

Integrality of Kauffman brackets of trivalent graphs

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Abstract. We show that Kauffman brackets of colored framed graphs (also known as quantum spin networks) can be renormalized to a Laurent polynomial with integer coefficients by multiplying it by a coefficient which is a product of quantum factorials depending only on the abstract combinatorial structure of the graph. Then we compare the shadow-state sums and the state-sums based on R -matrices and Clebsch–Gordan symbols, reprove their equivalence and comment on the integrality of the weight of the states. We also provide short proofs of most of the standard identities satisfied by quantum $6j$ -symbols of $U_q(\mathfrak{sl}_2)$.

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Contents

1	Introduction	143
2	Kauffman brackets via representations of $U_q(\mathfrak{sl}_2)$	145
3	Integrality	162
4	Shadow state-sums and integrality	169
5	R -matrices vs. $6j$ -symbols	179
	References	183

1. Introduction

The family of $U_q(\mathfrak{sl}_2)$ -quantum invariants of knotted objects in S^3 as knots, links and more in general trivalent graphs, can be defined via the recoupling theory ([9]) as well as via the theory of representations of the quantum group $U_q(\mathfrak{sl}_2)$ ([10] and [15]) and the so-called theory of “shadows” ([16], Chapter IX). These invariants are defined for framed objects (framed links or graphs, see Definition 2.1) equipped with a “coloring” on the set of 1-dimensional strata of the object satisfying certain

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admissibility conditions (Definition 2.4), and take values in $\mathbb{Q}(q^{\frac{1}{2}})$. As customary in the literature, we shall call these invariants the “Kauffman brackets” of the colored, framed graph G and denote them by $\langle G, \text{col} \rangle$. In particular, if G is a framed knot k the set of admissible colorings is the set of half-natural numbers (here we use the so-called “spin” notation) and they coincide with the unreduced colored Jones polynomials of the knot: $\langle k, s \rangle = J_{2s+1}(k)$.

Although the definition of Kauffman brackets via recoupling theory is simple and appealing, the definition based on the theory of representations of $U_q(\mathfrak{sl}_2)$ happens to be more useful for our purposes. In Section 2 we will sketch the proof of the equivalence of the two definitions (the relations have been already studied by S. Piunikhin [14] and is fully detailed in [3]).

It is known that, in general, $\langle G, \text{col} \rangle$ is a rational function of the variable $q^{\frac{1}{2}}$ (there are various notations in the literature, for instance our $q^{\frac{1}{2}}$ is A in [9]). If L is a framed link, it was shown by T. Le ([12]) that, up to a factor of the form $q^{\pm \frac{n}{2}}$, $\langle L, n \rangle$ is a Laurent polynomial in q (actually in [12] a much stronger result is proved which holds for general polynomial invariants issued from quantum group representations).

On contrast it is well known that $\langle G, \text{col} \rangle$ is not in general a Laurent polynomial if G is a trivalent graph. The main result of the present paper is Theorem 3.2, restated here in a simpler form (we refer to Section 3 for the notation).

Theorem 1.1 (integrality of the renormalized Kauffman brackets). *There exist $m, n \in \mathbb{Z}$ such that*

$$\langle\langle G, \text{col} \rangle\rangle \doteq \frac{\langle G, \text{col} \rangle \prod [2 \text{col}(e)]!}{\prod [a_v + b_v - c_v]! [b_v + c_v - a_v]! [c_v + a_v - b_v]!} \in (\sqrt{-1})^m q^{\frac{n}{4}} \mathbb{Z}[q, q^{-1}],$$

where the products are taken over the non-closed edges e of G and the vertices v of G .

It turns out that $\langle\langle G, \text{col} \rangle\rangle = \langle G, \text{col} \rangle$ if G is a link. This normalization was proposed and conjectured to be integral by S. Garoufalidis and R. van der Veen (in [8], where they also proved the integrality in the classical case when $q = \pm 1$) in order to define generating function for classical spin networks evaluations.

We hope that our result will allow further development in that direction and in the understanding of the categorification of $U_q(\mathfrak{sl}_2)$ -quantum invariants for general knotted objects: a categorification of tensor products of the tensor products of quantum $\mathfrak{sl}(2)$ -modules with their quantum group action and the Jones–Wenzl projector was constructed in [7] using categories of representations of the Lie algebra $\mathfrak{gl}(n)$ for various n (see also [1]). We expect that the proposed renormalization will turn out to fit into this categorial framework (as well into the framework introduced in [2]) and allow to build a categorial refinement of Kauffman brackets.

The last sections are almost independent from the preceding ones. In Section 4 we recall the definition of shadow-state sums to compute $\langle\langle G, \text{col} \rangle\rangle$, give a new self-contained proof of the equivalence (first proved in [11]) between the shadow-state formulation and the R -matrix formulation of the invariants, and comment on the non-integrality of the single shadow-state weights. Using shadow state-sums we also provide short proofs of the most famous identities for $6j$ -symbols (e.g. Racah, Biedenharn–Elliot, orthogonality). In the last section we will quickly comment on the case when G has non-empty boundary and on the algebraic meaning of the shadow-state sums with respect to the state-sum based on R -matrices and Clebsch–Gordan symbols.

1.1. Structure of the paper. In Section 2 we will recall the definition of Kauffman brackets of colored graphs and the basic facts on representation theory of $U_q(\mathfrak{sl}_2)$ (for generic q). We then show how to compute Kauffman brackets via morphisms associated to tangles, and provide explicit formulas for the elementary morphisms. In Section 3 we will define $\langle\langle G, \text{col} \rangle\rangle$ and prove its integrality. Section 4 is almost independent from the first sections (basically it depends only on Lemma 3.9); there we explain how to compute $\langle\langle G, \text{col} \rangle\rangle$ via shadow state-sums and provide short proofs of some well known identities for $6j$ -symbols. In Section 5 we comment on the algebraic meaning of shadow-state-sums.

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2. Kauffman brackets via representations of $U_q(\mathfrak{sl}_2)$

2.1. The definition of Kauffman brackets

Definition 2.1 (KTG). A *Knotted Trivalent Graph* (KTG) is a finite trivalent graph $G \subset S^3$ equipped with a “framing,” i.e. the germ of an orientable smooth surface $S \subset S^3$ such that S retracts on G .

Remark 2.2. Note that this is not a “fat graph” as S is required to exist around all G and not only around its vertices. In contrast we require G to be embedded in S^3 . Also, starting from the end of Subsection 2.3 we will drop the assumption for S to be orientable.

In order to specify a framing S on a graph G we will only specify (via thin lines as in the leftmost drawing of Figure 2) the edges around which it twists with respect to the blackboard framing in a diagram of G , implicitly assuming that S will be lying horizontally (i.e. parallel to the blackboard) around G out of these twists. Let us also remark that if D is a diagram of G there is a framing S_D (called the *blackboard*

framing) induced on G simply by considering a surface containing G and lying almost parallel to the projection plane. Pulling back the orientation of \mathbb{R}^2 shows that S_D is orientable. The following result is a converse.

Lemma 2.3. *If G is a framed graph and S is an orientable framing on G then there exists a diagram D of G such that $S_D = S$.*

Proof. The idea of the proof is to fix a diagram D and count the number of half twists of difference on each edge of G between S_D and S . The reduction mod 2 of these numbers forms an explicit cochain in $H^1(G; \mathbb{Z}_2)$ which is null cohomologous because S and S_D are orientable. The coboundary reducing it to the 0 cochain corresponds to a finite number of moves as those in Lemma 2.17 which change D and isotope G into a position such that the number of half twists of difference between S and S_D is even on every edge. Then up to adding a suitable number of kinks to each edge of G this difference can be reduced to 0 everywhere. \square

Let now G be a KTG, E the set of its edges, V the set of its vertices.

Definition 2.4 (Admissible coloring). An *admissible coloring* of G (see Figure 1 for an example) is a map $\text{col}: E \rightarrow \frac{\mathbb{N}}{2}$ (whose values are called *colors*) such that for all $v \in V$ the following conditions are satisfied:

- (1) $a_v + b_v + c_v \in \mathbb{N}$,
- (2) $a_v + b_v \geq c_v, b_v + c_v \geq a_v, c_v + a_v \geq b_v$,

where a_v, b_v , and c_v are the colors of the edges touching v .

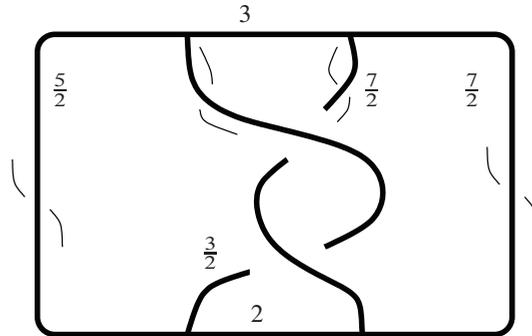


Figure 1. An admissibly colored KTG containing 4 vertices and 6 edges.

Let us now fix diagram D of G such that the blackboard framing coincides with that of G (it exists by Lemma 2.3), and an admissible coloring col on G , and recall how the Kauffmann bracket $\langle G, \text{col} \rangle$ is defined.

Let $q \in \mathbb{C}$,

$$[n] \doteq \frac{q^n - q^{-n}}{q - q^{-1}}$$

and

$$[n]! \doteq \prod_{j=1}^n [j], \quad [0]! = 1.$$

Let also

$$\begin{bmatrix} n \\ k \end{bmatrix} \doteq \frac{[n]!}{[k]![n-k]!}.$$

If G is a framed link L and col is $\frac{1}{2}$ on all the components of G , the Kauffman bracket $\langle L, \frac{1}{2} \rangle \in \mathbb{Z}[q^{\pm \frac{1}{2}}]$, is defined by applying recursively Kauffman's rules to D :

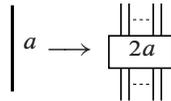
$$\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = q^{\frac{1}{2}} \left| \begin{array}{c} | \\ | \\ | \end{array} \right| + q^{-\frac{1}{2}} \begin{array}{c} \frown \\ \smile \end{array} \quad \text{and} \quad \bigcirc = -[2]$$

To define the general $\langle G, \text{col} \rangle$ for trivalent colored graphs let's first define the Jones–Wenzl projectors $\text{JW}_{2a} \in C(q^{\frac{1}{2}})[B(2a)]$:

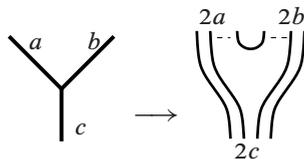
$$\text{JW}_{2a} = \begin{array}{c} | \dots | \\ | \dots | \\ \boxed{2a} \\ | \dots | \\ | \dots | \end{array} \doteq \sum_{\sigma \in \mathfrak{S}_{2a}} \frac{q^{-a(2a-1) + \frac{3}{2}T(\sigma)}}{[2a]!} \begin{array}{c} | \dots | \\ | \dots | \\ \hat{\sigma} \\ | \dots | \\ | \dots | \end{array}$$

where $\hat{\sigma}$ is the positive braid containing the minimal number ($T(\sigma)$) of crossings and inducing the permutation σ on its endpoints (it is a standard fact that such braid is well defined). Actually JW_{2a} is defined as an element of the Temperley–Lieb algebra, but for the purpose of this section we will just consider it as a formal sum of braids; in the next section a more precise interpretation will be provided. One defines $\langle G, \text{col} \rangle$ by the following algorithm.

- (1) Cable each edge e of G by $\text{JW}_{2\text{col}(e)}$, i.e. in D replace an edge e colored by a by a formal sum of braids in $B(2a)$ according to the above definition of JW_{2a} :



- (2) Around each vertex, connect the (yet free) endpoints of the so-obtained strands in the unique planar way without self returns:



- (3) This way one associates to (G, col) a formal sum with coefficients $c_i \in \mathbb{Q}(q^{1/2})$ of links L_i contained in a small neighborhood of the framing of G and therefore framed by annuli running parallel to it. Define

$$\langle G, \text{col} \rangle \doteq \sum_i c_i \langle L_i, \frac{1}{2} \rangle.$$

Theorem 2.5 (Kauffman, [9]). $\langle G, \text{col} \rangle$ is an invariant up to isotopy of (G, col) .

2.2. Basic facts on $U_q(\mathfrak{sl}_2)$

Definition 2.6. $U_q(\mathfrak{sl}_2)$ (we will also use the notation U_q) is the algebra generated by E, F, K and K^{-1} with relations

$$[E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

$$KE = qEK,$$

$$KF = q^{-1}FK,$$

$$KK^{-1} = K^{-1}K = 1.$$

Its Hopf algebra structure is given by

$$\Delta(E) = E \otimes K + K^{-1} \otimes E,$$

$$\Delta(F) = F \otimes K + K^{-1} \otimes F,$$

$$\Delta(K) = K \otimes K,$$

$$S(E) = -qE,$$

$$S(F) = -q^{-1}F,$$

$$S(K) = K^{-1},$$

$$\epsilon(E) = \epsilon(F) = 0,$$

$$\epsilon(K) = 1.$$

Remark 2.7. To make clear the relation with other notations note that ours is coherent with that of [10] after replacing their s with q ; our q corresponds to $q = A^2$ in [3] and our E, F respectively to X and Y .

