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The logarithms of Dehn twists

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Abstract. By introducing an invariant of loops on a compact oriented surface with one boundary component, we give an explicit formula for the action of Dehn twists on the completed group ring of the fundamental group of the surface. This invariant can be considered as "the logarithms" of Dehn twists. The formula generalizes the classical formula describing the action on the first homology of the surface, and Morita's explicit computations of the extended first and the second Johnson homomorphisms. For the proof we use a homological interpretation of the Goldman Lie algebra in the framework of Kontsevich's formal symplectic geometry. As an application, we prove that the action of the Dehn twist of a simple closed curve on the k-th nilpotent quotient of the fundamental group of the surface depends only on the conjugacy class of the curve in the k-th quotient.

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Contents

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1. Introduction

Let Σ be a compact oriented C^∞-surface of genus $g > 0$ with one boundary component, and $\mathcal{M}_{g,1}$ the mapping class group of Σ relative to the boundary. In other words, $\mathcal{M}_{g,1}$ is the group of diffeomorphisms of Σ fixing the boundary $\partial\Sigma$ pointwise, modulo isotopies fixing the boundary pointwise. Choose a basepoint $*$ on the boundary $\partial\Sigma$. The group $\mathcal{M}_{g,1}$ (faithfully) acts on $\pi = \pi_1(\Sigma, *)$, hence on the nilpotent quotients of π. For example, $\mathcal{M}_{g,1}$ acts on the first homology group $H_1(\Sigma; \mathbb{Z}) \cong \pi/[\pi, \pi]$, and this gives rise to the classical representation

$$\mathcal{M}_{g,1} \longrightarrow \mathrm{Sp}(2g; \mathbb{Z}),$$

whose kernel is called the Torelli group $\mathcal{I}_{g,1}$. Let $\mathcal{M}_{g,1}[k]$, $k \geq 1$, be the kernel of the action of $\mathcal{M}_{g,1}$ on the k-th nilpotent quotient of π. In particular, $\mathcal{M}_{g,1}[1] = \mathcal{I}_{g,1}$. The quotients $\mathcal{M}_{g,1}/\mathcal{M}_{g,1}[k]$ serve as approximations of $\mathcal{M}_{g,1}$, and the successive quotients $\mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1]$ can be seen as particles of them. For a systematic study of these particles Johnson [11] and [12] introduced a series of group homomorphisms

$$\tau_k \colon \mathcal{M}_{g,1}[k] \longrightarrow \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_k^{\mathbb{Z}}), \quad k \geq 1,$$

which induce the injections

$$\tau_k \colon \mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1] \hookrightarrow \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_k^{\mathbb{Z}}), \quad k \geq 1.$$

Here $H_{\mathbb{Z}} = H_1(\Sigma; \mathbb{Z})$ and $\mathcal{L}_k^{\mathbb{Z}}$ is the degree k-part of the free Lie algebra generated by $H_{\mathbb{Z}}$. The homomorphism τ_k is nowadays called *the k-th Johnson homomorphism*. He extensively studied the first and the second Johnson homomorphisms, and in [13] he proved that τ_1 gives the free part of the abelianization of $\mathcal{I}_{g,1}$.

One of several significant developments which followed the initial works of Johnson is about extensions of the Johnson homomorphisms to the whole mapping class group. In [22], Morita showed that the first Johnson homomorphism τ_1 extends to the whole group $\mathcal{M}_{g,1}$ as a crossed homomorphism, denoted by $\tilde{k} \in Z^1(\mathcal{M}_{g,1}; \frac{1}{2}\Lambda^3 H_{\mathbb{Z}})$. Here, $\Lambda^3 H_{\mathbb{Z}}$ is the third exterior product of $H_{\mathbb{Z}}$. He also showed that the extension is unique up to coboundaries. The arguments in [22] are supported by many explicit computations on Humphreys generators, which are generators of $\mathcal{M}_{g,1}$ consisting of several Dehn twists. In [23] and [24], Morita further showed that the second Johnson homomorphism τ_2 also extends to the whole $\mathcal{M}_{g,1}$ as a crossed homomorphism, and again did many explicit computations.

After the works of Morita, there have been known several studies including Hain [10] and Day [7] and [8] about extensions of the Johnson homomorphisms to the whole mapping class group. Another approach by using the notion of *generalized Magnus expansions* is developed in [14]. Hereafter, let $H = H_1(\Sigma; \mathbb{Q})$.

Roughly speaking, a Magnus expansion in the sense of [14] is an identification

$$\theta\colon \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}$$

as complete augmented algebras, where $\widehat{\mathbb{Q}\pi}$ is the completed group ring of π and $\widehat{T}$ is the completed tensor algebra generated by H. Once we choose a Magnus expansion θ, then we have an injective homomorphism

$$T^\theta\colon \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}(\widehat{T})$$

called *the total Johnson map associated to* θ. The map T^θ can be understood as a tensor expression of the action of $\mathcal{M}_{g,1}$ on the completed group ring of π, since $\theta\colon \widehat{\mathbb{Q}\pi} \overset{\cong}{\to} \widehat{T}$ is an isomorphism. For details, see §2.5. As was clarified in [14], T^θ induces θ-dependent extensions of all the Johnson homomorphisms τ_k, denoted by τ_k^θ where $k \geq 1$, to the whole mapping class group. The maps τ_k^θ's are no longer homomorphisms, and are not crossed homomorphisms if $k \geq 2$, but satisfy an infinite sequence of coboundary conditions.

Note that the fundamental group π is a free group of rank $2g$. Actually the treatment in [14] is on $\mathrm{Aut}(F_n)$, the automorphism group of a free group of rank n, rather than the mapping class group. As long as we just regard π as a free group, it does not seem a matter of concern which Magnus expansion we choose. However, recently Massuyeau [20] introduced the notion of *a symplectic expansion*, which seems suitable for the study of $\mathcal{M}_{g,1}$ from the view point of Magnus expansions. A symplectic expansion is a Magnus expansion of π recording the fact that π has a particular element corresponding to the boundary of Σ. For precise definition, see §2.4.

In this paper, we begin a quantitative approach to the topology of Σ and the mapping class group $\mathcal{M}_{g,1}$ via a symplectic expansion. The primary theme is the Dehn twist formula for the total Johnson map associated to a symplectic expansion. As was stated above, Dehn twists generate the mapping class group $\mathcal{M}_{g,1}$. We introduce an invariant of loops on Σ, and derive a formula for the values of T^θ on Dehn twists in terms of this invariant. It is classically known that the action of a Dehn twist on the homology of an oriented surface is given by transvection. Our formula can be seen as a generalization of this fact. Moreover, it gives formulas for the extensions τ_k^θ and recovers some computations of Morita on the extended τ_1, and τ_2. Behind our proof of the above formula, a close relationship between the Goldman Lie algebra of Σ and formal symplectic geometry plays a vital role. The relationship is established via a symplectic expansion, and this is another theme of this paper keeping pace with the first.

1.1. Statement of the main results. Let us briefly introduce several notations. The completed tensor algebra $\widehat{T} = \prod_{m=0}^{\infty} H^{\otimes m}$ has a decreasing filtration of two-sided ideals given by $\widehat{T}_p \stackrel{\text{def}}{=} \prod_{m\geq p}^{\infty} H^{\otimes m}$, for $p \geq 1$. For a Magnus expansion θ, let $\ell^\theta \stackrel{\text{def}}{=} \log\theta$. Then ℓ^θ is a map from π to $\widehat{T}_1$. Define a linear map $N\colon \widehat{T}_1 \to \widehat{T}_1$ by $N|_{H^{\otimes p}} = \sum_{m=0}^{p-1} \nu^m$, for $p \geq 1$, where $\nu\colon H^{\otimes p} \to H^{\otimes p}$ is the map induced from the cyclic permutation. For $x \in \pi$, let

$$L^\theta(x) \stackrel{\text{def}}{=} \frac{1}{2} N(\ell^\theta(x)\ell^\theta(x)) \in \widehat{T}_2.$$

It turns out that $L^\theta(x^{-1}) = L^\theta(x)$, and $L^\theta(yxy^{-1}) = L^\theta(x)$ for $x, y \in \pi$. Thus if γ is an unoriented loop on Σ, $L^\theta(\gamma) \in \widehat{T}_2$ is well-defined by taking a representative of γ in π. Using the Poincaré duality, we make an identification $\widehat{T}_1 = H \otimes \widehat{T} \cong \mathrm{Hom}(H, \widehat{T})$ and regard $L^\theta(\gamma)$ as a derivation of $\widehat{T}$ by applications of the Leibniz rule.

Let C be a simple closed curve on Σ. We denote by $t_C \in \mathcal{M}_{g,1}$ the right handed Dehn twist along C (see Figure 1).

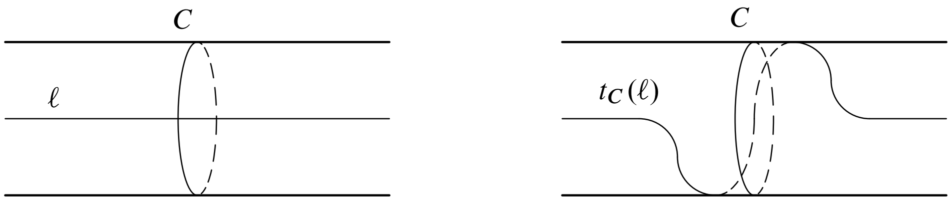

Figure 1. The right handed Dehn twist.

By the remark above, $L^\theta(C) \in \widehat{T}_2$ is defined. This invariant turns out to be "the logarithm" of t_C:

Theorem 1.1.1. *Let θ be a symplectic expansion and C a simple closed curve on Σ. Then the total Johnson map $T^\theta(t_C)$ is described as*

$$T^\theta(t_C) = e^{-L^\theta(C)}. \tag{1.1}$$

Here, the right hand side is the algebra automorphism of $\widehat{T}$ defined by the exponential of the derivation $-L^\theta(C)$.

The formula does not hold for a group-like Magnus expansion which is not symplectic. It should be remarked here that whether C is non-separating or separating, the formula holds. Note that (1.1) is an equality as filter-preserving automorphisms of $\widehat{T}$. If we compute (1.1) modulo $\widehat{T}_2$, we get the well-known formula for the action on the homology:

$$t_C(X) = X - (X \cdot [C])[C], \quad X \in H. \tag{1.2}$$

Here $(\cdot)$ is the intersection form on H and $[C]$ is the homology class of C with a fixed orientation. By computing (1.1) modulo higher tensors, we will get formulas for $\tau_k^\theta(t_C)$ in terms of $L^\theta(C)$. These formulas match the computations of the extended τ_1 for Humphreys generators and $\tau_2(t_C)$ for separating C by Morita [21] and [22]. See §6.

The classical formula (1.2) tells us that the action of t_C on $H_1(\Sigma; \mathbb{Z})$ depends only on the class $\pm[C]$. As an application of Theorem 1.1.1, we get a generalization of this fact. Let $N_k = N_k(\pi)$ be the k-th nilpotent quotient of π. We number the indices so that $N_1 = \pi^{\mathrm{abel}} \cong H_1(\Sigma; \mathbb{Z})$. The mapping class group $\mathcal{M}_{g,1}$ naturally acts on N_k. Let $\overline{N}_k$ be the quotient set of N_k by conjugation and the relation $x \sim x^{-1}$. Then any simple closed curve C defines an element of $\overline{N}_k$, which we denote by $\overline{C}_k$.

Theorem 1.1.2. *For each $k \geq 1$, the action of t_C on N_k depends only on the class $\overline{C}_k \in \overline{N}_k$. If C is separating, it depends only on the class $\overline{C}_{k-1} \in \overline{N}_{k-1}$.*

1.2. The Goldman Lie algebra and formal symplectic geometry. The key ingredients for our proof of Theorem 1.1.1 are *the Goldman Lie algebra* of Σ, see Goldman [9], and its homological interpretation in the framework of *formal symplectic geometry* by Kontsevich [17].

The Goldman Lie algebra is a Lie algebra associated to an oriented surface, and regarded as an origin of string topology by Chas and Sullivan [5]. It was introduced in [9] as a universal object for describing the Poisson brackets of coordinate functions on the space $\mathrm{Hom}(\pi, G)/G$, using his notation, with a natural symplectic structure. Here π is the fundamental group of a closed oriented surface (hence is not our π) and G is a Lie group satisfying very general conditions.

Let $\mathbb{Q}\hat{\pi}$ be the Goldman Lie algebra of Σ. Here, $\hat{\pi}$ is the set of conjugacy classes of π. In §3, we show that $\mathbb{Q}\hat{\pi}$ acts on the group ring $\mathbb{Q}\pi$ by derivations. Namely, we show that there is a Lie algebra homomorphism

$$\sigma \colon \mathbb{Q}\hat{\pi} \longrightarrow \mathrm{Der}(\mathbb{Q}\pi).$$

On the other hand, let $\mathfrak{a}_g^- = \mathrm{Der}_\omega(\widehat{T})$ be the space of derivations of $\widehat{T}$ killing the symplectic form ω. Here ω is a certain tensor of degree two coming from the intersection form on H. This is a variant of "associative," one of the three Lie algebras in formal symplectic geometry. In fact, we have a canonical isomorphism $\mathfrak{a}_g^- = N(\widehat{T}_1)$. For details, see §2.7.

Then we have the following two theorems. The slogan is: a symplectic expansion builds a bridge from 2-dimensional topology to formal symplectic geometry in the sense of Kontsevich.

Theorem 1.2.1. *Let θ be a symplectic expansion. Then the map*

$$-\lambda_\theta\colon \mathbb{Q}\hat{\pi} \longrightarrow N(\widehat{T}_1) = \mathfrak{a}_g^-, \quad x \longmapsto -N\theta(x)$$

is a Lie algebra homomorphism. The kernel is the subspace $\mathbb{Q}1$ spanned by the constant loop 1, and the image is dense in $N(\widehat{T}_1) = \mathfrak{a}_g^-$ with respect to the $\widehat{T}_1$-adic topology.

Theorem 1.2.2. *Let θ be a symplectic expansion. Then, for $u \in \mathbb{Q}\hat{\pi}$ and $v \in \mathbb{Q}\pi$, we have the equality*

$$\theta(\sigma(u)v) = -\lambda_\theta(u)\theta(v).$$

Here the right hand side means minus the action of $\lambda_\theta(u) \in \mathfrak{a}_g^-$ on the tensor $\theta(v) \in \widehat{T}$ as a derivation. In other words, the diagram

$$\begin{tikzcd} \mathbb{Q}\hat{\pi} \times \mathbb{Q}\pi \arrow[r, "\sigma"] \arrow[d, "-\lambda_\theta \times \theta"'] & \mathbb{Q}\pi \arrow[d, "\theta"] \\ \mathfrak{a}_g^- \times \widehat{T} \arrow[r] & \widehat{T}, \end{tikzcd} \tag{1.3}$$

where the bottom horizontal arrow means the action by derivations, commutes.

In fact, we can derive Theorem 1.1.1 from these two theorems and some care about convergence. See §5.5. For another application of these theorems, see [16].

1.3. Organization of the paper. This paper is organized as follows. In Section 2 we start by recalling Magnus expansions, symplectic expansions, and the total Johnson map associated to a Magnus expansion. Then we introduce the invariant L^θ and prove some properties of it. We close Section 2 by showing connections to formal symplectic geometry.

In Section 3, we look at the Goldman Lie algebra of Σ, and we show that it acts on the group ring of π by derivations. We also give a homological interpretation of this action. In Section 4, we construct a counterpart of the story in Section 3, in the framework of formal symplectic geometry. In particular, we give homological interpretations of $\mathfrak{a}_g^-$ and its action on $\widehat{T}$. To do this we need a (co)homology theory of (complete) Hopf algebras, to which Appendix A is devoted. We mention the relative homology of a pair, cap products, Kronecker products, and relation to (co)homology of groups.

The theorems in the Introduction are proved in Sections 5 and 6. In Section 5, we compare the stories in Sections 3 and 4 by a symplectic expansion, and prove Theorems 1.2.1 and 1.2.2. In Section 6 we prove Theorems 1.1.1 and 1.1.2, and derive some formulas for $\tau_k^\theta(t_C)$, which recover some computations by Morita. Finally in Section 7 we consider the case of the mapping class group of a once punctured surface and derive results similar to Theorems 1.1.1 and 1.1.2.

1.4. Conventions. Here we list the conventions of this paper.

(1) Let G be a group. For $x, y \in G$, we denote by $[x, y]$ their commutator $xyx^{-1}y^{-1} \in G$.

(2) As usual, we often ignore the distinction between a path and its homotopy class.

(3) For continuous paths γ_1, γ_2 on Σ such that the endpoint of γ_1 coincides with the start point of γ_2, their product $\gamma_1\gamma_2$ means the path traversing γ_1 first, then γ_2. The product in the fundamental group is the induced one.

(4) Sometimes we omit $\otimes$ to express tensors as well as products in the completed tensor algebra $\widehat{T}$. For example, if $X, Y, Z \in H$, then XYZ means $X \otimes Y \otimes Z \in H^{\otimes 3}$. If $u \in H^{\otimes k}$ and $X \in H$, then uX means $u \otimes X \in H^{\otimes k+1}$.

(5) Throughout the paper we basically work over $\mathbb{Q}$, although several results hold over the integers, especially in §3, and it would be possible to present all the main results with the coefficients in an integral domain including the rationals $\mathbb{Q}$.

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2. Magnus expansions and total Johnson map

2.1. Surface and mapping class group. As in the Introduction, Σ is a compact oriented C^∞-surface of genus $g > 0$ with one boundary component. We choose a basepoint $*$ on the boundary $\partial\Sigma$. The fundamental group $\pi \overset{\text{def}}{=} \pi_1(\Sigma, *)$ is a free group of rank $2g$. Let $H \stackrel{\text{def}}{=} H_1(\Sigma; \mathbb{Q})$ be the first homology group of Σ. The space H is naturally isomorphic to $H_1(\pi; \mathbb{Q}) \cong \pi^{\text{abel}} \otimes_{\mathbb{Z}} \mathbb{Q}$, the first homology group of π. Here $\pi^{\text{abel}} = \pi/[\pi, \pi]$ is the abelianization of π. Under this identification, we write

$$[x] \stackrel{\text{def}}{=} (x \bmod [\pi, \pi]) \otimes_{\mathbb{Z}} 1 \in H, \quad x \in \pi.$$

Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ relative to the boundary, namely the group of orientation-preserving diffeomorphisms of Σ fixing $\partial\Sigma$ pointwise, modulo isotopies fixing $\partial\Sigma$ pointwise.

Let $\zeta \in \pi$ be a based loop parallel to $\partial\Sigma$ and going by counter-clockwise manner. Explicitly, if we take symplectic generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \in \pi$ as shown in Figure 2, $\zeta = \prod_{i=1}^{g} [\alpha_i, \beta_i]$. By the classical theorem of Dehn-Nielsen, the natural action of $\mathcal{M}_{g,1}$ on $\pi = \pi_1(\Sigma, *)$ is faithful and we can identify $\mathcal{M}_{g,1}$ as a subgroup of $\mathrm{Aut}(\pi)$:

$$\mathcal{M}_{g,1} = \{\varphi \in \mathrm{Aut}(\pi);\ \varphi(\zeta) = \zeta\}. \tag{2.1}$$

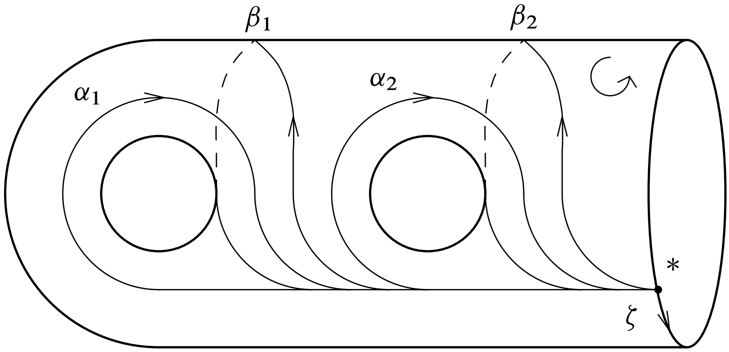

Figure 2. Symplectic generators of π for $g = 2$.

2.2. Group ring and tensor algebra. Let $\mathbb{Q}\pi$ be the group ring of π. It has an augmentation given by

$$\varepsilon \colon \mathbb{Q}\pi \longrightarrow \mathbb{Q},$$
$$\sum_i n_i x_i \longmapsto \sum_i n_i,$$

where $n_i \in \mathbb{Q}$, $x_i \in \pi$. Let $I\pi$ be the augmentation ideal, namely the kernel of ε. The powers of $I\pi$ give a decreasing filtration of $\mathbb{Q}\pi$. The completed group ring of π, or more precisely the $I\pi$-adic completion of $\mathbb{Q}\pi$, is

$$\widehat{\mathbb{Q}\pi} \stackrel{\text{def}}{=} \varprojlim_m \mathbb{Q}\pi/(I\pi)^m.$$

It naturally has a structure of a complete augmented algebra (in the sense of Quillen [25], Appendix A) with respect to the decreasing filtration given by

$$\mathrm{Ker}(\widehat{\mathbb{Q}\pi} \longrightarrow \mathbb{Q}\pi/(I\pi)^p), \quad p \geq 1.$$

Let $\widehat{T}$ be the completed tensor algebra generated by H. Namely

$$\widehat{T} = \prod_{m=0}^{\infty} H^{\otimes m},$$

where $H^{\otimes m}$ is the tensor space of degree m. Choosing a basis for H, it is isomorphic to the ring of non-commutative formal power series in $2g$ indeterminates. We can write elements of $\widehat{T}$ uniquely as

$$u = \sum_{m=0}^{\infty} u_m = u_0 + u_1 + u_2 + \dots, \quad u_m \in H^{\otimes m}.$$

The algebra $\widehat{T}$ has an augmentation given by

$$\begin{aligned} \varepsilon\colon \widehat{T} &\longrightarrow \mathbb{Q}, \\ u = \sum_{m=0}^{\infty} u_m &\longmapsto u_0, \end{aligned}$$

and it is a complete augmented algebra with respect to the decreasing filtration defined by

$$\widehat{T}_p \stackrel{\mathrm{def}}{=} \prod_{m \geq p} H^{\otimes m}, \quad p \geq 1.$$

Both $\mathbb{Q}\pi$ and $\widehat{T}$ have a structure of (complete) Hopf algebra. For simplicity, we use the same letters Δ and ι for the coproducts and the antipodes of both Hopf algebras. In the case of $\mathbb{Q}\pi$, these are given by

$$\Delta(x) = x \otimes x \quad \text{and} \quad \iota(x) = x^{-1}, \quad x \in \pi,$$

and in the case of $\widehat{T}$, the formulas are

$$\Delta(X) = X \widehat{\otimes} 1 + 1 \widehat{\otimes} X \quad \text{and} \quad \iota(X) = -X, \quad X \in H.$$

Here $\widehat{\otimes}$ means the completed tensor product. The Hopf algebra structure of $\mathbb{Q}\pi$ induces a structure of a complete Hopf algebra on $\widehat{\mathbb{Q}\pi}$.

By definition the set of group-like elements of $\widehat{T}$ is the set of $u \in \widehat{T}$ satisfying $\Delta(u) = u\widehat{\otimes}u$ and $u \neq 0$, and the set of primitive elements is

$$\widehat{\mathcal{L}} \stackrel{\text{def}}{=} \{u \in \widehat{T}; \Delta(u) = u\widehat{\otimes}1 + 1\widehat{\otimes}u\}.$$

As is well-known, $\widehat{\mathcal{L}}$ has a structure of a Lie algebra with the bracket $[u,v] \stackrel{\text{def}}{=} uv - vu$. The degree p-part

$$\mathcal{L}_p \stackrel{\text{def}}{=} \widehat{\mathcal{L}} \cap H^{\otimes p}$$

is described successively as $\mathcal{L}_1 = H$, and $\mathcal{L}_p = [H, \mathcal{L}_{p-1}]$ for $p \geq 2$. By the exponential map

$$\exp(u) = \sum_{n=0}^{\infty} \frac{1}{n!} u^n, \quad u \in \widehat{\mathcal{L}},$$

$\widehat{\mathcal{L}}$ is bijectively mapped to the set of group-like elements and the inverse is given by the logarithm

$$\log(u) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (u-1)^n.$$

Since the set of group-like elements constitutes a group with respect to the multiplication of $\widehat{T}$, the above bijection endows the underlying set of $\widehat{\mathcal{L}}$ with a group structure, which is described by the Baker–Campbell–Hausdorff series,

$$\begin{aligned} u \cdot v &= \log(\exp(u)\exp(v)) \\ &= u + v + \frac{1}{2}[u,v] + \frac{1}{12}[u-v,[u,v]] + \dots, \quad u,v \in \widehat{\mathcal{L}}. \end{aligned}$$

2.3. Magnus expansion. We recall the notion of a Magnus expansion in our generalized sense. Remark that the subset $1 + \widehat{T}_1$ constitutes a group with respect to the multiplication of $\widehat{T}$.

Definition 2.3.1 (Kawazumi [14]). A map $\theta \colon \pi \to 1 + \widehat{T}_1$ is a ($\mathbb{Q}$-*valued*) *Magnus expansion* of π if

(1) $\theta \colon \pi \to 1 + \widehat{T}_1$ is a group homomorphism, and

(2) $\theta(x) \equiv 1 + [x] \bmod \widehat{T}_2$ for any $x \in \pi$.

As was shown in [14], Theorem 1.3, any Magnus expansion θ induces a filter-preserving isomorphism of augmented algebras

$$\theta\colon \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}. \tag{2.2}$$

Since π is a free group, any Magnus expansion is (uniquely) defined by its values on free generators of π, hence we have many choices of Magnus expansions (see also §2.8). For example, let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \in \pi$ be symplectic generators (see §2.1) and write them as $x_1, \ldots, x_{2g}$. The Magnus expansion defined by $\theta(x_i) = 1 + [x_i]$, for $1 \le i \le 2g$, is called the standard Magnus expansion. This is introduced by Magnus [19].

Among all the Magnus expansions, group-like expansions respect the Hopf algebra structure of $\mathbb{Q}\pi$ and $\widehat{T}$. For a Magnus expansion θ, let $\ell^\theta \overset{\text{def}}{=} \log\theta$. Here it should be remarked the logarithm is defined on the set $1 + \widehat{T}_1$. A priori, ℓ^θ is a map from π to $\widehat{T}_1$.

Definition 2.3.2. A Magnus expansion θ is called group-like if $\theta(\pi)$ is contained in the set of group-like elements of $\widehat{T}$, or equivalently, $\ell^\theta(\pi) \subset \widehat{\mathcal{L}}$.

If θ is group-like, (2.2) turns out to be an isomorphism of complete Hopf algebras (see Massuyeau [20], Proposition 2.10). The standard Magnus expansion is not group-like. For example, let $x_1, \ldots, x_{2g}$ be as above, then the Magnus expansion defined by $\theta(x_i) = \exp([x_i])$, for $1 \le i \le 2g$, is group-like because of the Baker–Campbell–Hausdorff formula. Another example is given by Bene, Kawazumi, and Penner [1], where they constructed a group-like Magnus expansion canonically associated to any trivalent marked fatgraph.

2.4. Symplectic expansion. So far we have only used the fact that π is a free group. Here we recall the notion of a symplectic expansion, which is a Magnus expansion recording the fact that π is the fundamental group of a surface.

We identify H and its dual $H^* = \mathrm{Hom}_{\mathbb{Q}}(H, \mathbb{Q})$ by the Poincaré duality:

$$H \xrightarrow{\cong} H^*, \quad X \longmapsto (Y \longmapsto (Y \cdot X)). \tag{2.3}$$

Here $(\,\cdot\,)$ is the intersection pairing on $H = H_1(\Sigma; \mathbb{Q})$. Let $\omega \in \mathcal{L}_2 \subset H^{\otimes 2}$ be the symplectic form, namely the tensor corresponding to $-1_H \in \mathrm{Hom}_{\mathbb{Q}}(H, H) = H^* \otimes H = H \otimes H$. Explicitly, ω is given by

$$\omega = \sum_{i=1}^{g} A_i B_i - B_i A_i,$$

where $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ are symplectic generators and $A_i = [\alpha_i]$, $B_i = [\beta_i] \in H$.

