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The Witten–Reshetikhin–Turaev invariants of finite order mapping tori II

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Abstract. We identify the leading order term of the asymptotic expansion of the Witten–Reshetikhin–Turaev invariants for finite order mapping tori with classical invariants for all simple and simply-connected compact Lie groups. The square root of the Reidemeister torsion is used as a density on the moduli space of flat connections and the leading order term is identified with the integral over this moduli space of this density weighted by a certain phase for each component of the moduli space. We also identify this phase in terms of classical invariants such as Chern–Simons invariants, eta invariants, spectral flow and the ρ-invariant. As a result, we show agreement with the semiclassical approximation as predicted by the method of stationary phase.

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Keywords. Witten–Reshetikhin–Turaev quantum invariant, asymptotics, semiclassical approximation, Reidemeister torsion, ρ-invariant, eta invariant, spectral flow.

Contents

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1. Introduction

The Witten–Reshetikhin–Turaev quantum invariants were first proposed by Witten in his seminal paper [75], where he studied the Chern–Simons quantum field theory for a simple, simply connected compact Lie group G. He did so using path integral techniques, which let him to propose a combinatorial surgery formula for the invariants.

Shortly thereafter Reshetikhin and Turaev gave a rigorous construction of these quantum invariants using the representation theory of quantum groups. In fact, they subsequently constructed the whole topological quantum field theory (TQFT) $Z_G^{(k)}$ in [60], [61], and [70] for $G = \mathrm{SU}(2)$. The other classical groups were treated in [71] and [72]. The TQFT for $G = \mathrm{SU}(2)$ were also constructed using skein theory by Blanchet, Habegger, Masbaum and Vogel in [23] and [24]. Since then these constructions have been extended to other Lie groups G through the effort of many people. For a complete list we refer to the references in [70].

Witten also analyzed the Chern–Simons path integral from a perturbative point of view. The identification of the leading order asymptotics of the invariants in terms of classical topological invariants in the case of an isolated, irreducible flat connection, was proposed by Witten in [75]. There has been subsequent proposals for refinements and generalizations to this, for example by Freed and Gompf [36], eq. (1.3), and by Jeffrey [49], eq. (5.1), partially supported by computations of the quantum invariants, as well as solely from path integral techniques; see the works of Axelrod, Lawrence, Mariño, Rozansky, Singer, and Zagier in [17], [18], [63], [65], [52], and [53].

Both the perturbative expansion conjecture, Conjecture 7.6 in [1], and the asymptotic expansion conjecture, Conjecture 7.7 in [1] address the asymptotic behavior of the quantum invariants which we expect from the perturbative point of view. The first conjecture attempts to give a detailed description of the asymptotic expansion in terms of an integral of certain classical topological invariants over the moduli space of flat connections and Feynman diagrams, which come from stationary phase approximation and the perturbative expansion respectively. Let us refer to the part of this conjecture which is concerned with the leading order term as the *the semiclassical approximation conjecture*. Since the statement of the perturbative expansion conjecture requires some interpretation, the conjecture has been reduced to the mathematically precise asymptotic expansion conjecture, Conjecture 7.7 in [1]. This work is part of a series of papers analyzing the asymptotic behavior of the quantum invariants in the case of finite order mapping tori. While the first part [6] focuses solely on the asymptotic expansion conjecture, we establish the semiclassical approximation conjecture by starting from the results in [6].

The asymptotic expansion of the Witten–Reshetikhin–Turaev quantum invariants has been studied by a number of authors, and various results have been obtained for certain classes of closed three manifolds; see [36], [40, 49], [63], [52], [53], [65], [54], [43], [42], [44], [45], [22], [28], [29], [27], [26], and [30]. An overview can be found in the introduction of [6]. Let us mention the ones, which take the extra step of expressing the terms in the asymptotic expansion of the quantum invariants geometrically. Freed and Gompf [36] considered lens spaces and certain Brieskorn spheres, and they used computer calculations to confirm the semiclassical approximation conjecture. In a subsequent paper, Lisa Jeffrey [49] formally proved this conjecture for lens spaces as well as mapping tori of genus one surfaces with restrictions either on the choice of monodromy map or the structure group. A few missing details about the spectral flow contribution are formulated as Conjecture 5.8 in [49], which has later been partially confirmed; see [50] and [46]. Garoufalidis did work similar to [49] in his thesis [40]. In particular, he found formulas for the quantum invariants for certain Seifert manifolds and rewrote them so it was obvious that they satisfy the asymptotic expansion conjecture. Rozansky [63] studied the asymptotic expansion of the $\mathrm{SU}(2)$ quantum invariants for Seifert manifolds with non-zero orbifold Euler characteristic. In particular, he confirms the perturbative expansion conjecture up to the 2-loop contributions. Rozansky [65] also studied general Seifert manifolds in the case $\mathrm{SU}(2)$ and expresses the contributions to the asymptotic expansion of the quantum invariants in terms of intersection pairings on the moduli space, but it is unclear to what extent his calculations are rigorous. Nevertheless, his formulas bear a formal resemblance to the ones in [6]. Beasley and Witten [22] considered the path integral formula for these quantum invariants for Seifert manifolds with non-zero orbifold Euler characteristic; see eq. (3.20) in [22]. Since they are working with path integrals, their work is *per se* not rigorous, however they provide path integral arguments for the fact that the perturbation expansion of these invariants are finite (modulo the framing correction term). For a mathematical proof of that result please consult [6]. Recently, Charles and Marché in [27] and [29] proved the semiclassical approximation conjecture for Dehn fillings of torus knots and the figure eight knot for $\mathrm{SU}(2)$.

In contrast, our results are for finite order mapping tori of surfaces of genus greater than one, which are Seifert manifolds with vanishing orbifold Euler characteristic. Therefore our family of mapping tori is disjoint from the families considered by Jeffrey and Rozansky. Furthermore, we would like to point out that their approach relies on explicit computations of the quantum invariants and the Poisson resummation trick, while we identify the emerging spectral invariants on a more conceptual level based on geometric quantization. Note that it has recently been confirmed in a series of papers, [11], [12], [10], and [13] that the gauge theory construction of the quantum invariants for $G = \mathrm{SU}(n)$ coincides with the combinatorial or equivalently skein theory construction.

Since we have explicit combinatorial expressions for the quantum invariants, it is sensible to extract the perturbation expansion from these exact formulas. To this end we need an ansatz for the kind of asymptotic expansion we can expect based on Witten's path integral formula for the invariants. This leads us to the asymptotic expansion conjecture, Conjecture 7.7 in [1], Conjecture 1.1 in [6], and Conjecture 1 in [8].

Conjecture 1.1 (Asymptotic expansion conjecture). *Let X be a closed 3-manifold. There exist constants (depending on X) $d_{j,r} \in \mathbb{Q}$ and $b_{j,r} \in \mathbb{C}$ and $a^l_{j,r} \in \mathbb{C}$ for $r = 1, \dots, u_j$, $j = 0, \dots, n$, $l = 1, 2, \dots$ such that the asymptotic expansion of $Z^{(k)}_G(X)$ in the limit $k \to \infty$ is given by*

$$Z^{(k)}_G(X) \sim \sum_{j=0}^{n} e^{2\pi i k q_j} \sum_{r=1}^{u_j} k^{d_{j,r}} b_{j,r} \Big(1 + \sum_{l=1}^{\infty} a^l_{j,r} k^{-l}\Big), \tag{1.1}$$

where $q_0 = 0, q_1, \dots, q_n$ are finitely many different values of the Chern–Simons functional on the space of flat G-connections on X.

Here $\sim$ denotes *asymptotic equivalence* in the Poincaré sense, which means the following: let

$$d = \max\{d_{j,r}\};$$

then for any non-negative integer L, there is a $c_L \in \mathbb{R}$ such that

$$\Big|Z^{(k)}_G(X) - \sum_{j=0}^{n} e^{2\pi i k q_j} \sum_{r=1}^{u_j} k^{d_{j,r}} b_{j,r} \Big(1 + \sum_{l=0}^{L} a^l_{j,r} k^{-l}\Big)\Big| \le c_L k^{d-L+1}$$

for all levels k. Of course such a condition only puts limits on the large k behavior of $Z^{(k)}_G(X)$.

Through the previous definition we can make the following definition of the *leading order term* of the asymptotics.

Definition 1.2. If $Z^{(k)}_G(X)$ satisfies Conjecture 1.1, then we write

$$Z^{(k)}_G(X) \overset{\cdot}{\sim} \sum_{j=0}^{n} e^{2\pi i k q_j} k^{d_j} b_j$$

and we call the sum on the right *the leading order term* of (the asymptotic expansion of) $Z^{(k)}_G(X)$.

It is this leading order term for which there conjecturally is a classical topological expression. In fact, let $\mathcal{M}(X)$ be the moduli space flat G-connections on X and let us write the component decomposition as

$$\mathcal{M}(X) = \bigcup_{c \in C_X} \mathcal{M}(X)_c.$$

One expects that a square root of the Reidemeister torsion produces a measure on $\mathcal{M}(X)$; see [49], [63], [48], and [55]. Combining results from [17], [18], [63], [64], [65], [16], [52], [53], and the references therein we arrive at the following conjectured formula for the leading order term.

Conjecture 1.3 (Semiclassical approximation conjecture). *The leading order term of $Z_G^{(k)}(X)$ with respect to the Atiyah 2-framing* [15] *is given by*

$$Z_G^{(k)}(X) \dot{\sim} \sum_{c \in C_X} \frac{1}{|Z(G)|} e^{\pi i \dim G(1+b^1(X))/4} \int_{A \in \mathcal{M}(X)_c} \sqrt{\tau_X(A)}\, e^{2\pi i \,\mathrm{CS}_X(A)(k+h)} e^{2\pi i (\mathrm{SF}(\theta,A)/4 - (\dim(H^0(X,d_A)) + \dim(H^1(X,d_A)))/8)} k^{d_c} \tag{1.2}$$

and

$$d_c = \frac{1}{2} \max_{A \in \mathcal{M}(X)_c} (\dim(H^1(X, d_A)) - \dim(H^0(X, d_A))),$$

where max *here means the maximum value* $\dim(H^1(X, d_A)) - \dim(H^0(X, d_A))$ *attained on a Zariski open subset of* $\mathcal{M}(X)_c$.

Note that the exact solution of the path integral depends on a framing of twice the tangent bundle as a Spin(6) bundle; see eq. (2.24) and eq. (2.25) in [75]. The dependence on the 2-framing explained in the skein theoretic definition in [24] and in more general setting of quantum invariants for general modular categories in Turaev's exposition [70]. See also the discussion on this point by Freed and Gompf in [36].

In Appendix A we review the heuristics by which the method of stationary phase applies to the Chern–Simons path integral and produces this conjecture. Note that Appendix A contains the only non-rigorous part of this paper, which we decided to keep for motivational purposes. In the conjectured formula (1.2) we see expressions for the constants b_j and d_j in terms of Reidemeister torsion, spectral flow and dimensions of twisted cohomology groups.

In this paper we consider the Witten–Reshetikhin–Turaev quantum invariants of finite order mapping tori. Let Σ be a closed oriented surface. Then the mapping torus Σ_f of a diffeomorphism $f : \Sigma \to \Sigma$ is defined to be

$$\Sigma_f = \Sigma \times I/(x, 1) \sim (f(x), 0)$$

with the orientation given by the product orientation with the standard orientation on the interval $I = [0, 1]$. Let $\mathcal{M}(\Sigma)$ be the moduli space of flat G connections on Σ. This is a stratified symplectic space on which the mapping class group acts. We assume that f is of finite order, and denote by $|\mathcal{M}(\Sigma)| \subset \mathcal{M}(\Sigma)$ the fixed point set of f^*. Denote by C an indexing set for the set of all connected components $\{|\mathcal{M}(\Sigma)|_c\}_{c\in C}$ of $|\mathcal{M}(\Sigma)|$. There is a map $r : \mathcal{M}(\Sigma_f) \to |\mathcal{M}(\Sigma)|$ given by restricting a flat connection on Σ_f to $\Sigma \times \{0\}$. We will write $\mathcal{M}(\Sigma_f)_c = r^{-1}(|\mathcal{M}(\Sigma)|_c)$. Let the prime superscript denote the part which is irreducible in $\mathcal{M}(\Sigma)$: $\mathcal{M}(\Sigma)' \subset \mathcal{M}(\Sigma)$ denotes the irreducible subset, while $\mathcal{M}(\Sigma_f)'_c = r^{-1}(|\mathcal{M}(\Sigma)|'_c)$.

Choose a complex structure σ on Σ, which is fixed by f. Consider the moduli space $\mathcal{M}_\sigma$ of semi-stable $G^{\mathbb{C}}$ bundles over Σ_σ. We identify $\mathcal{M}_\sigma$ and $\mathcal{M}(\Sigma)$ as stratified symplectic spaces, but $\mathcal{M}_\sigma$ has the additional structure of a normal projective variety. We write

$$|\mathcal{M}_\sigma| = \bigcup_{c \in C} |\mathcal{M}_\sigma|_c$$

for the component decomposition of the fixed point set of f. Following the notation of [20] and [21] we denote the Grothendieck group of all equivariant coherent sheaves on $\mathcal{M}_\sigma$ by $K_0^{\mathrm{eq}}(\mathcal{M}_\sigma)$ and the Grothendieck group of all coherent sheaves on $|\mathcal{M}_\sigma|_c$ by $K_0^{\mathrm{alg}}(|\mathcal{M}_\sigma|_c)$. Let

$$L_\bullet^c \colon K_0^{\mathrm{eq}}(\mathcal{M}_\sigma) \longrightarrow K_0^{\mathrm{alg}}(|\mathcal{M}_\sigma|_c) \otimes \mathbb{C}$$

be the localizing homomorphism defined in §2 in [21], and

$$\tau_\bullet \colon K_0^{\mathrm{alg}}(|\mathcal{M}_\sigma|_c) \longrightarrow H_\bullet(|\mathcal{M}_\sigma|_c)$$

the homomorphism defined in the theorem on p. 180 in [20]. The Lefschetz–Riemann–Roch formula of Baum, Fulton, McPherson and Quart then states that

$$\mathrm{tr}(f \colon H^0(\mathcal{M}_\sigma, \mathcal{L}^k) \longrightarrow H^0(\mathcal{M}_\sigma, \mathcal{L}^k)) = \sum_{c \in C} a_c^k \, \mathrm{ch}(\mathcal{L}^k|_{|\mathcal{M}_\sigma|_c}) \cap \tau_\bullet(L_\bullet^c(\mathcal{O}_{\mathcal{M}_\sigma}))$$

where a_c is the complex number by which f acts on $\mathcal{L}|_{|\mathcal{M}_\sigma|_c}$. For the convenience of the reader we review the Lefschetz–Riemann–Roch theorem for singular varieties in Appendix B. From Theorem 2.19 in [35] we see that f acts on $\mathcal{L}_{[A]}$ by multiplication with $\exp(2\pi i \, \mathrm{CS}_{\Sigma_f}(A))$ and that $\mathrm{CS}_{\Sigma_f}(A) \mod \mathbb{Z}$ is constant for $A \in \mathcal{M}(\Sigma_f)_c$. If we write $\exp(2\pi i \, \mathrm{CS}_{\Sigma_f}(c)) = \exp(2\pi i \, \mathrm{CS}_{\Sigma_f}(A))$ for $A \in \mathcal{M}(\Sigma_f)_c$, we get

$$a_c = \exp(2\pi i \, \mathrm{CS}_{\Sigma_f}(c)).$$

Clearly

$$\mathrm{ch}(\mathcal{L}^k|_{|\mathcal{M}_\sigma|_c}) = \exp(k\omega_c),$$

where ω_c is the the restriction of $c_1(\mathcal{L})$ to $|\mathcal{M}_\sigma|_c$. In summary, we get the following theorem of [6], which proves the asymptotic expansion conjecture for finite order mapping tori.

