

Cohomology of mapping class groups and the abelian moduli space

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Abstract. We consider a surface Σ of genus $g \geq 3$, either closed or with exactly one puncture. The mapping class group Γ of Σ acts symplectically on the abelian moduli space $M = \text{Hom}(\pi_1(\Sigma), \text{U}(1)) = \text{Hom}(H_1(\Sigma), \text{U}(1))$, and hence both $L^2(M)$ and $C^\infty(M)$ are modules over Γ . In this paper, we prove that both the cohomology groups $H^1(\Gamma, L^2(M))$ and $H^1(\Gamma, C^\infty(M))$ vanish.

Mathematics Subject Classification (2010). Primary 57M60; Secondary 57M50, 20J06, 53D30.

Keywords. Mapping class groups, group cohomology, moduli space, property (T).

1. Introduction

Let Σ be a compact surface of genus g , which is either closed or with one boundary component. The mapping class group of Σ is the group

$$\Gamma = \pi_0(\text{Diff}^+(\Sigma, \partial\Sigma))$$

of orientation-preserving diffeomorphisms of Σ fixing the boundary (if any) pointwise, up to isotopy.

There are many natural ways to generate infinite dimensional unitary representations of the mapping class group via representation varieties of compact Lie groups. Let us here briefly recall the construction. Let G be a compact Lie group and consider the moduli space M of flat G -connections on Σ , i.e

$$M = \text{Hom}(\pi_1(\Sigma), G)/G.$$

If we choose a set of $2g$ generators for the fundamental group, we get an induced identification

$$M \cong G^{2g}/G \tag{1}$$

¹The authors were supported by QGM (Centre for Quantum Geometry of Moduli Spaces, funded by the Danish National Research Foundation).

if Σ has a boundary component and

$$M \cong \left\{ (A_i, B_i) \in G^{2g} \mid \prod_{i=1}^g [A_i, B_i] = 1 \right\} / G \quad (2)$$

if Σ is closed.

We use these presentations to define the space of smooth functions $C^\infty(M)$ on M . In the case of a surface with boundary, $C^\infty(M)$ is simply the G -invariant smooth functions on G^{2g} . The set $S = \{(A_i, B_i) \in G^{2g} \mid \prod_{i=1}^g [A_i, B_i] = 1\}$ is, in general, not a smooth manifold; we define $C^\infty(S)$ to be $C^\infty(G^{2g})$ modulo the ideal of functions vanishing on S . Then $C^\infty(M) = C^\infty(S)^G$. That these definitions are independent of the choice of generators follows from [11]. The mapping class group Γ clearly acts on M , and this way $C^\infty(M)$ becomes a module over Γ . In the case where Σ has one boundary component, we also observe that both $\text{Aut}(F_{2g})$ and $\text{Out}(F_{2g})$ acts on M , where F_{2g} denotes the free group on $2g$ generators.

The biinvariant Haar measure on G induces a measure on M via (1) in case Σ has one boundary component. In case G is closed, Goldman [8] has constructed a symplectic form ω on M , which induces the Liouville measure $\omega^n/n!$. In both cases, the mapping class group action preserves the measure on M , so $L^2(M)$, the space of complex-valued, square integrable functions on M , becomes an infinite-dimensional unitary representation of Γ .

By work of Goldman ([10] and [9]) and Gelander ([7]), the action of $\text{Aut}(F_n)$ on $\text{Hom}(F_n, G)$ and the action of $\text{Out}(F_n)$ on $\text{Hom}(F_n, G)/G$ are both ergodic for $n \geq 3$. Furthermore, Pickrell and Xia ([13] and [14]), based on Goldman's results, showed that the action of Γ on M is ergodic when Σ is closed. When Σ has boundary, the mapping class group preserves the subsets of M defined by requiring a representation $\rho: \pi_1(\Sigma) \rightarrow G$ to map each boundary component into a prescribed conjugacy class in G ; the action of Γ on each such subset is ergodic.

Ergodicity in particular means that the only invariant functions are the constants. Hence, letting L_0^2 denote the subspace of L^2 corresponding to functions with mean value 0, the above results may be interpreted as the vanishing of certain 0'th cohomology groups, such as $H^0(\text{Aut}(F_n), L_0^2(G^n))$ and $H^0(\Gamma, L_0^2(M))$.

It is very natural to ask if $H^1(\Gamma, L^2(M))$ vanishes both in case where Σ is closed and in the case where Σ has one boundary component. In the latter case, we can also ask if $H^1(\text{Aut}(F_{2g}), L^2(M))$ and $H^1(\text{Out}(F_{2g}), L^2(M))$ vanishes. As it is well known, answering any of these questions in the negative implies that the corresponding group does not have Kazhdan's property (T); see [6]. In case Σ is closed, Andersen has established in [3] that the mapping class group does not have Kazhdan's property (T) by using the TQFT quantum representations of Γ . We, however, do not expect that any of these cohomology groups are non-vanishing and so will not shed light on this question.

In this paper we answer the first question affirmatively in the abelian case, where $G = \text{U}(1)$; see Theorem 7.3.

Theorem. For $G = \mathrm{U}(1)$ we have that $H^1(\Gamma, L^2(M)) = 0$.

The proof of this theorem uses the fact that for $g \geq 2$, the group $\mathrm{Sp}(2g, \mathbb{Z})$ is known to have property (T), the Hochschild–Serre exact sequence, along with the following result (Theorem 5.1) which holds for all unitary representations.

Theorem. Let $\Gamma \rightarrow \mathrm{U}(V)$ be a unitary representation of the mapping class group on a real or complex Hilbert space V . For a Dehn twist τ_γ , let V_γ denote the subspace of V fixed under τ_γ , and let $p_\gamma: V \rightarrow V_\gamma$ denote the orthogonal projection. Then $p_\gamma u(\tau_\gamma) = 0$ for any cocycle $u: \Gamma \rightarrow V$ and any simple closed curve γ .

We are also able to prove the analogous result of Theorem 7.3 when we replace L^2 -functions by smooth functions (Theorem 7.4).