Definition 2.4.1 (Massuyeau [20]). A Magnus expansion θ is called a symplectic expansion if

(1) θ is group-like, and

(2) $\theta(\zeta) = \exp(\omega)$, or equivalently, $\ell^\theta(\zeta) = \omega$.

Unfortunately, the examples of group-like expansions in §2.3 are not symplectic. But symplectic expansions do exist, and they are infinitely many (see §2.8). Kawazumi [15] gave the first example of a symplectic expansion (with coefficients in $\mathbb{R}$), called the harmonic Magnus expansion, by a transcendental method. Massuyeau [20] gave the second example using the LMO functor. There is also a purely combinatorial way of associating a symplectic expansion with any (not necessary symplectic) free generators of π. See Kuno [18].

2.5. Total Johnson map. We denote by $\mathrm{Aut}(\widehat{T})$ the set of filter-preserving algebra automorphisms of $\widehat{T}$, which clearly constitutes a group. Let θ be a Magnus expansion of π. For $\varphi \in \mathcal{M}_{g,1}$ we use the same letter φ for the induced automorphism of π, in view of (2.1). As a consequence of the isomorphism (2.2), for each $\varphi \in \mathcal{M}_{g,1}$ there uniquely exists $T^\theta(\varphi) \in \mathrm{Aut}(\widehat{T})$ such that

$$T^\theta(\varphi) \circ \theta = \theta \circ \varphi.$$

Let $|\varphi|\colon H \to H$ be the automorphism of H induced by the action of φ on the first homology of Σ. We also denote by $|\varphi| \in \mathrm{Aut}(\widehat{T})$ the automorphism induced by $|\varphi|$. Then $\tau^\theta(\varphi) \overset{\text{def}}{=} T^\theta(\varphi) \circ |\varphi|^{-1} \in \mathrm{Aut}(\widehat{T})$ acts on $\widehat{T}_1/\widehat{T}_2 \cong H$ as the identity. Therefore the restriction of $\tau^\theta(\varphi)$ to H is uniquely written as

$$\tau^\theta(\varphi)|_H = 1_H + \sum_{k=1}^{\infty} \tau_k^\theta(\varphi),$$

where $\tau_k^\theta(\varphi) \in \mathrm{Hom}(H, H^{\otimes k+1})$.

Definition 2.5.1 ([14]). The automorphism $T^\theta(\varphi) \in \mathrm{Aut}(\widehat{T})$ is called the total Johnson map of φ associated to θ, and $\tau_k^\theta(\varphi)$ is called the k-th Johnson map of φ associated to θ.

The group homomorphism

$$T^\theta\colon \mathcal{M}_{g,1} \to \mathrm{Aut}(\widehat{T})$$

is also called the total Johnson map. It is injective since the natural map $\pi \to \widehat{\mathbb{Q}\pi}$ is injective by the classical fact $\bigcap_{m=1}^\infty (I\pi)^m = 0$. Note that our terminology here is different from [14], where $\tau^\theta(\varphi)$ is called the total Johnson map of φ.

2.6. The invariant L^θ. We introduce an invariant of unoriented loops on Σ associated with a Magnus expansion.

Definition 2.6.1. Define a linear map $N\colon \widehat{T} \to \widehat{T}$ by

$$N|_{H^{\otimes p}} = \sum_{m=0}^{p-1} \nu^m, \text{ for } p \geq 1,$$

where ν is the cyclic permutation given by $X_1X_2\dots X_p \mapsto X_2X_3\dots X_1$ $(X_i \in H)$, and $N|_{H^{\otimes 0}} = 0$.

The operator N also appeared in [15]. The following lemma will be used frequently.

Lemma 2.6.2. (1) *For $u, v \in \widehat{T}$, $N(uv) = N(vu)$.*

(2) *For $u, v, w \in \widehat{T}$, $N([u, v]w) = N(u[v, w])$.*

(3) *For $u \in \widehat{T}_1$, $u \in N(\widehat{T}_1)$ is equivalent to $\nu(u) = u$.*

(4) *Under the identification $\widehat{T}_1 \cong H \otimes \widehat{T}$,*

$$N(\widehat{T}_1) = \mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{T} \longrightarrow \widehat{T}). \tag{2.4}$$

Proof. The first assertion is clear if u and v are homogeneous, since $N(\nu(w)) = N(w)$ for a homogeneous $w \in \widehat{T}$. The general case follows by bi-linearity. Using (1), we compute

$$N([u, v]w) = N(uvw - vuw) = N(uvw - uwv) = N(u[v, w]),$$

which proves (2). If $u \in \widehat{T}_1$ is homogeneous of degree p and $\nu(u) = u$, then $u = N(\frac{1}{p}u) \in N(\widehat{T}_1)$. This proves (3). Finally,

$$\nu(X \otimes u) - X \otimes u = uX - Xu = -[X, u], \quad X \otimes u \in H \otimes \widehat{T}.$$

Combining this with (3), we have (4). □

Definition 2.6.3. Let θ be a Magnus expansion. Define

$$L^\theta\colon \pi \longrightarrow \widehat{T}_2$$

by

$$L^\theta(x) = \frac{1}{2} N(\ell^\theta(x)\ell^\theta(x)).$$

Lemma 2.6.4. *For any $x, y \in \pi$, we have*

(1) $L^\theta(x^{-1}) = L^\theta(x)$,

(2) $L^\theta(yxy^{-1}) = L^\theta(x)$.

Proof. The first part follows by $\ell^\theta(x^{-1}) = -\ell^\theta(x)$. For the second part note that $\ell^\theta(yxy^{-1}) = e^{\ell^\theta(y)}\ell^\theta(x)e^{-\ell^\theta(y)} = \theta(y)\ell^\theta(x)\theta(y^{-1})$. By Lemma 2.6.2 (1),

$$\begin{aligned} L^\theta(yxy^{-1}) &= \frac{1}{2}N(\theta(y)\ell^\theta(x)\theta(y^{-1})\theta(y)\ell^\theta(x)\theta(y^{-1})) \\ &= \frac{1}{2}N(\theta(y)\ell^\theta(x)\ell^\theta(x)\theta(y^{-1})) \\ &= \frac{1}{2}N(\ell^\theta(x)\ell^\theta(x)\theta(y^{-1})\theta(y)) \\ &= \frac{1}{2}N(\ell^\theta(x)\ell^\theta(x)) = L^\theta(x). \end{aligned}$$ □

Let γ be an (un)oriented loop on Σ, and take a representative x of γ in π. Then $L^\theta(\gamma) \stackrel{\text{def}}{=} L^\theta(x) \in \widehat{T}_2$ is independent of the choice of x by Lemma 2.6.4.

2.7. Formal symplectic geometry. The space $N(\widehat{T}_1)$ is closely related to formal symplectic geometry. In [17], Kontsevich introduced three Lie algebras "commutative," "associative," and "Lie." We recall two of the three, namely "associative" and "Lie."

First we recall "associative." By definition, a derivation of $\widehat{T}$ is a linear map $D: \widehat{T} \to \widehat{T}$ satisfying the Leibniz rule

$$D(u_1u_2) = D(u_1)u_2 + u_1D(u_2), \quad u_1, u_2 \in \widehat{T}.$$

The space $\mathrm{Der}(\widehat{T})$ of the derivations of $\widehat{T}$ has the structure of Lie algebra given by $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$, $D_1, D_2 \in \mathrm{Der}(\widehat{T})$. Since $\widehat{T}$ is freely generated by H as a complete algebra, any derivation of $\widehat{T}$ is uniquely determined by its values on H, and $\mathrm{Der}(\widehat{T})$ is identified with $\mathrm{Hom}(H, \widehat{T})$.

By (2.3), $\widehat{T}_1 \cong H \otimes \widehat{T}$ is identified with $\mathrm{Hom}(H, \widehat{T})$:

$$\widehat{T}_1 \cong H \otimes \widehat{T} \xrightarrow{\cong} \mathrm{Hom}(H, \widehat{T}), \quad X \otimes u \longmapsto (Y \longmapsto (Y \cdot X)u). \tag{2.5}$$

Let $\mathfrak{a}_g^- = \mathrm{Der}_\omega(\widehat{T})$ be the Lie subalgebra of $\mathrm{Der}(\widehat{T})$ consisting of derivations killing the symplectic form ω. We call such derivations *symplectic derivations of* $\widehat{T}$. In view of (2.5) any derivation D is written as

$$D = \sum_{i=1}^{g} B_i \otimes D(A_i) - A_i \otimes D(B_i) \in \widehat{T}_1. \tag{2.6}$$

Since $D(\omega) = \sum_{i=1}^{g}[D(A_i), B_i] + [A_i, D(B_i)]$ we can write

$$\mathfrak{a}_g^- = \mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{T} \longrightarrow \widehat{T}) = N(\widehat{T}_1) \tag{2.7}$$

(see also (2.4)). The Lie subalgebra $\mathfrak{a}_g \overset{\text{def}}{=} N(\widehat{T}_2)$ is nothing but (the completion of) what Kontsevich [17] calls a_g.

We next recall "Lie." A *derivation* of $\widehat{\mathcal{L}}$ is a linear map

$$D\colon \widehat{\mathcal{L}} \longrightarrow \widehat{\mathcal{L}}$$

satisfying

$$D([u_1, u_2]) = [D(u_1), u_2] + [u_1, D(u_2)], \quad u_1, u_2 \in \widehat{\mathcal{L}}.$$

Let $\mathfrak{l}_g = \mathrm{Der}_\omega(\widehat{\mathcal{L}})$ be the space of derivations of $\widehat{\mathcal{L}}$ killing $\omega \in \mathcal{L}_2$. By the same reason as above, we have

$$\mathfrak{l}_g = \mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{\mathcal{L}} \longrightarrow \widehat{\mathcal{L}}). \tag{2.8}$$

The Lie algebra $\mathfrak{l}_g$ is a Lie subalgebra of $\mathfrak{a}_g$.

Lemma 2.7.1. *Let $m \geq 1$, and $X, Y_1, \dots, Y_m \in H$. Set*

$$u = [Y_1, [Y_2, [\dots, [Y_{m-1}, Y_m]\dots]]] \in \mathcal{L}_m.$$

Then

$$N(X \otimes u) = X \otimes u + \sum_{i=1}^{m} Y_i \otimes [[Y_{i+1}, \dots [Y_{m-1}, Y_m]\dots], [\dots[[X, Y_1], Y_2], \dots, Y_{i-1}]].$$

In particular, we have $N(H \otimes \widehat{\mathcal{L}}) \subset H \otimes \widehat{\mathcal{L}}$.

Proof. Consider the tensor algebra T' generated by the letters $X, Y_1, \dots, Y_m$. The operator N is naturally defined on T'. There is a homomorphism $T' \to \widehat{T}$ coming from the universality of T'. This homomorphism is compatible with N. Thus it suffices to show the formula on T'. Let H' be the $\mathbb{Q}$-vector space spanned by $X, Y_1, \dots, Y_m$. The formula we want to show is an equality in $H'^{\otimes m+1}$. There is a direct sum decomposition

$$H'^{\otimes m+1} = X \otimes H'^{\otimes m} \oplus \bigoplus_{i=1}^{m} Y_i \otimes H'^{\otimes m}.$$

Let $p_X\colon H'^{\otimes m+1} \to X \otimes H'^{\otimes m} \cong H'^{\otimes m}$ be the projection according to this direct sum decomposition. Similarly, define p_{Y_i}, $1 \leq i \leq m$. Note that for any $v \in H'^{\otimes m+1}$ we have $v = Xp_X(v) + \sum_{i=1}^m Y_i p_{Y_i}(v)$. Now, set

$$v \stackrel{\text{def}}{=} N(X \otimes u).$$

It is clear that $p_X(v) = u$. For each $1 \leq i \leq m$, we denote

$$v_i' = [Y_{i+1}, \dots [Y_{m-1}, Y_m] \dots].$$

By Lemma 2.6.2, we compute

$$\begin{aligned} v &= N(X[Y_1, [Y_2, \dots, [Y_i, v_i'] \dots]]) \\ &= N([X, Y_1][Y_2, \dots, [Y_i, v_i'] \dots]) \\ &\;\;\vdots \\ &= N(v_i''[Y_i, v_i']) \\ &= N(v_i'' Y_i v_i' - v_i'' v_i' Y_i) \\ &= N(Y_i v_i' v_i'' - Y_i v_i'' v_i') \\ &= N(Y_i [v_i', v_i'']), \end{aligned}$$

where $v_i'' = [\dots[[X, Y_1], Y_2], \dots, Y_{i-1}]$. This shows $p_{Y_i}(v) = [v_i', v_i'']$, and completes the proof. $\square$

Lemma 2.7.2. $N(\widehat{\mathcal{L}}\widehat{\otimes}\widehat{\mathcal{L}}) = \mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{\mathcal{L}} \to \widehat{\mathcal{L}}) = \mathfrak{l}_g$.

Proof. Using Lemma 2.6.2 (2), we have $N(\widehat{\mathcal{L}}\widehat{\otimes}\widehat{\mathcal{L}}) = N(H \otimes \widehat{\mathcal{L}})$, and $N(H \otimes \widehat{\mathcal{L}})$ is contained in $(H \otimes \widehat{\mathcal{L}}) \cap N(\widehat{T}_1)$ by Lemma 2.7.1. Therefore we get

$$N(\widehat{\mathcal{L}}\widehat{\otimes}\widehat{\mathcal{L}}) \subset \mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{\mathcal{L}} \longrightarrow \widehat{\mathcal{L}})$$

by (2.4). On the other hand, if $u \in H \otimes \widehat{\mathcal{L}} \subset \widehat{T}_1$ is homogeneous of degree $p \geq 2$ and $\nu(u) = u$, then $u = N(u/p) \in N(H \otimes \widehat{\mathcal{L}}) = N(\widehat{\mathcal{L}}\widehat{\otimes}\widehat{\mathcal{L}})$. By Lemma 2.6.2 (3)(4), we get the other inclusion. $\square$

Thus, if θ is group-like, our invariant L^θ is considered as a map $L^\theta\colon \pi \to \mathfrak{l}_g$.

2.8. The space of symplectic expansions. There are infinitely many Magnus expansions and symplectic expansions. Here we consider the spaces that parametrize them. Let Θ be the set of Magnus expansions of π, and let $\Theta^{\mathrm{symp}} \subset \Theta$ be the set of symplectic expansions.

Let $\mathrm{IA}(\widehat{T})$ be the subgroup of $\mathrm{Aut}(\widehat{T})$ consisting of the automorphisms acting on $\widehat{T}_1/\widehat{T}_2 \cong H$ as the identity. This group acts on Θ freely and transitively by

$$U \cdot \theta \stackrel{\text{def}}{=} U \circ \theta, \quad U \in \mathrm{IA}(\widehat{T}), \theta \in \Theta.$$

See [14], Theorem 1.3. Thus if θ and θ' are Magnus expansions, there uniquely exists $U = U(\theta, \theta') \in \mathrm{IA}(\widehat{T})$ such that $\theta' = U \circ \theta$. The group $\mathrm{IA}(\widehat{T})$ is identified with $H \otimes \widehat{T}_2$ by the composition of the logarithm

$$\mathrm{IA}(\widehat{T}) \longrightarrow \mathrm{Hom}(H, \widehat{T}_2), \quad U \longmapsto (\log U)|_H$$

(note that $\log U$ converges since U acts on $\widehat{T}_1/\widehat{T}_2 \cong H$ as the identity) and the isomorphism (2.5). Thus if we fix a Magnus expansion θ, the correspondence $\theta' \mapsto U(\theta, \theta')$ induces a bijection

$$\Theta \cong H \otimes \widehat{T}_2. \tag{2.9}$$

Now if θ and θ' are symplectic expansions, then $U = U(\theta, \theta')$ satisfies $U(H) \subset \widehat{\mathcal{L}}$ and $U(\omega) = \omega$. Then by (2.9) the element θ' is mapped to an element in $\mathrm{Ker}([\ ,\ ]\colon H \otimes (\widehat{\mathcal{L}} \cap \widehat{T}_2) \to \widehat{\mathcal{L}})$ (see also (2.8)). By these discussions we conclude:

Proposition 2.8.1. *The set Θ^{symp} is not empty. Once we choose a symplectic expansion θ, the restriction of (2.9) to Θ^{symp} gives a bijection*

$$\Theta^{\mathrm{symp}} \cong \mathrm{Ker}([\ ,\ ]\colon H \otimes (\widehat{\mathcal{L}} \cap \widehat{T}_2) \longrightarrow \widehat{\mathcal{L}}).$$

3. The Goldman Lie algebra

In this section, we recall the Goldman Lie algebra [9]. In particular, we show that the Goldman Lie algebra of Σ acts on the group ring $\mathbb{Q}\pi$ by derivations. We will work over the rationals, but all the statements in this section except Proposition 3.4.3 holds over the integers.

All of the loops that we consider are piecewise differentiable.

3.1. The Goldman Lie algebra. Let S be a connected oriented 2-manifold and let $\hat{\pi}(S) = [S^1, S]$ be the set of free homotopy classes of oriented loops on S. In other words, $\hat{\pi}(S)$ is the set of conjugacy classes of the fundamental group of S. Let $|\ |\colon \pi_1(S) \to \hat{\pi}(S)$ be the natural quotient map. For a loop $\alpha\colon S^1 \to S$ and a simple point $p \in \alpha$, let α_p be the oriented loop α based at p.

Let $\mathbb{Q}\hat{\pi}(S)$ be the vector space spanned by $\hat{\pi}(S)$. We first recall the Goldman bracket on $\mathbb{Q}\hat{\pi}(S)$. Let α, β be immersed loops in S such that $\alpha \cup \beta \colon S^1 \cup S^1 \to S$ is an immersion with at worst transverse double points. For each intersection $p \in \alpha \cap \beta$, the conjunction $\alpha_p \beta_p \in \pi_1(S, p)$ is defined. Let $\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ be the local intersection number of α and β at p and set

$$[\alpha, \beta] \stackrel{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) |\alpha_p \beta_p| \in \mathbb{Q}\hat{\pi}(S).$$

Let $1 \in \hat{\pi}(S)$ be the homotopy class of the constant loop and $\hat{\pi}'(S) = \hat{\pi}(S) \setminus \{1\}$.

Theorem 3.1.1 (Goldman [9]). *The above bracket induces a Lie bracket*

$$[\ ,\ ] \colon \mathbb{Q}\hat{\pi}(S) \otimes \mathbb{Q}\hat{\pi}(S) \longrightarrow \mathbb{Q}\hat{\pi}(S).$$

Moreover, $\mathbb{Q}\hat{\pi}'(S)$ *is an ideal of* $\mathbb{Q}\hat{\pi}(S)$ *and* $\mathbb{Q}\hat{\pi}(S) = \mathbb{Q}\hat{\pi}'(S) \oplus \mathbb{Q}1$ *is a direct sum decomposition as Lie algebras.*

Remark 3.1.2. It is true that $[\mathbb{Q}\hat{\pi}(S), \mathbb{Q}\hat{\pi}(S)] \subset \mathbb{Q}\hat{\pi}'(S)$ for any S. But Goldman's proof for it [9] pp.294-295 is, unfortunately, not correct. In fact, his assertion $[\alpha, \alpha^{-1}] = 0$ for $\alpha \in \hat{\pi}(S)$ is not true in general. If we choose $\alpha = \alpha_1 \alpha_2$ as in Figure 3, then $[\alpha, \alpha^{-1}] = \alpha_1 \alpha_2 {\alpha_1}^{-1} {\alpha_2}^{-1} - \alpha_2 \alpha_1 {\alpha_2}^{-1} {\alpha_1}^{-1}$. Here we suppose the fundamental group of S is a finitely generated free group. Then we can take a Magnus expansion $\theta \colon \pi_1(S) \to \widehat{T}$, and define a map $\mathbb{Q}\hat{\pi}(S) \to \widehat{T}$ by $|x| \mapsto N\theta(x)$ for $x \in \pi_1(S)$. See Lemma 2.6.2 (1). Now we have

$$N\theta[\alpha, \alpha^{-1}] = \frac{1}{3} N([X_1, X_2][X_1, X_2][X_1, X_2]) + \text{higher terms} \neq 0$$

(see Theorem 1.2.1). Here we denote $X_1 = [\alpha_1]$ and $X_2 = [\alpha_2] \in H$. Hence $[\alpha, \alpha^{-1}] \neq 0$.

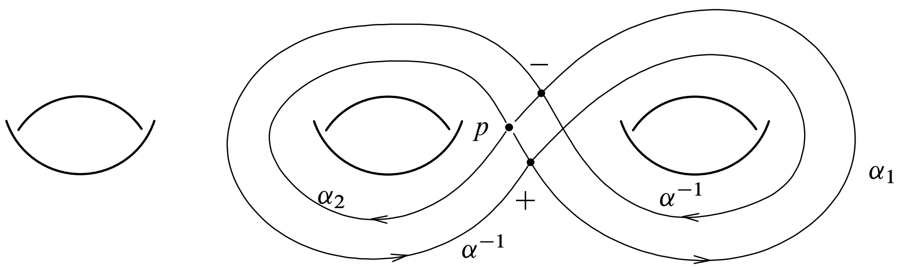

Figure 3. The Goldman bracket $[\alpha, \alpha^{-1}]$ is not zero.

But we can prove $[\alpha, \alpha^{-1}] \in \mathbb{Q}\hat{\pi}'(S)$ for any connected oriented 2-manifold S and any $\alpha \in \hat{\pi}$, as follows. Represent α by a generic immersion and let α^{-1} be a generic immersion such that $\alpha \cup \alpha^{-1}$ cobounds a narrow annulus, as in [9], p. 295. Let p be a double point of the loop α. It divides the loop α into two based loops α_1 and α_2 with basepoint p as in Figure 3. The two intersection points derived from p contributes $\alpha_1\alpha_2{\alpha_1}^{-1}{\alpha_2}^{-1}$ and $\alpha_2\alpha_1{\alpha_2}^{-1}{\alpha_1}^{-1}$, respectively, with the opposite sign. Then the following three conditions are equivalent:

(1) $|\alpha_1\alpha_2{\alpha_1}^{-1}{\alpha_2}^{-1}| = 1 \in \hat{\pi}(S)$;

(2) $\alpha_1\alpha_2 = \alpha_2\alpha_1 \in \pi_1(S, p)$;

(3) $|\alpha_2\alpha_1{\alpha_2}^{-1}{\alpha_1}^{-1}| = 1 \in \hat{\pi}(S)$.

This implies that the contributions of the two points cancel, or are in $\mathbb{Q}\hat{\pi}'(S)$. Hence we have $[\alpha, \alpha^{-1}] \in \mathbb{Q}\hat{\pi}'(S)$. As is observed by Goldman [9] loc.cit., $[\alpha, \beta] \in \mathbb{Q}\hat{\pi}'(S)$ if $\beta \neq \alpha^{-1}$. Hence we obtain $[\mathbb{Q}\hat{\pi}(S), \mathbb{Q}\hat{\pi}(S)] \subset \mathbb{Q}\hat{\pi}'(S)$. This completes the proof of the second half of Theorem 3.1.1.

3.2. The action on the group ring. Let S be as above, and choose a basepoint $* \in S$. Let $\alpha\colon S^1 \to S \setminus \{*\}$ be an immersed loop and $\beta\colon S^1 \to S$ an immersed loop based at $*$, and suppose $\alpha \cup \beta$ has at worst transverse double points. For each intersection $p \in \alpha \cap \beta$, let α_p and $\varepsilon(p; \alpha, \beta)$ be the same as before and let β_{*p} (resp. β_{p*}) be the path along β from $*$ to p (resp. p to $*$). Then the conjunction $\beta_{*p}\alpha_p\beta_{p*} \in \pi_1(S, *)$ is defined.

Definition 3.2.1. For such α and β, let

$$\sigma(\alpha)\beta \overset{\text{def}}{=} \sum_{p \in \alpha\cap\beta} \varepsilon(p; \alpha, \beta)\beta_{*p}\alpha_p\beta_{p*} \in \mathbb{Q}\pi_1(S, *).$$

Let $\mathrm{Der}(\mathbb{Q}\pi_1(S, *))$ be the Lie algebra of the derivations of the group ring $\mathbb{Q}\pi_1(S, *)$.

Proposition 3.2.2. *This definition of σ gives rise to a well-defined Lie algebra homomorphism*

$$\sigma\colon \mathbb{Q}\hat{\pi}(S \setminus \{*\}) \longrightarrow \mathrm{Der}(\mathbb{Q}\pi_1(S, *)).$$

Proof. One way to prove that σ is well-defined is to show that $\sigma(\alpha)\beta$ is unchanged if α and β are replaced by one of the standard moves (see Goldman [9], Lemma 5.6). This can be done by the same argument as Goldman did, so we omit details. Another way to see this is using our homological interpretation of σ, see Proposition 3.5.2.