Lemma 2.8. *For each $a \in \frac{\mathbb{N}}{2}$ there is a simple representation V^a of $U_q(\mathfrak{sl}_2)$ of dimension $2a + 1$ whose basis is g_u^a , $u = -a, -a + 1, \dots, a$ and on which the action of E, F, K is*

$$E(g_u^a) = [a - u][a + u + 1]g_{u+1}^a,$$

$$F(g_u^a) = g_{u-1}^a, \quad K(g_u^a) = q^u g_u^a.$$

Remark 2.9. In [3] and [10] different bases e_u^a and f_u^a for V^a were used. The changes of basis are in both cases diagonal and are

$$f_m^a = \frac{[2a]!}{[a - u]!} g_u^a$$

and

$$e_u^a = [a + u]! g_u^a.$$

Let also recall that, by Clebsch–Gordan decomposition theorem, $V^a \otimes V^b$ is isomorphic to $V^{a+b} \oplus V^{a+b-1} \oplus \dots \oplus V^{|a-b|}$. Hence by Schur’s lemma, the space $\text{Hom}(V^c, V^a \otimes V^b)$ has dimension 1 if (a, b, c) is admissible and 0 otherwise; in the next subsection, for each three-uple a, b, c we will choose explicit elements $Y_c^{a,b} \in \text{Hom}(V^c, V^a \otimes V^b)$ which are “induced from the topology.”

2.3. Computing Kauffman brackets invariants via U_q . The standard construction of quantum invariants via the representation theory of U_q ([10] and [15]) allows one to associate to each diagram of an (n, m) -colored framed tangle G (possibly containing some vertices) an operator between representations of U_q (i.e. a morphism of vector spaces commuting with the action of U_q). More explicitly if the bottom strands of G are colored by a_1, \dots, a_n and the top strands by b_1, \dots, b_m then one can associate to a diagram D of G a morphism

$$\text{op}(G, \text{col}, D): V^{a_1} \otimes \dots \otimes V^{a_n} \longrightarrow V^{b_1} \otimes \dots \otimes V^{b_m}.$$

To do this, one defines the operators associated to each “elementary” subdiagram (shown in Figure 2) equipped with any admissible coloring and then decomposes D into a vertical stacking of these subdiagrams: $\text{op}(G, \text{col}, D)$ is then defined as a composition of the operators associated to the elementary blocks. If one can choose the elementary operators so that the resulting morphism $\text{op}(G, \text{col}, D)$ does not depend on D , then in particular, if G is closed, $\text{op}(G, \text{col}, D)$ will be an invariant up to isotopy of (G, col) with values in $C(q^{\frac{1}{4}})$.

Proposition 2.10. *There exist choices of operators for all admissible colorings of the elementary diagrams of Figure 2 such that for each closed, colored KTG (G, col) , and each diagram D of G such that the framing of G coincides with the blackboard framing, it holds*

$$\text{op}(G, \text{col}, D) = \langle G, \text{col} \rangle.$$

Let us define operators

$$U_{\frac{1}{2}}: V^0 \longrightarrow V^{\frac{1}{2}} \otimes V^{\frac{1}{2}}$$

and

$$\cap_{\frac{1}{2}}: V^{\frac{1}{2}} \otimes V^{\frac{1}{2}} \longrightarrow V^0$$

which will allow us to compute $\langle G, \text{col} \rangle$ whenever G is a framed link and $\text{col} = \frac{1}{2}$.

We define it explicitly in the bases $g_u^{\frac{1}{2}}$ and g_0^0 by

$$\cap_{\frac{1}{2}}(g_u^{\frac{1}{2}} \otimes g_v^{\frac{1}{2}}) = \delta_{u,-v} \sqrt{-1}^{2u} q^u g_0^0 \quad (1)$$

$$U_{\frac{1}{2}}(g_0^0) = \sum_{u=-\frac{1}{2}}^{\frac{1}{2}} \sqrt{-1}^{2u} q^u g_u^{\frac{1}{2}} \otimes g_{-\frac{1}{2}-u}^{\frac{1}{2}} \quad (2)$$

Then we define the morphism

$$\frac{1}{2}R: V^{\frac{1}{2}} \otimes V^{\frac{1}{2}} \longrightarrow V^{\frac{1}{2}} \otimes V^{\frac{1}{2}}$$

associated to a positive crossing of $\frac{1}{2}$ -colored strands by

$$q^{\frac{1}{2}} \text{Id} + q^{-\frac{1}{2}} U_{\frac{1}{2}} \circ \cap_{\frac{1}{2}}.$$

Similarly, for a negative crossing we define

$$\frac{1}{2}R_-: V^{\frac{1}{2}} \otimes V^{\frac{1}{2}} \longrightarrow V^{\frac{1}{2}} \otimes V^{\frac{1}{2}}$$

by

$$q^{-\frac{1}{2}} \text{Id} + q^{+\frac{1}{2}} U_{\frac{1}{2}} \circ \cap_{\frac{1}{2}}.$$

Since

$$\cap_{\frac{1}{2}} \circ U_{\frac{1}{2}} = -[2],$$

and by the definition of the crossing operator, it is evident that for each framed link G colored by $\frac{1}{2}$ on each component and for any diagram D of it, it will hold

$$\text{op}(G, \text{col}, D) = \langle G, \text{col} \rangle.$$

Remark that

$$(\cap_{\frac{1}{2}} \otimes \text{Id}_{\frac{1}{2}})(\text{Id}_{\frac{1}{2}} \otimes \cup_{\frac{1}{2}}) = \text{Id}_{\frac{1}{2}}$$

and this encodes an isotopy invariance for planar diagrams; therefore for every framed (n, m) -tangle (implicitly colored by $\frac{1}{2}$) one has an associated morphism

$$(V^{\frac{1}{2}})^{\otimes n} \longrightarrow (V^{\frac{1}{2}})^{\otimes m}$$

well defined up to isotopy preserving the endpoints and the framing.

To treat the general case we define now $\text{JW}_{2a} \in \text{End}((V^{\frac{1}{2}})^{\otimes 2a})$ exactly as in Subsection 2.1, but now interpreting it as a morphism of representations of U_q (via the definition of the R -operators we already gave). The following holds (see [3], Section 3.5).

Theorem 2.11 (Jones, Wenzl). *The operators JW_{2a} are projectors over the unique submodule of $(V^{\frac{1}{2}})^{\otimes 2a}$ isomorphic to V^a .*

Therefore let us fix once and for all morphisms $\phi_a : V^a \rightarrow (V^{\frac{1}{2}})^{\otimes 2a}$ by¹

$$\phi_a(g_u^a) \doteq \frac{q^{\frac{a^2-u^2}{2}}}{[a+u]!} \text{JW}_{2a}((g_{\frac{1}{2}}^{\frac{1}{2}})^{\otimes (a+u)} \otimes (g_{-\frac{1}{2}}^{\frac{1}{2}})^{\otimes (a-u)})$$

and projectors

$$\mu_a : V^{\otimes 2a} \rightarrow V^a$$

so that

$$\mu_a \circ \phi_a = \text{Id}_a \quad \text{and} \quad \phi_a \circ \mu_a \circ \text{JW}_{2a} = \text{JW}_{2a}.$$

These operators allow to define the morphisms associated to a trivalent vertex

$$Y_c^{a,b} : V^c \longrightarrow V^a \otimes V^b$$

by

$$\begin{array}{c} a \quad b \\ \diagdown \quad / \\ | \\ c \end{array} \doteq Y_c^{a,b} \doteq (\mu_a \otimes \mu_b) \circ (\text{JW}_{2a} \otimes \text{JW}_{2b}) \circ \begin{array}{c} 2a \quad 2b \\ \cup \\ | \quad | \\ 2c \end{array} \circ \text{JW}_{2c} \circ \phi_c$$

¹See [3], Definition 3.5.6, and recall that our base is related to that of [3] by a diagonal change

$$g_u^a = \frac{e_u^a}{[a+u]}.$$

where the drawing on the right represents the morphism

$$(V^{\frac{1}{2}})^{\otimes 2c} \longrightarrow (V^{\frac{1}{2}})^{\otimes 2a} \otimes (V^{\frac{1}{2}})^{\otimes 2b}$$

obtained by tensoring suitably $\cup_{\frac{1}{2}}$ and identity maps. Similarly one can define maps

$$P_{a,b}^c : V^a \otimes V^b \longrightarrow V^c$$

by

$$\begin{array}{c} c \\ | \\ a \text{---} \text{---} b \end{array} \doteq P_{a,b}^c \doteq \mu_c \circ \text{JW}_{2c} \circ \begin{array}{c} 2c \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ 2a \quad \text{---} \quad 2b \end{array} \circ (\text{JW}_{2a} \otimes \text{JW}_{2b}) \circ (\phi_a \otimes \phi_b).$$

From these, one gets new morphisms

$$\cup_a : V^0 \longrightarrow V^a \otimes V^a$$

(as $Y_0^{(a,a)}$) and

$$\cap_a : V^a \otimes V^a \longrightarrow V^0$$

(as $P_{a,a}^0$), as well as

$$W^{a,b,c} : V^0 \longrightarrow V^a \otimes V^b \otimes V^c$$

(as $(\text{Id}_a \otimes Y_a^{b,c}) \circ \cup_a$) and

$$M_{a,b,c} : V^a \otimes V^b \otimes V^c \longrightarrow V^0$$

(as $\cap_c \circ (P_{a,b}^c \otimes \text{Id}_c)$) (see Figure 2).

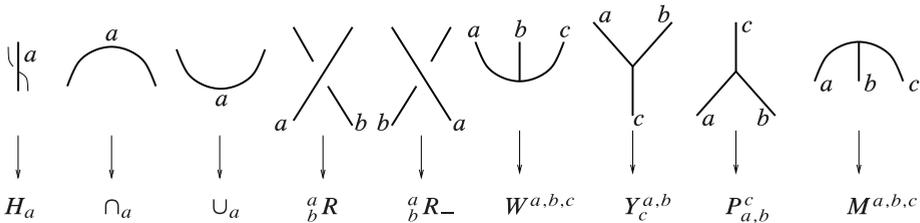
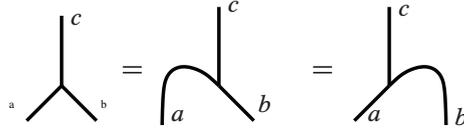


Figure 2. The elementary graphs and the associated morphisms. In all the drawings except the leftmost, the framing is the blackboard framing.

Composing these elementary morphisms one can associate morphisms to any diagram of a planar colored trivalent tangle.

Proposition 2.12. *The following morphisms $V^a \otimes V^b \rightarrow V^c$ coincide:*

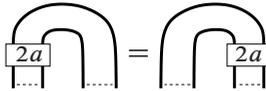


i.e. we have

$$P_{a,b}^c = (\text{Id}_c \otimes \cap_b) \circ Y_a^{c,b} = (\cap_a \otimes \text{Id}_c) \circ Y_b^{a,c}.$$

The proof is a direct consequence of isotopy invariance of morphisms induced by framed tangles colored by $\frac{1}{2}$, the definitions of $P_{a,b}^c$ and $Y_c^{a,b}$, the identities $\text{JW}_{2a}^2 = \text{JW}_{2a}$ and the following ones.

Lemma 2.13. *The morphisms from $(V^{\frac{1}{2}})^{\otimes 4a} \rightarrow V^0$ represented by the following diagrams coincide:*



Proof. Each minimal positive braid in the definition of JW_{2a} of the left hand side, can be slid through an isotopy over the max to a minimal positive braid in the definition of JW_{2a} in the right hand side. The morphisms two such braids induce are the same because the tangle they are represented by are isotopic, and their coefficients in the sum expressing JW_{2a} are the same because they contain the same number of crossings. \square

One also defines operators ${}^a_b R$ (resp. ${}^b_a R_-$) $V^a \otimes V^b \rightarrow V^b \otimes V^a$ associated to a colored positive (resp. negative) crossing as

$${}^a_b R \doteq (\mu_b \otimes \mu_a) \circ \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \hline \end{array} \circ (\phi_a \otimes \phi_b)$$

and

$${}^b_a R_- \doteq (\mu_b \otimes \mu_a) \circ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \hline \end{array} \circ (\phi_a \otimes \phi_b)$$

where the diagrams represent $2a$ parallel strands passing over (resp. under) $2b$ parallel strands.

Proposition 2.15. *It holds*

$$C_{u,v,t}^{a,b,c} = \sqrt{-1}^{c-a-b} q^{C_1} f_1(-1)^{\chi_1} \sum_{z,w: z+w=c-t} (-1)^z q^{C_2} Q_1, \quad (3)$$

where

$$C_1 = C_1(a, b, c, u, v) = \frac{(b-v)(b+v+1) - (a-u)(a+u+1)}{2},$$

$$C_2 = C_2(c, t, w, z) = \frac{(z-w)(c+t+1)}{2}$$

$$\chi_1 = \chi_1(b, c, t, v) = (b-v) - (c-t)$$

$$f_1 = f_1(a, b, c) = \frac{[a+b-c]![b+c-a]![c+a-b]!}{[2c]!},$$

$$Q_1 = Q_1(a, b, c, t, u, v, w, z) = \begin{bmatrix} a+u+z \\ a+c-b \end{bmatrix} \begin{bmatrix} b+v+w \\ b+c-a \end{bmatrix} \begin{bmatrix} c-t \\ z \end{bmatrix},$$

and where the sum is taken over all $z, w \in \mathbb{N}$ such that $z+w=c-t$ and all the arguments of the factorials are non-negative integers.

