Quantum Topol. 5 (2014), 259–287
DOI 10.4171/QT/52

Quantum Topology

A link invariant with values in the Witt ring

Gaël Collinet and Pierre Guillot

Abstract. Using Maslov indices, we show the existence of oriented link invariants with values in the Witt rings of certain fields. Various classical invariants are closely related to this construction. We also explore a surprising connection with the Weil representation.

Mathematics Subject Classification (2010). 57M25, 19G12.

Keywords. Link invariants, Witt rings.

Contents

1. Introduction

In this paper we show that, given an appropriate field K, one can associate to any oriented link L in Euclidean 3-space an element $\Theta_K(L) \in W(K)$, where $W(K)$ is the *Witt ring* of K. This element $\Theta_K(L)$ is an isotopy invariant.

Recall that an element in $W(K)$ is given by a (non-degenerate) quadratic form q, over a finite dimensional K-vector space. However, two such quadratic forms q and q' may define the same element in $W(K)$ even if they are not isomorphic; in this case we call them Witt equivalent. Witt equivalence can be more or less subtle, depending strongly on the field K. The easiest example is that of $K = \mathbb{R}$, the field of real numbers, for in this case q and q' are Witt equivalent precisely when they have the same signature. Accordingly one has an isomorphism $W(\mathbb{R}) \cong \mathbb{Z}$. By contrast, $W(\mathbb{Q})$ is considerably more complicated (see below).

There have been many efforts to use quadratic forms in order to define link invariants, and the pattern has frequently been as follows: one has a procedure to obtain a quadratic form from a link, but it is not itself an isotopy invariant, so that one is reduced to extracting coarser information. As early as 1932, Reidemeister in [8, §7 and §8] uses the so-called "Minkowski units" of a certain quadratic form of his design, and proves that they are invariant. Better known is the construction of the *signature* of a link, which is really the signature of a certain non-invariant quadratic form (see [5], Theorem 8.9 with $\omega = -1$). In retrospect it may be said, rather pedantically, that the quadratic form is replaced by its Witt equivalence class in $W(\mathbb{R})$ in order to get an invariant. In a sense, *in this paper we use the same strategy of working with the Witt ring, but over other fields*. However our construction is not as direct, and breaks the above pattern.

Indeed we shall follow in the footsteps of Ghys and Gambaudo (see [2]), who were explicitly thinking of the signature as taking its values in $W(\mathbb{R})$ in order to solve the following problem. Let $s(L)$ denote the signature of the oriented link L. If we consider the *braid group* B_n on n strands, then we may define a map $f_n \colon B_n \to \mathbb{Z}$ by $f_n(\beta) = s(\hat{\beta})$. Here we use $\hat{\beta}$ to denote the *closure* of the braid β, which is an oriented link in $\mathbb{R}^3$. One may ask whether f_n is a group homomorphism; it is not, and indeed Ghys and Gambaudo obtain an explicit formula for

$$c(\beta, \gamma) \stackrel{\text{def}}{=} f_n(\beta\gamma) - f_n(\beta) - f_n(\gamma).$$

Their formula is in terms of quadratic forms (as opposed to plain integers). In spite of the complicated notation, let us give it here

$$f_n(\beta\gamma) - f_n(\beta) - f_n(\gamma) = \tau(\Gamma_1,\ \Gamma_{r(\beta)},\ \Gamma_{r(\beta\gamma)}). \qquad (*)$$

Here $r \colon B_n \to \mathrm{GL}_n(\mathbb{R})$ is the *Burau representation* ("at $t = -1$"), the notation Γ_g is used for the graph of g, and τ is the *Maslov index*, an algebraic construction which produces quadratic forms (up to Witt equivalence).

What we do in this paper is *to take* $(*)$ *as a definition of a link invariant* instead. More precisely, we construct for each n a map $f_n \colon B_n \to W(K)$, for a suitable field K, such that the analog of $(*)$ holds. This makes sense since the Maslov index is a very general procedure, not constrained to $K = \mathbb{R}$. Then, we show that $(f_n)_{n\geq 2}$ is a *Markov function*, that is, it is compatible with the Markov moves. The celebrated theorems of Alexander and Markov then imply that $f_n(\beta) = \Theta_K(\hat{\beta})$ for some link invariant Θ_K. In particular, we have the following result.

Theorem 1.1. *Let $K = \mathbb{R}$, or $\mathbb{Q}$, or a finite field, or $\mathbb{Q}(t)$, the field of rational fractions in t. Then there exists a unique oriented link invariant Θ_K with values in the Witt ring $W(K)$, which takes the zero value for disjoint unions of unknots, and with the following extra property. Defining $f_n\colon B_n \to W(K)$ by $f_n(\beta) = \Theta_K(\hat{\beta})$, one has*

$$f_n(\beta\gamma) - f_n(\beta) - f_n(\gamma) = \tau(\Gamma_1,\ \Gamma_{r_n(\beta)},\ \Gamma_{r_n(\beta\gamma)}). \qquad (**)$$

Here $r_n\colon B_n \to \mathrm{GL}_n(K)$ is the appropriate version of the Burau representation. (We caution the reader who may glance at the results in the text now that for $K = \mathbb{Q}(t)$ we actually mention a link invariant with values in a ring written $\mathrm{WH}(\mathbb{Q}(t))$ and called the Hermitian Witt ring; luckily $\mathrm{WH}(\mathbb{Q}(t)) \subset W(\mathbb{Q}(t))$ in this case and the theorem holds as stated. These details need not distract us now.) This Theorem appears in the text as Theorem 3.4.

When it comes to computing $\Theta_K(L)$ explicitly, we have to rely on $(**)$, after having found a braid group element whose closure is L. We hasten to add that we have made a Sage script available, which can perform the calculations automatically. It outputs a diagonal matrix representing the quadratic form $\Theta_K(L)$. In the rest of this Introduction, we assume that the computational side of things is thus taken care of, and comment on the results. Before anything else though, this is as good a place as any to point out that $(**)$ implies that $f_n(\beta) = -f_n(\beta^{-1})$, so that $\Theta_K(L') = -\Theta_K(L)$ if L' is the mirror-image of L. In particular $2\Theta_K(L) = 0$ if L and L' are isotopic.

First and foremost, for $K = \mathbb{R}$ one can interpret the result by Ghys and Gambaudo as saying that $\Theta_{\mathbb{R}}(L)$ agrees with the signature of L. Things are already more interesting with $K = \mathbb{Q}$. In this case (see [7]) there is an exact sequence (where the arrows are completely explicit)

$$0 \longrightarrow \mathbb{Z} \longrightarrow W(\mathbb{Q}) \longrightarrow \bigoplus_p W(\mathbb{F}_p) \longrightarrow 0.$$

This sequence is split by the homomorphism $W(\mathbb{Q}) \to W(\mathbb{R}) \cong \mathbb{Z}$. Moreover, for p odd the group $W(\mathbb{F}_p)$ is either $\mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/4$ according as p is 1 mod 4 or not, while $W(\mathbb{F}_2) = \mathbb{Z}/2$. For each oriented link L, we obtain a set of primes which is an invariant of L, namely the set of those p for which $\Theta_{\mathbb{Q}}(L)$ maps to a non-zero element via the residue map $W(\mathbb{Q}) \to W(\mathbb{F}_p)$. Of course, for each p the value in $W(\mathbb{F}_p)$ is also an invariant.

The truly interesting case is $K = \mathbb{Q}(t)$. The ring $W(\mathbb{Q}(t))$ is very rich, so the first thing we should do is extract easily computable information from $\Theta_{\mathbb{Q}(t)}(L)$. We do this by showing that the above theorem yields a Laurent-polynomial invariant akin to the Alexander–Conway polynomial. We also exploit our method to produce a “signature” for each complex number ω of module 1, that is a $\mathbb{Z}$-valued invariant. When ω is a root of unity, this invariant is related to the Levine–Tristram signature, as follows again from [2]. Thus $\Theta_{\mathbb{Q}(t)}(L)$ seems to "contain" many other invariants, and its first virtue is unification.

However, there is more to $\Theta_{\mathbb{Q}(t)}(L)$ than the polynomial and the signatures. One has an exact sequence

$$0 \longrightarrow W(\mathbb{Q}) \longrightarrow W(\mathbb{Q}(t)) \longrightarrow \bigoplus_P W(\mathbb{Q}[t]/(P)) \longrightarrow 0.$$

Here the direct sum runs over all irreducible polynomials, so that $\kappa = \mathbb{Q}[t]/(P)$ is a number field. Finally, the Witt ring $W(\kappa)$ fits into yet another exact sequence, similar to that for $\mathbb{Q}$ but involving the Witt ring of the ring of integers in κ (for $\kappa = \mathbb{Q}$, this is the ring $\mathbb{Z}$, and $W(\mathbb{Z}) = \mathbb{Z}$; we shall not encounter Witt rings of rings which are not fields elsewhere in this paper). All the arrows are quite explicit, so even though $W(\mathbb{Q}(t))$ appears to be huge, it is in principle always possible to decide in finite time whether $\Theta_{\mathbb{Q}(t)}(L)$ is zero (and thus possibly show that L is not the trivial knot).

So far we have described the contents of the three sections of the paper following this Introduction. In Section 2 we present background material and give simple, sufficient conditions for an invariant as above to be defined out of representations of the braid groups. In Section 3 it is shown that these conditions are satisfied in the case of the Burau representation. Examples are provided in Section 4.

Let us now say a word about Section 5, which explores the ideas behind the proof of Theorem 1.1 in the case $K = \mathbb{R}$, rather than its statement, and connects them to the so-called Weil representation. In summarizing Section 5 we shall presently provide a sketch of the key steps in the proof of the Theorem (the assumption $K = \mathbb{R}$ allowing for simpler arguments).

The first ingredient is the observation that the Burau representation at $t = -1$ carries a B_{2n}-invariant symplectic form, thus we get a map $r_{2n}\colon B_{2n} \to \mathbf{Sp}_{2n}(\mathbb{R})$. Now, we have $\pi_1(\mathbf{Sp}_{2n}(\mathbb{R})) = \mathbb{Z}$, so that $\mathbf{Sp}_{2n}(\mathbb{R})$ possesses many covers; we shall be particularly interested in the simply-connected cover $\widetilde{\mathbf{Sp}}_{2n}(\mathbb{R})$ and the 2-fold cover M_{2n}, also known as the metaplectic group.

