which contains only one move, Ω3b, of type three. The set S generates all Reidemeister moves if and only if S contains Ω2c and Ω2d, and one of the pairs (Ω1a, Ω1b), (Ω1a, Ω1c), (Ω1b, Ω1d), or (Ω1c, Ω1d).*
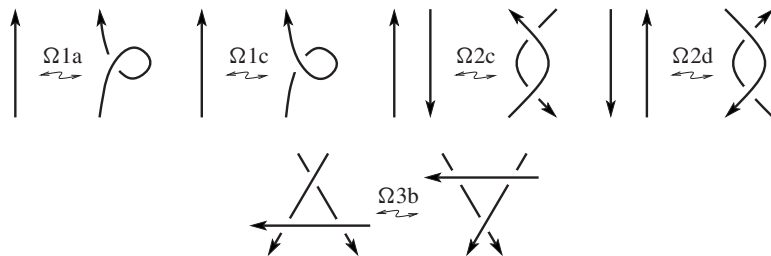

Figure 6. A generating set of Reidemeister moves containing Ω3b.

One of these generating sets is shown in Figure 6. It is interesting to note that while (by Markov's theorem) the set Ω1a, Ω1c, Ω2a, Ω2b and Ω3b shown in Figure 7 allows one to pass between any two braids with isotopic closures, this set is not sufficient to connect any pair of general diagrams representing the same link. This means that some extra moves should appear in the process of transforming a general link diagram into a closed braid. And indeed, in all known algorithms of such a transformation the additions moves occur. For example, in Vogel's algorithm [8] the moves Ω2c and Ω2d are the main steps of the algorithm.

Even more unexpected is the fact that all type one moves together with Ω2a, Ω2c (or Ω2d) and Ω3b are also insufficient; see Figure 8.

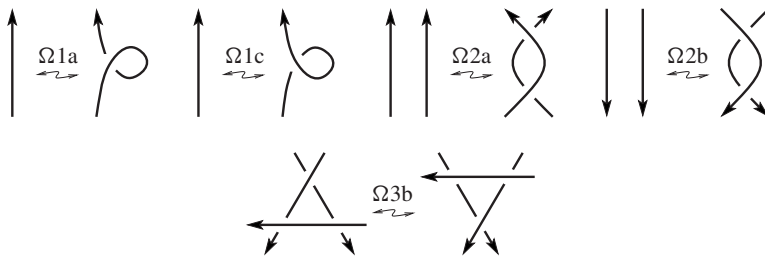[2]This is the only Ω3 move with all three positive crossings.
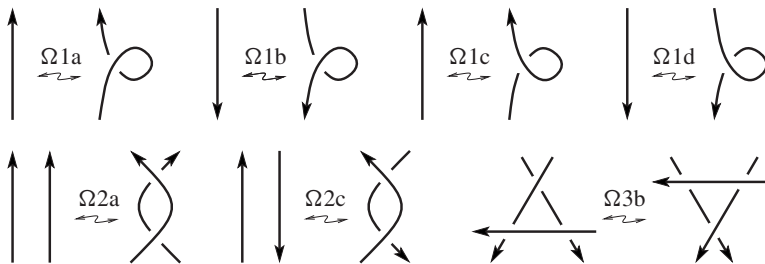
Figure 7. This is not a generating set.

Figure 8. Another set which is not generating.

Remark 1.3. For non-oriented links a sequence of Reidemeister moves can be arranged in such a form that first a number of $\Omega 1$ moves are performed, then $\Omega 2$ moves are performed, after this $\Omega 3$ moves are performed, and finally $\Omega 2$ moves have to be performed again; see [2]. It would be interesting to find such a theorem for oriented case.

All our considerations are local, and no global realization restrictions are involved. Therefore all our results hold also for virtual links.

Section 2 is dedicated to the proof of Theorem 1.1. In Section 3 we discuss various generating sets which contain $\Omega 3b$ and prove Theorem 1.2

2. A minimal set of oriented Reidemeister moves

In this section we prove Theorem 1.1 in several easy steps. The first step is to obtain $\Omega 2c$, $\Omega 2d$:

Lemma 2.1. *The move $\Omega 2c$ may be realized by a sequence of $\Omega 1a$, $\Omega 2a$ and $\Omega 3a$ moves. The move $\Omega 2d$ may be realized by a sequence of $\Omega 1b$, $\Omega 2a$ and $\Omega 3a$ moves.*
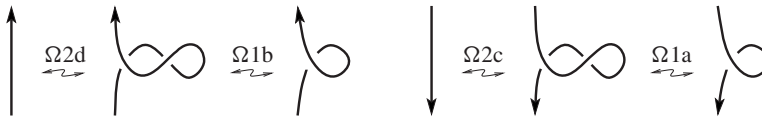
Proof.
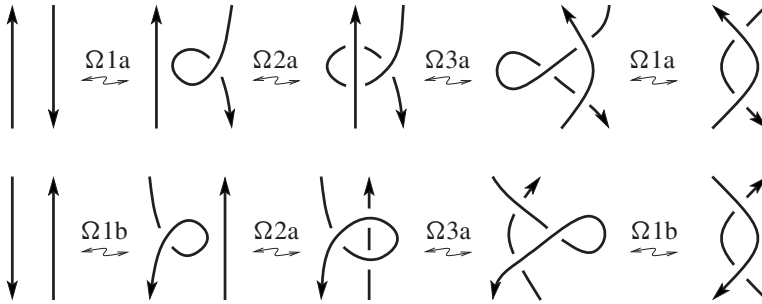
Now the remaining moves of type one may be obtained as in [5]:

Lemma 2.2 ([5]). *The move* Ω1c *may be realized by a sequence of* Ω1b *and* Ω2d *moves. The move* Ω1d *may be realized by a sequence of* Ω1a *and* Ω2c *moves.*
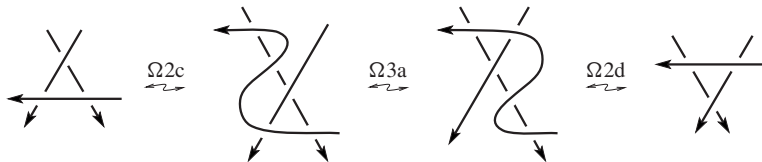
Proof.

This concludes the treatment of all Ω1 and Ω2 moves, except for Ω2b; we will take care of it later. Having in mind Section 3, where we will deal with Ω3b instead of Ω3a, we will first consider Ω3b:

Lemma 2.3. *The move* Ω3b *may be realized by a sequence of* Ω2c*,* Ω2d*, and* Ω3a *moves.*

Proof.
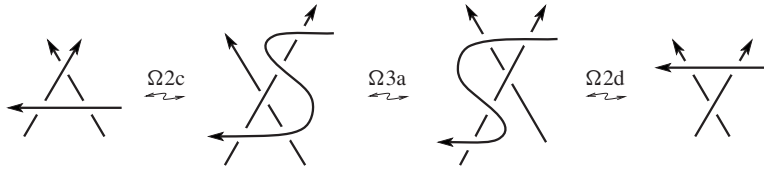

To deal with Ω2b we will need another move of type three:

Lemma 2.4. *The move* Ω3c *may be realized by a sequence of* Ω2c, Ω2d, *and* Ω3a *moves.*

Proof.
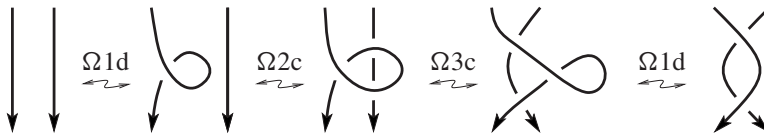

□

At this stage we can obtain the remaining move Ω2b of type two:

Lemma 2.5. *The move* Ω2b *may be realized by a sequence of* Ω1d, Ω2c *and* Ω3c *moves.*

Proof.

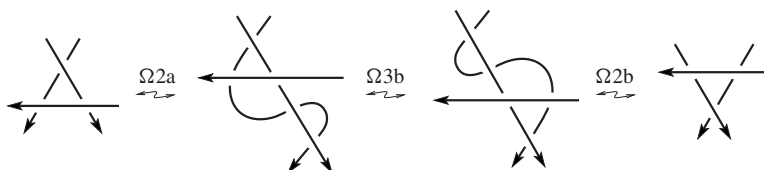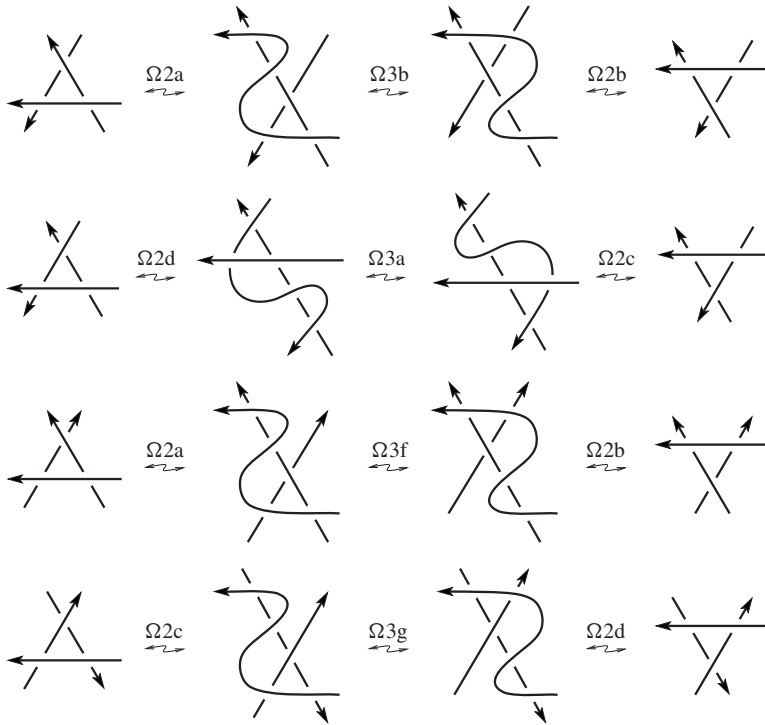□