Proof. In [3], Lemma 3.6.10, using the basis $e_u^a = [a+u]!g_u^a$ for V^a , the following formula was provided for the Clebsch–Gordan coefficients (where we are rewriting the formula via q -binomials and correcting a missing factor of $\sqrt{-1}^{(t-c)}$):

$$C_{u,v,t}^{a,b,c} = \sqrt{-1}^{c-a-b} (-1)^{\chi_1} q^{C_1} f_2 \sum_{z,w: z+w=c-t} (-1)^z q^{C_2} Q_2, \quad (4)$$

where

$$f_2 = f_2(c, t) = \frac{[c+t]![c-t]!}{[2c]!},$$

$$Q_2 = Q_2(a, b, c, u, v, w, z) = \begin{bmatrix} a+b-c \\ a-u-z \end{bmatrix} \begin{bmatrix} a+u+z \\ z \end{bmatrix} \begin{bmatrix} b+v+w \\ w \end{bmatrix},$$

and where the sum is taken over all z, w such that $z+w=c-t$ and all the arguments of the quantum factorials are non-negative integers. To get our statement it is then sufficient to multiply by $\frac{[a+u]![b+v]!}{[c+t]!}$ (to operate the change of basis from e_u^a to g_u^a), to single out of the factorials the terms $\frac{[a+b-c]![b+c-a]![c+a-b]!}{[2c]!}$ and to pair the factorials in the denominators of the summands so that their sums match $a+u+z$, $b+v+w$ and $c-t$ (recall that $u+v=t$). \square

2.4.3. Cup and Cap. Let

$$\cup_a: \mathbb{C} = V^0 \longrightarrow V^a \otimes V^a$$

be defined as $Y_0^{a,a}$. An explicit computation using (3) gives in the base g_u^a

$$\cup_a(g_0^0) = \sum_{u=-a}^a [2a]! \sqrt{-1}^{2u} q^u g_u^a \otimes g_{-u}^a. \quad (5)$$

This, together with the invariance under isotopy which forces the identity

$$(\cap_a \otimes \text{Id}_a) \circ (\text{Id}_a \otimes \cup_a) = \text{Id}_a$$

uniquely determines

$$\cap_a: V^a \otimes V^a \longrightarrow V^0$$

(defined as $P_{a,a}^0$) as

$$\cap_a(g_u^a \otimes g_v^a) = \delta_{u,-v} \frac{\sqrt{-1}^{2u} q^u}{[2a]!} g_0^0. \quad (6)$$

2.4.4. Morphisms associated to 3-valent vertices. To compute the coefficients of the projectors

$$P_{a,b}^c: V^a \otimes V^b \longrightarrow V^c$$

out of $Y_c^{a,b}$ we use Proposition 2.12. So letting

$$P_{a,b}^c(g_u^a \otimes g_v^b) = \sum_t P_{u,v,t}^{a,b,c} g_t^c,$$

it holds

$$P_{u,v,t}^{a,b,c} = C_{-u,t,v}^{a,c,b} \frac{\sqrt{-1}^{2u} q^u}{[2a]!} \quad (7)$$

Similarly, “non-smooth minima” operators

$$W^{a,b,c} \in \text{Hom}(V^0, V^a \otimes V^b \otimes V^c)$$

are defined by “pulling up the bottom leg in $Y_c^{a,b}$,” i.e. by setting

$$W^{a,b,c} \doteq (Y_c^{a,b} \otimes \text{Id}_c) \circ \cup_c.$$

So, letting

$$W^{a,b,c}(g_0^0) \doteq \sum_{u=-a}^a \sum_{v=-b}^b \sum_{t=-c}^c W_{u,v,t}^{a,b,c} g_u^a \otimes g_v^b \otimes g_t^c$$

the coefficients are

$$\begin{aligned} W_{u,v,t}^{a,b,c} &\doteq C_{u,v,-t}^{a,b,c} \sqrt{-1}^{-2t} q^{-t} [2c]! \\ &= \sqrt{-1}^{-a-b-c} (-1)^{\chi_2} q^{C_1} f_3 \sum_{z,w: z+w=c+t} (-1)^z q^{C_3} Q_3, \end{aligned} \quad (8)$$

where

$$\chi_2 = (b - v) - 2t,$$

$$f_3 = [a + b - c]![b + c - a]![c + a - b]!,$$

$$Q_3 = Q_1(a, b, c, -t, u, v, w, z) = \begin{bmatrix} a + u + z \\ a + c - b \end{bmatrix} \begin{bmatrix} b + v + w \\ b + c - a \end{bmatrix} \begin{bmatrix} c + t \\ z \end{bmatrix},$$

and

$$C_3 = -t + C_2(c, -t, w, z) = -t + \frac{(z - w)(c - t + 1)}{2}.$$

Finally,

$$M^{a,b,c} \in \text{Hom}(V^a \otimes V^b \otimes V^c, V^0)$$

defined by

$$M^{a,b,c} \doteq \cap_c \circ (P_{a,b}^c \otimes \text{Id}_c)$$

has coefficients

$$M_{u,v,t}^{a,b,c} \doteq M^{a,b,c}(g_u^a \otimes g_v^b \otimes g_t^c) = P_{u,v,-t}^{a,b,c} \frac{\sqrt{-1}^{-2t} q^{-t}}{[2c]!}. \quad (9)$$

2.4.5. R-matrix. A positive crossing corresponds to the action of Drinfeld's universal R -matrix.

Lemma 2.16. *The morphism ${}_b^a R: V^a \otimes V^b \rightarrow V^b \otimes V^a$ is given by the composition of Drinfeld's universal R -matrix with the flip of the coordinates and in the basis $g_u^a \otimes g_v^b$ it is*

$$R(g_u^a \otimes g_v^b) = \sum_{n \geq 0} [n]!(q - q^{-1})^n Q_4 q^{C_5} g_{v-n}^b \otimes g_{u+n}^a \quad (10)$$

where

$$Q_4 = Q_4(a, n, u) = \begin{bmatrix} a - u \\ n \end{bmatrix} \begin{bmatrix} a + u + n \\ n \end{bmatrix},$$

$$C_5 = C_5(u, v, n) = 2uv - n(u - v) - \frac{n(n + 1)}{2},$$

and where the sum is taken over all the n such that $|u + n| \leq a$ and $|v - n| \leq b$. We will denote ${}_b^a R_{u,v}^{h,k}$ the coefficient of $R(g_u^a \otimes g_v^b)$ with respect to $g_h^b \otimes g_k^a$.

Proof. The first statement is a direct consequence of the definition of ${}^a_b R$ and of the fact that $\frac{1}{2} R$ coincides with the composition of the action of Drinfeld's R -matrix and the flip. To compute the explicit entries of ${}^a_b R$, we use the formulas provided in [10] (Corollary 2.32: recall that $\bar{t} = q^{-\frac{1}{2}}$) in the basis f_m^j and the diagonal change of basis $f_m^j = \frac{[2j]!}{[j-m]!} g_m^j$:

$$\begin{aligned} R(g_u^a \otimes g_v^b) &= \sum_{n \geq 0} \frac{(q - q^{-1})^n}{[n]!} f_4 q^{C_5} g_{v-n}^b \otimes g_{u+n}^a \\ &= \sum_{n \geq 0} [n]! (q - q^{-1})^n Q_4 q^{C_5} g_{v-n}^b \otimes g_{u+n}^a \end{aligned}$$

where

$$\begin{aligned} f_4 &= f_4(a, b, n, u, v) \\ &= \frac{[a + u + n]! [b - v + n]! [a - u]! [b - v]!}{[a + u]! [b - v]!} \frac{[2a]! [2b]!}{[2a]! [2b]! [a - u - n]! [b - v + n]!}. \quad \square \end{aligned}$$

The morphism associated to a negative crossing whose upper strand is colored by a and whose lower strand is colored by b is the inverse of ${}^a_b R$ and can be computed in terms of the one we just gave and two extrema:

$$(R_-) \doteq (\text{Id}_a \otimes \text{Id}_b \otimes \cap_a) \circ (\text{Id}_a \otimes {}^a_b R \otimes \text{Id}_a) \circ (\cup_a \otimes \text{Id}_b \otimes \text{Id}_a).$$

An explicit formula is then computed out of (10), (5) and (6)):

$${}^a_b R_-(g_v^b \otimes g_u^a) = \sum_{n \geq 0} [n]! (q^{-1} - q)^n q^{-C_5} Q_4 g_{u+n}^a \otimes g_{v-n}^b. \quad (11)$$

where the sum is taken over all the n such that $|u + n| \leq a$ and $|v - n| \leq b$. Remark that ${}^a_b R_- = {}^a_b R^{-1}$ because ${}^a_b R_- \circ {}^a_b R = \text{Id}_a \otimes \text{Id}_b$. We will denote ${}^a_b (R^{-1})_{u,v}^{h,k}$ the coefficient of $R^{-1}(g_u^a \otimes g_v^b)$ with respect to $g_h^b \otimes g_k^a$. The following well-known lemma relates R -matrices to Y -morphisms.

Lemma 2.17 (half-twist around a vertex). *For every admissible 3-uple (a, b, c) it holds*

$$\begin{array}{c} \text{b} \curvearrowright \\ \circlearrowleft \\ \text{c} \end{array} \text{a} = {}^a_b R \circ Y_c^{a,b} = (H_b^{-1} \otimes H_a^{-1}) \circ Y_c^{b,a} \circ (H^c) = \begin{array}{c} \text{b} \curvearrowright \\ \text{c} \text{---} \text{a} \end{array}$$

Proof. It is sufficient to prove the equality for the highest weight vector $g_c^c \in V^c$; to do this it is sufficient to check that ${}^a_b R \circ Y_c^{a,b}(g_c^c)$ and $(H_b^{-1} \otimes H_a^{-1}) \circ Y_c^{b,a} \circ (H^c)(g_c^c)$ have the same coefficient with respect to the element $g_b^b \otimes g_{c-b}^{c-b}$ of the basis $g_j^b \otimes g_i^a$, $|i| \leq a$, $|j| \leq b$ of $V^b \otimes V^a$. By (10) and (3) this coefficient is for the left hand side

$$\begin{aligned} \sum_{u+v=c} {}^a_b R_{u,v}^{b,c-b} \times C_{u,v,c}^{a,b,c} &= {}^a_b R_{c-b,b}^{b,c-b} \times C_{c-b,b,c}^{a,b,c} \\ &= q^{C_6} \times \sqrt{-1}^{c-a-b} (-1)^0 q^{-C_7} f_5, \end{aligned}$$

where

$$f_5 = f_5(a, b, c) = \frac{[2b]![a+c-b]!}{[2c]!},$$

$$C_6 = C_6(b, c) = 2(c-b)b,$$

and

$$C_7 = C_7(a, b, c) = \frac{1}{2}(a - (c - b))(a + (c - b) + 1).$$

The coefficient on the right hand side is $(H_b^b)^{-1}(H_a^a)^{-1}C_{b,c-b,c}^{b,a,c}(H^c)$ which equals

$$\sqrt{-1}^{3c-3a-3b} q^{C_8} (-1)^{a-(c-b)} q^{C_7} f_5 = \sqrt{-1}^{c-b-a} q^{C_8} q^{C_7} f_5$$

where

$$C_8 = C_8(a, b, c) = c^2 + c - (a^2 + a) - (b^2 + b).$$

A straightforward computation shows that the two coefficients are indeed equal. \square

2.5. The state-sum computing $\langle G, \text{col} \rangle$. Let G be a closed KTG, E be the set of its edges, V the set of its vertices and $\text{col}: E \rightarrow \frac{\mathbb{N}}{2}$ an admissible coloring. Let also D be a diagram of G and for every $e \in E$, let $g_e \in \frac{\mathbb{N}}{2}$ be the difference between the framing of e in G and the blackboard framing on it (it is half integer because the two framings may differ of an odd number of half twists). Let C, M, N be respectively the set of crossings, maxima and minima in D (recall that we are fixing a height function on \mathbb{R}^2 to decompose D into elementary subgraphs). Then let f_1, \dots, f_n be the connected components of $D \setminus (V \cup M \cup N \cup C)$. Remark that each f_j is a substrand of an edge of G therefore it inherits a color which we will denote c_j . To express $\langle G, \text{col} \rangle$ as a state-sum, let us first define a state.

Definition 2.18 (States). A *state* is a map $s: \{f_1, \dots, f_n\} \rightarrow \frac{\mathbb{Z}}{2}$ such that $|s(f_j)| \leq c_j$, for all $j = 1, \dots, n$ and $s(f_j)$ is integer if and only if c_j is. Given a state s , we will call the value $s(f_j)$ the *state of f_j* . (Equivalently, a state is a choice of one vector $g_{s(f_j)}^{c_j}$ of the base of V^{c_j} for each component of $D \setminus (V \cup M \cup N \cup C)$.)

The *weight* $w(s)$ of a state s is the product of a factor $w_s(x)$ per each x crossing, vertex, maximum and minimum of D . To define these factors, in Table 1 use the letters a, b, c for the colors of the strands and u, t, v, w for their states.