Theorem. For $G = \mathrm{U}(1)$ we have that $H^1(\Gamma, C^\infty(M)) = 0$.

These two results should be compared to the main result from [5]. In that paper, we considered the case $G = \mathrm{SL}_2(\mathbb{C})$ and the space $\mathcal{O} = \mathcal{O}(\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})})$ of regular functions on the moduli space (this makes sense, since (1) and (2) give the moduli space the structure of an algebraic variety). The conclusion in that case was that $H^1(\Gamma, \mathcal{O}) = 0$. In [4] we considered the algebraic dual module, $\mathcal{O}^* = \mathrm{Hom}(\mathcal{O}, \mathbb{C})$, and found that $H^1(\Gamma, \mathcal{O}^*)$ can be written as a countable direct product of finite-dimensional components, of which at least one is non-zero.

This paper is organized as follows. In the next section, we briefly describe our motivation for studying this problem, apart from its connection to Property (T). In Section 4, we briefly recall certain well-known facts about mapping class groups: relations between Dehn twists, the action of a twist on a homology element, and generation of the Torelli group by bounding pair maps. The main purpose of section 5 is to prove that for $g \geq 3$, a certain necessary condition for the vanishing of the cohomology group $H^1(\Gamma, V)$ is always satisfied, for any unitary representation V of Γ (this is the above-mentioned Theorem 5.1). We also quote the results about $\mathrm{Sp}(2g, \mathbb{Z})$ and property (T) which we need. Section 6 is devoted to describing an orthonormal basis for the space of L^2 -functions on the abelian moduli space. This basis has two nice properties: the mapping class group acts by permuting basis elements, and there is a simple condition for determining if an L^2 -function is smooth in terms of its coefficients in this basis. Finally, in Section 7, we prove the two main theorems quoted above.

2. Motivation

The motivation for studying the first cohomology group of the mapping class group with coefficients in a space of functions on the moduli space came from [2]. In

that paper, the first author studied deformation quantizations, or star products, of the Poisson algebra of smooth functions on the moduli space M_G for $G = \text{SU}(n)$. The construction uses Toeplitz operator techniques and produces a family of star products parametrized by Teichmüller space. In [2] the problem of turning this family into one mapping class group invariant star product was reduced to a question about the first cohomology group of the mapping class group with various twisted coefficients. Specifically, one of the results in [2] (Proposition 6) is that, provided the cohomology group $H^1(\Gamma, C^\infty(M_G))$ vanishes, one may find a Γ -invariant equivalence between any two equivalent star products. Since it is easy to see that the only Γ -invariant equivalences are the multiples of the identity, this immediately implies that within each equivalence class of star products, there is at most one Γ -invariant star product.

Considering the results of [1], [2], and the present paper, we get the following application.

Theorem 2.1. *For $G = \text{U}(1)$, there is a unique mapping class group invariant star product on M_G .*

Existence follows from [1] and uniqueness from [2] and Theorem 7.4 below.

3. Group cohomology

In this section we will introduce some terminology and basic results which will be used throughout the rest of the paper. Let G be a group. A G -module is a module over the integral group ring $\mathbb{Z}G$, or equivalently, an abelian group M together with a homomorphism $\pi : G \rightarrow \text{Aut}(M)$.

A cocycle on G with values in M is a map $u : G \rightarrow M$ satisfying the *cocycle condition*

$$u(gh) = u(g) + gu(h) \tag{3}$$

for all $g, h \in G$. Here, and elsewhere, we suppress the homomorphism defining the action from the notation; the last term in (3) should be read $\pi(g)u(h)$. The space of all cocycles is denoted $Z^1(G, M)$. It is easy to see from (3) that a cocycle is determined by its values on a set of generators of G . If $1 \in G$ denotes the identity element, it is easy to see that $u(1) = 0$. From this it follows that $u(g^{-1}) = -g^{-1}u(g)$ for any $g \in G$. It is also easy to deduce the formula $u(ghg^{-1}) = (1 - ghg^{-1})u(g) + gu(h)$. These observations will be used without further comment.

A cocycle is said to be a *coboundary* if it is of the form $g \mapsto v - gv = (1 - g)v$ for some $v \in M$. The space of coboundaries is denoted $B^1(G, M)$, and the first cohomology group of G with coefficients in M is the quotient

$$H^1(G, M) = Z^1(G, M)/B^1(G, M). \tag{4}$$

Notice that in the special case where the group acts trivially on M , the cocycle condition simply means that u is a homomorphism, and the space of coboundaries

vanishes. Hence, in that case we have $H^1(G, M) = \text{Hom}(G, M)$. If

$$\pi : G \longrightarrow \text{Aut}(M)$$

is the homomorphism defining the action, we may also denote $H^1(G, M)$ by $H^1(G, \pi)$.

If G is a topological group, M is a topological vector space and the action of G on M is continuous, one may equip $Z^1(G, M)$ with the topology of uniform convergence over compact subsets. In this topology, $B^1(G, M)$ may or may not be closed in $Z^1(G, M)$; in any case, the quotient

$$\overline{H^1}(G, M) = Z^1(G, M) / \overline{B^1(G, M)} \quad (5)$$

is known as the reduced cohomology of G with coefficients in M .

Proposition 3.1. *Assume $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ is a short exact sequence of groups, and that M is a G -module on which K acts trivially (hence making M a Q -module). Then there is an exact sequence*

$$0 \longrightarrow H^1(Q, M) \longrightarrow H^1(G, M) \longrightarrow H^1(K, M)^G. \quad (6)$$

Here $H^1(K, M)^G$ denotes the subset of $H^1(K, M) = \text{Hom}(K, M)$ which is invariant under G . The action is given by $(g \cdot u)(k) = g^{-1}u(gkg^{-1})$, so an invariant homomorphism is one that satisfies the equivariance condition

$$u(gkg^{-1}) = gu(k) \quad (7)$$

for all $g \in G$ and $k \in K$.