To prove that $\mathbb{Q}\hat{\pi}(S \setminus \{*\})$ acts on $\mathbb{Q}\pi_1(S,*)$ by derivations via σ, it suffices to show

$$\sigma(\alpha)(\beta\gamma) = (\sigma(\alpha)\beta)\gamma + \beta\sigma(\alpha)\gamma,$$

where α is an immersed loop on S, and β, γ are immersed based loops on S. We may assume that α intersects the conjunction $\beta\gamma$ at worst transverse double points. Then

$$\alpha \cap (\beta\gamma) = (\alpha \cap \beta) \cup (\alpha \cap \gamma),$$

and

$$\begin{aligned}
\sigma(\alpha)(\beta\gamma) &= \sum_{p \in \alpha \cap (\beta\gamma)} \varepsilon(p;\alpha,\beta\gamma)(\beta\gamma)_{*p}\alpha_p(\beta\gamma)_{p*} \\
&= \sum_{p \in \alpha \cap \beta} \varepsilon(p;\alpha,\beta\gamma)(\beta\gamma)_{*p}\alpha_p(\beta\gamma)_{p*} \\
&\qquad + \sum_{p \in \alpha \cap \gamma} \varepsilon(p;\alpha,\beta\gamma)(\beta\gamma)_{*p}\alpha_p(\beta\gamma)_{p*} \\
&= \sum_{p \in \alpha \cap \beta} \varepsilon(p;\alpha,\beta)\beta_{*p}\alpha_p\beta_{p*}\gamma + \sum_{p \in \alpha \cap \gamma} \varepsilon(p;\alpha,\gamma)\beta\gamma_{*p}\alpha_p\gamma_{p*} \\
&= (\sigma(\alpha)\beta)\gamma + \beta\sigma(\alpha)\gamma.
\end{aligned}$$

To prove that σ is a homomorphism of Lie algebras it suffices to show

$$\sigma([\alpha,\beta])\gamma = \sigma(\alpha)\sigma(\beta)\gamma - \sigma(\beta)\sigma(\alpha)\gamma,$$

where α, β are immersed loops on S, and γ is an immersed based loop on S. We may assume that $\alpha \cup \beta \cup \gamma$ is an immersion with at worst transverse double points. We compute

$$\begin{aligned}
&\sigma(\alpha)\sigma(\beta)\gamma \\
&\quad = \sigma(\alpha)\Bigg(\sum_{p \in \beta \cap \gamma} \varepsilon(p;\beta,\gamma)\gamma_{*p}\beta_p\gamma_{p*}\Bigg) \\
&\quad = \sum_{p \in \beta \cap \gamma} \varepsilon(p;\beta,\gamma)\sigma(\alpha)\gamma_{*p}\beta_p\gamma_{p*} \\
&\quad = \sum_{p \in \beta \cap \gamma}\sum_{q \in \alpha \cap \beta} \varepsilon(p;\beta,\gamma)\varepsilon(q;\alpha,\beta)(\gamma_{*p}\beta_p\gamma_{p*})_{*q}\alpha_q(\gamma_{*p}\beta_p\gamma_{p*})_{q*} \\
&\qquad + \sum_{p \in \beta \cap \gamma}\sum_{r \in \alpha \cap \gamma} \varepsilon(p;\beta,\gamma)\varepsilon(r;\alpha,\gamma)(\gamma_{*p}\beta_p\gamma_{p*})_{*r}\alpha_r(\gamma_{*p}\beta_p\gamma_{p*})_{r*},
\end{aligned} \tag{3.1}$$

and similarly,

$$\begin{aligned}
&-\sigma(\beta)\sigma(\alpha)\gamma \\
&\quad= -\sum_{r\in\alpha\cap\gamma}\sum_{p\in\beta\cap\gamma} \varepsilon(r;\alpha,\gamma)\varepsilon(p;\beta,\gamma)(\gamma_{*r}\alpha_r\gamma_{r*})_{*p}\beta_p(\gamma_{*r}\alpha_r\gamma_{r*})_{p*} \\
&\qquad\quad -\sum_{r\in\alpha\cap\gamma}\sum_{q\in\beta\cap\alpha} \varepsilon(r;\alpha,\gamma)\varepsilon(q;\beta,\alpha)(\gamma_{*r}\alpha_r\gamma_{r*})_{*q}\beta_q(\gamma_{*r}\alpha_r\gamma_{r*})_{q*}.
\end{aligned} \tag{3.2}$$

Then the second term of (3.1) and the first term of (3.2) cancel and we have

$$\begin{aligned}
&\sigma(\alpha)\sigma(\beta)\gamma - \sigma(\beta)\sigma(\alpha)\gamma \\
&\quad= \sum_{p\in\beta\cap\gamma}\sum_{q\in\alpha\cap\beta} \varepsilon(p;\beta,\gamma)\varepsilon(q;\alpha,\beta)(\gamma_{*p}\beta_p\gamma_{p*})_{*q}\alpha_q(\gamma_{*p}\beta_p\gamma_{p*})_{q*} \\
&\qquad\quad +\sum_{r\in\alpha\cap\gamma}\sum_{q\in\alpha\cap\beta} \varepsilon(r;\alpha,\gamma)\varepsilon(q;\alpha,\beta)(\gamma_{*r}\alpha_r\gamma_{r*})_{*q}\beta_q(\gamma_{*r}\alpha_r\gamma_{r*})_{q*}.
\end{aligned}$$

Here we use $\varepsilon(q;\beta,\alpha) = -\varepsilon(q;\alpha,\beta)$. Now

$$(\gamma_{*p}\beta_p\gamma_{p*})_{*q}\alpha_q(\gamma_{*p}\beta_p\gamma_{p*})_{q*} = \gamma_{*p}|\alpha_q\beta_q|_p\gamma_{p*}$$

for $p \in \beta\cap\gamma$, $q \in \alpha\cap\beta$ and

$$(\gamma_{*r}\alpha_r\gamma_{r*})_{*q}\beta_q(\gamma_{*r}\alpha_r\gamma_{r*})_{q*} = \gamma_{*r}|\alpha_q\beta_q|_r\gamma_{r*}$$

for $r \in \alpha\cap\gamma$, $q \in \alpha\cap\beta$. Therefore, we have

$$\begin{aligned}
&\sigma(\alpha)\sigma(\beta)\gamma - \sigma(\beta)\sigma(\alpha)\gamma \\
&\quad= \sum_{x\in(\alpha\cup\beta)\cap\gamma}\sum_{y\in\alpha\cap\beta} \varepsilon(x;\alpha\cup\beta,\gamma)\varepsilon(y;\alpha,\beta)\gamma_{*x}|\alpha_y\beta_y|_x\gamma_{x*} \\
&\quad= \sigma\Big(\sum_{y\in\alpha\cap\beta}\varepsilon(y;\alpha,\beta)|\alpha_y\beta_y|\Big)\gamma \\
&\quad= \sigma([\alpha,\beta])\gamma.
\end{aligned}$$

This completes the proof. □

Note that to make Definition 3.2.1 work, we need to delete the basepoint $*$. A simple illustration of this fact is Figure 4. Here the loops α_1 and α_2 are homotopic as free loops on S, but not homotopic as loops on $S \setminus \{*\}$. Following Definition 3.2.1, we have $\sigma(\alpha_1)\gamma = \alpha\gamma - \gamma\alpha$, and clearly $\sigma(\alpha_2)\gamma = 0$. Hence $\sigma(\alpha_1)\gamma \neq \sigma(\alpha_2)\gamma$.

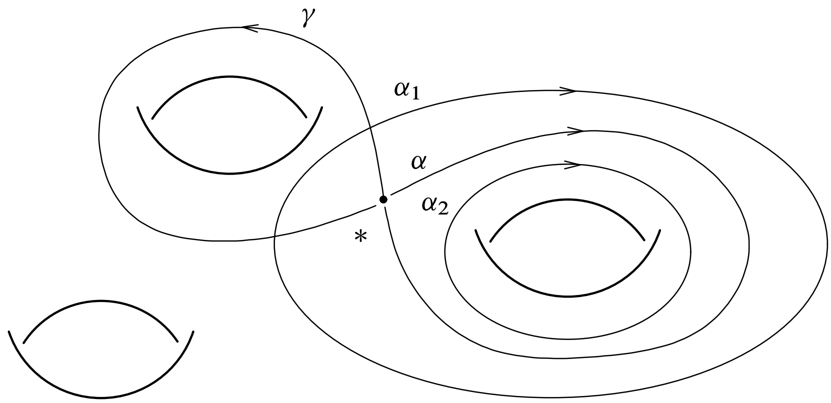

Figure 4. We need to delete the basepoint to define σ.

But if S and its basepoint are the surface Σ fixed in §2.1 and $* \in \partial\Sigma$, then the inclusion $\Sigma \setminus \{*\} \hookrightarrow \Sigma$ is a homotopy equivalence. Thus $\mathbb{Q}\hat{\pi}(\Sigma \setminus \{*\}) = \mathbb{Q}\hat{\pi}(\Sigma)$. Writing $\hat{\pi}(\Sigma) = \hat{\pi}$ for simplicity, we have a Lie algebra homomorphism

$$\sigma\colon \mathbb{Q}\hat{\pi} \longrightarrow \mathrm{Der}(\mathbb{Q}\pi). \tag{3.3}$$

Remark 3.2.3. Let M be a d-dimensional oriented C^∞-manifold, and choose a basepoint $* \in M$. We regard $S^1 = [0,1]/(0 \sim 1)$, and denote

$$\Omega M = \mathrm{Map}((S^1, 0), (M, *)),$$

the based loop space of M. The evaluation map $\mathrm{ev}\colon \Omega M \to M$, $\gamma \mapsto \gamma(1/2)$, is a Hurewicz fibration, whose fiber $\mathrm{ev}^{-1}(*)$ is naturally identified with $\Omega M \times \Omega M$. The map

$$\rho\colon \Omega M \times [0,1] \longrightarrow \Omega M,$$

given by

$$\rho(\gamma, s)(t) \stackrel{\text{def}}{=} \begin{cases} \gamma(2st) & \text{if } t \leq 1/2, \\ \gamma(s + (1-s)(2t-1)) & \text{if } t \geq 1/2, \end{cases}$$

induces a map of pairs

$$\rho\colon \Omega M \times ([0,1],\{0,1\}) \longrightarrow (\Omega M, \Omega M \times \Omega M).$$

We define a map

$$\Delta'\colon H_*(\Omega M) \longrightarrow H_{*+1}(\Omega M, \Omega M \times \Omega M)$$

by the composite

$$H_*(\Omega M) \xrightarrow{\times [I]} H_{*+1}(\Omega M \times ([0,1],\{0,1\})) \xrightarrow{\rho_*} H_{*+1}(\Omega M, \Omega M \times \Omega M),$$

where $[I] \in H_1([0,1],\{0,1\})$ is the fundamental class. In a way similar to Chas and Sullivan [5], we can define a loop product

$$\bullet\colon H_i(L(M \setminus \{*\})) \otimes H_j(\Omega M, \Omega M \times \Omega M) \longrightarrow H_{i+j-d}(\Omega M).$$

Here we denote

$$L(M \setminus \{*\}) = \mathrm{Map}(S^1, M \setminus \{*\}),$$

the free loop space of $M \setminus \{*\}$. Let $x\colon K_x \to L(M \setminus \{*\})$ be an i-cell, and $y\colon K_y \to \Omega M$ a j-cell. We denote by $K_{x\bullet y}$ a transversal preimage of the diagonal under the map

$$K_x \times K_y \longrightarrow (M \setminus \{*\}) \times M, \quad (k_x, k_y) \longmapsto (x(k_x)(0), y(k_y)(1/2)).$$

The $(i+j-d)$-cell

$$x \bullet y\colon K_{x\bullet y} \longrightarrow \Omega M$$

is defined by

$$(x \bullet y)(k_x, k_y) = \begin{cases} y(k_y)(2t) & \text{if } t \le 1/4, \\ x(k_x)(2(t-1/4)) & \text{if } 1/4 \le t \le 3/4, \\ y(k_y)(2t-1) & \text{if } 3/4 \le t. \end{cases}$$

Taking the composite of the loop product with the map

$$\Delta\colon H_*(L(M \setminus \{*\})) \longrightarrow H_{*+1}(L(M \setminus \{*\}))$$

introduced by Chas and Sullivan [5] and the map Δ', we obtain a map

$$H_i(L(M \setminus \{*\})) \otimes H_j(\Omega M) \longrightarrow H_{i+j+2-d}(\Omega M), \quad u \otimes v \longmapsto (\Delta u) \bullet (\Delta' v).$$

This coincides with our action σ in the case $d = 2$ and $i = j = 0$.

3.3. Conventions about (co)homology of groups. In the next two subsections, we give a homological interpretation of the Goldman bracket and the action σ for Σ. To state these we fix some conventions about (co)homology of groups. We basically follow Brown [3].

Let G be a group and M a left $\mathbb{Q}G$-module. For simplicity we use the term *G-module* for left $\mathbb{Q}G$-module and sometimes write $\otimes_G$ instead of $\otimes_{\mathbb{Q}G}$. We can always regard M as a right G-module by setting

$$mg = g^{-1}m, \quad m \in M, g \in G.$$

The homology group $H_*(G; M)$ and the cohomology group $H^*(G; M)$ are defined by

$$H_*(G; M) \stackrel{\text{def}}{=} \mathrm{Tor}_*^{\mathbb{Q}G}(M, \mathbb{Q})$$

and

$$H^*(G; M) \stackrel{\text{def}}{=} \mathrm{Ext}_{\mathbb{Q}G}^*(\mathbb{Q}, M),$$

respectively. Namely taking a $\mathbb{Q}G$-projective resolution $F_* \xrightarrow{\varepsilon} \mathbb{Q}$ of the trivial G-module $\mathbb{Q}$, we have

$$H_*(G; M) = H_*(M \otimes_G F_*)$$

and

$$H^*(G; M) = H^*(\mathrm{Hom}_{\mathbb{Q}G}(F_*, M)).$$

Let $\bar{F}_*(G)$ be the normalized standard complex. For each $n \geq 0$, $\bar{F}_n(G)$ is the quotient of the free G-module with $\mathbb{Q}G$-basis $\{[g_1|g_2|\ldots|g_n]; g_i \in G\}$ by the $\mathbb{Q}G$-submodule spanned by $\{[g_1|g_2|\ldots|g_n];\ g_i = 1 \text{ for some } i\}$. It gives a $\mathbb{Q}G$-projective resolution of $\mathbb{Q}$. For a G-module M, set

$$C_n(G; M) \stackrel{\text{def}}{=} M \otimes_G \bar{F}_n(G).$$

Of course we have $H_*(C_*(G; M)) = H_*(G; M)$. The boundary maps of $C_*(G; M)$ in low degrees are given by

$$\partial_1(m \otimes [g]) = g^{-1}m - m \in M \cong M[\,];$$

$$\partial_2(m \otimes [g_1|g_2]) = g_1^{-1}m \otimes [g_2] - m \otimes [g_1g_2] + m \otimes [g_1].$$

Here $\otimes$ means $\otimes_G$.

In this paper, we consider the (co)homology of groups of π for various π-modules M. One way to describe these is to use the normalized complex $C_*(\pi; M)$, and another way is to use the following particular resolution. Since π is a free group of rank $2g$, the augmentation ideal $I\pi$ is a free $\mathbb{Q}\pi$-module of rank $2g$. Thus

$$0 \to I\pi \longrightarrow \mathbb{Q}\pi \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0$$

gives a $\mathbb{Q}\pi$-projective resolution of $\mathbb{Q}$. In this view point, the homology group of π is computed as

$$H_*(\pi; M) = H_*(M \otimes_{\mathbb{Q}\pi} I\pi \longrightarrow M,\ m \otimes v \longmapsto \iota(v)m).$$

The canonical isomorphism $\bar{F}_0(\pi) \cong \mathbb{Q}\pi$ and the $\mathbb{Q}\pi$-linear map

$$\bar{F}_1(\pi) \longrightarrow I\pi, \quad [x] \longmapsto x - 1, \quad x \in \pi,$$

are compatible with the boundary maps of the two resolutions. In particular, we have a chain-level description of the canonical isomorphism

$$H_1(C_*(\pi; M)) \xrightarrow{\cong} H_1(M \otimes_{\mathbb{Q}\pi} I\pi \longrightarrow M). \tag{3.4}$$

Finally we mention the relative version of homology of groups. See also §A.5. Let G be a group and K a subgroup of G, and M a left G-module. Then $C_*(K; M)$ is a subcomplex of $C_*(G; M)$. We define the relative homology group as the homology of the quotient complex:

$$H_*(G, K; M) \stackrel{\text{def}}{=} H_*(C_*(G; M)/C_*(K; M)).$$

Note that since $C_0(G; M) = C_0(K; M)$, any 1-chain of $C_*(G; M)/C_*(K; M)$ is a cycle.

3.4. Homological interpretation of the Goldman Lie algebra. Let $\mathbb{Q}\pi^c$ be the following π-module. As a vector space, $\mathbb{Q}\pi^c = \mathbb{Q}\pi$, and the π-action is given by the conjugation: $xu \stackrel{\text{def}}{=} xux^{-1}$ for $x \in \pi, u \in \mathbb{Q}\pi^c$.

Definition 3.4.1. Define a $\mathbb{Q}$-linear map

$$\lambda\colon \mathbb{Q}\pi \longrightarrow H_1(\pi; \mathbb{Q}\pi^c)$$

by

$$\lambda(x) \stackrel{\text{def}}{=} x \otimes [x], \quad x \in \pi.$$

Here we understand $H_1(\pi; \mathbb{Q}\pi^c)$ as the homology of $C_*(\pi; \mathbb{Q}\pi^c)$.

We need to verify that this is well-defined, i.e., $\lambda(x)$ is a cycle.

Lemma 3.4.2. *For $x, y \in \pi$, we have*

(1) $x \otimes [x] \in C_1(\pi; \mathbb{Q}\pi^c)$ *is a cycle,*

(2) $\lambda(yxy^{-1}) = \lambda(x) \in H_1(\pi; \mathbb{Q}\pi^c)$.

Proof. The first part follows by $\partial_1(x \otimes [x]) = x^{-1} \cdot x - x = x^{-1}xx - x = 0$. For the second part, note that for any $x_1, x_2 \in \pi$, the 1-chain $[x_1x_2]$ is homologous to $x_1[x_2] + [x_1]$ since $\partial_2([x_1|x_2]) = x_1[x_2] - [x_1x_2] + [x_1]$, and for any $x \in \pi$, the 1-chain $[x^{-1}]$ is homologous to $-x^{-1}[x]$ since $\partial_2([x^{-1}|x]) = x^{-1}[x]-[1]+[x^{-1}] = x^{-1}[x] + [x^{-1}]$. Therefore, we compute

$$\begin{aligned}\lambda(yxy^{-1}) &= yxy^{-1} \otimes [yxy^{-1}] \equiv yxy^{-1} \otimes (y[xy^{-1}] + [y]) \\ &\equiv yxy^{-1} \otimes (y(x[y^{-1}] + [x]) + [y]) \\ &\equiv yxy^{-1} \otimes (y[x] - yxy^{-1}[y] + [y]) \\ &= x \otimes [x] = \lambda(x).\end{aligned}$$

Here $\equiv$ stands for "homologous." This proves (2). □

By Lemma 3.4.2, λ descends to a map $\lambda\colon \mathbb{Q}\hat{\pi} \to H_1(\pi; \mathbb{Q}\pi^c)$. We introduce a π-module map $\mathcal{B}\colon \mathbb{Q}\pi^c \otimes \mathbb{Q}\pi^c \to \mathbb{Q}\hat{\pi}$ by $\mathcal{B}(u \otimes v) = |uv|$. Here $|\,|\colon \mathbb{Q}\pi \to \mathbb{Q}\hat{\pi}$ is the linear map induced by the projection $|\,|\colon \pi \to \hat{\pi}$, the action of π on $\mathbb{Q}\pi^c \otimes \mathbb{Q}\pi^c$ is diagonal, and $\mathbb{Q}\hat{\pi}$ is regarded as a trivial π-module. Let $\mathcal{S}^c$ be the local system on Σ corresponding to the left π-module $\mathbb{Q}\pi^c$. Since Σ is a $K(\pi, 1)$-space, there is a canonical isomorphism $H_*(\pi; \mathbb{Q}\pi^c) \cong H_*(\Sigma; \mathcal{S}^c)$. Using the intersection form of the surface, we have the bilinear form

$$\begin{aligned}(\,\cdot\,)\colon H_1(\pi; \mathbb{Q}\pi^c) \times H_1(\pi; \mathbb{Q}\pi^c) &\cong H_1(\Sigma; \mathcal{S}^c) \times H_1(\Sigma; \mathcal{S}^c) \\ &\longrightarrow H_0(\Sigma; \mathcal{S}^c \otimes \mathcal{S}^c) \cong H_0(\pi; \mathbb{Q}\pi^c \otimes \mathbb{Q}\pi^c).\end{aligned}$$

Proposition 3.4.3. (1) *The sequence*

$$0 \longrightarrow \mathbb{Q}\hat{\pi}' \xrightarrow{\lambda} H_1(\pi; \mathbb{Q}\pi^c) \longrightarrow H \longrightarrow 0,$$

where the map $H_1(\pi; \mathbb{Q}\pi^c) \to H_1(\pi; \mathbb{Q}) = H$ is induced by the augmentation, is exact and canonically splits.

(2) *For $u, v \in \mathbb{Q}\hat{\pi}$,*

$$[u, v] = \mathcal{B}_*(\lambda(u) \cdot \lambda(v)).$$

Here $\mathcal{B}_*\colon H_0(\pi; \mathbb{Q}\pi^c \otimes \mathbb{Q}\pi^c) \to H_0(\pi; \mathbb{Q}\hat{\pi}) = \mathbb{Q}\hat{\pi}$ is the map induced by $\mathcal{B}$.

Proof. First of all, we remark that this proposition holds for any oriented aspherical surface S whose fundamental group is a free group or a non-commutative surface group. Then S is homotopic to a hyperbolic surface without cusps. For $x \in \pi$, let $Z(x) \subset \pi$ be the centralizer of x. $Z(x)$ is a cyclic group of infinite order if $x \neq 1$. In fact, the centralizer of a hyperbolic element in $\mathrm{PSL}_2(\mathbb{R})$ is an abelian subgroup isomorphic to $\mathbb{R}_{>0}$.

We have a direct decomposition

$$\mathbb{Q}\pi^c \cong \bigoplus_{|x| \in \hat{\pi}} \mathbb{Q}(\pi/Z(x))$$

as left π-modules. Here the action of π on $\mathbb{Q}(\pi/Z(x))$ is given by the multiplication from the left: $y(z \bmod Z(x)) = yz \bmod Z(x)$. Using the Shapiro lemma (see [3], p. 73), we have a canonical decomposition

$$\begin{aligned} H_1(\pi; \mathbb{Q}\pi^c) &= \bigoplus_{|x| \in \hat{\pi}} H_1(\pi; \mathbb{Q}(\pi/Z(x))) \\ &\cong \bigoplus_{|x| \in \hat{\pi}} H_1(Z(x); \mathbb{Q}) \\ &= H_1(\pi; \mathbb{Q}) \oplus \bigoplus_{|x| \in \hat{\pi}'} H_1(Z(x); \mathbb{Q}). \end{aligned}$$

Moreover if $x \neq 1$, the cycle $\lambda(x) = x \otimes [x]$ corresponds to a generator of $H_1(Z(x); \mathbb{Q}) \cong \mathbb{Q}$ (note: this part does not hold over the integers; in this case $\lambda(x)$ corresponds to a non-zero multiple of a generator of $H_1(Z(x); \mathbb{Z}) \cong \mathbb{Z}$). This proves the first part.

Next we proceed to the second part. As in Goldman [9] §2, we regard local systems as flat vector bundles and their (co)homology as the (co)homology of singular chains with values in flat vector bundles. Following this description the fiber of $\mathcal{S}^c$ at $p \in \Sigma$ is the group ring $\mathbb{Q}\pi_1(\Sigma, p)$, and the parallel transport along a path $\ell\colon [0,1] \to \Sigma$ is given by $\mathbb{Q}\pi_1(\Sigma, \ell(0)) \xrightarrow{\cong} \mathbb{Q}\pi_1(\Sigma, \ell(1))$, $\alpha \mapsto \ell^{-1}\alpha\ell$.

Let α be a based loop on Σ. Under the canonical isomorphism $H_1(\pi; \mathbb{Q}\pi^c) \cong H_1(\Sigma; \mathcal{S}^c)$, the 1-cycle $\alpha \otimes [\alpha] \in C_1(\pi; \mathbb{Q}\pi^c)$ corresponds to the flat section $s_\lambda(\alpha)$ of $\alpha^*\mathcal{S}^c$ over α, whose value at $p \in \alpha$ is just $\alpha_p \in \pi_1(\Sigma, p)$ (to be more precise, we need to write $p = p(t)$ for some $t \in S^1$). The homology class of the section $s_\lambda(\alpha)$ in $H_1(\Sigma; \mathcal{S}^c)$ depends only on the free homotopy class of the loop α, because of the homotopy equivalence of twisted homology. Let β be another free loop on Σ and suppose α and β intersect with at worst transverse double points. Similarly $\beta \otimes [\beta]$ is regarded as a section $s_\lambda(\beta)$.

Using the same letter let $\mathcal{B}\colon \mathcal{S}^c \otimes \mathcal{S}^c \to \mathbb{Q}\hat{\pi}$ be the pairing of local systems on Σ corresponding to the π-module map $\mathcal{B}$. Here $\mathbb{Q}\hat{\pi}$ is considered as a trivial local system. For each $p \in \Sigma$, this pairing is just the conjunction

$$\mathbb{Q}\pi_1(\Sigma, p) \otimes \mathbb{Q}\pi_1(\Sigma, p) \longrightarrow \mathbb{Q}\hat{\pi}, \quad u \otimes v \longmapsto |uv|.$$

By the formula in [9], p. 276, we have

$$\mathcal{B}_*(\lambda(\alpha) \cdot \lambda(\beta)) = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \mathcal{B}(s_\lambda(\alpha)_p \otimes s_\lambda(\beta)_p).$$

Since $\mathcal{B}(s_\lambda(\alpha)_p \otimes s_\lambda(\beta)_p) = |\alpha_p \beta_p|$, this is nothing but the Goldman bracket $[\alpha, \beta]$. This completes the proof. □

Remark 3.4.4. It should be remarked that λ is related to Chas-Sullivan's operator $\Delta = ME$ [5]. More precisely, let $L\Sigma$ be the free loop space of the surface Σ, $L\Sigma = \mathrm{Map}(S^1, \Sigma)$. The evaluation map at $0 \in S^1 = [0,1]/(0 \sim 1)$, $\mathrm{ev}\colon L\Sigma \to \Sigma$, $\ell \mapsto \ell(0)$, is a Hurewicz fibration with fiber $\Omega\Sigma$, the based loop space of Σ. Since Σ is a $K(\pi, 1)$-space, the homology group $H_*(\Omega\Sigma; \mathbb{Q})$ vanishes in positive degree. The 0-th homology group $H_0(\Omega\Sigma; \mathbb{Q})$ constitutes the local system $\mathcal{S}^c$ stated above. Hence we have an isomorphism $\mathrm{ev}_*\colon H_*(L\Sigma; \mathbb{Q}) \cong H_*(\Sigma; \mathcal{S}^c)$. The diagram

$$\begin{tikzcd} H_0(L\Sigma; \mathbb{Q}) \arrow[r, "\Delta"] \arrow[d, equal] & H_1(L\Sigma; \mathbb{Q}) \arrow[d, "\mathrm{ev}_*"] \\ \mathbb{Q}\hat{\pi} \arrow[r, "\lambda"'] & H_1(\Sigma; \mathcal{S}^c) \end{tikzcd}$$

commutes by the definition of $\Delta = ME$ and λ.

3.5. Homological interpretation of the action. Let $\mathbb{Q}\pi^r$ (resp. $\mathbb{Q}\pi^l$) be the following π-module. As a vector space, $\mathbb{Q}\pi^r = \mathbb{Q}\pi^l = \mathbb{Q}\pi$, and the π-action is given by the multiplication from the right (resp. the left): $xu \overset{\text{def}}{=} ux^{-1}$ (resp. $xu \overset{\text{def}}{=} xu$) for $x \in \pi$, and $u \in \mathbb{Q}\pi^r$ (resp. $\mathbb{Q}\pi^l$).

Let $\langle\zeta\rangle$ be the cyclic subgroup of π generated by ζ. We consider the relative homology of the pair $(\pi, \langle\zeta\rangle)$ as the homology of the relative complex (see §3.3).

Definition 3.5.1. Define a $\mathbb{Q}$-linear map

$$\xi\colon \mathbb{Q}\pi \longrightarrow H_1(\pi, \langle\zeta\rangle; \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l)$$

by

$$\xi(x) = (1 \otimes x) \otimes [x], \quad x \in \pi.$$

We denote by $\mathbb{Q}\pi^t$ the vector space $\mathbb{Q}\pi$ with the trivial π-action. We introduce a π-module map $\mathcal{C}\colon \mathbb{Q}\pi^c \otimes \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l \to \mathbb{Q}\pi^t$ by $\mathcal{C}(u \otimes v \otimes w) = vuw$. Here we consider the diagonal π-action on $\mathbb{Q}\pi^c \otimes \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l$. Let $\mathcal{S}^r$, $\mathcal{S}^l$, and $\mathcal{S}^t$ be the local system on Σ corresponding to the π-modules $\mathbb{Q}\pi^r$, $\mathbb{Q}\pi^l$, and $\mathbb{Q}\pi^t$, respectively. Then we have the canonical isomorphism $H_1(\pi, \langle\zeta\rangle; \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l) \cong H_1(\Sigma, \partial\Sigma; \mathcal{S}^r \otimes \mathcal{S}^l)$, etc. Using the intersection form of the surface, we have the bilinear form

$$\begin{aligned}(\,\cdot\,)\colon H_1(\pi; \mathbb{Q}\pi^c) \times H_1(\pi, \langle\zeta\rangle; \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l) &\cong H_1(\Sigma; \mathcal{S}^c) \times H_1(\Sigma, \partial\Sigma; \mathcal{S}^r \otimes \mathcal{S}^l)\\ &\longrightarrow H_0(\Sigma; \mathcal{S}^c \otimes \mathcal{S}^r \otimes \mathcal{S}^l) \cong H_0(\pi; \mathbb{Q}\pi^c \otimes \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l).\end{aligned}$$

Proposition 3.5.2. *For $u \in \mathbb{Q}\hat{\pi}$ and $v \in \mathbb{Q}\pi$, we have*

$$\sigma(u)v = \mathcal{C}_*(\lambda(u) \cdot \xi(v)).$$

Here $\mathcal{C}_*\colon H_0(\pi; \mathbb{Q}\pi^c \otimes \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l) \to H_0(\pi; \mathbb{Q}\pi^t) = \mathbb{Q}\pi$ is the map induced by $\mathcal{C}$.