The second ingredient is the cohomological fact that $H^2(B_n, \mathbb{Z}) = 0$, which implies that $r_{2n}\colon B_{2n} \to \mathbf{Sp}_{2n}(\mathbb{R})$ can be lifted to a map $r'_{2n}\colon B_{2n} \to \widetilde{\mathbf{Sp}}_{2n}(\mathbb{R})$. The kernel of the map $\widetilde{\mathbf{Sp}}_{2n}(\mathbb{R}) \to \mathbf{Sp}_{2n}(\mathbb{R})$ is $\mathbb{Z}$, and once we describe $\widetilde{\mathbf{Sp}}_{2n}(\mathbb{R})$ explicitly using a two-cocycle c with values in $\mathbb{Z}$, then finding r'_{2n} amounts to finding a one-cocycle on B_{2n} whose coboundary is c. That one-cocycle is the map f_{2n} which appears in Theorem 1.1.

These two ingredients must be slightly refined in the case of a general field K, but the spirit of the construction of the map f_n is always the same. Topological arguments are replaced by the apparatus of Maslov indices (which are needed in order to make precise statements anyway).

In Section 5, the emphasis is on the induced map $B_{2n} \to M_{2n}$, for M_{2n} is known to act on an infinite-dimensional Hilbert space via the Weil representation. Thus B_{2n} also acts on this space, and the representation is also known to have a "trace" in some technical sense. This trace has been computed by Thomas ([9]), who provides explicit formulae involving Maslov indices. Comparing these with the material in Section 2, we end up proving that the trace is a link invariant, which can be expressed in terms of the Alexander–Conway polynomial and the signature (which appears in the guise of $\Theta_{\mathbb{R}}$, see above). Note that strictly speaking our result about the trace of the Weil representation is not deduced from Theorem 1.1; rather, it constitutes a variant on its proof.

We conclude the paper with some remarks about the Weil representation of finite fields and the work of Goldschmidt and Jones.

In a subsequent paper it will be established that, at the price of more machinery including a recent theorem of Barge and Lannes on Maslov indices over rings, we can follow the above outline over $\mathbb{Z}[\frac{1}{2}, t, t^{-1}]$ instead of a field. There results a single link invariant which specializes to all the others, thus pushing the unification a step further. What is more, it will be shown that our method extends to the case of *coloured links*, for which the braid groups have to be replaced by an appropriate groupoid.

Acknowledgments. The authors wish to thank Étienne Ghys, Jean Barge, and Christian Kassel for their interest in the paper. Our thanks extend to Hubert Rubenthaler for helpful discussions on the harmonic analysis underlying the Weil representation. Pierre Torasso pointed out the reference [9], and we are grateful for his help. Finally, we are indebted to Ivan Marin for discovering an embarrassing mistake in an earlier version of the paper. Also, we would like to thank the referee for raising subtle technical points about §5 and generally encouraging us to develop that section.

2. Background material

2.1. The braid groups. The *braid group on n strands B_n* is the group generated by $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| > 2$, while

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

The well known interpretation of B_n in terms of geometric braids (see [4], Theorem 1.12) allows one to define the operation of *closure* $\beta \mapsto \hat{\beta}$ (*loc. cit.*, §2.2): here $\beta \in B_n$ and $\hat{\beta}$ is an oriented link in Euclidean 3-space. The celebrated theorem of Alexander (*loc. cit.*, §2.3) asserts that any oriented link in $\mathbb{R}^3$ is isotopic to one of the form $\hat{\beta}$ for some β belonging to some B_n.

This process defines an equivalence relation on the disjoint union $\coprod_{n\geq 2} B_n$, according to which $\beta \sim \gamma$ whenever the links $\hat{\beta}$ and $\hat{\gamma}$ are isotopic. Markov's theorem (*loc. cit.*, §2.5) describes this relation explicitly. Here we shall state the result in the following form: a map $f = \coprod_{n\geq 2} f_n$ on the above disjoint union, with values in any set E, is constant on the equivalence classes if and only if the two following properties are satisfied:

(i) each f_n is invariant under conjugation, that is $f_n(\gamma^{-1}\beta\gamma) = f_n(\beta)$ for all $\beta, \gamma \in B_n$.

(ii) for all $n \geq 2$ and all $\beta \in B_n$ one has $f_{n+1}(\iota_n(\beta)\sigma_n^{\pm 1}) = f_n(\beta)$, where ι_n denotes the inclusion of B_n into B_{n+1}.

Such a map is usually called a *Markov function*. It follows that the value of a Markov function on a braid β only depends on the closure $\hat{\beta}$, and in view of Alexander's theorem we see that a Markov function gives an oriented link invariant (and conversely).

Because of condition (i), a traditional strategy in order to produce Markov functions is to start with a collection of representations $r_n\colon B_n \to \mathrm{GL}(V_n)$, where V_n is a module over some ring R, and then rely on functions which are known to be conjugation-invariant on the group of invertible matrices, like the trace or determinant. A standard example is the Alexander–Conway polynomial, which relies on the Burau representation and the determinant (*loc. cit.*, §3.4). Here $R = \mathbb{Z}[t, t^{-1}]$ and the invariant takes its values in R.

2.2. Witt rings and Maslov indices. Let K be a field of characteristic different from 2. Suppose that K is endowed with an involution σ, and let $k = K^\sigma$ denote the field of fixed elements. We shall write $\bar{x}$ instead of $\sigma(x)$.

Let V be a vector space over K. A map $h\colon V \times V \to K$ is called an *anti-Hermitian form* (resp. a Hermitian form) when it is linear in one variable and satisfies

$$h(y, x) = -\overline{h(x, y)} \quad \text{(resp. } h(y, x) = \overline{h(x, y)}\text{)}.$$

In this case V is called an anti-Hermitian space (resp. a Hermitian space). The form h is called non-degenerate when the determinant of the corresponding matrix (in any basis) is non-zero.

Let V be anti-Hermitian. A *Lagrangian* is a subspace $\ell \subset V$ such that $\ell = \ell^\perp$. We say that V is *hyperbolic* when it is the direct sum of two Lagrangians.

Now given a hyperbolic, non-degenerate, anti-Hermitian space V with form h and three Lagrangians ℓ_1, ℓ_2 and ℓ_3, we shall describe their *Maslov index*, which is a certain element

$$\tau(\ell_1,\ \ell_2,\ \ell_3) \in \mathrm{WH}(K, \sigma).$$

Here $\mathrm{WH}(K, \sigma)$ is the Hermitian Witt ring of K: see [7]. For example $\mathrm{WH}(K, \sigma)$ may be defined as the quotient of the Grothendieck ring of the category of non-degenerate Hermitian spaces by the ideal consisting of all hyperbolic spaces.

The Maslov index $\tau(\ell_1,\ \ell_2,\ \ell_3)$ is then the non-degenerate space corresponding to the following Hermitian form on $\ell_1 \oplus \ell_2 \oplus \ell_3$,

$$H(\mathbf{v}, \mathbf{w}) = h(v_1, w_2 - w_3) + h(v_2, w_3 - w_1) + h(v_3, w_1 - w_2).$$

More precisely, if this Hermitian space is degenerate, we take the quotient by its kernel.

We claim that this construction enjoys the following properties.

(i) Dihedral symmetry:

$$\tau(\ell_1,\ \ell_2,\ \ell_3) = -\tau(\ell_3,\ \ell_2,\ \ell_1) = \tau(\ell_3,\ \ell_1,\ \ell_2).$$

(ii) Cocycle condition:

$$\tau(\ell_1,\ \ell_2,\ \ell_3) + \tau(\ell_1,\ \ell_3,\ \ell_4) = \tau(\ell_1,\ \ell_2,\ \ell_4) + \tau(\ell_2,\ \ell_3,\ \ell_4).$$

(iii) Additivity: if ℓ_1, ℓ_2 and ℓ_3 are Lagrangians in V, while ℓ_1', ℓ_2' and ℓ_3' are Lagrangians in V', then $\ell_i \oplus \ell_i'$ is a Lagrangian in the orthogonal direct sum $V \oplus V'$ and we have

$$\tau(\ell_1 \oplus \ell_1',\ \ell_2 \oplus \ell_2',\ \ell_3 \oplus \ell_3') = \tau(\ell_1,\ \ell_2,\ \ell_3) + \tau(\ell_1',\ \ell_2',\ \ell_3').$$

(iv) Invariance: for any $g \in \mathbf{U}(V)$ (the unitary group), one has

$$\tau(g \cdot \ell_1,\ g \cdot \ell_2,\ g \cdot \ell_3) = \tau(\ell_1,\ \ell_2,\ \ell_3).$$

In fact, in the particular case when $\sigma = \mathrm{Id}$, and thus $k = K$, an anti-Hermitian form is nothing but a symplectic form, and a Hermitian form is just a symmetric, bilinear form. In this setting, with the Maslov index taking its values in the classical Witt ring $W(k)$, the properties above have been established in [6] (over the reals, but the proof is not different for other fields).

To deal with the general case, first note that the real part of an anti-Hermitian form h, that is the map $s(x,y) = \frac{1}{2}(h(x,y) + \overline{h(x,y)})$, is a symplectic form on the space V_k (with scalars restricted to k). Likewise, Hermitian forms give rise to symmetric, bilinear forms, thus yielding a map $R_{K,k}\colon \mathrm{WH}(K,\sigma) \to W(k)$ which is known to be injective ([7]).

Lagrangians for h are Lagrangians for s, and it is immediate that the Maslov index computed in V corresponds to the Maslov index computed in the symplectic space V_k under the map $R_{K,k}$. It follows that the properties (i), (ii), (iii) and (iv) hold in the general situation as well.