This exact sequence comes from an abstract beast known as the Hochschild–Serre spectral sequence, and really continues with two H^2 terms. Also, in the more general case one does not need to require that K acts trivially on M ; instead, the cohomology of Q is taken with coefficients in the submodule M^K invariant under K . However, we only need the part of the exact sequence shown in (6), and we are able to give an explicit hands-on proof of this proposition which does not involve a spectral sequence.

Proof. The first map above is given by precomposing a cocycle $u : Q \rightarrow M$ with the projection map $\pi : G \rightarrow Q$. This clearly maps cocycles to cocycles. If $u \in Z^1(Q, M)$ is the coboundary of some element $v \in M$, then the cocycle $u \circ \pi \in Z^1(G, M)$ is also the coboundary of v . Hence the first map above is well-defined. Furthermore, if $u \circ \pi$ is a coboundary, then $u(q) = u(\pi(\tilde{q})) = (1 - \tilde{q})v = (1 - q)v$, where \tilde{q} is any element of G mapping to q under π . This proves that the first map above is injective, and hence proves exactness at $H^1(Q, M)$.

The second map above is given by restricting a cocycle $u : G \rightarrow M$ to K . It is easy to see that the restricted map is a homomorphism from K , and that restricting a coboundary gives the zero map, so that the map is well-defined. To see that the map

actually takes values in the space of invariant homomorphisms follows from the little calculation

$$\begin{aligned}
 (g \cdot u)(k) &= g^{-1}u(gkg^{-1}) \\
 &= g^{-1}((1 - gkg^{-1})u(g) + gu(k)) \\
 &= u(k) + (1 - k)g^{-1}u(g) \\
 &= u(k)
 \end{aligned}$$

since k acts trivially on M .

Clearly, if u is a cocycle $Q \rightarrow M$, the composition $K \rightarrow G \rightarrow Q \rightarrow M$ is zero, so the image of the first map is contained in the kernel of second. Conversely, assume that $u: G \rightarrow M$ is a cocycle which satisfies $u(k) = 0$ for any $k \in K$. For any $q \in Q$, choose some $g \in G$ mapping to q , and put $\tilde{u}(q) = u(g)$. This is well-defined, as another choice g' of lift would differ from g by an element $k \in K$, and then $u(g') = u(gk) = u(g) + gu(k) = u(g)$. If $q_1, q_2 \in Q$, choose lifts $g_1, g_2 \in G$. Then the product g_1g_2 is a lift of q_1q_2 , and we have

$$\tilde{u}(q_1q_2) = u(g_1g_2) = u(g_1) + g_1u(g_2) = \tilde{u}(q_1) + q_1\tilde{u}(q_2),$$

so \tilde{u} is a cocycle on Q . This proves exactness at $H^1(G, M)$. \square

4. Twists and relations

Lemma 4.1. *Dehn twists on disjoint curves commute.*

Lemma 4.2. *If α and β are simple closed curves intersecting transversely in a single point, the associated Dehn twists are braided. That is, $\tau_\alpha\tau_\beta\tau_\alpha = \tau_\beta\tau_\alpha\tau_\beta$.*

Lemma 4.3. *If α is a simple closed curve on Σ and $f \in \Gamma$, we have $f\tau_\alpha f^{-1} = \tau_{f(\alpha)}$.*

Lemma 4.4 (Chain relation). *Let α, β and γ be simple closed curves in a two-holed torus as in Figure 1, and let δ, ε denote curves parallel to the boundary components of the torus. Then $(\tau_\alpha\tau_\beta\tau_\gamma)^4 = \tau_\delta\tau_\varepsilon$.*

Lemma 4.5 (Lantern relation). *Consider the surface $\Sigma_{0,4}$, i.e. a sphere with four holes. Let γ_i denote the i 'th boundary component, $0 \leq i \leq 3$, and γ_{ij} a loop enclosing the i 'th and j 'th boundary components, $1 \leq i < j \leq 3$. Let $\tau_i = \tau_{\gamma_i}$ and $\tau_{ij} = \tau_{\gamma_{ij}}$. Then*

$$\tau_0\tau_1\tau_2\tau_3 = \tau_{12}\tau_{13}\tau_{23}. \quad (8)$$

For a picture of the lantern relation, see the left-hand part of Figure 3.

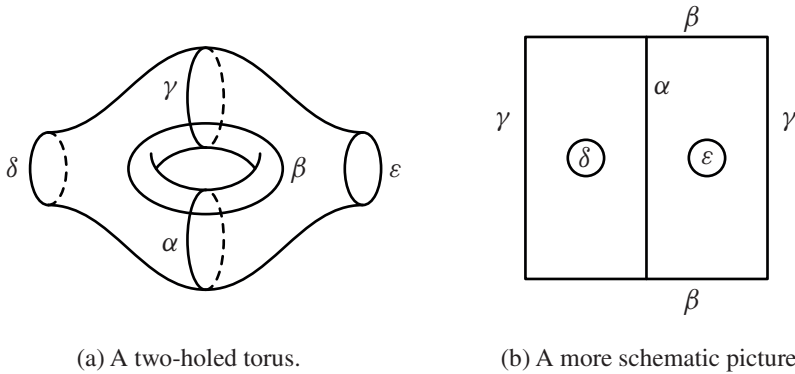


Figure 1. The chain relation.