Proof. The proof is similar to the proof of Proposition 3.4.3. Let α be an immersed loop and β an immersed based loop. Suppose they intersect with at worst transverse double points. The fiber of the local system $\mathcal{S}^r$ (resp. $\mathcal{S}^l$) at $p \in \Sigma$ is $\mathbb{Q}\pi(\Sigma, *, p)$ (resp. $\mathbb{Q}\pi(\Sigma, p, *)$). Here $\pi(\Sigma, p, q)$ is the set of homotopy classes of paths from p to q, and $\mathbb{Q}\pi(\Sigma, p, q)$ is the $\mathbb{Q}$-vector space spanned by $\pi(\Sigma, p, q)$.

By the canonical isomorphism $H_1(\pi, \langle\zeta\rangle; \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l) \cong H_1(\Sigma, \partial\Sigma; \mathcal{S}^r \otimes \mathcal{S}^l)$, the relative cycle $\xi(\beta) = (1 \otimes \beta) \otimes [\beta]$ corresponds to the flat section $s_\xi(\beta)$ of $\beta^*(\mathcal{S}^r \otimes \mathcal{S}^l)$ whose value at $p \in \beta$ is just $(\beta_{*p} \otimes \beta_{p*})$. Let $\mathcal{C}\colon \mathcal{S}^c \otimes \mathcal{S}^r \otimes \mathcal{S}^l \to \mathcal{S}^t$ be the pairing of local systems on Σ corresponding to the π-module map $\mathcal{C}$ (using the same letter). For each $p \in \Sigma$, this pairing is just the conjunction

$$\mathbb{Q}\pi_1(\Sigma, p) \otimes \mathbb{Q}\pi(\Sigma, *, p) \otimes \mathbb{Q}\pi(\Sigma, p, *) \longrightarrow \mathbb{Q}\pi, \quad u \otimes v \otimes w \longmapsto vuw.$$

By the formula in [9], p. 276, we have

$$\mathcal{C}_*(\lambda(\alpha), \xi(\beta)) = \sum_{p \in \alpha\cap\beta} \varepsilon(p; \alpha, \beta)\mathcal{C}(s_\lambda(\alpha)_p \otimes s_\xi(\beta)_p).$$

Since $\mathcal{C}(s_\lambda(\alpha)_p \otimes s_\xi(\beta)_p) = \beta_{*p}\alpha_p\beta_{p*}$, this equals $\sigma(\alpha)\beta$. This completes the proof. □

4. (Co)homology theory of $\widehat{T}$

Following Appendix A, we define $H_*(\widehat{T}; M)$, $H^*(\widehat{T}; M)$ and $H_*(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M)$ for any $\widehat{T}$-module M. Here $\mathbb{Q}[\![\omega]\!]$ is the ring of formal power series in the symplectic form ω, which is regarded as a complete Hopf subalgebra of $\widehat{T}$ in an obvious way. In this section we describe them in an explicit way, prove the Poincaré duality for the pair $(\widehat{T}, \mathbb{Q}[\![\omega]\!])$, and give a homological interpretation of symplectic derivations of the algebra $\widehat{T}$.

4.1. Explicit description of (co)homology of $\widehat{T}$. Let S be a (complete) Hopf algebra over $\mathbb{Q}$. We denote by IS the augmentation ideal of S, namely,

$$IS \overset{\text{def}}{=} \operatorname{Ker}(\varepsilon\colon S \longrightarrow \mathbb{Q}),$$

and by ∂ the inclusion map $IS \hookrightarrow S$. Then

$$P_*(S) \overset{\text{def}}{=} (IS \xrightarrow{\partial} S)$$

is a left S-resolution of the trivial S-module $\mathbb{Q}$. For a left S-module M we denote

$$\begin{aligned} D_*(S; M) &\overset{\text{def}}{=} M \otimes_S P_*(S) \\ &= (M \otimes_S IS \longrightarrow M \otimes_S S), \end{aligned}$$

and

$$\begin{aligned} D^*(S; M) &\overset{\text{def}}{=} \operatorname{Hom}_S(P_*(S), M) \\ &= (\operatorname{Hom}_S(IS, M) \longleftarrow \operatorname{Hom}_S(S, M)). \end{aligned}$$

Let $f\colon R \to S$ be a homomorphism of (complete) Hopf algebras. It induces a natural homomorphism $f\colon IR \to IS$ and natural (co)chain maps

$$f\colon D_*(R; M) \longrightarrow D_*(S; M) \quad \text{and} \quad f^*\colon D^*(S; M) \longrightarrow D^*(R; M).$$

The mapping cone

$$D_*(S, R; M) \overset{\text{def}}{=} D_*(S; M) \rtimes_f D_{*-1}(R; M)$$

has an acyclic subcomplex $M \otimes_R R = M \xrightarrow{1_M} M = M \otimes_S S$ (for the definition of the mapping cone, see §A.1). We denote the quotient complex by $\overline{D}_*(S, R; M)$, which is given by

$$\overline{D}_*(S, R; M) = \begin{cases} M \otimes_R IR & \text{if } * = 2, \\ M \otimes_S IS & \text{if } * = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\partial_2 = 1_M \otimes f \colon \bar{D}_2(S, R; M) = M \otimes_R IR \longrightarrow M \otimes_S IS = \bar{D}_1(S, R; M).$$

The natural projection $\varpi \colon D_*(S, R; M) \to \bar{D}_*(S, R; M)$ is a quasi-isomorphism.

We call the (complete) Hopf algebra S *free*, if IS is a left S-free module. For example, the algebras $\widehat{T}$, $\mathbb{Q}[\![\omega]\!]$, $\mathbb{Q}\pi$ and $\mathbb{Q}\langle\zeta\rangle$ are free. Then $P_*(S)$ is a left S-projective resolution of $\mathbb{Q}$. Hence we have

$$\begin{aligned} H_*(S; M) &= H_*(D_*(S; M)) \\ &= H_*(M \otimes_S IS \xrightarrow{1_M \otimes \partial} M \otimes_S S) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} H^*(S; M) &= H^*(D^*(S; M)) \\ &= H^*(\mathrm{Hom}_S(IS, M) \xleftarrow{\partial^*} \mathrm{Hom}_S(S, M)) \end{aligned} \tag{4.2}$$

as in (3.4). If R is also free, then we have

$$\begin{aligned} H_*(S, R; M) &= H_*(\bar{D}_*(S, R; M)) \\ &= H_*(M \otimes_R IR \xrightarrow{1_M \otimes f} M \otimes_S IS \longrightarrow 0). \end{aligned} \tag{4.3}$$

Lemma 4.1.1. *Let S and R be free (complete) Hopf algebras, $f \colon R \to S$ a homomorphism of (complete) Hopf algebras, and M a trivial S-module.*

(1) *Then*

$$H_*(S; M) = \begin{cases} M & \text{if } * = 0, \\ M \otimes (IS/(IS)^2) & \text{if } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *Further, if $f(IR) \subset (IS)^2$, then*

$$H_*(S, R; M) = \begin{cases} H_1(R; M), & \text{if } * = 2, \\ H_1(S; M), & \text{if } * = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\partial_* \colon H_2(S, R; M) \to H_1(R; M)$ is an isomorphism.

Proof. Since M is a trivial module, $1_M \otimes \partial\colon M \otimes_S IS \to M \otimes_S S$ is a zero map. Hence $H_0(S; M) = M$ and $H_1(S; M) = M \otimes_S IS$. The map

$$M \otimes_S IS \longrightarrow M \otimes (IS/(IS)^2), \quad u \otimes a \longmapsto u \otimes (a \bmod (IS)^2),$$

is a well-defined isomorphism. This proves the first part.

By the assumption $f(IR) \subset (IS)^2$,

$$f_*\colon M \otimes (IR/(IR)^2) \longrightarrow M \otimes (IS/(IS)^2)$$

is a zero map. Hence the homology exact sequence (A.4) implies the second part. □

Consider the case $S = \widehat{T}$ and $R = \mathbb{Q}[\![\omega]\!]$. The inclusion map $i\colon \mathbb{Q}[\![\omega]\!] \to \widehat{T}$ is a homomorphism of complete Hopf algebras. Then we have $IS = \widehat{T}_1 = \widehat{T} \otimes H$ as a left $\widehat{T}$-module, so that $M \otimes_S IS = M \otimes_{\widehat{T}} \widehat{T} \otimes H = M \otimes H$ and $\mathrm{Hom}_S(IS, M) = \mathrm{Hom}_{\widehat{T}}(\widehat{T} \otimes H, M) = \mathrm{Hom}(H, M)$. Under these isomorphisms, the operators $1_M \otimes \partial$ and ∂^* are given by

$$\begin{aligned} \partial_M\colon M \otimes H &\longrightarrow M, \\ m \otimes X &\longmapsto \iota(X)m, \end{aligned}$$

and

$$\begin{aligned} \delta_M\colon M &\longrightarrow \mathrm{Hom}(H, M), \\ m &\longmapsto (X \mapsto Xm), \end{aligned}$$

respectively. Hence we have

$$H_*(\widehat{T}; M) = H_*(M \otimes H \xrightarrow{\partial_M} M), \tag{4.4}$$

and

$$H^*(\widehat{T}; M) = H^*(\mathrm{Hom}(H, M) \xleftarrow{\delta_M} M). \tag{4.5}$$

A similar result holds for $R = \mathbb{Q}[\![\omega]\!]$. Under the isomorphism $M \otimes_R IR = M \otimes \mathbb{Q}\omega = M$, the boundary operator in $\bar{D}_*(S, R; M)$ is given by

$$\begin{aligned} d_M\colon M &\longrightarrow M \otimes_{\widehat{T}} \widehat{T}_1 = M \otimes H, \\ m &\longmapsto \; m \otimes \omega \; = \sum_{i=1}^{g} -(A_i m) \otimes B_i + (B_i m) \otimes A_i. \end{aligned}$$

Hence we have

$$\bar{D}_*(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M) = (M \xrightarrow{d_M} M \otimes H \longrightarrow 0). \tag{4.6}$$

Now we recall that the space H and its dual H^* are identified by the map

$$\begin{aligned}\vartheta : H &\xrightarrow{\cong} H^*,\\ X &\longmapsto (Y \mapsto Y \cdot X),\end{aligned}$$

as in (2.3), and introduce the isomorphisms

$$\begin{aligned}\vartheta : \bar{D}_1(\hat{T}, \mathbb{Q}[\![\omega]\!]; M) = M \otimes H &\xrightarrow{\cong} H^* \otimes M = D^1(\hat{T}; M),\\ m \otimes X &\longmapsto -\vartheta(X) \otimes m,\end{aligned}$$

and

$$\begin{aligned}\vartheta : \bar{D}_2(\hat{T}, \mathbb{Q}[\![\omega]\!]; M) = M &\xrightarrow{\cong} M = D^0(\hat{T}; M),\\ m &\longmapsto -m.\end{aligned}$$

It is easy to check that they constitute a chain map up to sign, and induce an isomorphism of cochain complexes

$$\vartheta : \bar{D}_{2-*}(\hat{T}, \mathbb{Q}[\![\omega]\!]; M) \xrightarrow{\cong} D^*(\hat{T}; M). \tag{4.7}$$

Hence we have an isomorphism $H_{2-*}(\hat{T}, \mathbb{Q}[\![\omega]\!]; M) \cong H^*(\hat{T}; M)$. In the next subsection we interpret this isomorphism as a certain kind of the Poincaré duality.

4.2. Poincaré duality for the pair $(\hat{T}, \mathbb{Q}[\![\omega]\!])$. We begin by introducing the fundamental class $[\hat{\mathcal{L}}] \in H_2(\hat{T}, \mathbb{Q}[\![\omega]\!]; \mathbb{Q})$, which is a counterpart of the fundamental class $[\Sigma] \in H_2(\Sigma, \partial\Sigma; \mathbb{Q})$ of the surface Σ. For $R = \mathbb{Q}[\![\omega]\!]$,

$$IR/(IR)^2 = \mathbb{Q}[\![\omega]\!]\omega/\mathbb{Q}[\![\omega]\!]\omega^2 = \mathbb{Q}\omega.$$

By Lemma 4.1.1(1),

$$H_1(\mathbb{Q}[\![\omega]\!]; \mathbb{Q}) = \mathbb{Q}\omega \cong \mathbb{Q}.$$

Since $i(IR) \subset (IS)^2$ for $S = \hat{T}$, the connecting homomorphism

$$\partial_* : H_2(S, R; \mathbb{Q}) \longrightarrow H_1(R; \mathbb{Q})$$

is an isomorphism by Lemma 4.1.1 (2). We define

$$[\hat{\mathcal{L}}] \stackrel{\text{def}}{=} -\partial_*{}^{-1}(\omega) \in H_2(\hat{T}, \mathbb{Q}[\![\omega]\!]; \mathbb{Q}), \tag{4.8}$$

which spans $H_2(\hat{T}, \mathbb{Q}[\![\omega]\!]; \mathbb{Q}) \cong \mathbb{Q}$ and is represented by $(0, -\omega)$ in

$$D_2(\hat{T}, \mathbb{Q}[\![\omega]\!]; \mathbb{Q}) = 0 \oplus \mathbb{Q} \otimes_{\mathbb{Q}[\![\omega]\!]} \mathbb{Q}[\![\omega]\!]\omega.$$

We call it *the fundamental class of the pair* $(\hat{T}, \mathbb{Q}[\![\omega]\!])$. We have a certain kind of the Poincaré duality with respect to this fundamental class $[\hat{\mathcal{L}}]$.

Proposition 4.2.1. *The cap product by the fundamental class $[\widehat{\mathcal{L}}]$ gives an isomorphism*

$$[\widehat{\mathcal{L}}]\cap\colon H^*(\widehat{T};M) \xrightarrow{\cong} H_{2-*}(\widehat{T},\mathbb{Q}[\![\omega]\!];M)$$

for any left $\widehat{T}$-module M. In particular, the cochain map ϑ in (4.7) induces the inverse of the map $[\widehat{\mathcal{L}}]\cap$.

Proof. We begin by computing the diagonal map (A.7)

$$\Delta_*\colon H_*(\widehat{T},\mathbb{Q}[\![\omega]\!];M) \longrightarrow H_*((M\otimes_S (P_*\widehat{\otimes}P_*)) \rtimes_{i\otimes i} (M\otimes_R (F_*\widehat{\otimes}F_*)_{*-1})) \tag{4.9}$$

explicitly. Here we write simply $P_* = P_*(\widehat{T})$ and $F_* = P_*(\mathbb{Q}[\![\omega]\!])$. It should be remarked the completed tensor product $P_*(\widehat{T})\widehat{\otimes}P_*(\widehat{T})$ given by

$$\begin{array}{ccccc} \widehat{T}_1\widehat{\otimes}\widehat{T}_1 & \xrightarrow{\partial_2} & \widehat{T}_1\widehat{\otimes}\widehat{T}\oplus\widehat{T}\widehat{\otimes}\widehat{T}_1 & \xrightarrow{\partial_1} & \widehat{T}\widehat{\otimes}\widehat{T}, \\ u & \longmapsto & (-u,u), & & \\ & & (v,w) & \longmapsto & v+w, \end{array}$$

is acyclic. We construct a chain map

$$\Delta\colon P_*(\widehat{T}) \longrightarrow P_*(\widehat{T})\widehat{\otimes}P_*(\widehat{T})$$

respecting the coproduct Δ as follows. In degree 0 we define

$$\Delta\colon P_0(\widehat{T}) = \widehat{T} \longrightarrow (P_*(\widehat{T})\widehat{\otimes}P_*(\widehat{T}))_0 = \widehat{T}\widehat{\otimes}\widehat{T}$$

by the coproduct Δ itself. In degree 1 we define

$$\Delta(X) \overset{\text{def}}{=} (X\widehat{\otimes}1, 1\widehat{\otimes}X) \in \widehat{T}_1\widehat{\otimes}\widehat{T}\oplus\widehat{T}\widehat{\otimes}\widehat{T}_1 = (P_*(\widehat{T})\widehat{\otimes}P_*(\widehat{T}))_1, \quad X\in H,$$

and extend it to the whole $\widehat{T}_1 = \widehat{T}\otimes H$ as a left $\widehat{T}$-homomorphism. Since

$$X\widehat{\otimes}1 + 1\widehat{\otimes}X = \Delta(X) \in \widehat{T}\widehat{\otimes}\widehat{T},$$

this map Δ is a $\widehat{T}$-chain map. We define a $\mathbb{Q}[\![\omega]\!]$-chain map

$$\Delta\colon P_*(\mathbb{Q}[\![\omega]\!]) \longrightarrow P_*(\mathbb{Q}[\![\omega]\!])\widehat{\otimes}P_*(\mathbb{Q}[\![\omega]\!])$$

in a similar way. We have

$$\Delta(\omega) = (\omega\widehat{\otimes}1, 1\widehat{\otimes}\omega)$$

for $\omega\in\mathbb{Q}[\![\omega]\!]\omega = P_1(\mathbb{Q}[\![\omega]\!])$. Then the (homotopy commutative) diagram

$$\begin{array}{ccc} P_*(\mathbb{Q}[\![\omega]\!]) & \xrightarrow{\Delta} & P_*(\mathbb{Q}[\![\omega]\!])\widehat{\otimes}P_*(\mathbb{Q}[\![\omega]\!]) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i\widehat{\otimes}i} \\ P_*(\widehat{T}) & \xrightarrow[\Delta]{} & P_*(\widehat{T})\widehat{\otimes}P_*(\widehat{T}) \end{array}$$

does *not* commute. If we denote

$$\hat{\omega} \overset{\text{def}}{=} \sum_{i=1}^{g} A_i \hat{\otimes} B_i - B_i \hat{\otimes} A_i \in \widehat{T}_1 \hat{\otimes} \widehat{T}_1,$$

then

$$\Delta i(\omega) = (\omega \hat{\otimes} 1 - \hat{\omega},\ \hat{\omega} + 1 \hat{\otimes} \omega) = (i \hat{\otimes} i)\Delta(\omega) + \partial_2 \hat{\omega}.$$

This means that the $\mathbb{Q}[\![\omega]\!]$-homomorphism

$$\Phi\colon P_*(\mathbb{Q}[\![\omega]\!]) \longrightarrow (P_*(\widehat{T}) \hat{\otimes} P_*(\widehat{T}))_{*+1}$$

defined by

$$\Phi|_{P_0} = 0 \quad \text{and} \quad \Phi|_{P_1}(\omega) = -\hat{\omega}$$

satisfies the relation

$$(i \hat{\otimes} i)\Delta - \Delta i = d\Phi + \Phi d.$$

Hence the diagonal map (4.9) is given by

$$h(\Delta, \Phi, \Delta) = \begin{pmatrix} \Delta & -\Phi \\ 0 & \Delta \end{pmatrix}.$$

In particular, the homology class $\Delta_*[\widehat{\mathcal{L}}]$ is represented by the cycle

$$\begin{pmatrix} \Delta & -\Phi \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} 0 \\ -\omega \end{pmatrix}$$
$$= \begin{pmatrix} -\hat{\omega} \\ (-\omega \hat{\otimes} 1,\ -1 \hat{\otimes} \omega) \end{pmatrix} \in (\mathbb{Q} \otimes_{\widehat{T}} (P_* \hat{\otimes} P_*)) \rtimes_{i \hat{\otimes} i} (\mathbb{Q} \otimes_{\mathbb{Q}[\![\omega]\!]} (F_* \hat{\otimes} F_*)_{*-1}).$$

By the explicit definition of the cap product (A.9), we have

$$(\Delta_*[\widehat{\mathcal{L}}]) \cap m = (0, -m \otimes \omega)$$

and

$$(\Delta_*[\widehat{\mathcal{L}}]) \cap v = \left(\sum_{i=1}^{g} -v(A_i) \otimes B_i + v(B_i) \otimes A_i, \sum_{i=1}^{g} A_i v(B_i) - B_i v(A_i) \right)$$

for $m \in M = D^0(\widehat{T}; M)$ and $v \in \mathrm{Hom}_{\widehat{T}}(\widehat{T}_1, M) = D^1(\widehat{T}; M)$. Hence

$$\varpi \circ ((\Delta_*[\widehat{\mathcal{L}}])\cap)\colon D^*(\widehat{T}; M) \longrightarrow \overline{D}_{2-*}(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M)$$

is exactly the inverse of the map ϑ; see (4.7). This proves the proposition. $\square$

In a way similar to the surface Σ we can introduce the intersection form

$$\begin{aligned}(\,\cdot\,)\colon H_1(\widehat{T}; M_1) \otimes H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M_2) &\longrightarrow M_1 \otimes_{\widehat{T}} M_2,\\ u \otimes v &\longmapsto \langle u, ([\widehat{\mathcal{L}}]\cap)^{-1} v\rangle,\end{aligned} \tag{4.10}$$

for any left $\widehat{T}$-modules M_1 and M_2. Here $\langle\ ,\ \rangle$ is the Kronecker product (A.12). Under the identifications (4.4) and (4.6), the intersection form coincides with the pairing

$$\begin{aligned}(\,\cdot\,)\colon\ M_1 \otimes H \otimes M_2 \otimes H &\longrightarrow M_1 \otimes_{\widehat{T}} M_2,\\ m_1 \otimes X_1 \otimes m_2 \otimes X_2 &\longmapsto (X_1 \cdot X_2) m_1 \otimes m_2.\end{aligned} \tag{4.11}$$

In fact, we have

$$\begin{aligned}\langle m_1 \otimes X_1, \vartheta(m_2 \otimes X_2)\rangle &= -\langle m_1 \otimes X_1, \vartheta(X_2) \otimes m_2\rangle\\ &= (X_1 \cdot X_2) m_1 \otimes m_2.\end{aligned}$$

The inclusion homomorphism

$$j_*\colon H_1(\widehat{T}; M_1) \longrightarrow H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M_1)$$

is induced by the composite

$$H_1(\widehat{T}; M_1) \hookrightarrow M_1 \otimes H \twoheadrightarrow H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M_1).$$

Hence the intersection form

$$\begin{aligned}(\,\cdot\,)\colon H_1(\widehat{T}; M_1) \otimes H_1(\widehat{T}; M_2) &\longrightarrow M_1 \otimes_{\widehat{T}} M_2,\\ u \otimes v &\longmapsto \langle u, ([\widehat{\mathcal{L}}]\cap)^{-1} j_* v\rangle,\end{aligned}$$

also coincides with the pairing (4.11).

In the succeeding subsections we use these intersections to give a homological interpretation of the Lie algebras $\mathfrak{a}_g^-$ and $\mathfrak{l}_g$ and the action of $\mathfrak{a}_g^-$ on the algebra $\widehat{T}$ by derivations.

4.3. Homological interpretation of $\mathfrak{a}_g^-$ and $\mathfrak{l}_g$. The space H acts on the spaces $\widehat{T}$ and $\widehat{\mathcal{L}}$ by

$$Xu \overset{\text{def}}{=} [X, u] \quad \text{and} \quad Xv \overset{\text{def}}{=} [X, v], \quad X \in H, u \in \widehat{T}, v \in \widehat{\mathcal{L}},$$

respectively. This action extends to the whole algebra $\widehat{T}$. In fact, we introduce an action of the algebra $\widehat{T}\widehat{\otimes}\widehat{T}$ on the space $\widehat{T}$ by

$$\begin{aligned}\mathcal{C}'\colon (\widehat{T}\widehat{\otimes}\widehat{T}) \otimes \widehat{T} &\longrightarrow \widehat{T},\\ (v'\widehat{\otimes}v'') \otimes u &\longmapsto v' u \iota(v''),\end{aligned}$$

for $u, v', v'' \in \widehat{T}$. The space $\widehat{T}$ is a left $\widehat{T}\widehat{\otimes}\widehat{T}$-module by the map $\mathcal{C}'$. We have

$$\mathcal{C}'(\Delta(X_1 \dots X_n) \otimes u) = [X_1, [X_2, [\dots [X_n, u] \dots]]], \quad X_i \in H, u \in \widehat{T}.$$

Hence the action

$$\widehat{T} \otimes \widehat{T} \longrightarrow \widehat{T}, \quad v \otimes u \longmapsto \mathcal{C}'((\Delta v) \otimes u),$$

is exactly an extension of the action of H stated above. We denote by $\widehat{T}^c$ and $\widehat{\mathcal{L}}^c$ the left $\widehat{T}$-modules defined by this action. In particular, if $v \in \widehat{T}$ is group-like, we have $\mathcal{C}'((\Delta v) \otimes u) = vu\iota(v)$. Hence these modules correspond to the $\mathbb{Q}\pi$-modules $\widehat{\mathbb{Q}\pi}^c$ and the set of primitive elements of $\widehat{\mathbb{Q}\pi}$ with the conjugate action of π, respectively. We denote by $\widehat{T}_1^c$ the $\widehat{T}$-submodule of $\widehat{T}^c$ whose underlying subspace is $\widehat{T}_1$.

As was stated in (2.7), the Lie algebra $\mathfrak{a}_g^- = \mathrm{Der}_\omega(\widehat{T})$ is identified with

$$\mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{T} \longrightarrow \widehat{T}) = N(\widehat{T}_1),$$

and the Lie algebra $\mathfrak{l}_g = \mathrm{Der}_\omega(\widehat{\mathcal{L}})$ with

$$\mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{\mathcal{L}} \longrightarrow \widehat{\mathcal{L}}) = N(\widehat{\mathcal{L}}\widehat{\otimes}\widehat{\mathcal{L}}).$$

Hence, by (4.4), we obtain

$$\mathfrak{a}_g^- = N(\widehat{T}_1) = H_1(\widehat{T}; \widehat{T}^c), \tag{4.12}$$

$$\mathfrak{a}_g = N(\widehat{T}_2) = H_1(\widehat{T}; \widehat{T}_1^c), \tag{4.13}$$

and

$$\mathfrak{l}_g = N(\widehat{\mathcal{L}}\widehat{\otimes}\widehat{\mathcal{L}}) = H_1(\widehat{T}; \widehat{\mathcal{L}}^c). \tag{4.14}$$

The brackets on the Lie algebras $\mathfrak{a}_g^-$ and $\mathfrak{l}_g$ can be interpreted as intersection forms on the homology introduced in (4.10). We introduce a map

$$\begin{aligned} \mathcal{B}\colon \widehat{T}^c \otimes_{\widehat{T}} \widehat{T}^c &\longrightarrow N(\widehat{T}_1) = \mathfrak{a}_g^-, \\ u \otimes v &\longmapsto N(uv), \end{aligned}$$

which is well-defined, since

$$N([u, X]v) = N(u[X, v]), \quad X \in H,$$

(Lemma 2.6.2 (2)).

For positive integers n and m, by a straightforward computation, we have:

Lemma 4.3.1. *For $X_i, Y_j \in H$,*

$$\begin{aligned}
&[N(X_1 \dots X_n), N(Y_1 \dots Y_m)] \\
&\quad = N((N(X_1 \dots X_n))(Y_1 \dots Y_m)) \\
&\quad = -\mathcal{B}(N(X_1 \dots X_n) \cdot N(Y_1 \dots Y_m)) \\
&\quad = -\sum_{i=1}^{n}\sum_{j=1}^{m}(X_i \cdot Y_j) N(X_{i+1} \dots X_n X_1 \dots X_{i-1} Y_{j+1} \dots Y_m Y_1 \dots Y_{j-1}).
\end{aligned}$$

Here the bracket $[\ ,\ ]$ is that as derivations of $\widehat{T}$, and $(N(X_1 \dots X_n))(Y_1 \dots Y_m)$ is the action of $N(X_1 \dots X_n)$ on the tensor $Y_1 \dots Y_m$ as a derivation. The third term is minus the pairing $(\cdot)$ in (4.11) of $N(X_1 \dots X_n)$ and $N(Y_1 \dots Y_m) \in \widehat{T} \otimes H$ applied by the map $\mathcal{B}$.

