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Quantum Topology

Dualizability and index of subfactors

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Abstract. In this paper, we develop the theory of bimodules over von Neumann algebras, with an emphasis on categorical aspects. We clarify the relationship between dualizability and finite index. We also show that, for von Neumann algebras with finite dimensional centers, the Haagerup L^2-space and Connes fusion are functorial with respect to homorphisms of finite index. Along the way, we describe a string diagram notation for maps between bimodules that are not necessarily bilinear.

Mathematics Subject Classification (2010). 46L10, 46L37.

Keywords. Subfactors, Connes fusion, dualizability, Haagerup L^2-space, index.

Contents

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1. Introduction

The operation $({}_AH_B, {}_BK_C) \mapsto {}_AH \boxtimes_B K_C$ of Connes fusion is an associative product on bimodules between von Neumann algebras [5, 31, 33, 42]. It behaves formally like a tensor product, but its construction is somewhat involved and relies heavily on the notion of non-commutative L^2-space [7, 16, 43]. Connes fusion is designed so as to have the L^2-space as its identity: ${}_AL^2A \boxtimes_A H_B \cong {}_AH_B \cong {}_AH \boxtimes_B L^2B_B$. Altogether, von Neumann algebras, their bimodules, and bimodule intertwiners form a symmetric monoidal bicategory [20]. As in any bicategory, one can talk about a morphism being dualizable[1] [27, 39]: a bimodule ${}_AH_B$ is called dualizable, with dual ${}_B\bar{H}_A$, if it comes equipped with maps

$$R^*: {}_AH \boxtimes_B \bar{H}_A \longrightarrow {}_AL^2(A)_A, \quad S: {}_BL^2(B)_B \longrightarrow {}_B\bar{H} \boxtimes_A H_B \tag{1.1}$$

subject to the duality equations $(R^*\otimes 1)(1\otimes S) = 1$, $(1\otimes R^*)(S\otimes 1) = 1$. The dual bimodule ${}_B\bar{H}_A$ is well defined up to unique isomorphism. In fact, under suitable normalization conditions on the duality maps R^* and S, the dual bimodule is well defined up to unique *unitary* isomorphism. If A and B are factors one can then define the *statistical dimension* of ${}_AH_B$ as $R^*R = S^*S$ [25].

A subfactor $N \subset M$ has an invariant called the index $[M : N] \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ [17, 19], and this index is finite if and only if the bimodule ${}_NL^2M_M$ is dualizable. When that bimodule is dualizable, the index may be defined as the square of the statistical dimension of ${}_NL^2M_M$. We show that this definition agrees with the traditional notion of index, by comparing the squared statistical dimension with the optimal bound of a Pimsner–Popa inequality for the subfactor [9, 21, 30].

Given two von Neumann algebras A and B that have finite-dimensional centers (in other words are finite direct sums of factors), we call a homomorphism $f: A \to B$ *finite* if the bimodule ${}_AL^2B_B$ is dualizable. Restricting attention to these finite homomorphisms makes the L^2 construction functorial:

Theorem. *The assignment*

$$A \longmapsto L^2(A)$$

is a functor from the category

$$\begin{cases} \textit{objects:} & \textit{von Neumann algebras with finite-dimensional center} \\ \textit{morphisms:} & \textit{finite homomorphisms} \end{cases}$$

to the category of Hilbert spaces and bounded linear maps.

[1] As written, equation (1.1) corresponds to the notion of *left* dualizability, but since our bicategory has a $*$-structure, there is no difference between left and right dualizability.

We conjecture that this functor in fact extends to the category of all von Neumann algebras and finite homomorphisms.

The Connes fusion $H \boxtimes_A K$ is certainly functorial in H and K. We show that it is moreover simultaneously functorial in the three variables H, K and A:

Theorem. *The assignment*

$$(H, A, K) \longmapsto H \boxtimes_A K$$

is a functor from the category

$$\begin{cases} \textit{objects:} & \textit{triples } (H, A, K) \textit{ where } A \textit{ is a von Neumann algebra} \\ & \textit{with finite-dimensional center, } H \textit{ is a right } A\textit{-module,} \\ & \textit{and } K \textit{ is a left } A\textit{-module} \\ \textit{morphisms:} & \textit{triples } (h, \alpha, k) \textit{ where } \alpha \textit{ is a finite homomorphism} \\ & A_1 \to A_2\textit{, } h \textit{ is a module map } H_1 \to H_2\textit{, and } k \textit{ is} \\ & \textit{a module map } K_1 \to K_2 \end{cases}$$

to the category of Hilbert spaces and bounded linear maps.

Note that our techniques and results all apply equally well to type I, II, and III von Neumann algebras.

Outlook. Our motivation for studying von Neumann algebras and Connes fusion comes from their relationship to quantum field theory and to the Stolz–Teichner program on elliptic cohomology. Their relevance to quantum field theory is evident in Wassermann's work [42] where Connes fusion is used to model the fusion rules of superselection sectors of the chiral Wess–Zumino–Witten conformal field theory with gauge group $\mathrm{SU}(N)$. Moreover, for those theories, Wassermann computes the Connes fusion explicitly, and recovers the Verlinde formulas.

The ongoing program of Stolz and Teichner aims to construct elliptic cohomology using *local* quantum field theories [37, 38]. Motivated by [42], Stolz and Teichner proposed the use of Connes fusion in their description local quantum field theories. Moreover, they asked asked whether there exits an interesting 3-category that deloops the bicategory of von Neumann algebras, their bimodules, and bimodule intertwiners. Here, interesting can be taken to mean that the 3-category should have many dualizable objects: as a consequence of the cobordism hypothesis [26] every dualizable object corresponds to a 3-dimensional local quantum field theory. The present paper arose as a byproduct of our ongoing construction of such a 3-category using conformal nets [2, 3].

The construction of our 3-category of conformal nets depends very much on the theory of von Neumann algebras, in particular non-commutative L^2-spaces and the index for subfactors. We hope that the present treatment of these topics will help make our future papers more accessible for readers who do not have a strong background in von Neumann algebras. This paper is *not* a complete survey of the index for subfactors; we mostly only discuss what we will need later on. Many of the results in this paper are surely well-known to experts; for example the identification of the index, defined using statistical dimension, with what we later call the minimal index, is no doubt known, but we are not aware of a reference.

Outline. Our new graphical notation is described in Section 2, along with preliminaries concerning von Neumann algebras and Haagerup's L^2-space. We emphasize the fact that it is not necessary to chose a state $\phi\colon A \to \mathbb{C}$ in order to define $L^2(A)$ [16]. In Section 3, we discuss Connes fusion and some of its elementary properties. In Section 4, we investigate the concept of dualizable bimodules. We prove that the endomorphism algebra $\mathrm{End}({}_A H_B)$ of a dualizable bimodule is finite-dimensional and is equipped with a canonical trace. Moreover, we show the dual is well defined up to unique unitary isomorphism. In Section 5, we define the statistical dimension of a dualizable bimodule and introduce the categorical definition of the index of a subfactor, namely $[M : N] \stackrel{\text{def}}{=} \dim({}_N L^2 M_M)^2$. In Section 6, we present our new results: the functoriality of L^2 and of Connes fusion. Finally, in Section 7, we use the Pimsner–Popa inequality to show that the categorical definition of the index agrees with other definitions [9, 17, 18, 21, 30]. We end the paper with some useful inequalities for the index.

2. Preliminaries

Von Neumann algebras. Given a complex Hilbert space H, let $\mathbf{B}(H)$ denote its algebra of bounded operators. The ultraweak topology on $\mathbf{B}(H)$ is the topology of pointwise convergence with respect to the pairing with its predual, the trace class operators.

Definition 2.1. A von Neumann algebra is a topological $*$-algebra that is embeddable as a closed subalgebra of $\mathbf{B}(H)$ with respect to the ultraweak topology.

Definition 2.2. Let A be a von Neumann algebra. A left (right) A-module is a Hilbert space H equipped with a continuous homomorphism from A (resp. A^{op}) to $\mathbf{B}(H)$. We will use the notation ${}_A H$ (respectively H_A) to denote the fact that H is a left (right) A-module.

