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Quantum Topology

A quaternionic braid representation (after Goldschmidt and Jones)

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Abstract. We show that the braid group representations associated with the $(3,6)$-quotients of the Hecke algebras factor over a finite group. This was known to experts going back to the 1980s, but a proof has never appeared in print. Our proof uses an unpublished quaternionic representation of the braid group due to Goldschmidt and Jones. Possible topological and categorical generalizations are discussed.

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1. Introduction

Jones analyzed the images of the braid group representations obtained from Temperley–Lieb algebras in [11] where, in particular, he determined when the braid group images are finite or not. Braid group representations with finite image were also recognized in [12] and [8]. Some 15 years later the problem of determining the closure of the image of braid group representations associated with Hecke algebras played a critical role in analyzing the computational power of the topological model for quantum computation [6]. Following these developments the author and collaborators analyzed braid group representations associated with BMW-algebras [15] and twisted doubles of finite groups [5]. Partially motivated by empirical evidence the author conjectured that the braid group representations associated with an object X in a braided fusion category $\mathcal{C}$ has finite image if, and only if, the Frobenius–Perron dimension of $\mathcal{C}$ is integral (see e.g. Conjecture 6.6 of [22]). In [18], [25] various instances of this conjecture were verified. This current work verifies this conjecture for the braided fusion category $\mathcal{C}(\mathfrak{sl}_3, 6)$ obtained from the representation category of the quantum group $U_q\mathfrak{sl}_3$ at $q = e^{\pi i/6}$ (see [23] for details and notation). More

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generally, Jimbo's [10] quantum Schur–Weyl duality establishes a relationship between the modular categories $\mathcal{C}(\mathfrak{sl}_k,\ell)$ obtained from the quantum group $U_q\mathfrak{sl}_k$ at $q=e^{\pi\,\mathrm{i}/\ell}$ and certain semisimple quotients $\mathcal{H}_n(k,\ell)$ of specialized Hecke algebras $\mathcal{H}_n(q)$ (defined below). That is, if we denote by $X\in\mathcal{C}(\mathfrak{sl}_k,\ell)$ the simple object analogous to the vector representation of $\mathfrak{sl}_k$ then there is an isomorphism $\mathcal{H}_n(k,\ell)\cong\mathrm{End}(X^{\otimes n})$ induced by $g_i\to I_X^{\otimes i-1}\otimes c_{X,X}\otimes I_X^{\otimes n-i-1}$. In particular, the braid group representations associated with the modular category $\mathcal{C}(\mathfrak{sl}_3,6)$ are the same as those obtained from $\mathcal{H}_n(3,6)$. It is known that braid group representations obtained from $\mathcal{H}_n(3,6)$ have finite image (mentioned in [6], [13], [18]), but a proof has never appeared in print. This fact was discovered by Goldschmidt and Jones during the writing of [8] and independently by Larsen during the writing of [6]. We benefitted from the notes of Goldschmidt and Jones containing the description of the quaternionic braid representation below. Our techniques follow closely those of [11], [12], [14]. The rest of the paper is organized into three sections. In Section 2 we recall some notation and facts about Hecke algebras and their quotients. The main results are in Section 3, and in Section 4 we indicate how the category $\mathcal{C}(\mathfrak{sl}_3,6)$ is exceptional from topological and categorical points of view.

2. Hecke algebras

We extract the necessary definitions and results from [27] that we will need in the sequel.

Definition 2.1. The *Hecke algebra* $\mathcal{H}_n(q)$ for $q\in\mathbb{C}$ is the $\mathbb{C}$-algebra with generators $g_1,\dots,g_{n-1}$ satisfying relations

(H1)′ $g_ig_{i+1}g_i=g_{i+1}g_ig_{i+1}$ for $1\leq i\leq n-2$,

(H2)′ $g_ig_j=g_jg_i$ for $|i-j|>1$, and

(H3)′ $(g_i+1)(g_i-q)=0$.

