

HOMFLY-PT polynomial and normal rulings of Legendrian solid torus links

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Abstract. We show that for any Legendrian link L in the 1-jet space of S^1 the 2-graded ruling polynomial, $R_L^2(z)$, is determined by the Thurston–Bennequin number and the HOMFLY-PT polynomial. Specifically, we recover $R_L^2(z)$ as a coefficient of a particular specialization of the HOMFLY-PT polynomial. Furthermore, we show that this specialization may be interpreted as the standard inner product on the algebra of symmetric functions that is often identified with a certain subalgebra of the HOMFLY-PT skein module of the solid torus.

In contrast to the 2-graded case, we are able to use 0-graded ruling polynomials to distinguish many homotopically non-trivial Legendrian links with identical classical invariants.

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1. Introduction

The study of Legendrian knots in standard contact \mathbb{R}^3 up to the equivalence relation of Legendrian isotopy provides an interesting variation on the classical theory of smooth knots in 3-space. Each smooth knot type has Legendrian representatives. However, Legendrian knots of the same underlying smooth knot type need not be equivalent as Legendrian knots.

There are two “classical invariants” capable of distinguishing between Legendrian knots with the same underlying smooth knot type. They are known as the Thurston–Bennequin number, $tb(L)$, and rotation number, $r(L)$. Beginning in the late 1990’s, several stronger invariants of Legendrian knots have been developed. Of particular interest for this article are invariants arising from counts of certain decompositions of front diagrams known as normal rulings. Normal rulings arose independently in the work of Fuchs [4] in connection with augmentations of the Chekanov–Eliashberg DGA and also in the work of Chekanov and Pushkar [2] who were motivated by generating families. Chekanov and Pushkar defined for each divisor p of $2r(L)$ an invariant which can be neatly encoded as the p -graded ruling polynomial, $R_L^p(z)$.

A particularly elegant aspect of Legendrian knot theory is the interplay between Legendrian invariants and invariants of the underlying smooth knot type. For instance, the values of the classical invariants are constrained by invariants which depend only on the smooth knot type via “Bennequin type inequalities” (see for instance [10]). As an example, Fuchs and Tabachnikov [5] proved that for any Legendrian link L in standard contact \mathbb{R}^3

$$tb(L) + |r(L)| \leq -\deg_a P_L(a, z), \tag{1}$$

where $P_L \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ is the HOMFLY-PT polynomial.¹ In turn, Legendrian knots can shed light on topological knot invariants. It is shown in [12] that the coefficient of $a^{-tb(L)}$ in $P_L(a, z)$ is precisely $R_L^2(z)$, and hence may be viewed as counting 2-graded normal rulings.

The main purpose of the present article is to investigate the relationship between 2-graded normal rulings and the HOMFLY-PT polynomial of Legendrian links in the 1-jet space of the circle, $J^1(S^1)$. $J^1(S^1)$ is a contact manifold diffeomorphic to an open solid torus, and Legendrian links in $J^1(S^1)$ can be represented diagrammatically via their front projections to the annulus.

The solid torus case is more interesting than \mathbb{R}^3 due to the nature of the HOMFLY-PT polynomial. Unlike link diagrams in the plane, annular link diagrams can not always be reduced to a multiple of the unknot via repeated applications of the skein relations. Instead there are sequences of oriented diagrams $A_{\pm 1}, A_{\pm 2}, \dots$ whose products (defined by stacking, see Section 2) form the base cases for evaluating P_L . This gives rise to a HOMFLY-PT polynomial with many new variables,

$$P_L \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}, A_{\pm 1}, A_{\pm 2}, \dots].$$

More systematically, one considers the skein module \mathcal{C} obtained by imposing the HOMFLY-PT relations on formal linear combinations of link diagrams. Turaev [17] showed that \mathcal{C} is a free module with linear basis consisting of products of the A_i . For a given monomial $A_{i_1} \dots A_{i_N}$ we collect the terms with positive and negative indices to write $A_{i_1} \dots A_{i_N} = A_\lambda A_{-\mu}$ for partitions λ and μ (see Section 4). P_L is simply a normalization of the expansion of L in Turaev’s basis $\{A_\lambda A_{-\mu}\}$. Chmutov and Goryunov [3] extended the estimate (1) to the $J^1(S^1)$ setting (see Theorem 6.1).

One of our main results is the following

Theorem 6.3. *For any Legendrian link $L \subset J^1(S^1)$,*

$$R_L^2(z) = \text{coefficient of } a^{-tb(L)} \text{ in } \widehat{P}_L(a, z).$$

Here, $\widehat{P}_L(a, z)$ is obtained from the HOMFLY-PT polynomial by replacing each occurrence of $A_\lambda A_{-\mu}$ with $R_{A_\lambda A_{-\mu}}^2(z)$. We will view these particular ruling polynomials as defining a bilinear form, $\langle \cdot, \cdot \rangle$, on the subalgebra $\mathcal{C}^+ \subset \mathcal{C}$ generated by

¹For consistency with the main body of this article P_L is normalized so that the unknot has the value $(a - a^{-1})/z$. This differs from [5] and [12].

A_i with $i > 0$ as follows

$$\langle A_\lambda, A_\mu \rangle = R_{A_\lambda A_{-\mu}}^2(z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}].$$

Theorem 4.2 gives a computation of $\langle A_\lambda, A_\mu \rangle$ in terms of the partitions λ and μ as a sum over a particular class of matrices.

In the literature, there is a traditional way of identifying \mathcal{C}^+ with the algebra Λ of symmetric functions (see [1], [7], [9] and the discussion in Section 5.1) where the Schur functions s_λ correspond to skein elements Q_λ . Λ has a standard inner product arising from taking the Schur functions as an orthonormal basis. In Section 5, we show that the corresponding inner product $\langle Q_\lambda, Q_\mu \rangle = \delta_{\lambda, \mu}$ agrees with the bilinear form $\langle \cdot, \cdot \rangle$ used to define $\widehat{P}_L(a, z)$.

Theorem 5.6. *For any partitions λ and μ ,*

$$(A_\lambda, A_\mu) = R_{A_\lambda A_{-\mu}}^2(z) = \langle A_\lambda, A_\mu \rangle.$$

Thus, with respect to Turaev’s basis A_λ the inner product on Λ has a skein theoretic interpretation using Legendrian links.

Our method for proving Theorem 6.3 is inductive as in [12], but several interesting complications arise. For starters, a more subtle measure of the complexity of a front diagram is required and an additional algorithm is necessary to reduce the complexity of front diagrams lacking cusps. More notably, the base case for the induction needs to be enlarged to include all products of Legendrian versions of the A_i . An interesting wrinkle occurs here. In contrast to the case of smooth link diagrams, the product of Legendrian front diagrams in the annulus is not commutative. This phenomenon was first observed by Traynor who showed that the two components of the Legendrian link $L = L_0 \sqcup L_1$ cannot be interchanged via a Legendrian isotopy. Here, L_0 and L_1 denote the 1-jets of the constant functions 0 and 1 on S^1 . In Theorem 4.3 we provide many further examples by showing that for any $i, j \in \mathbb{Z} \setminus \{0\}$ the locations of disjoint A_i and A_j in the z direction cannot be interchanged by a Legendrian isotopy. Nevertheless, we are able to establish in Lemma 4.5 that the 2-graded ruling polynomial of a product of the A_i does not depend on the ordering of the factors.

We have included a proof of Chmutov and Goryunov’s estimate (Theorem 6.1) at the end of Section 6. The versions for the HOMFLY-PT polynomial and $tb(L)$ used here and in [3] differ in a significant way (see Section 6.2 and Remark 2.2 for details). While we believe that the estimate appearing in [3] should indeed be equivalent to the one used here, it is straightforward to provide here a self-contained proof of Theorem 6.1. Our proof is based on the inductive method used in the proof of Theorem 6.3 and is similar in spirit to Ng’s approach to Bennequin type inequalities in \mathbb{R}^3 [10].

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2. Legendrian links in $J^1(S^1)$

The 1-jet space of S^1

$$J^1(S^1) = T^*S^1 \times \mathbb{R} = \{(x, y, z) \mid x \in S^1, y, z \in \mathbb{R}\}$$

is diffeomorphic to an open solid torus and is equipped with the contact structure $\xi = \ker(dz - y dx)$. A smooth (oriented) link $L \subset J^1(S^1)$ is called *Legendrian* if it is everywhere tangent to ξ . Two Legendrian links are *Legendrian isotopic* if they are isotopic through other Legendrian links. A Legendrian link L is determined by its *front projection* (also denoted L) to the annulus,

$$J^1(S^1) \rightarrow S^1 \times \mathbb{R}, \quad (x, y, z) \mapsto (x, z),$$

because the y -coordinate of L is recovered as the slope dz/dx . Viewing S^1 as $[0, 1]/\{0, 1\}$, we visualize the front projection of L as a collection of arcs in $[0, 1] \times \mathbb{R}$ with identifications at the boundary.

A front projection of a Legendrian link is called *generic* if it is immersed away from semi-cubical cusp points and the only self intersections are transverse double points. L is called *σ -generic* if in addition the double points and cusps all have distinct x -coordinates. Any collection of closed curves in the annulus without vertical tangencies and satisfying the conditions of a generic front projection may be lifted to a unique Legendrian link in $J^1(S^1)$. It is not necessary to indicate the over/under relationship between two strands at a crossing of a front projection. The y -axis is oriented away from the observer, so the strand with lesser slope always appears on top. See Figure 4 below for an example of a front projection.

The equivalence relation of Legendrian isotopy may be formulated in a somewhat combinatorial fashion using front projections [14]. Any Legendrian isotopy class has representatives with generic front projections. Furthermore, if two generic front projections represent Legendrian isotopic links then one may be transformed into the other via a combination of the *Legendrian Reidemeister moves* indicated in Figure 1 and isotopies of the plane which do not introduce vertical tangencies. Note that moves corresponding to reflections of I and II across horizontal or vertical axes are allowed as well.

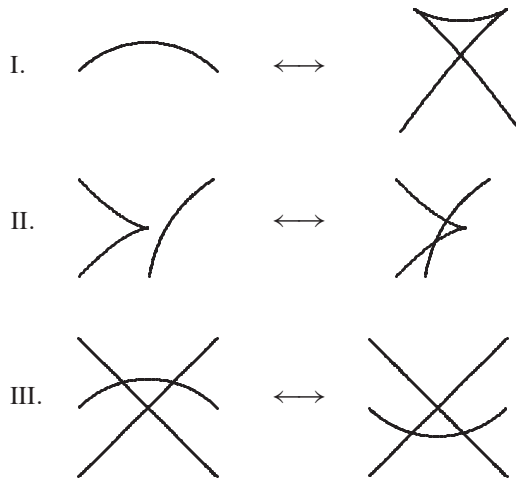


Figure 1. Legendrian Reidemeister moves.