To conclude the proof of Theorem 1.1, it remains to obtain Ω3d–Ω3h. Since by now we have in our disposal all moves of type two, this becomes an easy exercise:

Lemma 2.6. *The moves* Ω3d–Ω3h *of type three may be realized by a sequence of type two moves,* Ω3a, *and* Ω3b.

Proof. We realize moves Ω3d–Ω3h as shown in rows 1–5 of the figure below, using Ω3f to get Ω3g, and Ω3g to get Ω3h:

Remark 2.7. There are other generating sets which include Ω3a. In particular, Ω1a, Ω1b, Ω2b and Ω3a also give a generating set. To adapt the proof of Theorem 1.1 to this case, one needs only a slight modification of Lemma 2.1. All other lemmas do not change.

3. Other sets of Reidemeister moves

In this section we discuss other generating sets and prove Theorem 1.2. Unexpectedly, different Ω3 moves have different properties as far as generating sets of Reidemeister moves are concerned. Let us study the case of Ω3b in more detail, due to its importance for braid theory.

In a striking contrast to Theorem 1.1 which involves Ω3a, Theorem 1.2 implies that there does not exist a generating set of four moves which includes Ω3b. It is natural to ask where does the proof in Section 2 breaks down if we attempt to replace Ω3a with Ω3b.

The only difference between Ω3a and Ω3b may be pinpointed to Lemma 2.1: it does not have an analogue with Ω3b replacing Ω3a, as we will see in the proof of Lemma 3.8 below.

An analogue of Lemma 2.3 is readily shown to exist. Indeed, $\Omega3a$ may be realized by a sequence of $\Omega2c$, $\Omega2d$ and $\Omega3b$ moves, as illustrated below:
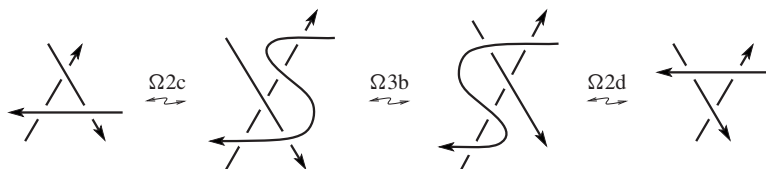
Using this fact instead of Lemma 2.3, together with the rest of Lemmas 2.2–2.6, implies that $\Omega1a$ and $\Omega1b$, taken together with $\Omega2c$, $\Omega2d$, and $\Omega3b$, indeed provide a generating set. Moreover, a slight modification of Lemma 2.2 shows that any of the other three pairs of $\Omega1$ moves in the statement of Theorem 1.2 may be used instead of $\Omega1a$ and $\Omega1b$. Thus we see that all sets described in Theorem 1.2 are indeed generating and obtain the "if" part of the theorem. It remains to prove the "only if" part of Theorem 1.2, i.e., to show that other combinations of four $\Omega1$ and $\Omega2$ moves, taken together with $\Omega3b$, do not result in generating sets. We will proceed in three steps:

Step 1: Prove that any such generating set should contain at least two $\Omega1$ moves and to eliminate two remaining pairs ($\Omega1a$, $\Omega1d$) and ($\Omega1b$, $\Omega1c$) of $\Omega1$ moves.

Step 2: Prove that any such generating set should contain at least two $\Omega2$ moves and to eliminate pairs ($\Omega2a$,$\Omega2c$), ($\Omega2a$,$\Omega2d$), ($\Omega2b$,$\Omega2c$), and ($\Omega2b$,$\Omega2d$).

Step 3: Eliminate the remaining pair ($\Omega2a$,$\Omega2b$).