Technically, $\mathcal{H}_n(q)$ is the Hecke algebra of type A, but we will not be considering other types so we suppress this distinction. One immediately observes that $\mathcal{H}_n(q)$ is the quotient of the braid group algebra $\mathbb{C}\mathcal{B}_n$ by the relation (H1)′. $\mathcal{H}_n(q)$ may also be described in terms of the generators $e_i=\frac{(q-g_i)}{(1+q)}$, which satisfy

(H1) $e_i^2=e_i$,

(H2) $e_ie_j=e_je_i$ for $|i-j|>1$, and

(H3) $e_ie_{i+1}e_i-q/(1+q)^2e_i=e_{i+1}e_ie_{i+1}-q/(1+q)^2e_{i+1}$ for $1\leq i\leq n-2$.

For any $\eta\in\mathbb{C}$, Ocneanu [7] showed that one may uniquely define a linear functional tr on $\mathcal{H}_\infty(q)=\bigcup_{n=1}^{\infty}\mathcal{H}_n(q)$ satisfying

(1) $\mathrm{tr}(1)=1$,

(2) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, and

(3) $\mathrm{tr}(xe_n) = \eta\,\mathrm{tr}(x)$ for any $x \in \mathcal{H}_n(q)$.

Any linear function on $\mathcal{H}_\infty$ satisfying these conditions is called a *Markov trace* and is determined by the value $\eta = \mathrm{tr}(e_1)$. Now suppose that $q = e^{2\pi \mathrm{i}/\ell}$ and $\eta = \frac{(1-q^{1-k})}{(1+q)(1-q^k)}$ for some integers $k < \ell$. Then, for each n, the (semisimple) quotient of $\mathcal{H}_n(q)$ by the annihilator of the restriction of the trace $\mathcal{H}_n(q)/\,\mathrm{Ann(tr)}$ is called the (k,ℓ)-*quotient*. We will denote this quotient by $\mathcal{H}_n(k,\ell)$ for convenience. Wenzl [27] has shown that $\mathcal{H}_n(k,\ell)$ is semisimple and described the irreducible representations $\rho_\lambda^{(k,\ell)}$ where λ is a (k,ℓ)-*admissible* Young diagrams of size n. Here a Young diagram λ is (k,ℓ)-admissible if λ has at most k rows and $\lambda_1 - \lambda_k \leq \ell - k$ where λ_i denotes the number of boxes in the ith row of λ. The (faithful) Jones–Wenzl representation is the sum $\rho^{(k,\ell)} = \bigoplus_\lambda \rho_\lambda^{(k,\ell)}$. Wenzl [27] has shown that $\rho^{(k,\ell)}$ is a C^*-representation, i.e. the representation space is a Hilbert space (with respect to a Hermitian form induced by the trace tr) and $\rho_\lambda^{(k,\ell)}(e_i)$ is a self-adjoint operator. One important consequence is that each $\rho_\lambda^{(k,\ell)}$ induces an irreducible unitary representation of the braid group $\mathcal{B}_n$ via composition with $\sigma_i \to g_i$, which is also called *the Jones–Wenzl representation of* $\mathcal{B}_n$.

3. A quaternionic representation

Consider the $(3,6)$-quotient $\mathcal{H}_n(3,6)$. The $(3,6)$-admissible Young diagrams have at most 3 rows and $\lambda_1 - \lambda_3 \leq 3$. For $n \geq 3$ there are either 3 or 4 Young diagrams of size n that are $(3,6)$-admissible, and $\eta = \frac{(1-q^{1-3})}{(1+q)(1-q^3)} = 1/2$ in this case. Denote by φ_n the unitary Jones–Wenzl representation of $\mathcal{B}_n$ induced by $\rho^{(3,6)}$. Our main goal is to prove the following:

Theorem 3.1. *The image $\varphi_n(\mathcal{B}_n)$ is a finite group.*

We will prove this theorem by embedding $\mathcal{H}_n(3,6)$ into a finite dimensional algebra (Lemma 3.2) and then showing that the group generated by the images of $g_1, \ldots, g_{n-1}$ is finite (Lemma 3.3). Denote by $[\ ,\ ]$ the multiplicative group commutator and let $q = e^{2\pi \mathrm{i}/6}$. Consider the $\mathbb{C}$-algebra Q_n with generators $u_1, v_1, \ldots, u_{n-1}, v_{n-1}$ subject to the relations

(G1) $u_i^2 = v_i^2 = -1$,

(G2) $[u_i, v_j] = -1$ if $|i-j| \leq 1$,

(G3) $[u_i, v_j] = 1$ if $|i-j| \geq 2$, and

(G4) $[u_i, u_j] = [v_i, v_j] = 1$.