2.1. Product of fronts. Given front projections K and L we define their product $K \cdot L$ by stacking K vertically above L in $S^1 \times \mathbb{R}$,

$$K \cdot L = \begin{array}{|c|} \hline K \\ \hline L \\ \hline \end{array} .$$

This product is well defined on Legendrian isotopy classes.

Remark 2.1. The corresponding product on smooth knot types is commutative. However, it follows from work of Traynor [16] that this is not always the case for Legendrian knot types. See Subsection 4.1.

2.2. Classical invariants. The simplest invariants capable of distinguishing between Legendrian links with the same underlying smooth link type are the *Thurston–Bennequin number* tb and the *rotation number* r . For a Legendrian link $L \subset J^1(S^1)$ the Thurston–Bennequin number is computed from a front projection of L as

$$tb(L) = \text{writhe}(L) - \frac{1}{2}(\# \text{ of cusps}),$$

and the rotation number is given by

$$r(L) = \frac{1}{2}((\# \text{ of downward oriented cusps}) - (\# \text{ of upward oriented cusps})).$$

Remark 2.2. If L is homologically trivial then $tb(L)$ is the linking number of L with a link L^+ obtained by a small shift in the oriented normal direction to the contact planes. There exist differing conventions for extending the definition of tb to homologically non-trivial links in $J^1(S^1)$. We follow the definition used in [11] which is natural when working with front projections. Geometrically, $tb(L)$ is the index of intersection of L^+ with an oriented surface bounded by L and an appropriate number of copies of A_1 or A_{-1} (see Section 4) located far away from L in the z -direction. Alternatively, Tabachnikov defined in [15] a “Bennequin affine invariant” for Legendrian links in $ST^*\mathbb{R}^2$ using instead an oriented surface bounded by L and some number of distant fibers of the projection $ST^*\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Under the standard contactomorphism $ST^*\mathbb{R}^2 \cong J^1(S^1)$ the front diagrams of these fibers appear as phase shifted cosine functions with large amplitudes. [3] follows the latter convention.

3. Normal rulings in $J^1(S^1)$

In this section we review Legendrian isotopy invariants introduced by Chekanov and Pushkar in [2]. The invariants depend on a choice of divisor $p|2r(L)$ and are computed as counts of additional combinatorial structures associated to a front diagram which we will call p -graded normal rulings. This terminology follows Fuchs [4] who, in connection with augmentations of the Chekanov–Eliashberg DGA, independently introduced similar combinatorial structures for front diagrams of Legendrian knots in standard contact \mathbb{R}^3 .

3.1. Maslov potentials. After removing cusp points a front diagram is divided into a union of immersed curves which we will call *strands*. Note that along each strand the orientation of L either entirely agrees or entirely disagrees with the orientation of $S^1 = [0, 1]/\{0, 1\}$. In this regard we may view the orientation of L as a function from the strands of L to $\mathbb{Z}/2\mathbb{Z}$ where we make the convention that the value 0 (1) indicates a strand oriented to the right (left).

Definition 3.1. A *Maslov potential* for a generic front diagram of a Legendrian knot L is a function μ from the strands of L to $\mathbb{Z}/(2r(L)\mathbb{Z})$ so that at cusps the value increases by 1 when moving from the lower half of the cusp to the upper half. See Figure 2. For convenience, we require that reducing $\mu \bmod 2$ gives the orientation of L . A Maslov potential for a multi-component link is a choice of Maslov potential for each component.

During any of the three Legendrian Reidemeister moves there is a unique way to extend a Maslov potential so that μ is unchanged outside of the region where the move occurs. The case of a Type I move is most interesting as it is the only one which

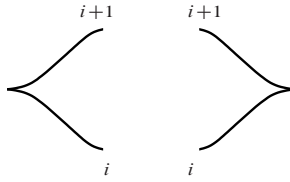


Figure 2. A Maslov potential near cusps.

can create new strands. In this manner, Maslov potentials are extended in a unique way along any generic Legendrian isotopy.

3.2. p -graded normal rulings. Suppose that L is a σ -generic front projection. Under this assumption the subset $\Sigma \subset S^1$ of x -values where L has double points or cusps is finite and for each $x_0 \in \Sigma$ the subset $\{x = x_0\} \subset S^1 \times \mathbb{R}$ intersects a single crossing or cusp of L .

Let $\pi: S^1 \times \mathbb{R} \rightarrow S^1$ denote the projection and, for each $x \in S^1$, $L_x = L \cap \pi^{-1}(x)$.

Definition 3.2. A continuous function f from a subset $N \subset S^1$ to the front projection $L \subset S^1 \times \mathbb{R}$ is called a *section* if $\pi \circ f = \text{id}_N$.

Definition 3.3. A *normal ruling* of a front projection L is a continuous function $\rho: L \setminus \pi^{-1}(\Sigma) \rightarrow L \setminus \pi^{-1}(\Sigma)$ satisfying the following conditions.

- (1) $\pi \circ \rho = \pi|_{L \setminus \pi^{-1}(\Sigma)}$, so for each $x \in S^1 \setminus \Sigma$ there is a restriction $\rho_x: L_x \rightarrow L_x$.
- (2) Each ρ_x is a fixed point free involution.

This condition together with the continuity of ρ implies that on any interval of $S^1 \setminus \Sigma$ the strands of L are divided into pairs. The remaining requirements give restrictions on this pairing near crossings and cusps.

- (3) Strands meeting at a cusp are paired by the involutions ρ_x in a neighborhood of the cusp point. The pairing of the remaining non-cusp strands should agree before and after the cusp.
- (4) Near a crossing the two strands that meet should not be paired together by ρ_x .
- (5) The pairing of strands arising from ρ_x can be continuously extended along a crossing in the following sense. Let $x_0 \in \Sigma$ such that L_{x_0} contains a crossing of L . In a neighborhood $N \subset S^1$ of x_0 one should be able to find a number of sections $f_1, \dots, f_n: N \rightarrow L$ so that every point of $L \cap \pi^{-1}(N)$ is in the image of exactly one of the f_i with the exception of the double point which is in the image of two of the f_i . Furthermore, these sections should be preserved by the involutions ρ_x , so that on $N \setminus \Sigma$ for each $f_i, \rho \circ f_i = f_j$ for some $f_j, 1 \leq j \leq n$. For the two sections meeting at the crossing there are two possibilities. Either they follow the diagram and cross transversally at the crossing, or they switch

strands at the crossing in a non-smooth manner. In the latter case, the crossing is called a *switch* of ρ .

Finally, we have a restriction at switches known as the *normality condition*.

- (6) Near switches of ρ the two intervals on the z -axis arising from connecting crossing strands to their companion strands are either disjoint or one is contained in the other.

Remark 3.4. (i) Three of the six possible arrangements of the switching strands and their companions along the vertical axis are prohibited by the normality condition. See Figure 3.

(ii) For Legendrian links in \mathbb{R}^3 it is usual to describe a normal ruling as a decomposition of the front diagram into pairs of paths with monotonic x -coordinates and each pair beginning and ending at shared left and right cusps (see for instance [4]). The continuity condition on ρ , together with (3) and (6), amount to requiring that at least locally we may still view a normal ruling as such a decomposition. This is the way the rulings are presented in Figure 5 below. However, note that the pairs of paths do not need to line up as one winds all the way around the annulus.

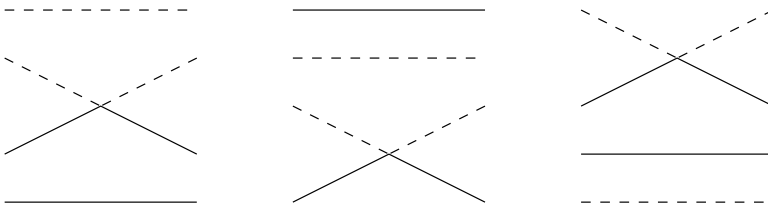


Figure 3. The three configurations for switching strands and companion strands allowed by the normality condition.

Suppose that a Legendrian link L has components L_1, \dots, L_N , and let p be a common divisor of $2r(L_i), i = 1, \dots, N$.

Definition 3.5. A normal ruling ρ of L is called *p-graded* with respect to a particular Maslov potential μ for L if, after reducing μ modulo p , whenever two strands are paired by the involutions ρ_x the strand with the larger z -coordinate has Maslov potential 1 larger than the strand with smaller z -coordinate. That is,

$$\rho(x, z) = (x, z') \text{ and } z' > z \Rightarrow \mu(x, z') = \mu(x, z) + 1 \pmod{p}.$$

Remark 3.6. (i) Every normal ruling is 1-graded.

(ii) We are most interested in the case $p = 2$. Note that, a normal ruling is 2-graded exactly when ρ reverses orientation. Choosing a Maslov potential is unnecessary.

(iii) If a normal ruling is p -graded, then at each of the switches the Maslov potentials of the crossing strands must agree modulo p . However, in contrast to

Legendrian links in \mathbb{R}^3 this condition is no longer sufficient for a normal ruling to be p -graded.

(iv) For a single component link the p -graded condition is independent of the choice of Maslov potential since any two Maslov potentials will differ by a constant.

(v) If p is even, then the involutions ρ_x reverse the orientation of L . It follows that only null-homologous links can have p -graded normal rulings when p is even.

Given a Legendrian link L with σ -generic front projection and chosen Maslov potential μ , we let $\Gamma^p(L, \mu)$ denote the set of normal rulings of L which are p -graded with respect to μ . To each $\rho \in \Gamma^p(L, \mu)$ we associate the integer²

$$j(\rho) = \#(\text{switches}) - \#(\text{right cusps}).$$

Finally, we define the p -graded ruling polynomial $R_{L,\mu}^p(z)$ as

$$R_{L,\mu}^p(z) = \sum_{\rho \in \Gamma^p(L,\mu)} z^{j(\rho)}.$$

As remarked above, if $p = 1, 2$ or L has a single component, then the choice of μ is not relevant and will be suppressed from the notation.

Given a sufficiently generic Legendrian isotopy between links L_1 and L_2 with σ -generic front projections, Chekanov and Pushkar provide a bijection between $\Gamma^1(L_1)$ and $\Gamma^1(L_2)$ which preserves the integers $j(\rho)$. Assuming the isotopy takes a Maslov potential μ_1 for L_1 to the corresponding Maslov potential μ_2 for L_2 their bijection takes $\Gamma^p(L_1, \mu_1)$ to $\Gamma^p(L_2, \mu_2)$.