Theorem 1.4 (Theorem 8.2 in [6]). *The Witten–Reshetikhin–Turaev invariants of Σ_f are given by*

$$Z_G^{(k)}(\Sigma_f) = \det(f)^{-\frac{1}{2}\zeta} \sum_{c \in C} \exp(2\pi i k \, \mathrm{CS}_{\Sigma_f}(c)) \exp(k\omega_c) \cap \tau_\bullet(L_\bullet^c(\mathcal{O}_{\mathcal{M}(\Sigma)})), \tag{1.3}$$

where

$$\zeta = \frac{k \dim G}{k + h}$$

is the central charge of the theory, h is the dual Coxeter number of G, $\det(f)^{-\frac{1}{2}\zeta}$ is the framing correction defined in Section 3.

Note that the restriction of $c_1(\mathscr{L})$ to the smooth part of the moduli space $\mathcal{M}_\sigma$ can be represented by the Kähler form on $\mathcal{M}_\sigma$. The evaluation of the top power of the class $c_1(\mathscr{L}^k)$ on $\tau_\bullet(L^c_\bullet(\mathcal{O}_{\mathcal{M}(\Sigma)}))$ is just the integration of this top form over the smooth part of $|\mathcal{M}(\Sigma)|_c$, when this component has the property that it has an open dense part of irreducibles. This follows from the lemma on p. 129 of [19] and part (6) of the Riemann–Roch Theorem of [39] (see also Appendix B below). Starting with Theorem 1.4, our main result is the following theorem, which applies to all finite order elements f of the mapping class group of Σ.

Theorem 1.5. *For each $c \in C$ such that $\mathcal{M}(\Sigma_f)'_c$ is nonempty we have*

$$k^{-d_c} \det(f)^{-\frac{1}{2}\zeta} e^{2\pi i k \operatorname{CS}_{\Sigma_f}(c)} \frac{1}{d_c!}(\exp(k\omega_c) \cap \tau_\bullet(L^c_\bullet(\mathcal{O}_{\mathcal{M}(\Sigma)}))) \\ = \frac{1}{|Z(G)|} \int_{A \in \mathcal{M}(\Sigma_f)'_c} e^{2\pi i k \operatorname{CS}_{\Sigma_f}(A)} \sqrt{\tau_{\Sigma_f}(A)}\, e^{2\pi i \frac{\rho_A(\Sigma_f)}{8}} + O\Big(\frac{1}{k}\Big), \tag{1.4}$$

where $\rho_A(\Sigma_f)$ is the classical ρ-invariant.

In Section 8 we give a proof of a well-known formula relating the spectral flow, the ρ-invariant and the Chern–Simons invariant for an arbitrary Lie group G, since the only proof we have found in the literature is for SU(2); see Section 7 in [51]. The precise relation – stated in Theorem 8.1 – shows, in particular, that Theorem 1.5 has an equivalent formulation in terms of spectral flow, which has the following theorem as an immediate consequence, once combined with Theorem 1.4.

Theorem 1.6. *If $\mathcal{M}(\Sigma)'_c$ is nonempty for every $c \in C$, then the above conjecture for the leading order term is correct, i.e.*

$$Z_G^{(k)}(\Sigma_f) \doteq\sim \sum_{c \in C} \frac{1}{|Z(G)|} e^{\pi i \dim G(1+b^1(\Sigma_f))/4} \\ \int_{A \in \mathcal{M}(\Sigma_f)'_c} k^{d_c} \sqrt{\tau_{\Sigma_f}(A)}\, e^{2\pi i \operatorname{CS}_{\tilde{A}_f}(A)(k+h)} \\ e^{2\pi i (\operatorname{SF}(\theta,A)/4 - (\dim(H^0(\Sigma_f,d_A)) + \dim(H^1(\Sigma_f,d_A)))/8)}.$$

Recall that $\mathcal{M}(\Sigma_f)'_c$ consists of the irreducible representation whose restriction to Σ is irreducible. By Theorem 2.3 the hypothesis of Theorem 1.6 is satisfied in the case $G = \mathrm{SU}(n)$ and $g(\Sigma/\langle f \rangle) > 1$. Note that unlike for lens spaces, the stationary phase approximation is in general not exact: for example, for $f = \mathrm{Id}$ lower order terms in the asymptotic expansion do not in general vanish, as one easily sees, since the Todd class of the moduli spaces are in general none trivial.

This paper is organized as follows. Section 2 contains a preliminary discussion about the Chern–Simons functional and the moduli spaces of flat connections. In Section 3 we express the leading order term of the Witten–Reshetikhin–Turaev invariants for each $c \in C$ as certain integrals of differential geometric data. In Section 4 and 5 we review Reidemeister torsion and compute it for mapping tori. In Section 6 we review the ρ-invariant and an essential result for finite order mapping tori by Bohn [25]. Section 7 combines the main results from Sections 3 and 5 to identify the classical invariants in the leading order term of the $Z_G^{(k)}(X)$ in the limit $k \to \infty$. Section 8 gives an equivalent formulation of this identification in Section 3 in terms of spectral flow. In Appendix A we present the heuristics which lead to the conjectured identification of the leading order term with classical topological invariants. In Appendix B we review the Lefschetz–Riemann–Roch theorem for singular varieties.

The results of this paper relies on the results of [6], which were obtained by using the gauge theory approach to the Witten–Reshetikhin–Turaev TQFT. The first named author has obtained other results about this TQFT using the gauge theory approach, such as the asymptotic faithfulness of the quantum representations [3] and the determination of the Nielsen–Thurston classification via these same representations [4]; see also [9]. He has further related these quantum representations to deformation quantization of moduli spaces both in the abelian and in the non-abelian case, please see [2], [7], and [5]. The second named author has answered some open questions by Jeffrey [49] about this TQFT for torus-bundles over S^1 by using cut-and-paste methods to perform spectral flow computations; see [46].

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2. The Chern–Simons invariant and moduli spaces of flat connections

In this section we give some necessary definitions and make some remarks regarding normalizations before we consider the moduli space and recall its decomposition into connected components.

2.1. Normalizations for the Chern–Simons functional and Poincaré duality. Let $\langle \cdot, \cdot \rangle$ be a multiple of the Killing form on the Lie Algebra $\mathfrak{g}$ of a simple and simply-connected compact Lie group G normalized so that $-\frac{1}{6}\langle \theta \wedge [\theta \wedge \theta] \rangle$ is a minimal integral generator of $H^3(G, \mathbb{R})$, where θ is the Maurer–Cartan form. A connection on a principal G-bundle P is a G-equivariant, horizontal Lie algebra valued 1-form on P. The group of gauge transformations $\mathcal{G}$ consists of all bundle automorphism $P \to P$, which acts on connections by pull-back. Let X be an oriented, closed 3-manifold. Since G is simply-connected, every principal G-bundle over X is trivializable; therefore let us fix a trivialization to simplify notation, which allows us to identify the affine space of connections with Lie algebra valued 1-forms $\mathcal{A}_X = \Omega^1(X;\mathfrak{g})$. Furthermore, the moduli space of flat G-connections on X, denoted by $\mathcal{M}(X)$, can be identified with the moduli space of flat connections in the trivial G-bundle. The Chern–Simons invariant is the map $\mathcal{A}_X \to \mathbb{R}$ given by

$$\mathrm{CS}_X(A) = \int_X \langle A \wedge dA + \frac{1}{3} A \wedge [A \wedge A]\rangle. \tag{2.1}$$

It is not difficult to see that – with our choice of normalization for the inner product on $\mathfrak{g}$ – CS_X factors through $\mathcal{G}$ as an $\mathbb{R}/\mathbb{Z}$-valued map. It is also not difficult to see that the map $\mathcal{G} \to \mathbb{Z}$ given by $\Phi \mapsto \mathrm{CS}_X(\Phi^* A) - \mathrm{CS}_X(A)$ is onto.

Let Σ be a closed oriented surface and consider the space of connections $\mathcal{A}_\Sigma$ in a trivial principal G-bundle over Σ. The symplectic structure on $\mathcal{A}_\Sigma$ is naturally given by

$$\omega(a,b) = -2\int_\Sigma \langle a \wedge b\rangle. \tag{2.2}$$

This gives a (stratified) symplectic structure on the moduli space $\mathcal{M}(\Sigma)$ of flat G-connections on Σ.

In order to view the square root of Reidemeister torsion as a density, we need to identify $H^2(\Sigma_f, d_A)$ with $(H^1(\Sigma_f, d_A))^*$ using Poincaré duality, which depends on a choice of inner product on $\mathfrak{g}$. For $\mathfrak{g}$-valued differential forms a and b we set

$$\mathrm{PD}(a)(b) = \int_{\Sigma_f} 2\langle a \wedge b\rangle,$$

which descends to the Poincaré duality isomorphism

$$H^k(\Sigma_f, d_A) \longrightarrow (H^{3-k}(\Sigma_f, d_A))^*.$$

Note that the factor 2 might seem unnatural, but as we mention in Appendix A, there is a choice involved, and the correct choice is the one which satisfies $\mathrm{PD}(a)(b) = -\omega(a,b)$.

2.2. Connected components of $\mathcal{M}(\Sigma_f)$ and $|\mathcal{M}(\Sigma)|$. Recall that $r\colon \mathcal{M}(\Sigma_f)'_c \to |\mathcal{M}(\Sigma)|'_c$ is a $|Z(G)|$-sheeted covering map; see [6], Section 7. We get a complete description of the leading order term of the Witten–Reshetikhin–Turaev invariants in terms of a sum of integrals over the components $\mathcal{M}(\Sigma_f)'_c$ of $\mathcal{M}'(\Sigma_f)$, if every $|\mathcal{M}(\Sigma)|_c$ contains an irreducible representation. The connected components of $\mathcal{M}(\Sigma)$ have been studied by Goldman [41]. The components of the fixed point set $|\mathcal{M}(\Sigma)|$ of f are analyzed in [6], Section 6. In this section we will see, in which situations all $\mathcal{M}(\Sigma_f)'_c$ are nonempty.

For a chosen diffeomorphism $f\colon \Sigma \to \Sigma$ of order m consider the projection $\pi\colon \Sigma \to \widetilde{\Sigma}$ to the quotient surface $\widetilde{\Sigma} = \Sigma/\langle f\rangle$. Σ is an m-fold branched cover over $\widetilde{\Sigma}$ with branch points $\tilde{p}_1, \ldots, \tilde{p}_n$, for which

$$\pi^{-1}(\tilde{p}_i) = \{p_i, f(p_i), \ldots, f^{m_i-1}(p_i)\} \quad \text{with } m_i < m.$$

Choose small disjoint closed discs D_i around each p_i, $i = 1, \dots, n$, such that $f^j(D_i)$, $j = 0, \dots, m_i$, $i = 0, \dots, n$, are disjoint. Let Σ' be the complement of the interior of all these discs and $\tilde{\Sigma}' = \Sigma'/\langle f \rangle$.

The indexing set for the connected components of $|\mathcal{M}'(\Sigma)|$ is shown in Proposition 6.1 in [6] to be

$$\Delta = \{(z, c_1, \dots, c_n) \in Z(G) \times \mathrm{Cl}^n \mid z \in c_i^{l_i}\}/Z(G),$$

where Cl is the set of conjugacy classes of G and $l_i = \frac{m}{m_i}$. We have a surjective map $\Delta \to C$ if $|\mathcal{M}(\Sigma)| = \overline{|\mathcal{M}'(\Sigma)|}$. By Proposition 6.3 in [6] this is the case for $G = \mathrm{SU}(n)$. For $c(\delta) = (c_1^{-k_1}, \dots, c_n^{-k_n})$ let $\mathcal{M}(\tilde{\Sigma}', c(\delta))$ be the moduli space of flat G-connections on $\tilde{\Sigma}'$ with holonomy around $\partial_i \tilde{\Sigma}'$ in $c_i^{k_i}$, $i = 1, \dots, n$. By Theorem 6.1 in [6], a component $|\mathcal{M}'(\Sigma)|_\delta$ can be described as the space $\mathcal{M}''(\tilde{\Sigma}', c(\delta))/Z_\delta$, where $\mathcal{M}''(\tilde{\Sigma}', c(\delta))$ consists of the flat G-connections in $\mathcal{M}'(\tilde{\Sigma}', c(\delta))$, which remain irreducible when pulled back via π.

Proposition 2.1. *Let G be a connected compact Lie group, $\tilde{\Sigma}'$ a genus two surface with one boundary circle $\partial_i \tilde{\Sigma}'$, and $\pi : \Sigma' \to \tilde{\Sigma}'$ a covering map. Then $\mathcal{M}''(\tilde{\Sigma}', c)$ is nonempty for every $c \in \mathrm{Cl}$.*

Proof. Write

$$\pi_1(\tilde{\Sigma}') = \langle x_1, y_1, x_2, y_2 \rangle,$$

then the moduli space $\mathcal{M}(\tilde{\Sigma}', c)$, $c \in \mathrm{Cl}$, consists of all conjugacy classes of ρ satisfying

$$\rho([x_1, y_1][x_2, y_2]) \in c.$$

By Auerbach's Generation Theorem, see [47], Theorem 6.82, we have $G = \overline{\langle g_1', h_1' \rangle}$ for some g_1', h_1'. Choose g_1, h_1 such that $g_1^m = g_1'$ and $h_1^m = h_1$. By Gotô's Commutator Theorem, see [47], Theorem 6.55, we find $g_2, h_2 \in G$ such that

$$[g_1, h_1][g_2, h_2] \in c.$$

Consider the representation determined on the generators by

$$\begin{aligned} \tilde{\rho} \colon \pi_1(\tilde{\Sigma}') &\longrightarrow G \\ x_i &\longmapsto g_i \\ y_i &\longmapsto h_i \end{aligned}$$

(see Figure 1). Clearly, $x_1^m, y_1^m \in \mathrm{im}(\pi : \pi_1(\Sigma') \to \pi_1(\tilde{\Sigma}'))$ and therefore $g_1', h_1' \in \mathrm{im}(\rho)$. We claim that $\rho = \pi^* \tilde{\rho}$ is irreducible, i.e. $\mathrm{Stab}_G(\rho) = Z(G)$. We automatically have $Z(G) \subset \mathrm{Stab}_G(\rho)$. Let $g \in \mathrm{Stab}_G(\rho)$. Then g is in particular in the centralizer $C_G(\{g_1', h_1'\}) = C_G(\langle g_1', h_1' \rangle)$. Therefore by continuity

$$g \in C_G(\overline{\langle g_1', h_1' \rangle}) = C_G(G) = Z(G).$$

Therefore ρ is irreducible. ☐

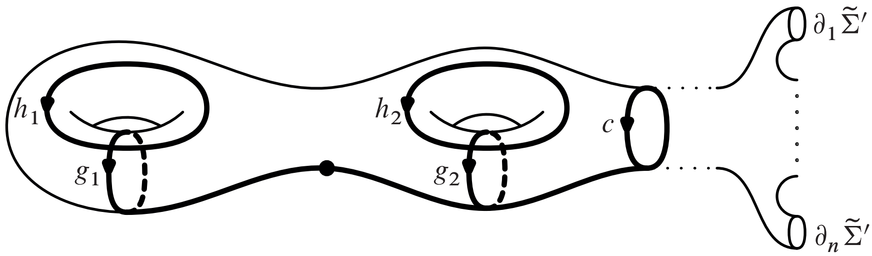

Figure 1. Images of the representation $\rho\colon \pi_1(\tilde{\Sigma}') \to G$.

A glance at Figure 1 gives the following.

Corollary 2.2. *Let* $\tilde{\Sigma} = \Sigma/\langle f \rangle$ *be a surface of genus greater than* 1. *Then* $\mathcal{M}''(\tilde{\Sigma}', c(\delta))$ *is nonempty for all* $\delta \in \Delta$.

As mentioned above, $\Delta \to C$ is surjective for $G = \mathrm{SU}(n)$, so that together with the above corollary we get the following.