Notice that the group $\{\pm 1, \pm u_i, \pm v_i, \pm u_i v_i\}$ is isomorphic to the group of quaternions. We see from these relations that $\dim(Q_n) = 2^{2n-2}$ since each word in the u_i, v_i has a unique normal form

$$\pm\, u_1^{\epsilon_1} \ldots u_{n-1}^{\epsilon_{n-1}} v_1^{\nu_1} \ldots v_{n-1}^{\nu_{n-1}} \tag{1}$$

with $\nu_i, \epsilon_i \in \{0, 1\}$. Observe that a basis for Q_n is given by taking all $+$ signs in (1). We define a $\mathbb{C}$-valued trace Tr on Q_n by setting $\mathrm{Tr}(1) = 1$ and $\mathrm{Tr}(w) = 0$ for any non-identity word in the u_i, v_i. One deduces that Tr is faithful from the uniqueness of the normal form (1). Define

$$s_i = \frac{-1}{2q}(1 + u_i + v_i + u_i v_i), \tag{2}$$

for $1 \leq i \leq n-1$.

Lemma 3.2. *The subalgebra $\mathcal{A}_n \subset Q_n$ generated by $s_1, \ldots, s_{n-1}$ is isomorphic to $\mathcal{H}_n(3, 6)$.*

Proof. It is a straightforward computation to see that the s_i satisfy

(B1) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$,

(B2) $s_j s_i = s_i s_j$ if $|i - j| \geq 2$, and

(E1) $(s_i - q)(s_i + 1) = 0$.

Indeed, relation (B2) is immediate from relations (G3) and (G4). It is enough to check (B1) and (E1) for $i = 1$. For this we compute

$$\begin{aligned} s_1^{-1} &= -\frac{q}{2}(1 - u_1 - v_1 - u_1 v_1), \\ s_1^{-1} u_1 s_1 &= u_1 v_1, \quad s_1^{-1} v_1 s_1 = u_1, \\ s_1^{-1} u_2 s_1 &= u_2 v_1, \quad s_1^{-1} v_2 s_1 = -u_1 v_1 v_2, \end{aligned} \tag{3}$$

from which (B1) and (E1) are deduced. Thus $\varphi(g_i) = s_i$ induces an algebra homomorphism $\varphi : \mathcal{H}_n(q) \to Q_n$ with $\varphi(\mathcal{H}_n(q)) = \mathcal{A}_n$. Set $f_i = \varphi(e_i) = \frac{(q - s_i)}{(1+q)}$ and let $b \in Q_{n-1}$, that is b is in the span of the words in $\{u_1, v_1, \ldots, u_{n-2}, v_{n-2}\}$. The constant term of $f_{n-1} b$ is the product of the constant terms of b and f_{n-1} since f_{n-1} is in the span of $\{1, u_{n-1}, v_{n-1}, u_{n-1} v_{n-1}\}$, so $\mathrm{Tr}(f_{n-1} b) = \mathrm{Tr}(f_{n-1}) \mathrm{Tr}(b)$. For each $a \in \mathcal{H}_n(q)$ we define $\varphi^{-1}(\mathrm{Tr})(a) = \mathrm{Tr}(\varphi(a))$, and conclude that $\varphi^{-1}(\mathrm{Tr})$ is a Markov trace on $\mathcal{H}_n(q)$. Computing, we see that $\mathrm{Tr}(f_{n-1}) = 1/2$, so that by uniqueness $\varphi^{-1}(\mathrm{Tr}) = \mathrm{tr}$ as functionals on $\mathcal{H}_n(q)$. Now if $a \in \ker(\varphi)$ we see that $\mathrm{tr}(ac) = \mathrm{Tr}(\varphi(ac)) = 0$ for any c so that $\ker(\varphi) \subset \mathrm{Ann}(\mathrm{tr})$. On the other hand, if $a \in \mathrm{Ann}(\mathrm{tr})$ we must have $\mathrm{Tr}(\varphi(ac)) = \mathrm{tr}(ac) = 0$ for all $c \in \mathcal{H}_n(q)$. If $\varphi(a) \neq 0$ then, by definition of Tr and φ, there exists an $a^\dagger \in \mathcal{H}_n(q)$ such that $\mathrm{Tr}(\varphi(a)\varphi(a^\dagger)) \neq 0$ since Tr is faithful. Therefore $\mathrm{Ann}(\mathrm{tr}) \subset \ker(\varphi)$. In particular, we see that φ induces