Corollary 4.6. *If $g \geq 2$, the Dehn twist on a boundary component of $\Sigma_{g,r}$ can be written in terms of Dehn twists on non-separating curves.*

Proof. The assumption on the genus implies that we may find an embedding of $\Sigma_{0,4} \rightarrow \Sigma_{g,r}$ such that γ_0 is mapped to the boundary component in question and the remaining six curves involved in the lantern relation are mapped to non-separating curves (think of $\Sigma_{g,r}$ as being obtained by gluing three boundary components of $\Sigma_{g-2,r+2}$ to γ_1, γ_2 and γ_3 , respectively). Then the relation $\tau_0 = \tau_{12}\tau_{13}\tau_{23}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$ also holds in $\Gamma_{g,r}$. \square

Corollary 4.7. *When $g \geq 3$, $\Gamma_{g,r}$ is generated by Dehn twists on non-separating curves.*

Proof. We already know that the mapping class group is generated by Dehn twists. If γ is a separating curve in Σ , cut Σ along γ and apply Corollary 4.6 to the component which has genus ≥ 2 , showing that τ_γ can be written in terms of twists on non-separating curves in Σ . \square

4.1. Action on homology. Let γ be a simple closed curve on Σ , let $\vec{\gamma}$ denote one of its oriented versions, and let $[\vec{\gamma}] \in H_1(\Sigma)$ denote the homology class of $\vec{\gamma}$. Then for any homology class $m \in H_1(\Sigma)$, the action of τ_γ on m is given by the formula

$$\tau_\gamma m = m + i([\vec{\gamma}], m)[\vec{\gamma}], \tag{9}$$

where $i(\cdot, \cdot)$ denotes the intersection pairing on homology. Clearly, the right-hand side of (9) is independent of the choice of orientation of γ . By induction and using linearity and antisymmetry of i , eq. (9) may be generalized to

$$\tau_\gamma^n m = m + ni([\vec{\gamma}], m)[\vec{\gamma}] \tag{10}$$

This formula immediately implies an important fact.

Lemma 4.8. *If τ_γ acts non-trivially on m , the orbit $\{\tau_\gamma^n m \mid n \in \mathbb{Z}\}$ is infinite.*

Let $(x_1, y_1, \dots, x_g, y_g)$ be a $2g$ -tuple of oriented simple closed curves representing a symplectic basis for $H_1(\Sigma)$; that is, $i(x_j, y_j) = 1$ and $i(x_j, x_k) = i(y_j, y_k) = 0$ for all j, k and $i(x_j, y_k) = 0$ for $j \neq k$. Such a basis induces a norm on $H_1(\Sigma)$ by putting

$$|m| = |a_1| + |b_1| + \dots + |a_g| + |b_g| \tag{11}$$

for $m = a_1x_1 + b_1y_1 + \dots + a_gx_g + b_gy_g$.

We will need the following little technical result later.

Lemma 4.9. *Given any symplectic basis and any non-zero homology element m , there exists a curve γ such that at least one of the sequences $|\tau_\gamma^n m|, |\tau_\gamma^{-n} m|, n = 0, 1, 2, \dots$, is strictly increasing.*

Proof. Let $(a_1, b_1, \dots, a_g, b_g) \in \mathbb{Z}^{2g}$ be the coordinates of m with respect to the given basis. At least one of these coordinates is non-zero. Assume without loss of generality that $a_1 \neq 0$ and put $\gamma = b_1$. Then, for any $n \in \mathbb{Z}$, the coordinates of $\tau_\gamma^n m$ are

$$(a_1, b_1 + na_1, a_2, b_2, \dots, a_g, b_g)$$

by (10) above. Then clearly if a_1 and b_1 have the same sign (b_1 may be 0), the sequence $|\tau_\gamma^n m|$ is increasing, while if a_1 and b_1 have opposite signs the sequence $|\tau_\gamma^{-n} m|$ is increasing. □

Note that we may in fact in all cases choose the Dehn twist from a finite collection of twists.

4.2. The Torelli group. An important subgroup of Γ is the *Torelli group* \mathcal{T} , which by definition is the kernel of the homomorphism $\Gamma \rightarrow \text{Sp}(H_1(\Sigma)) \cong \text{Sp}_{2g}(\mathbb{Z})$. By work of Johnson [12], it is known that the Torelli group is generated by *genus 1 bounding pair maps*. By definition, a bounding pair is a pair (γ, δ) of non-isotopic, non-separating simple closed curves γ, δ , such that the union $\gamma \cup \delta$ separates the surface. The genus of such a pair is, in the case of a closed surface, the minimum of the genera of the two subsurfaces separated by $\gamma \cup \delta$, and in the case of a once-puncture surface, the genus of the subsurface not containing the puncture. The bounding pair map (or BP map) associated to (γ, δ) is the map $\tau_\gamma \tau_\delta^{-1}$. Since γ and δ are homologous, τ_γ and τ_δ acts identically on the homology of Σ , so it is trivial that bounding pair maps belong to the Torelli group.

5. Unitary representations

In this section, we will observe some general facts about cocycles on the mapping class group with values in a unitary representation. Throughout this section, let V

be a real or complex Hilbert space endowed with an action of Γ preserving the inner product.

For a simple closed curve γ , we let $V_\gamma = V^{\tau_\gamma}$ denote the set of vectors fixed under the action of the twist τ_γ , and we let $p_\gamma: V \rightarrow V_\gamma$ denote the orthogonal projection onto the (obviously closed) subspace V_γ . If α and γ are disjoint simple closed curves, the unitary actions τ_α and τ_γ on V commute. Hence the associated projections p_α and p_γ commute with each other and with τ_α, τ_γ . If $\varphi\tau_\alpha\varphi^{-1} = \tau_\beta$, then $\varphi p_\alpha\varphi^{-1} = p_\beta$ for $\varphi \in \Gamma$.

5.1. A satisfied coboundary condition. From now on, let u denote a fixed cocycle. We will now investigate a certain condition for u to be a coboundary, which will turn out to be satisfied whenever $g \geq 3$. If $u(\varphi) = (1 - \varphi)v$ for some vector v , it is clear that $u(\varphi)$ is killed by the projection onto the subspace V^φ fixed by φ . Hence if α is a simple closed curve, it is natural to consider the entity $p_\alpha u(\tau_\alpha)$. The main result of this section is the following theorem.

Theorem 5.1. *Let Σ be a surface of genus at least 3 and let V be a unitary representation of the mapping class group Γ of Σ . For any cocycle $u: \Gamma \rightarrow V$ and any simple closed curve α we have $p_\alpha u(\tau_\alpha) = 0$.*

The proof of this theorem only requires the simple relations in the mapping class group mentioned in Section 4.

We will use the shorthand notation s_α for $p_\alpha u(\tau_\alpha)$.