The remainder of this section is dedicated to these three steps. Step 1 is the simplest and is given by Lemma 3 below. Step 2 is the most complicated; it is given by Corollaries 3.5 and 3.6. Step 3 is relatively simple and is given by Lemma 3.8.

To show that a certain set of Reidemeister moves is not generating, we will construct an invariant of these moves which, however, is not preserved under the set of all Reidemeister moves. The simplest classical invariants of this type are the *writhe* w and the *winding number* rot of the diagram. The winding number of the diagram grows (respectively drops) by one under $\Omega1b$ and $\Omega1d$ (respectively $\Omega1a$ and $\Omega1c$). The writhe of the diagram grows (respectively drops) by one under $\Omega1a$ and $\Omega1b$ (respectively $\Omega1c$ and $\Omega1d$). Moves $\Omega2$ and $\Omega3$ do not change w and rot. These simple invariants suffice to deal with moves of type one (see e.g. [5]):

Lemma 3.1 ([5]). *Any generating set of Reidemeister moves contains at least two $\Omega1$ moves. None of the two pairs ($\Omega1a$, $\Omega1d$) or ($\Omega1b$, $\Omega1c$), taken together with all $\Omega2$ and $\Omega3$ moves, gives a generating set.*

Proof. Indeed, both $\Omega1a$ and $\Omega1d$ preserve $w + \mathrm{rot}$, so this pair (or any of them separately) together with $\Omega2$ and $\Omega3$ moves cannot generate all Reidemeister moves. The case of $\Omega1b$ and $\Omega1c$ is obtained by the reversal of an orientation (of all components) of the link. □

This concludes step 1 of the proof. Let us proceed with step 2. Here the situation is quite delicate since the standard algebraic/topological invariants, reasonably well behaved under compositions, cannot be applied. The reason can be explained by a simple example: suppose that we want to show that $\Omega 2d$ cannot be obtained by a sequence of Reidemeister moves which includes $\Omega 2c$. Then our invariant should be preserved under $\Omega 2c$ and distinguish two tangles shown in Figure 9a. However, if we compose them with a crossing, as shown in Figure 9b, we may pass from one to another by $\Omega 2c$. Thus the invariant should not survive composition of tangles.
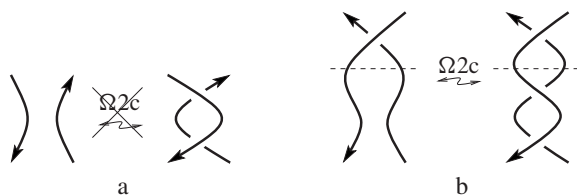

Figure 9. Composition destroys inequivalence.

Instead we will use a certain notion of positivity, which is indeed destroyed by such compositions. It is defined as follows. Let D be a $(2, 2)$-tangle diagram with two oriented ordered components D_1, D_2. Decorate all arcs of both components of D with an integer weight by the following rule. Start walking on D_1 along the orientation. Assign zero to the initial arc. Each time when we pass an overcrossing (we do not count undercrossings) with D_2, we add a sign (the local writhe) of this overcrossing to the weight of the previous arc. Now start walking on D_2 along the orientation. Again, assign zero to the initial arc. Each time when we pass an undercrossing (now we do not count overcrossings) with D_1, we add a sign of this undercrossing to the weight of the previous arc. See Figure 10a. Two simple examples are shown in Figure 10b, c.
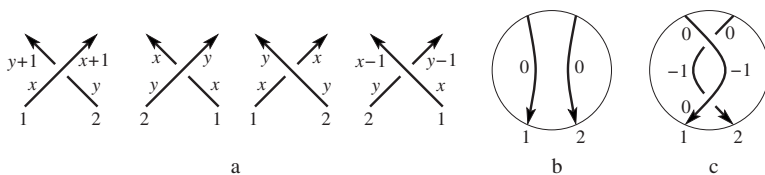
Figure 10. Weights of diagrams.

We call a component *positively weighted* if weights of all its arcs are non-negative. For instance, both components of the (trivial) tangle in Figure 10b are positively weighted. None of the components of a diagram in Figure 10c are positively weighted (since the weights of the middle arcs on both components are -1). The behavior of positivity under Reidemeister moves is considered in the next lemmas.

Denote by S_b the set which consists of all $\Omega 1$ moves and $\Omega 3b$.