Hence we obtain:

Proposition 4.3.2. *Under the identifications (4.12) and (4.14), the brackets on the Lie algebras $\mathfrak{a}_g^-$ and $\mathfrak{l}_g$ coincide with minus the intersection forms*

$$-\mathcal{B}(\,\cdot\,) \colon H_1(\widehat{T}; \widehat{T}^c) \otimes H_1(\widehat{T}; \widehat{T}^c) \longrightarrow N(\widehat{T}_1) = \mathfrak{a}_g^-,$$

and

$$-\mathcal{B}(\,\cdot\,) \colon H_1(\widehat{T}; \widehat{\mathcal{L}}^c) \otimes H_1(\widehat{T}; \widehat{\mathcal{L}}^c) \longrightarrow N(\widehat{\mathcal{L}} \otimes \widehat{\mathcal{L}}) = \mathfrak{l}_g,$$

respectively.

4.4. Homological interpretation of symplectic derivations of $\widehat{T}$. In order to interpret the action of $\mathfrak{a}_g^-$ on the algebra $\widehat{T}$ by derivations, we introduce three left $\widehat{T}$-modules $\widehat{T}^r$, $\widehat{T}^l$ and $\widehat{T}^t$, which correspond to the left $\mathbb{Q}\pi$-modules $\mathbb{Q}\pi^r$, $\mathbb{Q}\pi^l$ and $\mathbb{Q}\pi^t$. As vector spaces these three modules are the same $\widehat{T}$. The action of the algebra $\widehat{T}$ is given by the multiplication

$$u(v^r) \overset{\text{def}}{=} v^r \iota(u),$$

$$u(v^l) \overset{\text{def}}{=} uv^l,$$

and

$$u(v^t) \overset{\text{def}}{=} \varepsilon(u) v^t,$$

for $u \in \widehat{T}$, $v^r \in \widehat{T}^r$, $v^l \in \widehat{T}^l$ and $v^t \in \widehat{T}^t$. Denote by T the tensor algebra of H,

$$T \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} H^{\otimes n}.$$

We define a map

$$\xi \colon T \longrightarrow (\widehat{T}^r \widehat{\otimes} \widehat{T}^l) \otimes_{\widehat{T}} \widehat{T}_1$$

by

$$\xi(u) \stackrel{\text{def}}{=} 1 \otimes (1 \otimes (1 - \varepsilon))(\Delta u) = 1 \otimes (\Delta u - u \otimes 1)$$

for $u \in T$. In this expression, we regard $(\widehat{T}^r \widehat{\otimes} \widehat{T}^l) \otimes_{\widehat{T}} \widehat{T}_1$ as the natural quotient of $(\widehat{T}^r \widehat{\otimes} \widehat{T}^l) \otimes \widehat{T}_1$. Then we have

Lemma 4.4.1.

$$\xi(X_1 \dots X_n) = \sum_{i=1}^{n} (X_1 \dots X_{i-1} \otimes X_{i+1} \dots X_n) \otimes_{\widehat{T}} X_i$$

for $n \geq 1$ *and* $X_i \in H$.

Proof. First note that $\widehat{T} \otimes \widehat{T}$ acts on $(\widehat{T}^r \widehat{\otimes} \widehat{T}^l) \otimes \widehat{T}_1$ from the right, by

$$(u \otimes v \otimes w)(x \otimes y) = u \otimes vx \otimes wy,$$

and this action is compatible with the quotient map

$$(\widehat{T}^r \widehat{\otimes} \widehat{T}^l) \otimes \widehat{T}_1 \longrightarrow (\widehat{T}^r \widehat{\otimes} \widehat{T}^l) \otimes_{\widehat{T}} \widehat{T}_1.$$

In the below, $1 \otimes (\Delta w - w \otimes 1)(1 \otimes X_n)$ means the result of the application of $1 \otimes X_n$ to $1 \otimes (\Delta w - w \otimes 1)$ with respect to this action, etc.

We prove the lemma by induction on $n \geq 1$. If $n = 1$, we have

$$\xi(X_1) = 1 \otimes (\Delta X_1 - X_1 \otimes 1) = 1 \otimes 1 \otimes X_1.$$

Suppose $n \geq 2$. Denote

$$w \stackrel{\text{def}}{=} X_1 \dots X_{n-1}.$$

By the inductive assumption, we compute

$$\begin{aligned}
&1 \otimes (\Delta w - w \otimes 1)(1 \otimes X_n) \\
&\quad = \sum_{i=1}^{n-1} X_1 \dots X_{i-1} \otimes X_{i+1} \dots X_{n-1} \otimes X_i X_n \\
&\quad = -\sum_{i=1}^{n-1} (\Delta X_i)(X_1 \dots X_{i-1} \otimes X_{i+1} \dots X_{n-1}) \otimes X_n \\
&\quad = -\sum_{i=1}^{n-1} (X_i \otimes 1 + 1 \otimes X_i)(X_1 \dots X_{i-1} \otimes X_{i+1} \dots X_{n-1}) \otimes X_n \\
&\quad = \sum_{i=1}^{n-1} (X_1 \dots X_i \otimes X_{i+1} \dots X_{n-1}) \otimes X_n - (X_1 \dots X_{i-1} \otimes X_i \dots X_{n-1}) \otimes X_n \\
&\quad = X_1 \dots X_{n-1} \otimes 1 \otimes X_n - 1 \otimes X_1 \dots X_{n-1} \otimes X_n \\
&\quad = w \otimes 1 \otimes X_n - 1 \otimes w \otimes X_n.
\end{aligned}$$

Hence we have $1 \otimes (\Delta w)(1 \otimes X_n) = w \otimes 1 \otimes X_n$. Using the inductive assumption again, we compute

$$\begin{aligned}
&\xi(wX_n) \\
&\quad = 1 \otimes (\Delta(wX_n) - wX_n \otimes 1) \\
&\quad = 1 \otimes (\Delta w(X_n \otimes 1 + 1 \otimes X_n) - wX_n \otimes 1) \\
&\quad = 1 \otimes (\Delta w - w \otimes 1)(X_n \otimes 1) + 1 \otimes (\Delta w)(1 \otimes X_n) \\
&\quad = \sum_{i=1}^{n-1} X_1 \dots X_{i-1} \otimes X_{i+1} \dots X_{n-1} X_n \otimes X_i + X_1 \dots X_{n-1} \otimes 1 \otimes X_n \\
&\quad = \sum_{i=1}^{n} X_1 \dots X_{i-1} \otimes X_{i+1} \dots X_n \otimes X_i.
\end{aligned}$$

This completes the induction. $\square$

By this lemma, the map ξ is a graded homomorphism of degree 0. Hence it extends to the whole $\widehat{T}$ and induces a map

$$\xi \colon \widehat{T} \longrightarrow (\widehat{T}^r \widehat{\otimes} \widehat{T}^l) \otimes_{\widehat{T}} \widehat{T}_1 = \overline{D}_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; \widehat{T}^r \widehat{\otimes} \widehat{T}^l) \longrightarrow H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; \widehat{T}^r \widehat{\otimes} \widehat{T}^l),$$

which corresponds to the map in Definition 3.5.1. Consider the map

$$\begin{aligned}\mathcal{C}\colon \widehat{T}^c \otimes_{\widehat{T}} (\widehat{T}^r \widehat{\otimes} \widehat{T}^l) &\longrightarrow \widehat{T}^t,\\ w \otimes u \otimes v &\longmapsto uwv,\end{aligned}$$

which is well-defined, since, for all $X \in H$,

$$\mathcal{C}(Xw \otimes u \otimes v) + \mathcal{C}(w \otimes X(u \otimes v)) = u(Xw - wX)v - uXwv + uwXv = 0.$$

Then we have:

Lemma 4.4.2.

$$\mathcal{C}(w \cdot \xi(u)) = (\vartheta w)(u) \in \widehat{T}$$

for any $w \in \widehat{T}^c \otimes H$ *and* $u \in \widehat{T}$. *Here*

$$\begin{aligned}\vartheta\colon \widehat{T}^c \otimes H &\longrightarrow H^* \otimes \widehat{T}^c,\\ m \otimes Y &\longmapsto -(\vartheta Y) \otimes m,\end{aligned}$$

is the map given in (4.7). *The right hand side means the action of* ϑw *on the tensor* u *as a derivation.*

Proof. It suffices to prove the lemma for $u = X_1 \dots X_n$, $X_i \in H$, $w = m \otimes Y$, $m \in \widehat{T}^c$ and $Y \in H$. By Lemma 4.4.1,

$$\begin{aligned}&\mathcal{C}(w \cdot \xi(u))\\ &\quad= \mathcal{C}\Big((m \otimes Y) \cdot \sum_{i=1}^{n} X_1 \dots X_{i-1} \otimes X_{i+1} \dots X_n \otimes X_i\Big)\\ &\quad= \sum_{i=1}^{n} (Y \cdot X_i)\mathcal{C}(m \otimes X_1 \dots X_{i-1} \otimes X_{i+1} \dots X_n)\\ &\quad= -\sum_{i=1}^{n} (X_i \cdot Y)(X_1 \dots X_{i-1} m X_{i+1} \dots X_n)\\ &\quad= \sum_{i=1}^{n} X_1 \dots X_{i-1} \vartheta(m \otimes Y)(X_i) X_{i+1} \dots X_n = (\vartheta w)(u).\end{aligned}$$

This proves the lemma. □

Hence we obtain:

Proposition 4.4.3. *Under the identification* (4.12) *and the map*

$$\xi\colon \widehat{T} \longrightarrow H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; \widehat{T}^r \widehat{\otimes} \widehat{T}^l),$$

the action of the Lie algebra $\mathfrak{a}_g^-$ on the algebra $\widehat{T}$ by derivations coincides with minus the intersection form

$$-\mathcal{C}(\,\cdot\,)\colon H_1(\widehat{T}; \widehat{T}^c) \otimes H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; \widehat{T}^r \widehat{\otimes} \widehat{T}^l) \longrightarrow \widehat{T}^t = \widehat{T}.$$

In other words, for any $w \in H_1(\widehat{T}; \widehat{T}^c)$ and $u \in \widehat{T}$, we have

$$\mathcal{C}(w \cdot \xi(u)) = -w(u).$$

5. Comparison via a symplectic expansion

In this section we prove Theorems 1.2.1 and 1.2.2 in the Introduction.

5.1. Comparison via a Magnus expansion. Let $F_n = \langle x_1, \ldots, x_n \rangle$ be a free group of rank $n \geq 1$ with standard generators $x_1, \ldots, x_n$, $\widehat{T}$ the completed tensor algebra of the rational homology group $H_1(F_n; \mathbb{Q})$, and $\theta\colon F_n \to \widehat{T}$ a Magnus expansion of F_n as in Definition 2.3.1. Then θ induces an algebra homomorphism $\theta\colon \mathbb{Q}F_n \to \widehat{T}$. We regard a left $\widehat{T}$-module M as a left $\mathbb{Q}F_n$-module via θ.

Lemma 5.1.1. *For any right $\widehat{T}$-module M_1 and left $\widehat{T}$-module M_2, θ induces isomorphisms*

$$\theta_*\colon H_*(F_n; M_1) \xrightarrow{\cong} H_*(\widehat{T}; M_1),$$

and

$$\theta^*\colon H^*(\widehat{T}, M_2) \xrightarrow{\cong} H^*(F_n; M_2).$$

Proof. There exists a filter-preserving automorphism U of the algebra $\widehat{T}$, such that $\theta(x_i) = U(1 + [x_i])$ for any $1 \leq i \leq n$ (see [14], Theorem 1.3). Since $\{[x_i]\}_{i=1}^n \subset H_1(F_n; \mathbb{Q})$ is a free basis of the left $\widehat{T}$-module $\widehat{T}_1$, the set $\{\theta(x_i) - 1\}_{i=1}^n$ is also a free basis of $\widehat{T}_1$. Hence we have a decomposition

$$M_1 \otimes_{\widehat{T}} \widehat{T}_1 = \bigoplus_{i=1}^n M_1 \otimes (\theta(x_i) - 1).$$

On the other hand, by using Fox' free differential, we find out $\{x_i - 1\}_{i=1}^n$ is a free basis of the left $\mathbb{Q}\pi$-module IF_n. This implies a decomposition

$$M_1 \otimes_{\mathbb{Q}F_n} IF_n = \bigoplus_{i=1}^n M_1 \otimes (x_i - 1).$$

Hence we obtain an isomorphism of chain complexes

$$\theta_* \colon D_*(\mathbb{Q}F_n; M_1) \cong M_1 \otimes_{\widehat{T}} P_*(\widehat{T}),$$

and so the isomorphism $\theta_* \colon H_*(F_n; M_1) \cong H_*(\widehat{T}; M_1)$. A similar argument holds for $H^*(\widehat{T}; M_2)$ and $H^*(F_n; M_2)$. □

Let $\theta \colon \pi \to \widehat{T}$ be a symplectic expansion of the fundamental group π of the surface Σ. Then the restriction of θ to the subgroup $\langle\zeta\rangle$ is a Magnus expansion of the infinite cyclic group $\langle\zeta\rangle$. Hence, by Lemma 5.1.1 and the five-lemma, we obtain

Corollary 5.1.2. *Let θ be a Magnus expansion of the fundamental group π of the surface Σ satisfying $\theta(\zeta) = \exp(\omega)$. Then the algebra homomorphism θ induces an isomorphism*

$$\theta_* \colon H_*(\pi, \langle\zeta\rangle; M) \xrightarrow{\cong} H_*(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M)$$

for any left $\widehat{T}$-module M. Here we write simply θ_* for $(\theta, \theta)_*$.

5.2. Symplectic expansion. Hereafter suppose that θ is a symplectic expansion of the fundamental group π. Then we have a commutative diagram of (complete) Hopf algebras

$$\begin{tikzcd} \mathbb{Q}\langle\zeta\rangle \arrow[r, "\theta"] \arrow[d, "i"'] & \mathbb{Q}[\![\omega]\!] \arrow[d, "i"] \\ \mathbb{Q}\pi \arrow[r, "\theta"'] & \widehat{T}. \end{tikzcd} \tag{5.1}$$

As was proved in Lemma 5.1.1 and Corollary 5.1.2, we have isomorphisms

$$\theta_* \colon H_*(\pi; M) \xrightarrow{\cong} H_*(\widehat{T}; M),$$

$$\theta^* \colon H^*(\widehat{T}; M) \xrightarrow{\cong} H^*(\pi; M),$$

and

$$\theta_* \colon H_*(\pi, \langle\zeta\rangle; M) \xrightarrow{\cong} H_*(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M)$$

for any left $\widehat{T}$-module M. Now we have:

Lemma 5.2.1. $\theta_*[\Sigma] = [\widehat{\mathcal{L}}] \in H_2(\widehat{T}, \mathbb{Q}[\![\omega]\!]; \mathbb{Q})$.

Proof. In the commutative diagram

$$\begin{tikzcd}
H_2(\Sigma, \partial\Sigma; \mathbb{Q}) \arrow[r, equal] \arrow[d, "\partial_*"'] & H_2(\pi, \langle\zeta\rangle; \mathbb{Q}) \arrow[r, "\theta_*"] \arrow[d, "\partial_*"'] & H_2(\widehat{T}, \mathbb{Q}[\![\omega]\!]; \mathbb{Q}) \arrow[d, "\partial_*"'] \\
H_1(\partial\Sigma; \mathbb{Q}) \arrow[r, equal] & H_1(\langle\zeta\rangle; \mathbb{Q}) \arrow[r, "\theta_*"] & H_1(\mathbb{Q}[\![\omega]\!]; \mathbb{Q}),
\end{tikzcd}$$

the fundamental class $[\Sigma]$ is mapped to $-[\zeta] \in H_1(\langle\zeta\rangle; \mathbb{Q})$. In fact, the loop ζ goes around the boundary $\partial\Sigma$ in the opposite direction. Since θ is a symplectic expansion, we have $-\theta_*[\zeta] = -\omega = \partial_*[\widehat{\mathcal{L}}] \in H_1(\mathbb{Q}[\![\omega]\!]; \mathbb{Q})$. This implies $\theta_*[\Sigma] = [\widehat{\mathcal{L}}]$, as was to be shown. □

Hence, by Proposition A.3.2,

Corollary 5.2.2. *We have a commutative diagram*

$$\begin{tikzcd}
H^1(\pi; M) \arrow[d, "{[\Sigma]\cap}"'] & H^1(\widehat{T}; M) \arrow[l, "\theta^*"'] \arrow[d, "{[\widehat{\mathcal{L}}]\cap}"] \\
H_1(\pi, \langle\zeta\rangle; M) \arrow[r, "\theta_*"'] & H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M)
\end{tikzcd}$$

for any left $\widehat{T}$-module M.

Here it should be remarked that the cap product on the pair of spaces $(\Sigma, \partial\Sigma)$ coincides with that on the pair of Hopf algebras $(\mathbb{Q}\pi, \mathbb{Q}\langle\zeta\rangle)$ by what we proved in §A.5. Thus the intersection form on the pair $(\widehat{T}, \mathbb{Q}[\![\omega]\!])$ is directly related to that on the surface Σ.

Proposition 5.2.3. *For any left $\widehat{T}$-modules M_1 and M_2, we have a commutative diagram*

$$\begin{tikzcd}
H_1(\pi; M_1) \otimes H_1(\pi, \langle\zeta\rangle; M_2) \arrow[r, "(\,\cdot\,)"] \arrow[d, "\theta_*\otimes\theta_*"'] & M_1 \otimes_{\mathbb{Q}\pi} M_2 \arrow[d] \\
H_1(\widehat{T}; M_1) \otimes H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; M_2) \arrow[r, "(\,\cdot\,)"'] & M_1 \otimes_{\widehat{T}} M_2.
\end{tikzcd}$$

5.3. Proof of Theorem 1.2.1. Recall the map $\lambda\colon \mathbb{Q}\hat{\pi} \to H_1(\pi;\mathbb{Q}\pi^c)$ in §3.4, whose kernel is the subspace $\mathbb{Q}1$ spanned by the constant loop $1 = |1| \in \hat{\pi}$. Since the group π is free, the map $H_*(\theta)\colon H_1(\pi;\mathbb{Q}\pi^c) \to H_1(\pi;\widehat{T}^c)$ induced by the injection $\theta\colon \mathbb{Q}\pi^c \to \widehat{T}^c$ is injective. As was proved in Lemma 5.1.1, $\theta_*\colon H_1(\pi;\widehat{T}^c) \to H_1(\widehat{T};\widehat{T}^c)$ is an isomorphism. Let λ_θ be the composite

$$\lambda_\theta\colon \mathbb{Q}\hat{\pi} \xrightarrow{\lambda} H_1(\pi;\mathbb{Q}\pi^c) \xrightarrow{H_*(\theta)} H_1(\pi;\widehat{T}) \xrightarrow{\theta_*} H_1(\widehat{T};\widehat{T}^c) = N(\widehat{T}_1) = \mathfrak{a}_g^-.$$

By what we showed above, the kernel of λ_θ is the subspace $\mathbb{Q}1$.

In order to describe the map λ_θ explicitly, we introduce some notation around the algebra $\widehat{T}$. Let

$$\hat{N}\colon \widehat{T} \longrightarrow \widehat{T}_1$$

be the map defined by

$$\hat{N}|_{H^{\otimes 0}} = 0$$

and

$$\hat{N}|_{H^{\otimes n}} = \frac{1}{n}N|_{H^{\otimes n}} = \sum_{i=0}^{n-1}\frac{1}{n}\nu^i, \quad n \geq 1.$$

Clearly we have $\hat{N}|_{N(\widehat{T}_1)} = 1_{N(\widehat{T}_1)}$. We denote by χ the composite

$$\chi\colon \widehat{T}^c \otimes_{\widehat{T}} \widehat{T}_1 = \widehat{T}^c \otimes H \hookrightarrow \widehat{T}_1.$$

We have

$$\chi(\mathrm{Ker}(\widehat{T}^c \otimes_{\widehat{T}} \widehat{T}_1 \xrightarrow{1\otimes\partial} \widehat{T}^c \otimes_{\widehat{T}} \widehat{T})) = N(\widehat{T}_1).$$

Let

$$\Phi\colon \widehat{T} \longrightarrow \widehat{\mathcal{L}}$$

be the map defined by

$$\Phi(X_1 \dots X_n) = [X_1, [\dots[X_{n-1}, X_n]\dots]] \quad X_i \in H, n \geq 1.$$

We have $\Phi(u) = nu$ and $[u, \Phi(v)] = \Phi(uv)$ for any $u \in \widehat{\mathcal{L}} \cap H^{\otimes n}$ and $v \in \widehat{T}_1$. See [26] Part I, Theorem 8.1, p. 28. $\frac{1}{n}\Phi|_{H^{\otimes n}}$ is exactly the Dynkin idempotent.

Lemma 5.3.1. *For any $u \in \widehat{T}$ and $v \in \widehat{T}_1$, we have*

$$\hat{N}\chi(u \otimes v) = \hat{N}(u\Phi(v)).$$

Proof. It suffices to prove the lemma for $v \in H^{\otimes q}$ by induction on $q \geq 1$. If $q = 1$, then $\hat{N}\chi(u \otimes v) = \hat{N}(uv) = \hat{N}(u\Phi(v))$. Suppose $q \geq 2$ and $v \in H^{\otimes q-1}$. For any $X \in H$ we have

$$\begin{aligned}
\hat{N}\chi(u \otimes Xv) &= \hat{N}\chi([u, X] \otimes v) \\
&= \hat{N}([u, X]\Phi(v)) \\
&= \hat{N}(u[X, \Phi(v)]) \\
&= \hat{N}(u\Phi(Xv)).
\end{aligned}$$

This completes the induction. □

All we need for the following lemma is the group-like condition (see Definition 2.4.1 (1)) on the expansion θ. In particular, we do not need the symplectic condition (see Definition 2.4.1 (2)) here.

Lemma 5.3.2. *For any $x \in \pi$, we have*

$$\lambda_\theta(x) = N\theta(x) = N(\theta(x) - 1) \in N(\widehat{T}_1) = \mathfrak{a}_g^-.$$

Proof. The homology class $\lambda(x) = x \otimes [x] \in H_1(\pi; \mathbb{Q}\pi^c)$ is represented by $x \otimes (x - 1) \in \mathbb{Q}\pi^c \otimes_{\mathbb{Q}\pi} I\pi = D_1(\mathbb{Q}\pi; \mathbb{Q}\pi^c)$. Hence

$$\begin{aligned}
\lambda_\theta(x) &= \chi\theta_* H_*(\theta)(x \otimes (x - 1)) \\
&= \chi\theta_*(\theta(x) \otimes (x - 1)) \\
&= \chi(\theta(x) \otimes \theta(x - 1)).
\end{aligned}$$

Since $[\ell^\theta(x), \theta(x)] = 0$, we have

$$\begin{aligned}
\theta(x) \otimes \theta(x - 1) &= \sum_{k=1}^{\infty} \frac{1}{k!}\theta(x) \otimes \ell^\theta(x)^k \\
&= \theta(x) \otimes \ell^\theta(x) \in \widehat{T}^c \otimes_{\widehat{T}} \widehat{T}_1.
\end{aligned}$$

Clearly we have $N\ell^\theta(x) = [x] = \hat{N}\Phi\ell^\theta(x)$. We denote by

$$\ell_p^\theta(x) \in \mathcal{L}_p = \widehat{\mathcal{L}} \cap H^{\otimes p}$$

the degree p-part of $\ell^\theta(x) \in \widehat{\mathcal{L}}$. For $n \geq 2$, we have

$$
\begin{aligned}
& n\hat{N}(\ell^\theta(x)^{n-1}\Phi\ell^\theta(x)) \\
&\quad = \sum_{i=1}^{n}\sum_{p_1,\dots,p_n} \hat{N}(\ell^\theta_{p_{i+1}}(x)\dots\ell^\theta_{p_n}(x)\ell^\theta_{p_1}(x)\dots\ell^\theta_{p_{i-1}}(x)\Phi\ell^\theta_{p_i}(x)) \\
&\quad = \sum_{i=1}^{n}\sum_{p_1,\dots,p_n} \frac{p_i}{p_1+\cdots+p_n} N(\ell^\theta_{p_{i+1}}(x)\dots\ell^\theta_{p_n}(x)\ell^\theta_{p_1}(x)\dots\ell^\theta_{p_{i-1}}(x)\ell^\theta_{p_i}(x)) \\
&\quad = \sum_{i=1}^{n}\sum_{p_1,\dots,p_n} \frac{p_i}{p_1+\cdots+p_n} N(\ell^\theta_{p_1}(x)\dots\ell^\theta_{p_n}(x)) \\
&\quad = \sum_{p_1,\dots,p_n} N(\ell^\theta_{p_1}(x)\dots\ell^\theta_{p_n}(x)) \\
&\quad = N(\ell^\theta(x)^n).
\end{aligned}
$$

Hence, by Lemma 5.3.1, we have

$$
\begin{aligned}
& \lambda_\theta(x) \\
&\quad = \chi(\theta(x)\otimes\ell^\theta(x)) \\
&\quad = \hat{N}\chi(\theta(x)\otimes\ell^\theta(x)) \\
&\quad = \hat{N}(\Phi\ell^\theta(x)) + \sum_{k=1}^{\infty}\frac{1}{(k+1)!}(k+1)\hat{N}(\ell^\theta(x)^k\Phi\ell^\theta(x)) \\
&\quad = N\ell^\theta(x) + \sum_{k=1}^{\infty}\frac{1}{(k+1)!}N(\ell^\theta(x)^{k+1}) \\
&\quad = N(\theta(x)-1). \qquad \square
\end{aligned}
$$

With respect to the $\widehat{T}_1$-adic topology, the image of the map $\theta\colon \mathbb{Q}\pi \to \widehat{T}$ is dense in the space $\widehat{T}$. Clearly the map $N : \widehat{T} \to N(\widehat{T}_1)$ is a continuous surjection. Hence the image of the map $\lambda_\theta\colon \mathbb{Q}\hat{\pi} \to N(\widehat{T}_1)$ is dense in $N(\widehat{T}_1)$. Summing up Propositions 3.4.3 (2), 4.3.2, 5.2.3 and Lemma 5.3.2, we have a commutative diagram

$$
\begin{array}{ccccccc}
(\mathbb{Q}\hat{\pi})^{\otimes 2} & \xrightarrow{\lambda^{\otimes 2}} & (H_1(\pi;\mathbb{Q}\pi^c))^{\otimes 2} & \xrightarrow{(\theta_*\circ H_*(\theta))^{\otimes 2}} & (H_1(\widehat{T};\widehat{T}^c))^{\otimes 2} & = & (\mathfrak{a}_g^-)^{\otimes 2} \\
{\scriptstyle [\ ,\ ]}\downarrow & & {\scriptstyle \mathcal{B}_*(\,\cdot\,)}\downarrow & & {\scriptstyle \mathcal{B}(\,\cdot\,)}\downarrow & & \downarrow{\scriptstyle -[\ ,\ ]} \\
\mathbb{Q}\hat{\pi} & = & \mathbb{Q}\hat{\pi} & \xrightarrow{\lambda_\theta} & N(\widehat{T}_1) & = & \mathfrak{a}_g^-.
\end{array}
$$

This means that the map $-\lambda_\theta\colon \mathbb{Q}\hat{\pi} \to N(\widehat{T}_1) = \mathfrak{a}_g^-$ is a Lie algebra homomorphism. This completes the proof of Theorem 1.2.1.

By Theorem 1.2.1, we may regard the formal symplectic geometry $\mathfrak{a}_g^-$ as a completion of the Goldman Lie algebra $\mathbb{Q}\hat{\pi}'$. In our paper [16] we use this idea to compute the center of the Goldman Lie algebra of an oriented surface of infinite genus.

5.4. Proof of Theorem 1.2.2.

We need some lemmas.

