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Quantum Topology

The embedding theorem for finite depth subfactor planar algebras

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Abstract. We define a canonical planar $*$-algebra from a strongly Markov inclusion of finite von Neumann algebras. In the case of a connected unital inclusion of finite dimensional C^*-algebras with the Markov trace, we show this planar algebra is isomorphic to the bipartite graph planar algebra of the Bratteli diagram of the inclusion. Finally, we show that a finite depth subfactor planar algebra is a planar subalgebra of the bipartite graph planar algebra of its principal graph.

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Contents

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1. Introduction

A powerful method of construction of subfactors is the use of *commuting squares,* which are systems of four finite dimensional von Neumann algebras

$$\begin{array}{ccc} A_{1,0} & \subset & A_{1,1} \\ \cup & & \cup \\ A_{0,0} & \subset & A_{0,1} \end{array}$$

included as above, with a faithful trace on $A_{1,1}$ so that $A_{1,0}$ and $A_{0,1}$ are orthogonal modulo their intersection $A_{0,0}$.

One iterates the basic construction of [8] for the inclusions $A_{i,j} \subset A_{i,j+1}$ and $A_{i,j} \subset A_{i+1,j}$ to obtain a tower of inclusions $A_{0,n} \subset A_{1,n}$. By a lovely compactness argument of Ocneanu, [12] and [5], the standard invariant, or higher relative commutants, of the inductive limit inclusion $A_{0,\infty} \subset A_{1,\infty}$ are the algebras $A'_{0,1} \cap A_{n,0}$. Thus once bases have been chosen, the calculation of the relative commutants is a matter of elementary linear algebra.

It was to formalise this calculation that planar algebras were first introduced [9]. Finite dimensional inclusions are given by certain graphs (Bratteli diagrams), and, in [10], a planar algebra associated purely combinatorially to a bipartite graph was introduced so that it is rather obviously the tower of relative commutants for an inclusion $B_0 \subset B_1$ having the graph as its Bratteli diagram. But because Ocneanu's notion of *connection* was never completely formalised in [9], it was *not* proved that the planar algebra coming from a commuting square via Ocneanu compactness is a planar subalgebra of the one defined in [10] for the graph of the inclusion $A_{0,0} \subset A_{1,0}$.

Meanwhile the theory of planar algebras grew in its own right and a new method of constructing subfactors evolved by looking at planar subalgebras of a given planar algebra, [17] and [2]. Now if a subfactor is of *finite depth*, then by [20], there is a commuting square that constructs a hyperfinite model of it. Moreover the inclusion $A_{0,0} \subset A_{1,0}$ for this canonical commuting square has Bratteli diagram given by the so-called *principal graph,* which is a powerful subfactor invariant. Thus if the result of the previous paragraph had been proved, it would have implied the following theorem, which is the main result of this paper.

Theorem. *A finite depth subfactor planar algebra is a planar subalgebra of the bipartite graph planar algebra of its principal graph.*

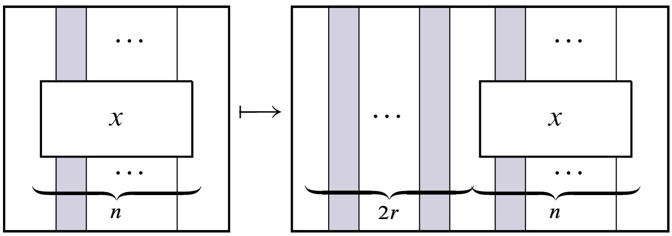
(See [14] for the definition of the principal graph of a planar algebra.)

We prove this result with the interesting twist of not using connections. In particular, our proof does not invoke the *dual principal graph,* which is perhaps rather surprising.

There are three steps to our proof. The first step, Section 2, is to define a canonical planar $*$-algebra structure on the tower of relative commutants from a connected unital inclusion of finite dimensional C^*-algebras whose Bratteli diagram is a given graph. We call this the *canonical planar $*$-algebra* associated to the inclusion. We do this in more generality, replacing finite dimensionality by a *strong Markov* property (see Definition 2.8), because it is no harder and should have applications.

The second step, Section 3, is to identify the canonical planar $*$-algebra with the bipartite graph planar algebra of [10] in the finite dimensional case. Loops on the Bratteli diagram for the inclusion give bases for the relative commutants, so the isomorphism is constructed by choosing bases for the vector spaces in the canonical planar $*$-algebra.

Finally, in Section 4, we construct the embedding map as follows: given a finite depth subfactor planar algebra $Q_\bullet$, pick $2r$ suitably large so that the inclusion $Q_{2r,+} \subset Q_{2r+1,+} \subset (Q_{2r+2,+}, e_{2r+1})$ is standard, i.e. isomorphic to the basic construction. Set $M_0 = Q_{2r,+}$ and $M_1 = Q_{2r+1,+}$, and let $P_\bullet$ be the canonical planar $*$-algebra $P_\bullet$ associated to the inclusion $M_0 \subset M_1$. We prove in Theorem 4.1 that the map $Q_\bullet \to P_\bullet$ given by adding $2r$ or $2r+1$ strings on the left, depending on whether we are in $Q_{n,+}$ or $Q_{n,-}$ respectively, is an inclusion of planar algebras.



While this paper was being written, Morrison and Walker in [15] produced a totally different proof which constructs an embedding directly from the planar algebra $Q_\bullet$ without the use of algebra towers and centralisers. Their method also has the advantage that it applies to *infinite depth* subfactor planar algebras without alteration!

2. The canonical planar $*$-algebra of a strongly Markov inclusion of finite von Neumann algebras

After defining the notion of a strongly Markov inclusion of finite von Neumann algebras, we show the basic construction is also strongly Markov with the same (Watatani) index. We then define the canonical planar $*$-algebra associated to a strongly Markov inclusion.

Many results of this section can be found in [8], [18], [23], [7], [21], [3], and [4], but our treatment differs slightly, so we provide some proofs for the reader's convenience.

2.1. Bases, traces, and strongly Markov inclusions

Notation 2.1. Throughout this paper, a *trace* on a finite von Neumann algebra means a faithful, normal, tracial state unless otherwise specified. We will write $M_0 \subset (M_1, \mathrm{tr}_1)$ to mean $M_0 \subset M_1$ is an inclusion of finite von Neumann algebras where tr_1 is a trace on M_1. We set $\mathrm{tr}_0 = \mathrm{tr}_1|_{M_0}$.

Let $M_0 \subset (M_1, \mathrm{tr}_1)$. Let $M_2 = \langle M_1, e_1 \rangle = JM_0'J \subset B(L^2(M_1, \mathrm{tr}_1))$ be the basic construction, where e_1 is the Jones projection with range $L^2(M_0, \mathrm{tr}_0)$, and $J : L^2(M_1, \mathrm{tr}_1) \to L^2(M_1, \mathrm{tr}_1)$ is the antilinear unitary given by the antilinear extension of $x\Omega \mapsto x^*\Omega$, where $\Omega \in L^2(M_1, \mathrm{tr}_1)$ is the image of $1 \in M_1$.

Recall from [22] that there is a unique trace-preserving conditional expectation $E_{M_0} : M_1 \to M_0$ determined by $\mathrm{tr}_1(xy) = \mathrm{tr}_0(E_{M_0}(x)y)$ for all $x \in M_1$ and $y \in M_0$, i.e. E_{M_0} is the (Banach) adjoint of the inclusion of preduals $(M_0)_* \to (M_1)_*$. The conditional expectation satisfies $e_1(x\Omega) = E_{M_0}(x)\Omega$ for all $x \in M_1$.

The following proposition is straightforward.

Proposition 2.2. *The following are equivalent for a finite subset $B = \{b\} \subset M_1$:*

(i) $1 = \sum_{b \in B} b e_1 b^*$,

(ii) $x = \sum_{b \in B} b E_{M_0}(b^* x)$ *for all $x \in M_1$, and*

(iii) $x = \sum_{b \in B} E_{M_0}(xb) b^*$ *for all $x \in M_1$.*

Definition 2.3. A *Pimsner–Popa basis* for M_1 over M_0 is a finite subset $B = \{b\} \subset M_1$ for which the conditions in Proposition 2.2 hold.

We refer the reader to [23] for the proof of the following result.

Proposition 2.4. *The following are equivalent:*

(i) *there is a Pimsner–Popa basis for M_1 over M_0,*

(ii) $M_1 \otimes_{M_0} M_1 \to M_2$ *by $x \otimes y \mapsto x e_1 y$ is an $M_1 - M_1$ bimodule isomorphism, and*

(iii) $M_2 = M_1 e_1 M_1$.

Remark 2.5. $M_1 \otimes_{M_0} M_1$ is a $*$-algebra with multiplication $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 \otimes E_{M_0}(y_1 x_2) y_2$ and adjoint $(x \otimes y)^* = y^* \otimes x^*$. If there is a Pimsner–Popa basis for M_1 over M_0, the sum $\sum_{b \in B} b \otimes b^*$ is independent of the choice of Pimsner–Popa basis B, as it is the identity. (We will renormalize in Proposition 2.25.)

Definition 2.6 ([23]). If there is a Pimsner–Popa basis $B = \{b\}$ for M_1 over M_0, then we define the (*Watatani*) *index*

$$[M_1 : M_0] = \sum_{b \in B} bb^*,$$

which is independent of the choice of basis.

Definition 2.7. Recall from [21] that M_2 has a canonical faithful, normal, semifinite trace Tr_2 which is the extension of the map $xe_1y \mapsto \mathrm{tr}_1(xy)$ for $x, y \in M_1$.

Definition 2.8. An inclusion $M_0 \subset (M_1, \mathrm{tr}_1)$ of finite von Neumann algebras is called *Markov* if it satisfies the Markov property:

(1) Tr_2 is finite with $\mathrm{Tr}_2(1)^{-1} \mathrm{Tr}_2 |_{M_1} = \mathrm{tr}_1$.

A Markov inclusion is called *strongly Markov* if

(2) there is a Pimsner–Popa basis for M_1 over M_0.

Remark 2.9. Markov inclusions have been studied by Jolissaint [7], Pimsner and Popa [18] and [21], and more. Jolissaint [7] showed that condition (1) implies condition (2) when the centers are atomic and the inclusion is *connected*, i.e. $Z(M_0) \cap Z(M_1) = M_1' \cap M_0$ is one dimensional. It is unknown to the authors at this point whether condition (1) implies condition (2) for connected inclusions with diffuse centers.

The adjective "strongly" in the term "strongly Markov" comes from Definition 3.6 in [1], where they define the notion of "fortement d'indice fini" for a conditional expectation. This notion translates as the existence of a finite Pimsner–Popa basis.

Remark 2.10. Recall from [21] that $\mathrm{Tr}_2(1)^{-1} \mathrm{Tr}_2$ extends tr_1 if and only if $\mathrm{Tr}_2(1) = [M_1 : M_0] \in [1, \infty)$.

Examples 2.11. (1) A finite Jones index inclusion of II_1-factors with the unique trace is strongly Markov, and the Watatani index is equal to the Jones index.

(2) A connected, unital inclusion of finite dimensional C^*-algebras with the Markov trace is strongly Markov, and the index is equal to $\|\Lambda^T \Lambda\|$ where Λ is the bipartite adjacency matrix for the Bratteli diagram of the inclusion.

Suppose $M_0 \subset (M_1, \mathrm{tr}_1)$ is strongly Markov. Then M_2 is finite and $\mathrm{tr}_2 = [M_1 : M_0]^{-1} \mathrm{Tr}_2$ extends tr_1, so we may iterate the basic construction for $M_1 \subset (M_2, \mathrm{tr}_2)$. Let $M_3 = \langle M_2, e_2 \rangle \subset B(L^2(M_2, \mathrm{tr}_2))$, where e_2 is the Jones projection with range $L^2(M_1, \mathrm{tr}_1)$. Let Tr_3 be the canonical faithful, normal, semifinite trace on M_3 (see Definition 2.7). The following lemma is straightforward.

Lemma 2.12. (1) *The conditional expectation* $E_{M_1}: M_2 \to M_1$ *is given by*

$$E_{M_1}(xe_1y) = xy,$$

(2) $e_1e_2e_1 = [M_1 : M_0]^{-1}e_1$ *and* $e_2e_1e_2 = [M_1 : M_0]^{-1}e_2$, *and*

(3) *if* B *is a Pimsner–Popa basis for* M_1 *over* M_0, *then* $\{[M_1 : M_0]^{1/2}be_1 \mid b \in B\}$ *is a Pimsner–Popa basis for* M_2 *over* M_1.

Theorem 2.13. $M_1 \subset (M_2, \mathrm{tr}_2)$ *is strongly Markov and* $[M_2 : M_1] = [M_1 : M_0]$.