The term "cocycle condition" is employed because the map

$$c\colon \mathbf{U}(V) \times \mathbf{U}(V) \longrightarrow \mathrm{WH}(K,\sigma)$$

defined by $c(g,h) = \tau(\ell,\ g\cdot\ell,\ gh\cdot\ell)$ is then a two-cocycle on the unitary group $\mathbf{U}(V)$, for any choice of Lagrangian ℓ. There is a corresponding central extension

$$0 \longrightarrow \mathrm{WH}(K,\sigma) \longrightarrow \widetilde{\mathbf{U}(V)} \longrightarrow \mathbf{U}(V) \longrightarrow 1,$$

in which the group $\widetilde{\mathbf{U}(V)}$ can be seen as the set $\mathbf{U}(V) \times \mathrm{WH}(K,\sigma)$ endowed with the twisted multiplication

$$(g,a)\cdot(h,b) = (gh, a+b+c(g,h)).$$

We conclude these definitions with a simple trick. The constructions above, particularly the definition of the two-cocycle, involve choosing a Lagrangian in an arbitrary fashion. Moreover, the anti-Hermitian space V needs to be hyperbolic, while many spaces arising naturally are not. Thus it is useful to note the following. Starting with any anti-Hermitian space (V,h), put $\mathcal{D}(V) = (V,-h) \oplus (V,-h)$, where the sum is orthogonal. Then $\mathcal{D}(V)$ is non-degenerate if V is, and it is automatically hyperbolic. Indeed, for any $g \in \mathbf{U}(V)$, let Γ_g denote its graph. Then Γ_g is a Lagrangian in $\mathcal{D}(V)$, and in fact $\mathcal{D}(V) = \Gamma_1 \oplus \Gamma_{-1}$. From now on, we will see Γ_1 as our preferred Lagrangian. Note that there is a natural homomorphism $\mathbf{U}(V) \to \mathbf{U}(\mathcal{D}(V))$ which sends g to $1 \times g$.

2.3. Some two-cocycles on the braid groups. Let us consider a homomorphism $r\colon B_n \to \mathbf{U}(V)$ for some anti-Hermitian space V, and let us compose it with the map $\mathbf{U}(V) \to \mathbf{U}(\mathcal{D}(V))$ just described. Let us call $\rho\colon B_n \to \mathbf{U}(\mathcal{D}(V))$ the resulting map. We obtain a two-cocycle on B_n by the formula

$$c(\beta,\gamma) = \tau(\Gamma_1,\ \rho(\beta)\cdot\Gamma_1,\ \rho(\beta\gamma)\cdot\Gamma_1) = \tau(\Gamma_1,\ \Gamma_{r(\beta)},\ \Gamma_{r(\beta\gamma)});$$

indeed this is the pull-back of the two-cocycle on $\mathbf{U}(\mathcal{D}(V))$ defined above.

We show below that this two-cocycle must be a coboundary, so that there must exist a map $f: B_n \to \mathrm{WH}(K)$ such that

$$f(\beta\gamma) = f(\beta) + f(\gamma) + c(\beta, \gamma). \tag{2.1}$$

In other words, we shall see that ρ can be lifted to a map $\tilde{\rho}\colon B_n \to \widetilde{\mathbf{U}(\mathcal{D}(V))}$.

A word of terminology. Given $g, h \in \mathbf{U}(V)$, the particular Maslov index

$$\tau(\Gamma_1,\ \Gamma_g,\ \Gamma_{gh}),$$

which involves the hyperbolic space $\mathcal{D}(V)$, is often called the *Meyer index* of g and h. By extension, we shall also call $c(\beta, \gamma)$ the Meyer index of the braids β and γ, with respect to r. A map $f: B_n \to \mathrm{WH}(K)$ satisfying eq. (2.1) will be called *Meyer-additive* for obvious reasons.

The first thing to notice is:

Proposition 2.1. *Any Meyer-additive function f is conjugation-invariant.*

Proof. We shall need the following simple property of Maslov indices. Let ℓ_1, ℓ_2, ℓ_3 be Lagrangians in some hyperbolic, anti- Hermitian space W with form h. Assume that $\alpha\colon W \to W$ is a linear map such that $h(\alpha(x), \alpha(y)) = -h(x, y)$ (in other words, α is a homomorphism $(W, h) \to (W, -h)$). Then $\alpha \cdot \ell_i$ is a Lagrangian in W, and

$$\tau(\alpha \cdot \ell_1,\ \alpha \cdot \ell_2,\ \alpha \cdot \ell_3) = -\tau(\ell_1,\ \ell_2,\ \ell_3).$$

This is clear.

Now apply this to $W = \mathcal{D}(V) = (-V) \oplus V$ and $\alpha(x, y) = (y, x)$. We obtain in particular

$$\tau(\Gamma_{g^{-1}},\ \Gamma_1,\ \Gamma_h) = -\tau(\Gamma_g,\ \Gamma_1,\ \Gamma_{h^{-1}}),$$

for any two $g, h \in \mathbf{U}(V)$. Using the formal properties of the Maslov index, we may rewrite this

$$\tau(\Gamma_1,\ \Gamma_g,\ \Gamma_{gh}) = \tau(\Gamma_1,\ \Gamma_h,\ \Gamma_{hg}).$$

Thus we see that the Meyer index of g and h is in fact equal to that of h and g. As a result, we see that $f(\beta\gamma)$ is symmetric in β, γ, as we wanted. □

We are now in position to prove:

Proposition 2.2. *In the situation above, there exists a unique Meyer-additive function f on B_n such that $f(\sigma_i) = 0$ for $1 \leq i \leq n-1$.*

We shall talk of the normalized Meyer-additive function associated to the representation r.

Proof. We prove the existence first. Let F_n denote the free group on $n-1$ generators written $\sigma_1, \dots, \sigma_{n-1}$, so that there is a projection map $\pi\colon F_n \longrightarrow B_n$. We put $R = \ker \pi$. The two-cocycle above is certainly trivial when pulled-back to F_n, since the latter has no non-trivial central extensions. Therefore there exists a Meyer-additive function $\bar{f}\colon F_n \to \mathrm{WH}(K)$; what is more, we may (and we do) impose $\bar{f}(\sigma_i) = 0$. We prove now that $\bar{f}(\beta)$ depends only on the class of $\beta \in F_n$ modulo R, so that $\bar{f}$ factors through B_n.

Since the representation r does factor through B_n, the two-cocycle $c(\beta, \gamma)$ vanishes for $\beta \in R$ and any $\gamma \in F_n$. It follows that $\bar{f}(\beta\gamma) = \bar{f}(\beta) + \bar{f}(\gamma)$ in this situation. Therefore, it suffices to show that $\bar{f}$ vanishes on R. Note also that $\bar{f}(\beta\gamma) = \bar{f}(\beta) + \bar{f}(\gamma)$ whenever $\beta\gamma \in R$, for similar reasons.

The previous Proposition applies to $\bar{f}$, and shows that $\bar{f}$ is conjugation-invariant. Thus it is sufficient to show that $\bar{f}$ vanishes on a set of generators for R *as a normal subgroup*. We take for those the commutators $[\sigma_i, \sigma_j]$ for $|i-j| \geq 2$, and $(\sigma_i\sigma_{i+1}\sigma_i)(\sigma_{i+1}\sigma_i\sigma_{i+1})^{-1}$ for $1 \leq i < n-1$.

Meyer-additivity implies, as the reader will check, that $\bar{f}(1) = 0$ and $\bar{f}(\beta^{-1}) = -\bar{f}(\beta)$. As a result $\bar{f}(\beta\gamma^{-1}) = \bar{f}(\beta) - \bar{f}(\gamma)$ whenever $\beta\gamma^{-1} \in R$. Therefore we have reduced the proof to checking that $\bar{f}(\sigma_i\sigma_j) = \bar{f}(\sigma_j\sigma_i)$ and that $\bar{f}(\sigma_i\sigma_{i+1}\sigma_i) = \bar{f}(\sigma_{i+1}\sigma_i\sigma_{i+1})$ for the relevant indices. However $\bar{f}(\sigma_i\sigma_j) = \bar{f}(\sigma_j\sigma_i)$ trivially holds for any pair i, j since we know that $\bar{f}$ is conjugation-invariant.

Going back to the definitions, we see after a little calculation that we need to prove that

$$\tau(\Gamma_1,\ \Gamma_{r(\sigma_i\sigma_{i+1})},\ \Gamma_{r(\sigma_i\sigma_{i+1}\sigma_i)}) = \tau(\Gamma_1,\ \Gamma_{r(\sigma_{i+1})},\ \Gamma_{r(\sigma_{i+1}\sigma_i\sigma_{i+1})}). \qquad (\star)$$

In B_n there is an element α such that $\alpha\sigma_i\alpha^{-1} = \sigma_{i+1}$ (for example $\alpha = \sigma_1\sigma_2\cdots\sigma_{n-1}$). Moreover, for any $\beta \in B_n$ we note that $r(\beta) \times r(\beta)$ is an automorphism of $\mathcal{D}(V)$ which satisfies $r(\beta) \times r(\beta) \cdot \Gamma_{r(\gamma)} = \Gamma_{r(\beta\gamma\beta^{-1})}$. By applying property (iv) of Maslov indices with the automorphism $r(\alpha) \times r(\alpha)$, we see that we only need to prove ($\star$) for $i = 1$, that is, we need to show

$$\tau(\Gamma_1,\ \Gamma_{r(\sigma_1\sigma_2)},\ \Gamma_{r(\sigma_1\sigma_2\sigma_1)}) = \tau(\Gamma_1,\ \Gamma_{r(\sigma_2)},\ \Gamma_{r(\sigma_2\sigma_1\sigma_2)}).$$

Of course $r(\sigma_1\sigma_2\sigma_1) = r(\sigma_2\sigma_1\sigma_2)$, so there are four Lagrangians involved in this equation. Appealing to the cocycle property (ii) of Maslov indices, we obtain the equivalent equation

$$\tau(\Gamma_1,\ \Gamma_{r(\sigma_1\sigma_2)},\ \Gamma_{r(\sigma_2)}) = \tau(\Gamma_{r(\sigma_2)},\ \Gamma_{r(\sigma_1\sigma_2\sigma_1)},\ \Gamma_{r(\sigma_1\sigma_2)}). \qquad (\star\star)$$

Finally consider the element $\beta = \sigma_2^{-1}\sigma_1^{-1}$. In B_n, conjugation by β takes σ_2 to σ_1, it takes $\sigma_1\sigma_2\sigma_1$ to $\sigma_1^2\sigma_2$, and $\sigma_1\sigma_2$ to itself. After applying $r(\beta) \times r(\beta)$ to the right hand side of ($\star\star$), one obtains thus $\tau(\Gamma_{r(\sigma_1)},\ \Gamma_{r(\sigma_1^2\sigma_2)},\ \Gamma_{r(\sigma_1\sigma_2)})$. Now apply $1 \times r(\sigma_1^{-1})$ and you get the left hand side of ($\star\star$). This concludes the proof of the existence of f.