Lemma 5.2. *The entity s is natural in the sense that $s_{\varphi(\alpha)} = \varphi s_\alpha$ for $\varphi \in \Gamma$ and any simple closed curve α .*

Proof. Since $\tau_{\varphi(\alpha)} = \varphi\tau_\alpha\varphi^{-1}$, it is easy to see that $p_{\varphi(\alpha)} = \varphi p_\alpha\varphi^{-1}$. Hence

$$\begin{aligned} s_{\varphi(\alpha)} &= p_{\varphi(\alpha)}u(\tau_{\varphi(\alpha)}) \\ &= \varphi p_\alpha\varphi^{-1}u(\varphi\tau_\alpha\varphi^{-1}) \\ &= \varphi p_\alpha\varphi^{-1}((1 - \varphi\tau_\alpha\varphi^{-1})u(\varphi) + \varphi u(\tau_\alpha)) \\ &= \varphi p_\alpha u(\tau_\alpha) \\ &= \varphi s_\alpha \end{aligned}$$

as claimed. □

Lemma 5.3. *Let α be a simple closed curve, and let $\varphi \in \Gamma$ be any element commuting with τ_α . Then $\varphi s_\alpha = s_\alpha$.*

Proof. We have $\varphi\tau_\alpha = \tau_\alpha\varphi$. Applying u and the cocycle condition we obtain the equation $u(\varphi) + \varphi u(\tau_\alpha) = u(\tau_\alpha) + \tau_\alpha u(\varphi)$. Applying p_α on both sides, the terms involving $u(\varphi)$ cancel (since obviously $p_\alpha\tau_\alpha = p_\alpha$), so we obtain $p_\alpha\varphi u(\tau_\alpha) = s_\alpha$. The claim then follows from the fact that p_α and φ commute. □

Assume α and β are two non-separating simple closed curves such that $\alpha \cup \beta$ is non-separating, and consider the number $c_{\alpha\beta} = \langle s_\alpha, s_\beta \rangle$.

Lemma 5.4. *The number $c_{\alpha\beta}$ only depends on the cocycle u , not on the pair (α, β) used to compute it.*

Proof. Let (α', β') be any other pair such that $\alpha' \cup \beta'$ does not separate Σ . Then, by the classification of surfaces, there is a diffeomorphism $\varphi \in \Gamma$ such that $\varphi(\alpha) = \alpha'$ and $\varphi(\beta) = \beta'$. Then by the naturality from Lemma 5.2 we have

$$\langle s_{\alpha'}, s_{\beta'} \rangle = \langle s_{\varphi(\alpha)}, s_{\varphi(\beta)} \rangle = \langle \varphi s_\alpha, \varphi s_\beta \rangle = \langle s_\alpha, s_\beta \rangle$$

since φ acts unitarily. □

The vector $s_\alpha = p_\alpha u(\tau_\alpha) \in V$ obviously only depends on the cohomology class $[u] \in H^1(\Gamma, V)$ of u . Hence, we have essentially proved that there exists a well-defined map $c: H^1(\Gamma, V) \rightarrow \mathbb{C}$, whose value on $[u]$ is given by picking any two jointly non-separating simple closed curves α, β and computing the number $c([u]) = \langle p_\alpha u(\tau_\alpha), p_\beta u(\tau_\beta) \rangle$.

Lemma 5.5. *When $g \geq 3$, the map c is identically 0.*

Proof. In any surface of genus at least 2, one may embed the two-holed torus relation (Lemma 4.4) in such a way that γ and δ are non-separating (the curves α, β, γ occurring in the two-holed torus relation are always non-separating). If the genus of the surface is at least 3, the complement of the two-holed torus is a surface of genus at least 1. Hence, in that subsurface we may find a sixth non-separating curve η . Observe that η makes a non-separating pair with each of the other five curves. See Figure 2.

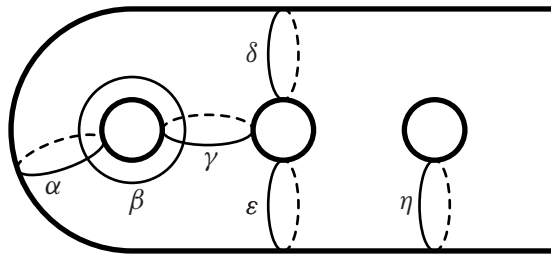


Figure 2. A two-holed torus embedded in a surface of genus ≥ 3 .

Applying u and the cocycle condition repeatedly to the two-holed torus relation yields the equation

$$u(\tau_\alpha) + \tau_\alpha u(\tau_\beta) + \dots = u(\tau_\delta) + \tau_\delta u(\tau_\epsilon). \tag{12}$$

The dots on the left-hand side represent 10 terms involving various actions of $\tau_\alpha, \tau_\beta, \tau_\gamma$ on the values of u on these twists. Since each of the five curves is disjoint from η , we have $\tau_\alpha^{\pm 1} s_\eta = s_\eta$, and similarly for $\beta, \gamma, \delta, \varepsilon$. Now we take the inner product of (12) with s_η to obtain

$$4\langle u(\tau_\alpha), s_\eta \rangle + 4\langle u(\tau_\beta), s_\eta \rangle + 4\langle u(\tau_\gamma), s_\eta \rangle = \langle u(\tau_\delta), s_\eta \rangle + \langle u(\tau_\varepsilon), s_\eta \rangle \quad (13)$$

using the fact that $\langle \varphi x, y \rangle = \langle x, \varphi^{-1} y \rangle$. But since $\tau_\alpha s_\eta = s_\eta$, we also have $p_\alpha s_\eta = s_\eta$, and since the projection p_α is self-adjoint, the first term in (13) is equal to $4\langle s_\alpha, s_\eta \rangle = 4c$. Similar remarks apply to the other terms, so (13) reduces to $12c = 2c$, so $c = 0$. □

Now we are ready to prove the main result of this section.