Proof. Note $M_3 = M_2e_2M_2$ by Proposition 2.4 and Lemma 2.12, so the canonical trace Tr_3 on M_3 is finite. By Definition 2.7 and Lemma 2.12, if $x \in M_2$,

$$\begin{aligned}\mathrm{Tr}_3(x) &= [M_1 : M_0] \sum_{b\in B} \mathrm{Tr}_3(xbe_1e_2e_1b^*) = [M_1 : M_0] \sum_{b\in B} \mathrm{tr}_2(xbe_1b^*)\\ &= [M_1 : M_0]\, \mathrm{tr}_2(x).\end{aligned}$$

Hence $[M_2 : M_1] = \mathrm{Tr}_3(1) = [M_1 : M_0]$, and $\mathrm{tr}_3 = [M_1 : M_0]^{-1}\,\mathrm{Tr}_3$ extends tr_2. □

Definition 2.14. Suppose $P \subset B(L^2(M_1, \mathrm{tr}_1))$ is a von Neumann algebra containing M_1, tr_P is a trace on P extending tr_1, and p is a projection in P. We say the inclusion $M_0 \subset M_1 \subset (P, \mathrm{tr}_P, p)$ is *standard* if there is an isomorphism of von Neumann algebras $\varphi\colon P \to M_2$ such that $\varphi|_{M_1} = \mathrm{id}_{M_1}$, $\mathrm{tr}_P = \mathrm{tr}_2 \circ \varphi$, and $\varphi(p) = e_1$.

The following lemma, which is an alteration of Lemma 5.8 of [7] and uses ideas from Lemma 5.3.1 in [12], allows us to identify when inclusions are standard.

Lemma 2.15. *Suppose* $M_0 \subset M_1 \subset (P, \mathrm{tr}_P, p)$ *such that*

(1) $pmp = E_{M_0}(m)p$ *for all* $m \in M_1$, *and*

(2) $E_{M_1}(p) = [M_1 : M_0]^{-1}$.

Then $\psi\colon M_1 \otimes_{M_0} M_1 \to M_1pM_1$ *by* $x \otimes y \mapsto xpy$ *is an* M_1*-bilinear isomorphism of* $*$*-algebras. Hence* $\varphi\colon M_1e_1M_1 \to M_1pM_1$ *by* $xe_1y \mapsto xpy$ *is an isomorphism of* $*$*-algebras. Moreover, if*

(3) $P = M_1pM_1$,

then $M_0 \subset M_1 \subset (P, \mathrm{tr}_P, p)$ *is standard via* φ. *Conversely, if* $M_0 \subset M_1 \subset (P, \mathrm{tr}_P, p)$ *is standard, then* (1), (2), *and* (3) *hold.*

Proof. First, note that $px = xp$ for all $x \in M_0$ by (1), and the map $M_1 \to M_1p$ by $y \mapsto yp$ is injective by (2). Clearly ψ is surjective and preserves the $*$-algebra structure. Suppose

$$\psi\Big(\sum_{i=1}^{k} x_i \otimes y_i\Big) = \sum_{i=1}^{k} x_ipy_i = 0.$$

Then for all $x, y \in M_1$,

$$px\Big(\sum_{i=1}^{k} x_i p y_i\Big) yp = \Big(\sum_{i=1}^{k} E_{M_0}(xx_i) E_{M_0}(y_i y)\Big) p = 0,$$

which implies $\sum_{i=1}^{k} E_{M_0}(xx_i)E_{M_0}(y_i y) = 0$. If $B = \{b\}$ is a Pimsner–Popa basis for M_1 over M_0, by Remark 2.5,

$$\begin{aligned}\sum_{i=1}^{k} x_i \otimes y_i &= \sum_{a \in B} a \otimes a^*\Big(\sum_{i=1}^{k} x_i \otimes y_i\Big) \sum_{b \in B} b \otimes b^* \\ &= \sum_{a,b \in B} \sum_{i=1}^{k} a \otimes E_{M_0}(a^* x_i) E_{M_0}(y_i b) b^* = 0.\end{aligned}$$

The remaining claims follow as in [7]. □

2.2. The Jones tower and tensor products. We give the background necessary to define the canonical planar $*$-algebra associated to a Markov inclusion and to prove its uniqueness. Many facts stated without proof in Subsection 2.6 rely on the results of this subsection. In particular, the multistep basic construction described in this subsection helps us understand tangles which cap off on the left (see Proposition 2.47), which are crucial to the proof of Theorem 4.1, the main result of this paper.

For the rest of this section, let $M_0 \subset (M_1, \mathrm{tr}_1)$ be a strongly Markov inclusion of finite von Neumann algebras, and set $d = [M_1 : M_0]^{1/2}$. For $n \in \mathbb{N}$, inductively define the basic construction

$$M_{n+1} = \langle M_n, e_n \rangle = M_n e_n M_n = J_n M_{n-1}' J_n \subset B(L^2(M_n, \mathrm{tr}_n))$$

with canonical trace tr_{n+1} extending tr_n and satisfying $\mathrm{tr}_{n+1}(xe_n) = d^{-2}\,\mathrm{tr}_n(x)$ for all $x \in M_n$ where $e_n \in B(L^2(M_n, \mathrm{tr}_n))$ is the Jones projection with range $L^2(M_{n-1}, \mathrm{tr}_{n-1})$. For $n \in \mathbb{N}$, set $E_n = de_n$.

Fact 2.16. *The E_i's satisfy the Temperley–Lieb relations*

(i) $E_i^2 = dE_i = dE_i^*$,

(ii) $E_i E_j = E_j E_i$ *for* $|i - j| > 1$, *and*

(iii) $E_i E_{i\pm1} E_i = E_i$.

Proposition 2.17. *Suppose $N \subset (M, \mathrm{tr}_M)$ and $M \subset (P, \mathrm{tr}_P)$ such that $\mathrm{tr}_P|_M = \mathrm{tr}_M$. Suppose $A = \{a\}$ is a Pimsner–Popa basis for P over M and $B = \{b\}$ is a Pimsner–Popa basis for M over N. Then*

(1) *$AB = \{ab \mid a \in A$ and $b \in B\}$ is a Pimsner–Popa basis for P over N,*

(2) *$[P : N] = [P : M][M : N]$, and*

(3) $\sum_{b\in B} be_N^P b^* = e_M^P \in B(L^2(P,\mathrm{tr}_P))$, *where* e_N^P *is the projection* $L^2(P,\mathrm{tr}_P) \to L^2(N,\mathrm{tr}_N)$ *and* e_M^P *is the projection* $L^2(P,\mathrm{tr}_P) \to L^2(M,\mathrm{tr}_M)$.

Proof. (1) For all $x \in P$,

$$\sum_{ab\in AB} abE_N^P(b^*a^*x) = \sum_{a,b} abE_N^M(E_M^P(b^*a^*x)) = \sum_{a,b} abE_N^M(b^*E_M^P(a^*x))$$
$$= \sum_a aE_M^P(a^*x) = x.$$

(2) Immediate from (1).
(3) If $p \in P$ and $\Omega \in L^2(P,\mathrm{tr}_P)$ is the image of $1 \in P$, then

$$\sum_{b\in B} be_N^P b^* p\Omega = \sum_{b\in B} bE_N^P(b^*p)\Omega = \sum_{b\in B} bE_N^M(b^*(E_M^P(p))\Omega$$
$$= E_M^P(p)\Omega = e_M^P p\Omega.$$
□

Corollary 2.18. $M_k \subset (M_n,\mathrm{tr}_n)$ *is strongly Markov for all* $0 \le k \le n$.

The following technical lemma will be used to define the multistep basic construction of [19], [7], and [3] in Proposition 2.20.

Lemma 2.19. *For all* $0 \le k \le n$*, define the following element of* M_{k+n} *(see Remark* 2.45*):*

$$f_{n-k}^n = d^{k(k-1)}(e_n e_{n-1}\dots e_{n-k+1})(e_{n+1}e_n\dots e_{n-k+2})\dots(e_{n+k-1}e_{n+k-2}\dots e_n).$$

If $0 \le j \le k \le n$ *and* B *is a Pimsner–Popa basis for* M_{n-j} *over* M_{n-k}*, then* $\sum_{b\in B} bf_{n-k}^n b^* = f_{n-j}^n$.

Proof. For $j+1 \le i \le k$, let A_i be a Pimsner–Popa basis for M_{n-i+1} over M_{n-i}. Then $A = A_{j+1}\dots A_k$ is a Pimsner–Popa basis for M_{n-j} over M_{n-k} by Proposition 2.17, and

$$\sum_{\substack{a_i\in A_i\\ j+1\le i\le k}} a_{j+1}\dots a_k f_{n-k}^n a_k^*\dots a_{j+1}^* = \sum_{\substack{a_i\in A_i\\ j+1\le i\le k-1}} a_{j+1}\dots a_{k-1} f_{n-k+1}^n a_{k-1}^*\dots a_{j+1}^*$$
$$= \dots = \sum_{a_{j+1}\in A_{j+1}} a_{j+1} f_{n-j-1}^n a_{j+1}^* = f_{n-j}^n.$$

Let B be another Pimsner–Popa basis for M_{n-j} over M_{n-k} and let us define $U \in \mathrm{Mat}_{|A|\times|B|}(M_{n-k})$ by $U_{a,b} = E_{M_{n-k}}^{M_{n-j}}(a^*b)$. If we consider A as a row vector in $\mathrm{Mat}_{1\times|A|}(M_{n-j})$, then $B = AU$ and $A = BU^*$. For $\ell \in \mathbb{N}$, let $F_\ell = f_{n-k}^n I_\ell \in$

$\mathrm{Mat}_{\ell\times\ell}(M_{n+k})$, i.e. F_ℓ is the $\ell\times\ell$ diagonal matrix with all diagonal entries equal to f^n_{n-k}. Then since f^n_{n-k} commutes with M_{n-k}, we have

$$\sum_{b\in B} bf^n_{n-k}b^* = BF_{|B|}B^* = AUF_{|B|}U^*A^* = AUU^*F_{|A|}A^* = AF_{|A|}A^*$$
$$= \sum_{a\in A} af^n_{n-k}a^* = f^n_{n-j}. \qquad \square$$

Proposition 2.20 (Multistep basic construction). *The inclusion $M_{n-k} \subset M_n \subset (M_{n+k}, \mathrm{tr}_{n+k}, f^n_{n-k})$ is standard.*

Proof. Let B be a Pimsner–Popa basis for M_n over M_{n-k}. Then by Lemma 2.19, $\sum_{b\in B} bf^n_{n-k}b^* = 1$, so $M_n f^n_{n-k} M_n = M_{n+k}$. It is straightforward to check $f^n_{n-k}xf^n_{n-k} = E_{M_{n-k}}(x)f^n_{n-k}$ for all $x \in M_n$ and $E_{M_n}(f^n_{n-k}) = d^{-2k}$, and the result follows by Lemma 2.15. $\square$

Remark 2.21. Note that $L^2(M_n, \mathrm{tr}_n)$ has left and right actions of $M_0, \ldots, M_{2n}$, where as usual, the right action of M_i is the left action of $J_nM_iJ_n \cong M_i^{\mathrm{op}}$. Note that $M_i' = J_nM_{2n-i}J_n$, so we define a canonical trace on $M_i' \cap B(L^2(M_n, \mathrm{tr}_n))$ by $\mathrm{tr}_i'(x) = \mathrm{tr}_{2n-i}(J_nx^*J_n)$ for all $x \in M_i' \cap B(L^2(M_n, \mathrm{tr}_n))$.

Proposition 2.22 (Shifts). *For all $0 \le k \le n$, there is a canonical isomorphism $M_k' \cap M_n \cong M_{k+2}' \cap M_{n+2}$.*

Proof. On $B(L^2(M_n, \mathrm{tr}_n))$, the map $x \mapsto J_nx^*J_n$ gives an anti-isomorphism $M_k' \cap M_n \cong M_n' \cap M_{2n-k}$. On $B(L^2(M_{n+1}, \mathrm{tr}_{n+1}))$, the map $x \mapsto J_nx^*J_n$ gives an anti-isomorphism $M_n' \cap M_{2n-k} \cong M_{k+2}' \cap M_{n+2}$. $\square$

Proposition 2.23. *The canonical trace-preserving conditional expectation $M_{n+k} \to M_{n+k-i}$ is given by $xf^n_{n-k}y \mapsto d^{-2i}xf^n_{n-k+i}y$ where $x, y \in M_n$. The canonical trace-preserving conditional expectation $M_{n-k}' = J_nM_{n+k}J_n \to J_nM_{n+k-i}J_n = M_{n-k+i}'$ is given by the same formula, only with $x, y \in M_n' = J_nM_nJ_n$.*

Proof. We prove the first statement, as the second is similar. By the Markov property, for all $x, y \in M_n$,

$$\mathrm{tr}_{n+k}(xf^n_{n-k}y) = d^{-2k}\,\mathrm{tr}_n(xy) = d^{-2i}\,\mathrm{tr}_{n+k-i}(xf^n_{n-k+i}y),$$

so the map is trace-preserving. Now M_{n+k-i}-bilinearity follows from the following two facts:

(i) for all $1 \le i \le k$, $M_{n-k} \subset M_{n-k+i}$, so $f^n_{n-k+i}f^n_{n-k} = f^n_{n-k}$, and

(ii) $E^{M_{n+k}}_{M_{n+k-i}}(f^n_{n-k}) = d^{-2i}f^n_{n-k+i}$. $\square$

We can now strengthen Proposition 2.7 from [3], versions of which also appear in [4]. This is the main proposition describing left-capping tangles.

Proposition 2.24. *Let $0 \leq k \leq \ell \leq n$, and let B be a Pimsner–Popa basis for M_ℓ over M_k. The conditional expectation $E^{M_k'}_{M_\ell'} : (M_k' \cap B(L^2(M_n, \mathrm{tr}_n)), \mathrm{tr}_k') \to (M_\ell' \cap B(L^2(M_n, \mathrm{tr}_n)), \mathrm{tr}_\ell')$ is given by*

$$E^{M_k'}_{M_\ell'}(x) = \frac{1}{d^{2(\ell-k)}} \sum_{b \in B} bxb^*.$$

In particular, this map is independent of n and the choice of basis.