Table 1

$w_s\left(\begin{array}{c} \text{arc} \\ u \quad a \quad v \end{array}\right) = \delta_{u,-v} \frac{\sqrt{-1}^{2u} q^u}{[2a]!}$	$w_s\left(\begin{array}{c} \text{arc} \\ u \quad a \quad v \end{array}\right) = \delta_{v,-u} [2a]! \sqrt{-1}^{2u} q^u$
$w_s\left(\begin{array}{c} \text{crossing} \\ t \quad w \\ a \quad b \\ u \quad v \end{array}\right) = \frac{a}{b} R_{u,v}^{t,w}$ (see (10))	$w_s\left(\begin{array}{c} \text{crossing} \\ t \quad w \\ a \quad b \\ u \quad v \end{array}\right) = \frac{a}{b} (R^{-1})_{u,v}^{t,w}$ (see (11))
$w_s\left(\begin{array}{c} \text{Y-junction} \\ a \quad b \\ c \end{array}\right) = C_{u,v,t}^{a,b,c}$ (see (3))	$w_s\left(\begin{array}{c} \text{Y-junction} \\ c \\ a \quad b \end{array}\right) = P_{u,v,t}^{a,b,c}$ (see (7))
$w_s\left(\begin{array}{c} \text{X-junction} \\ u \quad v \quad t \\ a \quad b \quad c \end{array}\right) = W_{u,v,t}^{a,b,c}$ (see (8))	$w_s\left(\begin{array}{c} \text{X-junction} \\ a \quad b \quad c \\ u \quad v \quad t \end{array}\right) = M_{u,v,t}^{a,b,c}$ (see (9))

Finally, to take into account the action of the half-twist operators H_j on the edges of G , let

$$F(D, \text{col}) \doteq \prod_{e \in \text{edges}} \sqrt{-1}^{4g_e \text{col}(e)} q^{2g_e(\text{col}(e)^2 + \text{col}(e))}$$

be the *framing factor* (note that it does not depend on any state but it does depend on the diagram D). The weight of a state s of $\langle G, \text{col} \rangle$ is then defined as

$$w(s) = F(D, \text{col}) \prod_{M \in \text{maxima}} w_s(M) \prod_{m \in \text{minima}} w_s(m) \prod_{c \in \text{crossings}} w_s(c) \prod_{v \in \text{vertices}} w_s(v) \quad (12)$$

The value of $\langle G, \text{col} \rangle$ is then given by

$$\langle G, \text{col} \rangle = \sum_{s \in \text{states}} w(s) \quad (13)$$

since the state-sum represents nothing else than the composition of the elementary morphisms associated to G as a morphism

$$\text{op}(G, \text{col}, D): V^0 \longrightarrow V^0.$$

Remark 2.19. The above state-sum shows that one can extend this way the definition of Kauffman brackets to colored KTG's whose framing is a non-orientable surface: diagrams of such KTG's will always contain some half-twists which will contribute through a constant multiplicative factor (included in $F(D, \text{col})$).

Example 2.20. If $L = L_1, \dots, L_n$ is an unlink with (possibly half-integral) framings g_1, \dots, g_n and colored by colors c_1, \dots, c_n then

$$\langle L, \text{col} \rangle = \prod_{i=1}^n (-1)^{2c_i} [2c_i + 1] (\sqrt{-1})^{4g_i c_i} q^{2g_i(c_i^2 + c_i)}$$

It is sufficient to prove it for the case of an unknot colored by c_j and with framing g_j . In that case the value is the trace of the operator $\cap_{c_j} \circ (\text{Id} \otimes H_{c_j}^{2g_j}) \circ \cup_{c_j}$ which equals

$$\begin{aligned} \text{tr}(\cap_{c_j} \circ (\text{Id} \otimes H_{c_j}^{2g_j}) \circ \cup_{c_j}) &= (\sqrt{-1})^{4g_j c_j} q^{2g_j(c_j^2 + c_j)} \sum_{u=-c_j}^{c_j} \sqrt{-1}^{4u} q^{2u} \\ &= (\sqrt{-1})^{4g_j c_j} q^{2g_j(c_j^2 + c_j)} (-1)^{2c_j} \sum_{u=-c_j}^{c_j} q^{2u} \\ &= (-1)^{2c_j} [2c_j + 1] (\sqrt{-1})^{4g_j c_j} q^{2g_j(c_j^2 + c_j)}. \end{aligned}$$

Example 2.21 (the theta graph). Consider a theta graph as in Figure 3. Equip it with the blackboard framing and color the edges by V^a, V^b and V^c . Its invariant is then

$$\theta(a, b, c) \doteq \sum_{m_1=-a}^a \sum_{m_2=-b}^b \sum_{m=-c}^c W_{m_1, m_2, m}^{a, b, c} M_{m_1, m_2, m}^{a, b, c} \tag{14}$$

By Proposition 2.10 the above formula equals the standard skein theoretical value:

$$\theta(a, b, c) = (-1)^{a+b+c} \frac{[a + b + c + 1]! [a + b - c]! [a + c - b]! [b + c - a]!}{[2a]! [2b]! [2c]!} \tag{15}$$

Remark 2.22. The above example shows that in general $\langle G, \text{col} \rangle$ is not a Laurent polynomial: consider for instance the case $a = b = c = 1$ in (15).

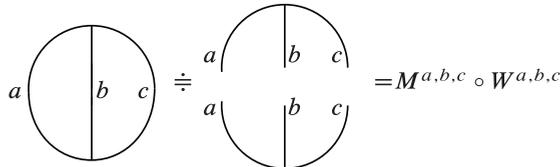


Figure 3. The operator associated to a colored theta graph.

3. Integrality

Let G be a closed KTG, E be the set of its edges, V the set of its vertices and $\text{col}: E \rightarrow \frac{\mathbb{N}}{2}$ an admissible coloring. We define the *Euler characteristic* $\chi(e)$ of an edge $e \in E$ as 1 if e touches a vertex and 0 otherwise (some edges of G may be knots). Let also D be a diagram of G and for every $e \in E$, let $g_e \in \frac{\mathbb{N}}{2}$ be the difference between the framing of e in G and the blackboard framing on it (it is half integer because the two framings may differ of an odd number of half twists). We define a renormalization for $\langle G, \text{col} \rangle$ as

$$\langle\langle G, \text{col} \rangle\rangle \doteq \langle G, \text{col} \rangle \frac{\prod_{e \in E, \chi(e)=1} [2 \text{col}(e)]!}{\prod_{v \in V} [a_v + b_v - c_v]! [b_v + c_v - a_v]! [c_v + a_v - b_v]!},$$

where by a_v, b_v, c_v we denote the colors of the three edges surrounding v .

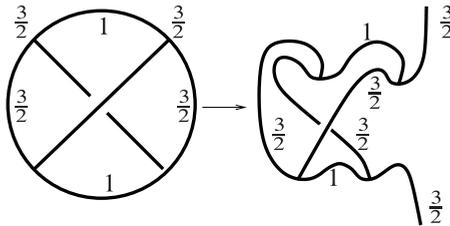
Remark 3.1. The renormalization factor depends only on the abstract combinatorial structure of $\langle G, \text{col} \rangle$, therefore $\langle\langle G, \text{col} \rangle\rangle$ is an invariant of colored KTG's.

Theorem 3.2 (Integrality of the renormalized Kauffman brackets). *There exist $m, n \in \mathbb{Z}$ such that*

$$\langle\langle G, \text{col} \rangle\rangle \in (\sqrt{-1})^m q^{\frac{n}{4}} \mathbb{Z}[q, q^{-1}]$$

Moreover, $\langle\langle G, \text{col} \rangle\rangle (\sqrt{-1})^{-m} q^{-\frac{n}{4}}$ is divisible in $\mathbb{Z}[q, q^{-1}]$ by $[2 \text{col}(e) + 1]$ for each edge e of G . If the framing of G is orientable, then $m = 0$ and n is even.

Proof. Up to isotopy, we can suppose that the diagram D of G is the closure of a $(1, 1)$ -tangle G' whose boundary strands are included in the same edge e and also (by small isotopies around vertices and crossings) that D contains only maxima, minima, positive crossings and vertices with 3 top legs. *As an example (written in italic to allow the reader to skip the example part easily), we will follow the general proof on the leftmost graph of Example 3.8, with a fixed choice of a coloring; this first step amounts to the operation*



The factor

$$F(D, \text{col}) = \sqrt{-1}^{\sum_{e \in E} -4g_e \text{col}(e)} \prod_{e \in E} q^{-2g_e (\text{col}(e)^2 + \text{col}(e))}$$

in the state-sum (12)) changes the value of $\langle G, \text{col} \rangle$ only by a factor of the form $(\sqrt{-1})^k q^{\frac{h}{4}}$, $k, h \in \mathbb{Z}$, therefore, up to dividing by $F(D, \text{col})$ we may suppose that the framing of G is the blackboard framing. *In the above example the factor is 1 as we started with a graph framed by the blackboard framing.* By Schur's lemma the morphism represented by G' is λId_{V^n} . We claim that there exists an integer s such that

$$\mu \doteq \lambda \frac{\prod_{e \in E, \chi(e)=1} [2 \text{col}(e)]!}{\prod_{v \in V} [a_v + b_v - c_v]! [b_v + c_v - a_v]! [c_v + a_v - b_v]!}$$

belongs to $q^{\frac{s}{2}} \mathbb{Z}[q, q^{-1}]$; this will conclude because

$$\langle \langle G, \text{col} \rangle \rangle = (-1)^{2 \text{col}(e)} [2 \text{col}(e) + 1] \mu.$$

In our example, this is to say that

$$\lambda \frac{[3]!^4 [2]!^2}{([2]! [1]! [1]!)^4} \in q^{\frac{s}{2}} \mathbb{Z}[q, q^{-1}]$$

for some integer s .

To prove our claim let us define “renormalized operators” associated to each maximum, minimum, crossing and vertex of D equipped with a state as

$$\begin{aligned} NH_j &\doteq H_j, \\ (N \cup_a)_{u,v} &\doteq \delta_{u,-v} \frac{1}{[2a]!} \begin{array}{c} u \quad v \\ \cup \quad a \end{array}, \\ (N \cap_a)_{u,v} &\doteq \delta_{u,-v} [2a]! \begin{array}{c} u \quad v \\ \cap \quad a \end{array}, \\ ({}^a N R)_{u,v}^{t,w} &:= {}^a R_{u,v}^{t,w}, \\ (NW)_{u,v,t}^{a,b,c} &\doteq \frac{W_{u,v,t}^{a,b,c}}{[a+b-c]! [b+c-a]! [c+a-b]!}. \end{aligned}$$

Since the morphism represented by G' is diagonal, the only non-zero weight states are those where the states of the top and bottom strand of G' are equal. Therefore, if in (13)) one fixes the same state u on the top and bottom strand of G' and replaces each weight by its “normalized version” defined above, the result will be

$$\langle \langle G', \text{col} \rangle \rangle = \langle G', \text{col} \rangle \times \frac{\prod_{e \in E} ([2 \text{col}(e)]!)^{(\text{cap}(e) - \text{cup}(e))}}{\prod_{v \in V} [a_v + b_v - c_v]! [b_v + c_v - a_v]! [c_v + a_v - b_v]!},$$

where for each edge, $\text{cup}(e)$ (resp. $\text{cap}(e)$) are the number of minima (resp. maxima) on e . The above formula coincides with normalization factor as in the claim since by our hypothesis on D all the vertices have 3 top legs and so for each edge e different from the top strand it holds $\chi(e) = \text{cap}(e) - \text{cup}(e)$, and that for the top

strand $\text{cap}(e) - \text{cup}(e) = 0$ (the top and bottom strands are part of the same edge in G therefore only one of them should be counted in the renormalization). *In our example this is self-evident: every edge of the open graph contains exactly one maximum and no minimum, with the only exception of the topmost edge which contains no maxima or minima. For the sake of completeness, let us also fix the state $u = \frac{3}{2}$ on the open strands.*

Remark now that the coefficients of each “renormalized operator” belong to $\mathbb{Z}[\sqrt{-1}][q^{\pm\frac{1}{2}}]$. This is straightforward because of Lemma 3.4 and (10) and (8). Let us first show that actually all the coefficients are non-imaginary. Let us remark that in the state-sum for $\langle\langle G, \text{col} \rangle\rangle$, since each edge with $\chi(e) = 1$ has two endpoints and each $NW^{a,b,c}$ belongs in particular to $\sqrt{-1}^{-a-b-c} \mathbb{Z}[q^{\pm\frac{1}{2}}]$, the product of the factors $\sqrt{-1}^{\text{col}(e)}$ coming from these vertices is $\sqrt{-1}^{-2\text{col}(e)}$. Similarly the product of the factors $\sqrt{-1}^{-2u_i}$ coming from the cups and caps on e is $\pm\sqrt{-1}^{2\text{col}(e)(\text{cap}(e)-\text{cup}(e))} = \pm\sqrt{-1}^{2\text{col}(e)}$ (because each u_i is a half integer if and only if $\text{col}(e_i)$ is) and this cancels with the previous imaginary phase.

In our example, the operators $NW^{\frac{3}{2},\frac{3}{2},1}$ have coefficients in $\sqrt{-1}^{-\frac{3}{2}-\frac{3}{2}-1} \mathbb{Z}[q^{\pm\frac{1}{2}}]$ (this is seen by inspecting (8)) and each edge (except the top-most) contains one maximum whose associated operator is either \cap_1 or $\cap_{\frac{3}{2}}$. In particular, \cap_1 has coefficients in $\sqrt{-1}^{2t} [2]! \mathbb{Z}[q^{\pm 1}]$, $t \in \{-1, 0, 1\}$ (see (6)) and so in $\sqrt{-1}^2 \mathbb{Z}[q^{\pm 1}]$ and $\cap_{\frac{3}{2}}$ has coefficients in $[3]! \sqrt{-1}^{2t} \mathbb{Z}[q^{\pm 1}]$, $t \in \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$ and so in $\sqrt{-1}^3 \mathbb{Z}[q^{\pm 1}]$. Now we can redistribute the $\sqrt{-1}$ factors on each edge so that the $\frac{3}{2}$ -colored edges get $2\sqrt{-1}^{\frac{3}{2}}$ factors from the $NW^{\frac{3}{2},\frac{3}{2},1}$ operators at their endpoints, and one $\sqrt{-1}^3$ from the operator $\cap_{\frac{3}{2}}$ at their maximum, and so the overall imaginary factor associate to each $\frac{3}{2}$ -colored edge is 1; the same holds for the other 2 edges.