We turn to the uniqueness of Meyer-additive functions. Such a map is clearly determined by its values on the generators σ_i of the braid group. What is more, these generators are all conjugate, as we have seen. Thus f is determined by, say, the value $f(\sigma_1)$, and there can only be one Meyer-additive function vanishing on σ_1, which is stronger than the statement in the Proposition. □

2.4. The Markov conditions. As announced at the beginning of this section, one can hope to produce a Markov function by using a sequence of representations $r_n \colon B_n \to \mathbf{U}(V_n)$, and using for f_n the corresponding normalized, Meyer-additive function, whose coboundary is the two-cocycle c_n. The chief example seems to be the Burau representation, to be described next. Other (unsuccessful) attempts by the authors include the Lawrence–Krammer–Bigelow representation, the representations afforded by Hecke algebras, and those related to the modules of the quantum group $U_q(\mathfrak{sl}_2)$.

It is easy to write down the conditions for the functions f_n to combine into a Markov function. Let us do this now in the special case, covering the Burau representation, when

$$V_{n+1} = V_n \oplus (triv)$$

as B_n-modules, where $(triv)$ refers to a trivial B_n-module. The additivity of the Maslov index immediately implies that $c_{n+1}(\beta,\gamma) = c_n(\beta,\gamma)$ for $\beta,\gamma \in B_n$ (here we see B_n as a subgroup of B_{n+1}, suppressing any inclusion map from the notation). It follows that f_{n+1}, when restricted to B_n, coincides with f_n.

The collection (f_n) is then a Markov function if and only if

$$c_{n+1}(\beta,\sigma_n^{\pm}) = \tau(\Gamma_1,\ \Gamma_{r_{n+1}(\beta)},\ \Gamma_{r_{n+1}(\beta\sigma_n^{\pm})}) = 0, \qquad (2.2)$$

for $\beta \in B_n$.

Remark 2.3. It may (and it will) happen that we have at our disposal a collection of representations $r_{d_n}\colon B_{d_n} \to \mathbf{U}(V_{d_n})$ for some increasing sequence of integers $d_1, d_2, \ldots$, but that r_n is not initially defined for all n. In this case, given an integer n we shall pick the smallest d_m such that $n \leq d_m$, and define r_n to be the composition of the inclusion $B_n \to B_{d_m}$ followed by r_{d_m}.

(In practice this will happen with the Burau representation at $t = -1$, for which the anti-Hermitian form is only non-degenerate for B_{2n}; so for B_{2n-1} we have to consider its inclusion into B_{2n}.)

3. The case of the Burau representation

3.1. Definitions. Initially, the Burau representation is the homomorphism

$$B_n \longrightarrow \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$$

mapping σ_i to the matrix

$$\Sigma_i = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & 1 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

(Some authors use the transpose of this matrix, for example in [4].)

The ring $\mathbb{Z}[t, t^{-1}]$ has an involution σ_0 with $\sigma_0(t) = t^{-1}$. As above we write $\bar{x}$ instead of $\sigma_0(x)$. Now put

$$\Omega_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1-t & 1 & 0 & \cdots & 0 \\ 1-t & 1-t & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-t & 1-t & 1-t & \cdots & 1 \end{pmatrix}.$$

For each i, $1 \leq i < n$, one has ${}^T\overline{\Sigma_i}\Omega_n\Sigma_i = \Omega_n$, where ${}^TA$ denotes the transpose of the matrix A (see [4], Theorem 3.1).

Let K be a field with involution σ, and pick a homomorphism $\alpha\colon \mathbb{Z}[t, t^{-1}] \to K$ compatible with σ_0 and σ. Typical examples will be (i) $K = \mathbb{Q}(t)$ or $\mathbb{F}_p(t)$ with involution σ defined by $\sigma(t) = t^{-1}$ and $\alpha(t) = t$, and (ii) $K = \mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{F}_p$ with trivial involution and $\alpha(t) = -1$.

We let $V_n = K^n$ and view it as a B_n-module by applying α to the coefficients of the Burau representation. Likewise, we may see Ω_n as a matrix with coefficients in K. Now we put

$$H_n = \Omega_n - {}^T\overline{\Omega_n},$$

so that ${}^T H_n = -\overline{H_n}$, and also ${}^T\overline{\Sigma_i} H_n \Sigma_i = H_n$.

The space V_n is thus anti-Hermitian when equipped with the form h_n given by (identifying vectors of V_n with $n \times 1$ matrices)

$$h_n(x, y) = {}^T x H_n y.$$

We are interested in cases when this form is non-degenerate. A simple calculation leads to

Lemma 3.1. *The determinant of H_n is given by*

$$\det H_n = (-1)^n \Big[(1 - \alpha(t))^{n-1} - \Big(\frac{1}{\alpha(t)} - 1\Big)^{n-1}\Big].$$

In particular, when $\alpha(t) \neq 1$, at most one of $\det H_n$ *and* $\det H_{n+1}$ *can be zero.*

From now on we assume that $\alpha(t) \neq 1$ so that this determinant is non-zero for infinitely many values of n.

The form h_n is preserved by B_n, so we end up with a map $r_n\colon B_n \to \mathbf{U}(V_n)$, as requested in the previous section (and bearing Remark 2.3 in mind). Thus we have maps

$$f_n\colon B_n \longrightarrow \mathrm{WH}(K, \sigma),$$

and we shall prove presently that together they give a Markov function, and thus a link invariant.

Our first step is to prove the existence of an auxiliary Markov function:

Proposition 3.2. *For each $n \geq 2$ and each $\beta \in B_n$, put*

$$d_n^K(\beta) = d_n(\beta) = \dim_K \ker(r_n(\beta) - \mathrm{Id}_n).$$

Then $(d_n)_{n\geq 2}$ is a Markov function.

Note that the Hermitian structure does not come into play here. Also, the characteristic of K may very well be 2 in this proposition.

Proof. That d_n is conjugation-invariant is obvious. We claim that, for any $n \times n$ matrix M, the rank of $\widetilde{M}\Sigma_n^{\pm} - \mathrm{Id}_{n+1}$ is one more than the rank of $M - \mathrm{Id}_n$, where $\widetilde{M}$ is the $(n+1) \times (n+1)$ matrix obtained from M by adding a 1 in the bottom right corner. This will imply the result.

It is enough to prove the claim with Σ_n replaced by its transpose ${}^T\Sigma_n$. We observe that, adding the last column of $\widetilde{M}\,{}^T\Sigma_n - \mathrm{Id}_{n+1}$ to the one immediately on its left, we obtain

$$\begin{pmatrix} M - \mathrm{Id}_n & * \\ 0 & -1 \end{pmatrix}.$$

This takes care of Σ_n. The same operation on $\widetilde{M}\,{}^T\Sigma_n^{-1} - \mathrm{Id}_{n+1}$ gives

$$\begin{pmatrix} M - \mathrm{Id}_n & * \\ 0 & \frac{-1}{t} \end{pmatrix}.$$

□

3.2. The main theorem. Before we turn to the proof of the main result, we quote Thomas's criterion for the vanishing of Maslov indices:

Lemma 3.3 (Thomas). *Let ℓ_1, ℓ_2 and ℓ_3 be Lagrangians in the anti-Hermitian space U. If*

$$\dim(\ell_1 \cap \ell_2) + \dim(\ell_2 \cap \ell_3) + \dim(\ell_3 \cap \ell_1) = \dim \ell_1 + 2\dim(\ell_1 \cap \ell_2 \cap \ell_3),$$

then

$$\tau(\ell_1,\ \ell_2,\ \ell_3) = 0.$$

(Of course $\dim \ell_1 = \dim \ell_2 = \dim \ell_3 = \frac{1}{2}\dim U$, so the equality is symmetric in ℓ_1, ℓ_2, ℓ_3.) For a proof see [9], Proposition 4.1. We may now state

Theorem 3.4. *For each $n \geq 2$, let $r_n\colon B_n \to \mathbf{U}(V_n)$ be the homomorphism obtained from the Burau representation as above.*

There is a unique map

$$f_n\colon B_n \longrightarrow \mathrm{WH}(K, \sigma)$$

with the property that $f_n(\sigma_i) = 0$ and

$$f_n(\beta\gamma) = f_n(\beta) + f_n(\gamma) + \tau(\Gamma_1,\ \Gamma_{r_n(\beta)},\ \Gamma_{r_n(\beta\gamma)}),$$

for $\beta, \gamma \in B_n$.

The collection $(f_n)_{n\geq 2}$ is a Markov function.

Given a link L which is isotopic to the closure $\hat{\beta}$ of the braid $\beta \in B_n$, the invariant $f_n(\beta)$ will be written $\Theta_K(L)$ (the dependence on σ and α being implicit).

Proof. At this stage, it remains to prove

$$\tau(\Gamma_1,\ \Gamma_{r_{n+1}(\beta)},\ \Gamma_{r_{n+1}(\beta\sigma_n^{\pm})}) = 0$$

for all $\beta \in B_n$, see eq. (2.2).

We use Thomas's criterion (Lemma 3.3). Note that, as explained in Remark 2.3, the representation r_{n+1} is the composition of an inclusion $B_{n+1} \to B_N$ for a certain N, followed by the Burau representation of B_N. However for the sake of applying Thomas's lemma, all we need is to prove an equality involving the dimensions of certain subspaces made up from

$$\ell_1 = \Gamma_1, \quad \ell_2 = \Gamma_{r_{n+1}(\beta)}, \quad \ell_3 = \Gamma_{r_{n+1}(\beta\sigma_n^{\pm})}.$$

For this, we can and we will assume that $N = n+1$ and so that r_{n+1} is really the Burau representation of B_{n+1}: indeed going into a larger braid group only adds a common summand to the subspaces in sight, which does not affect the equality of dimensions to be proved. So we work with the space $\mathcal{D}(V_{n+1})$ of dimension $2n+2$.