Theorem 2.3. *If* $\tilde{\Sigma} = \Sigma/\langle f \rangle$ *is a surface of genus greater than* 1 *and* $G = \mathrm{SU}(N)$, *then* $\mathcal{M}(\Sigma_f)'_c = r^{-1}(|\mathcal{M}(\Sigma)|'_c)$ *is nonempty for every* $c \in C$.

We will see in the next section that $|\mathcal{M}(\Sigma)|'_c$ being nonempty enables us to express the leading order term of the corresponding summand in the expression (1.3) as an integral over $|\mathcal{M}(\Sigma)|'_c$. Theorem 2.3 therefore shows, in which cases we get an integral expression for the entire leading order term of the asymptotic expansion.

3. The leading order term of the Witten–Reshetikhin–Turaev invariant

Let us now identify the leading order term of the asymptotic expansion (1.3) of the Witten–Reshetikhin–Turaev invariants of finite order mapping tori as an integral of differential geometric terms.

We first consider the framing correction term as defined in eq. (5) in [6]

$$\det(f)^\alpha = \mathrm{tr}(\tilde{f}\colon \mathcal{L}^\alpha_{D,\sigma} \to \mathcal{L}^\alpha_{D,\sigma}),$$

where $\tilde{f}$ is a lift of f to the rigged mapping class group determined by the Atiyah 2-framing. The rigged mapping class group is a central extension of the mapping class group constructed by [73] and [70] (see also [6], Section 2). This element $\tilde{f}$ acts on any power, say α, of the determinant line bundle $\mathcal{L}_D$ over Teichmüller space and σ is a point in Teichmüller space preserved by f. For the rest of this paper we denote Σ with the complex structure σ simply by Σ. The framing correction term is obtained by setting $\alpha = \zeta$.

Proposition 3.1. *For a finite order automorphism $f : \Sigma \to \Sigma$ of a surface Σ we have*

$$\det(f)^{\alpha} = \exp\left(\sum_{0 \neq \tilde{\omega} \in (-\frac{1}{2}, \frac{1}{2})} -2\pi i \alpha \tilde{\omega}_i \right),$$

where $e^{2\pi i \tilde{\omega}_j}$, $\tilde{\omega}_j \in [-\frac{1}{2}, \frac{1}{2})$, are the eigenvalues of $f^* \colon H^{1,0}(\Sigma, \bar{\partial}) \to H^{1,0}(\Sigma, \bar{\partial})$.

Proof. Let us identify $H^1(\Sigma, \mathbb{R})$ with $H^{1,0}(\Sigma, \bar{\partial})$ via

$$H^1(\Sigma, \mathbb{R}) \hookrightarrow H^1(\Sigma, \mathbb{C}) \xrightarrow{\text{pr}} H^{1,0}(\Sigma, \bar{\partial}),$$

where pr is the projection to the subspace. We get that the diagram

$$\begin{array}{ccc} H^{1,0}(\Sigma, \mathbb{R}) & \xleftarrow{f^*} & H^{1,0}(\Sigma, \mathbb{R}) \\ \cong \big\downarrow & & \big\downarrow \cong \\ H^1(\Sigma, \mathbb{R}) & \xleftarrow{f^*} & H^1(\Sigma, \mathbb{R}) \end{array}$$

commutes. By naturality of Poincaré duality, the diagram

$$\begin{array}{ccc} H^1(\Sigma, \mathbb{R}) & \xleftarrow{f^*} & H^1(\Sigma, \mathbb{R}) \\ \text{PD} \big\downarrow & & \big\downarrow \text{PD} \\ H_1(\Sigma, \mathbb{R}) & \xrightarrow{f_*} & H_1(\Sigma, \mathbb{R}) \end{array}$$

commutes. In particular, the eigenvalues of $\text{PD}^{-1} \circ f_* \circ \text{PD}$ and f^* are inverses of each other. In analogy to [6], Section 5, we get that

$$\det(f)^{\alpha} = \exp\left(\sum_{0 \neq \omega \in (-\frac{1}{2}, \frac{1}{2})} -2\pi i \alpha \tilde{\omega}_i \right)$$

where $e^{-2\pi i \tilde{\omega}_j}$, $\tilde{\omega}_j \in [-\frac{1}{2}, \frac{1}{2})$, are the eigenvalues of $\text{PD}^{-1} \circ f_* \circ \text{PD}$, or equivalently, where $e^{2\pi i \tilde{\omega}_j}$, $\tilde{\omega}_j \in [-\frac{1}{2}, \frac{1}{2})$, are the eigenvalues of the pull-back f^*. □

We now turn to the contribution from each component of the fixed point variety which contains irreducible connections. Let $c \in C$ with $|\mathcal{M}(\Sigma)|'_c$ nonempty and consider $\omega_c^{d_c} \cap \tau_i(L^c_\bullet(\mathcal{O}_{\mathcal{M}(\Sigma)})) \in H_*(|\mathcal{M}(\Sigma)|'_c)$. In order to give a formula for the top degree term of this element, we need to fix a complex structure on Σ which is preserved by f. This induces the structure of an algebraic projective variety on $\mathcal{M}(\Sigma)$ and hence also on $|\mathcal{M}(\Sigma)|_c$. Algebraic varieties have fundamental classes and we denote the fundamental class of $|\mathcal{M}(\Sigma)|_c$ by $[|\mathcal{M}(\Sigma)|_c]$. As described in Appendix B, the Lefschetz–Riemann–Roch Theorem in [21], Section 0.6, gives

$$\tau_\bullet(L^c_\bullet(\mathcal{O}_{\mathcal{M}(\Sigma)})) = \mathrm{Ch}^\bullet(\lambda^c_{-1}\mathcal{M}(\Sigma))^{-1} \cap [|\mathcal{M}(\Sigma)|_c]^{\mathrm{Td}} \in H_\bullet(|\mathcal{M}(\Sigma)|'_c),$$

where $[|\mathcal{M}(\Sigma)|_c]^{\mathrm{Td}}$ is the Todd fundamental class defined in [20] and $\lambda^c_{-1}\mathcal{M}(\Sigma)$ is a certain element in the K-theory of $|\mathcal{M}(\Sigma)|_c$ with complex coefficients also defined in [20]; see also [6], Section 8, for a computation of this element in the case at hand. We recall that the highest degree term of $[|\mathcal{M}(\Sigma)|_c]^{\mathrm{Td}}$ equals $[|\mathcal{M}(\Sigma)|_c]$. The top degree is $d_c = \dim_{\mathbb{C}} |\mathcal{M}(\Sigma)|'_c$, so the contribution from $\mathrm{Ch}(\lambda^c_{-1}\mathcal{M}(\Sigma))^{-1}$ to the top degree term of $\omega_c^{d_c} \cap \tau_i(L^c_\bullet(\mathcal{O}_{\mathcal{M}(\Sigma)}))$ will simply be its degree zero part. Following [6], Section 8, we have $\lambda^c_{-1}\mathcal{M}(\Sigma) = L^\bullet\left(\sum(-1)^i[\Lambda^i\mathcal{N}_c^*]\right)$, where $\mathcal{N}_c^*$ is the conormal sheaf to $|\mathcal{M}(\Sigma)|_c$ (thought of as an f-equivariant sheaf) and $L^\bullet$ is the homomorphism determined by $L^\bullet(E_a) = [E_a]\otimes a \in K^0(|\mathcal{M}(\Sigma)|_c)\otimes\mathbb{C}$ for an a-eigensheaf E_a of f. Since f is finite order, $\mathcal{N}_c^*$ splits as the direct sum $\mathcal{N}_c^* = \bigoplus_j \mathcal{N}_{c,j}^*$ of a_j-eigensheaves $\mathcal{N}_{c,j}^*$ of f, where $a_j = e^{2\pi i \frac{j}{m}}$ and $j = 1, \ldots, m-1$, we then have

$$L^\bullet(\mathcal{N}_c^*) = \sum_{j=1}^{m-1} \mathcal{N}_{c,j}^* \otimes a_j.$$

Then the degree zero part of $\mathrm{Ch}(\lambda^c_{-1}\mathcal{M}(\Sigma))^{-1}$ is

$$\lambda_{-1}(\mathrm{Rank}\,\mathcal{N}_c^*)^{-1} = \prod_{i=1}^{m-1}(1-a_i)^{-r_i} = \frac{1}{\det(1 - df|_{\mathcal{N}_c^*})}, \quad r_i = \mathrm{Rank}\,\mathcal{N}_{c,i}^*.$$

This shows the following.

Proposition 3.2.

$$\omega^{d_c} \cup \mathrm{Ch}^\bullet(\lambda^c_{-1}\mathcal{M}(\Sigma))^{-1} = \frac{\omega^{d_c}}{\det(1 - df|_{\mathcal{N}_c^*})}.$$

If $|\mathcal{M}|'_c$ is not empty, then it is open and dense in $|\mathcal{M}|_c$, so that we can integrate the above differential form over $|\mathcal{M}|'_c$. Furthermore, any sensible integral over $|\mathcal{M}|_c$ is equal to the integral over $|\mathcal{M}|'_c$. Therefore the expression of the leading order term of the Witten–Reshetikhin–Turaev invariants for each $c \in C$ in differential geometric terms is an immediate consequence of Proposition 3.1 and 3.2.

Theorem 3.3. *Let $|\mathcal{M}(\Sigma)|_c$ be a connected component of $|\mathcal{M}(\Sigma)|$ containing irreducible connections, then*

$$k^{-d_c}\det(f)^{\frac{1}{2}\zeta} e^{2\pi i k\,\mathrm{CS}_{\Sigma_f}(c)} \frac{1}{d_c!}(\exp(k\omega_c) \cap \tau_\bullet(L^c_\bullet(\mathcal{O}_{\mathcal{M}(\Sigma)})))$$
$$= \exp\Bigg(i\pi\zeta \sum_{0\neq\tilde{\omega}_j\in(-\frac{1}{2},\frac{1}{2})} \tilde{\omega}_j\Bigg) e^{2\pi i k\,\mathrm{CS}_{\Sigma_f}(c)} \int_{a\in|\mathcal{M}(\Sigma)|'_c} \frac{1}{d_c!} \frac{(\omega_c)^{d_c}_{[a]}}{\det(1 - df|_{\mathcal{N}^*_{[a]}})} + O\Big(\frac{1}{k}\Big),$$

where $e^{2\pi i\tilde{\omega}_j}$, $\tilde{\omega}_j \in [-\frac{1}{2}, \frac{1}{2})$, *are the eigenvalues of* $f^*: H^{1,0}(\Sigma, \bar{\partial}) \to H^{1,0}(\Sigma, \bar{\partial})$ *and* $\mathcal{N}^*_{[a]} = \mathcal{N}^*_{c,[a]}$ *is the fiber over* $[a] \in |\mathcal{M}(\Sigma)|_c$ *of the conormal sheaf* $\mathcal{N}^*_c$ *of* $|\mathcal{M}(\Sigma)|_c$.

Notice that a connected component may contain more than one irreducible component (in the Zariski topology). These components can be of different dimensions, but only the components of dimension d_c will contribute to the integral.

4. Reidemeister torsion

In this section we will summarize some basic facts about Reidemeister torsion, which is a term in the asymptotic expansion of the Witten–Reshetikin–Turaev invariants. To keep the proofs less technical we will consider it as a density. Note that, it is possible and could be interesting to lead this discussion in the context of sign-determined Reidemeister torsion as defined in [69] (see for example [31]).

4.1. Torsion of a complex. The notation has been adapted from [34] and [49]. Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let L be a 1-dimensional vector space over F. We will denote by L^{-1} the dual of a complex line L and by $l^{-1} \in L^{-1}$ the inverse of l given by $l^{-1}(l) = 1$. By a density on L we mean a function

$$|\cdot|: L \longrightarrow \mathbb{R}$$

such that

$$|c\omega| = |c||\omega|, \quad c \in F, \omega \in L.$$

We denote the densities on L by $|L^*|$. For an n-dimensional vector space V over F we let $\det V = \Lambda^n V$ and define a density on V to be an element of $|\det V^*|$. A density on a manifold M is a section of the density bundle $|\det T^*M|$. Every volume form ω on V gives a density $|\omega|$. If we choose an orientation, we can identify densities with volume forms.

Definition 4.1. Given a finite cochain complex $(C^\bullet, d)$ of finite-dimensional complex vector spaces, we denote

$$\det C^\bullet = \bigotimes_{j=0}^{n} (\det C^j)^{(-1)^j}.$$

Then the torsion

$$\tau_{C^\bullet,d} \in |(\det C^\bullet)^{-1} \otimes (\det H^\bullet(C, d))|$$

is given by

$$\tau_{C^\bullet,d} = \bigotimes_{j=0}^{n} (|ds^{j-1} \wedge s^j \wedge \hat{h}^j|^{(-1)^{j+1}} \otimes |h^j|^{(-1)^j}),$$

after an arbitrary choice of

- $s^j \in \bigwedge^{k_j} C^j$ with $ds^j \neq 0$, where k_j is the rank of $d\colon C^j \to C^{j+1}$,
- $h^j \in \det H^j(C)$ non-zero and
- a lift $\hat{h}^j \in \bigwedge^{l_j} C^j$ of h^j, where $l_j = \dim H^j(C^\bullet, d)$.

We will use the *Multiplicativity Lemma* as our main computational tool.

Lemma 4.2. *Let*

$$0 \longrightarrow C_1^\bullet \xrightarrow{\nu^\bullet} C_2^\bullet \xrightarrow{\mu^\bullet} C_3^\bullet \longrightarrow 0 \tag{4.1}$$

be a short exact sequence of cochain complexes, choose compatible volume elements $\omega_i^\bullet$ in $C_i^\bullet$ – that is, $\omega_2^j = \nu^*(\omega_1^j) \wedge \omega'^j$ with $\mu^*(\omega'^j) = \omega_3^j$ for $\omega_i^j \in \det C_i^j$ – and let $H^\bullet$ be the long exact sequence associated to (4.1). Then

$$\tau_{C_2^\bullet}(\omega_2) = \tau_{C_1^\bullet}(\omega_1) \cdot \tau_{C_3^\bullet}(\omega_3) \cdot \tau_{H^\bullet},$$

where $\omega_i = \prod (\omega_i^j)^{(-1)^j}$.

For a proof see Corollary 1.20 in [34] or Theorem 3.2 in [56].

4.2. The Wang exact sequence. In order to compute the Reidemeister torsion, we will employ the Wang exact sequence; see [74] and [67].

Let $(C^\bullet, d) = \bigoplus_{i=0}^n (C^i, d^i)$ be a chain complex and $f^\bullet = \{f^i\colon (C^i, d^i) \to (C^i, d^i)\}$ be a chain map. Then the algebraic mapping torus $(T^\bullet(f^\bullet), d_f)$ is the cochain complex with $T^i(f) = C^i \oplus C^{i-1}$ and boundary operator $d_f^i(x, y) = (d^i(x), -d^{i-1}(y) + \mu^i(x))$, where $\mu^\bullet = \mathrm{Id}^\bullet - f^\bullet\colon C^\bullet \to C^\bullet$. It is not difficult to confirm that we get a short exact sequence

$$0 \longrightarrow (C^{\bullet-1}, -d) \xrightarrow{\nu^{\bullet-1}} (T^\bullet(f^\bullet), d_f) \xrightarrow{\pi^\bullet} (C^\bullet, d) \longrightarrow 0 \tag{4.2}$$

of chain complexes, where $\nu^\bullet$ is the inclusion into first summand and $\pi^\bullet$ is the projection onto the second summand. Observe that $(C^\bullet, -d)$ and $(C^\bullet, d)$ are isomorphic chain complexes and that $H^i(C^{\bullet-1}, d) = H^{i-1}(C^\bullet, d)$. This yields a long exact sequence $H_W^\bullet$ by the name Wang exact sequence

$$\cdots \longrightarrow H_W^i(C^\bullet) \xrightarrow{\mu^i} H_W^i(C^\bullet) \xrightarrow{\nu^i} H_W^{i+1}(T^\bullet(f)) \xrightarrow{\pi^{i+1}} H_W^{i+1}(C^\bullet) \longrightarrow \cdots.$$

It is easy to check that the boundary map is indeed induced by $\mu^\bullet$. Together with the Multiplicativity Lemma 4.2 we get the following useful result.