The main distinguishing feature of the representation theory of von Neumann algebras is the following:

Proposition 2.3 ([10, Remark 2.1.3. (iii)]). *Let A be a von Neumann algebra and let H and K be two faithful left A-modules. Then $H \otimes \ell^2 \cong K \otimes \ell^2$. In particular, any A-module is isomorphic to a direct summand of $H \otimes \ell^2$.* □

If the Hilbert spaces H and K in this proposition are separable, then ℓ^2 can be taken to mean $\ell^2(\mathbb{N})$. Otherwise, the proposition is true for $\ell^2 = \ell^2(X)$, for X some set of sufficiently large cardinality.

The spatial tensor product $A_1 \bar{\otimes} A_2$ of von Neumann algebras $A_i \subset \mathbf{B}(H_i)$ is the closure in $\mathbf{B}(H_1 \otimes H_2)$ of the algebraic tensor product $A_1 \otimes_{\text{alg}} A_2$; by the above proposition, this closure is independent of the choices of Hilbert spaces H_1 and H_2. The spatial tensor product is a symmetric monoidal structure on the category of von Neumann algebras.

The Haagerup L^2-space. Given a von Neumann algebra A, the space of continuous linear functionals $A \to \mathbb{C}$ forms a Banach space $A_* = L^1(A)$ called the predual of A. It is equipped with two commuting A actions given by

$$(a\phi b)(x) \stackrel{\text{def}}{=} \phi(bxa)$$

and a cone

$$L^1_+(A) \stackrel{\text{def}}{=} \{\phi \in A_* \mid \phi(x) \geq 0, x \in A_+\}.$$

Here,

$$A_+ \stackrel{\text{def}}{=} \{a^*a \mid a \in A\}$$

is the set of positive elements of A.

The Haagerup L^2-space of A is an A-A-bimodule that is canonically associated to A. It is denoted $L^2(A)$ and its construction does not depend on any choices [16]. It is the completion of

$$\bigoplus_{\phi \in L^1_+(A)} \mathbb{C}\sqrt{\phi}$$

with respect to some pre-inner product. We will provide more details of the construction of $L^2(A)$ at the beginning of Section 6.

Remark 2.4. At this point, $\sqrt{\phi} \in L^2 A$ should be treated as a formal symbol. However, there exists a natural $*$-algebra structure on $\bigoplus_p L^p A$ in which $\sqrt{\phi}$ is the (unique positive) square root of $\phi \in L^1 A$ – see Remark 6.3 for further details.

As a consequence of that characterization, we learn that

$$u\sqrt{\phi}u^* = \sqrt{u\phi u^*} \tag{2.5}$$

for every $\phi \in L^1_+(A)$ and every unitary $u \in A$.

Remark 2.6. There is an isomorphism $L^2(A) \cong L^2(A^{\mathrm{op}})$ under which the left action of A on L^2A is equal to the right action of A^{op} on $L^2(A^{\mathrm{op}})$, and the right action of A on L^2A is equal to the left action of A^{op} on $L^2(A^{\mathrm{op}})$.

The L^2 construction is compatible with direct sums, in the sense that

$$L^2(A \oplus B) = L^2(A) \oplus L^2(B).$$

This is a corollary of the relationship expressed in the following lemma, between the L^2-space construction and the operation of taking the corner algebra pAp associated to a projection $p \in A$.

Lemma 2.7 ([7, Lemma 2.6]). *Given any projection $p \in A$, there is a canonical unitary isomorphism $L^2(pAp) \cong p(L^2A)p$ sending $\sqrt{\phi} \in L^2(pAp)$ to $\sqrt{\phi \circ E}$, where $E(a) = pap$.* □

The bimodule $L^2(A)$ may be characterized as follows. It is a Hilbert space H with faithful left and right actions of A, equipped with an antilinear isometric involution J and a self-dual cone $P \subset H$ subject to the properties

(i) $JAJ = A'$ on H,

(ii) $JcJ = c^*$ for all $c \in Z(A)$,

(iii) $J\xi = \xi$ for all $\xi \in P$,

(iv) $aJaJ(P) \subseteq P$ for all $a \in A$,

(v) $\xi a = Ja^*J\xi$ for all $\xi \in H$ and all $a \in A$.

Here,

$$A' \stackrel{\text{def}}{=} \{b \in \mathbf{B}(H) \mid [a,b] = 0, a \in A\}$$

is the commutant of A; $JAJ = \{JaJ \mid a \in A\}$; and the cone P is called self-dual if $P = \{\eta \in H \mid \langle \xi, \eta\rangle \geq 0, \xi \in P\}$. The operator J is called the modular conjugation. A Hilbert space H, so equipped with a modular conjugation J and a self-dual cone P, is called a *standard form*. Such a standard form is unique up to unique unitary isomorphism [7].

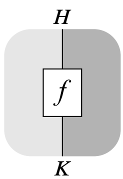
Remark 2.8. If ϕ is a faithful normal weight (an unbounded version of a state) on a von Neumann algebra A, then the GNS Hilbert space $L^2(A,\phi)$ is a standard form for A [1] and therefore serves as a particular construction of the bimodule $L^2(A)$. For example, taking ϕ to be the usual trace tr on $\mathbf{B}(H)$, we see that the ideal of Hilbert-Schmidt operators on H is a standard form for $\mathbf{B}(H)$.

Example 2.9. Let H be a Hilbert space and $\bar{H}$ its complex conjugate. Then $H \otimes \bar{H}$ is canonically identified with the ideal of Hilbert-Schmidt operators on H. Let $P \subseteq H \otimes \bar{H}$ correspond to the positive Hilbert-Schmidt operators, and J to the operation $x \mapsto x^*$, for x a Hilbert-Schmidt operator. Then $(H \otimes \bar{H}, J, P)$ is a standard form for $\mathbf{B}(H)$. We have $J(\xi \otimes \bar{\zeta}) = \zeta \otimes \bar{\xi}$, and $\xi \otimes \bar{\xi} \in P$ for all $\xi \in H$.[2]

Example 2.10. Let (H, J_A, P_A) and (K, J_B, P_B) be standard forms for von Neumann algebras A and B. Then there is a self-dual cone $P_{A \bar{\otimes} B}$ in $H \otimes K$ such that $(H \otimes K, J_A \otimes J_B, P_{A \bar{\otimes} B})$ is a standard form for $A \bar{\otimes} B$, and such that $\xi \otimes \zeta \in P_{A \bar{\otimes} B}$ whenever $\xi \in P_A$ and $\zeta \in P_B$ [28, 34]. Note that in general $P_{A \bar{\otimes} B}$ is strictly larger than the convex closure of $\{\xi \otimes \zeta \mid \xi \in P_A, \zeta \in P_B\}$.

String diagrams. String diagrams are a standard notation in monoidal categories and in bicategories [14, 35] and are often used in the context of von Neuman algebras [12, 6]. We briefly recall this notation and discuss an extension that will be useful later on.

In string diagrams, algebras are represented by shades, bimodules are represented by lines, and homomorphisms are nodes. For example, an A-B-bilinear map f between two bimodules ${}_AH_B$ and ${}_AK_B$ is depicted by the diagram



,

where the light shade corresponds to the algebra A and the darker shade corresponds to the algebra B. Other morphisms, such as

$$g\colon {}_AH \boxtimes_B K_C \longrightarrow {}_AM_C,$$

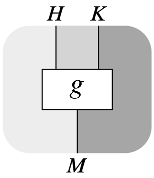
$$h\colon {}_AH_A \longrightarrow {}_AL^2A_A,$$

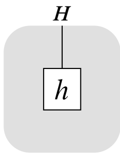
or

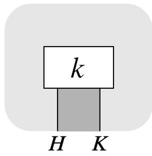
$$k\colon {}_AL^2A_A \longrightarrow {}_AH \boxtimes_B K_A$$

[2] Here, $\bar{\xi} \in \bar{H}$ is the image of $\xi \in H$ under the antilinear map $\mathrm{Id}_H : H \to \bar{H}$.

are drawn similarly:



, 

, and 

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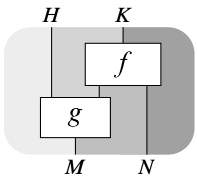
(Here, $\boxtimes$ is the operation of Connes fusion, which will be introduced in the following section, and ${}_AL^2A_A$ is the identity with respect to that operation.) The identity morphism between bimodules is drawn as a single vertical line . Horizontal juxtaposition of pictures corresponds to Connes fusion, and vertical concatenation corresponds to composition of morphisms. A more complicated composition of bimodule morphisms, such as

$$\underset{A}{}H\underset{B}{\boxtimes}K_{D}\xrightarrow{1_H\boxtimes f}\underset{A}{}H\underset{B}{\boxtimes}P\underset{C}{\boxtimes}N_{D}\xrightarrow{g\boxtimes 1_N}\underset{A}{}M\underset{C}{\boxtimes}N_{D}$$

is denoted



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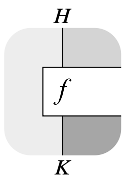
Our addition is the introduction of a notation for morphisms that are only left-linear, or only right-linear. We denote them by nodes that extend to the right and to the left of the diagram, respectively. Thus, an A-linear morphism f between bimodules ${}_AH_B$ and ${}_AK_C$ is denoted