$$\mathcal{H}_n(3, 6) = \mathcal{H}_n(q)/\,\mathrm{Ann}(\mathrm{tr}) \cong \varphi(\mathcal{H}_n(q)) = \mathcal{A}_n \subset Q_n. \qquad \square$$

Lemma 3.3. *The group G_n generated by $s_1, \ldots, s_{n-1}$ is finite.*

Proof. Consider the conjugation action of the s_i on Q_n. We claim that the conjugation action of s_i on the words in the u_i, v_i is by a signed permutation. Since s_i commutes with words in u_j, v_j with $j \notin \{i-1, i, i+1\}$, by symmetry it is enough to consider the conjugation action of s_1 on the four elements $\{u_1, v_1, u_2, v_2\}$, which is given in (3). Thus we see that G_n modulo the kernel of this action is a (finite) signed permutation group. The kernel of this conjugation action lies in the center $Z(Q_n)$ of Q_n. Using the normal form above we find that the center $Z(Q_n)$ is either 1-dimensional or 4-dimensional. Indeed, since the words

$$W = \{u_1^{\epsilon_1} \ldots u_{n-1}^{\epsilon_{n-1}} v_1^{\nu_1} \ldots v_{n-1}^{\nu_{n-1}}\}$$

for $(\epsilon_1, \ldots, \epsilon_{n-1}, \nu_1, \ldots, \nu_{n-1}) \in \mathbb{Z}_2^{2n-2}$ form a basis for Q_n and $tw = \pm wt$ for $w, t \in W$ we may explicitly compute a basis for the center as those words $w \in W$ that commute with u_i and v_i for all i. This yields two systems of linear equations over $\mathbb{Z}_2$:

$$\begin{cases} \epsilon_1 + \epsilon_2 = 0, \\ \epsilon_i + \epsilon_{i+1} + \epsilon_{i+2} = 0, \quad 1 \leq i \leq n-3, \\ \epsilon_{n-2} + \epsilon_{n-1} = 0, \end{cases} \tag{4}$$

and

$$\begin{cases} \nu_1 + \nu_2 = 0, \\ \nu_{i-1} + \nu_i + \nu_{i+1} = 0, \quad 1 \leq i \leq n-3, \\ \nu_{n-2} + \nu_{n-1} = 0. \end{cases} \tag{5}$$

Non-trivial solutions to (4) only exist if $3 \mid n$ since we must have $\epsilon_1 = \epsilon_2 = \epsilon_{n-2} = \epsilon_{n-1} = 1$ as well as $\epsilon_i = 0$ if $3 \mid i$ and $\epsilon_j = 1$ if $3 \nmid j$ and similarly for (5). Thus $Z(Q_n)$ is $\mathbb{C}$ if $3 \nmid n$ and is spanned by $1, U, V$ and UV where $U = \prod_{3 \nmid i} u_i$ and $V = \prod_{3 \nmid i} v_i$ if $3 \mid n$. The determinant of the image of s_i under any representation is a 6th root of unity and hence the same is true for any element $z \in Z(Q_n) \cap G_n$. Thus for $3 \nmid n$ the image of any $z \in Z(Q_n) \cap G_n$ under the left regular representation is a root of unity times the identity matrix, and thus has finite order. Similarly, if $3 \mid n$, the restriction of any $z \in Z(Q_n) \cap G_n$ to any of the four simple components of the left regular representation is a root of unity times the identity matrix and so has finite order. So the group G_n itself is finite. □

This completes the proof of Theorem 3.1.