Theorem 3.7 ([2]). *If there is a Legendrian isotopy between L_1 and L_2 which is compatible with corresponding Maslov potentials μ_1 and μ_2 then*

$$R_{L_1,\mu_1}^p(z) = R_{L_2,\mu_2}^p(z).$$

In particular, $R_L^1(z)$ and $R_L^2(z)$ are Legendrian isotopy invariants.

Example 3.8. In Figure 4 a Legendrian link $K \subset J^1(S^1)$ is presented via its front projection to $S^1 \times \mathbb{R}$. Two normal rulings of K are pictured in Figure 5. Both of the rulings are 0-graded with respect to the indicated Maslov potential μ . In case the value of μ on the lower component were altered to 3 the pictured rulings would remain 2-graded but would fail to be 0-graded. K has several other normal rulings, and its 2-graded and 0-graded ruling polynomials are given by

$$R_K^2(z) = 2 + 3z^2 + z^4, \quad R_{K,\mu}^0(z) = 2 + z^2.$$

²This differs by 1 from the convention for $j(\rho)$ used in [12].

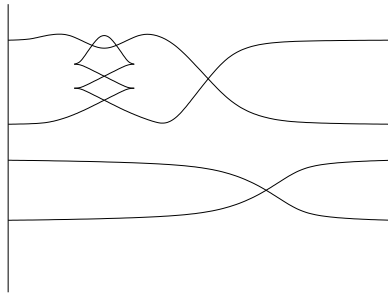


Figure 4. An annular front projection for a Legendrian link $K \subset J^1(S^1)$.

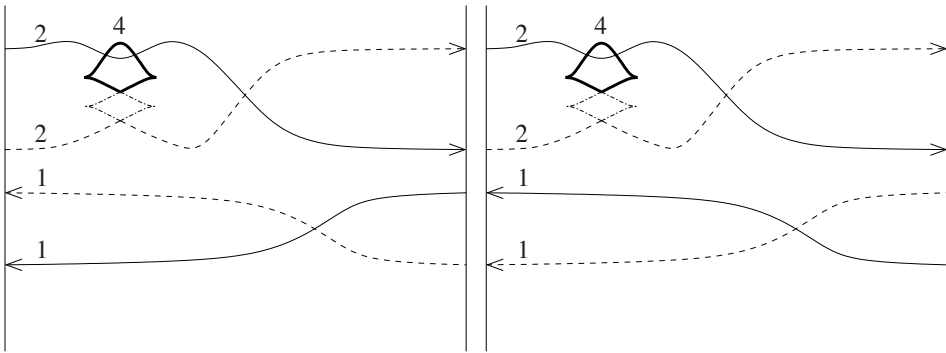


Figure 5. Two normal rulings of the front projection K .

4. Computation of R^2 for products of basic fronts

For each positive integer $m \geq 1$, we consider the front diagram A_m which consists of a single component wrapping m times around the annulus with $m - 1$ crossings. A_m is everywhere oriented to the right in $S^1 \times \mathbb{R}$ and can be viewed as the closure of the m -braid, $\sigma_1 \sigma_2 \dots \sigma_{m-1}$. Here, we compose braids from left to right and number strands from top to bottom. See Figure 6. We let A_{-m} denote A_m with its orientation reversed.

We will sometimes refer to the front diagrams A_m as *basic fronts*. The basic fronts will play a crucial role as their products form a basis for the HOMFLY-PT skein module of the annulus (See Section 5).

Recall that a finite non-increasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\lambda_i \geq \lambda_{i+1}, 1 \leq i \leq \ell - 1$ is called a *partition*. If $\sum \lambda_i = n$ we say that λ is a *partition of n* and write $\lambda \vdash n$. The integers $\lambda_i, 1 \leq i \leq \ell$ are called the parts of λ , and we write $\lambda = 1^{m_1} 2^{m_2} \dots r^{m_r}$ to indicate that λ is the partition with m_k parts equal to $k, 1 \leq k \leq r$, and no part larger than r . The total number of parts, $\ell = \ell(\lambda) = m_1 + \dots + m_r$, is called the *length of λ* .

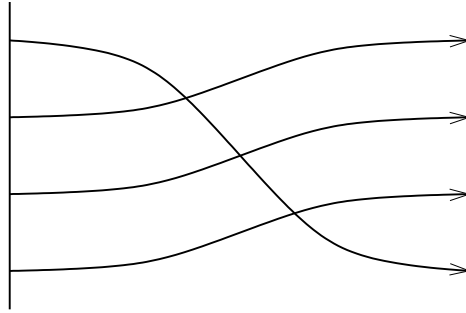


Figure 6. The front diagram A_4 .

Lemma 4.1. For any $m \geq 1$,

$$R_{A_m A_{-m}}^2(z) = \sum_{\lambda \vdash m} \left(\frac{\ell(\lambda)!}{m_1! m_2! \dots m_r!} \right) (1^{m_1} 2^{m_2} \dots r^{m_r}) z^{2(\ell(\lambda)-1)}.$$

Proof. First, note that there are precisely m normal rulings of $A_m A_{-m}$ which have no switches. The continuity conditions required in the definition of normal ruling show that such a ruling is uniquely determined by the value of the involution on a single strand of A_m near $x = 0$. Furthermore, an arbitrary choice of this value among the strands of A_{-m} may always be extended to a ruling without switches.

Now, given a ruling ρ of $A_m A_{-m}$ consider the front diagram, F_ρ , arising from resolving the switches of ρ into pairs of horizontal arcs as



ρ gives rise to a normal ruling of F_ρ without switches, and the continuity and normality conditions force that

$$F_\rho = (A_{i_1} \dots A_{i_\ell})(A_{-i_\ell} \dots A_{-i_1}), \quad i_1 + \dots + i_\ell = m$$

with the induced ruling on F_ρ pairing the factors on the left with those on the right in opposite order. Conversely, any such choice of decomposition $m = i_1 + \dots + i_\ell$, $i_1, \dots, i_\ell > 0$ and switchless rulings for $A_{i_j} A_{-i_j}$, $1 \leq j \leq \ell$, arises in this way from a unique ruling of $A_m A_{-m}$. The terms in the decomposition $m = i_1 + \dots + i_\ell$ may be reordered to give a partition $\lambda \vdash m$. In the statement of the lemma, the first term in the sum is the number of ways to rearrange the parts of λ to produce (i_1, \dots, i_ℓ) and the second term $(1^{m_1} 2^{m_2} \dots r^{m_r})$ accounts for the choices of switchless rulings. Finally, the number of switches in a ruling described by this data is $2(\ell(\lambda) - 1)$ which explains the power of z . \square

For $m \geq 1$ we introduce the notation

$$\langle m \rangle = R_{A_m A_{-m}}^2(z),$$

and in accordance with Lemma 4.1 we set

$$\langle 0 \rangle = z^{-2}.$$

Next we extend our computations of R^2 to products of the A_m . Given partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_k)$ we let A_λ and $A_{-\mu}$ denote the products

$$A_\lambda = A_{\lambda_1} \dots A_{\lambda_\ell} \quad \text{and} \quad A_{-\mu} = A_{-\mu_1} \dots A_{-\mu_k}.$$

Notice that Remark 3.6(v) implies that $R^2_{A_\lambda A_{-\mu}}(z) = 0$ unless $\sum \lambda_i = \sum \mu_i$.

Theorem 4.2. *Let $\lambda, \mu \vdash n$ with $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_k)$. Denote by $M_{\lambda, \mu}$ the set of $\ell \times k$ matrices with non-negative integer entries such that the entries in the i -th row sum to λ_i and the entries of the j -th column sum to μ_j . Then,*

$$R^2_{A_\lambda A_{-\mu}}(z) = z^{2\ell k - \ell - k} \sum_{(b_{ij}) \in M_{\lambda, \mu}} \prod_{i,j} \langle b_{ij} \rangle.$$

Proof. Let ρ be a normal ruling of $A_\lambda A_{-\mu}$. Divide each term A_{λ_i} into “blocks” B_{ij} where the block B_{ij} denotes the closure of the portion of A_{λ_i} paired with $A_{-\mu_j}$ by ρ . Distinct blocks can meet only at switches. The normality condition forces that if two blocks B_{ij} and B_{ik} meet at a switch with B_{ik} containing the upper half of the switching strands and B_{ij} the lower half then $j < k$. It follows that, after resolving the switches between distinct blocks into horizontal lines, A_{λ_i} becomes a product $A_{b_{ik}} \dots A_{b_{i2}} A_{b_{i1}}$ with the factor $A_{b_{ij}}$ corresponding to the block B_{ij} . (If the block $B_{i,j}$ is empty then we put $b_{ij} = 0$ and treat A_0 as an identity.) Clearly, $b_{i1} + b_{i2} + \dots + b_{ik} = \lambda_i$. Since the term $A_{-\mu_j}$ is the union of the closures of the images under ρ of blocks B_{ij} , we have also that $b_{1j} + b_{2j} + \dots + b_{\ell j} = \mu_j$. Therefore, $(b_{ij}) \in M_{\lambda, \mu}$. Notice that the ordering of $\rho(B_{ij})$ along the z -axis is likewise forced by the normality condition. Also, for each b_{ij} , ρ gives rise to a normal ruling of the front diagram comprised of the $A_{b_{ij}}$ factor of A_{λ_i} and the corresponding portion of $A_{-\mu_j}$ which may be viewed as $A_{-b_{ij}}$.

Conversely, a decomposition of each A_{λ_i} into $A_{b_{ik}} \dots A_{b_{i2}} A_{b_{i1}}$ and each $A_{-\mu_j}$ into $A_{-b_{\ell j}} \dots A_{-b_{2j}} A_{-b_{1j}}$ with $(b_{ij}) \in M_{\lambda, \mu}$ together with for each b_{ij} a choice of normal ruling for $A_{b_{ij}} A_{-b_{ij}}$ gives a unique ruling of $A_\lambda A_{-\mu}$. This justifies the terms in the summation. Notice that in each A_{λ_i} there are $k - 1 - \#\{j \mid b_{ij} = 0\}$ switches between distinct blocks. Similarly, in each $A_{-\mu_j}$ there are $\ell - 1 - \#\{i \mid b_{ij} = 0\}$ switches between the images of distinct blocks. Combined, these switches account for the $z^{2\ell k - \ell - k}$ term in front of the sum and the power of z arising from the product of $\langle b_{ij} \rangle$ with $b_{ij} = 0$. Remaining switches are accounted for in the $\langle b_{ij} \rangle$ with $b_{ij} \neq 0$ which correspond to the choices of rulings for each $A_{b_{ij}} A_{-b_{ij}}$. \square

4.1. Distinguishing $A_m A_n$ and $A_n A_m$ using 0-graded rulings. The product on smooth knot types arising from stacking knot diagrams is commutative. However, the

corresponding statement in the Legendrian setting fails to be true. For instance, using generating family methods Traynor [16] showed that it is not possible to interchange the two components of the Legendrian link A_1A_1 via a Legendrian isotopy. Using 0-graded ruling polynomials we are able to provide a generalization of Traynor's result.