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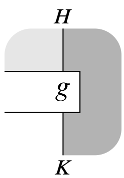
We will always use the color white for the algebra $\mathbb{C}$. For example, a B-linear map g from ${}_AH_B$ to some right B-module K_B is drawn like this:



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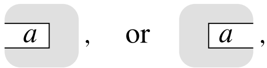
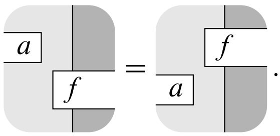
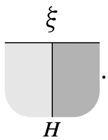
Our conventions also allow us to speak about algebra elements using the same graphical notation, as every right (left) A-linear morphism $L^2(A) \to L^2(A)$ is given by left (right) multiplication by an element $a \in A$. Such an element will be denoted



depending on whether we view it as acting on the left or on right on $L^2(A)$. The fact that an A-linear morphism $f\colon {}_AH_B \to {}_AH_B$ commutes with the left action of an element $a \in A$ is then nicely rendered by the equation



Finally, we can also denote vectors graphically, given that an element $\xi \in H$ is equivalent to a map $\mathbb{C} \to H$. For example, a vector in a bimodule ${}_AH_B$ is denoted



The node ξ extends both to the right and to the left, as the map $\xi\colon \mathbb{C} \to {}_AH_B$ is neither A- nor B-linear. Also, the space above ξ is white because the source of the above map is ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$.

3. Connes fusion

Definition 3.1. Given two modules H_A and ${}_AK$ over a von Neumann algebra A, their Connes fusion $H \boxtimes_A K$ is the completion of

$$\hom(L^2(A)_A, H_A) \otimes_A L^2(A) \otimes_A \hom({}_AL^2(A), {}_AK) \tag{3.2}$$

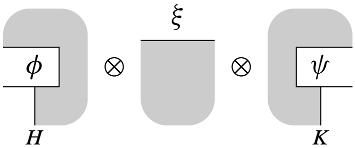
with respect to the inner product

$$\langle \phi_1 \otimes \xi_1 \otimes \psi_1, \phi_2 \otimes \xi_2 \otimes \psi_2 \rangle \stackrel{\text{def}}{=} \langle (\phi_2^*\phi_1)\xi_1(\psi_1\psi_2^*), \xi_2 \rangle,$$

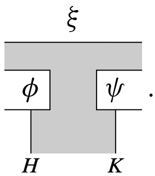
see [5, 31, 33, 42]. In the above equation, we have written the action of ψ_i on the right, which means that $\psi_1\psi_2^*$ stands for the composite

$$L^2(A) \xrightarrow{\psi_1} K \xrightarrow{\psi_2^*} L^2(A).$$

The image in the Connes fusion of an element

$\phi \otimes \xi \otimes \psi =$ 

is equal to



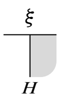
Strictly speaking, the latter picture refers to the morphism

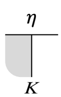
$$\mathbb{C} \xrightarrow{\xi} L^2 A \cong L^2 A \boxtimes_A L^2 A \xrightarrow{\phi \boxtimes \psi} H \boxtimes_A K,$$

but we can always identify a map from $\mathbb{C}$ to some vector space with the vector that is the image of 1 under that map.

Remark 3.3. It is useful to note that the completion map from (3.2) to $H \boxtimes_A K$ factors through both $H \otimes_A \hom({}_A L^2(A), {}_A K)$ and $\hom(L^2(A)_A, H_A) \otimes_A K$. The Hilbert space $H \boxtimes_A K$ therefore admits two alternative asymmetric definitions, as completions of either of those tensor products.

Remark 3.4. A pair of vectors $\xi \in H_A$, $\eta \in {}_A K$ does not represent anything in $H \boxtimes_A K$. This is nicely reflected by the fact that it is not possible to assemble the pictures



and


into a meaningful diagram.

Remark 3.5. There exist two algebraic alternatives to von Neumann bimodules and Connes fusion. In both cases, the Connes fusion is replaced by a simpler, purely algebraic operation.

In the first alternative, bimodules are replaced by homomorphism (often endomorphisms) of von Neumann algebras, and one usually restricts attention to type III factors [21, 22]. In this case, Connes fusion becomes merely the composition of homomorphisms. The translation from homomorphisms back to bimodules is as follows: given a homomorphism $\varphi\colon A \to B$, one precomposes the left action on $L^2(B)$ by φ to get a bimodule ${}_\varphi L^2(B)$. Given a second homomorphism $\psi\colon B \to C$, there is a canonical isomorphism ${}_\varphi L^2(B) \boxtimes_B {}_\psi L^2(C) \cong {}_{\psi\circ\varphi} L^2(C)$ of A-C-bimodules.

The second alternative has been pointed out by Thom [41]: the functor

$$ {}_A H_B \longmapsto \hom(L^2(B)_B, H_B) $$

provides an equivalence between the bicategory of dualizable bimodules (Definition 4.4) that are topologically finitely generated (i.e., for which there is a finite set that spans a dense submodule) both as left and as right modules, and the bicategory of algebraic bimodules (i.e., no Hilbert space structure) that are finitely generated projective both as left and as right modules. Under that equivalence, Connes fusion corresponds to the usual algebraic tensor product. Note that this does not provide a description of all dualizable bimodules. For example, the bimodule ${}_{\mathbf{B}(H)}H_{\mathbb{C}}$ is dualizable (it is even a Morita equivalence), but the corresponding algebraic bimodule is certainly not finitely generated.

Lemma 3.6. *Let H be a Hilbert space. View its complex conjugate $\bar{H}$ as a right $\mathbf{B}(H)$-module by*

$$ \bar{\xi}a \overset{\text{def}}{=} \overline{a^*\xi}. $$

Then there is a canonical isomorphism $\bar{H} \boxtimes_{\mathbf{B}(H)} H \cong \mathbb{C}$.

Proof. The Hilbert space $L^2(\mathbf{B}(H))$ is canonically isomorphic to the space of Hilbert-Schmidt operators on H, that is to $H \otimes \bar{H}$; see Example 2.9. Following Remark 3.3, the Connes fusion $\bar{H} \boxtimes_{\mathbf{B}(H)} H$ is obtained from

$$ \hom((H \otimes \bar{H})_{\mathbf{B}(H)}, \bar{H}_{\mathbf{B}(H)}) \otimes_{\mathbf{B}(H)} H \tag{3.7} $$

by completing it with respect to the inner product

$$ \langle \phi_1 \otimes \xi_1, \phi_2 \otimes \xi_2 \rangle \overset{\text{def}}{=} \langle (\phi_2^* \phi_1)\xi_1, \xi_2 \rangle. $$

There is an isomorphism

$$ \begin{aligned} \bar{H} &\longrightarrow \hom((H \otimes \bar{H})_{\mathbf{B}(H)}, \bar{H}_{\mathbf{B}(H)}) \\ \bar{\eta} &\longmapsto (\xi \otimes \bar{\zeta} \mapsto \overline{\langle \eta, \xi \rangle \zeta}). \end{aligned} $$

Applying the inverse of this isomorphism to the first term of (3.7), we obtain the vector space $\bar{H} \otimes_{\mathbf{B}(H)} H$ with inner product

$$ \langle \bar{\eta}_1 \otimes \xi_1, \bar{\eta}_2 \otimes \xi_2 \rangle \overset{\text{def}}{=} \langle \eta_2 \overline{\langle \eta_1, \xi_1 \rangle}, \xi_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \eta_2, \xi_2 \rangle. $$

The map $\bar{\eta} \otimes \xi \mapsto \langle \xi, \eta \rangle : \bar{H} \otimes_{\mathbf{B}(H)} H \to \mathbb{C}$ is therefore a unitary isomorphism. □

Remark 3.8. The functor $H \boxtimes_A -$ can be characterized by the existence of a right A-module isomorphism $H \boxtimes_A L^2(A) \cong H$ (see [36]).