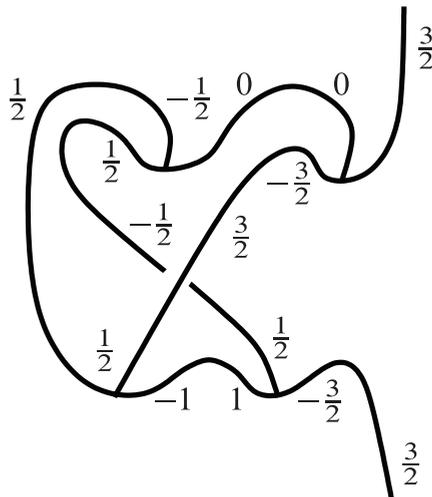
So now we are left to show that $\langle\langle G, \text{col} \rangle\rangle$ contains only all odd or all even powers of $q^{\frac{1}{2}}$. First of all remark that this is the case for the coefficients $NW_{u,v,t}^{a,b,c}$, $({}^a_b NR)_{u,v}^{v-n,u+n}$, $(N \cap_a)_{u,v}$, and $(N \cup_a)_{u,v}$; more specifically, an inspection of (8) and (10), reveals that the parities ($\in \frac{\mathbb{Z}}{2\mathbb{Z}}$) of the powers of $q^{\frac{1}{2}}$ in these coefficients are (beware: states may be half-integers but the values below are integers, then considered mod 2):

- in $NW_{u,v,t}^{a,b,c}$: $(a^2 + a - u^2 - u) + (b^2 + b - v^2 - v) + (c^2 + c - t^2 - t)$;
- in $({}^a_b NR)_{u,v}^{v-n,u+n}$: $i(a)n + i(b)n + i(a)i(b)$ where $i(x) \doteq 1$ if $2x$ is odd and 0 otherwise;
- in $(N \cup_a)_{u,v}$ and $(N \cap_a)_{u,v}$: $2a$ (not depending on any state).

Hence for each state s in the state-sum expressing $\langle\langle G, \text{col} \rangle\rangle$ the weight $w(s)$ contains either only even or only odd powers of $q^{\frac{1}{2}}$ because it is (by definition) a product of the above factors; we will therefore call s *even* or *odd* accordingly. Our goal is to

show that all the states have the same parity: for instance remark that if col has integers values, all the states are even. For each state s we will compute its parity “redistributing” on the edges the parities of the coefficients of the elementary operators and then summing them up over all the edges. For each edge e of G with $\chi(e) = 1$, orient e arbitrarily and let u and v the states of the substrands of e respectively at the beginning and at the end of e ; let the contributions of the endpoints to the parity of e be $2(\text{col}(e)^2 + \text{col}(e)) - u^2 - u - v^2 - v$. Similarly each \cup or \cap in e contributes by $2 \text{col}(e)$ and since e contains an odd number of such operators they contribute globally by $2 \text{col}(e)$. Finally to take into account the crossings, follow e and remark that each time e crosses another edge, say e' , the state on the substrand of e jumps from x to $x \pm n$ ($n \in \mathbb{Z}$) and the parity of the powers of $q^{\frac{1}{2}}$ in the R -matrix corresponding to the crossing is $ni(\text{col}(e)) + ni(\text{col}(e')) + i(\text{col}(e))i(\text{col}(e'))$; so we define the contribution of the crossing to the parity of e as $ni(\text{col}(e))$, dropping for the moment the term $i(\text{col}(e))i(\text{col}(e'))$ which does not depend on the state. Using the fact that on each $\cup_{\text{col}(e)}$ and $\cap_{\text{col}(e)}$ the state of the substrand of e changes sign (see (5) and (6)) and following e from its beginning to its end, one can check that the global contribution to the parity of e coming from the crossings is $-(u + v)i(\text{col}(e))$. Therefore summing up the parity of e is $2(\text{col}(e)^2 + \text{col}(e)) - u^2 - u - v^2 - v + 2 \text{col}(e) - (u + v)i(\text{col}(e))$, which modulo 2 is $2 \text{col}(e)^2 - u^2 - v^2$ if $i(\text{col}(e)) = 1$ or 0 if $i(\text{col}(e)) = 0$. In both cases it is constant mod 2 when u and v range in $\{-\text{col}(e), -\text{col}(e) + 1, \dots, \text{col}(e)\} \subset \frac{\mathbb{Z}}{2}$. Similarly, for edges with $\chi(e) = 0$ the parity is easily seen to be constant. Therefore the parity of the states is constant because it is the sum of constant contributions coming from the edges and the constant term $C = \sum_{\text{crossings}} i(a)i(b)$ (where a and b are the colors of the strands forming the crossing).

To decline the above arguments in our example, let us fix a state s on our graph as follows (recall Definition 2.18):



The contribution of s to the state sum is

$$w(s) = NW_{\frac{1}{2}, \frac{1}{2}, -1}^{\frac{3}{2}, \frac{3}{2}, 1} NW_{1, \frac{1}{2}, -\frac{3}{2}}^{1, \frac{3}{2}, \frac{3}{2}} NW_{\frac{1}{2}, -\frac{1}{2}, 0}^{\frac{3}{2}, \frac{3}{2}, 1} NW_{-\frac{3}{2}, 0, \frac{3}{2}}^{\frac{3}{2}, 1, \frac{3}{2}} \quad (16)$$

$$\left(\frac{3}{2}NR\right)_{\frac{1}{2}, \frac{1}{2}}^{-\frac{1}{2}, \frac{3}{2}} \cap_{-\frac{1}{2}, \frac{1}{2}}^{\frac{3}{2}} \cap_{\frac{3}{2}, -\frac{3}{2}}^{\frac{3}{2}} \cap_{-\frac{3}{2}, \frac{3}{2}}^{\frac{3}{2}} \cap_{\frac{1}{2}, -\frac{1}{2}}^{\frac{3}{2}} \cap_{0, 0}^1 \cap_{-1, 1}^1$$

Using (8) and (10), and reducing mod $\mathbb{Z}[q^{\pm 1}]$, so that for instance

$$NW_{\frac{1}{2}, \frac{1}{2}, -1}^{\frac{3}{2}, \frac{3}{2}, 1} \equiv kq^{\frac{3+3+2}{2}}$$

and

$$\left(\frac{3}{2}NR\right)_{\frac{1}{2}, \frac{1}{2}}^{-\frac{1}{2}, \frac{3}{2}} = [3](q - q^{-1})q^{\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - 1} \equiv hq^{C_9}$$

for some $k, h \in \mathbb{Z}$, where

$$C_9 = \frac{i\left(\frac{3}{2}\right)i\left(\frac{3}{2}\right)}{2} + \frac{i\left(\frac{3}{2}\right)}{2} + \frac{i\left(\frac{3}{2}\right)}{2},$$

we have

$$w(s) \equiv lq^{\frac{3+3+2}{2}} q^{\frac{0+3+3}{2}} q^{\frac{3+4+2}{2}} q^{\frac{3+2+0}{2}} q^{C_9} q^{\frac{3}{2}} q^{\frac{3}{2}} q^{\frac{3}{2}} q^{\frac{3}{2}} q^1 q^1 \pmod{\mathbb{Z}[q^{\pm 1}]}$$

for some $l \in \mathbb{Z}$, where we wrote the exponents so to match the forms used in the above bulleted list. To follow our above argument, we now need to re-aggregate the powers of $q^{\frac{1}{2}}$ corresponding to the same edges of the graph: for instance for the leftmost edge we have $q^{\frac{3}{2} + \frac{3}{2} + \frac{4}{2}}$ and so its parity is even. Acting similarly for all the edges we have

$$w(s) \equiv lq^{\frac{3}{2} + \frac{3}{2} + \frac{4}{2}} q^{\frac{3}{2} + \frac{i\left(\frac{3}{2}\right)}{2} + \frac{3}{2} + \frac{3}{2}} q^{\frac{3}{2} + 1 + \frac{0}{2}} q^{\frac{3}{2} + \frac{i\left(\frac{3}{2}\right)}{2} + \frac{3}{2} + \frac{3}{2}} q^{\frac{3}{2} + \frac{3}{2}} q^{\frac{3}{2} + 1 + \frac{2}{2}} q^0 q^{\frac{i\left(\frac{3}{2}\right)i\left(\frac{3}{2}\right)}{2}}$$

where we listed the edges counting them from the bottom left one, going to the right and then passing to the top part of the picture, left to right, and the very last term comes from the crossing (treated separately). In particular s is an odd state. It is easily checked that for each internal edge the exponent we found agrees mod 1 with the formula $\frac{1}{2}(2(\text{col}(e)^2 + \text{col}(e)) - u^2 - u - v^2 - v + 2\text{col}(e) - (u+v)i(\text{col}(e)))$ computed precedingly (where u and v are the states of the two substrands of e containing its endpoints); but as we observed above the overall parity of the formula depends only on the coloring of the graph and not on its state, so that all the other states will be odd. This agrees with the result of Example 3.8 which predicts that in our example $\langle\langle G, \text{col} \rangle\rangle \in q^{\frac{1}{2}}\mathbb{Z}[q^{\pm 1}]$.

To prove the last statement remark that by Lemma 2.3 one can choose D satisfying the additional requirement that the framing of G coincides with the blackboard framing, and so in this case one can suppose $F(D, \text{col}) = 1$. In the above proof, this factor was the only one responsible for possible terms of the form $\sqrt{-1}q^{\frac{1}{4}}$. \square

Corollary 3.3. *With the notation of Theorem 3.2, if a_1, \dots, a_k are colors of edges of G such that $2a_i + 1, i = 1, \dots, k$ are pairwise co-prime, $\langle\langle G, \text{col} \rangle\rangle \sqrt{-1}^{-m} q^{\frac{-m}{4}}$ is divisible by $\prod_j [2a_j + 1]$ in $\mathbb{Z}[q^{\pm \frac{1}{2}}]$.*

Lemma 3.4. *Let a_1, \dots, a_s be integers and let the q -multinomial be defined as*

$$\begin{bmatrix} a_1 + \dots + a_s \\ a_1 & a_2 & \dots & a_{s-1} & a_s \end{bmatrix} \doteq \frac{[a_1 + \dots + a_s]!}{[a_1]! \dots [a_s]!}.$$

Then the q -multinomial is a Laurent polynomial with positive, integer coefficients.

Proof. If $s = 2$ the statement is a direct consequence of the fact that, if $yx = q^2xy$ are two skew-commuting variables, then

$$(x + y)^n = \sum_{j=0}^n q^{\frac{n(n+1)}{2} - \frac{j(j+1)}{2} - \frac{(n-j)(n-j+1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix} x^j y^{n-j}$$

which is easily proved by induction. The general case follows by induction on s by remarking that

$$\begin{bmatrix} a_1 + \dots + a_s \\ a_1 & a_2 & \dots & a_{s-1} & a_s \end{bmatrix} = \begin{bmatrix} a_1 + \dots + a_s \\ (a_1 + a_2) & \dots & a_{s-1} & a_s \end{bmatrix} \begin{bmatrix} a_1 + a_2 \\ a_1 & a_2 \end{bmatrix}. \quad \square$$

3.1. Examples and properties. The following examples can be proved by re-normalizing the formulas provided in [13] for the standard skein invariants of tetrahedra and θ -graphs.

Example 3.5 (unknot). We have

$$\langle\langle \bigcirc^a \rangle\rangle = (-1)^{2a} [2a + 1]. \tag{17}$$

Example 3.6 (theta graph). We have

$$\langle\langle \bigcirc^{\frac{a}{b} \atop c} \rangle\rangle = (-1)^{a+b+c} [a + b + c + 1] \begin{bmatrix} a + b + c \\ a + b - c, b + c - a, c + a - b \end{bmatrix}. \tag{18}$$

Example 3.7 (the tetrahedron or symmetric $6j$ -symbol).

$$\langle\langle \bigcirc^{\frac{e^a \quad f}{c \quad d} \atop b} \rangle\rangle = \sum_{z=\max T_i}^{z=\min Q_j} (-1)^z [z + 1] Q_5, \tag{19}$$

where

$$\begin{aligned} Q_5 &= Q_5(T_1, T_2, T_3, T_4, U_1, U_2, U_3, z) \\ &= \begin{bmatrix} z \\ z - T_1, z - T_2, z - T_3, z - T_4, U_1 - z, U_2 - z, U_3 - z \end{bmatrix} \end{aligned}$$

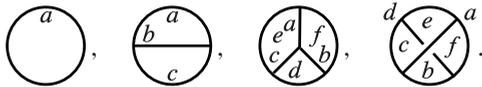
and

$$\begin{aligned}
 T_1 &= a + b + c, \\
 T_2 &= a + e + f, \\
 T_3 &= d + b + f, \\
 T_4 &= d + e + c, \\
 U_1 &= a + b + d + e, \\
 U_2 &= a + c + d + f, \\
 U_3 &= b + c + e + f.
 \end{aligned}$$

Example 3.8 (the crossed tetrahedron). Applying twice Lemma 2.17 to the preceding example one gets

$$\begin{aligned}
 \langle\langle \text{circle with } d, e, a \text{ at top and } c, f, b \text{ at bottom} \rangle\rangle &= \langle\langle \text{circle with } d, e, a \text{ at top and } c, f, b \text{ at bottom} \rangle\rangle \\
 &= \langle\langle \text{circle with } c, e, d, f \text{ at top and } a, b \text{ at bottom} \rangle\rangle \tag{20} \\
 &= \sqrt{-1}^{2(f+c-e-b)} q^{f^2+f+c^2+c-b^2-b-e^2-e} \langle\langle \text{circle with } c, e, d, f \text{ at top and } a, b \text{ at bottom} \rangle\rangle.
 \end{aligned}$$

From now on we will often drop the $\langle\langle \cdot \rangle\rangle$ and denote the values provided by the above formulas respectively by



All the invariants we will be dealing with will be the normalized version unless explicitly stated the contrary. The following lemma can be easily proved by starting from the analogous statements for the standard skein theoretical normalization of the invariants (see [9] or [13]).