Let us identify the terms in the lemma. First the intersection $\Gamma_1 \cap \Gamma_{r_{n+1}(\beta)}$ is isomorphic, as a vector space, with $\ker(r_{n+1}(\beta) - \mathrm{Id})$. So its dimension is $d_{n+1}(\beta)$, with the terminology of Proposition 3.2. In turn, $d_{n+1}(\beta) = 1 + d_n(\beta)$, clearly.

Proceeding in a similar fashion with the other terms, we see that the equality to check is really

$$1 + d_n(\beta) + d_{n+1}(\beta\sigma_n^{\pm}) + d_{n+1}(\sigma_n^{\pm}) = \\ n + 1 + 2\dim(\ker(r_{n+1}(\beta) - \mathrm{Id}) \cap \ker(r_{n+1}(\sigma_n^{\pm}) - \mathrm{Id})).$$

For simplicity let us now write $\sigma_n^{\pm}$ for $r_{n+1}(\sigma_n^{\pm})$. It is easy to describe $\ker(\sigma_n^{\pm} - \mathrm{Id})$. Indeed, let $v_n \in V_n$ be the vector with coordinates $(1, t, t^2, \ldots, t^{n-1})$; it is readily checked that $\gamma v_n = v_n$ for all $\gamma \in B_n$. We see that

$$\ker(\sigma_n^{\pm} - \mathrm{Id}) = V_{n-1} \oplus K v_{n+1};$$

in particular it has dimension n.

What is more, we see that the dimension of $\ker(r_{n+1}(\beta) - \mathrm{Id}) \cap \ker(\sigma_n^{\pm} - \mathrm{Id})$ is one more than the dimension of $\ker(r_{n+1}(\beta) - \mathrm{Id}) \cap V_{n-1}$. Comparing with the decomposition $V_n = V_{n-1} \oplus K v_n$, we conclude that

$$\dim\ker(r_{n+1}(\beta) - \mathrm{Id}) \cap \ker(\sigma_n^{\pm} - \mathrm{Id}) = \dim\ker(r_n(\beta) - \mathrm{Id}) = d_n(\beta).$$

Putting together these elementary observations, we find that Thomas's criterion reduces to

$$d_n(\beta) + d_{n+1}(\beta\sigma_n^{\pm}) = 2d_n(\beta),$$

which is guaranteed by Proposition 3.2. This concludes the proof. □

4. Examples & Computations

4.1. Signatures. Let us start with the example of $K = \mathbb{R}$, with trivial involution, and $\alpha(t) = -1$. The above procedure yields a link invariant with values in $W(\mathbb{R}) \cong \mathbb{Z}$ (the isomorphism being given by the signature of quadratic forms).

However, Gambaudo and Ghys have proved in [2] that the invariant which is classically called *the signature of a link* is in fact given by a normalized, Meyer-additive Markov function (in our terminology). As a result, $\Theta_{\mathbb{R}}(L)$ must always coincide with the signature of L.

An obvious refinement is obtained by taking $K = \mathbb{Q}$ (and still $\alpha(t) = -1$). The invariant $\Theta_{\mathbb{Q}}(L)$ lives in $W(\mathbb{Q})$. Recall from [7] that there is an exact sequence

$$0 \longrightarrow W(\mathbb{Z}) \longrightarrow W(\mathbb{Q}) \longrightarrow \bigoplus_p W(\mathbb{F}_p) \longrightarrow 0.$$

This sequence is split by the homomorphism $W(\mathbb{Q}) \to W(\mathbb{R}) \cong W(\mathbb{Z})$. Moreover, for p odd the group $W(\mathbb{F}_p)$ is either $\mathbb{Z}/2\times\mathbb{Z}/2$ or $\mathbb{Z}/4$ according as p is 1 mod 4 or not, while $W(\mathbb{F}_2) = \mathbb{Z}/2$. We obtain the invariants alluded to in the introduction.

We need not restrict ourselves to the case $\alpha(t) = -1$, however. For example we may take $K = \mathbb{C}$ with the usual complex conjugation, and $\alpha(t) = \omega$, a complex number of module 1. All of the above generalizes. We obtain a link invariant with values in $\mathrm{WH}(\mathbb{C}) \cong \mathbb{Z}$, whose value on L will be written $\Theta_\omega(L)$.

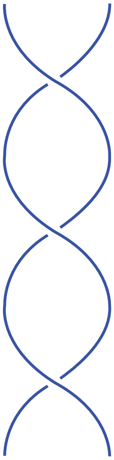
When ω is a root of unity at least, Gambaudo and Ghys also prove in *loc. cit.* that the so-called *Levine–Tristram signature* of a link is given by a normalized, Meyer-additive function, so that it must agree with $\Theta_\omega(L)$. Again, we obtain a refinement. Whenever ω is algebraic, the field $K = \mathbb{Q}(\omega)$ is a number field. There is an exact sequence

$$0 \longrightarrow W(\mathcal{O}) \longrightarrow W(K) \longrightarrow \bigoplus_{\mathfrak{p}} W(\mathcal{O}/\mathfrak{p}) \longrightarrow 0,$$

where $\mathcal{O}$ is the ring of integers in K, and the direct sum runs over the prime ideals $\mathfrak{p}$. Thus the invariant $\Theta_K(L) \in \mathrm{WH}(K) \subset W(K)$ yields invariants in the Witt rings of the various fields $\mathcal{O}/\mathfrak{p}$, which are finite.

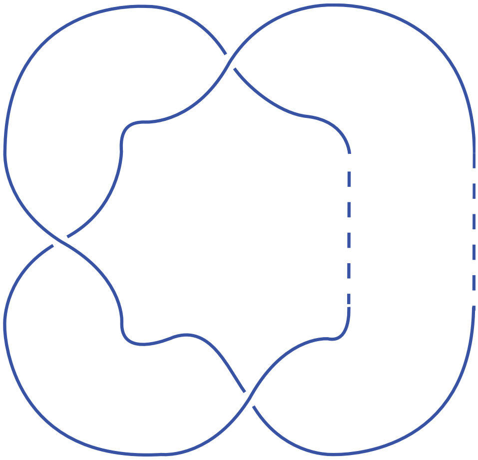
4.2. The case $K = \mathbb{Q}(t)$. Our favorite example is that of $K = \mathbb{Q}(t)$ with $\sigma(t) = t^{-1}$ and $\alpha(t) = t$; in some sense we shall be able recover the signatures of the previous examples from this one. We shall go into more computational considerations than above. The reader who wants to know more about the technical details should consult the accompanying Sage script, available on the authors' webpages. Conversely, this section is a prerequisite for understanding the code.

Consider $\beta = \sigma_1^3 \in B_2$ as a motivational example. Here $L = \hat{\beta}$ is the familiar trefoil knot.



LEFT: the braid σ_1^3.

RIGHT: its closure, the trefoil knot.



When computing $\Theta_{\mathbb{Q}(t)}(L)$ we are led to perform additions in $\mathrm{WH}(\mathbb{Q}(t), \sigma)$. Since a Hermitian form can always be diagonalized, we can represent any element in the Hermitian Witt ring by a sequence of scalars. In turn, these are in fact viewed in $k^\times/N(K^\times)$, where as above $k = K^\sigma$ and $N\colon K \to k$ is the norm map $x \mapsto x\bar{x}$. Summing two elements amounts to concatenating the diagonal entries.

Let us turn to the example of the trefoil knot. We relax the notation, and write f for f_n when n is obvious or irrelevant, and we write c for the two-cocycle $c(\beta, \gamma) = \tau(\Gamma_1, \Gamma_{r(\beta)}, \Gamma_{r(\beta\gamma)})$, so we have the formula $f(\beta\gamma) = f(\beta) + f(\gamma) + c(\beta, \gamma)$. Now,

$$\begin{aligned}\Theta_{\mathbb{Q}(t)}(L) = f(\sigma_1^3) &= f(\sigma_1) + f(\sigma_1^2) + c(\sigma_1, \sigma_1^2)\\ &= f(\sigma_1) + (f(\sigma_1) + f(\sigma_1) + c(\sigma_1, \sigma_1)) + c(\sigma_1, \sigma_1^2)\\ &= 0 + 0 + 0 + c(\sigma_1, \sigma_1) + c(\sigma_1, \sigma_1^2).\end{aligned}$$

Thus $\Theta_{\mathbb{Q}(t)}(L)$ is the sum of two Maslov indices, and direct computation shows that it is represented by

$$\left[1, -1, \frac{-2t^2 + 2t - 2}{t}, 1, -1, 2\right].$$

Now, the Hermitian form given by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is hyperbolic and so represents the trivial element in the Witt ring. We conclude that $\Theta_{\mathbb{Q}(t)}(L)$ is represented by the form whose matrix is

$$\begin{pmatrix} \frac{-2t^2+2t-2}{t} & 0 \\ 0 & 2 \end{pmatrix}.$$

Comparing elements in the Witt ring can be tricky. For example, we need to be able to tell quickly whether this last form is actually 0 or not. In general, link invariants need to be easy to compute and compare.

To this end, we turn to the construction of a Laurent polynomial invariant. There is a well-known homomorphism $D\colon \mathrm{WH}(K,\sigma) \to k^\times/N(K^\times)$ given by the *signed determinant*: given a non-singular, Hermitian $n \times n$-matrix A representing an element in the Witt ring, then $D(A) = (-1)^{\frac{n(n-1)}{2}} \det(A)$. This defines a link invariant with values in $k^\times/N(K^\times)$, and for the trefoil we have

$$D(\Theta_{\mathbb{Q}(t)}(L)) = \frac{-t^2+t-1}{t}.$$

(Note how we got rid of the factor $4 = N(2)$.) This happens to be the Alexander–Conway polynomial of L; see the next section for more on this.

For the sake of practicalities, let us indulge in some computational details.

Lemma 4.1. *Any element in $k^\times/N(K^\times)$ can be represented by a fraction of the form*

$$\frac{D(t)}{t^d},$$

where $D(t)$ is a polynomial in t, of degree $2d$, not divisible by t, and which is also palindromic.