Connes fusion shares the formal properties of the usual algebraic tensor product:

Proposition 3.9 ([20]). *There is a bicategory whose objects are von Neumann algebras, whose arrows are bimodules, and whose 2-morphisms are maps of bimodules. The composition of arrows is given by Connes fusion.*

Spatial tensor product of von Neumann algebras and tensor product of Hilbert spaces provides a symmetric tensor product on this bicategory, but since the formal definition of a symmetric monoidal bicategory is somewhat lengthy, we do not pursue this in detail here.

The invertible arrows of this bicategory are called *Morita equivalences*, and have the following alternative characterization:

Proposition 3.10. *A bimodule ${}_AH_B$ is invertible with respect to Connes fusion if and only if the two algebras act faithfully, and*

$$B' \stackrel{\text{def}}{=} \{x \in \mathbf{B}(H) \mid [x, B] = 0\} = A.$$

In that case, the inverse bimodule is given by the complex conjugate Hilbert space $\bar{H}$, with actions

$$b\bar{\xi} \stackrel{\text{def}}{=} \overline{\xi b^*} \quad \textit{and} \quad \bar{\xi} a \stackrel{\text{def}}{=} \overline{a^* \xi}.$$

Proof. We first assume that the two actions are faithful and that $B' = A$. Using a unitary A-module identification $H \otimes \ell^2 \cong L^2 A \otimes \ell^2$, we get isomorphisms

$$\begin{aligned} {}_AH \boxtimes_B \bar{H}_A &\cong {}_A(H \otimes \ell^2) \boxtimes_{B \bar{\otimes} \mathbf{B}(\ell^2)} \overline{H \otimes \ell^2}_A \\ &\cong {}_A(L^2 A \otimes \ell^2) \boxtimes_{A \bar{\otimes} \mathbf{B}(\ell^2)} \overline{L^2 A \otimes \ell^2}_A \\ &\cong {}_AL^2 A \boxtimes_A \overline{L^2 A}_A \cong {}_AL^2 A \boxtimes_A L^2 A_A \cong {}_AL^2 A_A. \end{aligned} \tag{3.11}$$

The first isomorphism follows from Lemma 3.6, and the fourth one is given by the modular conjugation on $L^2 A$. Similarly, we have ${}_B\bar{H} \boxtimes_A H_B \cong {}_BL^2(B)_B$, and so ${}_AH_B$ is invertible. Conversely, if ${}_AH \boxtimes_B \bar{H}_A \cong {}_AL^2 A_A$, then the A-action is faithful and we have

$$B' \subset \operatorname{End}(H \boxtimes_B \bar{H}_A) = \operatorname{End}(L^2 A_A) = A,$$

from which $B' = A$ follows. Similarly, the faithfulness of the right B-action follows from the isomorphism ${}_B\bar{H} \boxtimes_A H_B \cong {}_BL^2(B)_B$. □

Lemma 3.12. *Let ${}_AH$ be a faithful A-module and let $f: K_A \to L_A$ be an A-linear map. Then f is injective if and only if $f \otimes 1_H : K \boxtimes_A H \to L \boxtimes_A H$ is injective.*

Proof. Pick an A-module identification $H \otimes \ell^2 \cong L^2A \otimes \ell^2$. We then have

$$\begin{aligned} f \text{ is injective } &\iff K \boxtimes_A L^2A \xrightarrow{f\otimes 1} L \boxtimes_A L^2A \text{ is injective} \\ &\iff K \boxtimes_A L^2A \otimes \ell^2 \xrightarrow{f\otimes 1\otimes 1} L \boxtimes_A L^2A \otimes \ell^2 \text{ is injective} \\ &\iff K \boxtimes_A H \otimes \ell^2 \xrightarrow{f\otimes 1\otimes 1} L \boxtimes_A H \otimes \ell^2 \text{ is injective} \\ &\iff K \boxtimes_A H \xrightarrow{f\otimes 1} L \boxtimes_A H \text{ is injective.} \qquad \square \end{aligned}$$

Remark 3.13. The construction of ${}_AH \boxtimes_B \bar{H}_A \cong {}_AL^2A_A$ in (3.11) used the choice of an A-linear unitary $x: H \otimes \ell^2 \cong L^2A \otimes \ell^2$. Nevertheless, we claim that (3.11) is canonical. Note first that x enters (3.11) only through the isomorphism

$$ {}_A(H \otimes \ell^2) \boxtimes_{B\bar{\otimes}\mathbf{B}(\ell^2)} \overline{H \otimes \ell^2}_A \cong {}_A(L^2A \otimes \ell^2) \boxtimes_{A\bar{\otimes}\mathbf{B}(\ell^2)} \overline{L^2A \otimes \ell^2}_A. \tag{3.14}$$

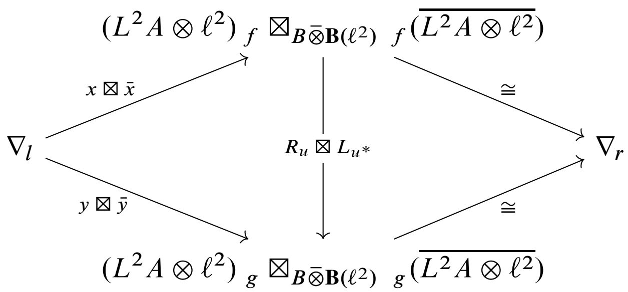
In order to understand (3.14) we provide a bit more notation. Conjugation by x yields an isomorphism $f: B\bar{\otimes}\mathbf{B}(\ell^2) \to A\bar{\otimes}\mathbf{B}(\ell^2)$. Using this, x can be viewed as an isomorphism of A-$B\bar{\otimes}\mathbf{B}(\ell^2)$-bimodules $H \otimes \ell^2 \cong (L^2A \otimes \ell^2)_f$, where the right action of $B\bar{\otimes}\mathbf{B}(\ell^2)$ on $L^2A \otimes \ell^2$ is defined by f. Similarly, the complex conjugate $\bar{x}$ yields an isomorphism of $B\bar{\otimes}\mathbf{B}(\ell^2)$-$A$-bimodules $\overline{H \otimes \ell^2} \cong {}_f(\overline{L^2A \otimes \ell^2})$. The map (3.14) is then the composition of $x \boxtimes \bar{x}$ with the isomorphism

$$(L^2A \otimes \ell^2)_f \boxtimes_{B\bar{\otimes}\mathbf{B}(\ell^2)} {}_f(\overline{L^2A \otimes \ell^2}) \cong (L^2A \otimes \ell^2) \boxtimes_{A\bar{\otimes}\mathbf{B}(\ell^2)} \overline{L^2A \otimes \ell^2}$$

that sends $\varphi \otimes \xi \otimes \psi$ to $(\varphi \circ L^2(f^{-1})) \otimes L^2(f)\xi \otimes (\psi \circ L^2(f^{-1}))$. Suppose that $y: H \otimes \ell^2 \cong L^2A \otimes \ell^2$ is another left A-module identification. Conjugation by y yields a second isomorphism $g: B\bar{\otimes}\mathbf{B}(\ell^2) \to A\bar{\otimes}\mathbf{B}(\ell^2)$. Now,

$$yx^*: L^2A \otimes \ell^2 \longrightarrow L^2A \otimes \ell^2$$

is left A-linear, and so there is a unitary $u \in A\bar{\otimes}\mathbf{B}(\ell^2)$ whose right action R_u on $L^2A \otimes \ell^2$ is yx^*. The left action L_{u^*} on $\overline{L^2A \otimes \ell^2}$ is then given by $\bar{y}\bar{x}^*$. Let also $v \in B\bar{\otimes}\mathbf{B}(\ell^2)$ be such that $R_v = x^*y$. We then have $f(v) = u$, and $g^{-1}f = \mathrm{ad}(v)$. Altogether, the two maps that we are trying to compare are along the top and along the bottom of the following diagram



where

$$\nabla_l \overset{\text{def}}{=} {}_A(H \otimes \ell^2) \boxtimes_{B\bar{\otimes}\mathbf{B}(\ell^2)} \overline{H \otimes \ell^2}_A$$

and

$$\nabla_r \overset{\text{def}}{=} {}_A(L^2A \otimes \ell^2) \boxtimes_{A\bar{\otimes}\mathbf{B}(\ell^2)} \overline{L^2A \otimes \ell^2}_A$$

It is not hard to check that the left triangle commutes. For the commutativity of the right triangle, we show that the vertical map $\varphi \otimes \xi \otimes \psi \mapsto (R_u \circ \varphi) \otimes \xi \otimes (L_{u^*} \circ \psi)$ agrees with the map that goes down the right side of the diagram. Indeed, recalling from (2.5) that $L^2(g^{-1}f) = L^2(\mathrm{ad}(v)) = L_v R_{v^*}$, that map is

$$\begin{aligned} \varphi \otimes \xi \otimes \psi \longmapsto &(\varphi \circ L^2(f^{-1}g)) \otimes L^2(g^{-1}f)\xi \otimes (\psi \circ L^2(f^{-1}g)) \\ &= (\varphi \circ L_{v^*} R_v) \otimes v\xi v^* \otimes (\psi \circ L_{v^*} R_v) \\ &= (\varphi \circ R_v) \otimes \xi \otimes (\psi \circ L_{v^*}) = (R_u \circ \varphi) \otimes \xi \otimes (L_{u^*} \circ \psi). \end{aligned}$$

It follows that (3.11) is independent of x.