Corollary 4.3. *Let $\omega^j \in \det C^j$ be a volume form for all j and let $\omega = \prod (\omega_i^j)^{(-1)^j}$. Then we have $\tau_{C^\bullet(M)}(\omega) = \tau_{C^{\bullet-1}}(\omega^{-1})$ and therefore for $\omega_f = \nu^*(\omega) \wedge \omega'$ with $\pi^*(\omega') = \omega^{-1}$*

$$\tau_{C^\bullet(M_f)}(\omega_f) = \tau_{H_W^\bullet}.$$

In particular, this is independent of the choice of ω.

4.3. Reidemeister torsion. If each C^j comes equipped with a volume form, then the torsion is an element of $|\det H^\bullet(C^\bullet, d)|$. If X is a smooth manifold, W an inner product space and $\rho\colon \pi \to \mathrm{GL}(W)$ a representation of $\pi = \pi_1(X)$, then we can consider the cellular chain complex with local coefficients in W twisted by ρ given by

$$C^\bullet(X, W_\rho) = \mathrm{Hom}_{\mathbb{Z}\pi}(C_\bullet(\widetilde{X}), W),$$

where $\widetilde{X}$ is the universal cover of X. Note that $C_\bullet(\widetilde{X})$ has a natural inner product, by which the cells are orthonormal. If furthermore ρ preserves the inner product on W, then $C^\bullet(X, W_\rho)$ carries an induced inner product and therefore volume forms. Then the *Reidemeister torsion of* X is a density given by

$$\tau_X(W_\rho) = \tau_{(C^\bullet(X,W_\rho),d)} \in |\det H^\bullet(C^\bullet, d)|$$

and is independent of the choice of the cell decomposition of X. The use of cochain complexes rather than chain complexes in defining Reidemeister torsion simplifies the notation in our arguments considerably when interpreting the torsion in terms of twisted de Rham cohomology. Even though we need to choose a multiple of the Killing form as a metric on $\mathfrak{g}$ in order to identify Reidemeister torsion defined through chains and Reidemeister torsion defined through cochains, it is not difficult to see that the identification is independent of this choice. If A is a G-connection and the representation $\mathrm{ad} \circ \mathrm{hol}(A) = \rho$ is associated to a flat G-connection A via the adjoint representation

$$\mathrm{ad}\colon G \longrightarrow \mathrm{O}(\mathfrak{g}^{\mathbb{C}}) \subset \mathrm{End}(\mathfrak{g}^{\mathbb{C}}),$$

which takes values in the orthogonal group with respect to the Killing form on $\mathfrak{g}$, we define

$$\tau_X(A) = \tau_X(\mathfrak{g}_\rho).$$

Note that we can also consider the complexified adjoint representation

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{U}(\mathfrak{g}^{\mathbb{C}}) \subset \mathrm{End}(\mathfrak{g}^{\mathbb{C}}),$$

where we have extended the Killing form to a sesquilinear form on $\mathfrak{g}^{\mathbb{C}}$. We then also have

$$\tau_X(A) = \tau_X(\mathfrak{g}^{\mathbb{C}}_{\mathrm{Ad} \circ \mathrm{hol}(A)}).$$

5. Reidemeister torsion of mapping tori

We will see in this section that for $c \in C$

$$\int_{\mathcal{M}(\Sigma_f)'_c} \tau_{\Sigma_f}(A)^{\frac{1}{2}} = |Z(G)| \int_{|\mathcal{M}(\Sigma)|'_c} \frac{|\omega_c^{d_c}|}{|\det(1 - df|_{\mathcal{N}^*_{[a]}})|}, \tag{5.1}$$

where $\mathcal{N}^*_{[a]} = \mathcal{N}^*_{c,[a]}$ and the conormal sheaf $\mathcal{N}^*_c$ is the dual of the normal sheaf

$$\mathcal{N}_c = \frac{T\mathcal{M}(\Sigma)|_{|\mathcal{M}(\Sigma)|_c}}{T|\mathcal{M}(\Sigma)|_c}.$$

In the above equation we identified $H^2(\Sigma_f, d_A)$ with $H^1(\Sigma_f, d_A)^*$ via PD. In fact, we will even show an equality for irreducible components on the level of densities. The factor $|Z(G)|$ then stems from the fact that $r\colon \mathcal{M}(\Sigma_f) \to |\mathcal{M}(\Sigma)|$ is a $|Z(G)|$-sheeted covering map; see Section 7 in [6].

Notice that $T|\mathcal{M}(\Sigma)|$ is simply the kernel of the bundle map

$$1 - df^*\colon T\mathcal{M}(\Sigma) \longrightarrow T\mathcal{M}(\Sigma)$$

and is therefore isomorphic to the bundle of 1-eigenspaces of

$$f^*\colon T_{[a]}\mathcal{M}(\Sigma) \longrightarrow T_{[a]}\mathcal{M}(\Sigma), \quad \text{where } [a] \in |\mathcal{M}(\Sigma)|_c.$$

We can fix an isomorphism $H^{0,1}(\Sigma, \bar{\partial}_a) \cong T_{[a]}\mathcal{M}(\Sigma)$ to get an equivalent statement for $H^{0,1}(\Sigma, \bar{\partial}_a)$. Also, note that the eigenvalues of $1 - df^*\colon \mathcal{N}_{[a]} \to \mathcal{N}_{[a]}$ and of $1 - df\colon \mathcal{N}^*_{[a]} \to \mathcal{N}^*_{[a]}$ are the same, where df is short for $(df^*)^*$.

5.1. General mapping tori. Consider a CW complex M and an orientation preserving simplicial homeomorphism $f\colon M \to M$. The torsion for the mapping torus M_f of f has been computed in Proposition 3 in [37] (see also [32], Section 6.2, and [57], Example 2.17) only when M_f is an acyclic CW complex. In this section we will give a generalization of the computation for mapping tori to the non-acyclic case. The computations in [31] of sign-determined Reidemeister torsion for fibered knots for the local coefficient systems $\mathfrak{su}(2)$ and $\mathfrak{sl}_2(\mathbb{C})$ use the same basic tools, namely the Wang exact sequence and the Multiplicity Lemma.

Let $\rho\colon \pi_1 M_f \to G$ be a G-representation of $\pi_1 M_f$ acting on $\mathfrak{g}$ by the adjoint representation. If we denote by $C_g\colon G \to G$ the conjugation action, then ρ is determined by a representation $\rho'\colon \pi_M \to G$ satisfying $\rho' = C_g \circ (f^*\rho')$ for some $g \in G$. The choice of g induces a chain map $f^\bullet = f^\bullet_g\colon C^\bullet(M, \mathfrak{g}_{\rho'}) \to C^\bullet(M, \mathfrak{g}_{\rho'})$. It is easy to check that the algebraic mapping torus $T(f^\bullet)$ is isomorphic – in fact, isometric – to $C^\bullet(M_f, \mathfrak{g}_\rho)$ induced by the cell decomposition of S^1 into two cells and $C^\bullet(M)$.

In this section let us from now on drop the coefficients in the cohomology and cochain groups entirely with the understanding that we consider coefficients twisted by representations compatible with the restriction. Instead of μ^i and π^i we will sometimes use the more familiar notation μ^* and π^*, when the grading is clear. Consider the diagram in cohomology induced by the Wang exact sequence and a positive multiple Θ of Poincaré duality on M

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{n-i}(M_f) & \xrightarrow{\pi^*} & H^{n-i}(M) & \xrightarrow{\mu^*} & H^{n-i}(M) & \xrightarrow{\nu^*} & H^{n-i+1}(M_f) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\Theta} & & \\
\cdots & \longrightarrow & (H^{i+1}(M_f))^* & \xrightarrow{\nu} & (H^i(M))^* & \xrightarrow{\mu'} & (H^i(M))^* & \xrightarrow{\pi} & (H^i(M_f))^* & \longrightarrow & \cdots,
\end{array}$$

where $\mu' = 1 - f^{-1}$ and we write $f^{-1} = ((f^{-1})^*)^*$. It is easy to check that the middle square commutes. Furthermore, since

$$\mu' = (1 - f) \circ (-f^{-1}) = \mu \circ (-f^{-1}) = (-f^{-1}) \circ \mu,$$

and $(-f^{-1})$ is an isomorphism, the exactness of the above sequence implies the exactness of the lower sequence. We can define isomorphisms $\Theta\colon \operatorname{im}\nu^* \to \operatorname{im}\pi$ so that the above diagram commutes, and we can extend these maps arbitrarily to isomorphisms $\Theta\colon H^{n-i}(M_f) \to H^{i+1}(M_f)$. We extend Θ to the exterior algebra by setting $\Theta(a \wedge b) = \Theta(a) \wedge \Theta(b)$.

Before we can compute Reidemeister torsion of a general mapping torus, we need a few technical facts. For finite order mapping tori the situation simplifies considerably and the result is more pleasing.

Lemma 5.1. *Let* $0 \neq h^{i+1} \in \det(\operatorname{im}(\pi^i))$. *Then we can find* $h^i_+ \wedge h^i_- \in \det(H^i(M))$ *such that* $\nu^*(h^i_-) = h^{i+1}$ *and* $\mu^*(h^i_+) \wedge h^i_- \neq 0$.

Proof. Let $h^i_+ \wedge h^i_- \in \det(H^i(M))$ such that $\nu^*(h^i_-) = h^{i+1}$. If $\mu^*(h^i_+) \wedge h^i_- = 0$, then let $k^i \in \Lambda(H^i(M))$ with $0 \neq \mu^*(k^i) \wedge h^i_- \in \det(H^i(M))$. Now choose $\lambda > 0$ small enough that for $\tilde{h}^i_+ = h^i_+ + \lambda k^i$

$$\tilde{h}^i_+ \wedge h^i_- \neq 0.$$

Then we also have

$$\mu(\tilde{h}^i_+) \wedge h^i_- = \lambda\mu(k^i) \wedge h^i_- \neq 0. \qquad \square$$

Proposition 5.2. *Let* M_f *be a mapping torus of a homeomorphism* $f\colon M \to M$, $\dim M = n$. *Then we may choose* $h^i \in \Lambda(H^i(M_f))$ *and* $h^i_-, h^i_+ \in \Lambda(H^i(M))$ *for all* i *with*

$$0 \neq \nu^*(h^{i-1}_-) \wedge h^i \in \det(H^i(M_f)),$$
$$0 \neq \pi^*(h^i) \wedge h^i_+ \in \det(H^i(M)), \tag{5.2}$$
$$0 \neq \mu^*(h^i_+) \wedge h^i_- \in \det(H^i(M)).$$

so that they satisfy

$$|\Theta(\nu^*(h^{n-i}_-))(h^i)| = 1 \quad \textit{and} \quad |h^i_- \wedge h^i_+| = |\pi^*(h^i) \wedge h^i_+|. \tag{5.3}$$

Furthermore, the Reidemeister torsion is

$$\tau(M_f) = \left|\bigotimes_{i=0}^{n+1}(\nu^*(h_-^{i-1}) \wedge h^i)^{(-1)^i}\right| \prod_{i=0}^{n} |\det(\tilde{\mu}^i)|^{(-1)^{i+1}},$$

where $\tilde{\mu}^i$ is determined by

$$\tilde{\mu}^i(h_-^i \wedge h_+^i) = h_-^i \wedge \mu^*(h_+^i).$$

Proof. The Wang exact sequence and Lemma 5.1 allow us to choose $h^i \in \Lambda(H^i(M_f))$ and $h_-^i, h_+^i \in \Lambda(H^i(M))$ for all i with

$$0 \neq \nu^*(h_-^{i-1}) \wedge h^i \in \det(H^i(M_f)),$$

$$0 \neq \pi^*(h^i) \wedge h_+^i \in \det(H^i(M)),$$

$$0 \neq \mu^*(h_+^i) \wedge h_-^i \in \det(H^i(M)),$$

$$0 \neq h_+^i \wedge h_-^i.$$

By rescaling we can assume $|\Theta(\nu^*(h_-^{n-i}))(h^i)| = 1$. Notice that, if h^i and h_-^{n-i} satisfy this condition, so do λh^i and $\frac{1}{\lambda} h_-^{n-i}$ for $\lambda > 0$. By choosing λ appropriately we may therefore assume that

$$|h_-^i \wedge h_+^i| = |\pi^*(h^i) \wedge h_+^i|.$$

Then

$$|\det \tilde{\mu}^i| \cdot |\pi^*(h^i) \wedge h_+^i| = |\det \tilde{\mu}^i| \cdot |h_-^i \wedge h_+^i| = |\tilde{\mu}^i(h_-^i \wedge h_+^i)| = |h_-^i \wedge \mu^*(h_+^i)|,$$

and therefore

$$\bigotimes_{i=0}^{n} |\pi^*(h^i) \wedge h_+^i|^{(-1)^i} \bigotimes_{i=0}^{n} |\mu^*(h_+^i) \wedge h_-^i|^{(-1)^{i+1}} = |\det \tilde{\mu}^i|^{i+1}.$$

By Corollary 4.3, the proposition follows. □

Note that even though the system of equations (5.3) seems to be overdetermined, half of them are equivalent to the other half, since the above diagram is commutative. Also observe that, even though our result seems to depend on $\Theta\colon H^{n-i}(M) \to H^{i+1}(M)$, we can use a different multiple of Poincaré duality without changing Reidemeister torsion: This can be easily verified by the skeptical reader by considering the cases n odd and n even separately.

5.2. Finite order mapping tori. We can enhance Theorem 5.2 and make it more useful for finite order mapping tori, if we put some restrictions on μ^*. Before we do that, let us state and prove a simple fact from linear algebra.

Lemma 5.3. *For a linear map $T: V \to V$ between finite-dimensional vector spaces, the following are equivalent:*

(1) $\overline{T}: V/\ker T \to V/\ker T$ *induced by* T *is an isomorphism;*

(2) $\widehat{T} = T|_{\operatorname{im} T}: \operatorname{im} T \to \operatorname{im} T$ *is an isomorphism.*

Furthermore $\det \overline{T} = \det \widehat{T}$.

Proof. Clearly, $\overline{T}$ is an isomorphism if and only if $\operatorname{im} T \hookrightarrow V \to V/\ker T$ is an isomorphism. The last statement is equivalent to $\operatorname{im} T \cap \ker T = 0$. This implies that $\widehat{T}: \operatorname{im} T \to \operatorname{im} T$ is an isomorphism. On the other hand, if $0 \neq v \in \operatorname{im} T \cap \ker T$, then $\widehat{T}$ is not injective, because $T(v) = 0$.