Remark 3.4. The proof of Lemma 3.3 shows that the projective image of G_n is a (non-abelian) subgroup of the full monomial group $G(2, 1, 4^{n-1})$ of signed $4^{n-1} \times 4^{n-1}$ matrices. The main goal of this paper is to verify [22], Conjecture 6.6, in this case, but with further effort one could determine the group G_n more precisely. It is suggested in [13] that G_n is an extension of $PSU(n-1, \mathbb{F}_2)$ so that

$$|G_n| \approx \frac{1}{3} 2^{(n-1)(n-2)/2} \prod_{i=1}^{n-1} (2^i - (-1)^i),$$

but that such a result has not appeared in print. Modulo the center, the generators s_i have order 3 so that $G_n/Z(G_n)$ is a quotient of the factor group $\mathcal{B}_n/\langle\sigma_1^3\rangle$ (here σ_i are the usual generators of $\mathcal{B}_n$). For $n \leq 5$, Coxeter [1] has shown that these quotients are finite groups and determined their structure. In particular, the projective image of $\mathcal{B}_5/\langle\sigma_1^3\rangle$ is $PSU(4, \mathbb{F}_2)$, so G_5 is an extension of this simple group. A strategy for showing G_n is an extension of $PSU(n-1, \mathbb{F}_2)$ for $n > 5$ would be to find an $(n-1)$-dimensional invariant subspace of Q_n so that the restricted action of the braid generators is by order 3 pseudo-reflections (projectively). A comparison of the dimensions of the simple $\mathcal{H}_n(3,6)$-modules with those of $PSU(n-1, \mathbb{F}_2)$ indicates that one must also restrict to those n not divisible by 3.

4. Concluding remarks, questions and speculations

The category $\mathcal{C}(\mathfrak{sl}_3, 6)$ does not seem to have any obvious generalizations. We discuss some of the ways in which $\mathcal{C}(\mathfrak{sl}_3, 6)$ appears to be exceptional by posing a number of (somewhat naïve) questions which we expect to have negative answers.

4.1. Link invariants. From any modular category one obtains (quantum) link invariants via Turaev's approach [26]. The link invariant $P'_L(q, \eta)$ associated with $\mathcal{C}(\mathfrak{sl}_k, \ell)$ is (a variant of) the HOMFLY-PT polynomial ([7], where a different choice of variables is used). For the choices $q = e^{2\pi\,\mathrm{i}/6}$ and $\eta = 1/2$ corresponding to $\mathcal{C}(\mathfrak{sl}_3, 6)$ the invariant has been identified [16]:

$$P'_L(e^{2\pi\,\mathrm{i}/6}, 1/2) = \pm\,\mathrm{i}(\sqrt{2})^{\dim H_1(T_L;\mathbb{Z}_2)},$$

where T_L is the triple cyclic cover of the three sphere S^3 branched over the link L. There is a similar series of invariants for any odd prime p: $\pm\,\mathrm{i}(\sqrt{p})^{\dim H_1(D_L;\mathbb{Z}_p)}$, where D_L is the double cyclic cover of S^3 branched over L (see [16] and [8]). It appears that this series of invariants can be obtained from modular categories $\mathcal{C}(\mathfrak{so}_p, 2p)$. This has been verified for $p = 3, 5$ (see [8] and [12]) and we have recently handled the $p = 7$ case (unpublished, using results in [29]).

Question 4.1. Are there modular categories with associated link invariant

$$\pm\,\mathrm{i}(\sqrt{p})^{\dim H_1(T_L;\mathbb{Z}_p)}?$$

In [15] it is suggested that if the braid group images corresponding to some ribbon category are finite then the corresponding link invariant is *classical*, i.e. equivalent to a homotopy-type invariant. Another formulation of this idea is found in [24] in which *classical* is interpreted in terms of computational complexity.