Proof. The result follows from Lemma 2.19 and Proposition 2.23, since for $x, y \in J_n M_n J_n \subset M_\ell'$,

$$\sum_{b \in B} bxf_k^n yb^* = \sum_{b \in B} xbf_k^n b^* y = xf_\ell^n y. \qquad \square$$

To define our planar $*$-algebra in Subsection 2.3, we need the following fact, which follows from Proposition 2.4 and a simple induction argument.

Proposition 2.25. *For $k \in \mathbb{N}$, let $v_k = E_k E_{k-1} \dots E_1$. For all $n \in \mathbb{N}$, there are isomorphisms of $M_1 - M_1$ bimodules*

$$\begin{aligned} \theta_n : \bigotimes_{M_0}^{n} M_1 &\longrightarrow M_n, \\ x_1 \otimes \cdots \otimes x_n &\longmapsto x_1 v_1 x_2 v_2 \dots v_{n-1} x_n. \end{aligned}$$

Remark 2.26. Recall that $L^2(M_n, \mathrm{tr}_n)$ is the completion of M_n with inner product $\langle x, y \rangle = \mathrm{tr}_n(y^* x)$. As usual, θ_n gives an isomorphism of Hilbert-bimodules

$$\bigotimes_{M_0}^{n} L^2(M_1, \mathrm{tr}_1) \longrightarrow L^2(M_n, \mathrm{tr}_n)$$

where the tensor product on the left is Connes' relative tensor product with inner product given inductively by

$$\begin{aligned} \langle x_1 \otimes u, y_1 \otimes v \rangle_n &= \langle E_{M_0}(y_1^* x_1) u, v \rangle_{n-1}, \\ \langle u \otimes x_n, v \otimes y_n \rangle_n &= \langle u, v E_{M_0}(y_n x_n^*) \rangle_{n-1}. \end{aligned}$$

The following operators will be useful in the definition of the rotation operators in Subsections 2.4 and 2.5.

Definition 2.27. Given $x \in M_1$, we get left and right multiplication operators

$$L(x), R(x)\colon \bigotimes_{M_0}^{n} L^2(M_1, \mathrm{tr}_1) \longrightarrow \bigotimes_{M_0}^{n} L^2(M_1, \mathrm{tr}_1)$$

by $L(x)(v) = xv$ and $R(x)(v) = vx$, and left and right creation operators

$$L_x, R_x\colon \bigotimes_{M_0}^{n} L^2(M_1, \mathrm{tr}_1) \longrightarrow \bigotimes_{M_0}^{n+1} L^2(M_1, \mathrm{tr}_1)$$

by $L_x(v) = x \otimes v$ and $R_x(v) = v \otimes x$.

Fact 2.28. *For $x, y_1, \dots, y_{n+1} \in M_1$, we have*

$$L_x^*(y_1 \otimes \cdots \otimes y_{n+1}) = E_{M_0}(x^* y_1) y_2 \otimes \cdots \otimes y_{n+1}$$

and

$$R_x^*(y_1 \otimes \cdots \otimes y_{n+1}) = y_1 \otimes \cdots \otimes y_n E_{M_0}(y_{n+1} x^*).$$

The following lemma will be instrumental in defining the action of tangles.

Lemma 2.29. *If A is a $\mathbb{C}$-algebra, V_1 is a right A-module, V_2 is an $A - A$ bimodule, and V_3 is a left A-module, then for each A-invariant $v_2 \in V_2$, the map*

$$v_1 \otimes v_3 \longmapsto v_1 \otimes v_2 \otimes v_3$$

defines a linear map $\varphi_{v_2}\colon V_1 \otimes_A V_3 \to V_1 \otimes_A V_2 \otimes_A V_3$. Moreover, the map $v \mapsto \varphi_v$ on $A' \cap V_2 = \{v \in V_2 \mid av = va \text{ for all } a \in A\}$ is $\mathbb{C}$-linear.

Proof. Middle A-linearity is satisfied as v_2 is A-invariant. □

Remark 2.30. This lemma gives an alternate proof that the map $E_{M_1'}^{M_0'}$ is well defined in Proposition 2.24. By Remark 2.5, $d^{-2} \sum_{b \in B} b \otimes b^*$ is independent of the choice of Pimsner–Popa basis B, so the composite map

$$x \longmapsto \varphi_x \longmapsto \varphi_x\Big(d^{-2} \sum_{b \in B} b \otimes b^*\Big) = d^{-2} \sum_{b \in B} b \otimes x \otimes b^* \longmapsto d^{-2} \sum_{b \in B} b x b^*$$

on $M_0' \cap B(L^2(M_n, \mathrm{tr}_n))$ is independent of the choice. Moreover, the result is M_1-invariant, since for any unitary $u \in M_1$, $\{ub \mid b \in B\}$ is another Pimsner–Popa basis for M_1 over M_0.

2.3. Definition of the canonical planar $*$-algebra. The definition of a planar $*$-algebra has evolved since its inception in [9]. We use the definition of [11] (see also [17]), but we do not reproduce it here.

In [9], it was shown how to endow the tower of relative commutants of an extremal, finite index II_1-subfactor with the structure of a *subfactor planar algebra*, i.e. a planar $*$-algebra $Q_\bullet = \{Q_{n,\pm}\}$ with $\dim(Q_{n,\pm}) < \infty$ for all $n \geq 0$ which is

- *spherical,* i.e $\dim(Q_{0,\pm}) = 1$ and any fully labelled 0-tangle is invariant under spherical isotopy. This implies shaded and unshaded contractible loops count for the same multiplicative factor of d, called the *modulus* of $Q_\bullet$, and
- *positive definite,* i.e. the bilinear form on $Q_{n,\pm}$ given by $\langle a, b\rangle = d^{-n}\operatorname{tr}(b^*a)$ is positive definite.

The only essential ingredient to the construction of [9] is a Pimsner–Popa basis, so the same construction applies to a strongly Markov inclusion $M_0 \subset (M_1, \operatorname{tr}_1)$. As we do not require the algebras to be factors or the inclusion to be extremal, the resulting planar algebra need not be spherical nor positive-definite nor have finite dimensional n-box spaces.

Below, we define a planar $*$-algebra structure on the vector spaces $P_{n,\pm}$ $(n \geq 0)$ given by $P_{n,+} = \theta_n^{-1}(M_0' \cap M_n)$ and $P_{n,-} = \theta_n^{-1}(M_1' \cap M_{n+1})$. This planar algebra is independent of any choices, so we call it the *canonical planar $*$-algebra* associated to $M_0 \subset (M_1, \operatorname{tr}_1)$.

We define the action of a planar tangle in *standard form*:

(1) all the input and output disks are horizontal rectangles with all strings (that are not closed loops) emanating from the top edges of the rectangles,

(2) all the input disks are in disjoint horizontal bands and all maxima and minima of strings are at different vertical levels, and not in the horizontal bands defined by the input disks, and

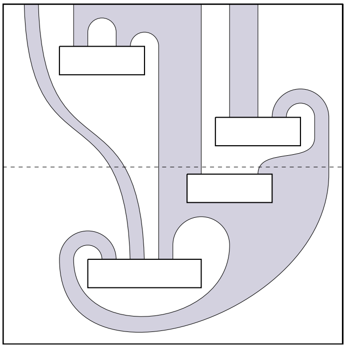
(3) the distinguished (starred) intervals of all the disks are at the left edges of the rectangles. (In the sequel, we will assume this convention and omit the $*$'s.)

We do not provide the proof of isotopy invariance, i.e. that the action is independent of the choice of standard form, as this proof is identical to that in [9]. However, in Subsection 2.4, we provide Burns' elegant proof that the rotation operator is well-defined.

Suppose we have a $(k, \pm)$-tangle T in standard form with s input rectangles, and input rectangle j has $2r_j$ strings emanating from the top. We define the action of T on an s-tuple $\xi = (\xi_1, \dots, \xi_s)$ where $\xi_j \in P_{r_j, \pm_j}$ and $\pm_j = \pm$ if the region just below input rectangle j is unshaded or shaded respectively.

We read the action of T on ξ by sliding a horizontal line through the tangle from bottom to top. For a fixed vertical y-value, off the input disks' horizontal bands and away from the relative extrema of the strings, the horizontal line will meet n_y shaded regions from left to right. One should think of the shaded regions along this line as elements of M_1 and the unshaded regions between shaded regions as the symbols

$\otimes_{M_0}$. Near the top, the line will meet k or $k+1$ shaded regions depending on whether the left-most region of T is unshaded or shaded respectively. We illustrate a typical $(3,+)$-tangle with the horizontal line about half way through its travel:



For each y coordinate of the horizontal line, one reads off an M_i-invariant element $\eta_y \in \bigotimes_{M_0}^{n_y} M_1$, where $i = 0$ if T is a $(k,+)$-tangle and $i = 1$ if T is a $(k,-)$-tangle.

The element η_y begins as $1 \in M_i$ near the bottom, and it remains constant as long as the horizontal line meets neither maxima, minima, nor rectangles. If the horizontal line passes input rectangle j for which exactly t shaded regions sit to the left, then we insert ξ_j into η_y as in Figure 1 by applying Lemma 2.29 with $v_2 = \xi_j$,

$$V_1 = \bigotimes_{M_0}^{t} M_1, \quad V_2 = P_{r_j,\pm_j}, \quad \text{and} \quad V_3 = \bigotimes_{M_0}^{n_y - t} M_1.$$

Note that V_1, V_3 are considered as M_ℓ-modules and $P_{r_j,\pm_j}$ is an $M_\ell - M_\ell$ bimodule, where $\ell = 0$ if $\pm_j = +$ and $\ell = 1$ if $\pm_j = -$. Note that inserting ξ_j into η_y gives an M_i-invariant vector.

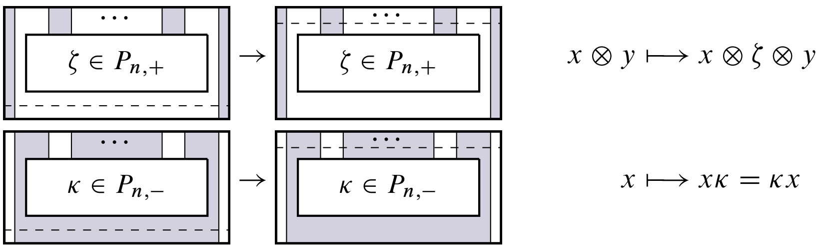
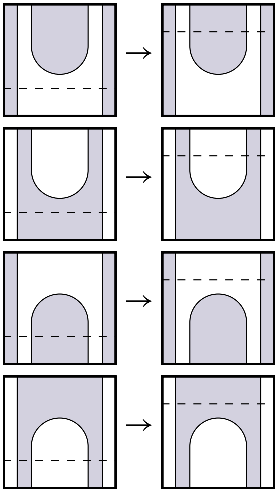


Figure 1. Inserting central vectors.

As the horizontal line passes a maximum or minimum, η_y changes according to Figure 2 where the changes indicated on the tensors are to be inserted into the position indicated by the shaded regions on the horizontal (dashed) line. With the exception of one case, each of these maps is a $(M_1 - M_1)$-bimodule map, so it will preserve M_i-invariant elements. The remaining case to consider is when the left-most or right-most shaded region is capped off by applying the third map pictured above, which is a $(M_0 - M_0)$-bimodule map. But this will only occur when the distinguished (starred) interval of the external disk meets an unshaded region, so i would have to be 0 from the beginning.



$$x \otimes y \longmapsto x \otimes 1 \otimes y$$

$$x \longmapsto d^{-1} \sum_{b \in B} xb \otimes b^* = d^{-1} \sum_{b \in B} b \otimes b^* x$$

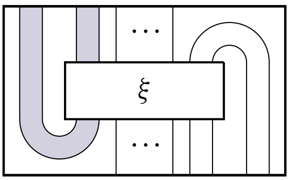
$$x \otimes y \otimes z \longmapsto dxE_{M_0}(y) \otimes z = dx \otimes E_{M_0}(y)z$$

$$x \otimes y \longmapsto xy.$$

Figure 2. Reading maxima and minima of planar tangles in standard form.

The action of the tangle on ξ is the element $\eta_y \in P_{k,\pm}$ read for horizontal lines sufficiently close to the top. The $*$-structure is the same as that of [9].

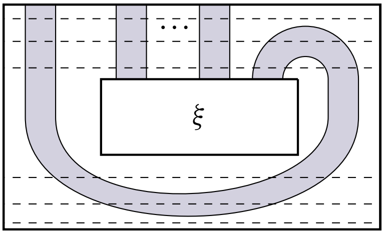
Example 2.31. To calculate



for

$$\xi = \sum_{i=1}^{k} x_1^i \otimes \cdots \otimes x_n^i \in \theta_n^{-1}(M_0' \cap M_n),$$

we first isotope the tangle into a standard form. The horizontal line travels upward as shown:



which we read as

$$1_{\mathbb{C}} \mapsto 1_M \mapsto d^{-1}\sum_{b\in B} b\otimes b^* \mapsto d^{-1}\sum_{b\in B} b\otimes \xi\otimes b^*$$
$$\mapsto d^{-1}\sum_{b\in B}\sum_{i=1}^{k} b\otimes x_1^i\otimes\cdots\otimes x_{n-1}^i\otimes x_n^i b^*$$
$$\mapsto \sum_{b\in B}\sum_{i=1}^{k} b\otimes x_1^i\otimes\cdots\otimes x_{n-1}^i E_{M_0}(x_n^i b^*),$$

the last line giving the output of the tangle applied to ξ.