Furthermore, if $\{b_i\}_i$ is a basis of $\operatorname{im} T$, then $\{[b_i]\}_i$ is a basis of $V/\ker T$. It follows immediately that $\det \overline{T} = \det \widehat{T}$. □

Proposition 5.4. *Assume that*

$$\bar{\mu}^i : H^i(M)/\ker(\mu^i) \longrightarrow H^i(M)/\ker(\mu^i)$$

is an isomorphism. Then we can choose h^i with $0 \neq \pi^*(h^i) \in \det(\operatorname{im} \pi^i)$ satisfying

$$\Theta(\nu^*(\pi^*(h^{n-i})))(h^i) = 1. \tag{5.4}$$

Furthermore, we have $\det(\bar{\mu}^i) = \det(\tilde{\mu}^i)$, *where $\tilde{\mu}^i$ is the map from Theorem 5.2. In particular, if $\bar{\mu}^i$ is an isomorphism for all i – for example for finite order mapping tori – we have*

$$\tau(M_f) = \left|\bigotimes_{i=0}^{n+1}(\nu^*(\pi^*(h^{i-1})) \wedge h^i)^{(-1)^i}\right| \prod_{i=0}^{n} |\det(\bar{\mu}^i)|^{(-1)^{i+1}}.$$

Proof. Suppose that $\bar{\mu}^i$ is an isomorphism. Choose h^i with $0 \neq \pi^*(h^i) \in \det(\operatorname{im} \pi^i)$. In view of Lemma 5.3 we can find $h^i_+ \in \det \operatorname{im} \mu^i$ with $0 \neq \pi^*(h^i) \wedge h^i_+ \in \det H^i(M)$. Since $h^i_+ \in \det \ker \nu^i$, we deduce $0 \neq \nu^* \circ \pi^*(h^i) \in \det \operatorname{im} \nu^i$, which allows us to rescale h^i so that it satisfies (5.4) above. If we set $h^i_- = \pi^*(h^i)$, it is straightforward to see that (5.3) is satisfied and that

$$\det \tilde{\mu}^i = \det \hat{\mu}^i.$$ □

5.3. Finite order mapping tori of surfaces. We will now focus on the case of a mapping torus Σ_f of finite order for a closed surface Σ. The goal of this section is to identify integral of the square root of Reidemeister torsion with the leading order term in formula (1.3) as predicted by the semiclassical approximation of the path integral. We will only do this for the case, when $c \in C$ contains an open, dense submanifold $|\mathcal{M}(\Sigma)|'_c$ of irreducible connections of $|\mathcal{M}(\Sigma)|$. More specifically, we will establish an identification on the level of densities for $|\mathcal{M}(\Sigma)|'_c$. Notice that while the square root of Reidemeister torsion is a density on a submanifold of the irreducible connections of $\mathcal{M}(\Sigma_f)$, ω^{d_c} is a density on top-dimensional component of $|\mathcal{M}(\Sigma)|'_c$. Therefore the density on $|\mathcal{M}(\Sigma)|$ needs to be pulled back to a density on $\mathcal{M}(\Sigma)$ via the natural restriction and $|Z(G)|$-sheeted covering map $r\colon \mathcal{M}(\Sigma_f) \to |\mathcal{M}(\Sigma)|$ before we can relate it to Reidemeister torsion. We also need to point out that by treating Reidemeister torsion as a density, we chose to identify $H^2(\Sigma_f, d_A)$ with $(H^1(\Sigma_f, d_A))^*$ for $A \in \mathcal{A}_{\Sigma_f}$ via PD.

Before we prove the main theorem, we would like to mention the following simple fact.

Lemma 5.5. *Let (V^{2n}, ω) be a symplectic vector space. We can identify V with V^* by*

$$\Theta(v)(w) = -\omega(v, w)$$

and extend this map to the exterior algebra by $\Theta(v \wedge w) = \Theta(v) \wedge \Theta(w)$. Then the volume form $\mathrm{vol} = \frac{1}{n!}\omega^n \in \det V^*$ *on V satisfies*

$$\Theta(\mathrm{vol}^{-1})(\mathrm{vol}^{-1}) = 1,$$

where $\mathrm{vol}^{-1} \in \det V$ *is given by* $\mathrm{vol}(\mathrm{vol}^{-1}) = 1$.

Proof. Form a symplectic basis $\{a_i, b_i\}_{i=1,\dots,n}$ of V, that is, $\omega(a_i, b_j) = -\omega(b_j, a_i) = \delta_{ij}$. Then we have

$$\omega = -\sum_{i=1}^{n} \Theta(b_i) \wedge \Theta(a_i) = \sum_{i=1}^{n} \Theta(a_i) \wedge \Theta(b_i).$$

Then

$$\mathrm{vol} = \frac{1}{n!}\omega^n = \frac{1}{n!} \bigwedge^{n} \sum_{i=1}^{n} \Theta(a_i) \wedge \Theta(b_i) = \bigwedge_{i=1}^{n} \Theta(a_i) \wedge \Theta(b_i)$$

as well as

$$vol^{-1} = (-1)^n \bigwedge_{i=1}^{n} b_i \wedge a_i = \bigwedge_{i=1}^{n} a_i \wedge b_i.$$

Therefore we get the desired equation

$$\Theta(\mathrm{vol}^{-1})(\mathrm{vol}^{-1}) = \Big(\bigwedge_{i=1}^{n} (\Theta(a_i) \wedge \Theta(b_i))\Big)\, \mathrm{vol}^{-1} = \mathrm{vol}(\mathrm{vol}^{-1}) = 1. \qquad \square$$

Theorem 5.6. *Let A be an irreducible flat connection on Σ_f such that $a = r(A)$ is irreducible on Σ and $c \subset |\mathcal{M}(\Sigma)|$ is a connected component containing a. Let ω be the usual symplectic form on $H^1(\Sigma, d_a)$ given by (2.2) and identify $H^2(\Sigma_f, d_A)$ with $H^1(\Sigma_f, d_A)^*$ via* PD. *Then we have*

$$\tau_{\Sigma_f}(A)^{\frac{1}{2}} = \frac{1}{d_c!}\frac{|r^*(\omega_c)^{d_c}|}{\sqrt{|\det(1-f^1)|}},$$

where $d_c = \frac{1}{2}(\dim_{\mathbb{R}} H^1(\Sigma_f, d_A) - \dim_{\mathbb{R}} H^0(\Sigma_f, d_A))$ and the restriction ω_c of ω to $\ker(1-f^1)$ *is a symplectic form on* $\ker(1-f^1)$.

Proof. The Reidemeister torsion

$$\tau_{\Sigma_f}(A) \in \det H^0(\Sigma_f, d_A) \otimes \det H^1(\Sigma_f, d_A)^* \otimes \det H^2(\Sigma_f, d_A) \otimes \det H^3(\Sigma_f, d_A)^*$$

has been computed in Theorem 5.4. Since we are only interested in A irreducible, we have $H^0(\Sigma_f, d_A) = H^3(\Sigma_f, d_A) = 0$. In contrast to Proposition 5.6 in [49], where f has isolated fixed points on $\mathcal{M}(\Sigma)$, we have to consider connected components $c \subset |\mathcal{M}(\Sigma)|$, which are positive-dimensional. Furthermore, PD identifies $H^2(\Sigma_f, d_A)$ with the dual of $H^1(\Sigma_f, d_A)$ and $\dim H^1(\Sigma_f, d_A) = \dim \mathcal{M}(\Sigma_f)_c = \dim |\mathcal{M}(\Sigma)|_c$. In summary we get

$$0 \neq \sqrt{\tau_{\Sigma_f}(A)} \in |\det H^1(\Sigma_f, d_A)^*|.$$

Since we also assume irreducibility of $a = r(A)$, $\ker(\pi^1) = \operatorname{im}(\nu^0) = 0$. Furthermore,

$$0 \neq \omega_c^{d_c} \in \det(E_1(f^1))^* = \det(\ker(1-f^1))^* = \det(\ker(\mu^1))^* = \det(\operatorname{im}(\pi^1))^*.$$

Since π^1 is injective, we can define an element $h^1 \in H^1(\Sigma)$ by requiring

$$\pi^*(h^1) = (\omega_c^{d_c})^{-1} \in \det(\operatorname{im}(\pi^1)).$$

All that is left to complete the proof of the theorem is that this choice of h^1 indeed satisfies condition (5.4). Since $H^2(\Sigma, d_a) = 0$ we have $\det(\operatorname{im}(\nu^1)) = \det(H^2(\Sigma_f, d_A))$. Since the map $\bar{\mu}^1$ from Theorem 5.4 is an isomorphism and $0 \neq \pi^*(h^1) \in \ker \mu^1$, we have

$$0 \neq \nu^*(\pi^*(h^1)) \in \det(H^2(\Sigma_f, d_A)).$$

We would like to apply Theorem 5.4. Since we chose PD to identify $H^2(\Sigma_f, d_A) = (H^1(\Sigma_f, d_A))^*$ we need to check that $\Theta = \mathrm{PD}$ indeed satisfies condition (5.4). We see that condition (5.4) is equivalent to

$$\begin{aligned}\mathrm{PD}((\omega_c^{d_c})^{-1})((\omega_c^{d_c})^{-1}) &= \Theta(\pi^*(h^1))(\pi^*(h^1)) = \pi(\Theta(\pi^*(h^1))(h^1)\\ &= \Theta(\nu^*(\pi^*(h^1)))(h^1) = 1,\end{aligned}$$

which is satisfied by Lemma 5.5. □

Geometrically, $T_{[a]}\mathcal{M}(\Sigma) \cong H^{0,1}(\Sigma, \bar{\partial}_a)$, and therefore

$$|\det(1 - f^1)| = |\det(1 - df_{\mathcal{N}^*_{r(A)}})|^2,$$

where we again understand df as $(d(f^*))^*$.

Theorem 5.7. *Let A be an irreducible flat connection on Σ_f such that $r(A)$ is irreducible on Σ. If we identify densities with volume forms using the orientation induced by $r^*(\omega_c)_A^{d_c}$, we have*

$$\sqrt{\tau_{\Sigma_f}(A)} = \frac{1}{d_c!}\frac{r^*(\omega_c)_A^{d_c}}{|\det(1 - df|_{\mathcal{N}^*_{r(A)}})|}.$$

This shows that over the moduli space of irreducible flat connections A with $r(A)$ irreducible we indeed have the identity (5.1).

6. The ρ-invariant

Another classical topological invariant, which appears in the expansion of the Witten–Reshetikin–Turaev invariants, is the ρ-invariant. We will briefly review the definition for 3-manifolds in the context of the adjoint representation and relate it to the original definition using the defining representation before we state the result from [25], which will be relevant for us.

6.1. The definition. For a formally self-adjoint, elliptic differential operator D of first order, acting on sections of a vector bundle over a closed manifold X, one defines the η-function

$$\eta(D, s) = \sum_{0 \neq \lambda \in \mathrm{Spec}(D)} \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}, \quad \mathrm{Re}(s) \text{ large.} \tag{6.1}$$

The function $\eta(D, s)$ admits a meromorphic continuation to the whole s-plane with no pole at the origin. Then $\eta(D) = \eta(D, 0)$ is called the η-invariant of D.

As a special case, let G be a compact, simple, simply-connected Lie group and A a G-connection on a Riemannian 3-manifold X. Then the odd signature operator coupled to A is the formally self-adjoint, elliptic, first order differential operator

$$\begin{aligned} D_A \colon \Omega^0(X; \mathfrak{g}) \oplus \Omega^1(X; \mathfrak{g}) &\longrightarrow \Omega^0(X; \mathfrak{g}) \oplus \Omega^1(X; \mathfrak{g}), \\ (\alpha, \beta) &\longmapsto (d_A^*\beta, d_A\alpha + *d_A\beta), \end{aligned} \tag{6.2}$$

where $d_A \colon \Omega^p(X; \mathfrak{g}) \to \Omega^{p+1}(X; \mathfrak{g})$ is the covariant derivative associated to A and G acts on $\mathfrak{g}$ via the adjoint action. If A is flat, the ρ-invariant is given by

$$\rho_A(X) = \eta(D_A) - \eta(D_\theta), \tag{6.3}$$

where θ is the trivial connection. The ρ-invariant is metric-independent and gauge-invariant. We write $\rho_{\mathrm{hol}(A)} = \rho_A$, where the representation $\mathrm{hol}(A)\colon \pi_1 X \to G$ is the holonomy of A.

In the original definition [14] by Atiyah, Patodi, and Singer, their ρ-invariant has been similarly defined for a $\mathrm{U}(n)$-representation, where $\mathrm{U}(n)$ acts on $\mathbb{C}^n$ by the defining representation. We will briefly describe its relationship to our definition of the ρ-invariant in (6.3). With respect to an ad-invariant metric on $\mathfrak{g}$ – for example the Killing form – on $\mathfrak{g}$, the adjoint representation takes values in the orthogonal endomorphisms of $\mathfrak{g}$

$$\mathrm{ad}\colon G \longrightarrow \mathrm{SO}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g}).$$

We can consider the complexified adjoint representation

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{SU}(\mathfrak{g}^{\mathbb{C}}) \subset \mathrm{End}(\mathfrak{g}^{\mathbb{C}}).$$

Then $\rho_{\mathrm{hol}(A)}$ is equal to the Atiyah–Patodi–Singer ρ-invariant of $\mathrm{Ad} \circ \mathrm{hol}(A)$.

6.2. The ρ-invariant of finite order mapping tori. Let Σ be a surface and P a principal G-bundle. In order to make use of the results in [25], we consider the bundle $\mathrm{Ad}\, P$ associated to the complexified adjoint representation, which is a Hermitian vector bundle of rank $\dim G$.

The chirality operator τ_Σ on $\Omega^p(\Sigma)$ is given by

$$\tau_\Sigma = (-1)^{\frac{p(p-1)}{2}+2p} i *_p,$$

where $*_p$ is the Hodge star operator on $\Omega^p(\Sigma)$. Note that the splitting into ± 1-eigenspaces of τ_Σ restricted to the harmonic forms $\mathcal{H}_a^\bullet(\Sigma; \mathrm{Ad}\, P) = \ker \Delta_a$ of $\Delta_a = d_a d_a^* + d_a^* d_a$

$$\mathcal{H}_a^\bullet(\Sigma; \mathrm{Ad}\, P) = \mathcal{H}_a^+(\Sigma; \mathrm{Ad}\, P) \oplus \mathcal{H}_a^-(\Sigma; \mathrm{Ad}\, P)$$

is invariant under Φf^* for any gauge transformation $\Phi\colon P \to P$ satisfying $\Phi f^* a = a$. Since the unitary structure on $\mathrm{Ad}\, P$ arises from $\mathrm{ad}\colon G \to \mathrm{O}(\mathfrak{g})$ (see [25], Remark (ii) on p. 136), we have

$$\begin{aligned}
&\mathrm{tr}\log[\Phi f^*|_{\mathcal{H}_a^+(\Sigma;\mathrm{Ad}\,P)\cap\Omega^1}] \\
&\quad = \mathrm{rk}[(\Phi f^* - \mathrm{Id})|_{\mathcal{H}_a^-(\Sigma;\mathrm{Ad}\,P)\cap\Omega^1}] - \mathrm{tr}\log[\Phi f^*|_{\mathcal{H}_a^-(\Sigma;\mathrm{Ad}\,P)\cap\Omega^1}],
\end{aligned}$$

$$\mathrm{rk}[(\Phi f^* - \mathrm{Id})|_{\mathcal{H}_a^+(\Sigma;\mathrm{Ad}\,P)\cap\Omega^1} = \mathrm{rk}[(\Phi f^* - \mathrm{Id})|_{\mathcal{H}_a^-(\Sigma;\mathrm{Ad}\,P)\cap\Omega^1},$$

$$\mathrm{rk}[(f^* - \mathrm{Id})|_{\mathcal{H}^+(\Sigma)\cap\Omega^1}] = \mathrm{rk}[(f^* - \mathrm{Id})|_{\mathcal{H}^-(\Sigma)\cap\Omega^1}],$$

where tr log is defined for a diagonalizable map T as

$$\mathrm{tr}\log T = \sum_{j=1}^{n} \theta_j \in \mathbb{R},$$

where $e^{2\pi i\theta_j}$ are the eigenvalues of T, and where we require $\theta_j \in [0,1)$. Also note that

$$\mathcal{H}_a^+(\Sigma; \mathrm{Ad}\, P) \cap \Omega^1 = H^{1,0}(\Sigma, \bar\partial_a)$$

and

$$\mathcal{H}_a^-(\Sigma; \mathrm{Ad}\, P) \cap \Omega^1 = H^{0,1}(\Sigma, \bar\partial_a).$$

A fixed isomorphism $H^{0,1}(\Sigma, \bar\partial_a) \cong T_{[a]}\mathcal{M}(\Sigma)$ gives the commutative diagram

$$\begin{array}{ccc} H^{0,1}(\Sigma, \bar\partial_a) & \xrightarrow{\Phi f^*} & H^{0,1}(\Sigma, \bar\partial_a) \\ \cong \downarrow & & \downarrow \cong \\ T_{[a]}\mathcal{M}(\Sigma) & \xrightarrow{df^*} & T_{[a]}\mathcal{M}(\Sigma), \end{array}$$

and we have $\mathrm{rk}[(df^* - \mathrm{Id})|_{T_{[a]}\mathcal{M}(\Sigma)}] = \mathrm{rk}\, \mathcal{N}_{[a]}$. We simplify Theorem 4.2.4 in [25] as follows.