Theorem 4.3. *Given non-zero integers m and n it is impossible to interchange the positions of the components of A_mA_n via a Legendrian isotopy. In particular, if $m \neq n$ then A_mA_n is not Legendrian isotopic to A_nA_m .*

In the following we may assume all Legendrian isotopies to be generic. This is done so that an initially chosen Maslov potential has a unique extension throughout the isotopy (see the remark after Definition 3.1).

Lemma 4.4. *Suppose $L_t, 0 \leq t \leq 1$ is a Legendrian isotopy, so that $L_0 = A_m$ and L_1 is a translation of A_m along the z -axis. If μ_0 is a Maslov potential for L_0 taking the value $\mu_0 = k \in \mathbb{Z}$ and μ_0 is extended during the isotopy to $\mu_t, 0 \leq t \leq 1$, then $\mu_1 = k$.*

Proof. Consider the Legendrian isotopy \tilde{L}_t arising from taking the products L_tA_{-m} . Here A_{-m} is placed sufficiently far along the negative z -axis to not intersect the fronts L_t at any point during the isotopy. We equip \tilde{L}_t with the Maslov potentials $\tilde{\mu}_t$ where

$$\tilde{\mu}_t|_{L_t} = \mu_t \quad \text{and} \quad \tilde{\mu}_t|_{A_{-m}} = k - 1.$$

Now using Theorem 3.7 and Lemma 4.1,

$$R_{\tilde{L}_1, \tilde{\mu}_1}^0(z) = R_{\tilde{L}_0, \tilde{\mu}_0}^0(z) = R_{A_mA_{-m}}^2(z) \neq 0.$$

However, if $\mu_1 \neq k$ then $R_{\tilde{L}_1, \tilde{\mu}_1}^0(z) = 0$ follows directly from Definition 3.5. \square

Proof of Theorem 4.3. Assume that $L_t, 0 \leq t \leq 1$, is a Legendrian isotopy with $L_0 = A_mA_n$ and $L_1 = A_nA_m$ so that during the course of the isotopy the two components are interchanged. Without loss of generality we can assume $|m| \geq |n|$ (if not reverse the isotopy) and that $m > 0$ and $n < 0$ (if not reverse orientations appropriately).

Now, consider the isotopy \tilde{L}_t arising from including an extra component A_{-m-n} far below the other two. Equip \tilde{L}_t with a Maslov potential μ_0 so that $\mu_0|_{A_m} = 1$ and $\mu_0|_{A_n} = \mu_0|_{A_{-m-n}} = 0$. μ_0 may be uniquely extended along the isotopy as $\mu_t, 0 \leq t \leq 1$. According to Lemma 4.4, μ_1 will take these same values on the respective components. (\tilde{L}_0, μ_0) has 0-graded rulings (they can be described as in Theorem 4.2), but (\tilde{L}_1, μ_1) does not. Thus, Theorem 3.7 gives the contradiction

$$0 = R_{\tilde{L}_1, \mu_1}^0(z) = R_{\tilde{L}_0, \mu_0}^0(z) \neq 0. \quad \square$$

However, 2-graded rulings cannot be used to distinguish products of the basic fronts A_m , and this will play a crucial role in the proof of Theorem 6.3.

Lemma 4.5. *If L_1 and L_2 are products of the basic fronts A_m which differ only in the ordering of factors then $R_{L_1}^2(z) = R_{L_2}^2(z)$.*

Proof. For such a link L suppose that the components of L are precisely $A_{\alpha_1}, \dots, A_{\alpha_\ell}$ and $A_{-\beta_1}, \dots, A_{-\beta_k}$ where $\alpha_1 \geq \dots \geq \alpha_\ell \geq 1$ and $\beta_1 \geq \dots \geq \beta_k \geq 1$. A slight variation of the proof of Theorem 4.2 shows that regardless of the order in which these factors appear we may compute

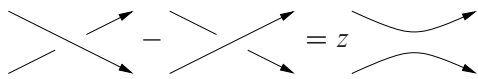
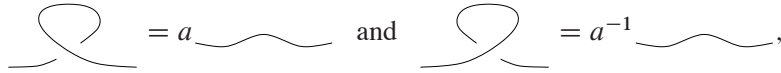
$$R_L = z^{2\ell k - \ell - k} \sum_{(b_{ij}) \in \mathcal{M}_{\alpha, \beta}} \prod_{i,j} \langle b_{ij} \rangle.$$

Once again, a ruling ρ of L divides each component A_{α_i} into “blocks” B_{ij} , $1 \leq j \leq k$ where $\rho(B_{ij}) \subset A_{-b_j}$. The key observation is that the normality condition still forces the ordering along the z -axis of both the blocks B_{ij} within A_{α_i} as well as their images $\rho(B_{ij})$ within $A_{-\beta_j}$. Specifically, the ordering of the B_{ij} within A_{α_i} must be as follows. Cut the z -axis just above A_{α_i} and glue the end at $+\infty$ to the end at $-\infty$. The factors $A_{-\beta_j}$ appear in some order along this now unbroken interval, and the ordering of the B_{ij} within A_{α_i} should be reverse to this. From here the calculation proceeds as in Theorem 4.2. \square

5. HOMFLY-PT skein module of the annulus

Let $R = \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ be the ring of Laurent polynomials in variables a and z . Denote by \mathcal{L} the set of equivalence classes of oriented link diagrams in the annulus up to regular isotopy. (That is, two diagrams are considered equivalent if they are related via Reidemeister moves of type II or III.) In addition, let $R\mathcal{L}$ denote the free R -module generated by \mathcal{L} .

The HOMFLY-PT skein module of the annulus, denoted \mathcal{C} , is the quotient of $R\mathcal{L}$ obtained by imposing the skein relations

- (i) 
- (ii) 
- (iii) $D \sqcup \text{loop} = \frac{a - a^{-1}}{z} D$.

Remark 5.1. (i) The third relation follows from the first two except in the case when D is the diagram of an empty link.

(ii) Here we consider usual diagrams of smooth knots and links rather than Legendrian front diagrams. However, a front diagram may be considered as a usual smooth knot diagram by rounding cusps and hence determines an element of \mathcal{C} .

(iii) \mathcal{C} inherits a multiplication from the stacking of diagrams as in Section 2. In contrast to the Legendrian case, the multiplication is commutative at the level of diagrams.

(iv) The diagrams appearing in the skein relations share the same homology class, so \mathcal{C} inherits a grading,

$$\mathcal{C} = \bigoplus_{x \in H_1(S^1)} \mathcal{C}_x.$$

Turaev introduced the skein module \mathcal{C} in [17] and proved that \mathcal{C} is free with linear basis, $\{A_\lambda A_{-\mu} \mid \lambda \vdash n_1, \mu \vdash n_2; n_1, n_2 \geq 0\}$ consisting of monomials in the basic fronts $A_{\pm m}$. \mathcal{C} has subalgebras

$$\mathcal{C}^+ = \bigoplus_{n \geq 0} \mathcal{C}_n^+, \quad \mathcal{C}_n^+ = \text{span}\{A_\lambda \mid \lambda \vdash n\},$$

$$\mathcal{C}^- = \bigoplus_{n \geq 0} \mathcal{C}_{-n}^-, \quad \mathcal{C}_{-n}^- = \text{span}\{A_{-\mu} \mid \mu \vdash n\}.$$

satisfying

$$\mathcal{C} = \mathcal{C}^+ \otimes \mathcal{C}^-. \tag{2}$$

Using Turaev’s basis, 2-graded ruling polynomials provide a linear map

$$\mathcal{C} \rightarrow R, \quad A_\lambda A_{-\mu} \mapsto R_{A_\lambda A_{-\mu}}^2(z)$$

which in view of eq. (2) may be considered as a bilinear form on \mathcal{C}^+ ,

$$\langle \cdot, \cdot \rangle: \mathcal{C}^+ \times \mathcal{C}^+ \rightarrow R, \quad \langle A_\lambda, A_\mu \rangle = R_{A_\lambda A_{-\mu}}^2(z).$$

Remark 5.2. $\langle \cdot, \cdot \rangle$ is symmetric as reversing the orientation of all components of a Legendrian link will not change the 2-graded ruling polynomial. We will see in the next section that $\langle \cdot, \cdot \rangle$ is actually a positive definite inner product.

5.1. Identification of \mathcal{C}^+ with the algebra of symmetric functions. \mathcal{C}^+ is a free algebra with unit possessing one generator A_m in each grading degree $m \geq 1$. Another well known graded algebra with this property is the algebra of symmetric functions Λ , and in this section we shall fix an isomorphism between them following existing conventions in the literature [1], [7], [9]. Turaev’s geometric basis A_λ will be identified with a deformation of the power sum symmetric functions. As the power sums form a rational basis for Λ it will be necessary to begin by enlarging our coefficient ring.

Let R' denote the smallest subring of rational functions in two variables a and s containing $\mathbb{Z}[a^{\pm 1}, s^{\pm 1}]$ as well as the denominators $s^r - s^{-r}$, $r \geq 1$. We set $z = s - s^{-1}$ so that $R \subset R'$. In this section, we consider the HOMFLY-PT skein module $\mathcal{C}_{R'} = R' \otimes_R \mathcal{C}$ over the coefficient ring R' although we will not continue to indicate this with our notation.

Let $\Lambda = \Lambda_{R'}$ denote the algebra of symmetric functions in a countably infinite set of variables $X = \{x_1, x_2, x_3, \dots\}$. Here we take coefficients in R' . Λ consists of formal polynomials in the x_i 's which are unchanged by permuting the variables. See for instance [8] or [13]. A grading, $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ arises where Λ_n consists of those symmetric functions which are homogeneous of degree n in the x_i 's.