Proof of Theorem 5.1. We first treat the case where α is non-separating. We cannot simply put $\alpha = \beta$ in the computation of c , since (α, α) is not a non-separating pair. But when the surface has genus at least 3, we may embed the lantern relation (Lemma 4.5) in such a way that all seven curves are non-separating. Furthermore, it can be done in such a way that γ_0 makes a non-separating pair with each of the other six curves. On Figure 3 this is shown for a genus 3 surface; note that the shown surface has been cut along γ_0 . The right-hand part of the cut surface (a sphere with four holes) could be replaced by a surface with any genus and four boundary components. Now the cocycle condition applied to the lantern relation gives

$$u(\tau_0) + \tau_0 u(\tau_1 \tau_2 \tau_3) = u(\tau_{12} \tau_{13} \tau_{23}). \quad (14)$$

Finally, taking the inner product with s_{γ_0} on both sides and applying computations similar to those above, we get $\langle s_{\gamma_0}, s_{\gamma_0} \rangle = \langle u(\tau_0), s_{\gamma_0} \rangle = 0$. Hence $s_{\gamma_0} = 0$, and by naturality (Lemma 5.2) this holds for any non-separating curve.

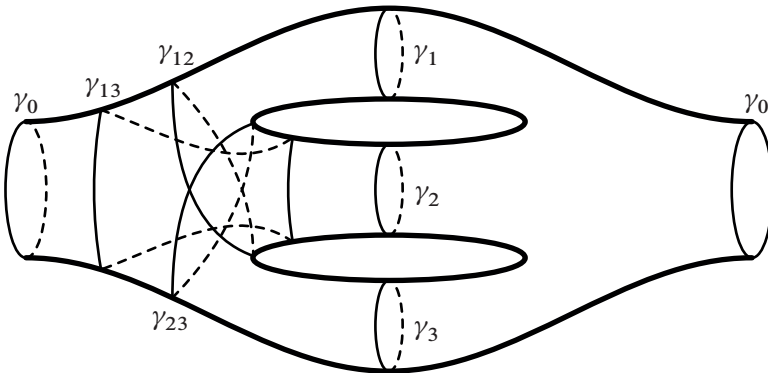


Figure 3. An embedding of the lantern relation such that all seven curves are non-separating. The γ_0 on the left is identified with that on the right.

If α is separating, we use the fact that one of the sides of α has genus at least 2 and Corollary 4.6 to write τ_α as a product of twists in six non-separating curves. For some appropriate choice of signs ε_j , we thus have $\tau_\alpha = \prod_{j=1}^6 \tau_j^{\varepsilon_j}$, where the τ_j are the twists in the appropriate non-separating curves disjoint from α . Now apply the cocycle condition and take the inner product with s_α to obtain

$$\langle s_\alpha, s_\alpha \rangle = \langle u(\tau_\alpha), s_\alpha \rangle = \langle u(\tau_1^{\varepsilon_1}), s_\alpha \rangle + \cdots + \langle \tau_1^{\varepsilon_1} \tau_2^{\varepsilon_2} \tau_3^{\varepsilon_3} \tau_4^{\varepsilon_4} \tau_5^{\varepsilon_5} u(\tau_6^{\varepsilon_6}), s_\alpha \rangle.$$

By Lemma 5.3, $\tau_j^{\pm 1} s_\alpha = s_\alpha$, so using the unitarity of the action this reduces to

$$\langle s_\alpha, s_\alpha \rangle = \sum_{j=1}^6 \langle u(\tau_j^{\varepsilon_j}), s_\alpha \rangle.$$

Finally, we conclude that each term on the right-hand side vanishes by writing s_α as $p_j s_\alpha$, moving the self-adjoint projection p_j to $u(\tau_j^{\varepsilon_j})$ and using that $s_\beta = 0$ for non-separating curves β . □

5.2. Property (T) and Property (FH). Two properties of topological groups, known as Property (T) and Property (FH), respectively, are intimately related to the cohomology of groups with coefficients in real or complex Hilbert spaces. A thorough exposition of these properties and their relationship to group cohomology is far beyond the scope of this paper. We instead refer the interested reader to the very comprehensive book [6]. In this short section we will simply outline the facts we need.

Proposition 5.6. *For $g \geq 2$, the discrete group $\text{Sp}(2g, \mathbb{Z})$ has Property (T).*

Proof. By Theorem 1.5.3 of [6], the locally compact group $\text{Sp}(2g, \mathbb{R})$ has Property (T), and by Theorem 1.7.1, Property (T) is inherited by lattices in locally compact groups. Finally, $\text{Sp}(2g, \mathbb{Z})$ is known to be a lattice in $\text{Sp}(2g, \mathbb{R})$. □

For finitely generated groups, a number of conditions are known to be equivalent to Property (T). The following is quoted from [6], Theorem 3.2.1.

Theorem 5.7. *Let G be a locally compact group which is second countable and compactly generated. The following conditions are equivalent:*

- (i) G has Property (T);
- (ii) $H^1(G, \pi) = 0$ for every irreducible unitary representation π of G ;
- (iii) $\overline{H^1}(G, \pi) = 0$ for every irreducible unitary representation π of G ;
- (iv) $\overline{H^1}(G, \pi) = 0$ for every unitary representation π of G .

In fact, one can add a fifth element to the list.

Lemma 5.8. *Let G be a group satisfying the conditions of Theorem 5.7. Then conditions (i)–(iv) are also equivalent to*

(v) $H^1(G, \pi) = 0$ for every unitary representation π of G .

Proof. Clearly (v) implies (ii) and hence the other conditions. By the Delorme–Guichardet Theorem (Theorem 2.12.4 in [6]), Property (T) and Property (FH) are equivalent for the class of groups considered, so Property (T) implies that $H^1(G, \pi) = 0$ for any orthogonal representation π . Any unitary representation is in particular an orthogonal representation, so $H^1(G, \pi) = 0$ for any unitary representation as well. □

Corollary 5.9. *For any unitary representation $\pi : \mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \mathrm{U}(V)$, the cohomology group $H^1(\mathrm{Sp}(2g, \mathbb{Z}), V)$ vanishes.*

6. Functions on the abelian moduli space

From now on, we let $M = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{U}(1)) = \mathrm{Hom}(H_1(\Sigma), \mathrm{U}(1))$ denote the modulo space of flat $\mathrm{U}(1)$ connections on Σ . The mapping class group acts on M by $(\varphi \cdot \varrho)(m) = \varrho(\varphi^{-1}m)$ for $\varphi \in \Gamma$, $\varrho \in M$ and $m \in H_1(\Sigma)$. This action is smooth and preserves the measure on M , so there are induced actions on $C^\infty(M)$ and $L^2(M)$ given by $(\varphi \cdot f)(\varrho) = f(\varphi^{-1}\varrho)$ for smooth or square integrable functions f .