Lemma 3.2. *Let D be a* $(2,2)$*-tangle diagram with both positively weighted components. Then both components of a diagram obtained from it by a sequence of moves which belong to* $S_b \cup \Omega 2a$ *are also positively weighted.*

Proof. Indeed, an application of a first Reidemeister move does not change this property since we count only intersections of two different components. An application of $\Omega 2a$ adds (or removes) two crossings on each component in such a way that walking along a component we first meet a positive crossing and then the negative one. So the weights of the middle arcs are either the same or larger than on the surrounding arcs; see Figure 11a. An application of $\Omega 3b$ preserves the weights since $\Omega 3b$ involves only positive crossings. □

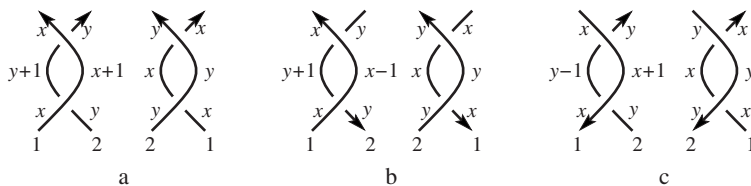Figure 11. Weights and Reidemeister moves of type two.

Lemma 3.3. *Let D be a* $(2,2)$*-tangle diagram with a positively weighted second component. Then any diagram obtained from it by* $\Omega 2c$ *also has a positively weighted second component.*

Proof. An application of $\Omega 2c$ may add (or remove) two undercrossings on D_2, but in such a way that we first meet a positive undercrossing and then the negative one, so the weight of a middle arc is larger than on the surrounding arcs, see Figure 11b. □

Lemma 3.4. *Let D be a* $(2,2)$*-tangle diagram with a positively weighted first component. Then any diagram obtained from it by* $\Omega 2d$ *also has a positively weighted first component.*

Proof. An application of $\Omega 2d$ may add (or remove) two overcrossings on D_1, but in such a way that we first meet a positive overcrossing and then the negative one. So the weight of a middle arc is larger than on the surrounding arcs; see Figure 11c. □

Comparing Figures 10b and 10c we conclude

Corollary 3.5. *None of the two sets* $S_b \cup \Omega 2a \cup \Omega 2c$ *and* $S_b \cup \Omega 2a \cup \Omega 2d$ *generates* $\Omega 2b$.

The reversal of orientations (of both components) of the tangle in the above construction gives

Corollary 3.6. *None of the two sets* $S_b \cup \Omega 2b \cup \Omega 2c$ *and* $S_b \cup \Omega 2b \cup \Omega 2d$ *generates* $\Omega 2a$.

Remark 3.7. In [7], Theorem 5.4 (and later [9]) the set $S_b \cup \Omega 2a \cup \Omega 2c$ is considered as a generating set. Fortunately (V. Turaev, personal communication), an addition of $\Omega 2d$ does not change the proof of the invariance in [7], Theorem 5.4.

Note that the above corollaries imply that any generating set S which contains only one move, $\Omega 3b$, of type three, should contain at least two $\Omega 2$ moves. This concludes step 2 of the proof.

Since at the same time such a set S should contain at least two $\Omega 1$ moves by Lemma , we conclude that if S consists of five moves, there should be exactly two $\Omega 2$ moves and two $\Omega 1$ moves. This simple observation allows us to eliminate the last remaining case:

Lemma 3.8. *Let S be a set which consists of two $\Omega 1$ moves, $\Omega 2a$, $\Omega 2b$, and $\Omega 3b$. Then S is not generating.*

Proof. Given a link diagram, smooth all double points of the diagram respecting the orientation, as illustrated in Figure 12.

Figure 12. Smoothing the diagram respecting the orientation.

Count the numbers C^- and C^+ of clockwise and counter-clockwise oriented circles of the smoothed diagram, respectively. Note that $\Omega 2a$, $\Omega 2b$, and $\Omega 3b$ preserve an isotopy class of the smoothed diagram, thus preserve both C^+ and C^-. On the other hand, $\Omega 1b$ and $\Omega 1d$ add one to C^+, and $\Omega 1a$, $\Omega 1c$ add one to C^-. Thus if S contains $\Omega 1a$ and $\Omega 1c$, all moves of S preserve C^+. The case of $\Omega 1b$ and $\Omega 1d$ is obtained by the reversal of an orientation (of all components) of the link. If S contains $\Omega 1a$ and $\Omega 1b$, all moves of S preserve $C^+ + C^- - w$. Similarly, if S contains $\Omega 1c$ and $\Omega 1d$, all moves of S preserve $C^+ + C^- + w$. In all the above cases, moves from S can not generate $\Omega 2c$, $\Omega 2d$ since each of $\Omega 2c$ and $\Omega 2d$ may change C^+ as well as $C^+ + C^- \pm w$ (while preserving w and $C^+ - C^- = $ rot). □

This concludes the proof of Theorem 1.2.

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