2.4. Burns' treatment of the rotation operator on $P_{n,+}$. The key to showing that the $P_{n,\pm}$'s define a planar algebra is isotopy invariance, which relies on the existence of the rotation on $P_{n,\pm}$. A particularly elegant treatment of this is due to Michael Burns, but it only appears in his thesis [4], so we include a proof below for the reader's convenience.

Definition 2.32. Let B be a Pimsner–Popa basis of M_1 over M_0. For all $x = x_1\otimes\cdots\otimes x_n \in \bigotimes_{M_0}^n M_1$, define

$$\rho(x) = \sum_{b\in B} L_b R_b^*(x) = \sum_{b\in B} b\otimes x_1\otimes\cdots\otimes x_{n-1}E_{M_0}(x_n b^*)$$

(see Example 2.31).

Proposition 2.33. *The map ρ preserves $P_{n,+}$, and its restriction to $P_{n,+}$ is independent of the choice of B.*

Proof. Middle linearity is respected by ρ, so it is well defined, though it may depend on B. By Lemma 2.29 and Remark 2.5, for all M_0-invariant x, the sum $\sum_{b\in B} b\otimes x\otimes b^*$ is independent of B. We obtain ρ by applying a $(M_0 - M_0)$-bilinear map which does not involve B, so the restriction of ρ is M_0-invariant and independent of B. □

Theorem 2.34 ([4]). *For $x\in P_{n,+}$ and $y_1,\ldots,y_n\in M_1$,*

$$\langle \rho(x), y_1\otimes\cdots\otimes y_n\rangle = \langle x, y_2\otimes\cdots\otimes y_n\otimes y_1\rangle,$$

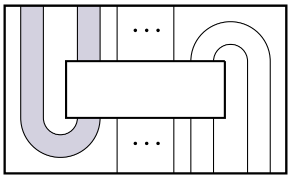
so $\rho^n = \mathrm{id}$ on $P_{n,+}$.

Proof. As $\rho(x) = \sum_{b \in B} L_b R_b^*(x)$, we have

$$\begin{aligned}
\langle \rho(x), y_1 \otimes \cdots \otimes y_n \rangle &= \sum_{b \in B} \langle L_b R_b^* x, y_1 \otimes \cdots \otimes y_n \rangle \\
&= \sum_{b \in B} \langle x, R_b L_b^* y_1 \otimes \cdots \otimes y_n \rangle \\
&= \sum_{b \in B} \langle x, E_{M_0}(b^* y_1) y_2 \otimes \cdots \otimes y_n \otimes b \rangle \\
&= \sum_{b \in B} \langle E_{M_0}(b^* y_1)^* x, y_2 \otimes \cdots \otimes y_n \otimes b \rangle \\
&= \sum_{b \in B} \langle x E_{M_0}(b^* y_1)^*, y_2 \otimes \cdots \otimes y_n \otimes b \rangle \\
&= \sum_{b \in B} \langle x, y_2 \otimes \cdots \otimes y_n \otimes b E_{M_0}(b^* y_1) \rangle \\
&= \langle x, y_2 \otimes \cdots \otimes y_n \otimes y_1 \rangle.
\end{aligned}$$

□

Corollary 2.35. *The rotation*



on $P_{n,+}$ is well defined.

2.5. The rotation on $P_{n,-}$. We mimic Burns' treatment of the rotation on $P_{n,+}$ to define the rotation on $P_{n,-}$.

Definition 2.36. Let B be a Pimsner–Popa basis of M_1 over M_0. For

$$x = x_1 \otimes \cdots \otimes x_{n+1} \in \bigotimes_{M_0}^{n+1} M_1,$$

define $\sigma(x) = \sum_{b \in B} R(b^*) R_1^* L_b(x) = \sum_{b \in B} b \otimes x_1 \otimes \cdots \otimes x_n E_{M_0}(x_{n+1}) b^*$.

Proposition 2.37. *The map σ preserves $P_{n,-}$, and its restriction to $P_{n,-}$ is independent of the choice of B.*

Proof. Similar to Proposition 2.33. □

Theorem 2.38. *For $x \in P_{n,-}$ and $y_1, \dots, y_{n+1} \in M_1$, we have*

$$\langle \sigma(x), y_1 \otimes \cdots \otimes y_{n+1} \rangle = \langle x, y_2 \otimes \cdots \otimes y_n \otimes y_{n+1} y_1 \otimes 1 \rangle.$$

Proof. Similar to Theorem 2.34. □

Corollary 2.39. $\sigma^n = \mathrm{id}$ *on* $P_{n,-}$.

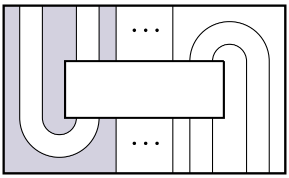
Proof. As σ preserves $P_{n,-}$, we repeatedly apply Theorem 2.38 for $x \in P_{n,-}$ to get

$$\begin{aligned} \langle \sigma^n(x), y_1 \otimes \cdots \otimes y_{n+1} \rangle &= \langle \sigma^{n-1}(x), y_2 \otimes \cdots \otimes y_n \otimes y_{n+1} y_1 \otimes 1 \rangle \\ &= \langle \sigma^{n-2}(x), y_3 \otimes \cdots \otimes y_n \otimes y_{n+1} y_1 \otimes y_2 \otimes 1 \rangle \\ &= \cdots = \langle x, y_{n+1} y_1 \otimes y_2 \otimes \cdots \otimes y_n \otimes 1 \rangle. \end{aligned}$$

We then invoke Burns' trick again to get

$$\begin{aligned} \langle x, y_{n+1} y_1 \otimes y_2 \otimes \cdots \otimes y_n \otimes 1 \rangle &= \langle y_{n+1}^* x, y_1 \otimes \cdots \otimes y_n \otimes 1 \rangle \\ &= \langle x y_{n+1}^*, y_1 \otimes \cdots \otimes y_n \otimes 1 \rangle \\ &= \langle x, y_1 \otimes \cdots \otimes y_n \otimes y_{n+1} \rangle. \end{aligned}$$ □

Corollary 2.40. *The rotation*



on $P_{n,-}$ is well defined.

2.6. Uniqueness of the canonical planar $*$-algebra. We have the following facts whose proofs are similar to those in [9] and will be omitted (they are straightforward from the results in Subsections 2.1 and 2.2). We shade tangles as much as possible, but sometimes we will not have enough information.

Proposition 2.41 (Multiplication). *Suppose $x, y \in M_n$ such that*

$$\theta_n^{-1}(x) = x_1 \otimes \cdots \otimes x_n \quad \text{and} \quad \theta_n^{-1}(y) = y_1 \otimes \cdots \otimes y_n$$

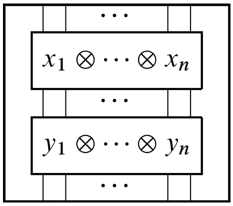
Then if $n = 2k$ for some $k \in \mathbb{N}$,

$$\theta_n^{-1}(xy) = x_1 \otimes \cdots \otimes x_k E_{M_0}(x_{k+1} E_{M_0}(x_{k+2}(\dots) y_{k-1}) y_k) \otimes y_{k+1} \otimes \cdots \otimes y_{2k},$$

while if $n = 2k+1$ for some $k \in \mathbb{N}$,

$$\theta_n^{-1}(xy) = x_1 \otimes \cdots \otimes x_{k+1} E_{M_0}(x_{k+2} E_{M_0}(x_{k+3}(\dots) y_{k-1}) y_k) y_{k+1} \otimes \cdots \otimes y_{2k+1}.$$

Remark 2.42. If x, y as above are in $M_i' \cap M_n$ where $i \in \{0, 1\}$, then

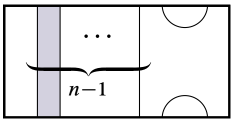
$\theta_n^{-1}(xy) =$ 

,

where the shading depends on i and the parity of n.

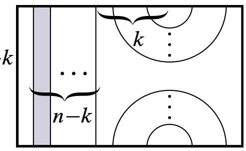
Proposition 2.43 ($*$-structure). *Suppose $x \in M_n$ such that $\theta_n^{-1}(x) = x_1 \otimes \cdots \otimes x_n$. Then $\theta_n^{-1}(x^*) = x_n^* \otimes \cdots \otimes x_1^*$.*

Proposition 2.44 (Jones projections). *For $n \geq 1$, the Jones projection $E_n \in P_{n+1,+}$ is given by*



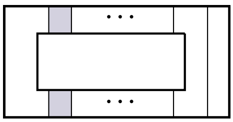
.

Remark 2.45. The multistep basic construction projection of Proposition 2.20 is given by

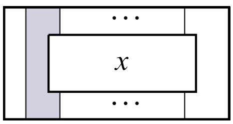
$f_{n-k}^n = d^{-k}$ 

.

Proposition 2.46 (Inclusions). (1) *Let $i_n : M_0' \cap M_n \to M_0' \cap M_{n+1}$ be the inclusion. Then the inclusion $\theta_{n+1}^{-1} \circ i_n \circ \theta_n : P_{n,\pm} \to P_{n+1,\pm}$ is given by*

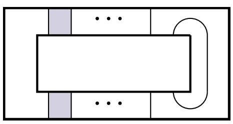

.

(2) *If $x \in P_{n,-}$, then*

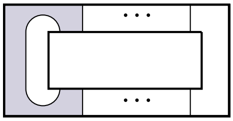


$= x \in P_{n+1,+}$.

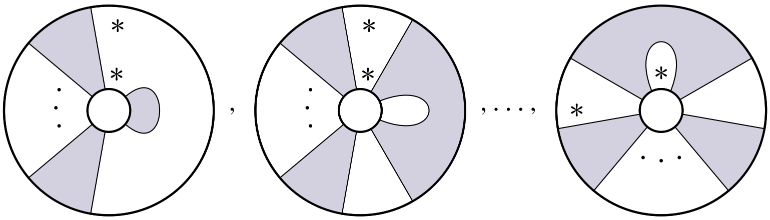
Proposition 2.47 (Conditional expectations). (1) *The conditional expectation $\theta_{n-1}^{-1} \circ E_{M_{n-1}} \circ \theta_n : P_{n,+} \to P_{n-1,+}$ is given by*

$d^{-1} \cdot$ 
.

(2) *The conditional expectation* $\theta_n^{-1} \circ E_{M_1'}^{M_0'} \circ \theta_n : P_{n,+} \to P_{n-1,-}$ (*see Proposition* 2.24) *is given by*

$d^{-1} \cdot$  .

Notation 2.48. We use the notation from [16]. (1) Denote the annular capping maps $P_{n,+} \to P_{n-1,+}$ by α_j as shown:



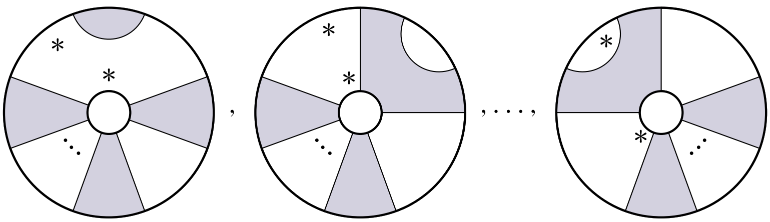
i.e. numbering the boundary points clockwise from $*$, the i^{th} and $(i+1)^{\text{th}}$ (modulo $2n$) internal boundary points are joined by a string and all other internal boundary points are connected to external boundary points such that

(i) if $i = 1$, then the first external point is connected to the third internal point;

(ii) if $1 < i < 2n$, then the first external point is connected to the first internal point;

(iii) if $i = 2n$, then the first external point is connected to the $(2n-1)^{\text{th}}$ internal point.

(2) Denote the annular cupping maps $P_{n-1,+} \to P_{n,+}$ by β_j as shown:

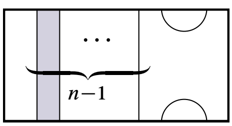


i.e. β_j is α_j turned inside out.

The following lemma is similar to a result in [13].

Lemma 2.49. *Suppose $P_\bullet$ is a planar $*$-algebra with modulus $d \neq 0$ and $Q_{n,\pm} \subset P_{n,\pm}$ are $*$-subalgebras which are closed under the following operations:*

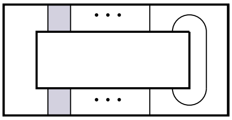
(1) *left and right multiplication by*

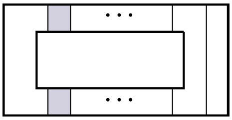
$E_n =$ 

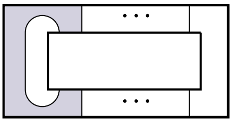
$\in P_{n+1,+}$

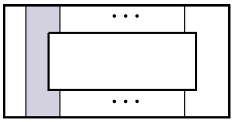
for $n \in \mathbb{N}$*;*

(2) *the maps from* $P_{n,+}$ *as follows:*

$\alpha_n =$  $: P_{n,+} \to P_{n-1,+},$

$\beta_{n+1} =$  $: P_{n,+} \to P_{n+1,+},$

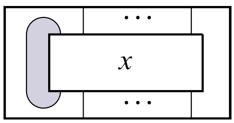
$\gamma_n^+ =$  $: P_{n,+} \to P_{n-1,-};$ *and*

(3) *the map* $i_n^- =$  $: P_{n,-} \to P_{n+1,+}.$

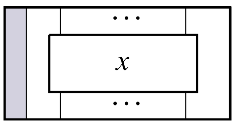
Then the $Q_{n,\pm}$ *define a planar* $*$*-subalgebra* $Q_\bullet \subset P_\bullet$.