4. Dualizable bimodules

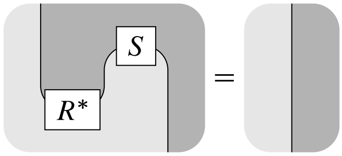
A von Neumann algebra whose center is one dimensional is called a *factor*. A von Neumann algebra has finite-dimensional center if and only if it is a finite direct sum of factors. Given an A-B-bimodule H over von Neumann algebras with finite-dimensional center, we say that a B-A-bimodule $\bar{H}$ is dual to H if it comes equipped with maps

$$R\colon {}_AL^2(A)_A \longrightarrow {}_AH \boxtimes_B \bar{H}_A, \tag{4.1a}$$

and

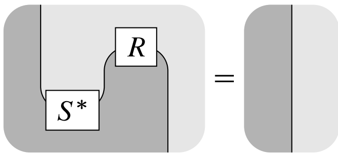
$$S\colon {}_BL^2(B)_B \longrightarrow {}_B\bar{H} \boxtimes_A H_B \tag{4.1b}$$

subject to the duality equations $(R^* \otimes 1)(1 \otimes S) = 1$, $(S^* \otimes 1)(1 \otimes R) = 1$, and to the normalization condition $R^*(pxq \otimes 1)R = S^*(1 \otimes pxq)S$ for all $x \in \mathrm{End}({}_A H_B)$ and all minimal central projections $p \in Z(A)$ and $q \in Z(B)$. The first two conditions are classical [27]. The latter was inspired by [25, Lemma 3.9]. The above equations are best depicted using string diagrams. The duality equations are given by



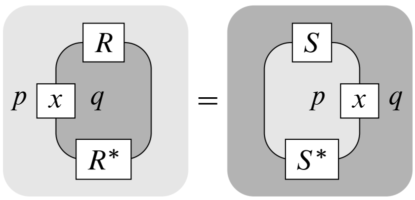
(4.2a)

and



(4.2b)

and the normalization condition is



(4.3)

The two shades stand for the algebras A and B, and the lines correspond to the bimodules ${}_A H_B$ and ${}_B \bar{H}_A$. Note that the two sides of (4.3) are in $p\mathrm{End}({}_A L^2(A)_A) \cong pZ(A) \cong \mathbb{C}$ and $q\mathrm{End}({}_B L^2(B)_B) \cong qZ(B) \cong \mathbb{C}$, respectively, and so it makes sense to ask for them to be equal.

Definition 4.4. A bimodule whose dual module exists is called *dualizable*.

We will show later, in Corollary 6.12, that the dual of a dualizable bimodule is canonically isomorphic to the complex conjugate of the bimodule. For the time being, we now reserve the notation ${}_B \bar{H}_A$ for the dual.

Remark 4.5. In the literature, the term *dual* typically refers to a solution of (4.2) only. (When the conditions (4.2) are re-expressed purely in terms of R and S^* the triple $(\bar{H}, R, S^*)$ is called a right dual, and when the conditions are re-expressed in terms of R^* and S the triple $(\bar{H}, S, R^*)$ is called a left dual.) Such a dual, if it exists, is well defined up to unique isomorphism. However, in our Hilbert space context, having an object that is well defined up to unique isomorphism is not sufficient, as the isomorphism might fail to be unitary. Condition (4.3) is added to ensure that the dual is well defined up to unique unitary isomorphism – see Theorem 4.22.

Lemma 4.6. *Let ${}_AH_B$ and ${}_BK_A$ be irreducible bimodules. If H is dualizable, then*

$$\hom_{A,A}(L^2(A), H \boxtimes_B K) \cong \begin{cases} \mathbb{C} & \text{if } {}_B\bar{H}_A \cong {}_BK_A, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The map $f \mapsto (S^* \otimes 1)(1 \otimes f)$ is an isomorphism between the vector spaces $\hom({}_AL^2(A)_A, {}_AH \boxtimes_B K_A)$ and $\hom({}_B\bar{H}_A, {}_BK_A)$. □

We will see later, in Lemma 4.20, that given two A-B-bimodules, their direct sum is dualizable if and only if they are both dualizable. One direction is straightforward, and is given presently as Lemma 4.7. Similarly, given a non-zero A-B-bimodule and a non-zero B-C-bimodule, their Connes fusion is dualizable if and only if they are both dualizable. Again one direction is easier, and is given here as Lemma 4.8. The other direction is established in Corollary 7.9.

Lemma 4.7. *Let ${}_AH_B$ and ${}_AK_B$ be dualizable bimodules, with respective structure maps R, S, $\tilde{R}$, and $\tilde{S}$. Then ${}_A(H \oplus K)_B$ is dualizable, with dual ${}_A(\bar{H} \oplus \bar{K})_B$, and structure maps*

$$\begin{pmatrix} R \\ 0 \\ 0 \\ \tilde{R} \end{pmatrix} : L^2(A) \longrightarrow (H \oplus K) \underset{B}{\boxtimes} (\bar{H} \oplus \bar{K}),$$

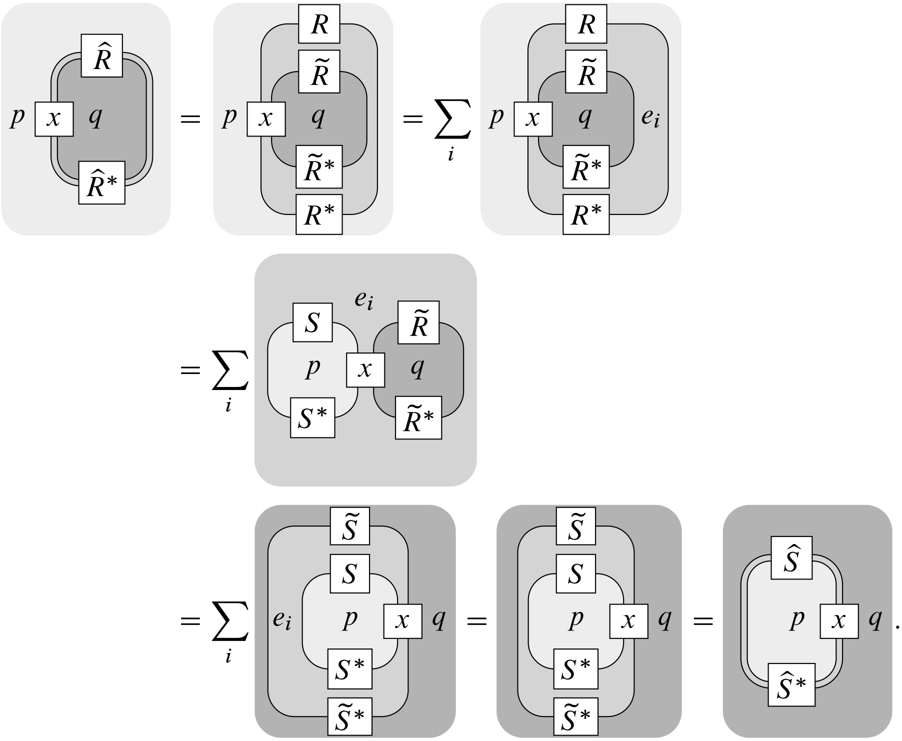
and

$$\begin{pmatrix} S \\ 0 \\ 0 \\ \tilde{S} \end{pmatrix} : L^2(B) \longrightarrow (\bar{H} \oplus \bar{K}) \underset{A}{\boxtimes} (H \oplus K).$$