What is more, if D has minimal degree among such polynomials, then it is uniquely defined up to a square in $\mathbb{Q}^\times$.

The proof will also indicate an algorithm to compute the minimal D.

Proof. By definition any element is represented by

$$\frac{F(t)}{G(t)},$$

for some polynomials F and G, so that we may multiply by the norm $G(t)G(t^{-1})$ to obtain a Laurent polynomial representative. Since it must be stable under σ, it has the form

$$\frac{R(t)}{t^d}$$

with R palindromic of degree $2d$, not divisible by t.

We turn to the uniqueness. Given a polynomial P, consider $\widetilde{P} = t^{\deg P} P(t^{-1})$, which is again a polynomial. In this notation we have $R = \widetilde{R}$. The assignment $P \mapsto \widetilde{P}$ is multiplicative, that is $\widetilde{PQ} = \widetilde{P}\widetilde{Q}$; moreover, performing this operation twice on a polynomial P not divisible by t gives again P (while in general the power of t dividing P disappears from $\widetilde{\widetilde{P}}$, as follows from $\tilde{t} = 1$). We conclude that, when P is an irreducible polynomial, prime to t, then $\widetilde{P}$ is also irreducible.

Now factor $R = \prod P_i^{\alpha_i}$ into a product of powers of prime polynomials. For a given P_i, we have one of two options. The first possibility is that $\widetilde{P_i}$ is prime to P_i (for example if $P_i = t^2 - 2$), so that $R = \widetilde{R}$ is divisible by $P_i \widetilde{P_i}$. In this case we divide R/t^d by the norm of $P_i^{\alpha_i}$, which has the effect of replacing R by a polynomial of smaller degree with all the same properties as R. Do this for all such factors P_i.

The remaining factors P_i of the new R are all of the second type, that is $\widetilde{P_i}$ is a scalar multiple of P_i (for example $P_i = t - 1$). Write $\alpha_i = 2v_i + \varepsilon_i$ with $\varepsilon_i = 0$ or 1, and divide R by the norm of $P_i^{v_i}$. Do this for all the factors.

There results a polynomial, which we still call R, with the same properties as above, and with the extra feature that it factors as a product $R = \prod P_i$ where each P_i is a scalar multiple of $\widetilde{P_i}$ (the various P_i's being pairwise coprime). We now show that this R can be taken for D.

Let Q be any polynomial such that Q/t^q and R/t^d represent the same element in $k^\times/N(K^\times)$. We prove that R divides Q, which certainly implies the uniqueness statement.

Indeed there must exist coprime polynomials F and G and an integer k such that

$$RG\widetilde{G} = t^k QF\widetilde{F}.$$

Pick a prime factor P_i in R. If P_i does not divide Q, then it must divide one of F or $\widetilde{F}$; hence $\widetilde{P_i}$ divides the other one. However since $\widetilde{P_i} = P_i$ up to a scalar, we see that P_i^2 divides $QF\widetilde{F}$, so that P_i^2 divides $RG\widetilde{G}$. We know that P_i^2 does not divide R, so P_i divides one of G or $\widetilde{G}$, hence both. We see that F and G have a factor in common, a contradiction which shows that P_i divides Q. □

Here is another simple invariant deduced from $\Theta_{\mathbb{Q}(t)}$. Suppose θ is a real number such that $e^{i\theta}$ is not algebraic (this excludes only countably many possibilities for θ). The assignment $t \mapsto e^{i\theta}$ gives a field homomorphism $\mathbb{Q}(t) \to \mathbb{C}$ which is compatible with the involutions (on the field of complex numbers we use the standard conjugation). There results a map

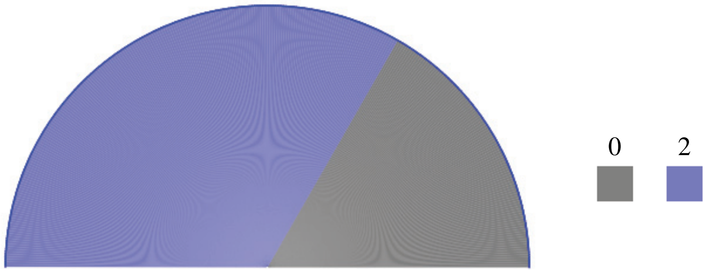
$$\mathrm{WH}(\mathbb{Q}(t), \sigma) \longrightarrow \mathrm{WH}(\mathbb{C}) \cong \mathbb{Z},$$

which we call the θ-signature. It is clear by construction that it agrees with the invariant $\Theta_{e^{i\theta}}$ presented in §4.1.

Looking at the trefoil again, we obtain the form over $\mathbb{C}$

$$\begin{pmatrix} 2-4\cos(\theta) & 0 \\ 0 & 2 \end{pmatrix}$$

whose signature is 0 if $0 < \theta < \frac{\pi}{3}$ and 2 if $\frac{\pi}{3} < \theta < \pi$ (the diagonal entries are always even functions of θ, so we need only consider the values between 0 and π.) We may present this information with the help of a camembert:



This figure is a link invariant.

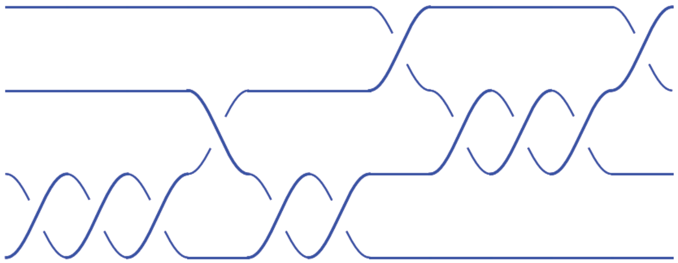
We may get rid of the restriction on θ. Given an element in $\mathrm{WH}(\mathbb{Q}(t))$, pick a diagonal matrix as representative, and arrange to have Laurent polynomials as entries. Now substitute $e^{i\theta}$ for t, obtaining a Hermitian form over $\mathbb{C}$, and consider the function which to θ assigns the signature of this form. This is a step function s, which is even and 2π-periodic.

Now, whenever θ is such that $e^{i\theta}$ is not algebraic, the value $s(\theta)$ is intrinsically defined by the procedure above, and thus does not depend on the choice of representative. Since such θ are dense in $\mathbb{R}$, the following is well-defined:

$$\hat{s}(\theta) = \lim_{\alpha \to \theta, \alpha > \theta} s(\alpha).$$

At least this provides a definition for all θ, though it is not so easy to work with it. This may well change in the future when we prove that it is possible to work with the ring $\mathbb{Z}[\frac{1}{2}, t, t^{-1}]$ rather than the field $\mathbb{Q}(t)$. This will prove that, at least when θ is not a “jump” for s, the value $\hat{s}(\theta)$ agrees with $\Theta_{e^{i\theta}}$.

To give a more complicated example, take $\sigma_1^3\sigma_2^{-1}\sigma_1^2\sigma_3^1\sigma_2^3\sigma_3^1 \in B_4$. The braid looks as follows:

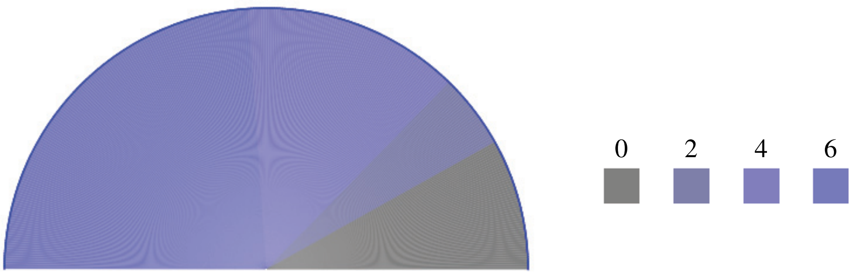


The signed determinant is

$$\frac{3t^6 - 9t^5 + 15t^4 - 17t^3 + 15t^2 - 9t + 3}{t^3},$$

where the numerator has minimal degree.

The camembert is



On the authors' webpages, the reader will find many examples of links for which the corresponding camemberts and polynomials are given.

4.3. Comparison with the Alexander–Conway polynomial. Let us expand a little on the above "coincidence" in the case when L is the trefoil knot, for which $D(\Theta_{\mathbb{Q}(t)}(L))$ happens to be equal to the Alexander–Conway polynomial of L. And first, since there are several (related) polynomials with that name, let us add that we consider the polynomial $\nabla_L(s)$ as defined in [4], §3.4.2. It is the only link invariant with values in $\mathbb{Z}[s, s^{-1}]$ satisfying the skein relation

$$\nabla_{L_+}(s) - \nabla_{L_-}(s) = (s^{-1} - s)\nabla_{L_0}(s),$$

in standard notation. It follows from this definition that $\nabla_L(s) = \nabla_L(-s^{-1})$, since $L \mapsto \nabla_L(-s^{-1})$ is a link invariant satisfying the same skein relation (and taking the value 1 on the unknot).

It is well known that when L is a knot, rather than just a link, one has $\nabla_L(s) = P_L(s^2)$ for some Laurent polynomial P_L. It follows that $P_L(t) = P_L(t^{-1})$. Thus P_L defines an element in $k^\times/N(K^\times)$, and we may compare it with $D(\Theta_{\mathbb{Q}(t)}(L))$.

We have carried a computer experiment, whose result is that *for all the knots we have tested, the Alexander–Conway polynomial and $D(\Theta_{\mathbb{Q}(t)}(L))$ are equal in $k^\times/N(K^\times)$*. The experiment was conducted on 157 (pairwise non-isotopic) knots. It is of course reasonable to conjecture that this holds for all knots, though we do not have a proof of this fact.

5. Examples related to the Weil representation

5.1. Background on the Weil representation. Consider $\mathbf{Sp}_{2n}(\mathbb{R})$, the symplectic group over the reals. Its fundamental group is $\mathbb{Z}$, so this group has a twofold cover, usually called the metaplectic group and denoted by $M_{2n}(\mathbb{R})$.