Theorem 6.1. *Let $f : \Sigma \to \Sigma$ be a finite order homeomorphism. Let A be a flat G-connection over Σ_f with $r([A]) = [a]$. Then*

$$\begin{aligned} \rho_A(\Sigma_f) = {} & -4 \,\mathrm{tr} \log[df^*|_{T_{[a]}\mathcal{M}(\Sigma)}] + 2\,\mathrm{rk}\, \mathcal{N}_{[a]} \\ & - 4 \dim G \,\mathrm{tr} \log[f^*|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}] \\ & + 2 \dim G \,\mathrm{rk}[(f^* - \mathrm{Id})|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}]. \end{aligned} \tag{6.4}$$

Remark 6.2. It follows from the proof of Theorem 4.2.4 in [25] that

$$\eta(D_A) = -4 \,\mathrm{tr} \log[df^*|_{T_{[a]}\mathcal{M}(\Sigma)}] + 2\,\mathrm{rk}\, \mathcal{N}_{[a]}$$

and

$$\eta(D_\theta) = -4 \dim G \,\mathrm{tr} \log[f^*|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}] + 2 \dim G \,\mathrm{rk}[(f^* - \mathrm{Id})|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}].$$

7. Identifying the classical invariants

In this section we identify the classical invariants in the leading order term of the Witten–Reshetikhin–Turaev invariants (1.3) of a finite order mapping torus $X = \Sigma_f$ as conjectured by the stationary phase approximation (A.5). More precisely, since the leading order term of

$$\zeta = \frac{k \dim G}{k + h} = \dim G - \frac{h \dim G}{k + h}$$

is simply dim G, we identify the classical invariants in the leading order term

$$\det(f)^{-\frac{1}{2}\dim G} e^{2\pi i k \operatorname{CS}_{\Sigma_f}(c)} \frac{1}{d_c!}(\omega_c^{d_c} \cap \tau_{d_c}(L_\bullet^c(\mathcal{O}_{\mathcal{M}(\Sigma)})))k^{d_c} \tag{7.1}$$

of (1.3) corresponding to an irreducible component $|\mathcal{M}(\Sigma)|_c$ of the variety $|\mathcal{M}(\Sigma)|$ containing an irreducible connection. Theorem 3.3 gives an expression of (7.1) in terms of an integral over $|\mathcal{M}(\Sigma)|_c'$. We reformulate this as an integration of classical invariants of 3-manifolds over $\mathcal{M}(\Sigma)_c'$.

By Theorem 5.7 we have, for $A \in \mathcal{M}(\Sigma_f)_c$,

$$\sqrt{\tau_{\Sigma_f}(A)} = \frac{1}{d_c!} \frac{r^*(\omega_c)_A^{d_c}}{|\det(1 - df|_{\mathcal{N}^*_{r(A)}})|},$$

keeping in mind that we have identified densities with volume forms in the orientation induced by $r^*(\omega_c)_A^{d_c}$. Notice that for a complex root of unity $\xi = e^{2\pi i\theta}$ with $\theta \in (0,1)$, we have $1 - \xi = \xi(\xi^{-1} - 1) = \xi(\bar{\xi} - 1)$. Therefore

$$\Big(\frac{1-\xi}{|1-\xi|}\Big)^2 = \frac{1-\xi}{1-\bar{\xi}} = -\xi.$$

By observing that the real part of $1 - \xi$ is always positive, we see that

$$\frac{1}{1-\xi} = \frac{1}{|1-\xi|} e^{2\pi i(\frac{1}{4} - \frac{\theta}{2})} = \frac{1}{|1-\xi|} e^{-2\pi i \frac{\theta}{2}} i. \tag{7.2}$$

For $a = r(A)$ the maps $df|_{T^*_{[a]}\mathcal{M}(\Sigma)}$ and $df^*|_{T_{[a]}\mathcal{M}(\Sigma)}$ have the same eigenvalues, and we have $\operatorname{rk} \mathcal{N}_{[a]} = \operatorname{rk} \mathcal{N}^*_{[a]}$. Therefore by Proposition 3.2, Equation (7.2) and Remark 6.2 we get

$$\begin{aligned}
&\frac{1}{d_c!} r^*(\omega_c^{d_c} \cup \operatorname{Ch}^\bullet(\lambda_{-1}^c \mathcal{M}(\Sigma))^{-1})_A \\
&= \frac{1}{d_c!} \frac{r^*(\omega_c)_A^{d_c}}{\det(1 - df)|_{\mathcal{N}^*_{[a]}}} \\
&= \tau_{\Sigma_f}(A)^{\frac{1}{2}} \exp\Big(-2\pi i \frac{\operatorname{tr}\log[df^*|_{T_{[a]}\mathcal{M}(\Sigma)}]}{2}\Big) i^{\operatorname{rk}\mathcal{N}_{[a]}} \\
&= \tau_{\Sigma_f}(A)^{\frac{1}{2}} e^{\frac{\pi i}{4}\eta(D_A)}.
\end{aligned}$$

In particular, we get

$$\begin{aligned}
\frac{1}{d_c!}(\omega_c^{d_c} \cap \tau_{d_c}(L_\bullet^c(\mathcal{O}_{\mathcal{M}(\Sigma)}))) &= \int_{|\mathcal{M}(\Sigma)|_c'} \frac{1}{d_c!} \omega_c^{d_c} \cup \operatorname{Ch}^\bullet(\lambda_{-1}^c \mathcal{M}(\Sigma))^{-1} \\
&= \int_{A \in \mathcal{M}(\Sigma_f)_c'} \tau_{\Sigma_f}(A)^{\frac{1}{2}} e^{\frac{\pi i}{4}\eta(D_A)}.
\end{aligned} \tag{7.3}$$

Observe that we can rewrite

$$\operatorname{tr}\log[f^*|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}] - \frac{\operatorname{rk}[(f^*-\mathrm{Id})|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}]}{2} = \sum_{0\neq\tilde\omega_i\in(-\frac{1}{2},\frac{1}{2})} \tilde\omega_i,$$

where $e^{2\pi i\tilde\omega_i} = \omega_i$, $\tilde\omega_i \in [-\frac{1}{2},\frac{1}{2})$, are the eigenvalues of the pull-back

$$f^*\colon \mathcal{H}^{1,0}(\Sigma,\bar\partial) \longrightarrow \mathcal{H}^{1,0}(\Sigma,\bar\partial).$$

By Proposition 3.1 we therefore have

$$\begin{aligned}&\det(f)^{-\frac{1}{2}\dim G}\\ &\quad = \exp\Big(i\pi\dim G\Big(\operatorname{tr}\log[f^*|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}] - \frac{\operatorname{rk}[(f^*-\mathrm{Id})|_{\mathcal{H}^{1,0}(\Sigma,\bar\partial)}]}{2}\Big)\Big).\end{aligned}$$

Therefore, it is easy to see from Remark 6.2 that the leading order term of $\det(f)^{-\frac{1}{2}\zeta}$ is given by

$$\det(f)^{-\frac{1}{2}\dim G} = e^{-\frac{\pi i}{4}\eta(D_\theta)}. \tag{7.4}$$

Together, (7.3) and (7.4) prove Theorem 1.5. In particular, we have the following.

Theorem 7.1. *Let each $\mathcal{M}(\Sigma_f)'_c$ be nonempty for every $c \in C$. Then*

$$Z_G^{(k)}(\Sigma_f) \mathrel{\dot\sim} \frac{1}{|Z(G)|}\sum_{c\in C}\int_{A\in\mathcal{M}(\Sigma_f)'_c} k^{d_c} e^{2\pi i k\,\mathrm{CS}_{\Sigma_f}(A)}\sqrt{\tau_{\Sigma_f}(A)}\, e^{2\pi i\frac{\rho_A(\Sigma_f)}{8}}, \tag{7.5}$$

and each factor of the integrand gets identified in the leading term of the Witten–Reshetikhin–Turaev invariants.

8. Spectral flow

The spectral flow along a path of formally self-adjoint, elliptic differential operators D_t is the algebraic intersection number in $[0,1]\times\mathbb{R}$ of the track of the spectrum

$$\{(t,\lambda) \mid t\in[0,1], \lambda\in\operatorname{Spec}(D_t)\}$$

and the line segment from $(0,-\varepsilon)$ to $(1,-\varepsilon)$. We choose the $(-\varepsilon,-\varepsilon)$-convention, which makes the spectral flow additive under concatenation of paths of connections.[1]

The main statement of this section relating spectral flow, the Chern–Simons invariant and the ρ-invariant for a compact Lie group seems to be well-known. Since it

[1] In the literature one also frequently finds the $(-\varepsilon,\varepsilon)$-convention (see for example [35] and [51]), so we need to be careful when relating to formulas found elsewhere.

depends on several conventions and we have not found a general proof anywhere in the literature, we decided to provide a proof in this paper in the hope that it may be a useful reference. With slightly different conventions, this has been proven in [51], Section 7, for SU(2). Even though the main proof is completely analogous, we give a detailed exposition for the convenience of the reader.

8.1. The dual Coxeter number. Let G be a simple Lie group of dimension n and rank r. Consider any positive definite normalization $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ of the Killing form on $\mathfrak{g}$. Given a basis $\{X_i\}_{i=1,\dots,n}$ of $\mathfrak{g}$ and its dual basis $\{X^i\}$ with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, the quadratic Casimir is the element

$$\Omega = \sum_i X_i \otimes X^i \in \mathfrak{g} \otimes \mathfrak{g}.$$

As an element of the universal enveloping algebra it commutes with all elements of $\mathfrak{g}$. The Casimir invariant in the adjoint representation is given by

$$\mathrm{ad}_*(\Omega) = \sum_i \mathrm{ad}_*(X_i)\,\mathrm{ad}_*(X^i) \in \mathrm{End}(\mathfrak{g}).$$

By Schur's Lemma we know that it is proportional to the identity with factor – by definition – the Casimir eigenvalue C_{ad} in the adjoint representation with respect to the normalization $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.

Therefore, we have for all $X, Y \in \mathfrak{g}$

$$\mathrm{tr}(\mathrm{ad}_*(X)\,\mathrm{ad}_*(Y)) = K\langle X, Y\rangle_{\mathfrak{g}},$$

where K is determined by

$$K = K\frac{1}{n}\sum_{i=1}^{n}\langle X_i, X^i\rangle_{\mathfrak{g}} = \frac{1}{n}\sum_{i=1}^{n}\mathrm{tr}(\mathrm{ad}_*(X_i)\,\mathrm{ad}_*(X^i)) = \frac{1}{n}C_{\mathrm{ad}}\,\mathrm{tr}(\mathrm{Id}) = C_{\mathrm{ad}}.$$

The inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ gives rise to the identification $\mathfrak{g}^* \to \mathfrak{g}$, $\beta \to X_\beta$, where $\beta(X) = \langle X_\beta, X\rangle_{\mathfrak{g}}$ for all $X \in \mathfrak{g}$. We also have an induced inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}^*$ given by $\langle\beta,\gamma\rangle_{\mathfrak{g}} = \langle X_\beta, X_\gamma\rangle_{\mathfrak{g}}$. Then we have $C_{\mathrm{ad}} = \langle\theta,\theta\rangle_{\mathfrak{g}}\cdot h$ for the maximal root θ (see for example [38], eq. (1.6.51)), where the dual Coxeter number h is independent of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.

Notice that for the inner product on SU(n) given by $\langle X, Y\rangle_{\mathfrak{su}(n)} = -\mathrm{tr}(XY)$, the maximal root $\tilde\theta$ satisfies $\langle\tilde\theta,\tilde\theta\rangle_{\mathfrak{su}(n)} = 2$. We therefore get

$$-\mathrm{tr}(\mathrm{ad}_*(X)\,\mathrm{ad}_*(Y)) = -2n\,\mathrm{tr}(XY), \quad X, Y \in \mathfrak{su}(n). \tag{8.1}$$

8.2. Relating Chern classes via the adjoint representation. We have seen in Section 6 that with respect to an ad-invariant metric on $\mathfrak{g}$, we can consider the complexified adjoint representation

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{SU}(n) \subset \mathrm{End}(\mathfrak{g}^{\mathbb{C}}),$$

and its differential

$$\mathrm{Ad}_*\colon \mathfrak{g} \longrightarrow \mathfrak{su}(n) \subset \mathrm{End}(\mathfrak{g}^{\mathbb{C}}).$$

It is easy to see that $C_{\mathrm{Ad}} = C_{\mathrm{ad}}$.

We can define the second Chern form $c_2(B)$ of a connection B in a principal G-bundle P over a 4-manifold Z by

$$c_2(B) = \langle F_B \wedge F_B \rangle_{\mathfrak{g}},$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is the normalization of the Killing form on $\mathfrak{g}$ introduced in Section 2. This normalization is given in terms of the Killing form by

$$\langle X, Y \rangle_{\mathfrak{g}} = \frac{1}{16\pi^2 h} \mathrm{tr}(\mathrm{Ad}_X, \mathrm{Ad}_Y), \tag{8.2}$$

which is shown in [33], p. 242, together with a list of the dual Coxeter numbers h. Note that in this normalization $c_2(B)$ represents an integral generator of the second cohomology.

Ad P, the complexified adjoint bundle of P, is a Hermitian vector bundle, which we view as a principal SU(n)-bundle via its frame bundle. Therefore, it makes sense to consider the adjoint bundle Ad(Ad P) of Ad P, whose fiber is $\mathfrak{u}(n)$. The connection B in P induces a connection Ad B in Ad P as follows. Given a section $s\colon U \to P$ for $U \subset M$ open, then s^*B is a $\mathfrak{g}$-valued 1-forms on U. B is uniquely determined by the family of 1-forms $B_g^s = (sg)^*B$, where $g \in C^\infty(U;G)$. In this way, Ad B is determined by $\{\mathrm{Ad}_* \circ B_g^s\}_{g \in C^\infty(U;G)}$. Similarly we get $F_{\mathrm{Ad}\,B} = \mathrm{Ad}\, F_B$, where $\mathrm{Ad}\, F_B \in \Omega^2(\mathrm{Ad}(\mathrm{Ad}\, P))$.

By the previous paragraph we can consider the second Chern form

$$c_2(\mathrm{Ad}\, B) = \langle F_{\mathrm{Ad}\,B} \wedge F_{\mathrm{Ad}\,B} \rangle_{\mathfrak{su}(n)} = \langle \mathrm{Ad}\, F_B \wedge \mathrm{Ad}\, F_B \rangle_{\mathfrak{su}(n)}$$

of Ad B in Ad P. By eq. (8.2) and (8.1) we get

$$\begin{aligned} c_2(\mathrm{Ad}\, B) &= \frac{1}{16\pi^2 n} \mathrm{tr}(\mathrm{Ad}(\mathrm{Ad}\, F_B) \wedge \mathrm{Ad}(\mathrm{Ad}\, F_B)) \\ &= \frac{2n}{16\pi^2 n} \mathrm{tr}(\mathrm{Ad}\, F_B \wedge \mathrm{Ad}\, F_B) \\ &= 2n \frac{16\pi^2 h}{16\pi^2 n} \langle F_B \wedge F_B \rangle_{\mathfrak{g}} \\ &= 2h c_2(B). \end{aligned}$$

Notice that $c_1(\mathrm{Ad}\, B) = 0$, because Ad P is the complexification of ad P, and therefore $\mathrm{ch}_2(\mathrm{Ad}\, B) = \frac{1}{2}c_1^2(\mathrm{Ad}\, B) - c_2(\mathrm{Ad}\, B) = -c_2(\mathrm{Ad}\, B)$. This gives

$$-\mathrm{ch}_2(\mathrm{Ad}\, B) = c_2(\mathrm{Ad}\, B) = 2h\, c_2(B). \tag{8.3}$$

8.3. The relationship to the ρ-invariant and the Chern–Simons function. We are ultimately interested in the spectral flow $\mathrm{SF}(D_{A_t})$ of the odd signature operator coupled to a path of connections A_t from the trivial connection θ to another flat connection A. Since the spectral flow only depends on the endpoints, we will call this the spectral flow from θ to A

$$\mathrm{SF}(\theta, A) = \mathrm{SF}(D_{A_t}).$$

Theorem 8.1. *Let G be a simple Lie group and A a flat G-connection, then we get*

$$\mathrm{SF}(\theta, A) = -4h\,\mathrm{CS}(A) + \frac{\rho_A(X)}{2} - \frac{\dim G(1 + b^1(X))}{2} + \frac{\dim(H^0(X, d_A)) + \dim(H^1(X, d_A))}{2}.$$

We note that this theorem combined with Theorem 7.1 implies Theorem 1.6 from the introduction.