Lemma 3.9 (properties of the renormalized invariant). *The following are some of the properties of $\langle\langle G, \text{col} \rangle\rangle$.*

- (1) (Erasing 0-colored strand). *If G' is obtained from (G, col) by deleting a 0-colored edge, then $\langle\langle G', \text{col}' \rangle\rangle = \langle\langle G, \text{col} \rangle\rangle$.*

(2) (Connected sum). If $G = G_1 \# G_2$ along an edge colored by a , then

$$\langle\langle G_1 \# G_2 \rangle\rangle = \frac{1}{(-1)^{2a}[2a + 1]} \langle\langle G_1 \rangle\rangle \langle\langle G_2 \rangle\rangle.$$

(3) (Whitehead move). If G and G' differ by a Whitehead move then

$$\langle\langle G \rangle\rangle = \sum_i \frac{\langle\langle \text{Whitehead move } i \rangle\rangle}{\langle\langle \text{Whitehead move } i \rangle\rangle} \langle\langle G' \rangle\rangle.$$

where i ranges over the all the admissible values.

(4) (Fusion rule). In particular, applying the preceding formula to the case $j = 0$ one has

$$\langle\langle a \rangle\rangle \langle\langle b \rangle\rangle = \sum_i \frac{\langle\langle \text{Fusion } i \rangle\rangle}{\langle\langle \text{Fusion } i \rangle\rangle} \langle\langle G' \rangle\rangle.$$

4. Shadow state-sums and integrality

In this section we will first provide a so-called shadow-state sum formula for the invariants $\langle\langle G, \text{col} \rangle\rangle$ and give some examples. Then we will show through explicit examples that the summands of the shadow state-sums are not Laurent polynomials (even though, of course, the global sum of the state-sums are); in Subsection 4.3, we will then provide short proofs of known identities on $6j$ -symbols.

4.1. Shadow state sums. Let (G, col) be a fixed colored graph, $D \subset \mathbb{R}^2$ be a diagram of it and V, E be the sets of vertices and edges of G , and C, F the sets of crossings and edges of D (each edge of D is a sub-arc of one of G therefore it inherits the coloring from col). Let the regions r_0, \dots, r_m of D be the connected components of $\mathbb{R}^2 \setminus D$ with r_0 the unbounded one; we will denote by R the set of regions and we will say that a region “contains” an edge of D or a crossing if its closure does.

Definition 4.1 (shadow-state). A shadow-state s is a map $s : R \cup F \rightarrow \frac{\mathbb{N}}{2}$ such that $s(f)$ equals the color of the edge of G containing f , $s(r_0) = 0$ and whenever two regions r_i and r_j contain an edge f of D then $s(r_i), s(r_j), s(f)$ form an admissible three-uple.

Given a shadow-state s , we can define its weight as a product of factors coming from the local building blocks of D i.e. the regions, the edges of D , the vertices of G and the crossings. To define these factors explicitly, in the following we will denote by a, b, c the colors of the edges of G (or of D) and by u, v, t, w the shadow-states of the regions and will use the examples given in Section 3.1.

- (1) If r is a region whose shadow-state is u and $\chi(r)$ is its Euler characteristic,

$$w_s(r) \doteq \textcircled{u}^{\chi(r)}.$$

- (2) If f is an edge of D colored by a and u, v are the shadow-states of the regions containing it then, letting $\chi(f)$ to be 0 if f is a closed component and 1 otherwise

$$w_s(f) \doteq \textcircled{\frac{a}{v}}^{\chi(f)}.$$

- (3) If v is a vertex of G colored by a, b, c and u, v, t are the shadow-states of the regions containing it then

$$w_s(v) \doteq \textcircled{\frac{va}{c} \frac{tb}{u}}.$$

- (4) If c is a crossing between two edges of G colored by a, b and u, v, t, w are the shadow-states of the regions surrounding c then

$$w_s(c) \doteq \textcircled{\frac{bt}{v} \frac{aw}{u}}.$$

From now on, to avoid a cumbersome notation, given a shadow-state s we will not explicitly write the colors of the edges of each graph providing the weight of the local building blocks of D as they are completely specified by the states of the regions and the colors of the edges of G surrounding the block. Then we may define the weight of the shadow-state s as

$$w(s) = \prod_{r \in R} \textcircled{}^{\chi(r)} \prod_{f \in F} \textcircled{\ominus}^{-\chi(f)} \prod_{v \in V} \textcircled{\text{Y}} \prod_{c \in C} \textcircled{\text{X}}. \tag{21}$$

Then, since the set of shadow-states of D is easily seen to be finite, we may define the shadow-state sum of (G, col) as

$$\text{shs}(G, \text{col}) \doteq \sum_{s \in \text{shadow states}} w(s).$$

As the following theorem says, the shadow state-sums provide a different approach to the computation of $\langle\langle G, \text{col} \rangle\rangle$.

Theorem 4.2. *It holds*

$$\langle\langle G, \text{col} \rangle\rangle = F(D, \text{col}) \text{shs}(G, \text{col}),$$

where (as in the preceding sections)

$$F(D, \text{col}) \doteq \prod_{e \in \text{edges}} \sqrt{-1}^{4g_e \text{col}(e)} q^{2g_e(\text{col}(e)^2 + \text{col}(e))}.$$

The original definition of the shadow state sums and proof of the above result (but for the standard normalization of the invariants) is due to Kirillov and Reshetikhin [11] and was later generalized to general shadows by Turaev [16]. We used this formulation to extend the definition of colored Jones polynomials to the case of graphs and links in connected sums of copies of $S^2 \times S^1$ ([5]) and to prove a version of the generalized volume conjecture for an infinite family of hyperbolic links called fundamental hyperbolic links ([4]). These links were already studied in [6] for their remarkable topological and geometrical properties.

Proof. Multiplying by $F(D, \text{col})^{-1}$ we can reduce to the case when the framing of G is the blackboard framing. Let D be a diagram of G ; we can add to G some 0-colored edges cutting the regions of D (except r_0) into discs (this changes G and D but not the value of the resulting invariant by Lemma 3.9); for each region we will need $\chi(r) - 1$ such arcs. Fix also a maximal connected sub-tree T in D and let $o \subset \mathbb{R}^2$ be a 0-colored unknot bounding a disc containing D ; it is clear that $\langle\langle G, \text{col} \rangle\rangle = \langle\langle (G, \text{col}) \cup (o, 0) \rangle\rangle$. Let also A be the trivalent graph defined as follows: $A \doteq (N(T) \cap D) \cup \partial N(T)$ (where $N(T)$ is the regular neighborhood of T in \mathbb{R}^2). The idea of the proof is to apply a sequence of fusion rules and inverse connected sums in order to express $\langle\langle G, \text{col} \rangle\rangle$ as a $\sum_{\text{col}_i} c(\text{col}_i) \langle\langle A, \text{col}_i \rangle\rangle$ for some colorings col_i of A and coefficients $c(\text{col}_i)$; then to show that each summand $c(\text{col}_i) \langle\langle A, \text{col}_i \rangle\rangle$ is the weight of exactly one shadow-state.

The unknot o is isotopic to $\partial N(T)$ and, while following the isotopy, at isolated moments it will cross some edges of $D \setminus (N(T) \cap D)$ (but no vertices or crossings because they are all contained in $N(T)$). Let us choose the isotopy so that every edge of $D \setminus (N(T) \cap D)$ is crossed exactly once (this can be done since each region is a disc and T is a maximal connected sub-tree of D). We say that u enters a region r if during the isotopy a subarc of u not contained in r crosses an edge contained in r . We claim that, since T is connected each region r_i , $i = 1, \dots, n$ will be “entered” by o exactly once during the isotopy. Indeed if u enters twice a region r_i , let α, β the subarcs of o in r_i , connecting them by an arc γ we may produce two unknots whose connected sum is u . Since T is connected and contains all the vertices and crossings of D , one of the two discs bounded by these unknots cannot contain vertices and so α and β cross the same edge of r_i , against our hypothesis on the isotopy.

We interpret each crossing moment as a fusion rule so that the isotopy of u progressively “erases” each arc of $D \setminus (N(T) \cap D)$ exactly when entering a region containing that arc, and the sum is taken over all admissible u :

$$a \left(b = \sum_u \frac{\text{circle}(u)}{\text{circle}(u^a/b)} \right) .$$

At the end of this isotopy, since for each $i \leq n$ u entered r_i only once, all the components of $\partial N(T) \cap r_i$ (which are arcs) will be colored by the same colors u_i (see Figure 4 for an example of this construction in the case of a planar graph). Therefore each summand in the final expression will be associated to a shadow-state s given by $s(r_0) = 0, s(r_i) = u_i$. Moreover, the other edges of A (i.e. those of $N(T) \cap D$) are included in those of G and therefore inherit the coloring col . Then we proved

$$\langle\langle G, \text{col} \rangle\rangle = \sum_{u_1, \dots, u_n} \prod_{r \in R} \text{circle}(u_i) \prod_{f \in F \setminus T} \text{circle}^{-1}$$

where the colors of the edges of the θ graphs are specified by the u_i ’s and col . Remark that the summation range is exactly the set of shadow-states because the colors of the arcs of $\partial N(T) \cap r_i$ are all u_i and the admissibility conditions for a three-uple of colors around an edge are satisfied at every moment we apply the fusion rule. Moreover, in the above formula we already got part of the weights of each shadow-state (i.e. those of the regions and of all the edges out of T). We are left to prove that what is missing equals $\langle\langle A, \text{col} \cup \{u_1, \dots, u_n\} \rangle\rangle$, i.e. we claim

$$\langle\langle A, \text{col}(u_1, \dots, u_n) \rangle\rangle = \prod_{f \in T} \text{circle}^{-1} \prod_{v \in V} \text{circle} \prod_{c \in C} \text{circle}$$

where in each factor the colors are specified by the combinatorics of D and by the state $u_1, \dots, u_n \cup \text{col}$ on $R \cup F$. To prove this, remark that

$$\langle\langle \begin{matrix} a \\ b \end{matrix} \rightrightarrows \rangle\rangle = \sum_i \frac{\text{circle}(i)}{\text{circle}(i^a/b)} \langle\langle \begin{matrix} a \\ b \end{matrix} \rightrightarrows \begin{matrix} i \\ u \end{matrix} \rangle\rangle = \text{circle}^{-1} \langle\langle \begin{matrix} a \\ b \\ c \end{matrix} \rangle\rangle \langle\langle \begin{matrix} a \\ b \\ u \end{matrix} \rangle\rangle$$

where the first equality is a fusion rule and the second is the inverse of a connected sum. Applying this identity on all the edges of T we split $\langle\langle A, \text{col} \cup \{u_1, \dots, u_n\} \rangle\rangle$ into the product of the graphs remaining in the neighborhoods of the crossings and vertices (which are respectively \otimes and \oplus) divided by the product of the \ominus ’s corresponding to the edges of T . This proves the claim and completes the proof when all the regions

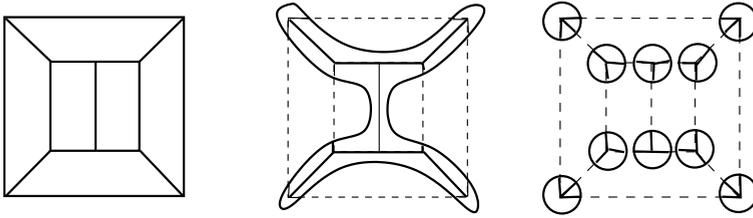


Figure 4. On the left a planar graph G to which we apply the construction of the proof of Theorem 4.2. In the middle the graph A constructed by shrinking on a maximal subtree of G an unknot bounding a disk containing G (the dotted parts are left just for reference). On the right the final union of planar tetrahedra (in this case there are no crossings).

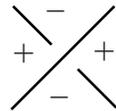
are discs. If in the beginning we added some 0-colored edge to G to cut D into a diagram D' whose regions are discs, then it is clear that every shadow state s' of D' can be lifted to a unique shadow state s of D : indeed the compatibility conditions around a 0-colored edge force the states of the neighboring regions to be the same. Moreover, since the only difference between D' and D is given by the presence of the 0-colored edge, it holds

$$w(s) = w(s') \prod_{f \in F_0} \left(\frac{u^u}{0} \right)^{-1} = w(s') \prod_{f \in F_0} \left(\bigcirc^u \right)^{-1} = w(s') \prod_{r \in R} \left(\bigcirc^u \right)^{\chi(r)-1} .$$

□

The formula given by Theorem 4.2 is often re-written by means of the so-called “gleams.”