4.2. Fusion categories and II_1 factors. The category $\mathcal{C}(\mathfrak{sl}_3, 6)$ is an *integral* fusion category, that is the simple objects have integral dimensions. The categories $\mathcal{C}(\mathfrak{sl}_k, \ell)$ are integral for $(k, \ell) = (3, 6)$ and $(k, k+1)$ but no other examples are known (or believed to exist). $\mathcal{C}(\mathfrak{sl}_3, 6)$ has six simple (isomorphism classes of) objects: $\{X_i, X_i^*\}_{i=1}^3$ of dimension 2 (dual pairs), three simple objects $\mathbf{1}, Z, Z^*$ of dimension 1, and one simple object Y of dimension 3. The Bratteli diagram for tensor powers of the generating object X_1 is given in Figure 1. It is shown in [4] that $\mathcal{C}$ is an integral fusion category if, and only if, $\mathcal{C} \cong \mathrm{Rep}(H)$ for some semisimple finite dimensional quasi-Hopf algebra H, so in particular $\mathcal{C}(\mathfrak{sl}_3, 6) \cong \mathrm{Rep}(H)$ for some quasi-triangular quasi-Hopf algebra H. One wonders if strict coassociativity can be achieved:

Question 4.2. Is there a (quasi-triangular) semisimple finite dimensional Hopf algebra H with $\mathcal{C}(\mathfrak{sl}_3, 6) \cong \mathrm{Rep}(H)$?

Other examples of integral categories are the representation categories $\mathrm{Rep}(D^\omega G)$ of twisted doubles of finite groups studied in [5] (here G is a finite group and ω is a 3-cocycle on G). Any fusion category $\mathcal{C}$ with the property that its Drinfeld center $\mathcal{Z}(\mathcal{C})$ is equivalent as a braided fusion category to $\mathrm{Rep}(D^\omega G)$ for some ω, G is called *group-theoretical* (see [4], [19]). The main result of [5] implies that if $\mathcal{C}$ is any braided group-theoretical fusion category then the braid group representations obtained from $\mathcal{C}$ must have finite image. In [18] we showed that $\mathcal{C}(\mathfrak{sl}_3, 6)$ is not group-theoretical and in fact has minimal dimension (36) among non-group-theoretical integral modular categories.

Question 4.3. Is there a family of non-group-theoretical integral modular categories that includes $\mathcal{C}(\mathfrak{sl}_3, 6)$?

Notice that $\mathcal{C}(\mathfrak{sl}_3, 6)$ has a ribbon subcategory $\mathcal{D}$ with simple objects $\mathbf{1}, Z, Z^*$ and Y. The fusion rules are the same as those of $\mathrm{Rep}(\mathfrak{A}_4)$: $Y \otimes Y \cong \mathbf{1} \oplus Z \oplus Z^* \oplus Y$. However $\mathcal{D}$ is not symmetric and $\mathcal{C}(\mathfrak{sl}_3, 6)$ has smallest dimension among modular categories containing $\mathcal{D}$ as a ribbon subcategory (what Müger would call a *minimal modular extension* [17]). One possible generalization of $\mathcal{C}(\mathfrak{sl}_3, 6)$ would be a minimal modular extension of a non-symmetric ribbon category $\mathcal{D}_n$ similar to $\mathcal{D}$ above. That is, $\mathcal{D}_n$ should be a non-symmetric ribbon category with n 1-dimensional simple objects $\mathbf{1} = Z_0, \ldots, Z_{n-1}$ and one simple n-dimensional object Y_n such that $Y_n \otimes Y_n \cong Y_n \oplus \bigoplus_{i=0}^{n-1} Z_i$ and the Z_i have fusion rules like $\mathbb{Z}_n$. For $\mathcal{D}_n$ to exist even at the generality of fusion categories one must have $n = p^\alpha - 1$ for some prime p and integer α by [3], Corollary 7.4. However, V. Ostrik [20] informs us that these categories do not admit non-symmetric braidings except for $n = 2, 3$. So this does not produce a generalization. A pair of hyperfinite II_1 factors $A \subset B$ with index $[B : A] = 4$ can be constructed from $\mathcal{C}(\mathfrak{sl}_3, 6)$ (see [28], Section 4.5). The corresponding principal graph is the Dynkin diagram $E_6^{(1)}$ the nodes of which we