Let \mathbb{C} denote the space of constant functions on M . Then there are splittings of Γ -modules

$$C^\infty(M) \cong C_0^\infty(M) \oplus \mathbb{C},$$

$$L^2(M) \cong L_0^2(M) \oplus \mathbb{C},$$

where $C_0^\infty(M)$ and $L_0^2(M)$ denotes the space of smooth, respectively square integrable, functions with mean value 0. The action of Γ on \mathbb{C} is obviously trivial, so $H^1(\Gamma, \mathbb{C}) = \mathrm{Hom}(\Gamma, \mathbb{C})$, but since the abelianization of Γ is known to be trivial for $g \geq 3$, the latter is trivial. This yields the isomorphisms

$$H^1(\Gamma, C^\infty(M)) \cong H^1(\Gamma, C_0^\infty(M)),$$

$$H^1(\Gamma, L^2(M)) \cong H^1(\Gamma, L_0^2(M)).$$

6.1. Pure phase functions. Topologically, M is simply a $2g$ -dimensional torus. There is a natural orthonormal basis for $L^2(M)$ parametrized by $H_1(\Sigma)$, which can be described in several different ways.

The intrinsic definition is rather simple. To a homology element $m \in H_1(\Sigma)$, we associate the function \tilde{m} on M given by evaluation in m , i.e we put

$$\tilde{m}(\varrho) = \varrho(m) \in \mathrm{U}(1) \subset \mathbb{C}$$

for $\varrho \in M = \text{Hom}(H_1(\Sigma), \text{U}(1))$.

A choice of basis $(x_1, y_1, \dots, x_g, y_g)$ for $H_1(\Sigma)$ induces a diffeomorphism $M \cong \text{U}(1)^{2g}$ given by

$$\varrho \mapsto (\varrho(x_1), \varrho(y_1), \dots, \varrho(x_g), \varrho(y_g)).$$

Under this identification, the function corresponding to the homology element $m = a_1x_1 + b_1y_1 + \dots + a_gx_g + b_gy_g$ is simply the trigonometric monomial

$$(z_1, w_1, \dots, z_g, w_g) \mapsto z_1^{a_1} w_1^{b_1} \dots z_g^{a_g} w_g^{b_g}$$

on $\text{U}(1)^{2g}$. From this description it is clear that the family $\{\tilde{m} \mid m \in H_1(\Sigma)\}$ constitutes an orthonormal basis for $L^2(M)$.

For any (discrete) set S , we use $\ell^2(S)$ to denote the set of square summable function $S \rightarrow \mathbb{C}$, that is, the set $\{f : S \rightarrow \mathbb{C} \mid \sum_{s \in S} |f(s)|^2 < \infty\}$. We will write such a function as a formal linear combination $\sum_{s \in S} f_s s$.

Lemma 6.1. *There is a mapping class group equivariant isomorphism*

$$L^2(M) \cong \ell^2(H_1(\Sigma)) \tag{15}$$

where $H_1(\Sigma)$ is considered as a discrete set.

Proof. We compute

$$(\varphi \cdot \tilde{m})(\varrho) = \tilde{m}(\varphi^{-1} \cdot \varrho) = (\varphi^{-1} \cdot \varrho)(m) = \varrho(\varphi \cdot m) = \widetilde{\varphi \cdot m}(\varrho),$$

proving the equivariance claim. \square

Since the element $0 \in H_1(\Sigma)$ clearly corresponds to the constant function 1 on M , we immediately obtain the following result.

Lemma 6.2. *Put $H' = H_1(\Sigma) - \{0\}$, considered as a discrete set. Then there is a mapping class group equivariant isomorphism*

$$L_0^2(M) \cong \ell^2(H'). \tag{16}$$

It is very convenient that the action of the mapping class group can be described by a permutation action on an orthonormal basis.

6.2. Smooth functions. Now we know that elements of $L_0^2(M)$ can be thought of as formal linear combinations $\sum_{m \in H'} c_m m$ with $\sum_{m \in H'} |c_m|^2 < \infty$. We will also need to know under which conditions a collection of coefficients (c_m) defines a smooth function. Choose a basis for $H_1(\Sigma)$, and let $|m|$ denote the norm of a homology element as defined by (11).

Proposition 6.3. *The formal sum $\sum_{m \in H_1(\Sigma)} f_m m$ defines a smooth function on M if and only if $|f_m|$ approaches 0 faster than any polynomial in $|m|^{-1}$, or equivalently, if and only if for each $k \in \mathbb{N}$, there is a constant F_k such that*

$$|m|^k |f_m| \leq F_k \tag{17}$$

for all $m \in H_1(\Sigma)$.

These conditions are independent of the chosen basis for $H_1(\Sigma)$.

7. Cohomology computation

In this final section, we will state and prove the main results of this paper.