Theorem 5.3 ([1], [7], [9]). *There is an isomorphism of graded algebras*

$$\begin{aligned} \mathcal{C}^+ &\cong \Lambda_{R'}, \\ Q_\lambda &\leftrightarrow s_\lambda \end{aligned}$$

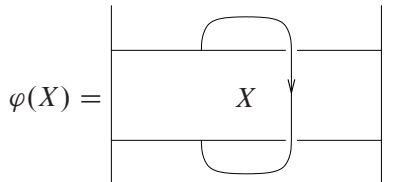
where s_λ denotes the Schur function and the Q_λ satisfy

$$A_m = \sum_{\substack{a+b=m-1 \\ a,b \geq 0}} (-1)^b s^{a-b} Q_{(a|b)} \tag{3}$$

for $m \geq 1$. Here, $(a | b)$ denotes the hook partition $(a + 1, \underbrace{1, \dots, 1}_b)$.

Remark 5.4. The skein elements Q_λ are described in [1]. They arise as closures (identify the boundaries) of linear combinations E_λ of link diagrams in the rectangle $[0, 1] \times \mathbb{R}$ with n boundary points on each of $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ oriented as inputs and outputs respectively. The E_λ are explicitly described in terms of the Young diagram of λ . The skein module generated by diagrams of this type in $[0, 1] \times \mathbb{R}$ is one version of the Hecke algebra \mathcal{H}_n (the product here is defined composing diagrams side to side rather than vertically) which specializes to the group algebra of the symmetric group S_n when $s = 1$. The E_λ are idempotents which specialize to appropriate multiples of the Young symmetrizers when $s = 1$.

Alternatively, in [7] the Q_λ are characterized up to scalars as the eigenvectors of the endomorphism $\varphi: \mathcal{C}^+ \rightarrow \mathcal{C}^+$ defined by adding an extra loop around a diagram



[7] provides as well a skein theoretic proof that the identification of the Q_λ with the Schur symmetric functions gives an algebra isomorphism between \mathcal{C}^+ and Λ . This

is remarked in [1] as a consequence of the fact that the $SU(N)_q$ quantum invariants of links in \mathbb{R}^3 with components decorated by irreducible representations V_λ may be computed from the HOMFLY-PT polynomial by satelliting each component with the corresponding Q_λ and then specializing the variables.

The relationship of Turaev’s basic fronts A_m with the Q_λ given in eq. (3) is found in [9]. Under this identification the basic fronts A_m specialize to the power sum functions when $s = 1$. [9] also contains formulas relating the A_m with other well known bases for Λ .

Remark 5.5. During the final preparation of this article the author noticed that a seemingly related deformation of the power sum symmetric functions has appeared in the literature on representation theory of Hecke algebras. The interested reader may wish to make a comparison of the A_μ described in the present paper with the symmetric functions $q_\mu(x; q)$ appearing in [6] keeping in mind that the versions of the Hecke algebra used there and in [9] differ a bit. We note that [6] contains a computation of the inner product $(q_\mu(x; q), q_\lambda(x; q))$ involving a sum of similar nature to the one appearing in our Theorem 4.2, and this result may be related to Theorem 5.6 below. However, no analog of the variable z is considered in [6], and the proofs seem to be quite different.

The algebra Λ has a standard inner product with respect to which the Schur functions form an orthonormal basis. Hence, it is natural to define an inner product on \mathcal{C}^+ so that the Q_λ form an orthonormal basis,

$$(\cdot, \cdot) : \mathcal{C}^+ \times \mathcal{C}^+ \rightarrow R', \quad (Q_\lambda, Q_\mu) = \delta_{\lambda, \mu}. \tag{4}$$

Note that the graded components $\mathcal{C}_n^+ \subset \mathcal{C}^+$ are orthogonal to one another since $Q_\lambda \in \mathcal{C}_n^+$ when $\lambda \vdash n$.

It turns out that (\cdot, \cdot) may be interpreted on Turaev’s basis A_λ in terms of ruling polynomials, and in fact agrees with the bilinear form $\langle \cdot, \cdot \rangle$ defined earlier in this section.

Theorem 5.6. *For any partitions λ and μ ,*

$$(A_\lambda, A_\mu) = R_{A_\lambda A_{-\mu}}^2(z) = \langle A_\lambda, A_\mu \rangle.$$

After providing some lemmas we complete this section with the proof of Theorem 5.6.

Lemma 5.7. *For $m \geq 1, (A_m, A_m) = \langle m \rangle$.*

By definition, we have $\langle m \rangle = \langle A_m, A_m \rangle$ which was computed in Lemma 4.1.

Proof. Consider the generating function $F(t) = z^2 \sum_{m \geq 0} \langle m \rangle t^m$ (we maintain here the convention that $\langle 0 \rangle = z^{-2}$). Standard calculations with formal power series show that

$$F(t) = \left(1 - \sum_{m \geq 1} m z^2 t^m\right)^{-1}.$$

Now, introduce the notation

$$\{m\} = (A_m, A_m) = \sum_{a=0}^{m-1} s^{2(2a-(m-1))} = \frac{s^{2m} - s^{-2m}}{s^2 - s^{-2}}$$

(second equality follows from eq. 3) and generating function

$$G(t) = 1 + \sum_{m \geq 1} z^2 \{m\} t^m.$$

To see that $F(t) = G(t)$ we show that in the product

$$G(t) \left(1 - \sum_{m \geq 1} m z^2 t^m\right) = 1 + \sum_{m \geq 1} \left(\left[\sum_{k=1}^{m-1} z^2 \{k\} (-(m-k)z^2) \right] - m z^2 + z^2 \{m\} \right) t^m$$

the coefficients of t^m vanish for $m \geq 1$. After removing a factor of $\frac{z^2}{s^2 - s^{-2}}$ the m -th coefficient becomes

$$\begin{aligned} & \left[\sum_{k=1}^{m-1} (s^{2k} - s^{-2k})(k-m)z^2 \right] + s^{2m} - s^{-2m} - m(s^2 - s^{-2}) \\ &= \left[\sum_{k=1}^{m-1} (k-m)(s^{2k} - s^{-2k})(s^2 - 2 + s^{-2}) \right] + s^{2m} - s^{-2m} - m(s^2 - s^{-2}). \end{aligned}$$

Expand the product in the summation. After collecting terms into pairs and reindexing the summations we have

$$\begin{aligned} & \left[\sum_{k=2}^m (k-m-1)(s^{2k} - s^{-2k}) \right] + (s^{2m} - s^{-2m}) \\ &+ (-m)(s^2 - s^{-2}) + \left[\sum_{k=0}^{m-2} (k-m+1)(s^{2k} - s^{-2k}) \right] \\ &+ ((m-1) - m + 1)(s^{2(m-1)} - s^{-2(m-1)}) \\ &+ \sum_{k=1}^{m-1} (-2)(k-m)(s^{2k} - s^{-2k}) \end{aligned}$$

$$= \sum_{k=1}^{m-1} [(k - m - 1) + (k - m + 1) - 2(k - m)](s^{2k} - s^{-2k}) = 0. \quad \square$$

To deduce the more general calculation of (A_λ, A_μ) from that of (A_m, A_m) we make use of a coproduct on Λ . As described, for instance in [8], p. 91, one can consider $\Lambda \otimes \Lambda$ as consisting of functions of two countably infinite sets of variables X and Y which are symmetric with respect to permutations of both X and Y . Due to the countable number of variables, given $f \in \Lambda$ one may define $\Delta(f) \in \Lambda \otimes \Lambda$ by using a bijection $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ to substitute

$$\Delta(f)(X, Y) = f(X, Y).$$

Properties of Δ which will be important for us include

- Δ is an algebra homomorphism. (In fact Λ may be given the structure of a Hopf algebra.)
- With respect to (\cdot, \cdot) and the induced inner product³ on $\Lambda \otimes \Lambda$, Δ is adjoint to multiplication. That is, for any $f, g, h \in \Lambda$

$$(f, g \cdot h) = (\Delta(f), g \otimes h).$$

- The coproduct of a generic Schur function Q_λ may be computed as

$$\Delta(Q_\lambda) = \sum_{\mu, \nu} c_{\mu\nu}^\lambda Q_\mu \otimes Q_\nu,$$

where $c_{\mu\nu}^\lambda$ are the Littlewood–Richardson coefficients.

Recall that $c_{\mu\nu}^\lambda$ is 0 unless the Young diagram of μ is contained in that of λ . In the latter case $c_{\mu\nu}^\lambda$ is the number of Littlewood–Richardson tableaux of shape $\lambda \setminus \mu$ consisting of ν_1 1’s, ν_2 2’s, etc. In turn, such a tableau T is given by removing those boxes in the Young diagram of λ which are contained in μ and then labeling the remaining boxes with positive integers so that

- rows are weakly increasing from left to right and columns are strictly increasing from top to bottom, and
- if a word $w_1 w_2 \dots w_n$ is formed from the entries of T by reading each row from right to left and working top to bottom, then for $k, l \geq 1$ the number of occurrences of k in the truncation $w_1 w_2 \dots w_l$ is greater than or equal to the number of occurrences of $k + 1$.

To simplify the next formula we make the convention that $Q_{(a|b)} = 0$ if one of a or b is negative and the other is positive.

³The induced inner product is characterized by bilinearity and $(f_1 \otimes f_2, g_1 \otimes g_2) = (f_1, g_1) \cdot (f_2, g_2)$.