Lemma 4.8. *Let ${}_AH_B$ and ${}_BK_C$ be dualizable bimodules, with respective structure maps R, S, and $\tilde{R}$, $\tilde{S}$. Their fusion ${}_AH \boxtimes_B K_C$ is then also dualizable, with dual ${}_C\bar{K} \boxtimes_B \bar{H}_A$, and structure maps*

$$\hat{R} \stackrel{\text{def}}{=} (1 \otimes \tilde{R} \otimes 1)R \quad \text{and} \quad \hat{S} \stackrel{\text{def}}{=} (1 \otimes S \otimes 1)\tilde{S}.$$

Proof. The duality equations (4.2) for $\hat{R}$ and $\hat{S}$ are straightforward. To verify the normalization condition (4.3), we make use of the graphical calculus introduced earlier:



Here, $e_i \in Z(B)$ are the minimal central projections of B. The shades correspond to the algebras A, B, C, and the lines stand for the bimodules H, $\bar{H}$, K, and $\bar{K}$. ☐

We henceforth often abbreviate the maps

$$R\colon {}_AL^2(A)_A \longrightarrow {}_AH \boxtimes_B \bar{H}_A$$

and

$$S\colon {}_BL^2(B)_B \longrightarrow {}_B\bar{H} \boxtimes_A H_B$$

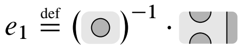
as

 and 

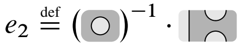
respectively. We will show, in Theorem 4.12, that a bimodule between von Neumann algebras with finite-dimensional centers is “non-normalized dualizable" if and only if it is dualizable. We first record two lemmas regarding consequences of the duality equations (4.2).

Lemma 4.9. *Let ${}_AH_B$ be a non-zero bimodule between factors. If R and S are maps as in* (4.1) *satisfying* (4.2)*, then R and S are injective and $(R^*R)(S^*S) \geq 1$.*

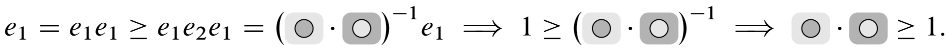
Proof. The expressions $R^*R =$  and $S^*S =$  are in $\mathbb{C}$ (in fact in $\mathbb{R}$) because A and B are factors. As H is non-zero and A and B are factors, H is faithful, both as an A-module and a B^{op}-module. By (4.2) and Lemma 3.12, this implies that S and R are injective. In particular,  and  are nonzero. Letting



and



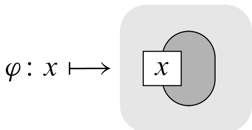
(these are the Jones projections), we have



$\square$

The next lemma is similar to [25, Lemma 3.2].

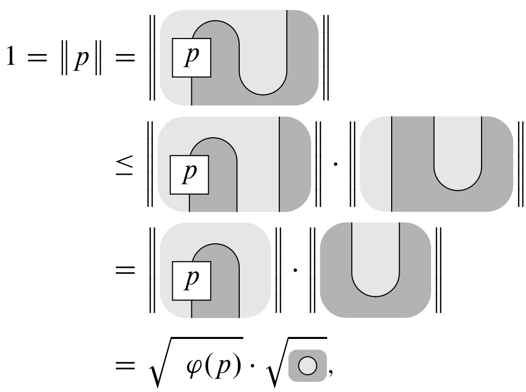
Lemma 4.10. *Let ${}_AH_B$ be a bimodule between factors. If there exist maps R and S as in* (4.1) *satisfying* (4.2)*, then ${}_AH_B$ is a finite direct sum of irreducible bimodules; its algebra of bimodule endomorphisms is therefore finite-dimensional. Moreover, the* (*non-normalized*) *state $\varphi \colon \mathrm{End}({}_AH_B) \to \mathbb{C}$ given by*



(4.11)

is faithful.

Proof. For any non-zero projection $p \in \mathrm{End}({}_AH_B)$, we have



where the last step follows from the general identity $\|a^*a\| = \|a\|^2$. Let

$$c \stackrel{\text{def}}{=} ($$  $$)^{-1}.$$

By the above estimate, we have $\varphi(p) \geq c$ for any non-zero projection p. In particular, φ is faithful. If H failed to be a finite direct sum of irreducible bimodules, we could pick countably many non-zero mutually orthogonal projections $p_n \in \mathrm{End}({}_AH_B)$, and get

$$\varphi(1) > \varphi\Big(\sum_{n=1}^{N} p_n\Big) = \sum_{n=1}^{N} \varphi(p_n) \geq \sum_{n=1}^{N} c = Nc$$

for every N. This is clearly impossible. Our bimodule is therefore a finite direct sum of irreducible ones and its endomorphism algebra is finite-dimensional. □

We can now prove that a bimodule that admits a not-necessarily normalized dual in fact admits a normalized dual:

Theorem 4.12. *Let ${}_AH_B$ and ${}_B\bar{H}_A$ be bimodules between von Neumann algebras with finite-dimensional center, and let*

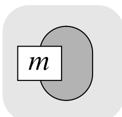
$$\tilde{R}\colon {}_AL^2(A)_A \longrightarrow {}_AH \boxtimes_B \bar{H}_A \tag{4.13a}$$

and

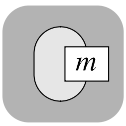
$$\tilde{S}\colon {}_BL^2(B)_B \longrightarrow {}_B\bar{H} \boxtimes_A H_B \tag{4.13b}$$

be bimodule maps satisfying (4.2). Then it is possible to find new maps R and S as in (4.1) that satisfy both (4.2) and (4.3).

Proof. We first assume that A and B are factors. For this proof, we write  for $\tilde{R}$ and  for $\tilde{S}$, and let $\varphi, \psi : \mathrm{End}({}_AH_B) \to \mathbb{C}$ be given by

$\varphi : m \longmapsto$ 

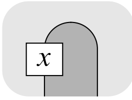
and

$\psi : m \longmapsto$ 

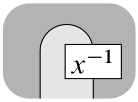
.

The state φ is faithful by the previous lemma, and so is ψ for similar reasons. Pick a trace $\tau\colon \mathrm{End}({}_A H_B) \to \mathbb{C}$; one exists because the algebra is finite-dimensional. Let $a, b \in \mathrm{End}({}_A H_B)$ be the unique solutions to the equations $\varphi = a\tau$ and $\psi = b\tau$; here, we use the action of the algebra $\mathrm{End}({}_A H_B)$ on its L^1-space, as introduced in Section 2. Since φ and ψ are positive and faithful, a and b are positive and invertible.

The new structure maps R and S are given in terms of the old ones $\tilde{R}$ and $\tilde{S}$ by

$$R \stackrel{\text{def}}{=} (x \otimes 1)\tilde{R} =$$ 

and

$$S \stackrel{\text{def}}{=} (1 \otimes x^{-1})\tilde{S} =$$ 

for some appropriately chosen positive element $x \in \mathrm{End}({}_A H_B)$. Clearly R and S satisfy the duality equations (4.2). To ensure that they also satisfy the normalization equation (4.3), the element x needs to satisfy $\varphi(xyx) = \psi(x^{-1}yx^{-1})$ for all $y \in \mathrm{End}({}_A H_B)$, which is to say $x\varphi x = x^{-1}\psi x^{-1}$ or, equivalently, $xax = x^{-1}bx^{-1}$. That equation has a unique positive solution:[3]

$$\begin{aligned} xax = x^{-1}bx^{-1} &\iff x^2ax^2 = b \\ &\iff \sqrt{a}x^2ax^2\sqrt{a} = \sqrt{a}b\sqrt{a} \\ &\iff \sqrt{a}x^2\sqrt{a} = \sqrt{\sqrt{a}b\sqrt{a}} \\ &\iff x^2 = \sqrt{a}^{-1}\sqrt{\sqrt{a}b\sqrt{a}}\sqrt{a}^{-1} \\ &\iff x = \sqrt{\sqrt{a}^{-1}\sqrt{\sqrt{a}b\sqrt{a}}\sqrt{a}^{-1}}. \end{aligned}$$

When $A = \bigoplus A_i$ and $B = \bigoplus B_j$ are direct sums of factors, then we can write H as a direct sum of A_i-B_j-bimodules $H = \bigoplus H_{ij}$, and similarly $\bar{H} = \bigoplus \bar{H}_{ji}$. We also have

$$L^2 A \cong \bigoplus L^2 A_i \quad \text{and} \quad L^2 B \cong \bigoplus L^2 B_j$$

by Lemma 2.7. The maps (4.13) induce structure maps

$$\tilde{R}_{ij}\colon L^2 A_i \longrightarrow H_{ij} \boxtimes_{B_j} \bar{H}_{ji}$$

and

$$\tilde{S}_{ij}\colon L^2 B_j \longrightarrow \bar{H}_{ji} \boxtimes_{A_i} H_{ij}$$

[3] Courtesy of http://mathoverflow.net/questions/70838.

to which we can apply the above argument and get

$$R_{ij}:{}_{A_i}L^2(A_i)_{A_i} \longrightarrow {}_{A_i}H_{ij} \boxtimes_{B_j} \bar{H}_{ji\,A_i}$$

and

$$S_{ij}:{}_{B_j}L^2(B_j)_{B_j} \longrightarrow {}_{B_j}\bar{H}_{ji} \boxtimes_{A_i} H_{ij\,B_j}$$

subject to (4.2) and (4.3).