Proof. As $Q_{n,\pm}$ is closed under multiplication and $*$, it suffices to show $Q_\bullet$ is closed under all annular maps. To show this, it suffices to show all α_j's, all β_j's, and both rotations by 1 preserve $Q_\bullet$.

First, note that the maps $\gamma_n^- : P_{n,-} \to P_{n-1,+}$ and $i_n^+ : P_{n,+} \to P_{n+1,-}$ by

$\gamma_n^-(x) =$ 

$$= \frac{1}{d}\alpha_{n+2}(E_n E_{n-1} \dots E_1 \cdot \beta_{n+2}(i_n^- x)) \cdot E_1 E_2 \dots E_n),$$

$i_n^+(x) =$ 

$$= \gamma_{n+2}^+(E_1 E_2 \dots E_n \cdot \beta_{n+2}\beta_{n+1}(x) \cdot E_{n+1} E_n \dots E_1)$$

preserve $Q_\bullet$.

We show all α_j's preserve $Q_\bullet$. For $j < n$ and $x \in Q_n$,

$$\alpha_j(x) = \frac{1}{d}\alpha_n \alpha_{n+1}((E_n E_{n-1} \dots E_j) \cdot \beta_{n+1}(x) \cdot (E_n))).$$

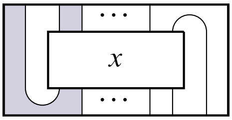
The case $n < j < 2n$ is similar. It is clear $\alpha_{2n}(x) = \alpha_{2n-1}(i_{n-1}^-(\gamma_n^+(x)))$.

We show all β_j's preserve $Q_\bullet$. If $j < n+1$, we have

$$\beta_j(x) = (E_j E_{j-1} \ldots E_n) \cdot \beta_{n+1}(x).$$

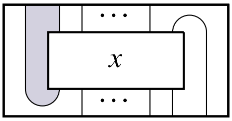
The case $n+1 < j < 2n+2$ is similar. It is clear $\beta_{2n+2}(x) = \alpha_2 \gamma_{n+1}^- \gamma_n^+(x)$.

We show both rotations by 1 preserve $Q_\bullet$. We have



$$= \frac{1}{d} \gamma_{n+1}^+ \alpha_{2n+2} i_{n+1}^- i_n^+ \alpha_n \beta_{n+1}(x)$$

and



$$= \alpha_{n+1} \beta_{n+2} \alpha_{2n+1} i_n^-(x).$$ □

Theorem 2.50. *Given a strongly Markov inclusion $M_0 \subset (M_1, \mathrm{tr}_1)$, there is a unique planar $*$-algebra $P_\bullet$ of modulus $d = [M_1 : M_0]^{1/2}$ where*

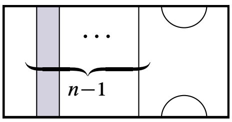
$$P_{n,+} = \theta_n^{-1}(M_0' \cap M_n) \quad \textit{and} \quad P_{n,-} = \theta_{n+1}^{-1}(M_1' \cap M_{n+1})$$

such that the multiplication is given by Remark 2.42,

(0) *for all tangles T with n input disks, $T(\xi_1^*, \ldots, \xi_n^*) = T^*(\xi_1, \ldots, \xi_n)^*$ where, for $\xi_i \in P_{n_i, \pm_i}$, ξ_i^* is as in Proposition 2.43 and T^* is the mirror image of T;*

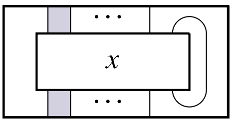
(1) *for $n \in \mathbb{N}$,*

$$E_n =$$

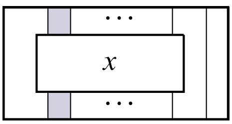


$$\in P_{n+1,+};$$

(2) *for $x \in P_{n,+}$ and B a Pimsner–Popa basis for M_1 over M_0,*

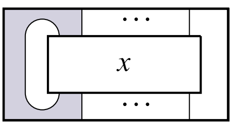


$$= d E_{M_{n-1}}(x),$$



$$= x \in P_{n+1,+},$$

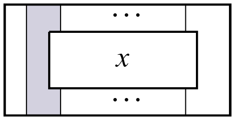
and



$$= d E_{M_1'}^{M_0'}(x) = d^{-1} \sum_{b \in B} b x b^*;$$

and

(3) *for* $x \in P_{n,-}$,

 $= x \in P_{n+1,+}$.

Proof. Uniqueness follows from Lemma 2.49. Existence follows from the existence of the canonical planar $*$-algebra associated to $M_0 \subset (M_1, \mathrm{tr}_1)$. □

Corollary 2.51. *The canonical planar $*$-algebra associated to an extremal, finite index II_1-subfactor is the subfactor planar algebra constructed in* [9].

3. The planar algebra isomorphism for finite dimensional C*-algebras

We now restrict our attention to a connected unital inclusion $M_0 \subset M_1$ of finite dimensional C*-algebras with the Markov trace. We show that in this case, the canonical planar $*$-algebra of Theorem 2.50 is isomorphic to the bipartite graph planar algebra [10] of the Bratteli diagram.

Many of the results in this section can be found in [6], [12], and [5], but we present them here for completeness and for the reader's convenience.

3.1. Loop algebras. We define loop algebras in the spirit of [10] which are another description of Evans, Ocneanu, and Sunder's path algebras [6], [12], and [5], with a more GNS (rather than spatial) flavor.

Notation 3.1. For this section, let Γ be a finite, connected, bipartite multi-graph. Let $\mathcal{V}_\pm$ denote the set of even/odd vertices of Γ, and let $\mathcal{E}$ denote the edge set of Γ. Usually we will denote edges by ε and ξ. All edges will be directed from even to odd vertices, so we have source and target functions $s\colon \mathcal{E} \to \mathcal{V}_+$ and $t\colon \mathcal{E} \to \mathcal{V}_-$. We will write ε^* to denote an edge ε traversed from an odd vertex to an even vertex, and we define source and target functions $s\colon \mathcal{E}^* = \{\varepsilon^* \mid \varepsilon \in \mathcal{E}\} \to \mathcal{V}_-$ and $t\colon \mathcal{E}^* \to \mathcal{V}_+$ by $s(\varepsilon^*) = t(\varepsilon)$ and $t(\varepsilon^*) = s(\varepsilon)$. Let $m_+\colon \mathcal{V}_+ \to \mathbb{N}$ be a dimension (row) vector for the even vertices. For $v \in \mathcal{V}_-$, define the dimension (row) vector for the odd vertices by

$$m_-(v) = \sum_{t(\varepsilon)=v} m_+(s(\varepsilon)).$$

Let Λ be the bipartite adjacency matrix for Γ ($\Lambda_{i,j}$ is the number of times the i^{th} vertex in $\mathcal{V}_+$ is connected to the j^{th} vertex in $\mathcal{V}_-$).

Remark 3.2. Given (Γ, m_+), we can associate a connected unital inclusion of finite dimensional C*-algebras $M_0 \subset M_1$. We set

$$M_0 = \bigoplus_{v\in\mathcal{V}_+} M_{m_+(v)}(\mathbb{C}) \quad \text{and} \quad M_1 = \bigoplus_{v\in\mathcal{V}_-} M_{m_-}(\mathbb{C}),$$

and the inclusion is such that Γ is the Bratteli diagram for the inclusion, and Λ is the inclusion matrix ($\Lambda_{i,j}$ is the number of times the i^{th} simple summand of M_0 is contained in the j^{th} simple summand of M_1). Conversely, given such an inclusion, we get a finite, connected, bipartite multi-graph (the Bratteli diagram) and a dimension vector m_+ (corresponding to the simple summands of M_0).

Definition 3.3. Let $G_{0,\pm}$ be the complex vector space with basis $\mathcal{V}_\pm$ respectively. For $n \in \mathbb{N}$, $G_{n,\pm}$ will denote the complex vector space with basis loops of length $2n$ on Γ based at a vertex in $\mathcal{V}_\pm$ respectively.

We discuss the vector spaces $G_{n,+}$. The spaces $G_{n,-}$ are similar, and it is clear what the corresponding notation should be and how they will behave.

Notation 3.4. Loops in $G_{n,+}$ will be denoted $[\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*]$. Any time we write such a loop, it is implied that

(i) $t(\varepsilon_i) = s(\varepsilon_{i+1}^*) = t(\varepsilon_{i+1})$ for all odd $i < 2n$,

(ii) $t(\varepsilon_i^*) = s(\varepsilon_i) = s(\varepsilon_{i+1})$ for all even $i < 2n$, and

(iii) $t(\varepsilon_{2n}^*) = s(\varepsilon_{2n}) = s(\varepsilon_1)$.

For a loop $\ell = [\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*] \in G_{n,+}$ and $1 \le k \le 2n$, we define the following paths in ℓ:

$$\ell_{[1,k]} = \begin{cases} \varepsilon_1\varepsilon_2^* \dots \varepsilon_{k-1}\varepsilon_k^* & k \text{ even,} \\ \varepsilon_1\varepsilon_2^* \dots \varepsilon_{k-1}^*\varepsilon_k & k \text{ odd,} \end{cases} \qquad \ell_{[k,2n]} = \begin{cases} \varepsilon_k\varepsilon_{k+1}^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^* & k \text{ odd,} \\ \varepsilon_k^*\varepsilon_{k+1} \dots \varepsilon_{2n-1}\varepsilon_{2n}^* & k \text{ even.} \end{cases}$$

Definition 3.5. Define an antilinear map $*$ on $G_{n,+}$ by the antilinear extension of the map

$$[\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*]^* = [\varepsilon_{2n}\varepsilon_{2n-1}^* \dots \varepsilon_2\varepsilon_1^*].$$

There is also an obvious notion of taking $*$ of a path in a loop. We define a multiplication on $G_{n,+}$ by

$$\ell_1 \cdot \ell_2 = \delta_{(\ell_1)^*_{[n+1,2n]},(\ell_2)_{[1,n]}}[(\ell_1)_{[1,n]}(\ell_2)_{[n+1,2n]}].$$

It is clear that $*$ is an involution, i.e. an anti-automorphism of period 2, for $G_{n,+}$ under this multiplication.

Remark 3.6. We can think of a loop in $G_{n,+}$ as a path up and down the multi-graph Γ_n corresponding to the Bratteli diagram for the inclusions

$$M_0 \subset M_1 \subset \dots \subset M_n,$$

which is obtained by reflecting Γ a total of $n-1$ times, as the inclusion matrix of $M_j \subset M_{j+1}$ is given by Λ or Λ^T if j is even or odd, respectively [8].

Definition 3.7. Let $\widetilde{\Gamma}$ be the augmentation of the bipartite graph Γ by adding a distinguished vertex $\star$ which is connected to each $v \in \mathcal{V}_+$ by $m_+(v)$ distinct edges. These edges are oriented so they begin at $\star$. We will denote these added edges by $\eta' s$ (and ζ's and κ's when necessary).

Definition 3.8. For $n \in \mathbb{Z}_{\geq 0}$, let A_n be the $\mathbb{C}$-algebra defined as follows: a basis of A_n will consist of loops of length $2n+2$ on $\widetilde{\Gamma}$ of the form

$$[\eta_1 \varepsilon_1 \varepsilon_2^* \dots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*]$$

i.e. the loops start and end at $\star$, but remain in Γ otherwise. Note that we have an obvious $*$-structure on each A_n. Multiplication will be given as follows: if one defines the similar path notation as in Notation 3.4, then we have

$$\ell_1 \cdot \ell_2 = \delta_{(\ell_1)^*_{[n+2,2n+2]},(\ell_2)_{[1,n+1]}} [(\ell_1)_{[1,n+1]}(\ell_2)_{[n+2,2n+2]}].$$

Remark 3.9. We can think of a loop in A_n as a path up and down the multi-graph $\widetilde{\Gamma}_n$ corresponding to the Bratteli diagram for the inclusions

$$\mathbb{C} \subset M_0 \subset M_1 \subset \cdots \subset M_n.$$

Definition 3.10 (Inclusions). The inclusion $A_n \to A_{n+1}$ is given by the linear extension of

$$[\eta_1 \varepsilon_1 \varepsilon_2^* \dots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*] \longmapsto \begin{cases} \displaystyle\sum_{s(\varepsilon)=s(\varepsilon_n)} [\eta_1 \varepsilon_1 \varepsilon_2^* \dots \varepsilon_n^* \varepsilon \varepsilon^* \varepsilon_{n+1} \dots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*] & n \text{ even,} \\ \displaystyle\sum_{s(\varepsilon)=t(\varepsilon_n)} [\eta_1 \varepsilon_1 \varepsilon_2^* \dots \varepsilon_n \varepsilon^* \varepsilon \varepsilon_{n+1}^* \dots \varepsilon_{2n-1} \varepsilon_{2n}^* \eta_2^*] & n \text{ odd.} \end{cases}$$

We identify A_n with its image in A_{n+1}.