The metaplectic group is famous for having a semi-simple representation on the Hilbert space $L^2(\mathbb{R}^n)$, called the Weil representation; we refer to [6] for a description. We are chiefly interested in the fact that this representation has a *trace*, in the following sense. Let us write $T(g)$ for the operator corresponding to g. For any smooth function f with compact support on $M_{2n}(\mathbb{R})$, we may define an operator $\bar{T}(f)$ by the obvious averaging process, that is

$$\bar{T}(f) = \int_{M_{2n}(\mathbb{R})} f(g)T(g)dg.$$

(Here a unimodular Haar measure is employed.) This operator has a trace in the naive sense, namely for any Hilbert basis $(e_r)_{r\in\mathbb{N}}$, one has

$$\sum_r \langle \bar{T}(f)e_r, e_r\rangle < +\infty.$$

What is more, the sum above can be computed as the integral

$$\int_{M_{2n}(\mathbb{R})} f(g)\theta_n(g)dg.$$

where θ_n is a smooth function defined on a dense, open set.

In [9], Thomas gives a description of the map θ_n, which we partially reproduce below. It turns out that θ_n^{-1} extends to a continuous function on the whole of $M_{2n}(\mathbb{R})$, which is conjugation-invariant.

Since our construction in the case $K = \mathbb{R}$, $\alpha(t) = -1$ already considered gives a map $r_{2n}\colon B_{2n} \to \mathbf{Sp}_{2n}(\mathbb{R})$, and since the cohomological considerations of §2.3 guarantee that it lifts to a map $B_{2n} \to M_{2n}(\mathbb{R})$, one may wish to use θ_n^{-1} in order to produce a Markov function. It turns out that this works! In fact the corresponding link invariant can be expressed in terms of the signature and the Alexander–Conway polynomial, so it is certainly not new. It is however striking to think that the Weil representation should have any relationship to links.

Let us turn to the proof. It will be concluded together with the precise statement of Theorem 5.5.

5.2. An alternative construction of Meyer-additive functions. Let us return to the setting of §2.3: we fix a representation $r\colon B_n \to \mathbf{U}(V)$, and consider the composition $\rho\colon B_n \to \mathbf{U}(\mathcal{D}(V))$. We keep the notation c for the two-cocycle produced.

In order to relate Thomas's construction to our own, we shall present an alternative description of the normalized, Meyer-additive function f corresponding to c. This involves choosing a Lagrangian ℓ in V in an arbitrary way, which is not only less satisfying but will also only work when V is hyperbolic. For extreme simplicity, we restrict to the case $K = \mathbb{R}$ with trivial involution, so $\mathrm{WH}(K,\sigma) = W(\mathbb{R}) \simeq \mathbb{Z}$. Also $\mathbf{U}(V) = \mathbf{Sp}(V)$ in this case. This is enough for our purposes, though the reader can easily generalize what follows.

Having chosen ℓ, we can consider the function μ defined on $\mathbf{Sp}(V) \times \mathbf{Sp}(V)$ by

$$\mu(g,h) = \tau(\ell,\ g\ell,\ gh\ell) \in W(\mathbb{R}).$$

This is again a two-cocyle on $\mathbf{Sp}(V)$, different from c, and we pull it back to a two-cocyle on B_n as before. Since $H^2(B_n,\mathbb{Z}) = 0$, we deduce the existence of a map $w\colon B_n \to \mathrm{WH}(K,\sigma)$ whose coboundary is μ. The relationship between f and w is given by the following Proposition, in which we write $\beta \mapsto \langle\beta\rangle$ for the homomorphism $B_n \to \mathbb{Z}$ sending each σ_i to 1.

Proposition 5.1. *There exists an integer k such that, for any $\beta \in B_n$ we have*

$$f(\beta) = w(\beta) + \tau(\Gamma_\beta,\ \Gamma_1,\ \ell \oplus \ell) + k\langle\beta\rangle.$$

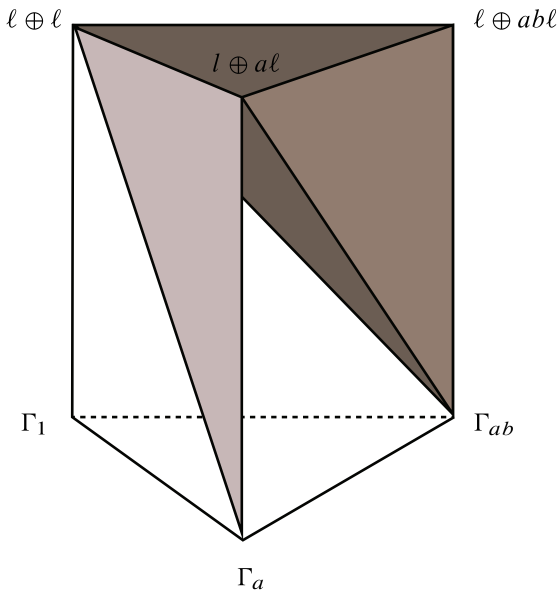
This should be compared to Proposition 1.2 in [9].

Proof. Put $f'(\beta) = w(\beta) + \tau(\Gamma_\beta,\ \Gamma_1,\ \ell\oplus\ell)$. We shall prove that the function f' is Meyer-additive with respect to c. As a result $f - f'$ is a homomorphism $B_n \to \mathbb{Z}$; however the abelianization B_n^{ab} of the braid group B_n is isomorphic to $\mathbb{Z}$ via the length map, so the Proposition follows.

By definition, after a minor rearrangement of the terms, we need to prove that

$$\tau(\Gamma_1,\ \Gamma_a,\ \Gamma_{ab}) - \tau(\ell \oplus \ell,\ \ell \oplus a\ell,\ \ell \oplus ab\ell) = \\ \tau(\Gamma_{ab},\ \Gamma_1,\ \ell \oplus \ell) + \tau(\Gamma_1,\ \Gamma_a,\ \ell \oplus \ell) + \tau(\Gamma_1,\ \Gamma_b,\ \ell \oplus \ell),$$

for all $a, b \in B_n$ (here we write Γ_a for $\Gamma_{\rho(a)}$, and so on). Applying the unitary automorphism $1 \times a$ (in $\mathcal{D}(V)$), we see that the very last term may be replaced by $\tau(\Gamma_a,\ \Gamma_{ab},\ \ell \oplus a\ell)$. The situation is summed up on the following diagram:



On this figure, a triangulation of a triangular prism is presented (it is meant to include 3-simplices). The vertices are decorated with Lagrangians, and any triangle with an orientation defines a Maslov index unambiguously (by the dihedral symmetry property (i)). Choose orientations consistently around the figure.

Now, the cocycle property of Maslov indices ensures that the sum of the indices on the boundary is zero. Moreover, three of these indices are already zero, by the lemma below: they are drawn in gray. The result follows. □

We have made use of the following:

Lemma 5.2. *Let $g \in \mathbf{U}(V)$ and ℓ, ℓ' be Lagrangians in V, then*

$$\tau(\Gamma_g, \ell \oplus g\ell, \ell \oplus \ell') = 0.$$

Proof. Thomas's representative (as in Lemma 3.3) is of dimension 0. □

5.3. Thomas's model. Since we are going to rely on the computations by Thomas in [9], we need to bridge our notation with his.

Identify once and for all $W(\mathbb{R})$ with $\mathbb{Z}$ via the signature. The "Weil character" is the homomorphism $\gamma\colon W(\mathbb{R}) \to \mathbb{C}^\times$ given by $\gamma(x) = e^{\frac{i\pi}{4}x}$; note that in general there is a Weil character for each embedding $\psi\colon \mathbb{R} \to \mathbb{C}^\times$, and our choice corresponds to $\psi(x) = e^{2i\pi x}$ (see [6] p. 112). The Weil representation itself also depends on ψ, and from now we shall only consider the representation corresponding to our choice of ψ. The consistency is important for Theorem 5.3 below.

Let V denote $\mathbb{R}^{2n}$ with the usual symplectic structure, so that $\mathbf{Sp}(V)$ refers to the group denoted $\mathbf{U}(V)$ in the general situation of §5.2, while in §5.1 we wrote $\mathbf{Sp}_{2n}(\mathbb{R})$ for the same group.

Fix a Lagrangian ℓ for the rest of the discussion. The cocycle μ of §5.2 is defined, and we compose it with γ to obtain a two-cocycle on $\mathbf{Sp}(V)$ with values in $\mathbb{C}^\times$ (in fact, in the 8-th roots of unity). We are interested in the corresponding central extension of $\mathbf{Sp}(V)$ which explicitly is the set $M_1(V) = \mathbf{Sp}(V) \times \mathbb{C}^\times$ with multiplication

$$(g, t) \cdot (h, s) = (gh, ts\gamma(\mu(g, h))).$$

The group $M_1(V)$ is "compatible" with our notation so far, and will be used for explicit constructions.

Consider now the set $Gr(V)$ of all Lagrangians in V. Any pair $(g, t) \in M_1(V)$ gives rise to a function $t_g : Gr(V) \to \mathbb{C}^\times$ defined by

$$t_g(\ell') = \gamma(\tau(\ell, g\ell, g\ell', \ell'))t. \tag{†}$$

Conversely given g and the function t_g, one can of course recover t as $t_g(\ell)$. So we can consider the group $M_2(V)$, canonically isomorphic to $M_1(V)$, whose elements are all pairs (g, t_g) where $t_g\colon Gr(V) \to \mathbb{C}^\times$ is a map satisfying (†) for some complex number t. A word of warning: we borrow the notation t_g from Thomas's paper already cited, even though it may be slightly misleading in suggesting that t_g can be obtained from g (there is *no* section $\mathbf{Sp}(V) \to M_2(V)$).

Finally within $M_2(V)$, Thomas considers the subgroup $\mathbf{Mp}(V)$ of pairs (g, t_g) such that $t_g^2 = m_g^2$, where m_g is some function on $Gr(V)$ whose definition is irrelevant for our purposes. He shows that $\mathbf{Mp}(V)$ is a two-fold cover of the group $\mathbf{Sp}(V)$ (see [9], definition in §5.2, Proposition 5.1 and Proposition 5.3). Since this cover is non-trivial, it must be a model of what is universally called the metaplectic group.

We are finally in position to state Thomas's main result, computing the values of the function θ_n introduced in §5.1. It is defined on $\mathbf{Mp}(V)$ and Thomas uses the notation $\mathrm{Tr}\,\rho(g, t_g)$ (our own usage of the letter ρ is unrelated). The following is Theorem 2A in [9].