The desired maps R and S are then given by

$$L^2A \cong \bigoplus_i L^2A_i \xrightarrow{\oplus_{ij} R_{ij}} \bigoplus_{ij}(H_{ij} \underset{B_j}{\boxtimes} \bar{H}_{ji}) \subset \bigoplus_{ijk}(H_{ij} \underset{B_j}{\boxtimes} \bar{H}_{jk}) \cong H \boxtimes_B \bar{H}$$

and

$$L^2B \cong \bigoplus_j L^2B_j \xrightarrow{\oplus_{ij} S_{ij}} \bigoplus_{ij}(\bar{H}_{ji} \underset{A_i}{\boxtimes} H_{ij}) \subset \bigoplus_{lij}(\bar{H}_{li} \underset{A_i}{\boxtimes} H_{ij}) \cong \bar{H} \boxtimes_A H. \qquad \square$$

Remark 4.14. We will see later, in Proposition 7.17, that when H is irreducible the mere existence of non-zero maps

$$\tilde{R}\colon L^2(A) \longrightarrow H \boxtimes_B \bar{H}$$

and

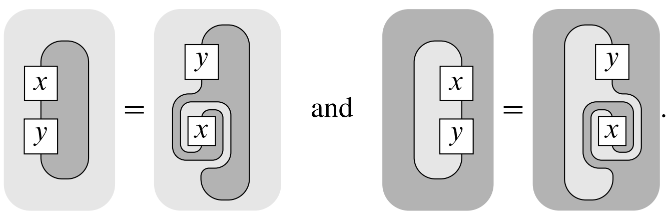
$$\tilde{S}\colon L^2(B) \longrightarrow \bar{H} \boxtimes_A H$$

implies that ${}_AH_B$ is dualizable.

We now discuss two lemmas that we will need in order to prove, in Theorem 4.22, that the dual is well defined up to unique unitary isomorphism.

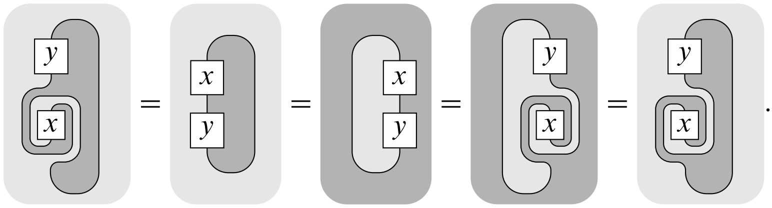
Lemma 4.15. *Let A and B be factors, and let ${}_AH_B$ be a dualizable bimodule, with structure maps R and S subject to* (4.2) *and* (4.3)*. Then the state φ defined in* (4.11) *is a trace.*

Proof. By a few applications of (4.2) and some planar isotopies, we get

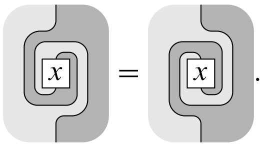


(4.16)

Combining these equations with (4.3) yields



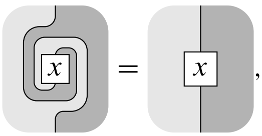
The latter being true for any $y \in \mathrm{End}({}_A H_B)$ and the state φ being faithful by Lemma 4.10, it follows that

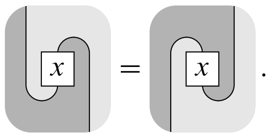
$\hat{x} \stackrel{\text{def}}{=}$ 

Equivalently, the map $x \mapsto \hat{x}$ is an involution.

As in the proof of the previous lemma, pick a trace τ and a positive invertible element a such that $\varphi = a\tau$. Our goal is to show that φ is a trace; this is true provided a is central. Equation (4.16) implies $a\hat{x} = xa$ for all x. Equivalently, we have $\hat{x} = a^{-1}xa$. Because the map $x \mapsto \hat{x}$ is an involution, we have $x = \hat{\hat{x}} = a^{-2}xa^2$. Since a is positive and its square is central, a is also central. ☐

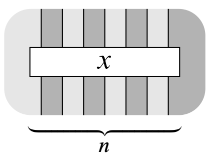
As a corollary of the above proof, we see

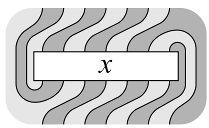


, and thus also $\bar{x} \stackrel{\text{def}}{=}$ 

. (4.17)

Remark 4.18. The first equation in (4.17) is essentially the same as [12, Theorem 4.1.18] or [13, Corollaries 2.35 and 2.39], which states that the nth power of the operation

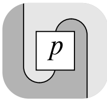
ρ_n : 

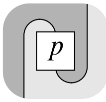
$\longmapsto$ 

(4.19)

is the identity. One should note that Jones' rotation ρ_n does not always agree with our way of interpreting figure (4.19). It agrees when the type II_1 subfactor $A \subset B$ is extremal, that is, when the normalized traces on A' and B coincide on $A' \cap B$ or, equivalently, when the minimal conditional expectation $B \to A$ is equal to the trace preserving one. See also Warning 5.11.

Lemma 4.20. *Let $_AH_B$ be a dualizable bimodule with dual $_B\bar{H}_A$, and let $p \in$* $\mathrm{End}(_AH_B)$ *be a projection. The A-B-bimodule pH is then dualizable and its dual is given by $\bar{p}\bar{H}$, where*

$$\bar{p} \stackrel{\text{def}}{=}$$ 

$=$ 

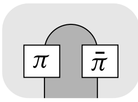
$$\in \mathrm{End}(_B\bar{H}_A). \qquad (4.21)$$

Moreover, its statistical dimension (see Definition 5.1) is given by

$$\dim(pH) =$$ 

.

Proof. Let $\pi\colon H \twoheadrightarrow pH$, $\bar{\pi}\colon \bar{H} \twoheadrightarrow \bar{p}\bar{H}$ be the orthogonal projections, so that $p = \pi^*\pi$ and $\bar{p} = \bar{\pi}^*\bar{\pi}$. The maps



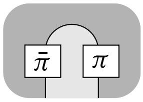
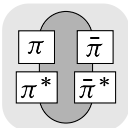
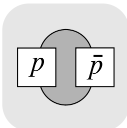
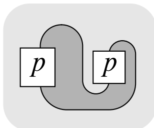
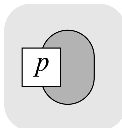
and


exhibit $\bar{p}\bar{H}$ as dual to pH. The statistical dimension is therefore given by

$$\dim\big(pH\big) =$$ 

$=$ 

$=$ 

$=$ 

. □

Theorem 4.22. *Let $_AH_B$ be a dualizable bimodule. Then its dual $(_B\bar{H}_A, R, S)$ is well defined up to unique unitary isomorphism.*

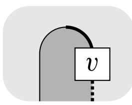
Proof. Let $_B\bar{H}_A$ and $_B\bar{H}'_A$ be two bimodules that are dual to $_AH_B$, with respective structure maps R, S, R', S':

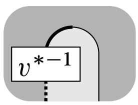
$R =$ , $S =$ , $R' =$ , $S' =$ .

Here, thick lines represent $\bar{H}$, and thick dotted lines represent $\bar{H}'$. The isomorphism between $\bar{H}$ and $\bar{H}'$ is given by

$$v \stackrel{\text{def}}{=} (S^* \otimes 1)(1 \otimes R') =$$ 
.