Definition 4.3 (gleam). The *gleam* of a diagram D of G is the map $g : R \rightarrow \frac{\mathbb{Z}}{2}$ which on a region r_i equals the sum over all the sectors of crossings contained in r_i of $\frac{1}{2}$ times the local contributions of the crossing determined according to the pattern on the right. (Here by “sector” we mean one of the four angular sectors identified around each crossing by the strands of the graph).



Remark 4.4. Do not confuse a shadow state with the gleam. In general for a given diagram D there is a unique gleam but many different shadow-states.

Corollary 4.5. Under the same hypotheses as Theorem 4.2, it holds

$$\langle\langle G, \text{col} \rangle\rangle = F(G, \text{col}) \sum_s \prod_{r \in R} \sqrt{-1}^{4g(r)u} q^{2g(r)(u^2+u)} \left(\bigcirc^u \right)^{\chi(r)} \prod_{f \in F} \left(\bigcirc^{\ominus} \right)^{-\chi(f)} \prod_{V \cup C} \left(\bigcirc^{\text{triple}} \right) .$$

(22)

Proof. Using Example 3.8 one can rewrite the factors coming from crossings in term of tetrahedra multiplied by extra factors of the form $\sqrt{-1}^{\pm 2u} q^{\pm(u^2+u)}$ for each of the 4 sectors around a crossing. To conclude, collect these factors according to the region containing the corresponding sectors and compare with Definition 4.3. \square

4.2. Simplifying formulas. Both (22) and (21) are far from being optimal: indeed most of the factors in the state-sum can be discarded from the very beginning. Instead of giving a general theorem for doing this let us show why this happens through some examples. We will say that an edge, vertex or crossing is *external* if it is contained in the closure of r_0 and a region is external if its closure contains an external edge. Then the following simplifications can be operated.

- (1) If an external region contains two distinct external edges whose colors are different, then $\langle\langle G, \text{col} \rangle\rangle = 0$.
- (2) If an external region r contains external edges f_1, \dots, f_k whose colors are all c , then, for every shadow-state s on D , the total contribution coming from $r \cup f_1, \dots, f_k$ is

$$\begin{array}{c} c \\ \circ \end{array} \chi^{(r) - \sum_i \chi(f_i)} .$$

- (3) If an external vertex v is the endpoint of a non-external edge f then their contribution simplify because

$$\begin{array}{c} a \\ \circ \\ c \end{array} \begin{array}{c} a \\ b \\ c \end{array} = \begin{array}{c} a \\ \circ \\ b \\ c \end{array} .$$

(Beware: if ∂f is composed of two external vertices only one of them can be simplified with f .)

- (4) For the same reason, if D contains a sequence

$$\begin{array}{c} a \\ \times \\ b \end{array} \dots \begin{array}{c} r_0 \\ \times \end{array}$$

of $n \in \mathbb{Z} \setminus \{0\}$ half-twists separating r_0 from a region r , then, in each summand of (22) the total contribution of the crossings and internal edges of the twist (excepted the initial and final ones) is

$$\begin{array}{c} a \\ \circ \\ b \\ u \end{array}$$

where u is the state of r and a, b are the colors of the strands (beware: the power of q coming from the gleams do not simplify).

where

$$s_1 = s_1(a, b, u, v) = 2(a + 1)u + 2(b + 1)v - 4n(a + b + 1)$$

where a box with $a \in \mathbb{Z}$ stands for a sequence



of a half twists and

$$g(u, v) \doteq (a + 1)(u^2 + u) + (b + 1)(v^2 + v) - 2(a + b + 1)(n^2 + n).$$

To get the above formula, before applying (22), remark that up to isotopy of the diagram, the region r_0 can be chosen freely. Therefore, it is better to pick r_0 as the region touching the two boxes contemporaneously. Again, the summands are not Laurent polynomials but the sum is (take for instance $a = b = n = u = v = 1$). More surprisingly, by Theorem 3.2, for every $a, b \geq 0$ the resulting invariant will be divisible by $[2n + 1]$ in $\mathbb{Z}[q, q^{-1}]$.

4.3.3. Some planar graphs. If G is a planar graph equipped with the blackboard framing, then the gleam of its regions are 0 and so by Corollary 4.5 $\langle\langle G, \text{col} \rangle\rangle$ has a simple expression. In the example of Figure 4, if all the edges of G are colored by $n \in \mathbb{N}$ then

$$\langle\langle G, n \rangle\rangle = \sum_{u, v=0, |u-v| \leq n \leq u+v}^{2n} \frac{\text{Diagram with 6 regions and weights } u, v, n, n, n, n}{\text{Diagram with 3 regions and weights } n, n, n}, \quad (23)$$

where u, v are the shadow-states of the two internal regions. If for instance in (23) one puts $n = 1$, and considers the shadow-state with $u = v = 1$ then its weight is

$$\frac{[3]^2([4]!([5] - 1))^6}{(-[4]!)^7} = -\frac{[3]^2([5] - 1)^6}{[4]!} \notin \mathbb{Z}[q].$$

4.4. Identities on $6j$ -symbols. Shadow state formulas provide a straightforward way to re-prove standard identities on $6j$ -symbols. (The normalization we are using here for the symbols is that of Example 3.7).

4.4.1. Normalizations of $6j$ -symbols. It holds

$$\delta_{b,0} \text{Diagram with regions } a, c = \sum_{u=|a-c|}^{a+c} \frac{\text{Diagram with regions } u, ca, u, b, c}{\text{Diagram with regions } a, u, c}. \quad (24)$$

This is proved by first checking that (22) applied to

$$(G, \text{col}) = \boxed{\begin{array}{|c|c|} \hline & b \\ \hline c & a \\ \hline \end{array}}$$

gives the right hand side. Indeed the diagram of G splits \mathbb{R}^2 in two regions (besides the external region r_0) both of which are discs equipped with 0 gleam; moreover in each shadow state the most exterior region is colored by c and the interior region is colored by u ranging in $\{|a - c|, \dots, a + c - 1, a + c\}$; then by (22) we have

$$\langle\langle G, \text{col} \rangle\rangle = \sum_{u=|c-a|}^{a+c} \frac{\begin{array}{c} \textcircled{c} \quad \textcircled{u} \quad \textcircled{\begin{array}{|c|c|} \hline 0c & c \\ \hline c & b \\ \hline \end{array}} \quad \textcircled{\begin{array}{|c|c|} \hline ua & c \\ \hline a & b \\ \hline \end{array}} \\ \hline \textcircled{\begin{array}{|c|} \hline c \\ \hline c \\ \hline \end{array}} \quad \textcircled{\begin{array}{|c|} \hline b \\ \hline c \\ \hline \end{array}} \quad \textcircled{\begin{array}{|c|} \hline a \\ \hline c \\ \hline \end{array}} \end{array}}$$

which equals the right hand side of eq. (24) because

$$\textcircled{c} = \textcircled{\begin{array}{|c|} \hline c \\ \hline 0 \\ \hline c \\ \hline \end{array}}$$

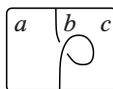
and

$$\textcircled{\begin{array}{|c|} \hline b \\ \hline c \\ \hline \end{array}} = \textcircled{\begin{array}{|c|c|} \hline 0c & c \\ \hline c & b \\ \hline \end{array}}.$$

Then recall that the invariant of a union of two unlinked graphs connected by a single arc is zero unless the color of the arc is 0, in which case the invariant is just the product of the invariants of the graphs (left hand side of (24)). Similarly, it holds

$$\textcircled{\begin{array}{|c|} \hline b \\ \hline a \\ \hline c \\ \hline \end{array}} = \sum_u \sqrt{-1}^{2(u+a-2c)} q^{u^2+u+a^2+a-2(c^2+c)} \frac{\textcircled{u} \quad \textcircled{\begin{array}{|c|c|} \hline b^a & c \\ \hline c & b \\ \hline \end{array}}}{\textcircled{\begin{array}{|c|} \hline c \\ \hline b \\ \hline u \\ \hline \end{array}}},$$

where u ranges between $|b - c|$ and $b + c$. This is proved by applying (22) to

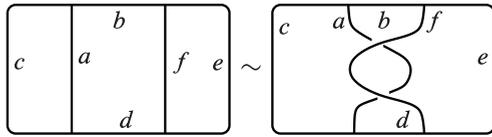


and to the same framed graph after undoing the kink on the b -colored edge.

4.4.2. Orthogonality relation. A direct corollary of (22) is the well-known orthogonality relation

$$\delta_{b,d} \frac{\begin{array}{c} \textcircled{b^a} \textcircled{f^e} \\ \textcircled{c} \quad \textcircled{b} \end{array}}{\textcircled{b}} = \sum_u \sqrt{-1}^{2(d-b)} q^{d^2+d-b^2-b} \frac{\begin{array}{c} \textcircled{u} \quad \textcircled{u^a} \textcircled{e} \textcircled{u} \\ \textcircled{c} \textcircled{f} \quad \textcircled{b} \textcircled{d} \end{array}}{\begin{array}{c} \textcircled{e^a} \textcircled{f^c} \\ \textcircled{u} \quad \textcircled{u} \end{array}},$$

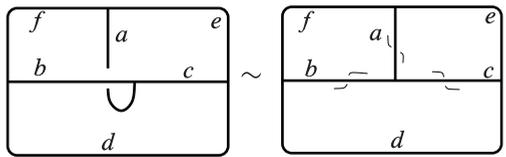
where u ranges over all the admissible colorings of the union of the two tetrahedral graphs on the right. To prove it, just apply (22) to the following two isotopic graphs and simplify the common factors:



4.4.3. Racah identity. It holds

$$\begin{aligned} & \sqrt{-1}^{2(a+b-c)} q^{a^2+a+b^2+b-c^2-c} \begin{array}{c} \textcircled{e^a} \textcircled{f^b} \\ \textcircled{c} \textcircled{d} \end{array} \\ &= \sum_u \sqrt{-1}^{2(u+f-e-d)} q^{u^2+u+f^2+f-d^2-d-e^2-e} \frac{\begin{array}{c} \textcircled{u} \quad \textcircled{d^a} \textcircled{u} \textcircled{u} \\ \textcircled{f} \textcircled{b} \quad \textcircled{e} \textcircled{c} \end{array}}{\begin{array}{c} \textcircled{b^a} \textcircled{d^b} \\ \textcircled{u} \quad \textcircled{u} \end{array}}, \end{aligned} \tag{25}$$

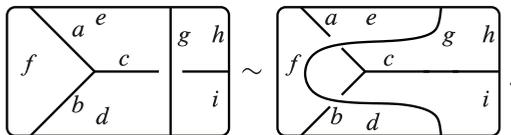
where u ranges over all the admissible colorings of the union of tetrahedra on the right. Indeed it is sufficient to apply (22) to the following two isotopic graphs and simplify the common factors:



4.4.4. Biedenharn–Elliot identity. Another direct corollary of (22) is the well-known Biedenharn–Elliot identity

$$\frac{\begin{array}{c} \textcircled{e^a} \textcircled{f^b} \textcircled{d^g} \textcircled{i^h} \\ \textcircled{c} \textcircled{d} \quad \textcircled{e} \textcircled{c} \end{array}}{\begin{array}{c} \textcircled{e^c} \\ \textcircled{d} \end{array}} = \sum_u \frac{\begin{array}{c} \textcircled{u} \quad \textcircled{u^a} \textcircled{h} \textcircled{u} \textcircled{u} \\ \textcircled{f} \textcircled{g} \quad \textcircled{e} \textcircled{b} \end{array}}{\begin{array}{c} \textcircled{h^a} \textcircled{i^b} \textcircled{g^f} \\ \textcircled{u} \quad \textcircled{u} \quad \textcircled{u} \end{array}}, \tag{26}$$

where u ranges over all the admissible colorings of the union of tetrahedra on the right. Indeed it is sufficient to apply (22) to the following two isotopic graphs and simplify the common factors:



5. R-matrices vs. 6j-symbols

In the preceding sections we showed how to compute invariants of colored graphs by means of two different state-sums. Although for practical computation shadow state-sums turn out to be easier to deal with, state-sums based on R -matrices and Clebsch–Gordan symbols allow one to prove integrality results. In this section we will compare the state-sums when applied to graphs with boundary. So let a (n, m) -KTG be a framed graph embedded in a square box which contains only 3-valent vertices (inside the box) and n (resp. m) 1-valent vertices on the bottom (resp. top) edge of the box. A typical example is a framed (n, m) -tangle. Let as before E, V be the set of edges and 3-valent vertices of G , $\partial G^\pm \doteq \partial G \cap \partial box$ and, once chosen a diagram D of G , let F, C be the set of edges, and crossings of D . Let also b_1, \dots, b_n be the bottom (univalent) vertices of G and t_1, \dots, t_m the top vertices. The definition of admissible coloring of a (n, m) is the same as the standard one, but in this case, a second coloring is needed to get a numerical invariant out of G , namely a coloring on ∂G .

Definition 5.1 (∂ -colorings for (n, m) -KTG’s). Let (G, col) be a colored (n, m) -KTG; a ∂ -coloring for G is a map $col_\partial: \partial G \rightarrow \frac{\mathbb{Z}}{2}$ such that if i_k (resp. j_k) is the color of the edge containing b_k (resp. t_k) then $|col_\partial(b_k)| \leq i_k$ and $col_\partial(b_k) - i_k \in \mathbb{Z}$.