Proof. The proof is analogous to the argument in [51], Section 7. Notice that because we have $\langle X, Y\rangle = -\frac{1}{8\pi^2}\operatorname{tr}(XY)$ for $X, Y \in \mathfrak{su}(n)$, our Chern–Simons function has a different sign than the Chern–Simons function used for example in [51, 58]. Let $S_B\colon \Omega^1 \to \Omega^0 \oplus \Omega^2_-$ be the self-duality operator on $Z = X \times I$ defined by $\omega \mapsto (d_B^*\omega, P_-(d_B\omega))$ for a connection B on Z, where P_- is the projection to the anti-self-dual 2-forms. We will use the "outward normal first" convention to orient Z, so that we do not have to introduce signs in Stokes' Theorem. Near the boundary we have $S_B \circ \Psi_2 = \Psi_1(D'_A + \frac{\partial}{\partial u})$, where $D'_A(a, b) = (-d_A^* b, *d_A b - da_A)$, $A = B|_{\partial Z}$, $\Psi_2(a, b) = a\,du + b$ and $\Psi_1(a, b) = (-a, P_-(b\,du))$. By the Atiyah-Patodi-Singer index theorem (see also [51], Theorem 7.1)) we get for the connection $B = A_t$ on Z

$$\begin{aligned}\mathrm{SF}(D_{A_t}) &= \operatorname{Index} S_B \\ &= \int_Z \widehat{A}(Z)\operatorname{ch}(V_-)\operatorname{ch}(\operatorname{Ad} B) + \frac{1}{2}(\eta(D_{A_1}) + \dim\ker D_{A_1}) \\ &\qquad - \frac{1}{2}(\eta(D_{A_0}) + \dim\ker D_{A_0}),\end{aligned}$$

where $\operatorname{ch}(\operatorname{Ad} B)$ is the total Chern character form of the connection $\operatorname{Ad} B$ in the trivial bundle $Z \times \mathfrak{g}^{\mathbb{C}}$ induced by B and V_- is the complex spinor bundle of $-\frac{1}{2}$-spinors on Z, whose rank is 2 for a 4-manifold. Consider $c_2(B) = \langle F_B \wedge F_B\rangle$. Then by Stokes' theorem

$$\mathrm{CS}(A_1) - \mathrm{CS}(A_0) = \int_Z c_2(B).$$

By eq. (8.3) we have

$$2h(\mathrm{CS}(A_1) - \mathrm{CS}(A_0)) = 2h \int_Z c_2(B) = -\int_Z \operatorname{ch}(\operatorname{Ad} B).$$

We have

$$\widehat{A}(Z) = 1 + \frac{1}{24}c_2(Z),$$

so that the integrand in the index theorem can be split up

$$\int_Z \widehat{A}(Z)\mathrm{ch}(V_-)\mathrm{ch}(\mathrm{Ad}\, B) = \int_Z \Big(\widehat{A}(Z)\mathrm{ch}(V_-)\,\mathrm{rk}(\mathrm{Ad}\, B) + 2\,\mathrm{ch}_2(\mathrm{Ad}\, B)\Big).$$

The first contribution can immediately be computed to be zero by applying the index theorem to a constant path at the trivial connection. The second contribution is precisely $-4h(\mathrm{CS}(A_1) - \mathrm{CS}(A_0))$. By definition, the difference of the η-invariants $\eta(D_A) - \eta(D_\theta)$ is the ρ-invariant. After identifying the cohomology with the kernel of the odd signature operator, the theorem follows. □

Appendix A. Heuristic discussion of the path integral

As a disclaimer, we would like to mention that this appendix reviews parts of [75] and is the only non-rigorous part in the paper, which we decided to include for motivational purposes. See Sawon's overview [66] on the perturbative expansion of Chern–Simons theory and Rozansky's work [63] in particular [64] for a detailed account.

For a function $f : \mathbb{R}^n \to \mathbb{R}$ with finitely many non-degenerate critical points and a compactly supported function $\varphi : \mathbb{R}^n \to \mathbb{R}$, we have the asymptotic behaviour

$$\int_{\mathbb{R}^n} e^{ikf(x)}\varphi(x)\,dx \sim_{k\to\infty} \Big(\frac{2\pi}{k}\Big)^{\frac{n}{2}} \sum_{x\in\mathrm{Crit}(f)} e^{\frac{\pi i}{4}\,\mathrm{sign}\,\mathrm{Hess}_x(f)} \frac{e^{ikf(x)}\varphi(x)}{\sqrt{|\det \mathrm{Hess}_x(f)|}}$$

by the method of stationary phase. We may assume that $\varphi(x) = 1$ for $x \in \mathrm{Crit}(f)$ and $\varphi \equiv 0$ outside of a compact set. Therefore, we will abuse the notation and eliminate the function φ from the formulas entirely. Let G be a simple, simply-connected, compact Lie group. If $G/Z(G)$ acts freely from the right on $\mathbb{R}^n$ and $e^{ikf(x)}$ is G-invariant, then the Jacobian J of the G action on x induces the measure $d[x] = |\det J(x)|dx$ on $\mathbb{R}^n/G$ and we get the leading order asymptotic behaviour

$$\frac{\mathrm{vol}\, G}{|Z(G)|}\int_{\mathbb{R}^n/G} e^{ikf(x)}\,d[x] \sim_{k\to\infty} \Big(\frac{2\pi}{k}\Big)^{\frac{n-\dim G}{2}} \sum_{[x]\in\mathrm{Crit}(f)/G} e^{\frac{\pi i}{4}\,\mathrm{sign}\,\mathrm{Hess}_x(f)} \frac{e^{ikf(x)}}{\sqrt{|\det \mathrm{Hess}_x(f)|}}\,\frac{|\det J(x)|\,\mathrm{vol}\, G}{|Z(G)|}. \tag{A.1}$$

Also see [76], Section 2.2, and [64], Section 2.2, for the appearance of the factor $\frac{1}{|Z(G)|}$.

According to Witten [75], the invariants $Z_G^{(k)}(X)$ can be written as the path integral characterizing the Chern–Simons theory

$$Z_G^{(k)}(X) = \int\limits_{A\in\mathcal{A}} e^{2\pi i k \operatorname{CS}(A)}\, dA,$$

where we have identified $\mathcal{A} = \Omega^1(X, \mathfrak{g})$. Even though the right-hand side is not mathematically rigorous, we would like to formally apply the above stationary phase approximation to this path integral. This procedure in quantum field theory is known as the Faddeev–Popov method (see for example [59] and [62] for more information). $\mathcal{G} = C^\infty(X, G)$ acts on $\mathcal{A}$. It can easily be seen that $|Z(\mathcal{G})| = |Z(G)|$, and we need to ignore $\operatorname{vol}\mathcal{G}$. In our case, $\det D$ is the zeta-regularized determinant of a formally self-adjoint elliptic differential operator D

$$\det D = e^{-\zeta_k'(0)},$$

where

$$\zeta(s) = \sum_{\lambda_j\neq 0} \lambda_j^{-s},$$

where λ_j are the eigenvalues of D. The differential of the $\mathcal{G}$ action on A can be seen to be d_A. Observe that

$$\|d_A\varphi\|^2 = \langle \Delta_A^{(0)}\varphi, \varphi\rangle_{L^2} = \lambda\|\varphi\|^2,$$

where the L^2 inner product on $\Omega^k(X; \mathfrak{g})$ is given by

$$\langle a, b\rangle_{L^2} = \int_X \langle a \wedge *b\rangle,$$

$\Delta_A^{(k)}$ is the twisted Laplacian on $\Omega^k(X, \mathfrak{g})$ and φ is an eigenvector of $\Delta_A^{(0)}$ with (positive) eigenvalue λ. Therefore we have

$$|\det J(A)| = \sqrt{\det \Delta_A^{(0)}}, \tag{A.2}$$

which is the Faddeev–Popov determinant in disguise.

On a finite-dimensional Riemannian manifold we have

$$\operatorname{Hess}_x(f)(X, Y) = \langle \nabla_X \operatorname{grad} f, Y\rangle$$

for a critical point x of a Morse-function f, where ∇ is the Levi-Civita connection. We can view the L^2 inner product on the space of connections $\mathcal{A}$ as a metric on $\mathcal{A}$. We can use it to identify vectors and covectors of $T_{[A]}(\mathcal{B}) = \operatorname{coker} d_A \cong \ker d_A^*$. With respect to this the linearization of $\operatorname{CS}\colon \mathcal{B} \to \mathbb{R}/\mathbb{Z}$ is given by the gradient $\operatorname{grad}\operatorname{CS}|_A = *F_A\colon \ker d_A^* \to \ker d_A^*$. Consider now the odd signature operator coupled to a connection A, as defined in (6.2) Notice that $D_A^2 = \Delta_A^{(0)} \oplus \Delta_A^{(1)}$ and therefore $(\det D_A)^2 = \det \Delta_A^{(0)} \det \Delta_A^{(1)}$. Let A be flat, then we have under the decomposition $\Omega^1(X;\mathfrak{g}) = \operatorname{im} d_A \oplus \ker d_A^*$

$$D_A = H_A \oplus S_A$$

where

$$\begin{aligned} S_A \colon \Omega^0(X;\mathfrak{g}) \oplus \operatorname{im} d_A &\longrightarrow \Omega^0(X;\mathfrak{g}) \oplus \operatorname{im} d_A, \\ (\alpha, \beta) &\longmapsto (d_A^*\beta, d_A\alpha) \end{aligned}$$

has symmetric spectrum and satisfies $|\det S_A| = \det \Delta_A^{(0)}$, while

$$H_A = \operatorname{proj}_{\ker d_A^*} * d_A \colon \ker d_A^* \to \ker d_A^*$$

is the linearization of $\operatorname{grad}_A \mathrm{CS}$ satisfying $\langle H_A(a), b\rangle = \operatorname{Hess}_A \mathrm{CS}(a,b)$. Therefore we have

$$|\det \operatorname{Hess}_A \mathrm{CS}\,| = |\det H_A| = \frac{|\det D_A|}{|\det S_A|} = \frac{|\det D_A|}{\det \Delta_A^{(0)}}. \tag{A.3}$$

The analytic torsion

$$T_X(A) = \prod_k (\det \Delta_A^{(k)})^{(-1)^{k+1}k/2}$$

is an invariant of Riemannian manifolds defined by Ray and Singer, which proved to be equal to the Reidemeister torsion $\tau_X(A)$ by work of Cheeger and Müller after choosing the volume form on cohomology induced by the metric on the manifold. Since $\det \Delta_A^{(k)} = \det \Delta_A^{(3-k)}$ by Poincaré duality, we deduce from (A.2) and (A.3)

$$\begin{aligned} \sqrt{\tau_X(A)} &= (\det \Delta_A^{(0)})^{3/4} (\det \Delta_A^{(1)})^{-1/4} \\ &= \frac{\det \Delta_A^{(0)}}{\sqrt{|\det D_A|}} \\ &= \frac{|\det J(A)|}{\sqrt{|\det \operatorname{Hess}_A \mathrm{CS}\,|}}. \end{aligned} \tag{A.4}$$

Let us turn to the analogue of the signature. In finite dimensions we have for a path x_t between two nondegenerate critical points x_0 and x_1 we get

$$\operatorname{sign}(\operatorname{Hess}_{x_1}(f)) - \operatorname{sign}(\operatorname{Hess}_{x_0}(f)) = 2\,\mathrm{SF}(\nabla \operatorname{grad}_{x_t} f),$$

where the spectral flow SF is defined in Section 8. Therefore, instead of the signature of the Hessian, we can use twice the spectral flow of H_{A_t} for a path of connections

A_t from the trivial connection θ to some flat connection $A = A_1$. Since S_{A_t} has symmetric spectrum and $D_{A_t} - H_{A_t}$ is a compact operator for all t, we can use $2\,\mathrm{SF}(D_{A_t}) = 2\,\mathrm{SF}(H_{A_t})$. Keep in mind that this procedure neglects the signature at the trivial connection. Note that this is the idea behind the gauge-theoretic version of Casson's invariant for homology 3-spheres by Taubes [68] and its generalizations. This turned out to be the perfect approach for the Casson invariant, because we needed an integer-valued analogue to the signature. The case of the Witten–Reshetikhin–Turaev invariants allows for an alternative approach.

We can consider the η-invariant defined in (6.1) as a generalized signature. This has the immediate merit of being defined for every connection, but it is metric-dependent and not necessarily an integer. Since the ρ-invariant defined in (6.3) is independent of the metric, we will choose it as a generalized signature, keeping in mind that we introduced $\eta(D_\theta)$. By following the arguments in [75], $\eta(D_\theta)$ can be altered into a prefactor, which is a topological invariant of a framed, oriented manifold. It was observed in [36] that this prefactor vanishes for the (canonical) Atiyah 2-framing. For further details we refer to [75], Section 2, and [36], Section 1.

It has been mentioned by Jeffrey [49], Section 5.2.2, that Reidemeister torsion can be used as a density, thereby extending the above use of Reidemeister torsion in the formal application of the stationary phase method to degenerate critical points. We need this idea to allow for critical components of positive dimension. To this end we identify $T_A\mathcal{M}(X)$ with $H^1(X, d_A)$ and $H^2(X, d_A)$ with $(H^1(X, d_A))^*$ using Poincaré duality. Note that Poincaré duality depends on a choice of inner product on $\mathfrak{g}$, which is possibly a multiple of our original choice of inner product on $\mathfrak{g}$. Furthermore, we need to choose a suitable volume form or density on $H^0(X, d_A)$ and $(H^3(X, d_A))^*$, for example we might take the one induced by the inner product on $\mathfrak{g}$ as suggested in [49], Section 5.2.2, or further normalized as suggested on [63], p. 284.

Since we allow higher-dimensional components in the moduli space of flat connections and the stationary phase approximation in finite dimensions (A.1) includes the factor $k^{-\frac{n-\dim G}{2}}$, we have to shift our result by the factor k^{d_c}, where d_c is half the real dimension of the critical component $\mathcal{M}(X)_c$ minus the dimension of the stabilizer of some generic $[A] \in \mathcal{M}_c$ under $\mathcal{G}$. Since the tangent space to the stabilizer is isomorphic to $H^0(X, d_A)$, we expect

$$d_c = \frac{1}{2} \max_{A \in \mathcal{M}(X)_c} (\dim(H^1(X, d_A)) - \dim(H^0(X, d_A))),$$

which is also known as the growth rate conjecture; see [6], Lemma 7.2, for evidence). By the same argument we may like to introduce the factor $\frac{1}{(2\pi)^{d_c}}$. For similar reasons Rozansky [64], eq. (2.33), includes such a factor. We will simply set every factor of the form K^{d_c} for a constant $K > 0$ to 1, because a change of normalization for Poincaré duality (used to treat Reidemeister torsion as a density) by a factor K results in the factor K^{-d_c} in the stationary phase approximation. The other factors, which only depend on n and the dimension of G, we need to omit, because both $\mathcal{A}$ and $\mathcal{G}$ are infinite-dimensional.