Lemma 5.8. *For the hook partition $(a \mid b) \vdash m$ we have*

$$\begin{aligned} \Delta(Q_{(a|b)}) &= \sum_{k=0}^{m-2} \left(\sum_{a'+b'=k} Q_{(a'|b')} \otimes Q_{(a-a'-1|b-b')} \right. \\ &\quad \left. + Q_{(a'|b')} \otimes Q_{(a-a'|b-b'-1)} \right) \\ &\quad + Q_{\emptyset} \otimes Q_{(a|b)} + Q_{(a|b)} \otimes Q_{\emptyset}. \end{aligned}$$

Proof. The convention guarantees that in the summation only $(a' \mid b')$ with $a' \leq a$ and $b' \leq b$ appear. When both inequalities are strict there are two Littlewood–Richardson tableaux. The top row of such a tableau must consist entirely of 1’s and the left hand column will consist of consecutive integers beginning with either 1 or 2. The first of these accounts for the $Q_{(a-a'|b-b'-1)}$ term and the second for $Q_{(a-a'-1|b-b')}$. If $a' = a$ or $b' = b$, then there is only one Littlewood–Richardson tableau of shape $(a \mid b) \setminus (a' \mid b')$ and according to the convention one of the terms in the sum will correspondingly vanish. The only remaining possibilities for μ are \emptyset or $(a \mid b)$ and these account for the other two terms. \square

Proposition 5.9. *Letting $A_0 = z^{-1}$ we have for $m \geq 1$*

$$\Delta(A_m) = z \sum_{i=0}^m A_i \otimes A_{m-i}.$$

Proof. Starting from eq. 3, we compute

$$\begin{aligned} \Delta(A_m) &= \sum_{a+b=m-1} (-1)^b s^{a-b} \Delta(Q_{(a|b)}) && \text{(by Lemma 5.8)} \\ &= \sum_{a+b=m-1} (-1)^b s^{a-b} \left(\sum_{k=0}^{m-2} \sum_{a'+b'=k} Q_{(a'|b')} \otimes Q_{(a-a'-1|b-b')} \right. \\ &\quad \left. + Q_{(a'|b')} \otimes Q_{(a-a'|b-b'-1)} \right) \\ &\quad + \sum_{a+b=m-1} Q_{\emptyset} \otimes ((-1)^b s^{a-b} Q_{(a|b)}) \\ &\quad + ((-1)^b s^{a-b} Q_{(a|b)}) \otimes Q_{\emptyset}. \end{aligned}$$

The final two terms are just $1 \otimes A_m + A_m \otimes 1 = z(A_0 \otimes A_m + A_m \otimes A_0)$. After putting $c = a - a'$ and $d = b - b'$ the first term becomes

$$\begin{aligned} &\sum_{k=0}^{m-2} \sum_{a'+b'=k} \sum_{c+d=m-1-k} (-1)^{b'+d} s^{a'+c-b'-d} (Q_{(a'|b')} \otimes Q_{(c-1|d)}) \\ &\quad + Q_{(a'|b')} \otimes Q_{(c|d-1)} \\ &= \sum_{k=0}^{m-2} \left(\sum_{a'+b'=k} (-1)^{b'} s^{a'-b'} Q_{(a'|b')} \right) \end{aligned}$$

$$\begin{aligned}
 & \otimes s \left(\sum_{(c-1)+d=(m-1-k)-1} (-1)^d s^{(c-1)-d} Q_{(c-1|d)} \right) \\
 & + \sum_{k=0}^{m-2} \left(\sum_{a'+b'=k} (-1)^{b'} s^{a'-b'} Q_{(a'|b')} \right) \\
 & \quad \otimes (-s^{-1}) \left(\sum_{c+(d-1)=(m-1-k)-1} (-1)^{d-1} s^{c-(d-1)} Q_{(c|d-1)} \right) \\
 & = \sum_{k=0}^{m-2} A_{k+1} \otimes (sA_{m-k-1} - s^{-1}A_{m-k-1}) = z \sum_{k=1}^{m-1} A_k \otimes A_{m-k}. \quad \square
 \end{aligned}$$

Proof of Theorem 5.6. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\mu = (\mu_1, \dots, \mu_k)$.

Inductively define operators

$$D_k: \mathcal{C}^+ \rightarrow (\mathcal{C}^+)^{\otimes k}, \quad D_1 = \text{id}, \quad D_{k+1} = (\Delta \otimes (\text{id})^{\otimes k-1}) \circ D_k.$$

From the properties of Δ and Proposition 5.9 we have that

- the D_k are algebra homomorphisms;
- $(f, g_1 g_2 \dots g_k) = (D_k(f), g_1 \otimes g_2 \otimes \dots \otimes g_k)$;
- again, letting $A_0 = z^{-1}$, $D_k(A_m) = z^{k-1} \sum_{i_1+\dots+i_k=m} A_{i_1} \otimes \dots \otimes A_{i_k}$, where the indices i_r are non-negative integers.

Now,

$$\begin{aligned}
 (A_\lambda, A_\mu) &= (D_k(A_{\lambda_1} \dots A_{\lambda_\ell}), A_{\mu_1} \otimes \dots \otimes A_{\mu_k}) \\
 &= (D_k(A_{\lambda_1}) \dots D_k(A_{\lambda_\ell}), A_{\mu_1} \otimes \dots \otimes A_{\mu_k}) \\
 &= \left(\left(z^{k-1} \sum_{b_{11}+\dots+b_{1k}=\lambda_1} A_{b_{11}} \otimes \dots \otimes A_{b_{1k}} \right) \right. \\
 & \quad \left. \dots \left(z^{k-1} \sum_{b_{\ell 1}+\dots+b_{\ell k}=\lambda_\ell} A_{b_{\ell 1}} \otimes \dots \otimes A_{b_{\ell k}} \right), A_{\mu_1} \otimes \dots \otimes A_{\mu_k} \right) \\
 &= z^{\ell k - \ell} \sum_{(b_{ij}) \in M_{\lambda, \mu}} \prod_{j=1}^k (A_{b_{1j}} \dots A_{b_{\ell j}}, A_{\mu_j}).
 \end{aligned} \tag{5}$$

We are able to restrict the sum to $(b_{ij}) \in M_{\lambda, \mu}$ because, as noted after eq. (4), the graded components of \mathcal{C}^+ are orthogonal with respect to (\cdot, \cdot) . To conclude, (5) becomes

$$\begin{aligned}
 & z^{\ell k - \ell} \sum_{(b_{ij}) \in M_{\lambda, \mu}} \prod_{j=1}^k (A_{b_{1j}} \otimes \dots \otimes A_{b_{\ell j}}, D_\ell(A_{\mu_j})) \\
 & = z^{\ell k - \ell} z^{k\ell - k} \sum_{(b_{ij}) \in M_{\lambda, \mu}} \prod_{i,j} (A_{b_{ij}}, A_{b_{ij}}),
 \end{aligned}$$



Figure 7. A positive crossing and a negative crossing.

where the final equality again uses orthogonality of graded components of \mathcal{C}^+ . According to Lemma 5.7 and Theorem 4.2 this is equal to $R^2_{A_\lambda A_{-\mu}}$. \square

6. 2-graded rulings and the Bennequin estimate

We define the HOMFLY-PT polynomial of a solid torus link L in two steps. First, using an annular diagram of L and Turaev’s basis we have

$$\mathcal{C} \cong \mathbb{Z}[a^{\pm 1}, z^{\pm 1}, A_{\pm 1}, A_{\pm 2}, \dots].$$

$$[L] \leftrightarrow H_L(a, z, A_i)$$

H_L is a regular isotopy invariant, and provides an invariant of L as a framed link (assume the framing is blackboard with respect to the projection of L used). The HOMFLY-PT polynomial of L is then defined using the normalization

$$P_L(a, z, A_i) = a^{-w(L)} H_L(a, z, A_i),$$

where $w(L)$ denotes the writhe of the diagram L . The writhe is a signed sum of crossings (see Figure 7) in the diagram used to compute H_L .

Chmutov and Goryunov established the following upper bound in $J^1(S^1)$.

Theorem 6.1 ([3]). *For any Legendrian link $L \subset J^1(S^1)$,*

$$tb(L) + |r(L)| \leq -\deg_a P_L.$$

Remark 6.2. A proof of Theorem 6.1 is given in Subsection 6.2.

In \mathbb{R}^3 there is a strong connection between an analogous bound and the 2-graded ruling polynomial [12]. Namely, $R^2(z)$ is equal to the coefficient of $a^{-tb(L)}$ in P_L . (Here we use the convention that the unknot is normalized to $(a - a^{-1})/z$.) As a consequence, L has a 2-graded ruling if and only if $tb(L) = -\deg_a P_L$.

Analogous results in $J^1(S^1)$ can be obtained provided we specialize the HOMFLY-PT polynomial using the inner product from Section 5. Specifically, for any $L \subset J^1(S^1)$ we let $\hat{P}_L(a, z)$ be the image of $P_L(a, z, A_i)$ under the $\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ -module morphism $\mathcal{C} \rightarrow \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ defined on Turaev’s basis according to

$$A_\lambda A_{-\mu} \mapsto (A_\lambda, A_\mu).$$

Explicitly,

$$\begin{aligned}
 P_L(a, z, A_i) = \sum_{\lambda, \mu} c_{\lambda, \mu}(a, z) A_\lambda A_{-\mu} &\mapsto \widehat{P}_L(a, z) = \sum_{\lambda, \mu} c_{\lambda, \mu}(a, z) (A_\lambda, A_\mu) \\
 &= \sum_{\lambda, \mu} c_{\lambda, \mu}(a, z) R_{A_\lambda A_{-\mu}}^2(z).
 \end{aligned}$$

Theorem 6.3. *For any Legendrian link $L \subset J^1(S^1)$*

$$R_L^2(z) = \text{coefficient of } a^{-tb(L)} \text{ in } \widehat{P}_L(a, z).$$

Example 6.4. For the Legendrian K with front diagram pictured in Figure 4 we have

$$tb(K) = 4,$$

$$P_K(a, z, A_i) = a^{-4}[(1 + z^2)A_2A_{-2}] + a^{-6}[zA_1^2A_{-2} + z^2A_2A_{-2}],$$

$$\widehat{P}_K(a, z) = a^{-4}(z^4 + 3z^2 + 2) + a^{-6}(z^4 + 3z^2), \text{ and}$$

$$R_K^2(z) = z^4 + 3z^2 + 2.$$

Corollary 6.5. *If a Legendrian link $L \subset J^1(S^1)$ has a 2-graded ruling, then $tb(L)$ is maximal among knots of the same smooth knot type as L .*

Proof. According to Theorem 6.3, if L admits a 2-graded ruling then the estimate from Theorem 6.1 is sharp. □

Corollary 6.6. *The 2-graded ruling polynomial $R^2(z)$ cannot distinguish Legendrian links in $J^1(S^1)$ with the same smooth knot type and Thurston–Bennequin number.*

Proof. Theorem 6.3 shows that $R_L^2(z)$ is a coefficient of \widehat{P}_L depending only on $tb(L)$, and \widehat{P}_L depends only on the underlying smooth knot type. □

6.1. Proof of Theorem 6.3. Let us introduce the notation $B_L(z)$ for the coefficient of $a^{-tb(L)}$ in $\widehat{P}_L(a, z)$. Note that $B_L(z)$ is a Legendrian isotopy invariant. Using a corresponding specialization of H_L , $B_L(z)$ is given as the coefficient of $a^{c(L)}$ in $\widehat{H}_L(a, z)$ where $c(L)$ is the number of right cusps of L .

The proof of Theorem 6.3 is based on several lemmas.