Equivalently a ∂ -coloring is a choice of a vector in $V^{i_1} \otimes \dots \otimes V^{i_n}$ of the form

$$g_{col_\partial(b_1)}^{i_1} \otimes \dots \otimes g_{col_\partial(b_n)}^{i_n}$$

and a vector in $V^{j_1} \otimes \dots \otimes V^{j_m}$ of the form

$$g_{col_\partial(t_1)}^{j_1} \otimes \dots \otimes g_{col_\partial(t_m)}^{j_m}$$

Given a (n, m) -KTG equipped with a coloring col and a ∂ -coloring col_∂ , one can compute $\langle\langle G, col \cup col_\partial \rangle\rangle$ exactly as in (13): it is sufficient to restrict the set of admissible states to those such that the state of the boundary edges coincide with col_∂ . Then G represents a morphism

$$Z(G, col): V^{i_1} \otimes \dots \otimes V^{i_n} \longrightarrow V^{j_1} \otimes \dots \otimes V^{j_m}$$

and $\langle\langle G, \text{col} \cup \text{col}_\partial \rangle\rangle$ is an entry in the matrix expressing $Z(G, \text{col})$ in the bases formed by tensor products of basis elements.

Most of the integrality result 3.2 still holds true (the idea of the proof is exactly the same).

Theorem 5.2 (integrality: case with boundary). *The following belongs to $\mathbb{Z}[q^{\pm\frac{1}{2}}]$:*

$$\langle\langle G, \text{col} \cup \text{col}_\partial \rangle\rangle \doteq \frac{\langle G, \text{col} \cup \text{col}_\partial \rangle F(G, \text{col}) \prod_{e \in E'} [2 \text{col}(e)]! \prod_{k=1}^m \sqrt{-1}^{jk}}{\prod_{v \in V} [a_v + b_v - c_v]! [a_v + c_v - b_v]! [c_v + b_v - a_v]! \prod_{k=1}^n \sqrt{-1}^{ik}}$$

where E' is the set of all the edges of G which do not intersect ∂G^+ and $F(G, \text{col})$ is defined as in the preceding sections.

What is interesting is that one can re-compute the invariant of (G, col) also via shadow-state sums and Clebsch–Gordan symbols. To explain this, we will use the following definition.

Definition 5.3. Given a finite sequence j_1, \dots, j_m a *Bratteli sequence* associated to it is a sequence s_0, s_1, \dots, s_m such that $s_0 = 0$ and for each $0 \leq k \leq m - 1$ the three-uple s_k, j_{k+1}, s_{k+1} is admissible.

It is not difficult to realize that the set of Bratteli sequences associated to j_1, \dots, j_m is in bijection with the set of irreducible submodules of $V^{j_1} \otimes \dots \otimes V^{j_m}$ and that the submodule $V(s)$ corresponding to a Bratteli sequence $s = (s_0, \dots, s_m)$ is isomorphic to V^{s_m} . Moreover, using the morphisms defined in Subsections 2.4.2 and 2.4.4, we may fix explicit maps

$$\pi(s): V^{j_1} \otimes \dots \otimes V^{j_m} \longrightarrow V^{s_m}$$

and

$$i(s): V^{s_m} \longrightarrow V^{j_1} \otimes \dots \otimes V^{j_m}.$$

Graphically they are expressed as

$$\pi(s) \doteq \begin{array}{c} s_2 \quad \dots \quad s_m \\ \diagup \quad | \quad | \quad \dots \quad | \\ j_1 \quad j_2 \quad j_3 \quad \dots \quad j_m \end{array}$$

and

$$i(s) \doteq \begin{array}{c} j_1 \quad j_2 \quad j_3 \quad \dots \quad j_m \\ \diagdown \quad | \quad | \quad \dots \quad | \\ s_2 \quad \dots \quad s_m \end{array}$$

More explicitly, to construct $\pi(s)$, apply the Clebsch–Gordan morphism $P_{j_1, j_2}^{s_2}$ to the first two factors of $V^{j_1} \otimes \dots \otimes V^{j_m}$ (the isomorphism is fixed by the choice of Clebsch–Gordan projectors made in Subsection 2.4.4). The result is in the space $V^{s_2} \otimes V^{j_3} \otimes \dots \otimes V^{j_m}$ and composing iteratively with $P_{s_k, j_{k+1}}^{s_{k+1}}$ one gets the desired map $\pi(s): V^{j_1} \otimes \dots \otimes V^{j_m} \rightarrow V^{s_m}$. Similarly, $i(s): V^{s_m} \rightarrow V^{j_1} \otimes \dots \otimes V^{j_m}$ is defined by composing recursively from right to left the Clebsch–Gordan morphisms of Subsection 2.4.2:

$$V^{s_m} \longrightarrow V^{s_{m-1}} \otimes V^{j_m} \longrightarrow V^{s_{m-2}} \otimes V^{j_{m-2}} \otimes V^{j_m} \longrightarrow \dots \longrightarrow V^{j_1} \otimes \dots \otimes V^{j_m}.$$

Let

$$\bar{G} \doteq G \cup \partial \text{ box}$$

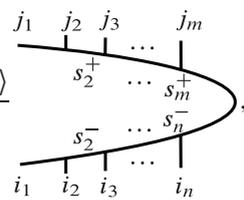
viewed as a framed graph in \mathbb{R}^2 by embedding the box in \mathbb{R}^2 with the blackboard framing around its boundary. Let $s^+ = (s_1^+, \dots, s_m^+)$ and $s^- = (s_1^-, \dots, s_n^-)$ be Brattelli sequences associated to j_1, \dots, j_m and i_1, \dots, i_n and suppose that $s_m^+ = s_n^- \doteq x$. We can extend col to a coloring $\text{col} \cup s^- \cup s^+$ of \bar{G} : the color of the edge in the top (resp. bottom) edge of the box bounded by j_k and j_{k+1} (resp. i_k and i_{k+1}) is s_k^+ (resp. s_k^-), the color of the left edge of the box is 0 and that of the right edge x (see Figure 5).

Theorem 5.4 (Shadow state-sums vs. R-matrices and Clebsch–Gordan symbols). *Let $\lambda(s^-, s^+) \in \mathbb{C}$ be defined by $\pi(s^+) \circ Z(G, \text{col}) \circ i(s^-) = \lambda(s^-, s^+) \text{Id}_{V^x}$. Then*

$$\lambda(s^-, s^+) \left(\bigcirc^x \right) = \langle \langle \bar{G}, \text{col} \cup s^- \cup s^+ \rangle \rangle$$

and

$$Z(G, \text{col})$$

$$= \sum_{s^-} \sum_{s^+} \prod_{t=1}^{n-1} \frac{\bigcirc_{s_t}^{s_t}}{\bigcirc_{s_{t+1}}^{s_t i_t}} \prod_{t=1}^{m-1} \frac{\bigcirc_{s_t}^{s_t}}{\bigcirc_{s_{t+1}}^{s_t j_t}} \langle \langle \bar{G}, \text{col} \cup s^- \cup s^+ \rangle \rangle \bigcirc_{s_n}^{s_n}$$


+

where the sum are taken over all the Brattelli sequences s^- and s^+ associated respectively to i_1, \dots, i_n and j_1, \dots, j_m .

Proof. The value on the right hand side is the invariant of the colored graph

$$(\bar{G}, \text{col} \cup s^+ \cup s^-)$$

depicted on the right of Figure 5. But, as shown in the picture, \bar{G} is the closure of the graph representing $\pi(s^+) \circ Z(G, \text{col}) \circ i(s^-)$.

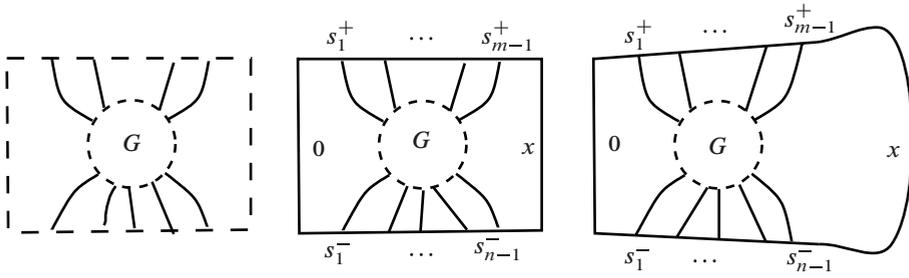


Figure 5. Using a pair of Bratteli sequences it is possible to close (G, col) to a colored closed KTG (in the middle).

The second statement follows by applying $n - 1$ times the fusion rule on the bottom legs of G and $m - 1$ times on the top legs to get

$$= \sum_{s^-} \sum_{s^+} \prod_{t=1}^{n-1} \frac{\text{circle}(s_t)}{\text{circle}(s_t, i_t, s_{t+1})} \prod_{t=1}^{m-1} \frac{\text{circle}(s_t)}{\text{circle}(s_t, j_t, s_{t+1})}$$

where s^- and s^+ range over all Bratteli sequences associated to i_1, \dots, i_n and j_1, \dots, j_n respectively and \bar{G}' is the $(1, 1)$ colored KTG obtained by opening

$$(\bar{G}, \text{col} \cup s^+ \cup s^-)$$

along the right edge of the box. □

Example 5.5 (R -matrices vs $6j$ -symbols). It holds

$${}_b^a R_{u,v}^{t,w} = \sum_{c=|a-b|}^{a+b} C_{t,w,v+u}^{b,a,c} \frac{\sqrt{-1}^{2(c-a-b)} q^{c^2+c-a^2-a-b^2-b} \text{circle}(c)}{\text{circle}(b, a, c)} P_{u,v,u+v}^{a,b,c}$$

Indeed it is sufficient to apply the preceding theorem to (G, col) being a crossing.

References

- [1] J. Bernstein, I. Frenkel, and M. Khovanov, A categorification of the Temperley–Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors. *Selecta Math. (N.S.)* **5** (1999), 199–241. [MR 1714141](#) [Zbl 0981.17001](#)
- [2] L. Crane and I. Frenkel, Four-dimensional topological quantum field theory, Hopf categories and the canonical bases. *J. Math. Phys.* **35** (1994), 5136–5154. [MR 1295461](#) [Zbl 0892.57014](#)
- [3] J. S. Carter, D. E. Flath, and M. Saito, *The classical and quantum 6j-symbols*. Mathematical Notes (Princeton) 43. Princeton University Press, Princeton, N.J., 1995. [MR 1366832](#) [Zbl 0851.17001](#)
- [4] F. Costantino, 6j-symbols hyperbolic structures and the volume conjecture. *Geom. Topol.* **11** (2007), 1831–1853. [MR 2350469](#) [Zbl 1132.57011](#)
- [5] F. Costantino, Coloured Jones invariants of links and the volume conjecture. *J. London Math. Soc. (2)* **76** (2007), 1–15. [MR 2351605](#) [Zbl 1153.57003](#)
- [6] F. Costantino and D. P. Thurston, 3-manifolds efficiently bound 4-manifolds. *J. Topology* **1** 3, (2008), 703–745. [MR 2417451](#) [Zbl 1166.57016](#)
- [7] I. Frenkel, M. Khovanov, and C. Stroppel, A categorification of finite-dimensional representations of quantum \mathfrak{sl}_2 and their tensor products. *Selecta Math. (N.S.)* **12** (2006), 379–431. [MR 2305608](#) [Zbl 1188.17011](#)
- [8] S. Garoufalidis and R. Van der Veen, Asymptotics of classical spin networks. *Geom. Topol.* **17** (2013), 1–37. [MR 3035322](#) [Zbl 1277.57020](#)
- [9] L. Kauffman and S. Lins, *Temperley–Lieb recoupling theory and invariants of 3-manifolds*. Annals of Mathematics Studies 134. Princeton University Press, Princeton, N.J., 1994. [MR 1280463](#) [Zbl 0821.57003](#)
- [10] R. Kirby and P. Melvin, The 3-manifold invariants of Witten and Reshetikhin–Turaev for $\mathfrak{sl}(2, \mathbb{C})$. *Invent. Math.* **105** (1991), 473–545. [MR 1117149](#) [Zbl 0745.57006](#)
- [11] A. N. Kirillov and N. Yu Reshetikhin, Representations of the algebra $U_q(\mathfrak{sl}(2))$, q -orthogonal polynomials and invariants of links. In V. G. Kac (ed.), *Infinite-dimensional Lie algebras and groups*. Proceedings of the conference held at Luminy-Marseille, July 4–8, 1988. Advanced Series in Mathematical Physics 7. World Scientific, Teaneck, N.J., 1989, 285–339. [MR 1026957](#) [MR 1026942](#) (collection) [Zbl 0742.17018](#) [Zbl 0741.00064](#) (collection)
- [12] T. Q. Le, Integrality and symmetry of quantum link invariants. *Duke Math. J.* **102** (2000), 273–306. [MR 1749439](#) [Zbl 0951.57004](#)
- [13] G. Masbaum and P. Vogel, 3-valent graphs and the Kauffman bracket. *Pacific J. Math.* **164** (1994), 361–381. [MR 1272656](#) [Zbl 0838.57007](#)
- [14] S. Piunikhin, Turaev–Viro and Kauffman–Lins invariants for 3-manifolds coincide. *J. Knot Theory Ramifications* **1** (1992), 105–135. [MR 1164112](#) [Zbl 0759.57011](#)
- [15] N. Yu. Reshetikhin and V. G. Turaev, Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.* **127** (1990), 1–26. [MR 1036112](#) [Zbl 0768.57003](#)

- [16] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*. de Gruyter Studies in Mathematics 18. Walter de Gruyter, Berlin, 1994. Second revised ed., Walter de Gruyter, Berlin, 2010. [MR 1292673](#) [MR 2654259](#) (second ed.) [Zbl 0812.57003](#) [Zbl 1213.57002](#) (second ed.)

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