If we therefore replace the signature of the Hessian by the ρ-invariant (6.3), replace the rest via (A.4) and normalize Poincaré duality appropriately (independently of X and G), the following conjecture is justified.

Conjecture 8.2. *Let G be a simple, simply-connected, compact Lie group. Let X be a closed 3-manifold and C the set of all connected components of $\mathcal{M}(X)$. Then, in the Atiyah 2-framing, the leading order asymptotic behavior of $Z_G^{(k)}(X)$ in the limit $k \to \infty$ is given by*

$$Z_G^{(k)}(X) \dot{\sim} \sum_{c \in C} \frac{1}{|Z(G)|} \int_{A \in \mathcal{M}(X)_c} \sqrt{\tau_X(A)}\, e^{2\pi i \,\mathrm{CS}_X(A) k} e^{\frac{\pi i}{4}\rho_A(X)} k^{d_c}. \tag{A.5}$$

Theorem 8.1 immediately yields the more familiar version (1.2) of (A.5) stated in the introduction. Observe that the Chern–Simons invariant is constant on connected components of flat connections, we could therefore put it in front of the integral. On reducible subsets it will be necessary to interpret these conjectures in a suitable way, however we do not consider the reducible case in this paper.

Appendix B. Review of the Lefschetz–Riemann–Roch theorem for singular varieties

We shall very quickly review the Lefschetz–Riemann–Roch theorem for singular varieties due to P. Baum, W. Fulton, R. MacPherson and G. Quart; see [20] for a proof of the Riemann–Roch theorem and the general theory and [21] for a proof of the Lefschetz–Riemann–Roch theorem. We will only state their theorems in the generalities we need.

Let X be a complex quasi-projective algebraic variety. Consider the Grothendieck group $K^0_{\mathrm{alg}}(X)$ of algebraic vector bundles (i.e. locally free sheaves) on X. This is a ring-valued contravariant functor. Let $K_0^{\mathrm{alg}}(X)$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_X$ modules on X. This is a covariant functor for proper morphism: if $f : X \to Y$ is a proper morphism, then

$$f_* \colon K_0^{\mathrm{alg}}(X) \longrightarrow K_0^{\mathrm{alg}}(Y),$$

is defined by setting $f_*[\mathcal{F}] = \sum (-1)^i [R^i f_* \mathcal{F}]$.

For a topological space X one considers the Grothendieck group $K^0_{\mathrm{top}}(X)$ of topological vector bundles on X, so $K^0_{\mathrm{top}}(X)$ is a ring-valued contravariant functor. Let $K_0^{\mathrm{top}}(X)$ be the Grothendieck group of complexes of vector bundles on $\mathbb{C}^N$ exact off X for some closed embedding of X in $\mathbb{C}^N$. (One is making Alexander duality a definition here.)

For any complex algebraic variety X there is a natural ring homomorphism

$$\alpha^\bullet\colon K^0_{\mathrm{alg}}(X) \longrightarrow K^0_{\mathrm{top}}(X),$$

which is a natural transformation of contravariant functors. Suppose X is a closed algebraic subset of a variety Y. Let $K_X^{\mathrm{alg}}(Y)$ be the Grothendieck group of complexes of algebraic vector bundles on Y which are exact off X. There is a natural homology map

$$h\colon K_X^{\mathrm{alg}}(Y) \longrightarrow K_0^{\mathrm{alg}}(X)$$

given by

$$h([E_\bullet]) = \sum (-1)^i [H_i(E_\bullet)],$$

where the $H_i(E_\bullet)$ are the homology sheaves of the complex $E_\bullet$ of locally free sheaves on Y. The map h is an isomorphism; see [20], Appendix 2. Suppose X is a closed subspace of Y, where Y is a C^∞-manifold. When we have a closed embedding of C^∞-manifolds $Y \hookrightarrow \mathbb{C}^N$ and the normal bundle of Y in $\mathbb{C}^N$ has a complex structure, we get the Thom–Gysin isomorphism

$$h\colon K_X^{\mathrm{top}}(Y) \longrightarrow K_X^{\mathrm{top}}(\mathbb{C}^N) = K_0^{\mathrm{top}}(X).$$

Again for any closed subset X of an algebraic variety Y we have homomorphism of abelian groups $\alpha^\bullet\colon K_X^{\mathrm{alg}}(Y) \to K_X^{\mathrm{top}}(Y)$. We shall now describe the key construction in the formulation of the Riemann–Roch theorem in [20]. There is a homomorphism

$$\alpha_\bullet\colon K_0^{\mathrm{alg}}(X) \longrightarrow K_0^{\mathrm{top}}(X)$$

of abelian groups, which is covariant for proper morphisms defined the following way: choose an embedding of X in a nonsingular variety Y; then $\alpha_\bullet$ is the composition

$$\alpha_\bullet\colon K_0^{\mathrm{alg}}(X) \xrightarrow{h^{-1}} K_X^{\mathrm{alg}}(Y) \xrightarrow{\alpha^\bullet} K_X^{\mathrm{top}}(Y) \xrightarrow{h} K_0^{\mathrm{top}}(X).$$

With this setup at hand the main theorem in [20] is stated; see [20], pp. 174–75. We are however only interested in the weaker version of this theorem where topological K-theory is replaces by ordinary homology theory with rational coefficients. Let $H^\bullet(X)$ be ordinary singular cohomology with rational coefficients. If X is closed in Y, let $H_X^\bullet(Y) = H^\bullet(Y, Y - X)$. Again, we use the Alexander duality to define the homology groups

$$H_i(X) = H_X^{2n-i}(\mathbb{C}^N).$$

Let

$$\mathrm{Ch}^\bullet\colon K^0_{\mathrm{top}}(X) \longrightarrow H^\bullet(X)$$

be the usual Chern character and let

$$\mathrm{Ch}^\bullet\colon K_X^{\mathrm{top}}(Y) \longrightarrow H_X^\bullet(Y)$$

be the canonical extension of the Chern character. We can define the homological Chern character by embedding X in some $\mathbb{C}^N$ and then define $\mathrm{Ch}_\bullet$ to be the composition

$$\mathrm{Ch}_\bullet\colon K_0^{\mathrm{top}}(X) = K_X^{\mathrm{top}}(\mathbb{C}^N) \xrightarrow{\mathrm{Ch}^\bullet} H_X^\bullet(\mathbb{C}^N) = H_\bullet(X).$$

We shall also use the notation $\mathrm{Ch}^\bullet$ for the composition $\mathrm{Ch}^\bullet\alpha^\bullet\colon K^0_{\mathrm{alg}} \to H^\bullet$ and we define $\tau_\bullet = \mathrm{Ch}_\bullet\alpha_\bullet\colon K_0^{\mathrm{alg}} \to H_\bullet$. The Riemann–Roch theorem for singular varieties can now be formulated as follows.

Theorem 8.3 ([20], p. 180). *The mapping*

$$\tau_\bullet\colon K_0^{\mathrm{alg}}(X) \longrightarrow H_\bullet(X)$$

is covariant for proper morphisms, compatible with cap products, cartesian products and restrictions to open subvarieties. If X is non-singular

$$\tau_\bullet[\mathcal{O}_X] = \mathrm{Td}(T_X) \cap [X].$$

We remark that for a projective variety

$$\tau_\bullet[\mathcal{O}_X] = [X]$$

modulo lover degree terms. This follows from the lemma on p. 129 of [19] and part (6) of the Riemann–Roch Theorem of [39].

Let us now press on with the Lefschetz–Riemann–Roch theorem. An equivariant variety X will be defined to be a quasi-projective algebraic variety with an automorphism $x\colon X \to X$, such that $x^m = \mathrm{Id}$. The fixed point subvariety we will denote $|X|$. We will for the rest of this section assume that the varieties we are considering are equivariant, unless otherwise stated.

An equivariant sheaf on X is a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules together with a homomorphism of sheaves

$$\varphi_{\mathcal{F}}\colon x^*\mathcal{F} \longrightarrow \mathcal{F}.$$

Let $K_0^{\mathrm{eq}}(X)$ be the Grothendieck group of all equivariant sheaves on X and let $K^0_{\mathrm{eq}}(X)$ be the Grothendieck group of all equivariant locally free sheaves on X.

If the automorphism x is the identity, then any equivariant sheaf $\mathcal{F}$ on X breaks up into a finite direct sum of sheaves $\mathcal{F}_a$, $a \in \mathbb{C}$, where $\mathcal{F}_a$ is the generalized a-eigensheaf for $\varphi_{\mathcal{F}}$. This gives maps

$$\begin{aligned} K^0_{\mathrm{eq}}(X) &\longrightarrow K^0_{\mathrm{alg}}(X) \otimes \mathbb{Z}[\mathbb{C}] \longrightarrow K^0_{\mathrm{alg}}(X) \otimes \mathbb{C}, \\ K_0^{\mathrm{eq}}(X) &\longrightarrow K_0^{\mathrm{alg}}(X) \otimes \mathbb{Z}[\mathbb{C}] \longrightarrow K_0^{\mathrm{alg}}(X) \otimes \mathbb{C}, \end{aligned} \tag{B.1}$$

by mapping $[\mathcal{F}]$ to $\sum[\mathcal{F}_a] \otimes a$ followed by the natural trace map $\mathrm{tr}\colon \mathbb{Z}[\mathbb{C}] \to \mathbb{C}$. In particular, if we in the general case compose the homomorphism in (B.1) with the homomorphism coming from the inclusion map $|X| \hookrightarrow X$, we get a natural homomorphism

$$L^{\bullet}\colon K^0_{\mathrm{eq}}(X) \longrightarrow K^0_{\mathrm{alg}}(|X|) \otimes \mathbb{C}.$$

If V is a component of $|X|$ and X is non-singular in a neighborhood of V, then V is also non-singular, and the conormal sheaf $\mathcal{N}$ to V in X is an equivariant locally free sheaf on V. Then $\lambda_{-1}(\mathcal{N}) = \sum(-1)^i[\Lambda^i\mathcal{N}]$ determines an element in $K^0_{\mathrm{eq}}(V)$, which in turn under (B.1) maps to the element, say

$$\lambda^V_{-1}X \in K^0_{\mathrm{alg}}(V) \otimes \mathbb{C}.$$

This element is clearly invertible in $K^0_{\mathrm{alg}}(V) \otimes \mathbb{C}$.

In order to state the Lefschetz–Riemann–Roch theorem, we need to discuss relative equivariant K-theory. Let X be a closed equivariant subvariety of Y. Define $K^{\mathrm{eq}}_X(Y)$ to be the Grothendieck group of equivariant complexes on Y which are exact off X. Suppose now that Y is non-singular and that $j\colon X \to Y$ is the inclusion map and that $|X|$ is projective. We have the homology isomorphism

$$h\colon K^{\mathrm{eq}}_X(Y) \longrightarrow K^{\mathrm{eq}}_0(X)$$

defined the same way as in the non-equivariant case. We also define the modified homology map

$$\tilde{h}\colon K^{\mathrm{eq}}_{|X|}(|Y|) \otimes \mathbb{C} \longrightarrow K^{\mathrm{alg}}_0(|X|) \otimes \mathbb{C}$$

by the formula

$$\tilde{h}(\xi) = |j|^*(\lambda^{|Y|}_{-1}Y)^{-1} \cap h(\xi).$$

There is a natural homomorphism (in the case $|X|$ is projective)

$$L\colon K^{\mathrm{eq}}_X(Y) \longrightarrow K^{\mathrm{eq}}_{|X|}(|Y|) \otimes \mathbb{C}$$

for X closed in Y, given by the pull-back homomorphism induced by the inclusion of $|Y|$ in Y. Now define a homomorphism

$$L_{\bullet}\colon K^{\mathrm{eq}}_0(X) \longrightarrow K^{\mathrm{alg}}_0(|X|) \otimes \mathbb{C}$$

to be the composition

$$L_{\bullet}\colon K^{\mathrm{eq}}_0(X) \xrightarrow{h^{-1}} K^{\mathrm{eq}}_X(Y) \xrightarrow{L} K^{\mathrm{eq}}_{|X|}(|Y|) \otimes \mathbb{C} \xrightarrow{\tilde{h}} K^{\mathrm{alg}}_0(|X|) \otimes \mathbb{C}$$

for some closed embedding of X in a non-singular Y.

The Lefschetz–Riemann–Roch theorem for singular varieties can now be stated as follows.

Theorem 8.4 (Baum, Fulton, and Quart). *The homomorphism*

$$L_{\bullet}\colon K^{\mathrm{eq}}_0(X) \longrightarrow K^{\mathrm{alg}}_0(|X|) \otimes \mathbb{C}$$

is independent of the embedding of X in a non-singular Y and is compatible with cap-products, cartesian products, restrictions to open equivariant sub-varieties. Moreover $L_\bullet$ is covariant for proper morphisms and if X is non-singular around a component V of $|X|$, then

$$L_\bullet^V[\mathcal{O}_X] = (\lambda_{-1}^V X)^{-1} \cap [\mathcal{O}_V] \in K^0_{\mathrm{alg}}(V) \otimes \mathbb{C}.$$

Suppose now that X is an equivariant projective algebraic variety and that $\mathcal{E}$ is an equivariant locally free sheaf on X. By pushing forward to a point and combining the two theorems above we get the following Lefschetz–Riemann–Roch formula due to Baum, Fulton, MacPherson, and Quart

$$\sum (-1)^i \operatorname{tr}(x\colon H^i(X,\mathcal{E}) \longrightarrow H^i(X,\mathcal{E})) = \mathrm{Ch}^\bullet(L^\bullet(\mathcal{E})) \cap \tau_\bullet L_\bullet(\mathcal{O}_X).$$

Here $\cap\colon H_\bullet(|X|,\mathbb{C}) \otimes H^\bullet(|X|,\mathbb{C}) \to \mathbb{C}$ is the cap product pairing between cohomology and homology. Let C be the finite set which indexes connected components of $|X|$, i.e.

$$|X| = \coprod_{c\in C} |X|_c.$$

Denote $\dim |X|_c = n_c$. Suppose furthermore that $\mathcal{E} = \mathcal{L}^k$ where $\mathcal{L}$ is an equivariant line bundle over X. Say $c_1(\mathcal{L}) = \alpha$ and denote $\alpha|_{|X|_c} = \alpha_c$. Let $a_c \in \mathbb{C}$ be such that

$$\mathrm{Ch}^\bullet(L_c^\bullet(\mathcal{L}^k)) = \exp(k\alpha_c) \otimes a_c^k.$$

The Lefschetz–Riemann–Roch formula then reads

$$\begin{aligned}
&\sum (-1)^i \operatorname{tr}(x\colon H^i(X,\mathcal{L}^k) \to H^i(X,\mathcal{L}^k)) \\
&\quad = \sum_{c\in C} a_c^k \exp(k\alpha_c) \cap \tau_\bullet(L_\bullet^c(\mathcal{O}_X)) \\
&\quad = \sum_{c\in C} a_c^k \Big(\sum_{i=0}^{n_c} \frac{1}{i!} (\alpha_c)^i \cap \tau_i(L_\bullet^c(\mathcal{O}_X)) k^i \Big).
\end{aligned}$$

If $|X|_c$ is contained in the non-singular part of X, we get that

$$\begin{aligned}
\exp(k\alpha_c) \cap \tau_\bullet(L_\bullet^c(\mathcal{O}_X)) &= \exp(k\alpha_c) \cap \mathrm{Ch}^\bullet(\lambda_{-1}^c X)^{-1} \cap \tau_\bullet([\mathcal{O}_{|X|_c}]) \\
&= (\exp(k\alpha_c) \cup \mathrm{Ch}^\bullet(\lambda_{-1}^c X)^{-1} \cup \mathrm{Td}(T_{|X|_c})) \cap [|X|_c].
\end{aligned}$$

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