Lemma 5.4.1. *If $u \in \widehat{T}$ is group-like, namely, u satisfies $\Delta(u) = u\widehat{\otimes}u$, then we have*

$$\xi(u) = (1\widehat{\otimes}u) \otimes (u-1) \in (\widehat{T}^r\widehat{\otimes}\widehat{T}^l) \otimes_{\widehat{T}} \widehat{T}_1.$$

Proof. When u is given by $u = \sum_{k=0}^\infty u_k$, $u_k \in H^{\otimes k}$, we denote

$$u_{\le m} \overset{\text{def}}{=} \sum_{k=0}^m u_k \in T, \quad m \ge 1.$$

Then we have $\xi(u) \equiv \xi(u_{\le m}) \equiv 1 \otimes u_{\le m} \otimes (u_{\le m} - 1) \equiv (1\widehat{\otimes}u) \otimes (u-1)$ modulo the elements $\in (\widehat{T}^r\widehat{\otimes}\widehat{T}^l) \otimes_{\widehat{T}} \widehat{T}_1$ whose degree are greater than m. Since we can choose m arbitrarily, we obtain $\xi(u) = (1\widehat{\otimes}u) \otimes (u-1)$. This proves the lemma. □

Lemma 5.4.2. *We have a commutative diagram*

$$\begin{tikzcd}
\mathbb{Q}\pi \arrow[r, "\xi"] \arrow[d, "\theta"'] & H_1(\pi, \langle\zeta\rangle; \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l) \arrow[d, "\theta_*\circ H_*(\theta)"] \\
\widehat{T} \arrow[r, "\xi"'] & H_1(\widehat{T}, \mathbb{Q}[\![\omega]\!]; \widehat{T}^r\widehat{\otimes}\widehat{T}^l).
\end{tikzcd}$$

Here

$$H_*(\theta)\colon H_1(\pi, \langle\zeta\rangle; \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l) \longrightarrow H_1(\pi, \langle\zeta\rangle; \widehat{T}^r\widehat{\otimes}\widehat{T}^l)$$

is the map induced by

$$\theta\colon \mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l \to \widehat{T}^r\widehat{\otimes}\widehat{T}^l.$$

Proof. For any $x \in \pi$, $\theta(x)$ is group-like, so that, by Lemma 5.4.1,

$$\begin{aligned}
\xi\theta(x) &= (1\widehat{\otimes}\theta(x)) \otimes (\theta(x) - 1) \\
&= \theta_*((1\widehat{\otimes}\theta(x)) \otimes [x]) \\
&= \theta_* H_*(\theta)((1 \otimes x) \otimes [x]) \\
&= \theta_* H_*(\theta)\xi(x).
\end{aligned}$$

□

By the definition of the two $\mathcal{C}$'s we have a commutative diagram

$$\begin{tikzcd} \widehat{\mathbb{Q}\pi}^c \otimes_{\mathbb{Q}\pi} (\mathbb{Q}\pi^r \otimes \mathbb{Q}\pi^l) \arrow[r, "\mathcal{C}"] \arrow[d, "\theta\otimes\theta\otimes\theta"'] & \widehat{\mathbb{Q}\pi}^t \arrow[d, "\theta"] \\ \widehat{T}^c \otimes_{\widehat{T}} (\widehat{T}^r \widehat{\otimes} \widehat{T}^l) \arrow[r, "\mathcal{C}"'] & \widehat{T}^t. \end{tikzcd} \tag{5.2}$$

By Propositions 3.5.2, 5.2.3 and Lemma 5.4.2, we have

$$\begin{aligned} \theta(\sigma(u)v) &= \theta\mathcal{C}_*(\lambda(u)\cdot\xi(v)) \\ &= \mathcal{C}((\theta_* H_*(\theta)\lambda(u))\cdot(\theta_* H_*(\theta)\xi(v))) \\ &= \mathcal{C}(\lambda_\theta(u)\cdot\xi\theta(v)), \end{aligned}$$

which equals $-\lambda_\theta(u)\theta(v)$ by Proposition 4.4.3. Hence we obtain

$$\theta(\sigma(u)v) = -\lambda_\theta(u)\theta(v).$$

This completes the proof of Theorem 1.2.2.

5.5. Key formula. Recall that, by §3, the map $|\cdot|\colon \mathbb{Q}\pi \to \mathbb{Q}\hat{\pi}$ and define

$$\sigma\colon \mathbb{Q}\pi \times \mathbb{Q}\pi \longrightarrow \mathbb{Q}\pi$$

by

$$\sigma(u,v) = \sigma(|u|)v.$$

Lemma 5.5.1. *For any integers $p, q \geq 0$, we have*

$$\sigma((I\pi)^p \times (I\pi)^q) \subset (I\pi)^{p+q-2}.$$

Proof. Since $\theta^{-1}(\widehat{T}_p) = (I\pi)^p$, it suffices to show the following: if $u \in (I\pi)^p$ and $v \in (I\pi)^q$, then $\theta(\sigma(u,v)) \in \widehat{T}_{p+q-2}$. By Lemma 5.3.2 and Theorem 1.2.2, $\theta(\sigma(u,v)) = -\lambda_\theta(u)\theta(v) = (N\theta(u))\theta(v)$. On the other hand, we have $N\theta(u) \in \widehat{T}_p$ and $\theta(v) \in \widehat{T}_q$, by assumption. Hence $(N\theta(u))\theta(v) \in \widehat{T}_{p+q-2}$. □

By this lemma, we see that σ naturally extends to $\sigma\colon \widehat{\mathbb{Q}\pi} \times \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi}$ and the diagram

$$\begin{tikzcd} \widehat{\mathbb{Q}\pi} \times \widehat{\mathbb{Q}\pi} \arrow[r, "\sigma"] \arrow[d, "-\lambda_\theta\times\theta"'] & \widehat{\mathbb{Q}\pi} \arrow[d, "\theta"] \\ \mathfrak{a}_g^- \times \widehat{T} \arrow[r] & \widehat{T}, \end{tikzcd} \tag{5.3}$$

which is an extension of the diagram (1.3), commutes. Let $f(x)$ be a power series in $x-1$. Then for $x \in \pi$, $N(\theta(f(x)) \in \mathfrak{a}_g^- = N(\widehat{T}_1)$ is defined. For example, if $f(x) = \log x$, then $N(\theta(f(x))) = N(\ell^\theta(x)) = [x]$. Let $f(x) = (\log x)^2$. Then $N(\theta(f(x))) = N(\ell^\theta(x)\ell^\theta(x)) = 2L^\theta(x)$. Therefore, by (5.3) we obtain a key formula to prove Theorem 1.1.1.

Theorem 5.5.2. *For $x, y \in \pi$,*

$$\theta(\sigma((\log x)^2)y) = -2L^\theta(x)\theta(y).$$

As an immediate consequence, we have the following:

Corollary 5.5.3. *Let α be a free loop and β a based loop on Σ. Suppose $\alpha \cap \beta = \emptyset$. Then $L^\theta(\alpha)\theta(\beta) = L^\theta(\alpha)\ell^\theta(\beta) = 0$.*

Proof. By assumption, $\sigma(\alpha^n)\beta = 0$ for each $n \geq 0$, hence $\sigma((\log \alpha)^2)\beta = 0$. Using Theorem 5.5.2, we have $L^\theta(\alpha)\theta(\beta) = 0$. Since $\ell^\theta(\beta) = \log \theta(\beta)$ and $L^\theta(\alpha)$ is a derivation, we also have $L^\theta(\alpha)\ell^\theta(\beta) = 0$. □

6. The logarithms of Dehn twists

In this section we prove Theorems 1.1.1 and 1.1.2 in the Introduction, and derive some formulas of $\tau_k^\theta(t_C)$, which matches the computations by Morita.

Let us recall some notations. As in §5.3, for a group-like expansion θ and $x \in \pi$ we denote by

$$\ell_p^\theta(x) \in \mathcal{L}_p = \widehat{\mathcal{L}} \cap H^{\otimes p}$$

the degree p-part of $\ell^\theta(x) \in \widehat{\mathcal{L}}$. For example, we have $\ell_1^\theta(x) = [x]$. Further we denote

$$L^\theta(x) = \sum_{i=2}^{\infty} L_i^\theta(x), \quad L_i^\theta(x) \in H^{\otimes i}.$$

Then we have

$$L_i^\theta(x) = \frac{1}{2} N\left(\sum_{p=1}^{i-1} \ell_p^\theta(x)\ell_{i-p}^\theta(x)\right).$$

The tensors $L^\theta(x)$ and $L_i^\theta(x)$ are regarded as derivations of the algebra $\widehat{T}$, and are elements of $\mathfrak{l}_g$ (see §2.7).

A simple closed curve C on Σ is called non-separating (resp. separating) if $\Sigma \setminus C$ is connected (resp. not connected). The proof of Theorem 1.1.1 is divided into two cases according to whether C is separating or not. We take symplectic generators suitable to C and compute $L^\theta(C)\theta(x_i)$, hence $e^{-L^\theta(C)}\theta(x_i)$ by using Theorem 5.5.2 where x_i is one of the generators. Next we observe this value coincides with $T^\theta(t_C)\theta(x_i) = \theta(t_C(x_i))$. Together with the fact that $\{\theta(x_i)\}_i$ generates $\widehat{T}$ as a complete algebra, we will get the conclusion.

6.1. Proof of Theorem 1.1.1. Suppose C is non-separating. We take symplectic generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ such that $|\alpha_1|$ is homotopic to C as unoriented loops. Then the action of t_C on π is given by

$$\begin{cases} t_C(\alpha_i) = \alpha_i, & 1 \leq i \leq g, \\ t_C(\beta_1) = \beta_1\alpha_1, & \\ t_C(\beta_i) = \beta_i, & 2 \leq i \leq g. \end{cases} \tag{6.1}$$

Lemma 6.1.1. *Notations are as above. Then*

$$\begin{cases} L^\theta(C)\theta(\alpha_i) = 0, & 1 \leq i \leq g, \\ L^\theta(C)\theta(\beta_1) = -\theta(\beta_1)\ell^\theta(\alpha_1), & \\ L^\theta(C)\theta(\beta_i) = 0, & 2 \leq i \leq g. \end{cases}$$

Proof. Since $C \cap \alpha_i = \emptyset$ for $1 \leq i \leq g$ and $C \cap \beta_i = \emptyset$ for $2 \leq i \leq g$, we have $L^\theta(C)\theta(\alpha_i) = 0$ for $1 \leq i \leq g$ and $L^\theta(C)\theta(\beta_i) = 0$ for $2 \leq i \leq g$ by Corollary 5.5.3.

It remains to prove $L^\theta(C)\theta(\beta_1) = -\theta(\beta_1)\ell^\theta(\alpha_1)$. We have

$$\sigma(\alpha_1^n)\beta_1 = n\beta_1\alpha_1^n, \quad n \geq 0.$$

Thus for $m \geq 0$, we compute

$$\begin{aligned} \sigma((\alpha_1 - 1)^m)\beta_1 &= \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \sigma(\alpha_1^n)\beta_1 \\ &= \sum_{n=1}^{m} (-1)^{m-n} n \binom{m}{n} \beta_1\alpha_1^n \\ &= m\beta_1\alpha_1(\alpha_1 - 1)^{m-1}. \end{aligned} \tag{6.2}$$

Here we use $n\binom{m}{n} = m\binom{m-1}{n-1}$. This implies if $f(\alpha_1)$ is a power series in $\alpha_1 - 1$, then $\sigma(f(\alpha_1))\beta_1 = \beta_1\alpha_1 f'(\alpha_1)$, where $f'(\alpha_1)$ is the derivative of $f(\alpha_1)$. If $f(\alpha_1) = (\log\alpha_1)^2$, then $\alpha_1 f'(\alpha_1) = 2\log\alpha_1$. Therefore, $\sigma((\log\alpha_1)^2)\beta_1 = 2\beta_1\log\alpha_1$. Substituting this into Theorem 5.5.2, we get

$$L^\theta(C)\theta(\beta_1) = -\frac{1}{2}\theta(\sigma((\log\alpha_1)^2)\beta_1) = -\theta(\beta_1\log\alpha_1) = -\theta(\beta_1)\ell^\theta(\alpha_1). \qquad \square$$

By Lemma 6.1.1, we have

$$e^{-L^\theta(C)}\theta(\alpha_i) = \theta(\alpha_i), \quad 1 \le i \le g,$$

and

$$e^{-L^\theta(C)}\theta(\beta_i) = \theta(\beta_i), \quad 2 \le i \le g.$$

Also Lemma 6.1.1 implies

$$L^\theta(C)^i\theta(\beta_1) = (-1)^i\theta(\beta_1)\ell^\theta(\alpha_1)^i, \quad i \ge 0.$$

Hence

$$\begin{aligned} e^{-L^\theta(C)}\theta(\beta_1) &= \sum_{i=0}^\infty (-1)^i \frac{1}{i!} L^\theta(C)^i \theta(\beta_1) \\ &= \theta(\beta_1)\sum_{i=0}^\infty \frac{1}{i!}\ell^\theta(\alpha_1)^i \\ &= \theta(\beta_1)\theta(\alpha_1) \\ &= \theta(\beta_1\alpha_1). \end{aligned}$$

On the other hand, (6.1) implies that the total Johnson map $T^\theta(t_C)$ satisfies

$$T^\theta(t_C)(\theta(\alpha_i)) = \theta(\alpha_i), \quad 1 \le i \le g,$$

$$T^\theta(t_C)(\theta(\beta_1)) = \theta(\beta_1\alpha_1),$$

and

$$T^\theta(t_C)(\theta(\beta_i)) = \theta(\beta_i), \quad 2 \le i \le g.$$

We have shown that the values of $e^{-L^\theta(C)}$ and $T^\theta(t_C)$ coincide on the set $\{\theta(\alpha_i), \theta(\beta_i)\}_i$, and this set generates $\widehat{T}$ as a complete algebra. Therefore we obtain the equality $e^{-L^\theta(C)} = T^\theta(t_C) \in \mathrm{Aut}(\widehat{T})$. This completes the proof of Theorem 1.1.1 for the case C is non-separating.

Next, suppose C is separating. Take symplectic generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ such that C is homotopic to $|\gamma_h|$ as unoriented loops, where $\gamma_h = \prod_{i=1}^h [\alpha_i, \beta_i]$ for some h. Then the action of t_C on π is given by

$$\begin{cases} t_C(\alpha_i) = \gamma_h^{-1}\alpha_i\gamma_h, & 1 \le i \le h, \\ t_C(\alpha_i) = \alpha_i, & h+1 \le i \le g, \\ t_C(\beta_i) = \gamma_h^{-1}\beta_i\gamma_h, & 1 \le i \le h, \\ t_C(\beta_i) = \beta_i, & h+1 \le i \le g. \end{cases} \tag{6.3}$$

Lemma 6.1.2. *With the above notations,*

$$L^\theta(C)\theta(\alpha_i) = \begin{cases} [\ell^\theta(\gamma_h), \theta(\alpha_i)], & \text{if } 1 \le i \le h, \\ 0, & \text{if } h+1 \le i \le g, \end{cases}$$

and

$$L^\theta(C)\theta(\beta_i) = \begin{cases} [\ell^\theta(\gamma_h), \theta(\beta_i)], & \text{if } 1 \le i \le h, \\ 0, & \text{if } h+1 \le i \le g. \end{cases}$$

Proof. Suppose $i \ge h+1$. Since $C \cap \alpha_i = C \cap \beta_i = \emptyset$, by Corollary 5.5.3, $L^\theta(C)\theta(\alpha_i) = L^\theta(C)\theta(\beta_i) = 0$.

If $i \le h$,

$$\sigma(\gamma_h^n)\alpha_i = -n\gamma_h^n\alpha_i + n\alpha_i\gamma_h^n, \quad n \ge 0.$$

By a computation similar to (6.2),

$$\sigma((\gamma_h - 1)^m)\alpha_i = m\alpha_i\gamma_h(\gamma_h-1)^{m-1} - m\gamma_h(\gamma_h-1)^{m-1}\alpha_i, \quad m \ge 0.$$

This implies if $f(\gamma_h)$ is a power series in $\gamma_h - 1$, then

$$\sigma(f(\gamma_h))\alpha_i = \alpha_i\gamma_h f'(\gamma_h) - \gamma_h f'(\gamma_h)\alpha_i.$$

Therefore, $\sigma((\log\gamma_h)^2)\alpha_i = 2(\alpha_i \log\gamma_h - (\log\gamma_h)\alpha_i)$. Substituting this into Theorem 5.5.2,

$$L^\theta(C)\theta(\alpha_i) = -\theta(\alpha_i\log\gamma_h - (\log\gamma_h)\alpha_i) = [\ell^\theta(\gamma_h), \theta(\alpha_i)].$$

The proof of $L^\theta(C)\theta(\beta_i) = [\ell^\theta(\gamma_h), \theta(\beta_i)]$ is similar. $\square$

By Lemma 6.1.2, if $i \geq h+1$, then

$$e^{-L^\theta(C)}\theta(\alpha_i) = \theta(\alpha_i) \quad \text{and} \quad e^{-L^\theta(C)}\theta(\beta_i) = \theta(\beta_i).$$

Suppose $i \leq h$. By Corollary 5.5.3, $L^\theta(C)\ell^\theta(\gamma_h) = 0$. Combining this with Lemma 6.1.2, $L^\theta(C)^m\theta(\alpha_i) = \mathrm{ad}(\ell^\theta(\gamma_h))^m\theta(\alpha_i)$ for $m \geq 0$. Hence,

$$\begin{aligned} e^{-L^\theta(C)}\theta(\alpha_i) &= \sum_{m=0}^{\infty} \frac{1}{m!}\mathrm{ad}(-\ell^\theta(\gamma_h))^m\theta(\alpha_i) \\ &= e^{-\ell^\theta(\gamma_h)}\theta(\alpha_i)e^{\ell^\theta(\gamma_h)} \\ &= \theta(\gamma_h^{-1}\alpha_i\gamma_h). \end{aligned}$$

Similarly, $e^{-L^\theta(C)}\theta(\beta_i) = \theta(\gamma_h^{-1}\beta_i\gamma_h)$ for $i \leq h$. On the other hand, by (6.3)

$$T^\theta(t_C)\theta(\alpha_i) = \theta(\gamma_h^{-1}\alpha_i\gamma_h), \quad T^\theta(t_C)\theta(\beta_i) = \theta(\gamma_h^{-1}\beta_i\gamma_h), \quad 1 \leq i \leq h,$$

and

$$T^\theta(t_C)\theta(\alpha_i) = \theta(\alpha_i), \qquad T^\theta(t_C)\theta(\beta_i) = \theta(\beta_i), \qquad h+1 \leq i \leq g.$$

Now the values of $e^{-L^\theta(C)}$ and $T^\theta(t_C)$ coincide on $\{\theta(\alpha_i), \theta(\beta_i)\}_i$, and as the proof for non-separating C, this leads to the equality $e^{-L^\theta(C)} = T^\theta(t_C)$. This completes the proof of Theorem 1.1.1 for the case C is separating.

We have established Theorem 1.1.1. Let us give an orientation on C and denote by $[C] \in H$ its homology class. Then the square of $L_2^\theta(C) = [C][C]$ acts on H trivially. See Lemma 6.4.1 and Proposition 6.5.1. Recall that, by §2.5,

$$T^\theta(t_C) = \tau^\theta(t_C) \circ |t_C|.$$

Let $X \in H$. Modulo $\widehat{T}_2$,

$$\begin{aligned} |t_C|X &\equiv \tau^\theta(t_C) \circ |t_C|X \\ &= T^\theta(t_C)X \\ &= e^{-L^\theta(C)}X \\ &\equiv X - L_2^\theta(C)X \\ &= X - (X \cdot [C])[C]. \end{aligned}$$

Namely,

$$|t_C|X = X - (X \cdot [C])[C], \ X \in H.$$

This is nothing but the classical formula as stated in the Introduction (1.2).

6.2. Action on the nilpotent quotients. We prove Theorem 1.1.2. Let $\Gamma_k = \Gamma_k(\pi)$, $k \geq 1$ be the lower central series of π. Namely $\Gamma_1 = \pi$, and define Γ_k successively by $\Gamma_k = [\Gamma_{k-1}, \pi]$ for $k \geq 2$. For $k \geq 0$, the k-th nilpotent quotient of π is defined as the quotient group $N_k = N_k(\pi) = \pi/\Gamma_{k+1}$. Note that $N_1 = \pi/[\pi, \pi]$ is nothing but the abelianization of π. Since any automorphism of π preserves Γ_k, the mapping class group $\mathcal{M}_{g,1}$ naturally acts on N_k for each k.

Let θ be a (not necessarily symplectic) Magnus expansion of π. For each $k \geq 1$ we have

$$\theta^{-1}(1 + \widehat{T}_k) = \Gamma_k. \tag{6.4}$$

See Bourbaki [2] ch.2, §5, no.4, Theorem 2. Therefore, θ induces an injective homomorphism $\theta\colon N_k \to (1 + \widehat{T}_1)/(1 + \widehat{T}_{k+1})$. Note that $1 + \widehat{T}_{k+1}$ is a normal subgroup of $1 + \widehat{T}_1$. By post-composing the natural injection

$$(1 + \widehat{T}_1)/(1 + \widehat{T}_{k+1}) \hookrightarrow \widehat{T}/\widehat{T}_{k+1},$$

we get an injection

$$\theta\colon N_k \hookrightarrow \widehat{T}/\widehat{T}_{k+1}. \tag{6.5}$$

Since the total Johnson map $T^\theta(\varphi)$ of $\varphi \in \mathcal{M}_{g,1}$ is filter-preserving, it naturally induces a filter-preserving automorphism of the quotient algebra $\widehat{T}/\widehat{T}_{k+1}$. Using the same letter we denote it by $T^\theta(\varphi)$. By construction the injection (6.5) is compatible with the action of $\mathcal{M}_{g,1}$: we have $T^\theta(\varphi) \circ \theta(x) = \theta \circ \varphi(x)$ for any $x \in N_k$.

For a group G, let $\overline{G}$ be the quotient set of G by conjugation and the relation $g \sim g^{-1}$, $g \in G$. Let C be a simple closed curve on Σ. Choose any $x \in \pi$ such that x is freely homotopic to C as unoriented loops. Then the element of $\bar{\pi}$ represented by x is independent of the choice of x. For each $k \geq 0$, let $\overline{C}_k \in \overline{N}_k$ be the image of this element under the natural surjection $\bar{\pi} \to \overline{N}_k$.

Proof of Theorem 1.1.2. Fix a symplectic expansion θ. By Theorem 1.1.1, we have $T^\theta(t_C) = e^{-L^\theta(C)} \in \mathrm{Aut}(\widehat{T})$. Remark that the action of $e^{-L^\theta(C)}$ on $\widehat{T}/\widehat{T}_{k+1}$ depends only on $L_i^\theta(C)$, $2 \leq i \leq k+1$.

Pick $x \in \pi$ such that x is freely homotopic to C as unoriented loops. Let $x' \in \pi$ such that $x^{-1}x' \in \Gamma_{k+1}$. By (6.4), it follows that $\ell_i^\theta(x) = \ell_i^\theta(x')$ for $1 \leq i \leq k$. Since $L^\theta(C) = L^\theta(x) = \frac{1}{2}N(\ell^\theta(x)\ell^\theta(x))$, this observation together with Lemma 2.6.4 shows that $L_i^\theta(C)$, $2 \leq i \leq k+1$, depend only on the class $\overline{C}_k \in \overline{N}_k$. This proves the first part.

If C is separating, $x \in \Gamma_2$ hence $\ell_1^\theta(x) = 0$. Thus if $x' \in \Gamma_2$ is a representative of another separating simple closed curve C', satisfying $x^{-1}x' \in \Gamma_k$, then $L_i^\theta(x) = L_i^\theta(x')$ for $2 \leq i \leq k+1$. Therefore, $L_i^\theta(C)$, $2 \leq i \leq k+1$ depend only on the class $\overline{C}_{k-1} \in \overline{N}_{k-1}$. This completes the proof. $\square$

Theorem 1.1.2 is a generalization of the following well-known facts: 1) the action of t_C on $N_1 = H_1(\Sigma; \mathbb{Z})$ depends only on the class $\pm[C]$; 2) if C is separating, then t_C belongs to the Johnson kernel $\mathcal{K}_{g,1} = \mathcal{M}_{g,1}[2]$, the subgroup of the mapping classes acting on N_2 as the identity.

6.3. The formula for $\boldsymbol{\tau_k^\theta(t_C)}$ for separating $\boldsymbol{C}$. In the rest of this section we derive formulas for the k-th Johnson map (see Definition 2.5.1) of t_C associated to a symplectic expansion. We often write simply $L^\theta(C) = L$, $L_k^\theta(C) = L_k$, etc.

In this subsection we treat the case of separating curves.

Theorem 6.3.1. *Let θ be a symplectic expansion and C a separating simple closed curve on Σ. Then for $k \geq 1$, the k-th Johnson map $\tau_k^\theta(t_C)$ is given by*

$$\tau_k^\theta(t_C) = \sum_{1 \leq n \leq [k/2]} \frac{(-1)^n}{n!} \sum_{\substack{(m_1,\dots,m_n), m_i \geq 4,\\ m_1+\cdots+m_n = 2n+k}} L_{m_1} \dots L_{m_n}.$$

For example, we have $\tau_1^\theta(t_C) = 0$, and

$$\tau_2^\theta(t_C) = -L_4;$$

$$\tau_3^\theta(t_C) = -L_5;$$

$$\tau_4^\theta(t_C) = -L_6 + \frac{1}{2} L_4^2;$$

$$\tau_5^\theta(t_C) = -L_7 + \frac{1}{2}(L_4 L_5 + L_5 L_4);$$

$$\tau_6^\theta(t_C) = -L_8 + \frac{1}{2}(L_4 L_6 + L_5^2 + L_6 L_4) - \frac{1}{6} L_4^3.$$

Here, L_4^2 is the composition $L_4 \circ L_4 \colon H \to H^{\otimes 3} \to H^{\otimes 5}$, etc.

Proof. Since C is separating, $|t_C| = \mathrm{id}$ hence $\tau^\theta(t_C) = T^\theta(t_C)$, and $L_2^\theta(C) = L_3^\theta(C) = 0$. Thus, $L^\theta(C) = L_4 + L_5 + \dots$. For $X \in H$, the degree $k+1$ part of $L(C)^n X$ is equal to

$$\sum_{\substack{(m_1,\dots,m_n), m_i \geq 4,\\ m_1+\cdots+m_n = 2n+k}} L_{m_1} \dots L_{m_n} X.$$

In particular if $n > [k/2]$, the degree $k+1$ part of $L(C)^n X$ is zero. The conclusion follows by Theorem 1.1.1 $\square$

Remark 6.3.2. In [21] Proposition 1.1, Morita computed $\tau_2(t_C)$ for separating C, and our formula $\tau_2^\theta(t_C) = -L_4^\theta$ coincides with his formula. In fact, we have $t_C \in \mathcal{K}_{g,1}$ as we remarked at the end of §6.4, and $\tau_2^\theta(t_C)$ does not depend on the choice of θ.

6.4. Computations of $L_k^\theta(x)$ for small k. Compared with the separating case, the non-separating case is more complicated because, $L_2^\theta(C) \neq 0$ for non-separating C. So, for non-separating C, we do not have a complete formula for $\tau_k^\theta(t_C)$, $k \geq 1$, and in this paper we only give formulas for $\tau_1^\theta(t_C)$ and $\tau_2^\theta(t_C)$. Even in these cases, we need considerable computations. This subsection is a preparation for the computations.

Let $\Lambda^k H$ be the k-th exterior product of H. We can realize $\Lambda^k H$ as a subspace of $H^{\otimes k}$ by the embedding

$$\Lambda^k H \longrightarrow H^{\otimes k}, \quad X_1 \wedge \cdots \wedge X_k \longmapsto \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma) X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}.$$

Note that $\Lambda^2 H = \mathcal{L}_2$ and $X \wedge Y = [X, Y]$. By a straightforward computation, we have

Lemma 6.4.1. *Let θ be a group-like expansion. Then for each $x \in \pi$,*

(1) $L_2^\theta(x) = [x][x]$*;*

(2) $L_3^\theta(x) = [x] \wedge \ell_2^\theta(x) \in \Lambda^3 H$.