Remark 3.11. The inclusion identifications allow us to define a multiplication

$$A_m \times A_n \to A_{\max\{m,n\}}$$

by including A_m, A_n into $A_{\max\{m,n\}}$ and using the regular multiplication. Explicitly, if $\ell_1 \in A_m$ and $\ell_2 \in A_n$ with $m \leq n$, then

$$\ell_1 \cdot \ell_2 = \delta_{(\ell_1)^*_{[m+2,2m+2]},(\ell_2)_{[1,m+1]}} [(\ell_1)_{[1,m+1]}(\ell_2)_{[m+2,2n+2]}].$$

The case $m \geq n$ is similar.

3.2. Towers of loop algebras. We provide an isomorphism of the tower $(M_n)_{n\geq 0}$ coming from a connected unital inclusion of finite dimensional C*-algebras with the Markov trace and the corresponding tower $(A_n)_{n\geq 0}$ of loop algebras. Assume the notation of Subsection 3.1.

For $n \geq 0$, if S_i is the i^{th} simple summand of M_n, then loops ℓ in A_n for which $\ell_{[1,n+1]}$ ends at the corresponding vertex of $\widetilde{\Gamma}_n$ form a system of matrix units for a simple algebra isomorphic to S_i. Hence for $n \in \mathbb{Z}_{\geq 0}$, there is a $*$-algebra isomorphism $A_n \cong M_n$, and $\dim(A_n) = \dim(M_n)$.

At this point, we only choose such isomorphisms $\varphi_n \colon A_n \to M_n$ for $n = 0, 1$ which respect the inclusion given in Definition 3.10. In Proposition 3.17, we will inductively define isomorphisms $\varphi_n \colon A_n \to M_n$ for $n \geq 2$ to identify the Jones projections.

Definition 3.12. Following [8], let λ_i be the Markov trace (column) vector for M_i for $i = 0, 1$ such that

$$m_+\lambda_0 = 1 = m_-\lambda_1,$$

so λ_i gives the traces of minimal projections in the simple summands of M_i for $i = 0, 1$. In order for the trace on M_1 to restrict to the trace on M_0, we must have $\Lambda\lambda_1 = \lambda_0$.

Recall that the inclusion matrix for $M_n \subset M_{n+1}$ is given by Λ if n is even and Λ^T if n is odd. This means that to extend the trace, we must have $\Lambda\Lambda^T\lambda_0 = d^{-2}\lambda_0$, $\Lambda^T\Lambda\lambda_1 = d^{-2}\lambda_1$, and $\lambda_n = d^{-2}\lambda_{n-2}$ for all $n \geq 2$, where λ_n is the Markov trace vector for M_n and $d = \sqrt{\|\Lambda^T\Lambda\|} = \sqrt{\|\Lambda\Lambda^T\|}$.

Definition 3.13. Let $\lambda = \binom{\lambda_0}{d\lambda_1}$, a Frobenius–Perron eigenvector for $\begin{pmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{pmatrix}$.

Definition 3.14 (Traces). We define a trace on A_0 by

$$\mathrm{tr}_0([\eta_1\eta_2^*]) = \begin{cases} \lambda(t(\eta_1)) = \lambda_0(t(\eta_1)) & \text{if } \eta_1 = \eta_2, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\ell = [\eta_1\varepsilon_1\varepsilon_2^* \ldots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] \in A_n$ with $n \geq 1$. We define a trace on A_n by

$$\mathrm{tr}_n(\ell) = \begin{cases} d^{-n}\lambda(s(\varepsilon_n)) & \text{if } n \text{ is even and } \ell = \ell^*, \\ d^{-n}\lambda(t(\varepsilon_n)) & \text{if } n \text{ is odd and } \ell = \ell^*, \\ 0 & \text{if } \ell \neq \ell^*. \end{cases}$$

Remark 3.15. The isomorphisms φ_n for $n = 0, 1$ preserve the trace. Moreover, $\mathrm{tr}_{n+1}|_{A_n} = \mathrm{tr}_n$ for all $n \in \mathbb{N}$ as λ is a Frobenius–Perron eigenvector.

Proposition 3.16 (Conditional expectations). *If* $\ell = [\eta_1\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] \in A_n$, *the conditional expectation* $A_n \to A_{n-1}$ *is given by*

$$
\begin{aligned}
&E_{A_{n-1}}(\ell)\\
&= \begin{cases} d^{-1}\delta_{\varepsilon_n,\varepsilon_{n+1}}\Big(\dfrac{\lambda(s(\varepsilon_n))}{\lambda(t(\varepsilon_n))}\Big)[\eta_1\varepsilon_1\varepsilon_2^*\dots\varepsilon_{n-1}\varepsilon_{n+2}^*\dots\varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] & n \text{ even,}\\ d^{-1}\delta_{\varepsilon_n,\varepsilon_{n+1}}\Big(\dfrac{\lambda(t(\varepsilon_n))}{\lambda(s(\varepsilon_n))}\Big)[\eta_1\varepsilon_1\varepsilon_2^*\dots\varepsilon_{n-1}^*\varepsilon_{n+2}\dots\varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] & n \text{ odd.}\end{cases}
\end{aligned}
$$

Proof. We consider the case n even. The case n odd is similar. We must show $\mathrm{tr}_n(xy) = \mathrm{tr}_{n-1}(E_{A_{n-1}}(x)y)$ for all $x \in A_n$ and $y \in A_{n-1}$. It suffices to check when x, y are loops. If

$$x = [\eta_1\varepsilon_1\varepsilon_2^*\dots\varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] \quad \text{and} \quad y = [\eta_3\xi_1\xi_2^*\dots\xi_{2n-3}\xi_{2n-2}^*\eta_4^*],$$

using the formula above, we have

$$
\begin{aligned}
\mathrm{tr}_{n-1}(E_{A_{n-1}}(x)y) &= d^{-1}\delta_{\varepsilon_n,\varepsilon_{n+1}}\delta_{x^*_{[n+2,2n+2]},y_{[1,n]}}\frac{\lambda(s(\varepsilon_n))}{\lambda(t(\varepsilon_n))}\\
&\qquad \mathrm{tr}_{n-1}([\eta_1\varepsilon_1\varepsilon_2^*\dots\varepsilon_{n-2}^*\varepsilon_{n-1}\xi_n^*\xi_{n+1}\dots\xi_{2n-3}\xi_{2n-2}^*\eta_4^*])\\
&= d^{-n}\delta_{x^*_{[n+2,2n+2]},y_{[1,n]}}\delta_{\varepsilon_n,\varepsilon_{n+1}}\delta_{y^*_{[n+1,2n-2]},x_{[1,n]}}\lambda(s(\varepsilon_n))\\
&= \mathrm{tr}_n(xy).
\end{aligned}
$$

□

Definition 3.17 (Jones projections). For $n \in \mathbb{N}$, define distinguished elements of A_{n+1} as follows: if n is odd, define

$$
\begin{aligned}
F_n = &\sum_{\vec{\imath}}\sum_{t(\eta)=s(\varepsilon_{i_1})}\frac{[\lambda(t(\varepsilon_{i_n}))\lambda(t(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{i_n}))}\\
&\qquad [\eta\varepsilon_{i_1}\varepsilon_{i_2}^*\dots\varepsilon_{i_{n-1}}^*\varepsilon_{i_n}\varepsilon_{i_n}^*\varepsilon_{i_{n+1}}\varepsilon_{i_{n+1}}^*\varepsilon_{i_{n-1}}\dots\varepsilon_{i_2}\varepsilon_{i_1}^*\eta^*]
\end{aligned}
$$

where the sum is taken over all vectors $\vec{\imath} = (i_1, i_2, \dots, i_{n+1})$ such that

$$[\varepsilon_{i_1}\varepsilon_{i_2}^*\dots\varepsilon_{i_{n-1}}^*\varepsilon_{i_n}\varepsilon_{i_n}^*\varepsilon_{i_{n+1}}\varepsilon_{i_{n+1}}^*\varepsilon_{i_{n-1}}\dots\varepsilon_{i_2}\varepsilon_{i_1}^*] \in G_{n+1,+}.$$

If n is even, then define

$$
\begin{aligned}
F_n = &\sum_{\vec{\imath}}\sum_{t(\eta)=s(\varepsilon_{i_1})}\frac{[\lambda(s(\varepsilon_{i_n}))\lambda(s(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(t(\varepsilon_{i_n}))}\\
&\qquad [\eta\varepsilon_{i_1}\varepsilon_{i_2}^*\dots\varepsilon_{i_{n-1}}\varepsilon_{i_n}^*\varepsilon_{i_n}\varepsilon_{i_{n+1}}^*\varepsilon_{i_{n+1}}\varepsilon_{i_{n-1}}^*\dots\varepsilon_{i_2}\varepsilon_{i_1}^*\eta^*]
\end{aligned}
$$

with a similar limitation on the vectors $\vec{\imath} = (i_1, i_2, \dots, i_{n+1})$.

Lemma 3.18. (1) $F_n x F_n = d E_{A_{n-1}}(x) F_n$, *for all* $x \in A_n$, *and*
(2) $\mathrm{tr}_{n+1}(x F_n) = d^{-1}\,\mathrm{tr}_n(x)$, *for all* $x \in A_n$, *i.e.* $E_{A_n}(F_n) = d^{-1}$.

Proof. We prove the case n odd. The case n even is similar.

(1) If $x = [\zeta_1 \xi_1 \xi_2^* \dots \xi_{n-1} \xi_n^* \dots \xi_{2n-1} \xi_{2n}^* \zeta_2^*] \in A_n$, then $F_n x F_n$ is given by

$$\sum_{\vec{i}} \sum_{t(\eta)=s(\varepsilon_{i_1})} \frac{[\lambda(t(\varepsilon_{i_n}))\lambda(t(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{i_n}))} [\eta \varepsilon_{i_1} \dots \varepsilon_{i_{n-1}}^* \varepsilon_{i_n} \varepsilon_{i_n}^* \varepsilon_{i_{n+1}} \varepsilon_{i_{n+1}}^* \varepsilon_{i_{n-1}} \dots \varepsilon_{i_1}^* \eta^*]$$

$$x \sum_{\vec{j}} \sum_{t(\kappa)=s(\varepsilon_{j_1})} \frac{[\lambda(t(\varepsilon_{j_n}))\lambda(t(\varepsilon_{j_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{j_n}))} [\kappa \varepsilon_{j_1} \dots \varepsilon_{j_{n-1}}^* \varepsilon_{j_n} \varepsilon_{j_n}^* \varepsilon_{j_{n+1}} \varepsilon_{j_{n+1}}^* \varepsilon_{j_{n-1}} \dots \varepsilon_{j_1}^* \kappa^*]$$

$$= \sum_{s(\xi)=s(\xi_{n-1})} \frac{[\lambda(t(\xi))\lambda(t(\xi_{n+1}))]^{1/2}}{\lambda(s(\xi))} [\zeta_1 \xi_1 \xi_2^* \dots \xi_{n-1}^* \xi \xi^* \xi_n \xi_{n+1}^* \dots \xi_{2n-1} \xi_{2n}^* \zeta_2^*]$$

$$\sum_{\vec{j}} \sum_{t(\kappa)=s(\varepsilon_{j_1})} \frac{[\lambda(t(\varepsilon_{j_n}))\lambda(t(\varepsilon_{j_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{j_n}))} [\kappa \varepsilon_{j_1} \dots \varepsilon_{j_{n-1}}^* \varepsilon_{j_n} \varepsilon_{j_n}^* \varepsilon_{j_{n+1}} \varepsilon_{j_{n+1}}^* \varepsilon_{j_{n-1}} \dots \varepsilon_{j_1}^* \kappa^*]$$

$$= \delta_{\xi_n, \xi_{n+1}} \frac{\lambda(t(\xi_n))}{\lambda(s(\xi_n))} \sum_{\substack{s(\xi)=s(\xi_{n-1}) \\ s(\varepsilon)=s(\xi_{n+2})}} \frac{[\lambda(t(\varepsilon))\lambda(t(\xi))]^{1/2}}{\lambda(s(\varepsilon))} [\zeta_1 \xi_1 \dots \xi_{n-1}^* \xi \xi^* \varepsilon \varepsilon^* \xi_{n+2} \dots \xi_{2n}^* \zeta_2^*]$$

$$= d E_{A_{n-1}}(x) F_n.$$

(2) Another straightforward calculation. □

Proposition 3.19 (Basic construction). *For* $n \in \mathbb{N}$, *the inclusion* $A_{n-1} \subset A_n \subset (A_{n+1}, \mathrm{tr}_{n+1}, d^{-1} F_n)$ *is standard. Hence for all* $k \geq 0$, *there are isomorphisms* $\varphi_k \colon A_k \to M_k$ *preserving the trace such that* $\varphi_{k+1}|_{A_k} = \varphi_k$ *and* $\varphi_m(F_n) = E_n$ *for all* $m > n$.