Theorem 5.3 (Thomas). *The trace of the Weil representation is given by*

$$\theta_n(g, t_g) = \frac{t_g(\ell) \cdot \gamma(\tau(\Gamma_g,\ \Gamma_1,\ \ell \oplus \ell))}{|\det(g-1)|^{\frac{1}{2}}}.$$

Let us introduce a few more groups. Let $M_1'(V)$ be the subgroup of $M_1(V)$ of pairs (g, t) with t an 8-th root of unity, and let $M_2'(V)$ be the corresponding subgroup of $M_2(V)$. Since the function m_g takes its values also in the 8-th roots of unity, it is clear that $\mathbf{Mp}(V) \subset M_2'(V)$. To finish with, recall the group $\widetilde{\mathbf{Sp}(V)}$, made of pairs (g, n) with $g \in \mathbf{Sp}(V)$ and $n \in \mathbb{Z}$ with the multiplication using the cocycle μ.

Lemma 5.4. *The group* $\mathbf{Mp}(V)$ *is normal in* $M_2'(V)$.

Proof. In [6] it is proved that $\widetilde{\mathbf{Sp}(V)}$ has four connected component (for some appropriate topology which is not the product topology on $\mathbf{Sp}(V) \times \mathbb{Z}$); more precisely in 1.7.11 in *loc. cit.* one finds the definition of a continuous character

$$s \colon \widetilde{\mathbf{Sp}(V)} \longrightarrow \mathbb{Z}/4$$

whose fibres are exactly the connected components. What is more, $s(g, n)$ depends only on n modulo 4.

The Weil character γ gives a continuous and surjective map

$$\widetilde{\mathbf{Sp}(V)} \longrightarrow M_1'(V),$$

and it is clear that s factors through $M_1'(V)$. As a result, the group $M_1'(V)$ also has four connected components. They are homeomorphic to each other, and each component is thus a two-fold cover of $\mathbf{Sp}(V)$, since $M_1'(V)$ is an eight-fold cover.

The connected component G of the identity is a normal subgroup of $M_1'(V)$, so it suffices to prove that $\mathbf{Mp}(V) = G$ (identifying $M_1'(V)$ and $M_2'(V)$). However $\mathbf{Mp}(V) \subset G$ since $\mathbf{Mp}(V)$ is connected, and since these groups are both two-fold covers of $\mathbf{Sp}(V)$ one must have $\mathbf{Mp}(V) = G$. □

5.4. Main result. Let the notation be as in §5.1, so that we work over $K = \mathbb{R}$ with the Burau representation at $t = -1$, written $r_{2n}\colon B_{2n} \to \mathbf{Sp}(V)$. As in §5.2, we pick a one-cocycle w whose boundary is the two-cocycle μ, a Lagrangian ℓ being chosen. The map $\beta \mapsto (r_{2n}(\beta), \gamma(w(\beta)))$ defines a lift

$$\widetilde{r_{2n}}\colon B_{2n} \longrightarrow M_1'(V)$$

of r_{2n}. Let us show that we can arrange for $\widetilde{r_{2n}}$ to take its values in the metaplectic group. Indeed, one may add any homomorphism $B_n \to \mathbb{Z}$ to w, so we can certainly have $w(\sigma_1)$ taking any convenient value, and in particular we can have $\widetilde{r_{2n}}(\sigma_1) \in \mathbf{Mp}(V)$ (as before we see $\mathbf{Mp}(V)$ as a subgroup of $M_1(V)$). Since σ_i is conjugated to σ_1 in B_n, and in virtue of Lemma 5.4, this forces the image of $\widetilde{r_{2n}}$ to lie entirely in $\mathbf{Mp}(V)$.

Theorem 5.5. *There exists an integer $k = k_n$ such that*

$$e^{\frac{ki\pi}{4}\langle\beta\rangle}\theta_n^{-1}(\widetilde{r_{2n}}(\beta)) = e^{-\frac{i\pi}{4}\Theta_{\mathbb{R}}(\hat{\beta})}|\det(r_{2n}(\beta) - \mathrm{Id})|^{\frac{1}{2}}.$$

In particular the collection $(e^{\frac{k_n i\pi}{4}}\theta_n^{-1} \circ \widetilde{r_{2n}})_{n\geq 1}$ can be extended to a Markov function.

In other words, after a simple renormalization involving the braid "exponent sum", the trace of the Weil representation yields an oriented link invariant. We shall not attempt to compute the value of k_n, as the link invariant does not contain "new" information anyway (see comments in the proof).

Proof. Note that the right hand side of the equation to prove defines a Markov function, since $|\det(r_{2n}(\beta) - \mathrm{Id})|$ is the absolute value of the Alexander–Conway polynomial at $\sqrt{-1}$. Of course $\Theta_{\mathbb{R}}(\hat{\beta})$ is the signature of $\hat{\beta}$, as already observed.

As for the equality itself, it now follows directly from Theorem 5.3 combined with Proposition 5.1. ☐

5.5. Finite fields. The symplectic groups over finite fields also have a Weil representation (with no need to go to a two-sheeted cover), so one may wish to try and use it to obtain more link invariants. This was investigated by Goldschmidt and Jones in [3]. Here is our interpretation of their results.

Pick $K = \mathbb{F}_p(t)$ for an odd prime p, with involution satisfying $\sigma(t) = t^{-1}$, and choose α so that $\alpha(t) = t$. Our general method produces an invariant $\Theta_{\mathbb{F}_p(t)}$ with values in $\mathrm{WH}(\mathbb{F}_p(t), \sigma)$. We would like to specialize t to a value in some finite field $\mathbb{F}_q$; however, there is no field homomorphism $\mathbb{F}_p(t) \to \mathbb{F}_q$, so this idea cannot be pursued literally. When we prove the existence of an invariant in the Witt ring of $\mathbb{Z}[\frac{1}{2}, t, t^{-1}]$, we shall be able to specialize t directly. For the time being, here is an *ad hoc* trick, which amounts to the description given by Goldschmidt and Jones.

Let $u = t + t^{-1}$, so that the fixed field of σ is $k = \mathbb{F}_p(u)$. Recall that we have an anti-Hermitian space V_n, over the field K, on which B_n acts unitarily. As explained in §2.2, we may consider the "real part" of the anti-Hermitian form, which is a symplectic form on $(V_n)_k$, that is V_n viewed as a k-vector space. This allows the formation of Maslov indices, and clearly yields a link invariant, whose value on a link L is simply the image of $\Theta_{\mathbb{F}_p(t)}(L)$ under the map $\mathrm{WH}(\mathbb{F}_p(t), \sigma) \to W(\mathbb{F}_p(u))$. What is noticeable at this point is that the matrices giving the action of B_n on $(V_n)_k$, as well as the symplectic form, all have their entries in $\mathbb{F}_p[u, u^{-1}]$ and thus may be specialized to a non-zero value of u chosen in any finite field $\mathbb{F}_q$ of characteristic p. If we define our Maslov indices *then*, we end up with an invariant with values in $W(\mathbb{F}_q)$, which we may call $\Theta_{\mathbb{F}_q}$.

Let us go back to the trace of the Weil representation of $\mathbf{Sp}_{2n}(\mathbb{F}_q)$. It was also computed by Thomas in [9]. His formula (Theorem 2B) shows that the trace of an element of the form $r_n(\beta)$, where $r_n\colon B_n \to \mathbf{Sp}_{2n}(\mathbb{F}_q)$ is the representation just defined, can be expressed in terms of $\Theta_{\mathbb{F}_q}(\hat{\beta})$ and the invariant of Proposition 3.2 (details will be omitted). In particular, it gives a link invariant.

References

[1] F. R. Cohen, Th. J. Lada, and J. P. May, *The homology of iterated loop spaces.* Lecture Notes in Mathematics, Vol. 533. Springer, Berlin etc., 1976. MR 0436146 Zbl 0334.55009

[2] J.-M. Gambaro and È. Ghys, Braids and signatures. *Bull. Soc. Math. France* **133** (2005), 541–579. MR 2233695 Zbl 1103.57001

[3] D. M. Goldschmidt and V. F. R. Jones, Metaplectic link invariants. *Geom. Dedicata* **31** (1989), 165–191. MR 1012438 Zbl 0678.57007

[4] Ch. Kassel and V. G. Turaev, *Braid groups.* With the graphical assistance of O. Dodane. Graduate Texts in Mathematics, 247. Springer, New York, 2008. MR 2435235 Zbl 2435235

[5] W. B. R. Lickorish, *An introduction to knot theory.* Graduate Texts in Mathematics, 175. Springer, New York, 1997. MR 1472978 Zbl 0886.57001

[6] G. Lion and M. Vergne, *The Weil representation, Maslov index and theta series.* Progress in Mathematics, 6. Birkhäuser, Boston, Mass., 1980. MR 0573448 Zbl 0444.22005

[7] J. W. Milnor and D. H. Husemoller, *Symmetric bilinear forms.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 73. Springer, Berlin etc., 1973 MR 0506372 Zbl 0292.10016

[8] K. Reidemeister, *Knotentheorie*. Ergebnisse der Mathematik und ihrer Grenzgebiete, I. Bd. Heft 1. Springer, Berlin, 1932. Reprint, Springer, Berlin etc., 1974. English transl., *Knot theory*. Translated from the German by L. F. Boron, Ch. O. Christenson, and B. A. Smith. BCS Associates, Moscow, Idaho, 1983. MR 0717222 (reprint) MR 0717222 (transl.) JFM 58.1202.04 Zbl 0277.55007 (reprint) Zbl 0519.57004 (transl.)

[9] T. Thomas, The character of the Weil representation. *J. Lond. Math. Soc.* (2) **77** (2008), 221–239. MR 2389926 Zbl 1195.11058

Received February 8, 2012

Gaël Collinet, Université de Strasbourg & CNRS,
Institut de Recherche Mathématique Avancée, 7 rue René Descartes, 67084
Strasbourg, France

e-mail: collinet@math.unistra.fr

Pierre Guillot, Université de Strasbourg & CNRS,
Institut de Recherche Mathématique Avancée, 7 rue René Descartes, 67084
Strasbourg, France

e-mail: guillot@math.unistra.fr