Let θ and θ' be symplectic expansions. As we saw in §2.8, there uniquely exists $U = U(\theta, \theta') \in \mathrm{IA}(\widehat{T})$ such that $\theta' = U \circ \theta$, $U(H) \subset \widehat{\mathcal{L}}$, and $U(\omega) = \omega$. The restriction of U to H is uniquely written as

$$U|_H = 1_H + \sum_{k=1}^{\infty} u_k, \; u_k \in \mathrm{Hom}(H, \mathcal{L}_{k+1}).$$

By (2.5) we regard $u_k \in H \otimes \mathcal{L}_{k+1}$.

Lemma 6.4.2. *With the above notations,*

(1) $u_1 \in \Lambda^3 H \subset H \otimes \mathcal{L}_2$*;*

(2) *for* $x \in \pi$*,*

$$\ell_2^{\theta'}(x) = \ell_2^\theta(x) + u_1([x]);$$
$$\ell_3^{\theta'}(x) = \ell_3^\theta(x) + u_1(\ell_2^\theta(x)) + u_2([x]).$$

Here, $u_1(\ell_2^\theta(x))$ *means* $(1_H \otimes u_1 + u_1 \otimes 1_H)\ell_2^\theta(x)$.

Proof. Modulo $\widehat{T}_4$, we compute

$$\begin{aligned}\omega = U(\omega) &= \sum_{i=1}^{g} U(A_i)U(B_i) - U(B_i)U(A_i)\\ &\equiv \sum_{i=1}^{g}(A_i + u_1(A_i))(B_i + u_1(B_i)) - (B_i + u_1(B_i))(A_i + u_1(A_i))\\ &\equiv \omega + \sum_{i=1}^{g}(A_i u_1(B_i) + u_1(A_i)B_i - B_i u_1(A_i) - u_1(B_i)A_i)\end{aligned}$$

By the same reason as the discussion in §2.8, this implies

$$u_1 \in \mathrm{Ker}([\ ,\ ]\colon H \otimes \mathcal{L}_2 \longrightarrow \mathcal{L}_3).$$

Also, we have

$$\mathrm{Ker}([\ ,\ ]\colon H \otimes \mathcal{L}_2 \longrightarrow \mathcal{L}_3) = \Lambda^3 H.$$

In fact, if $u \in \mathrm{Ker}([\ ,\ ]\colon H \otimes \mathcal{L}_2 \to \mathcal{L}_3)$ then $\nu(u) = u$ by Lemma 2.6.2, thus $u = \frac{1}{3}(u + \nu(u) + \nu^2(u))$. This shows $u \in \Lambda^3 H$. The other inclusion follows by the Jacobi identity. This proves the first part.

Again modulo $\widehat{T}_4$, we compute

$$\begin{aligned}\ell^{\theta'}(x) &\equiv U([x] + \ell_2^\theta(x) + \ell_3^\theta(x))\\ &\equiv [x] + u_1([x]) + u_2([x]) + \ell_2^\theta(x) + u_1(\ell_2^\theta(x)) + \ell_3^\theta(x).\end{aligned}$$

This proves (2). □

Corollary 6.4.3. *Notations are the same as Lemma 6.4.2. For $x \in \pi$,*

(1) $L_3^{\theta'}(x) - L_3^\theta(x) = [x] \wedge u_1([x])$, *and*

(2) $L_4^{\theta'}(x) - L_4^\theta(x) = N([x]u_1(\ell_2^\theta(x)) + N([x]u_2([x])) + N(\ell_2^\theta(x)u_1([x])) + \dfrac{1}{2}N(u_1([x])u_1([x]))$.

Proof. The first part is clear by Lemmas 6.4.1 and 6.4.2. The second part follows by

$$L_4^\theta(x) = N([x]\ell_3^\theta(x)) + \frac{1}{2}N(\ell_2^\theta(x)\ell_2^\theta(x))$$

and Lemma 6.4.2. □

6.5. The formulas for $\tau_1^\theta(t_C)$ and $\tau_2^\theta(t_C)$ for non-separating C. Let C be a non-separating simple closed curve on Σ. As we did in §6.2, we take symplectic generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ such that $|\alpha_1|$ is freely homotopic to C as unoriented loops. In this situation, Massuyeau [20], Example 2.19 gave a partial example of a symplectic expansion θ^0 whose values of $\ell^{\theta^0}(\alpha_1)$ and $\ell^{\theta^0}(\beta_1)$ modulo $\widehat{T}_5$ are

$$\ell^{\theta^0}(\alpha_1) \equiv A_1 + \frac{1}{2}[A_1, B_1] + \frac{-1}{12}[B_1, [A_1, B_1]] + \frac{1}{24}[A_1, [A_1, [A_1, B_1]]];$$

and

$$\begin{aligned} \ell^{\theta^0}(\beta_1) \equiv B_1 &- \frac{1}{2}[A_1, B_1] + \frac{1}{12}[A_1, [A_1, B_1]] + \frac{1}{4}[B_1, [A_1, B_1]] \\ &- \frac{1}{24}[B_1, [B_1, [B_1, A_1]]]. \end{aligned} \tag{6.6}$$

Here, $A_1 = [\alpha_1]$ and $B_1 = [\beta_1]$. Note that our conventions about symplectic generators and the boundary loop ζ are different from Massuyeau [20].

Proposition 6.5.1. *Let θ be a symplectic expansion and C a non-separating simple closed curve on Σ. Let $L_k = L_k^\theta(C)$. We regard them as derivations of $\widehat{T}$. Then $L_2^2 = L_2L_3 = L_3L_2 = 0$ on H. In particular, as linear endomorphisms of $\widehat{T}$, $L_2^{n+1}|_{H^{\otimes n}} = 0$ and $L_2L_3 = L_3L_2$.*

Proof. We take symplectic generators as above. Since $L_2 = A_1^2$,

$$L_2^2(X) = (X \cdot A_1)L_2A_1 = 0, \quad X \in H.$$

Therefore, $L_2^2 = 0$ on H.

Let θ^0 be the symplectic expansion (6.6). By Lemma 6.4.1 (2),

$$L_3^{\theta^0}(C) = \frac{1}{2}A_1 \wedge A_1 \wedge B_1 = 0.$$

Thus $L_2L_3 = L_3L_2 = 0$ for θ^0.

Let θ' be another symplectic expansion and let $U = U(\theta^0, \theta')$. We need to show $L_2^{\theta'}L_3^{\theta'} = L_3^{\theta'}L_2^{\theta'} = 0$ on H. If $U = \mathrm{id}$, this is true by what we have shown. Therefore, the proposition follows by Corollary 6.4.3 (1) and the following lemma. □

Lemma 6.5.2. *Let $L_2 = A_1^2$ and let $L_3'' = A_1 \wedge u_1(A_1)$, where $u_1 \in \Lambda^3 H$. We regard them as derivations of $\widehat{T}$. Then $L_2L_3'' = L_3''L_2 = 0$ on H.*

Proof. The proof is straightforward, so we omit the details. We just remark that by linearity, it suffices to prove the lemma when u_1 is of the form $u_1 = X \wedge Y \wedge Z$, where $X, Y, Z \in \{A_i, B_i\}_i$ are distinct. □

Theorem 6.5.3. *Let θ be a symplectic expansion and C a non-separating simple closed curve on Σ. Then we have*

$$\tau_1^\theta(t_C) = -L_3^\theta(C). \tag{6.7}$$

Proof. For $X \in H$, we have

$$\tau^\theta(t_C)X = T^\theta(t_C)(|t_C|^{-1}X) = e^{-L}(X + L_2X).$$

Thus $\tau_1^\theta(t_C)X$ is equal to the degree two-part of $e^{-L}(X + L_2X)$. Modulo $\widehat{T}_3$, we compute

$$e^{-L}(X + L_2X) \equiv X + L_2X - L_2(X + L_2X) - L_3(X + L_2X) = X - L_3X,$$

using Proposition 6.5.1. This completes the proof. □

This theorem is compatible with the computation by Morita [22], Proposition 4.2. One reason for the choice of our convention about the Poincaré duality (2.3) is to make our formula compatible with his computation.

We next compute $\tau_2^\theta(t_C)$ for non-separating C.

Proposition 6.5.4. *Let θ be a symplectic expansion and C a non-separating curve on Σ. Let $L_k = L_k^\theta(C)$. We regard them as derivations of $\widehat{T}$. Then $L_2L_2L_2L_4 = L_2L_2L_4L_2 = 0$, and $2L_2L_4L_2 = L_2L_2L_4$ on H.*

Proof. Let θ^0 be a symplectic expansion of (6.6). We first prove the proposition for $\theta = \theta^0$. We have $L_2 = A_1^2$. For simplicity we write $L_4^{\theta^0}(C) = L_4^0$. By a direct computation using Lemma 2.6.2, we have

$$L_4^0 = \frac{1}{24}N([A_1, B_1][A_1, B_1]]).$$

By this we can show that $L_2L_2L_4^0 = L_2L_4^0L_2 = 0$ on H, hence we obtain $L_2L_2L_2L_4^0 = L_2L_2L_4^0L_2 = 0$ and $2L_2L_4^0L_2 = L_2L_2L_4^0(= 0)$ on H.

We next consider the general case. Let θ' be another symplectic expansion and $U = U(\theta^0, \theta')$. Let $L_2 = A_1^2$ and $L_4'' = L_4^{\theta'}(C) - L_4^{\theta^0}(C)$. It suffices to prove $L_2^3L_4'' = L_2^2L_4''L_2 = 0$, and $2L_2L_4''L_2 = L_2^2L_4''$ on H.

By a direct computation using Corollary 6.4.3 (2) and Lemma 2.6.2, we obtain

$$L_4'' = N(A_1u_2(A_1)) + \frac{1}{2}N(u_1(A_1)u_1(A_1)).$$

Since $u_2(A_1) \in \mathcal{L}_3$ and $u_1 \in \Lambda^3H$, it follows that $L_4'' \in H^{\otimes 4}$ is a linear combination of monomials in $\{A_i, B_i\}_i$ with the number of the occurrences of B_1 at most two. By this observation we have $L_2^3L_4'' = L_2^2L_4''L_2 = 0$ on H.

It remains to prove the assertion $2L_2L_4''L_2 = L_2^2L_4''$ on H. Let

$$L_4''' = \frac{1}{2}N(u_1(A_1)u_1(A_1)).$$

Since $u_1 \in \Lambda^3 H$, L_4''' is a linear combination of monomials in $\{A_i, B_i\}_i$ with no occurrence of B_1. It follows that $2L_2L_4'''L_2 = L_2^2L_4''' = 0$ on H. Now the assertion follows by the following lemma. □

Lemma 6.5.5. *Let $u \in \mathcal{L}_3$, $X \in H$ and set $L_X = X^2$, $L_4 = N(Xu)$. We regard L_X and L_4 as a derivation of $\widehat{T}$. Then $2L_XL_4L_X = L_X^2L_4$ on H.*

Proof. The proof is straightforward, so we omit the details. We just remark that by linearity, we may assume that $u = [Y_1, [Y_2, Y_3]]$, $Y_i \in H$. □

Theorem 6.5.6. *Let θ be a symplectic expansion and C a non-separating simple closed curve on Σ. Then we have*

$$\tau_2^\theta(t_C) = -L_4 + \frac{1}{2}[L_2, L_4] + \frac{1}{2}L_3^2. \tag{6.8}$$

Proof. Let $X \in H$. Modulo $\widehat{T}_4$, we compute

$$\begin{aligned}
LX &\equiv L_2X + L_3X + L_4X,\\
LLX &\equiv L_2(L_2X + L_3X + L_4X) + L_3(L_2X + L_3X) + L_4L_2X\\
&= L_2L_4X + L_3L_3X + L_4L_2X,\\
LLLX &\equiv L_2L_2L_4X + L_2L_3L_3X + L_2L_4L_2X\\
&= L_2L_2L_4X + L_2L_4L_2X,\\
LLLLX &\equiv L_2L_2L_2L_4X + L_2L_2L_4L_2X = 0,
\end{aligned}$$

$L(L_2X) \equiv L_4L_2X$, $LL(L_2X) \equiv L_2L_4L_2X$, and $LLL(L_2X) \equiv 0$. Here we use Proposition 6.5.1 and the first part of Proposition 6.5.4. Note that $L_2L_3L_3X = L_3L_2L_3X = 0$. Therefore, the degree 4-part of $\tau^\theta(t_C)X = e^{-L}(X + L_2X)$ is

$$\begin{aligned}
&-L_4X - L_4L_2X + \frac{1}{2}(L_2L_4X + L_3L_3X + L_4L_2X)\\
&\quad + \frac{1}{2}L_2L_4L_2X - \frac{1}{6}(L_2L_2L_4X + L_2L_4L_2X).
\end{aligned}$$

Using the second part of Proposition 6.5.4, we obtain the formula. □

7. The case of $\mathcal{M}_{g,*}$

We close this paper by deriving similar results for the mapping class group of a once punctured surface. Let Σ_g be a closed oriented C^∞-surface of genus g. Choose a basepoint $*' \in \Sigma_g$ and let $\pi_1(\Sigma_g) = \pi_1(\Sigma_g, *')$.

7.1. The mapping class group $\mathcal{M}_{g,*}$. Let $\mathcal{M}_{g,*}$ be the mapping class group of Σ_g relative to $*'$, namely the group of orientation-preserving diffeomorphisms of Σ_g fixing $*'$, modulo isotopies fixing $*'$. By the theorem of Dehn–Nielsen, we have a natural identification

$$\mathcal{M}_{g,*} = \mathrm{Aut}^+(\pi_1(\Sigma_g)), \tag{7.1}$$

where $+$ means acting on $H_2(\pi_1(\Sigma_g); \mathbb{Z}) \cong \mathbb{Z}$ as the identity.

We take a small disk D around $*'$ and fix an identification

$$\Sigma_g \setminus \mathrm{Int}(D) \cong \Sigma.$$

We can extend any diffeomorphism of Σ to a diffeomorphism of Σ_g by defining the extension as the identity on D. In this way we have a natural surjective homomorphism

$$\mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,*}. \tag{7.2}$$

For simplicity let us write $\mathrm{Aut}_\zeta(\pi) = \{\varphi \in \mathrm{Aut}(\pi); \varphi(\zeta) = \zeta\}$ (see (2.1)). We have a natural surjection from $\pi = \pi_1(\Sigma, *)$ to $\pi_1(\Sigma_g) = \pi_1(\Sigma_g, *')$. This naturally induces a homomorphism $\mathrm{Aut}_\zeta(\pi) \to \mathrm{Aut}^+(\pi_1(\Sigma_g))$. This map is compatible with (2.1) and (7.1).

7.2. Action on the completed group ring of $\pi_1(\Sigma)$. Let $\mathcal{N}$ be the two-sided ideal of $\widehat{T}$ generated by ω, and $\widehat{T}/\mathcal{N}$ the quotient algebra. It naturally inherits a decreasing filtration $(\widehat{T}/\mathcal{N})_p$, $p \geq 1$ and a structure of complete Hopf algebra from $\widehat{T}$. We denote by ϖ the projection $\widehat{T} \to \widehat{T}/\mathcal{N}$.

If θ is a symplectic expansion of π, $\theta(\zeta) = \exp(\omega) \in 1 + \mathcal{N}$. Thus it induces a group homomorphism $\bar{\theta}\colon \pi_1(\Sigma_g) \to 1 + (\widehat{T}/\mathcal{N})_1$. The following lemma is essentially the same as [20] Proposition 2.18, so we omit the proof.

Lemma 7.2.1. *Let θ be a symplectic expansion of π. Then the induced map*

$$\bar{\theta}\colon \widehat{\mathbb{Q}\pi_1(\Sigma_g)} \longrightarrow \widehat{T}/\mathcal{N} \tag{7.3}$$

is an isomorphism of complete Hopf algebras. Here $\widehat{\mathbb{Q}\pi_1(\Sigma_g)}$ is the completed group ring of $\pi_1(\Sigma_g)$, namely the completion of $\mathbb{Q}\pi_1(\Sigma_g)$ by the augmentation ideal.

The isomorphism (7.3) leads to the definition of a counterpart of the total Johnson map $T^\theta\colon \mathcal{M}_{g,1} \to \mathrm{Aut}(\widehat{T})$. Let $\mathrm{Aut}(\widehat{T}/\mathcal{N})$ be the group of the filter-preserving algebra automorphisms of $\widehat{T}/\mathcal{N}$. Let $\bar{\varphi} \in \mathcal{M}_{g,*}$. As a consequence of (7.3) there uniquely exists $\bar{T}^\theta(\bar{\varphi}) \in \mathrm{Aut}(\widehat{T}/\mathcal{N})$ such that $\bar{T}^\theta(\bar{\varphi}) \circ \bar{\theta} = \bar{\theta} \circ \bar{\varphi}$. In this way we have the group homomorphism

$$\bar{T}^\theta\colon \mathcal{M}_{g,*} \longrightarrow \mathrm{Aut}(\widehat{T}/\mathcal{N}).$$

It is known that $\bigcap_{m=1}^\infty (I\pi_1(\Sigma_g))^m = 0$, where $I\pi_1(\Sigma_g)$ is the augmentation ideal. See, for example, Chen [6] p.193, Corollary 1 and p.197, Corollary 4. It follows that the natural map $\pi_1(\Sigma_g) \to \widehat{\mathbb{Q}\pi_1(\Sigma_g)}$ is injective, so is the homomorphism $\bar{T}^\theta$.

Let $\varphi \in \mathcal{M}_{g,1}$. Since θ is symplectic, $T^\theta(\varphi)(\omega) = T^\theta(\varphi)(\ell^\theta(\zeta)) = \ell^\theta(\varphi(\zeta)) = \ell^\theta(\zeta) = \omega$. Thus $T^\theta(\varphi) \in \mathrm{Aut}(\widehat{T})$ preserves $\mathcal{N}$. By construction, we have

$$\varpi \circ T^\theta(\varphi) = \bar{T}^\theta(\bar{\varphi}) \circ \varpi, \tag{7.4}$$

where $\bar{\varphi} \in \mathcal{M}_{g,*}$ is the image of φ by (7.2).

Let C be a simple closed curve on $\Sigma_g \setminus \{*'\}$. Then t_C, the Dehn twist along C, is defined as an element of $\mathcal{M}_{g,*}$. Since Σ is a deformation retract of $\Sigma_g \setminus \{*'\}$, we can regard C as a simple closed curve on Σ. Thus, t_C is also defined as an element of $\mathcal{M}_{g,1}$. By (7.4), we have

$$\varpi \circ T^\theta(t_C) = \bar{T}^\theta(t_C) \circ \varpi.$$

Also, $L^\theta(C) \in \widehat{T}_2$ is well-defined. Since $L^\theta(C) \in \mathfrak{l}_g$ by Lemma 2.7.2, $L^\theta(C)\omega = 0$. Therefore, $L^\theta(C)$ preserves $\mathcal{N}$ and it defines a derivation of $\widehat{T}/\mathcal{N}$. We denote it by $\bar{L}^\theta(C)$. By construction, we have $\varpi \circ L^\theta(C) = \bar{L}^\theta(C) \circ \varpi$ and moreover,

$$\varpi \circ e^{-L^\theta(C)} = e^{-\bar{L}^\theta(C)} \circ \varpi.$$

By Theorem 1.1.1, we have $\bar{T}^\theta(t_C) \circ \varpi = e^{-\bar{L}^\theta(C)} \circ \varpi$ and since ϖ is surjective, $\bar{T}^\theta(t_C) = e^{-\bar{L}^\theta(C)}$. In summary, we have proved the following theorem.

Theorem 7.2.2. *Let θ be a symplectic expansion and C a simple closed curve on $\Sigma_g \setminus \{*'\}$. Let $t_C \in \mathcal{M}_{g,*}$ be the Dehn twist along C. Then*

$$\bar{T}^\theta(t_C) = e^{-\bar{L}^\theta(C)}.$$

Here the right hand side is the algebra automorphism of $\widehat{T}/\mathcal{N}$ defined by the exponential of the derivation $-\bar{L}^\theta(C)$.

7.3. Action on $N_k(\pi_1(\Sigma_g))$. We prove a result similar to Theorem 1.1.2. Let C be a simple closed curve on $\Sigma_g \setminus \{*'\}$. As we saw in §7.2, we can regard C as a simple closed curve on Σ. As we did in §6.4, for each $k \geq 0$, $\bar{C}_k \in \bar{N}_k = \bar{N}_k(\pi)$ is defined. For each $k \geq 0$, let $N_k(\pi_1(\Sigma_g))$ be the k-th nilpotent quotient of $\pi_1(\Sigma_g)$, defined similarly to $N_k = N_k(\pi)$. The mapping class group $\mathcal{M}_{g,*}$ naturally acts on $N_k(\pi_1(\Sigma_g))$.

Theorem 7.3.1. *For each $k \geq 1$, the action of t_C on $N_k(\pi_1(\Sigma_g))$ depends only on $\bar{C}_k \in \bar{N}_k$. If C is separating, it depends only on $\bar{C}_{k-1} \in \bar{N}_{k-1}$.*

Proof. Let $N_k(\pi) \to N_k(\pi_1(\Sigma_g))$ be the natural surjection. This map is compatible with (7.2) and the actions of the two mapping class groups on the nilpotent quotients. The result follows by Theorem 1.1.2. □

A. Appendix. (Co)homology theory for Hopf algebras

We discuss a general theory of relative homology and cap products for (complete) Hopf algebras. Theory of cap products on the absolute (co)homology of a single (complete) Hopf algebra was already discussed in Cartan and Eilenberg [4], Chapter XI. But, unfortunately, the authors do not find an appropriate reference for cap products on the relative (co)homology of a pair of (complete) Hopf algebras. In §4 and §5, these notions relate the Goldman Lie algebra to symplectic derivations of the algebra $\widehat{T}$.

A.1. The mapping cone of a chain map. We begin by recalling the notion of the mapping cone of a chain map. See, for example, [3] pp.6-7. Throughout this section we suppose any chain complex $C_* = \{C_n\}$ satisfies $C_n = 0$ for $n < 0$. We call it *acyclic* if $H_n(C_*) = 0$ for $n > 0$. Let $f : (C_*, d') \to (D_*, d)$ be a chain map of chain complexes. The mapping cone $D_* \rtimes_f C_{*-1}$ of the map f is defined by

$$(D_* \rtimes_f C_{*-1})_n \overset{\text{def}}{=} D_n \oplus C_{n-1}, \quad \text{and} \quad d'' \overset{\text{def}}{=} \begin{pmatrix} d & f \\ 0 & -d' \end{pmatrix}.$$

Then we have a natural long exact sequence

$$\ldots \longrightarrow H_n(C_*) \xrightarrow{f_*} H_n(D_*) \longrightarrow H_n(D_* \rtimes_f C_{*-1}) \longrightarrow H_{n-1}(C_*) \longrightarrow \ldots . \tag{A.1}$$

The proof of the following lemma is an easy exercise, so we omit it.

Lemma A.1.1. *If the chain map f is injective, then the natural projection*

$$\varpi : D_* \rtimes_f C_{*-1} \longrightarrow D_*/fC_*, \quad (u, v) \longmapsto u \bmod fC_*$$

is a quasi-isomorphism.

The following lemma will play a fundamental role in this section.

Lemma A.1.2. (1) *If a chain homotopy $\Phi\colon C_* \to D_{*+1}$ connects f to another chain map $g\colon C_* \to D_*$, namely, $d\Phi + \Phi d' = g - f$, then the map*

$$h(\Phi) \overset{\text{def}}{=} \begin{pmatrix} 1 & -\Phi \\ 0 & 1 \end{pmatrix} : D_* \rtimes_f C_{*-1} \longrightarrow D_* \rtimes_g C_{*-1}$$

is a chain map and a quasi-isomorphism.

(2) *Assume another chain homotopy $\Phi'\colon C_* \to D_{*+1}$ connecting f to g is homotopic to Φ, in other words, there exists a map $\Psi\colon C_* \to D_{*+2}$ satisfying the relation*

$$\Phi'_n - \Phi_n = (-1)^n (d\Psi_n + \Psi_{n-1} d')\colon C_n \longrightarrow D_{n+1}$$

for each degree n. Then we have

$$h(\Phi) \simeq h(\Phi')\colon D_* \rtimes_f C_{*-1} \longrightarrow D_* \rtimes_g C_{*-1}.$$

Proof. By a straightforward computation, $h(\Phi)$ is a chain map. It defines a homomorphism between the long exact sequences (A.1). Hence it is a quasi-isomorphism by the five-lemma. We have

$$\begin{pmatrix} d & g \\ 0 & -d' \end{pmatrix} \begin{pmatrix} 0 & (-1)^{n-2}\Psi_{n-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & (-1)^{n-3}\Psi_{n-2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d & f \\ 0 & -d' \end{pmatrix}$$
$$= h(\Phi') - h(\Phi).$$

This implies the second part of the lemma. □

The followings are well-known.

Lemma A.1.3. *Let R be an associative algebra, C_* a left R-projective chain complex, and D_* a left R-acyclic chain complex. Then*

(1) *For any R-map $f : H_0(C_*) \to H_0(D_*)$, there exists an R-chain map*

$$\varphi\colon C_* \longrightarrow D_*$$

inducing the map f on H_0.

(2) *If two R-chain maps φ and $\psi\colon C_* \to D_*$ satisfy*

$$\varphi_* = \psi_*\colon H_0(C_*) \longrightarrow H_0(D_*),$$

then we have an R-chain homotopy

$$\varphi \simeq \psi\colon C_* \longrightarrow D_*.$$

(3) *Moreover, if Φ and Φ' are R-chain homotopies connecting φ to ψ, then Φ and Φ' are chain homotopic to each other. In other words, there exists an R-map $\Psi\colon C_* \to D_{*+2}$ satisfying the relation*

$$\Phi'_n - \Phi_n = (-1)^n(d\Psi_n + \Psi_{n-1}d')\colon C_n \longrightarrow D_{n+1}$$

for each $n \geq 0$.

Let R', S', R and S be associative algebras, and C'_*, D'_*, C_* and D_* chain complexes of left R', S', R and S modules, respectively. Suppose

$$\begin{tikzcd} R' \arrow[r, "\varphi"] \arrow[d, "f'"'] & R \arrow[d, "f"] \\ S' \arrow[r, "\psi"'] & S \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C'_* \arrow[r, "\varphi"] \arrow[d, "f'"'] & C_* \arrow[d, "f"] \\ D'_* \arrow[r, "\psi"'] & D_* \end{tikzcd} \tag{A.2}$$

are a commutative diagram of algebra homomorphisms and a *homotopy* commutative diagram of chain maps, respectively, such that the chain maps f', f, φ and ψ respect the algebra homomorphisms f', f, φ and ψ, respectively, and the augmentations. Then we have a left R'-chain homotopy $\Theta\colon C'_* \to D_{*+1}$ connecting $\psi f'$ to $f\varphi$. Let M be a right S-module. Then we define a chain map

$$\begin{aligned} h(\varphi, \Theta, \psi) \stackrel{\text{def}}{=} \begin{pmatrix} \psi & -\Theta \\ 0 & \varphi \end{pmatrix}\colon (M \otimes_{S'} D'_*) \rtimes_{f'} (M \otimes_{R'} C'_{*-1}) \\ \longrightarrow (M \otimes_S D_*) \rtimes_f (M \otimes_R C_{*-1}) \end{aligned}$$

by the composite

$$\begin{aligned} (M \otimes_{S'} D'_*) \rtimes_{f'} (M \otimes_{R'} C'_{*-1}) &\xrightarrow{\left(\begin{smallmatrix} \psi & 0 \\ 0 & 1 \end{smallmatrix}\right)} (M \otimes_S D_*) \rtimes_{\psi f'} (M \otimes_{R'} C'_{*-1}) \\ &\xrightarrow{h(\Theta)} (M \otimes_S D_*) \rtimes_{f\varphi} (M \otimes_{R'} C'_{*-1}) \\ &\xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varphi \end{smallmatrix}\right)} (M \otimes_S D_*) \rtimes_f (M \otimes_R C_{*-1}). \end{aligned}$$

Here we regard M as a module on which R', R and S' act through the homomorphisms $f \circ \varphi = \psi \circ f'$, f, and ψ, respectively.