7.1. Applying Hochschild–Serre. From now on, we fix a symplectic basis

$$(x_1, y_1, \dots, x_g, y_g)$$

for $H_1(\Sigma)$, and using this basis we identify $\text{Sp}(H_1(\Sigma))$ with $\text{Sp}(2g, \mathbb{Z})$. Consider the short exact sequence

$$1 \longrightarrow \mathcal{T} \longrightarrow \Gamma \longrightarrow \text{Sp}(2g, \mathbb{Z}) \longrightarrow 1.$$

Since the Torelli group, by definition, acts trivially on $H_1(\Sigma)$ and hence on $\ell^2(H')$, we are in a position to apply the exact sequence (6). This now takes the guise

$$0 \longrightarrow H^1(\text{Sp}(2g, \mathbb{Z}), \ell^2(H')) \longrightarrow H^1(\Gamma, \ell^2(H')) \longrightarrow H^1(\mathcal{T}, \ell^2(H'))^\Gamma. \tag{18}$$

Lemma 7.1. *The last map in (18) is the zero map.*

Proof. We must prove that any cocycle $u : \Gamma \rightarrow \ell^2(H')$ restricts to zero on the Torelli group. To this end, we use the fact that the Torelli group is generated by genus 1 bounding pair maps. Let $t = \tau_\gamma \tau_\delta^{-1}$ be such a generator for \mathcal{T} . Since t is invariant under conjugation by τ_γ , the equivariance (7) of u restricted to \mathcal{T} implies that

$$u(t) = u(\tau_\gamma t \tau_\gamma^{-1}) = \tau_\gamma u(t)$$

which in turn implies that $u(t) = p_\gamma u(t)$. Now, using the fact that τ_γ and τ_δ acts identically on $H_1(\Sigma)$, we know that $p_\gamma = p_\delta$ on $\ell^2(H')$. Hence using the fact that u is in fact defined on all of Γ , we obtain

$$u(t) = p_\gamma u(t) = p_\gamma (u(\tau_\gamma) - \tau_\gamma \tau_\delta^{-1} u(\tau_\delta)) = p_\gamma u(\tau_\gamma) - \tau_\gamma \tau_\delta^{-1} p_\delta u(\tau_\delta) = 0$$

by Theorem 5.1. □

Corollary 7.2. *The map*

$$H^1(\mathrm{Sp}(2g, \mathbb{Z}), \ell^2(H')) \longrightarrow H^1(\Gamma, \ell^2(H'))$$

is an isomorphism. □

Now the first main theorem.

Theorem 7.3. *The cohomology group*

$$H^1(\Gamma, L_0^2(M))$$

vanishes.

Proof. By Corollary 5.9, the cohomology group $H^1(\mathrm{Sp}(2g, \mathbb{Z}), \ell^2(H'))$ vanishes, and by Corollary 7.2 the same is true for $H^1(\Gamma, \ell^2(H'))$. Finally, $\ell^2(H')$ and $L_0^2(M)$ are isomorphic as Γ -modules by Lemma 6.2. □

7.2. Smooth coefficients. The second main result looks similar to the first, and its proof is also based on it.

Theorem 7.4. *The cohomology group*

$$H^1(\Gamma, C_0^\infty(M))$$

vanishes.

Proof. Let $u: \Gamma \rightarrow C_0^\infty(M)$ be a cocycle. Composing with the inclusion $C_0^\infty(M) \rightarrow L_0^\infty(M) \cong \ell^2(H')$ we may think of u as a cocycle $\Gamma \rightarrow \ell^2(H')$. Hence, by Theorem 7.3 there exists an element $f = \sum_{m \in H'} f_m m$ in $\ell^2(H')$ such that $u(\gamma) = f - \gamma f$ for each $\gamma \in \Gamma$. We claim that f is in fact a smooth function.

To see this, we must verify the condition (17) from Proposition 6.3. It is clearly enough to do this for all large enough k , so assume $k \geq 2$. We must find a constant F_k such that $|m|^k |f_m| \leq F_k$ for all $m \in H'$. Consider the $2g$ Dehn twists $\tau_1, \tau_2, \dots, \tau_{2g}$ in the simple closed curves representing our fixed basis for $H_1(\Sigma)$. By assumption, for each $j = 1, \dots, 2g$, the element

$$u(\tau_j^{\pm 1}) = f - \tau_j^{\pm 1} f = \sum_{m \in H'} (f_m - f_{\tau_j^{\mp 1} m}) m$$

defines a smooth function. Putting $g_{m,j}^\pm = f_m - f_{\tau_j^{\mp 1} m}$, there is a constant G_{k+1} such that

$$|m|^{k+1} |g_{m,j}^\pm| \leq G_{k+1}$$

for all $m \in H'$ and all $j = 1, 2, \dots, 2g$ (such a constant exist for each $\tau_j^{\pm 1}$; we may choose the largest of these $4g$ numbers). We claim that $F_k = G_{k+1}/k$ suffices.

To see this, observe that $|f_m| \rightarrow 0$ as $|m| \rightarrow \infty$ since the collection (f_m) is square summable. Now, let $m \in H'$ be any given element. Choose, by Lemma 4.9, a $j \in \{1, 2, \dots, 2g\}$ and $\varepsilon = \pm 1$ such that $|\tau_j^{\varepsilon n} m|$ is strictly increasing. Assume without loss of generality that $\varepsilon = +1$. For each $R \geq 1$, we have the telescoping sum

$$f_{\tau_j^R m} - f_m = g_{\tau_j^R m, j}^+ + g_{\tau_j^{R-1} m, j}^+ + \dots + g_{\tau_j m, j}^+ = \sum_{r=1}^R g_{\tau_j^r m, j}^+$$

and hence, since $f_{\tau_j^R m} \rightarrow 0$ for $R \rightarrow \infty$, we obtain

$$\begin{aligned} |f_m| &= \left| \sum_{r=1}^{\infty} g_{\tau_j^r m, j}^+ \right| \\ &\leq \sum_{r=1}^{\infty} |g_{\tau_j^r m, j}^+| \\ &\leq G_{k+1} \sum_{r=1}^{\infty} \frac{1}{|\tau_j^r m|^{k+1}} \\ &\leq G_{k+1} \sum_{r=|m|+1}^{\infty} \frac{1}{r^{k+1}} \\ &< G_{k+1} \int_{|m|}^{\infty} \frac{1}{r^{k+1}} dr = \frac{G_{k+1}}{k|m|^k} \end{aligned}$$

using the fact that $|\tau_j^r m|$ is a strictly increasing sequence of integers and elementary estimates.

In case $\varepsilon = -1$, we instead use the identity

$$f_m = \sum_{r=1}^{\infty} g_{\tau_j^{-r} m, j}^-$$

and proceed exactly as above. □

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Received November 30, 2010

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