Proof. We construct the isomorphisms φ_n for $n \geq 1$ by induction on n. The base case is finished. Suppose we have constructed φ_n for $n \geq 1$. We know that $M_{n+1} = M_n E_n M_n$ and $A_n \cong M_n$ via φ_n. By Lemmata 2.15 and 3.18, there is an algebra isomorphism $h_{n+1} \colon M_{n+1} = M_n E_n M_n \to A_n F_n A_n \subseteq A_{n+1}$ such that $E_n \mapsto F_n$. But $\dim(M_{n+1}) = \dim(A_{n+1})$, so $A_{n+1} = A_n F_n A_n$, and we set $\varphi_{n+1} = h_{n+1}^{-1}$, which extends φ_n. Finally, note the φ_m's preserve the trace by construction and the uniqueness of the Markov trace. □

3.3. Relative commutants are isomorphic to loop algebras. We provide isomorphisms between the relative commutants of the tower $(A_n)_{n \geq 0}$ and the spaces $G_{n,\pm}$.

Proposition 3.20 (Central vectors). *A basis for the central vectors $A_0' \cap A_n$ is given by*

$$S_{0,n} = \Big\{ \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*] \in A_n \mid [\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*] \in G_{n,+} \Big\}.$$

A basis for the central vectors $A_1' \cap A_{n+1}$ is given by

$$S_{1,n+1} = \Big\{ \sum_{\substack{t(\eta)=s(\varepsilon)\\ t(\varepsilon)=t(\varepsilon_1)}} [\eta\varepsilon\varepsilon_1^*\varepsilon_2 \dots \varepsilon_{2n-1}^*\varepsilon_{2n}\varepsilon^*\eta^*] \in A_{n+1} \mid [\varepsilon_1^*\varepsilon_2 \dots \varepsilon_{2n-1}^*\varepsilon_{2n}] \in G_{n,-} \Big\}.$$

Proof. Note that if $[\zeta_1\zeta_2^*] \in A_0$, then we have

$$\begin{aligned}
[\zeta_1\zeta_2^*] \cdot \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*]
&= \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\zeta_2,\eta}[\zeta_1\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*]\\
&= [\zeta_1\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*\zeta_2^*]\\
&= \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\eta,\zeta_1}[\eta\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*\zeta_2^*]\\
&= \Big(\sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*]\Big) \cdot [\zeta_1\zeta_2^*]
\end{aligned}$$

Hence $S_{0,n} \subset A_0' \cap A_n$. Similarly, $S_{1,n+1} \subset A_1' \cap A_n$.

Suppose now that $x \in A_0' \cap A_n$. Then since $1_{A_0} = \sum_\eta [\eta\eta^*]$, we have

$$x = \Big(\sum_\eta [\eta\eta^*]\Big)x = \Big(\sum_\eta [\eta\eta^*]\cdot[\eta\eta^*]\Big)x = \sum_\eta [\eta\eta^*]\cdot x \cdot [\eta\eta^*] \in \operatorname{span}(S_{0,n}).$$

Similarly, $A_1' \cap A_{n+1} \subseteq \operatorname{span}(S_{1,n+1})$. □

Corollary 3.21. *There are $*$-algebra isomorphisms $\varphi_{n,+}\colon G_{n,+} \to A_0' \cap A_n$ and $\varphi_{n,-}\colon G_{n,-} \to A_1' \cap A_{n+1}$. If $n = 0$, the isomorphisms are given by*

$$\varphi_{0,+}(v_+) = \sum_{t(\eta)=v_+} [\eta\eta^*] \quad \text{and} \quad \varphi_{0,-}(v_-) = \sum_{\substack{t(\eta)=s(\varepsilon)\\ t(\varepsilon)=v_-}} [\eta\varepsilon\varepsilon^*\eta^*].$$

For $n \in \mathbb{N}$, the isomorphisms are given by

$$\varphi_{n,+}([\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*]) = \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^* \dots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*]$$

and

$$\varphi_{n,-}([\varepsilon_1^*\varepsilon_2\dots\varepsilon_{2n-1}^*\varepsilon_{2n}]) = \sum_{\substack{t(\eta)=s(\varepsilon)\\ t(\varepsilon)=t(\varepsilon_1)}} [\eta\varepsilon\varepsilon_1^*\varepsilon_2\dots\varepsilon_{2n-1}^*\varepsilon_{2n}\varepsilon^*\eta^*].$$

It will be helpful to have an explicit Pimsner–Popa basis for A_1 over A_0.

Proposition 3.22 (Pimsner–Popa bases). *For each $v_+ \in \mathcal{V}_+$, pick a distinguished η_{v_+} with $t(\eta_{v_+}) = v_+$. Set*

$$B_1 = \Big\{\Big(\frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))}\Big)^{1/2} \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^*\eta^*] \mid [\varepsilon_1\varepsilon_2^*] \in G_{1,+}\Big\}$$

and

$$B_2 = \Big\{\Big(\frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))}\Big)^{1/2} [\eta_1\varepsilon_1\varepsilon_2^*\eta_{s(\varepsilon_2)}^*] \mid s(\varepsilon_1) \neq s(\varepsilon_2)\Big\}.$$

Then $B = B_1 \amalg B_2$ is a Pimsner–Popa basis for A_1 over A_0.

Proof. Suppose $x = [\zeta_1\xi_1\xi_2^*\zeta_2^*] \in A_1$. Case 1. Suppose that $s(\xi_1) = s(\xi_2)$, so $[\xi_1\xi_2^*] \in G_{1,+}$. If $b \in B_2$, then $E_{A_0}(b^*x) = 0$ as the formula will have delta functions $\delta_{\xi_i,\varepsilon_i}$ for $i = 1, 2$. Hence we have

$$\begin{aligned}
&\sum_{b\in B} bE_{A_0}(b^*x)\\
&= \sum_{b\in B_1} bE_{A_0}(b^*x)\\
&= \sum_{b\in B_1} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \sum_{\substack{t(\eta)=s(\varepsilon_1)\\ t(\zeta)=s(\varepsilon_1)}} [\eta\varepsilon_1\varepsilon_2^*\eta^*]E_{A_0}([\zeta\varepsilon_2\varepsilon_1^*\zeta^*]\cdot[\zeta_1\xi_1\xi_2^*\zeta_2^*])\\
&= \sum_{b\in B_1} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\zeta_1,\zeta}\delta_{\xi_1,\varepsilon_1}[\eta\varepsilon_1\varepsilon_2^*\eta^*]E_{A_0}([\zeta\varepsilon_2\xi_2^*\zeta_2^*])\\
&= \sum_{b\in B_1} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \sum_{t(\eta)=s(\xi_1)} [\eta\xi_1\varepsilon_2^*\eta^*]E_{A_0}([\zeta_1\varepsilon_2\xi_2^*\zeta_2^*])\\
&= \sum_{b\in B_1} \sum_{t(\eta)=s(\xi_1)} \delta_{\xi_2,\varepsilon_2}[\eta\xi_1\varepsilon_2^*\eta^*]\cdot[\zeta_1\zeta_2^*]\\
&= [\zeta_1\xi_1\xi_2^*\zeta_2^*]\\
&= x.
\end{aligned}$$

Case 2. Suppose that $s(\xi_1) \neq s(\xi_2)$. If $b \in B_1$, then, similarly, $E_{A_0}(b^*x) = 0$.

Hence

$$\sum_{b\in B} bE_{A_0}(b^*x) = \sum_{b\in B_2} bE_{A_0}(b^*x)$$
$$= \sum_{b\in B_2} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))}[\eta_1\varepsilon_1\varepsilon_2^*\eta_{s(\varepsilon_2)}^*]E_{A_0}([\eta_{s(\varepsilon_2)}\varepsilon_2\varepsilon_1^*\eta_1^*]\cdot[\zeta_1\xi_1\xi_2^*\zeta_2^*])$$
$$= [\zeta_1\xi_1\xi_2^*\eta_{s(\xi_2)}^*]\cdot[\eta_{s(\xi_2)}\zeta_2^*] = [\zeta_1\xi_1\xi_2^*\zeta_2] = x. \qquad \square$$

Remark 3.23. One could also take

$$B_2 = \left\{\left(\frac{d\lambda(s(\varepsilon_2))}{m_+(s(\varepsilon_2))\lambda(t(\varepsilon_2))}\right)^{1/2}[\eta_1\varepsilon_1\varepsilon_2^*\eta_2^*] \mid s(\varepsilon_1)\neq s(\varepsilon_2)\right\}.$$

Corollary 3.24 (Commutant conditional expectations). *If*

$$x = \sum_{t(\zeta)=s(\xi_1)} [\zeta\xi_1\xi_2^*\ldots\xi_{2n-1}\xi_{2n}^*\zeta^*] \in A_0'\cap A_n,$$

the conditional expectation $A_0'\cap A_n \to A_1'\cap A_n$ *is given by*

$$E_{A_1'}^{A_0'}(x) = d^{-1}\delta_{\xi_1,\xi_{2n}}\left(\frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))}\right)\sum_{\substack{t(\zeta)=s(\varepsilon)\\ t(\varepsilon)=t(\xi_2)}} [\eta\varepsilon\xi_2^*\xi_3\ldots\xi_{2n-2}^*\xi_{2n-1}\varepsilon^*\eta^*].$$

Proof. Let B be as in Proposition 3.22. By Proposition 2.24, we have

$$d^2E_{A_1'}^{A_0'}(x) = \sum_{b\in B} bxb^* = \sum_{b\in B_1} bxb^* + \sum_{b\in B_2} bxb^*.$$

We treat each sum separately. We have $\sum_{b\in B_1} bxb^*$ is given by

$$\sum_{b\in B_1}\frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))}\sum_{\substack{t(\eta)=s(\varepsilon_1)=t(\kappa)\\ t(\zeta)=s(\xi_1)}} [\eta\varepsilon_1\varepsilon_2^*\eta^*]\cdot[\zeta\xi_1\xi_2^*\ldots\xi_{2n-1}\xi_{2n}^*\zeta^*]\cdot[\kappa\varepsilon_2\varepsilon_1^*\kappa^*]$$

$$= d\sum_{\substack{s(\varepsilon)=s(\varepsilon_2)\\ t(\varepsilon)=t(\xi_2)}}\frac{\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))}\sum_{\substack{t(\eta)=s(\varepsilon_1)=t(\kappa)\\ t(\zeta)=s(\xi_1)}} \delta_{\eta,\zeta}\delta_{\zeta,\kappa}\delta_{\varepsilon_2,\xi_1}\delta_{\varepsilon_2,\xi_{2n}}[\eta\varepsilon\xi_2^*\ldots\xi_{2n-1}\varepsilon^*\kappa^*]$$

$$= d\sum_{\substack{t(\eta)=s(\varepsilon)=s(\xi_1)\\ t(\varepsilon)=t(\xi_2)}}\frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))}\delta_{\xi_1,\xi_{2n}}[\eta\varepsilon\xi_2^*\ldots\xi_{2n-1}\varepsilon^*\eta^*].$$

Similarly, we have

$$\sum_{b\in B_2} bxb^* = d \sum_{\substack{t(\eta)=s(\varepsilon)\neq s(\xi_1)\\ t(\varepsilon)=t(\xi_2)}} \frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))}\delta_{\xi_1,\xi_{2n}}[\eta\varepsilon\xi_2^*\dots\xi_{2n-1}\varepsilon^*\eta^*].$$

Putting these two together, we get the desired formula for $E_{A_1'}^{A_0'}$. □

3.4. The bipartite graph planar algebra and the isomorphism. We refer the reader to [10] for the full definition of the planar algebra of a bipartite graph.

Let $G_\bullet$ be the planar algebra of the bipartite graph Γ with spin vector λ as in Subsections 3.1 and 3.2. We briefly recall the action of tangles on the $G_{n,\pm}$ and we calculate some necessary examples.

A *state* σ of a tangle T is a way of assigning the regions and strings of T with compatible vertices and edges of Γ respectively, i.e. if a string S of T partitions the unshaded region R_+ from the shaded region R_-, then for $\sigma(S)\in\mathcal{E}$, $s(\sigma(S)) = \sigma(R_+)\in\mathcal{V}_+$ and $t(\sigma(S)) = \sigma(R_-)\in\mathcal{V}_-$.

Define the *output loop* ℓ_σ as the loop obtained by reading clockwise around the outer boundary of T once it has been labelled by σ.

Suppose now that T has n input disks, and $\ell = \ell_1\otimes\cdots\otimes\ell_n$ is a simple tensor of loops where ℓ_i is a loop in $G_{n_i,\pm_i}$. Then the action of T on ℓ is given by

$$T(\ell) = \sum_{\text{states }\sigma} c(\sigma,\ell)\ell_\sigma,$$

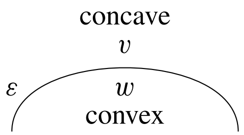
where $c(\sigma,\ell)$ is a correction factor defined as follows.

(1) First, label the regions and strings of T adjacent to the input disks with the edges and vertices which compose the ℓ_i's. If the labelling contradicts σ, then $c(\sigma,\ell)=0$.