Lemma A.1.4. *Assume C'_* is left R'-projective and D_* acyclic. Then the map*

$$(\varphi,\psi)_* \stackrel{\text{def}}{=} h(\varphi,\Theta,\psi)_* \colon H_*((M\otimes_{S'} D'_*)\rtimes_{f'}(M\otimes_{R'} C'_{*-1})) \longrightarrow H_*((M\otimes_S D_*)\rtimes_f (M\otimes_R C_{*-1}))$$

induced by the chain map $h(\varphi,\Theta,\psi)$ depends only on the homotopy classes of the chain maps φ and ψ.

Proof. Suppose $\varphi'\colon C'_*\to C_*$ and $\psi'\colon D'_*\to D_*$ are chain maps homotopic to φ and ψ, respectively, and Θ' a chain homotopy connecting $\psi' f'$ to $f\varphi'$. Take a chain homotopy $\Phi\colon C'_*\to C_{*+1}$ connecting φ to φ', and $\Psi\colon D'_*\to D_{*+1}$ connecting ψ to ψ'. Then the three diagrams

$$\begin{array}{ccc}
(M\otimes_{S'} D'_*)\rtimes_{f'}(M\otimes_{R'} C'_{*-1}) & \xrightarrow{\psi} & (M\otimes_S D_*)\rtimes_{\psi\circ f'}(M\otimes_{R'} C'_{*-1}) \\
\| & & \downarrow{\scriptstyle h(\Psi\circ f')} \\
(M\otimes_{S'} D'_*)\rtimes_{f'}(M\otimes_{R'} C'_{*-1}) & \xrightarrow[\psi']{} & (M\otimes_S D_*)\rtimes_{\psi'\circ f'}(M\otimes_{R'} C'_{*-1})
\end{array}$$

$$\begin{array}{ccc}
(M\otimes_S D_*)\rtimes_{\psi\circ f'}(M\otimes_{R'} C'_{*-1}) & \xrightarrow{h(\Theta)} & (M\otimes_S D_*)\rtimes_{f\circ\varphi}(M\otimes_{R'} C'_{*-1}) \\
{\scriptstyle h(\Psi\circ f')}\downarrow & & \downarrow{\scriptstyle h(f\circ\Phi)} \\
(M\otimes_S D_*)\rtimes_{\psi'\circ f'}(M\otimes_{R'} C'_{*-1}) & \xrightarrow[h(\Theta')]{} & (M\otimes_S D_*)\rtimes_{f\circ\varphi'}(M\otimes_{R'} C'_{*-1})
\end{array}$$

$$\begin{array}{ccc}
(M\otimes_S D_*)\rtimes_{f\circ\varphi}(M\otimes_{R'} C'_{*-1}) & \xrightarrow{h(\Theta)} & (M\otimes_S D_*)\rtimes_f(M\otimes_R C_{*-1}) \\
{\scriptstyle h(f\circ\Phi)}\downarrow & & \| \\
(M\otimes_S D_*)\rtimes_{f\circ\varphi'}(M\otimes_{R'} C'_{*-1}) & \xrightarrow[\varphi']{} & (M\otimes_S D_*)\rtimes_f(M\otimes_R C_{*-1})
\end{array}$$

commute up to homotopy. Here the horizontal ψ, ψ', φ and φ' mean the chain maps $\begin{pmatrix}\psi & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}\psi' & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & \varphi\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & \varphi'\end{pmatrix}$, respectively. In fact, the chain homotopies $\begin{pmatrix}\Psi & 0\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0\\ 0 & -\Phi\end{pmatrix}$ induce the homotopy commutativity of the first and the third diagrams, respectively. The maps $\Theta + f\circ\Phi$ and $\Theta' + \Psi\circ f'$ are both chain homotopies connecting $\psi\circ f'$ to $f\circ\varphi'$. Since C'_* is R'-projective and D_* acyclic, Lemma A.1.3 (3) implies there exists a homotopy connecting $\Theta + f\circ\Phi$ to $\Theta' + \Psi\circ f'$. Hence, by Lemma A.1.2 (2), we have

$$h(f\circ\Phi)h(\Theta) = h(\Theta + f\circ\Phi) \simeq h(\Theta' + \Psi\circ f') = h(\Theta')h(\Psi\circ f').$$

This means the homotopy commutativity of the second diagram. Hence we obtain

$$h(\varphi,\Theta,\psi) \simeq h(\varphi',\Theta',\psi')\colon (M\otimes_{S'} D'_*)\rtimes_{f'}(M\otimes_{R'} C'_{*-1}) \longrightarrow (M\otimes_S D_*)\rtimes_f (M\otimes_R C_{*-1}). \qquad \square$$

Moreover, suppose

$$\begin{array}{ccc} R'' & \xrightarrow{\alpha} & R' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} \\ S'' & \xrightarrow[\beta]{} & S' \end{array} \quad \text{and} \quad \begin{array}{ccc} C''_* & \xrightarrow{\alpha} & C'_* \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} \\ D''_* & \xrightarrow[\beta]{} & D'_* \end{array}$$

are a commutative diagram and a homotopy commutative diagram as in (A.2). Let Ξ be a chain homotopy connecting $\beta f''$ to $f'\alpha$, and Υ connecting $\psi\beta f''$ to $f\varphi\alpha$.

Lemma A.1.5. *Assume C''_* is left R''-projective and D_* acyclic. Then we have*

$$\begin{aligned} &h(\varphi, \Theta, \psi)_* h(\alpha, \Xi, \beta)_* = h(\varphi\alpha, \Upsilon, \psi\beta)_* \colon \\ &\quad H_*((M \otimes_{S''} D''_*) \rtimes_{f''} (M \otimes_{R''} C''_{*-1})) \\ &\qquad \longrightarrow H_*((M \otimes_S D_*) \rtimes_f (M \otimes_R C_{*-1})). \end{aligned} \tag{A.3}$$

Proof. By a straightforward computation,

$$h(\varphi, \Theta, \psi) h(\alpha, \Xi, \beta) = h(\varphi\alpha, \Theta\alpha + \psi\Xi, \psi\beta)$$

as chain maps. The chain homotopy $\Theta\alpha + \psi\Xi$ connects $\psi\beta f''$ to $f\varphi\alpha$. Hence, by Lemma A.1.4,

$$h(\varphi\alpha, \Upsilon, \psi\beta)_* = h(\varphi\alpha, \Theta\alpha + \psi\Xi, \psi\beta)_* = h(\varphi, \Theta, \psi)_* h(\alpha, \Xi, \beta)_*.$$

This proves the lemma. □

A.2. Relative homology of a pair of Hopf algebras. Let S be an augmented algebra over $\mathbb{Q}$ with the augmentation map $\varepsilon\colon S \to \mathbb{Q}$. We regard $\mathbb{Q}$ as a two-sided S-module via the map ε. Let $P_* \xrightarrow{\varepsilon} \mathbb{Q}$ be a left S-projective resolution. Then the homology group $H_*(S; M)$ is defined to be

$$\mathrm{Tor}^S_*(M, \mathbb{Q}) = H_*(M \otimes_S P_*)$$

for any *right* S-module M, and the cohomology group $H^*(S; M)$ to be

$$\mathrm{Ext}^*_S(M, \mathbb{Q}) = H^*(\mathrm{Hom}_S(P_*, S))$$

for any *left* S-module M.

Now let S be a (complete) Hopf algebra over $\mathbb{Q}$ with the augmentation ε, the antipode ι and the coproduct Δ. For a left S-module M, we can always regard it as a right S-module by $ms = \iota(s)m$, $s \in S$, $m \in M$.

Consider a homomorphism $f\colon R \to S$ of (complete) Hopf algebras. We regard M as a left R-module through the homomorphism f. Let $F_* \xrightarrow{\varepsilon} \mathbb{Q}$ be a left R-projective resolution of $\mathbb{Q}$. By Lemma A.1.3 (1), we can choose a chain map $f\colon F_* \to P_*$ which respects the homomorphism $f\colon R \to S$ and the augmentations. The (co)chain maps

$$f \stackrel{\text{def}}{=} 1_M \otimes f\colon M \otimes_R F_* \longrightarrow M \otimes_S P_*$$

and

$$f \stackrel{\text{def}}{=} \operatorname{Hom}(f, 1_M)\colon \operatorname{Hom}_S(P_*, M) \longrightarrow \operatorname{Hom}_R(F_*, M)$$

define the induced maps

$$f_*\colon H_*(R; M) \longrightarrow H_*(S; M) \quad \text{and} \quad f^*\colon H^*(S; M) \longrightarrow H^*(R; M),$$

which are independent of the choice of the chain map $f\colon F_* \to P_*$.

We define the relative homology group $H_*(S, R; M)$ by the homology group of the mapping cone

$$H_*(S, R; M) \stackrel{\text{def}}{=} H_*((M \otimes_S P_*) \rtimes_f (M \otimes_R F_{*-1})),$$

which we call *the relative homology of the pair* (S, R) *with coefficients in* M. Here to simplify the notation we drop the symbol f. We will use the case where R is a Hopf subalgebra of S and f is the inclusion. It does not depend on the choice of the resolutions P_*, F_*, and the chain map f. In fact, let P'_* and F'_* be other resolutions and $f'\colon F'_* \to P'_*$ a chain map respecting the homomorphism f and the augmentations. By Lemma A.1.3 (1)(2), we have homotopy equivalences $\varphi\colon F'_* \to F_*$ and $\psi\colon P'_* \to P_*$ respecting the identities 1_R and 1_S, respectively. Lemma A.1.3 (2) implies $f\varphi \simeq \psi f'\colon F'_* \to P_*$. Hence, by Lemma A.1.4, we obtain a uniquely determined map

$$\begin{aligned}(\varphi, \psi)_*\colon H_*((M \otimes_S P'_*) \rtimes_{f'} (M \otimes_R F'_{*-1}))\\ \longrightarrow H_*((M \otimes_S P_*) \rtimes_f (M \otimes_R F_{*-1})).\end{aligned}$$

Homotopy inverses of φ and ψ induce a uniquely determined map

$$\begin{aligned}H_*((M \otimes_S P_*) \rtimes_f (M \otimes_R F_{*-1}))\\ \longrightarrow H_*((M \otimes_S P'_*) \rtimes_{f'} (M \otimes_R F'_{*-1})).\end{aligned}$$

It is the inverse of the map $(\varphi, \psi)_*$ by Lemmas A.1.4 and A.1.5. Hence the relative homology group $H_*(S, R; M)$ is well-defined.

By the sequence (A.1), we have a natural exact sequence

$$\cdots \longrightarrow H_n(R;M) \xrightarrow{f_*} H_n(S;M) \xrightarrow{j_*} H_n(S,R;M) \xrightarrow{\partial_*} H_{n-1}(R;M) \longrightarrow \cdots. \tag{A.4}$$

We may choose $P_0 = S$ and $F_0 = R$. Then the boundary operator

$$\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix} = (d\ 1)\colon (M \otimes_S P_1) \oplus (M \otimes_R F_0) = (M \otimes_S P_1) \oplus M$$
$$\longrightarrow (M \otimes_S P_0) \oplus (M \otimes_R F_{-1}) = M$$

is surjective. Hence we have

$$H_0(S,R;M) = 0. \tag{A.5}$$

For a commutative diagram

$$\begin{array}{ccc} R' & \xrightarrow{\varphi} & R \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[\psi]{} & S \end{array} \tag{A.6}$$

of (complete) Hopf algebras, we can take a homotopy commutative diagram of resolutions as in (A.2). By Lemma A.1.4, it induces a well-defined map

$$(\varphi,\psi)_*\colon H_*(S',R';M) \longrightarrow H_*(S,R;M)$$

for any S-module M. By Lemma A.1.5 the relative homology of a pair of (complete) Hopf algebras satisfies a functoriality.

Next consider the coproducts $\Delta\colon S \to S \otimes S$ and $\Delta\colon R \to R \otimes R$. By the Künneth formula, $P_* \otimes P_*$ and $F_* \otimes F_*$ are acyclic. We regard them as left S- and left R- chain complexes by using the coproducts, respectively. In the case where S and R are complete Hopf algebras, we consider $P_* \widehat{\otimes} P_*$ and $F_* \widehat{\otimes} F_*$ instead, and *assume* they are acyclic. In any cases, by Lemma A.1.3, we have chain maps $\Delta\colon P_* \to P_* \otimes P_*$ and $\Delta\colon F_* \to F_* \otimes F_*$. By Lemma A.1.4, we can define a uniquely determined map

$$\begin{aligned} \Delta_* &\overset{\text{def}}{=} (\Delta,\Delta)_*\colon H_*(S,R;M) \\ &\longrightarrow H_*((M \otimes_S (P_* \otimes P_*)) \rtimes_{f\otimes f} (M \otimes_R (F_* \otimes F_*)_{*-1})), \end{aligned} \tag{A.7}$$

which we call *the diagonal map*. Consider a commutative diagram of (complete) Hopf algebras as in (A.6). Take resolutions P'_* and F'_* over S' and R', respectively. By Lemma A.1.3, we have chain maps $\varphi\colon F'_* \to F_*$ and $\psi\colon P'_* \to P_*$ respecting the Hopf algebra homomorphisms φ and ψ, respectively, and the augmentations. The homotopy commutative diagrams

$$\begin{CD} F'_* @>{\Delta}>> F'_* \otimes F'_* @>{\varphi\otimes\varphi}>> F_* \otimes F_* \\ @V{f'}VV @V{f'\otimes f'}VV @V{f\otimes f}VV \\ P'_* @>{\Delta}>> P'_* \otimes P'_* @>{\psi\otimes\psi}>> P_* \otimes P_* \end{CD} \qquad \text{and} \qquad \begin{CD} F'_* @>{\varphi}>> F_* @>{\Delta}>> F_* \otimes F_* \\ @V{f'}VV @V{f'\otimes f'}VV @V{f\otimes f}VV \\ P'_* @>{\psi}>> P_* @>{\Delta}>> P_* \otimes P_* \end{CD}$$

respect the same commutative diagram (A.6). Hence, by Lemma A.1.5, we obtain the commutative diagram

$$\begin{CD} H_*(S', R'; M) @>{\Delta_*}>> H_*((M \otimes_{S'} (P'_* \otimes P'_*)) \rtimes_{f'\otimes f'} (M \otimes_{R'} (F'_* \otimes F'_*)_{*-1})) \\ @V{(\varphi,\psi)_*}VV @VV{(\varphi\otimes\varphi,\psi\otimes\psi)_*}V \\ H_*(S, R; M) @>>{\Delta_*}> H_*((M \otimes_S (P_* \otimes P_*)) \rtimes_{f\otimes f} (M \otimes_R (F_* \otimes F_*)_{*-1})), \end{CD} \tag{A.8}$$

namely, the naturality of the map in (A.7). Here, if Θ is a chain homotopy connecting $\psi f'$ to $f\varphi$, the vertical map $(\varphi,\psi)_*$ is given by $h(\varphi,\Theta,\psi)$, and $(\varphi\otimes\varphi,\psi\otimes\psi)_*$ by $h(\varphi\otimes\varphi, (f\varphi)\otimes\Theta + \Theta\otimes(\psi f'), \psi\otimes\psi)$ as in (A.1).

A.3. Cap products on the relative (co)homology. Now we introduce the cap product on the relative homology of a pair of (complete) Hopf algebras. We remark our sign convention is different from [14] and [1], since our Poincaré duality in this paper is given by (2.3). See [14] §5, for details.

Let $f\colon R \to S$ be a homomorphism of (complete) Hopf algebras over $\mathbb{Q}$, M_1 and M_2 left S-modules, P_* and F_* projective resolutions of $\mathbb{Q}$ over S and R, respectively, and $f\colon F_* \to P_*$ a chain map respecting the homomorphism f and the augmentations. We define the cap product

$$\cap\colon M_1 \otimes_R (F_* \otimes F_*) \otimes \mathrm{Hom}_S(P_*, M_2) \longrightarrow (M_1 \otimes M_2) \otimes_R F_* \tag{A.9}$$

by $\cap(u\otimes x\otimes y\otimes v) = (u\otimes x\otimes y)\cap v \overset{\text{def}}{=} (-1)^{\deg(x\otimes y)\deg v} u\otimes v(f(x))\otimes y$ for $u \in M_1$, $x, y \in F_*$ and $v \in \mathrm{Hom}_S(P_*, M_2)$. Here $M_1 \otimes M_2$ is regarded as an R-module by the homomorphism f and the coproduct Δ. In the case R is a complete Hopf algebra, we consider the completed tensor product $M_1 \hat{\otimes} M_2$ instead. By a straightforward computation, we find out $\cap$ is a chain map. In the case where $R = S$ and $f = 1_S$, we have a chain map

$$\cap\colon M_1 \otimes_S (P_* \otimes P_*) \otimes \mathrm{Hom}_S(P_*, M_2) \longrightarrow (M_1 \otimes M_2) \otimes_S P_*,$$

which is compatible with the map (A.9). Hence we obtain a chain map

$$\begin{aligned}(M_1 \otimes_S (P_* \otimes P_*) \rtimes_{f\otimes f} M_1 \otimes_R (F_* \otimes F_*)_{*-1}) \otimes \operatorname{Hom}_S(P_*, M_2)\\ \longrightarrow (M_1 \otimes M_2) \otimes_S P_* \rtimes_f (M_1 \otimes M_2) \otimes_R F_{*-1}\end{aligned}$$

and the induced map

$$\begin{aligned}\cap\colon H_*((M_1 \otimes_S (P_* \otimes P_*) \rtimes_{f\otimes f} M_1 \otimes_R (F_* \otimes F_*)_{*-1})) \otimes H^*(S; M_2)\\ \longrightarrow H_*(S, R; M_1 \otimes M_2).\end{aligned} \tag{A.10}$$

We have to prove the naturality of the cap product (A.10). For the commutative diagram of (complete) Hopf algebras (A.6), choose chain maps $\varphi\colon F'_* \to F_*$ and $\psi\colon P'_* \to P_*$ of resolutions as in (A.8).

Lemma A.3.1. *For any* $\xi \in H_*((M_1 \otimes_{S'} (P'_* \otimes P'_*)) \rtimes_{f\otimes f} (M_1 \otimes_{R'} (F'_* \otimes F'_*)_{*-1}))$ *and* $\eta \in H^*(S; M_2)$*, we have*

$$(\varphi, \psi)_*(\xi \cap \psi^*\eta) = ((\varphi, \psi)_*\xi) \cap \eta \in H_*(S, R; M_1 \otimes M_2).$$

Here $(\varphi, \psi)_*\xi$ *in the right hand side means the homology class*

$$h(\varphi \otimes \varphi, (f\varphi) \otimes \Theta + \Theta \otimes (\psi f'), \psi \otimes \psi)_*\xi.$$

The lemma in the case where $R' = R$, $S' = S$, $\varphi = 1_R$ and $\psi = 1_S$ implies that the cap product is independent of the choice of the resolutions and the chain maps.

Proof. Let $u, u' \in M_1$, $x, y \in P'_*$, $x', y' \in F'_*$ and $v \in \operatorname{Hom}_S(P_*, M_2)$. We set

$$\Xi \overset{\text{def}}{=} (f\varphi) \otimes \Theta + \Theta \otimes (\psi f').$$

Then, by a straightforward computation,

$$\begin{aligned}
&(-1)^{\deg(x'\otimes y')\deg v}\begin{pmatrix}\psi\otimes\psi & -\Theta\\ 0 & \varphi\end{pmatrix}((u\otimes x\otimes y, u'\otimes x'\otimes y')\cap(\psi^*v))\\
&\quad-(-1)^{\deg(x'\otimes y')\deg v}\left(\begin{pmatrix}\psi\otimes\psi & -\Xi\\ 0 & \varphi\otimes\varphi\end{pmatrix}(u\otimes x\otimes y, u'\otimes x'\otimes y')\right)\cap v\\
&=(-(-1)^{\deg x'}u'\otimes(dv)(\Theta x')\otimes\Theta y', (-1)^{\deg v}u'\otimes(dv)(\Theta x')\otimes\varphi y')\\
&\quad-((-1)^{\deg v}u'\otimes(v\Theta\otimes\Theta)d(x'\otimes y'), u'\otimes(v\Theta\otimes\varphi)d(x'\otimes y'))\\
&\quad+\begin{pmatrix}d & f\\ 0 & -d\end{pmatrix}((-1)^{\deg x'}u'\otimes(v\Theta x')\otimes\Theta y', (-1)^{\deg v}u'\otimes(v\Theta x')\otimes\varphi y').
\end{aligned}$$

If v is a cocycle, and $(u\otimes x\otimes y, u'\otimes x'\otimes y')$ is a cycle, then the right hand side is null-homologous. This proves the lemma. □

Taking the composite of the map $\cap$ in (A.10) and the diagonal map Δ_* in (A.7), we obtain the cap product

$$\cap \overset{\text{def}}{=} \cap \circ \Delta_* \colon H_*(S, R; M_1) \otimes H^*(S; M_2) \longrightarrow H_*(S, R; M_1 \otimes M_2). \quad \text{(A.11)}$$

By the naturality of the diagonal map Δ_* (A.8) and Lemma A.3.1, this is independent of the choice of resolutions and chain maps. We also obtain the naturality of the cap product:

Proposition A.3.2. *In the situation of the commutative diagram* (A.6)*, let M_1 and M_2 be left S-modules. For any $\xi \in H_*(S', R'; M_1)$ and $\eta \in H^*(S; M_2)$, we have*

$$(\varphi, \psi)_*(\xi \cap \psi^* \eta) = ((\varphi, \psi)_* \xi) \cap \eta \in H_*(S, R; M_1 \otimes M_2).$$

A.4. Kronecker product. We recall the Kronecker product on the (co)homology of a Hopf algebra. Let S be a (complete) Hopf algebra over $\mathbb{Q}$, P_* an S-projective resolution of $\mathbb{Q}$, and M_1 and M_2 left S-modules. The Kronecker product on the (co)chain level

$$\langle\ ,\ \rangle \colon (M_1 \otimes_S P_*) \otimes \mathrm{Hom}_S(P_*, M_2) \longrightarrow M_1 \otimes_S M_2 \quad \text{(A.12)}$$

is defined by

$$\langle u \otimes x, v \rangle = (-1)^{\deg x \deg v} u \otimes v(x), \quad u \in M_1, x \in P_*, v \in \mathrm{Hom}_S(P_*, M_2).$$

Since $\langle d(u \otimes x), v \rangle = (-1)^{\deg x} \langle u \otimes x, dv \rangle$, we have the Kronecker product on the (co)homology level

$$\langle\ ,\ \rangle \colon H_*(S; M_1) \otimes H^*(S; M_2) \longrightarrow M_1 \otimes_S M_2.$$

If we regard $M_1 \otimes_S M_2$ as the 0-th homology group $H_0(S; M_1 \otimes M_2)$, the Kronecker product is just the cap product. Let $\psi \colon S' \to S$ be a homomorphism of (complete) Hopf algebras, P'_* an S'-projective resolution of $\mathbb{Q}$, and $\psi \colon P'_* \to P_*$ a chain map which respects the homomorphism ψ and the augmentations. Then we have

$$\langle \psi_* u, v \rangle = \langle u, \psi^* v \rangle \quad \text{(A.13)}$$

for any $u \in H_*(M_1 \otimes_{S'} P'_*)$ and $v \in H^*(\mathrm{Hom}_S(P_*, M_2))$. Hence the Kronecker product is independent of the choice of the resolution P_*, and has a naturality.

A.5. Homology of a pair of groups. Let G be a group, K a subgroup of G, and M a left $\mathbb{Q}G$-module. As was stated in §3.3, we have $H_*(G;M) = H_*(\mathbb{Q}G;M)$ and $H^*(G;M) = H^*(\mathbb{Q}G;M)$. The normalized standard complex $\bar{F}_*(G)$ is a $\mathbb{Q}G$-projective resolution of $\mathbb{Q}$. Since the inclusion map

$$C_*(K;M) = M \otimes_{\mathbb{Q}K} \bar{F}_*(K) \longrightarrow C_*(G;M) = M \otimes_{\mathbb{Q}G} \bar{F}_*(G)$$

is injective, the mapping cone $C_*(G;M) \rtimes C_{*-1}(K;M)$ is naturally quasi-isomorphic to the quotient complex $C_*(G;M)/C_*(K;M)$ by Lemma A.1.1. Hence we have a natural isomorphism

$$H_*(G,K;M) = H_*(\mathbb{Q}G, \mathbb{Q}K; M). \tag{A.14}$$

The standard complex $F_*(G) = \{F_n(G)\}$, where $F_n(G)$ is the free $\mathbb{Q}G$-module with $\mathbb{Q}G$-basis $\{[g_1|g_2|\dots|g_n]; g_i \in G\}$, is also a natural $\mathbb{Q}G$-projective resolution of $\mathbb{Q}$ (see [3] p. 18), so that it can be used for computing the relative homology $H_*(\mathbb{Q}G, \mathbb{Q}K; M)$. The Alexander–Whitney map $\Delta\colon F_*(G) \to F_*(G) \otimes F_*(G)$ is a $\mathbb{Q}G$-chain map respecting the augmentation maps. See, for example, [3] p. 108. Hence the cap product on the relative homology $H_*(\mathbb{Q}G, \mathbb{Q}K; M)$ of the pair $(\mathbb{Q}G, \mathbb{Q}K)$ introduced in §4.3 coincides with the usual cap product on the relative homology $H_*(G,K;M)$ of the pair (G,K) via the isomorphism (A.14).

On the other hand, consider the classifying spaces BG and BK. We assume BK is realized as a subspace of BG. Choose a basepoint $* \in BK$. Denote by Δ^n the standard n-simplex, and by $S_*(X)$ the rational singular chain complex of a topological space X. For any $g \in G$ we choose a continuous map $\rho(g)\colon \Delta^1 \to BG$ satisfying the conditions

(1) $\rho(g)(0) = \rho(g)(1) = *$ under the natural identification $\Delta^1 \approx [0,1]$,

(2) the based homotopy class of $\rho(g)$ is exactly $g \in G = \pi_1(BG, *)$, and

(3) $\rho(k)(\Delta^1) \subset BK$ if $k \in K$.

This assignment defines a $\mathbb{Q}G$-map

$$\rho\colon F_1(G) \longrightarrow S_1(EG)$$

and a $\mathbb{Q}K$-map

$$\rho\colon F_1(K) \longrightarrow S_1(EK),$$

where EG and EK are the universal covering spaces of BG and BK, respectively. Since the spaces BG and BK are aspherical, the map ρ extends to a $\mathbb{Q}G$-chain map $\rho\colon F_*(G) \to S_*(EG)$ and a $\mathbb{Q}K$-chain map $\rho\colon F_*(K) \to S_*(EK)$. Since the map ρ respects the augmentations, it induces a natural isomorphism

$$\rho_*\colon H_*(G,K;M) \longrightarrow H_*(BG,BK;M). \tag{A.15}$$

In the right hand side we regard M as the local system on the space BG associated with the G-module M. By the construction of the map ρ, we have a commutative diagram of $\mathbb{Q}G$-chain maps

$$\begin{tikzcd} F_*(G) \arrow[r, "\Delta"] \arrow[d, "\rho"'] & F_*(G) \otimes F_*(G) \arrow[d, "\rho\otimes\rho"] \\ S_*(EG) \arrow[r, "\Delta"'] & S_*(EG) \otimes S_*(EG), \end{tikzcd}$$

where the lower Δ is the Alexander-Whitney map on the singular chain complex. Hence the cap product on the relative homology $H_*(G, K; M)$ of the pair (G, K) coincides with the cap product on the relative homology $H_*(BG, BK; M)$ of the pair (BG, BK) of topological spaces via the isomorphism (A.15).

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