Lemma 6.7. $R_L^2(z) = B_L(z)$ whenever L is a product of the basic fronts $A_m, m = \pm 1, \pm 2 \dots$

Proof. From Lemma 4.5 we know that R_L^2 is independent of the ordering of the factors. This is immediate for B_L , so we may assume that $L = A_\lambda A_{-\mu}$. Then, $H_L = A_\lambda A_{-\mu}$ and so by the definition of the specialization we have

$$\widehat{H}_L = (A_\lambda, A_\mu) = R_{A_\lambda A_{-\mu}}^2(z).$$

Since L has no cusps the result follows. □

Lemma 6.8. *Both $R^2(z)$ and $B(z)$ satisfy skein relations*

$$\begin{aligned}
 \text{(i)} \quad & \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) - \left(\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) = z \left(\delta_1 \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} - \delta_2 \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right), \\
 \text{(ii)} \quad & \left(\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right) = \left(\begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right) = 0, \\
 \text{(iii)} \quad & K \sqcup \left(\begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} \right) = z^{-1} K.
 \end{aligned}$$

In (i), δ_1 (resp. δ_2) is 1 when the crossing in the first (resp. second) term on the LHS is positive and 0 if it is negative.

Remark 6.9. Although the orientations are not pictured they are assumed to agree (outside of the pictured portion) in the terms on the LHS of (i). Whichever term on the RHS has coefficient $\delta_i \neq 0$ is assumed to be oriented in agreement with the terms on the LHS.

Proof. The proof is the same as in [12] and will only be sketched here.

To see that R^2 satisfies (i), observe that for the two diagrams appearing on the LHS there is a bijection between those rulings where the visible crossing is not switched. Terms corresponding to these rulings cancel. Due to the 2-graded condition only one of the fronts on the LHS can have rulings with the crossing switched. These remaining rulings are in bijection with the rulings of the term on the RHS with $\delta_i \neq 0$.

For B_L , (i) and (iii) follow from the HOMFLY skein relations. For (ii), observe that replacing $A_\lambda A_{-\mu}$ with $R^2_{A_\lambda A_{-\mu}}(z)$ can only decrease the degree in a of P_L . Combining this with Theorem 6.1 gives

$$tb(L) \leq -\deg_a P_L \leq -\deg_a \hat{P}_L, \tag{6}$$

and if B_L is non-zero we must have equality throughout (6). This shows that if $B_L \neq 0$ then the estimate of Theorem 6.1 is sharp. For the front diagrams in (ii) this can never be the case because we can increase tb by removing the zig-zag. \square

The proof of Theorem 6.3 is then completed by

Lemma 6.10. *A Legendrian isotopy invariant function*

$$\mathcal{F} : \{ \text{Annular front diagrams} \} \rightarrow \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

satisfying the relations of Lemma 6.8 is uniquely determined by its values on products of the basic fronts, $A_i, i = \pm 1, \pm 2, \dots$

The proof of Lemma 6.10 is by induction on the value of a certain complexity function on front diagrams described in the following subsection. First we record some additional relations which follow from Lemma 6.8.

Lemma 6.11. *A Legendrian isotopy invariant satisfying the relations of Lemma 6.8 also satisfies*

$$\text{crossing} = \text{cup} + z \left(\delta_1 \text{crossing} - \delta_2 \text{cup} \right) \text{cup}.$$

where δ_1 (resp. δ_2) is 1 (resp. 0) when the crossings in the term on the LHS are positive and 0 (resp. 1) if they are negative.

Proof.

$$\begin{aligned} \text{crossing} &= \text{crossing} = \text{crossing} \\ &= \text{crossing} + z \left(\delta_1 \text{crossing} - \delta_2 \text{crossing} \right) \\ &= \text{cup} + z \left(\delta_1 \text{crossing} - \delta_2 \text{cup} \right) \text{cup}. \end{aligned}$$

The third equality is Lemma 6.8, and the rest are Legendrian isotopies. □

Remark 6.12. Actually the skein relations given in Lemma 6.11 and Lemma 6.8(i) are equivalent.

6.1.1. Setup for induction. For non-negative integers N and M we let $\text{Front}(N, M)$ denote the collection of front diagrams in a rectangle $[0, 1] \times \mathbb{R}$ with N boundary points on $\{0\} \times \mathbb{R}$ and M boundary points on $\{1\} \times \mathbb{R}$. Given front diagrams $f_1 \in \text{Front}(N_0, N_1)$ and $f_2 \in \text{Front}(N_1, N_2)$ we may form their product $f_1 f_2 \in \text{Front}(N_0, N_2)$ by rescaling the first coordinate and then identifying the right boundary of f_1 with the left boundary of f_2 . (This may involve modifying f_1 and f_2 a bit near their boundaries so that the boundary points fit together appropriately, but the result is well defined up to Legendrian isotopy.)

Definition 6.13. A front diagram in $\text{Front}(N, M)$ is called an *elementary tangle* if it contains a single crossing or cusp.

We adopt the convention of labeling the boundary points of a tangle in $\text{Front}(N, M)$ as $1, \dots, N$ and $1, \dots, M$ from top to bottom. We introduce notations for elementary tangles. A crossing between the strands with boundary points labeled m and $m + 1$ will be denoted as $\sigma_m \in \text{Front}(N, N)$, while a left (resp. right) cusp where the strands meeting at the cusp are labeled m and $m + 1$ at their boundary will be denoted as $l_m \in \text{Front}(N, N + 2)$ (resp. $r_m \in \text{Front}(N + 2, N)$), see Figure 8.

After cutting along vertical lines any σ -generic annular front diagram F may be decomposed into a product of elementary tangles,

$$F = f_1 f_2 \dots f_n, \quad f_i \in \text{Front}(N_{i-1}, N_i), \quad N_0 = N_n.$$

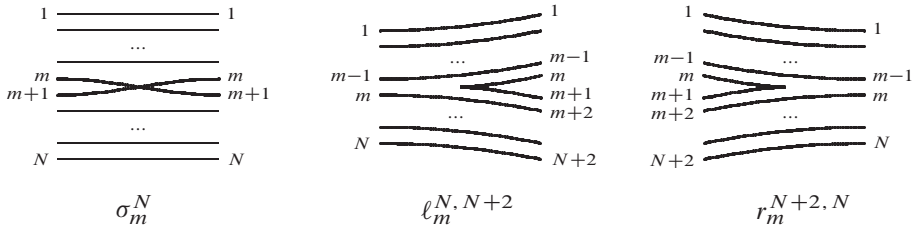


Figure 8. Elementary tangles.

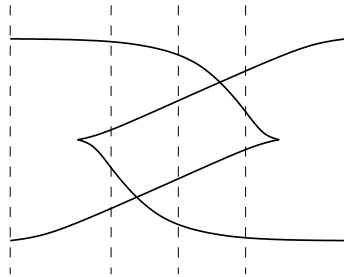


Figure 9. An annular front diagram F with $\text{Area}(F) = 4 + 4 + 4 + 2 = 14$.

Here each f_i is some σ_m , l_m , or r_m . The factors that appear in such a decomposition of F are unique up to cyclic reordering. n is called the *word length* of F . For our induction we need a slightly more refined measure of the complexity of a front.

Definition 6.14. Given a σ -generic annular front diagram F as above define the *word area* of F


$$\text{Area}(F) = \sum_{i=1}^n N_i.$$

Example 6.15. The basic front A_m has word area $m(m - 1)$. The front pictured in Figure 9 has word area 14. For each N there are fronts with word area 0 corresponding to the empty product in $\text{Front}(N, N)$. These are simply products of the basic fronts A_1 and A_{-1} .

Proof of Lemma 6.10. By induction on $\text{Area}(F)$. The base case follows from Lemma 6.7 since $\text{Area}(F) = 0$ implies F is a product of A_1 and A_{-1} .

For the inductive step, given an annular front F we need to show that $\mathcal{F}(F)$ may be evaluated in terms of the values of \mathcal{F} on basic fronts and fronts of lesser word area.

Case 1. F has no cusps.

We show that either F is a product of the A_i , or we can find a front F' Legendrian isotopic to F so that $\text{Area}(F) = \text{Area}(F')$ and part of F' has the form . In the latter case the result follows from Lemma 6.11 as all the front diagrams on the RHS have lesser area than F' .

Write F as a word in the σ_m . We describe an algorithm to transform F into the desired form using a combination of cyclic permutations and the braid relations

$$\sigma_k \sigma_l = \sigma_l \sigma_k, \quad |k - l| \geq 2, \quad \text{and} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

both of which correspond to word area preserving Legendrian isotopies.

Assume that we have successfully modified F to a front of the form

$$A_{i_1} \dots A_{i_r} \bar{F}, \quad r \geq 0 \tag{7}$$

(product of diagrams in the solid torus). If \bar{F} is empty then we have a product of basic fronts and the work is complete.

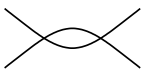
Else, write $\bar{F} = \sigma_1 \sigma_2 \dots \sigma_s W, s \geq 0$. We may always assume that W contains at least one σ_i with $i \leq s + 1$. If this is not the case than we could write $\bar{F} = A_{\pm(s+1)} G$ which allows us to absorb the first factor into the product in eq. (7) and replace \bar{F} with G .

Now, W has the form $\sigma_i W'$ and we proceed as follows.

- If $i > s + 1$, commute σ_i with $\sigma_1 \sigma_2 \dots \sigma_s$ and cyclically permute it to get

$$\sigma_1 \sigma_2 \dots \sigma_s \sigma_i W' \rightarrow \sigma_1 \sigma_2 \dots \sigma_s W' \sigma_i.$$

Replace W with $W' \sigma_i$ and repeat.

- If $i = s + 1$, increase s to $s + 1$ and repeat the argument with W replaced by W' .
- If $i = s$, then \bar{F} contains $\sigma_s \sigma_s =$  and the algorithm is complete.
- If $i < s$, then

$$\begin{aligned} & \sigma_1 \sigma_2 \dots \sigma_s \sigma_i W' \\ & \rightarrow \sigma_1 \dots \sigma_i \sigma_{i+1} \sigma_i \dots \sigma_s W' \\ & \rightarrow \sigma_1 \dots \sigma_{i+1} \sigma_i \sigma_{i+1} \dots \sigma_s W' \\ & \rightarrow \sigma_{i+1} \sigma_1 \dots \sigma_i \sigma_{i+1} \dots \sigma_s W' \\ & \rightarrow \sigma_1 \sigma_2 \dots \sigma_s W' \sigma_{i+1}. \end{aligned}$$

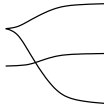
Now, replace W with $W' \sigma_{i+1}$ and repeat.

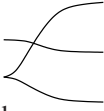
It is clear that this procedure cannot loop indefinitely. s is bounded above, and every time the case $i < s$ occurs the sum of the indices of the σ_i occurring in \bar{F} is increased.


Case 2. F has cusps.


The following is a slight modification of an argument from [12].