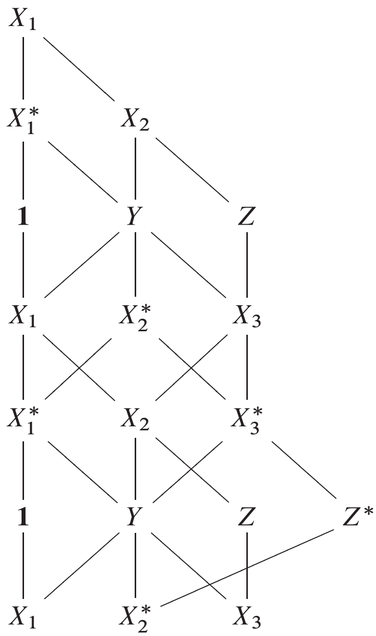
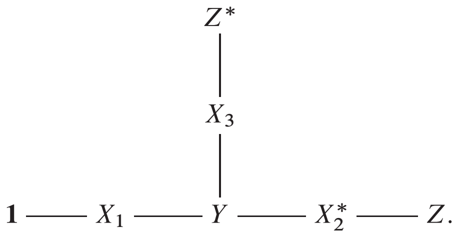


Figure 1. Bratteli diagram for $\mathcal{C}(\mathfrak{sl}_3, 6)$.

label by simple objects:



This principal graph can be obtained directly from the Bratteli diagram in Figure 1 as the nodes in the 6th and 7th levels and the edges between them. Hong [9] showed that any II_1 subfactor pair $M \subset N$ with principal graph $E_6^{(1)}$ can be constructed from some II_1 factor P with an outer action of $\mathfrak{A}_4$ as $M = P \rtimes \mathbb{Z}_3 \subset P \rtimes \mathfrak{A}_4 = N$. Subfactor pairs with principal graph $E_7^{(1)}$ and $E_8^{(1)}$ can also be constructed (see e.g. [21]). We ask:

Question 4.4. Is there a unitary non-group-theoretical integral modular category with principal graph $E_7^{(1)}$ or $E_8^{(1)}$?

Even a braided fusion category with such a principal graph would be interesting, and have interesting braid group image. Notice that the subcategory $\mathcal{D}$ mentioned above plays a role here as $\mathfrak{A}_4$ corresponds to the Dynkin diagram $E_6^{(1)}$ in the McKay correspondence. A modular category $\mathcal{C}$ with principal graph $E_7^{(1)}$ (resp. $E_8^{(1)}$) would contain a ribbon subcategory $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) with the same fusion rules as $\mathrm{Rep}(\mathfrak{S}_4)$ (resp. $\mathrm{Rep}(\mathfrak{A}_5)$). Using [2], Lemma 1.2, we find that such a category $\mathcal{C}$ must have dimension divisible by 144 (resp. 3600). The ribbon subcategory $\mathcal{F}_2$ must have symmetric braiding (D. Nikshych's proof: $\mathrm{Rep}(\mathfrak{A}_5)$ has no non-trivial fusion subcategories so if it has a non-symmetric braiding, the Müger center is trivial. But if the Müger center is trivial it is modular, which fails by [2], Lemma 1.2). This suggests that for $E_8^{(1)}$ the answer to Question 4.4 is "no." There is a non-symmetric choice for $\mathcal{F}_1$ (as V. Ostrik informs us [20]), with Müger center equivalent to $\mathrm{Rep}(\mathfrak{S}_3)$. By [17], Proposition 5.1, a minimal modular extension $\mathcal{C}$ of such an $\mathcal{F}_1$ would have dimension 144.

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