(2) If the labels agree, put the tangle in a standard form similar to Section 2.3, where the only difference is that the half the strings emanate from the top of the input rectangles, and half the strings emanate down, but the $*$ is still on the left side. Let $E(T)$ be the set of local extrema of the strings of the standard form of the tangle. For each $e\in E(T)$, let $\operatorname{conv}(e)$ be the vertex assigned by σ to the convex region of the extrema, and let $\operatorname{conc}(e)$ be the vertex assigned to the concave region. Set

$$k_e = \sqrt{\frac{\lambda(\operatorname{conv}(e))}{\lambda(\operatorname{conc}(e))}}.$$

Below is an example of an extrema e on a string S with $\sigma(S)=\varepsilon$, connecting vertices w, v:



$$\longrightarrow k_e = \sqrt{\frac{\lambda(w)}{\lambda(v)}}.$$

Note that conv(e) may be in either $\mathcal{V}_+$ or $\mathcal{V}_-$. Finally, set

$$c(\sigma, \ell) = \prod_{e \in E(T)} k_e.$$

The $*$-structure on the bipartite graph planar algebra is given as follows: if T, ℓ are as above, then

$$T(\ell_1^* \otimes \cdots \otimes \ell_n^*) = T^*(\ell_1 \otimes \cdots \otimes \ell_n)^*,$$

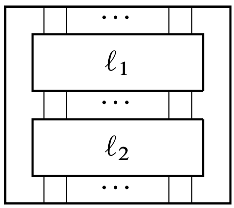
where T^* is the mirror image of T, and the adjoint of a loop is the loop traversed backwards as in Definition 3.5.

Remark 3.25. Contractible loops are traded for a multiplicative factor of d as λ is a Frobenius–Perron eigenvector (see Definition 3.13).

Remark 3.26. Note from Corollary 3.21 that there is a natural inclusion identification $G_{n,-} \to G_{n+1,+}$ given by

$$[\varepsilon_1^* \varepsilon_2 \ldots \varepsilon_{2n-1}^* \varepsilon_{2n}] \longmapsto \sum_{t(\varepsilon)=s(\varepsilon_1)} [\varepsilon \varepsilon_1^* \varepsilon_2 \ldots \varepsilon_{2n-1}^* \varepsilon_{2n} \varepsilon^*].$$

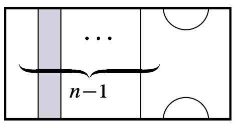
Examples 3.27. (0) If $\ell_1, \ell_2 \in G_{n,\pm}$, then

$\ell_1 \cdot \ell_2 =$ 

,

the shading depending on $n, \pm$.

(1) For $n \in \mathbb{N}$ odd,



is equal to

$$\sum_{\vec{\imath}} \frac{[\lambda(t(\varepsilon_{i_n}))\lambda(t(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{i_n}))} [\varepsilon_{i_1} \ldots \varepsilon_{i_{n-1}}^* \varepsilon_{i_n} \varepsilon_{i_n}^* \varepsilon_{i_{n+1}} \varepsilon_{i_{n+1}}^* \varepsilon_{i_{n-1}} \ldots \varepsilon_{i_1}^*],$$

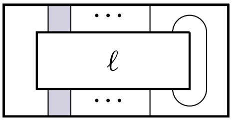
where the sum is taken over all vectors $\vec{\imath} = (i_1, i_2, \ldots, i_{n+1})$ such that

$$[\varepsilon_{i_1} \varepsilon_{i_2}^* \ldots \varepsilon_{i_{n-1}}^* \varepsilon_{i_n} \varepsilon_{i_n}^* \varepsilon_{i_{n+1}} \varepsilon_{i_{n+1}}^* \varepsilon_{i_{n-1}} \ldots \varepsilon_{i_2} \varepsilon_{i_1}^*] \in G_{n+1,+}.$$

There is a similar formula for n even. (Compare with Definition 3.17.)

(2) Suppose $\ell = [\varepsilon_1 \varepsilon_2^* \ldots \varepsilon_{2n-1} \varepsilon_{2n}^*] \in G_{n,+}$.

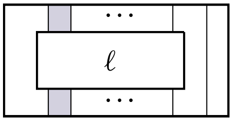
(i) If n is even, then



$$= \delta_{\varepsilon_n,\varepsilon_{n+1}} \frac{\lambda(s(\varepsilon_n))}{\lambda(t(\varepsilon_n))} [\varepsilon_1 \varepsilon_2^* \ldots \varepsilon_{n-1} \varepsilon_{n+2}^* \varepsilon_{2n-1} \ldots \varepsilon_{2n}^*],$$

with a similar formula for n odd. (Compare with Proposition 3.16.)

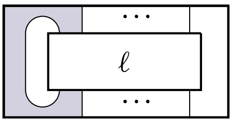
(ii) If n is even, then



$$= \sum_{s(\varepsilon)=s(\varepsilon_n)} [\varepsilon_1 \varepsilon_2^* \ldots \varepsilon_n^* \varepsilon \varepsilon^* \varepsilon_{n+1} \ldots \varepsilon_{2n-1} \varepsilon_{2n}^*],$$

with a similar formula for n odd. (Compare with Definition 3.10.)

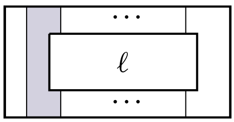
(iii)



$$= \delta_{\varepsilon_1,\varepsilon_{2n}} \frac{\lambda(s(\varepsilon_1))}{\lambda(t(\varepsilon_1))} [\varepsilon_2^* \varepsilon_3 \ldots \varepsilon_{2n-2}^* \varepsilon_{2n-1}].$$

(Compare with Proposition 3.24 and Remark 3.26.)

(3) If $\ell = [\varepsilon_1^* \varepsilon_2 \ldots \varepsilon_{2n-1}^* \varepsilon_{2n}] \in G_{n,-}$, then



$$= \sum_{t(\varepsilon)=s(\varepsilon_1)} [\varepsilon \varepsilon_1^* \varepsilon_2 \ldots \varepsilon_{2n-1}^* \varepsilon_{2n} \varepsilon^*],$$

which may be identified with $\ell \in G_{n+1,+}$ by Remark 3.26.

Theorem 3.28. *The canonical planar $*$-algebra $P_\bullet$ associated to $M_0 \subset (M_1, \mathrm{tr}_1)$ is isomorphic to the bipartite graph planar $*$-algebra $G_\bullet$ of the Bratteli diagram Γ for the inclusion.*

Proof. To show that the $*$-algebra isomorphisms

$$G_{n,+} \xrightarrow{\varphi_{n,+}} A_0' \cap A_n \xrightarrow{\varphi_n|_{A_0' \cap A_n}} M_0' \cap M_n \xrightarrow{\theta_n^{-1}|_{M_0' \cap M_n}} P_{n,+}$$

$$G_{n,-} \xrightarrow{\varphi_{n,-}} A_1' \cap A_{n+1} \xrightarrow{\varphi_{n+1}|_{A_1' \cap A_{n+1}}} M_1' \cap M_{n+1} \xrightarrow{\theta_{n+1}^{-1}|_{M_1' \cap M_{n+1}}} P_{n,-}$$

give an isomorphism of planar $*$-algebras $G_\bullet \to P_\bullet$, we must check that

(1) they map Jones projections in $G_\bullet$ to those in $P_\bullet$, and

(2) they preserve the action of annular tangles.

Both follow immediately from Examples 3.27 and the proof of Lemma 2.49. □

4. The embedding theorem

Let $Q_\bullet$ be a finite depth subfactor planar algebra of modulus d. Pick $r \geq 0$ minimal such that $Q_{2r,+} \subset Q_{2r+1,+} \subset (Q_{2r+2,+}, e_{2r+1})$ is standard (with the usual trace). Note this is possible if and only if $Q_\bullet$ has finite depth. In fact, $Q_{k,+} \subset Q_{k+1,+} \subset (Q_{k+2,+}, e_{k+1})$ is standard for all $k \geq 2r$. For $n \geq 0$, set $M_n = Q_{2r+n,+}$ and $F_{n+1} = E_{2r+n+1}$ (shifted Jones projections). Let $P_\bullet$ be the canonical planar $*$-algebra associated to the inclusion $M_0 \subset M_1$, i.e.

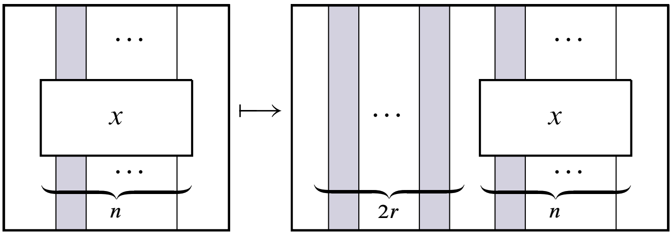
$$P_{n,+} = M_0' \cap M_n = Q_{2r,+}' \cap Q_{2r+n,+}$$

and

$$P_{n,-} = M_1' \cap M_{n+1} = Q_{2r+1,+}' \cap Q_{2r+n+1,+},$$

where we suppress the isomorphisms θ_n with the tensor products of $Q_{2r+1,+}$ over $Q_{2r,+}$.

Theorem 4.1. *Define $\Phi\colon Q_\bullet \to P_\bullet$ by adding $2r$ strings to the left for $x \in Q_{n,+}$ and adding $2r+1$ strings to the left for $x \in Q_{n,-}$:*



Then Φ is an inclusion of planar $*$-algebras.

Proof. We use Lemma 2.49. Note that $\Phi(x^*) = \Phi(x)^*$ and $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in Q_{n,\pm}$.

(1) Since $\Phi(E_j) = E_{2r+j} = F_j$ for all $j \in \mathbb{N}$, we have $\Phi(E_j x) = F_j \Phi(x)$ and $\Phi(x E_j) = \Phi(x) F_j$ for all $x \in Q_{n,\pm}$ and all $j \in \mathbb{N}$.

(2) Note that

(i) For $n \in \mathbb{N}$, $\Phi(E_{Q_{n-1,+}}(x)) = E_{P_{n-1,+}}(\Phi(x))$ since

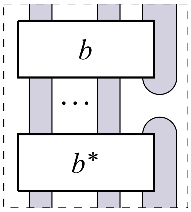
$$E_{Q_{2r+n-1,+}}|_{Q_{2r,+}' \cap Q_{2r+n,+}} = E_{Q_{2r+n-1,+}}|_{P_{n,+}} = E_{P_{n-1,+}}$$

(since $Q_{2r,+} \subset Q_{2r+n-1,+}$, we have that $E_{Q_{2r+n-1,+}}$ preserves $Q_{2r,+}$-central vectors as it is $Q_{2r+n-1,+}$-bilinear).

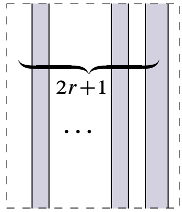
(ii) $\Phi(\beta_{n+1}(x)) = \beta_{n+1}(\Phi(x))$ for all $x \in Q_{n,+}$ since the inclusion $P_{n,+} \to P_{n+1,+}$ is the restriction of the inclusion $Q_{2r+n,+} \to Q_{2r+n+1,+}$.

(iii) Let $B = \{b\}$ be a Pimsner–Popa basis for $M_1 = Q_{2r+1,+}$ over $M_0 = Q_{2r,+}$. Since each $b \in B$ is an $(2r+1,+)$-box in $Q_{2r+1,+}$,

$$\frac{1}{d}\sum_{b\in B}$$



$$= \sum_{b\in B} b e_{2r+1} b^* = 1_{P_{2r+2,+}} =$$

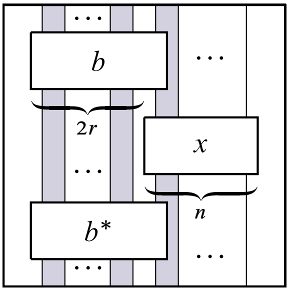


.

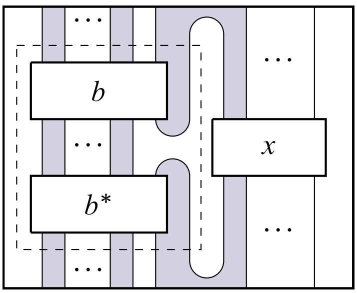
Then by Proposition 2.24 and Theorem 2.50, for all $x \in Q_{n,+}$,

$$\gamma_n^+(\Phi(x)) = \frac{1}{d}\sum_{b\in B} b\Phi(x)b^*$$

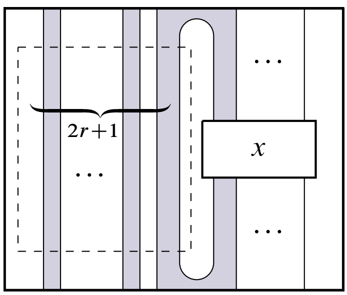
$$= \frac{1}{d}\sum_{b\in B}$$



$$= \frac{1}{d}\sum_{b\in B}$$

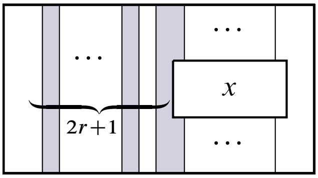


$$=$$



$$= \Phi(\gamma_n^+(x)).$$

(3) The inclusion $i_n^-: P_{n,-} \to P_{n+1,+}$ is the identity in the canonical planar $*$-algebra. If $x \in Q_{n,-}$, then we have

$$i_n^-(\Phi(x)) = \Phi(x) =$$


$$= \Phi(i_n^-(x)). \qquad \square$$

Corollary 4.2. *Let $N \subset M$ be a finite index, finite depth II_1-subfactor, and let $P_\bullet$ be the associated canonical subfactor planar algebra. Let Γ be the principal graph of $N \subset M$, and let $G_\bullet$ be the bipartite graph planar algebra of Γ. Then there is an embedding of planar algebras $P_\bullet \to G_\bullet$.*

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