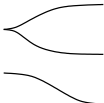

Note that if the result is known for fronts of lesser area then it is true for a diagram

of the form $\dots l_m \sigma_{m+1} \dots =$  if and only if it is true for $\dots l_{m+1} \sigma_m =$

. This follows from Lemma 6.8 since the diagrams appearing on the RHS have smaller area than the two on the LHS. We will refer to the interchanging of $l_m \sigma_{m+1}$ with $\dots l_{m+1} \sigma_m$ as a *skein move*. Note that performing a skein move does not change the word area of a front.

This case is dealt with by describing an algorithm which uses a combination of skein moves and Legendrian isotopies to reduce the word area of F or arrange the front diagram to contain a stabilization  or a disjoint unknot component

. Whenever skein moves are applied during the algorithm the word area will be such that the inductive hypothesis applies to the corresponding diagrams

 and  on the RHS of Lemma 6.8 so that they may be safely ignored.

In the case that the resulting diagram is stabilized the value of \mathcal{F} is 0 according to Lemma 6.8(ii), and in the case we arrive at an unknot component the value of \mathcal{F} is uniquely determined by Lemma 6.8(iii) together with the inductive hypothesis.

The algorithm is nearly identical to Statement A of [12], but for the reader's convenience we include the argument here. The reader is also referred to Figure 1 of [10] for an excellent pictorial description of the algorithm.

Begin by writing F as a product of elementary tangles. It must be the case that there exists a portion of this product which has the form $l_m W r_n$ where W is a word consisting entirely of crossings. (F has cusps. Therefore, it must contain both left cusps and right cusps, and one of these left cusps must appear adjacently to a right cusp.) A cyclic permutation then transforms F into a word of the form $l_m W r_n X$.

Now suppose we are given a word of the form $l_m W r_n X$ where W is a word in the σ_i which is written in the form

$$W = \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s} W' \text{ for some } s \geq 0.$$

Subcase 1. W' is non-empty.

Then $W' = \sigma_i W''$ and we proceed as follows.

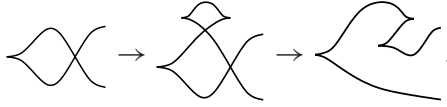
- (1) If $i < m - 1$, then a Legendrian isotopy commutes σ_i past $l_m \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s}$ and when it passes the cusp the word area decreases by 2.

- (2) If $i = m - 1$, we apply a Legendrian isotopy then a skein move

$$\begin{aligned} & l_m \sigma_{m+1} \dots \sigma_{m+s} \sigma_{m-1} W'' \dots \\ & \rightarrow l_m \sigma_{m-1} \sigma_{m+1} \dots \sigma_{m+s} W'' \dots \\ & \rightarrow l_{m-1} \sigma_m \sigma_{m+1} \dots \sigma_{m+s} W'', \end{aligned}$$

then repeat the argument with W' replaced by W'' .

- (3) If $i = m$, then when $s = 0$ the front is Legendrian isotopic to a stabilized front,



When $s > 0$ we can apply a Type II Legendrian Reidemeister move to decrease the word area

$$\begin{aligned} & l_m \sigma_{m+1} \dots \sigma_{m+s} r_n \dots \\ & \rightarrow l_m \sigma_{m+1} \dots \sigma_n \sigma_{n+1} r_n \dots \sigma_{m+s} \dots \\ & \rightarrow l_m \sigma_{m+1} \dots r_{n+1} \dots \sigma_{m+s} \dots \end{aligned}$$

- (4) If $m < i < m + s$ then we apply a Type III Reidemeister move and subsequently pass a crossing by the left cusp which decreases word area

$$\begin{aligned} & l_m \sigma_{m+1} \dots \sigma_{m+s} \sigma_i W'' \dots \\ & \rightarrow l_m \sigma_{m+1} \dots \sigma_i \sigma_{i+1} \sigma_i \dots \sigma_{m+s} W'' \dots \\ & \rightarrow l_m \sigma_{m+1} \dots \sigma_{i+1} \sigma_i \sigma_{i+1} \dots \sigma_{m+s} W'' \dots \\ & \rightarrow \sigma_{i+1} l_m \sigma_{m+1} \dots \sigma_{m+s} W''. \end{aligned}$$

- (5) If $i = m + s, s > 0$ then we can apply s successive skein moves followed by a Type II Reidemeister move to decrease word area,

$$\begin{aligned} & l_m \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s} \sigma_{m+s} \dots \\ & \rightarrow l_{m+1} \sigma_m \sigma_{m+2} \dots \sigma_{m+s} \sigma_{m+s} \dots \\ & \rightarrow l_{m+1} \sigma_{m+2} \dots \sigma_{m+s} \sigma_{m+s} \sigma_m \dots \\ & \rightarrow \dots \\ & \rightarrow l_{m+s} \sigma_{m+s-1} \sigma_{m+s} \sigma_{m+s-2} \dots \sigma_{m+1} \sigma_m \dots \\ & \rightarrow l_{m+s-1} \sigma_{m+s-2} \dots \sigma_{m+1} \sigma_m. \end{aligned}$$

- (6) If $i = m + s + 1$ replace W' with W'' and repeat the algorithm.
 (7) If $i > m + s + 1$, a Legendrian isotopy commutes σ_i past $l_m \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s}$ and when it passes the cusp the word area decreases by 2.

Subcase 2. W' is empty.

We are given a word of the form $l_m \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s} r_n \dots$, for some $s \geq 0$.

- If $n < m-1$ or $n > m+s+1$ then we can commute r_n past $l_m \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s}$ and reduce the word area.
- If $n = m-1$, then

$$l_m \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s} r_{m-1} \dots \rightarrow l_m r_{m-1} \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s} \dots$$

which is stabilized.

- If $n = m$ then when $s = 0$ we have $l_m r_m \dots$ which has an unknot component. When $s > 0$ we can apply a Type I Legendrian Reidemeister move and decrease word area.
- If $m < n < m+s$ then we can apply a type II Reidemeister to decrease word area,

$$\begin{aligned} l_m \sigma_{m+1} \dots \sigma_{m+s} r_n \dots \\ \rightarrow l_m \sigma_{m+1} \dots \sigma_n \sigma_{n+1} r_n \dots \sigma_{m+s} \dots \\ \rightarrow l_m \sigma_{m+1} \dots r_{n+1} \dots \sigma_{m+s} \dots \end{aligned}$$

- If $n = m+s$, $s > 0$ then because of the presence of $\sigma_{m+s} r_{m+s}$ the front is Legendrian isotopic to a stabilization.
- If $n = m+s+1$ we can apply the skein move s times to obtain

$$\begin{aligned} l_m \sigma_{m+1} \sigma_{m+2} \dots \sigma_{m+s} r_{m+s+1} \dots \\ \rightarrow l_{m+1} \sigma_m \sigma_{m+2} \dots \sigma_{m+s} r_{m+s+1} \dots \\ \rightarrow l_{m+1} \sigma_{m+2} \dots \sigma_{m+s} r_{m+s+1} \sigma_m \dots \\ \rightarrow \dots \\ \rightarrow l_{m+s} r_{m+s+1} \sigma_{m+s-1} \sigma_{m+s-2} \dots \sigma_m \dots, \end{aligned}$$

which is stabilized. □

6.2. Proof of Theorem 6.1. In [3] a contactomorphism $J^1(S^1) \cong ST^*(\mathbb{R}^2)$ is used to treat Legendrian links as co-oriented plane curves, and their proof of Theorem 6.1 is carried out in this context. However, the version of P_L used in [3] differs from ours in a non-trivial manner as the annulus within $J^1(S^1)$ which is used there for link projections differs from ours by a full twist. We conclude by giving a proof of Theorem 6.1 matching our conventions. Our proof uses the front projection perspective and is based on the inductive method used in the proof of Lemma 6.10. This is similar to the approach to Bennequin type inequalities in \mathbb{R}^3 appearing in [10].

First, observe that for a Legendrian link $L \subset J^1(S^1)$ the inequality

$$tb(L) + |r(L)| \leq -\deg_a P_L \tag{8}$$

is equivalent to

$$\deg_a H_L \leq c(L) - |r(L)|. \tag{9}$$

Here, H_L is computed using the front projection of L , and $c(L)$ denotes the number of right cusps appearing in the front projection.

Observe that this inequality trivially holds for products of the A_i as both sides equal 0. We now establish (9) for a general front diagram F by induction on $\text{Area}(F)$. The base case is covered by the previous remark.

Case 1. F has no cusps.

As in the proof of Lemma 6.10, after a Legendrian isotopy either F becomes a product of basic fronts, or we can modify F to a front diagram F' containing

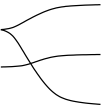
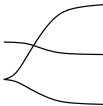
. In the latter case, use the HOMFLY-PT relations (i) and (iii) to compute $H_{F'}$ as

$$\text{Figure-eight} = \text{Saddle} + z(\delta_1 \text{Cross} + a^{-1}\delta_2 \text{Cusp}), \tag{10}$$



where exactly one of δ_1 and δ_2 is non-zero depending on the orientation of F' . Denote the 3 front diagrams appearing on the RHS as F_1 , F_2 , and F_3 . In general, $c(F) = c(F_1) = c(F_2) = c(F_3) - 1$ and $r(F) = r(F_1)$. Also, if $\delta_1 \neq 0$ (resp. $\delta_2 \neq 0$) then $r(F) = r(F_2)$ (resp. $r(F) = r(F_3)$). Thus, the inductive hypothesis applies to deduce that both non-zero terms on the RHS of (10) have degree in a less than or equal to $c(F) - |r(F)|$.

Case 2. F has cusps.

Provided the inductive hypothesis applies to fronts of lesser word area (9) holds


for a front containing  if and only if it holds for the front  obtained

from a skein move. The proof of Lemma 6.10 contains an algorithm which makes use of a combination of word area preserving Legendrian isotopies and skein moves to reduce the word area of F or arrange the front diagram to contain a stabilization

 or a disjoint unknot component . In the latter two cases, proceed as follows.

1. If F contains a stabilization the inductive hypothesis will apply to the front diagram F_0 obtained from removing the pair of cusps. We have $|r(F_0)| = |r(F)| \pm 1$, where the sign depends on the orientation of F . Then, $H_F = H_{F_0}$ and

$$\deg_a H_{F_0} \leq c(F_0) - |r(F_0)| = (c(F) - 1) - (|r(F)| \pm 1) \leq c(F) - |r(F)|.$$

2. If F contains a disjoint unknot component  then deduce (9) from the inductive hypothesis and HOMFLY-PT relation (iii).

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