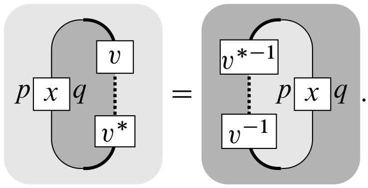
This isomorphism is certainly the unique isomorphism intertwining R and R', and S and S'. Our goal is to show that v is unitary. In other words, we need to show that v is equal to v^{*-1}; note that  is the inverse of v^*. We can rewrite R' and S' as

$R' =$ 

, $S' =$ 

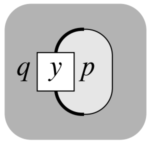
.

Equation (4.3) for R' and S' then reads



(4.23)

Given minimal central projections $p \in A$ and $q \in B$, the map

$\mathrm{tr}_{pq} : y \longmapsto$ 

is a trace on $\mathrm{End}({}_B\bar{H}_A)$, as can be seen by applying Lemma 4.15 to the bimodule ${}_{qB}(q\bar{H}p)_{pA}$. Applying Lemma 4.10 to each summand in the decomposition $\bar{H} = \bigoplus_{pq} q\bar{H}p$, and using the fact that $\mathrm{End}(\bar{H}) = \bigoplus_{pq} \mathrm{End}(q\bar{H}p)$, it follows that the traces tr_{pq} are jointly faithful. That is, given a positive element y, there exists at least one tr_{pq} such that $\mathrm{tr}_{pq}(y) \neq 0$. Letting $\bar{x}$ be as in (4.17), equation (4.23) implies

$$\mathrm{tr}_{pq}(v^* v \bar{x}) = \mathrm{tr}_{pq}(v^{-1} v^{*-1} \bar{x}), \quad x \in \mathrm{End}({}_A H_B).$$

This being true for all p, q, it follows that $v^* v = v^{-1} v^{*-1}$. In other words, $v^* v = (v^* v)^{-1}$. Since $v^* v$ is positive, we must have $v^* v = 1$. □

5. Statistical dimension and index

The following definition is well known. Our approach follows [25].

Definition 5.1. If A and B are factors, then the *statistical dimension* of a dualizable bimodule ${}_A H_B$ is given by

$$\dim({}_A H_B) \stackrel{\text{def}}{=} R^* R = S^* S \in \mathbb{R}_{\geq 0}.$$

For non-dualizable bimodules, one simply declares $\dim({}_A H_B)$ to be ∞.

The basic properties of the statistical dimension can be found in many places [18, 19, 21, 25]. We include some proofs for completeness.

Proposition 5.2. *The statistical dimension of a non-zero bimodule ${}_AH_B$ is always ≥ 1, and is equal to 1 if and only if H is invertible. The statistical dimension is additive under direct sums, and multiplicative under Connes fusion.*[4] *It is also multiplicative under external tensor product. In other words,*

$$\dim({}_AH_B) \in \{0\} \cup [1, \infty], \text{ and it is } 0 \text{ if and only if } H = 0, \tag{5.3}$$

$$\dim({}_AH_B) = 1 \text{ if and only if } A' = B, \tag{5.4}$$

$$\dim({}_A(H \oplus K)_B) = \dim({}_AH_B) + \dim({}_AK_B), \tag{5.5}$$

$$\dim({}_AH \boxtimes_B K_C) = \dim({}_AH_B)\dim({}_BK_C), \tag{5.6}$$

$$\dim(({}_AH_B) \otimes_{\mathbb{C}} ({}_CK_D)) = \dim({}_AH_B)\dim({}_CK_D) \tag{5.7}$$

Proof. (5.3). If $H \neq 0$, then $\dim({}_AH_B) \geq 1$ by Lemma 4.9. If $H = 0$, then clearly $R^*R = 0$.

(5.4). Let e_1, e_2 be as in Lemma 4.9. If $\dim({}_AH_B) = 1$, then $e_1 = e_1e_2e_1$ and $e_2 = e_2e_1e_2$. As e_1 and e_2 are projections, the first equation implies $e_2 \geq e_1$, while the second implies $e_1 \geq e_2$. Thus $e_1 = e_2$. >From this (and a reflection along a vertical axis of the argument so far), we get  =  = . As A is a factor and ${}_AH_B \neq 0$, the latter is a faithful A-module. Lemma 4.9 implies that the projection $RR^* =$  is non-trivial. Thus, the previous equation implies  = . The map R is therefore invertible, and similarly for S. Having shown ${}_B\bar{H} \boxtimes_A H_B \cong L^2B$ and ${}_AH \boxtimes_B \bar{H}_A \cong L^2A$, the result follows from Proposition 3.10.

Conversely, if H is invertible, there exist unitary maps $\tilde{R}\colon L^2(A) \to H \boxtimes_B \bar{H}$ and $S\colon L^2(B) \to \bar{H} \boxtimes_A H$. Since ${}_AH_B$ is irreducible,

$$\lambda \stackrel{\text{def}}{=} (\tilde{R}^* \otimes 1)(1 \otimes S)$$

is a scalar, and so

$$R \stackrel{\text{def}}{=} \lambda \tilde{R}$$

and S satisfy (4.2). Again because ${}_AH_B$ is irreducible (and R and S are unitary), the normalization condition (4.3) is satisfied as well. Thus $d = R^*R = 1$.

(5.5). If either H or K is not dualizable, then both sides of (5.5) are infinite by Lemma 4.20. If they are both dualizable, then Lemma 4.7 provides a description of the duality maps for $H \oplus K$, which we can use to compute

$$\dim(H \oplus K) = \begin{pmatrix} R \\ 0 \\ 0 \\ \tilde{R} \end{pmatrix}^* \begin{pmatrix} R \\ 0 \\ 0 \\ \tilde{R} \end{pmatrix} = R^*R + \tilde{R}^*\tilde{R} = \dim(H) + \dim(K).$$

[4] For this to always be true, it is appropriate to use the convention $0 \cdot \infty = 0$.

(5.6). If both H and K are dualizable, then using the duality maps described in Lemma 4.8, we compute

$$\dim(H \boxtimes_B K) = R^*(1 \otimes \tilde{R}^* \otimes 1)(1 \otimes \tilde{R} \otimes 1)R = R^* \dim(K) R = \dim(H)\dim(K).$$

If either H or K is zero, then the equation clearly holds. The remaining case $H \neq 0$, $\dim(K) = \infty$ requires different techniques[5] and will be treated later, in Corollary 7.9.

(5.7). Apply equation (5.6) to the decomposition

$$({}_A H_B) \otimes_{\mathbb{C}} ({}_C K_D) \cong (({}_A H_B) \otimes_{\mathbb{C}} ({}_C L^2 C_C)) \underset{B \bar{\otimes} C}{\boxtimes} (({}_B L^2 B_B) \otimes_{\mathbb{C}} ({}_C K_D)). \qquad \square$$

Remark 5.8. As was shown in the celebrated papers [11, 17], equation (5.3) can be improved: the statistical dimension of a bimodule can only take values in the set $\{2\cos(\frac{\pi}{n}); n = 2, 3, 4, \ldots\} \cup [2, \infty]$.

If the von Neumann algebras $A = \bigoplus A_i$ and $B = \bigoplus B_j$ are finite direct sums of factors (in other words have finite-dimensional centers), then any A-B-bimodule H can be written as a direct sum

$$H = \bigoplus H_{ij} \tag{5.9}$$

of A_i-B_j-bimodules. The statistical dimension of ${}_A H_B$ is then best taken to be a matrix of numbers [40]:

$$\dim({}_A H_B)_{ij} \stackrel{\text{def}}{=} \dim({}_{A_i} H_{ij\,B_j}).$$

This matrix-valued statistical dimension for bimodules between von Neumann algebras with finite center satisfies the same formal properties (5.3)–(5.7) as above, provided the right hand sides of (5.6) and (5.7) are interpreted in terms of matrix and Kronecker products, respectively.

As will be shown later, in Corollary 7.14, the following definition of index is equivalent to other definitions that exist in the literature [9, 17, 18, 21, 30]:

Definition 5.10. The *index* $[B : A]$ of an inclusion of factors $\iota \colon A \to B$ is the square of the statistical dimension of ${}_A L^2 B_B$.

Warning 5.11. The above definition does not always agree with Jones' original definition [11]. It agrees if and only if the type II_1 subfactor $A \subset B$ is extremal, that is, the normalized traces on A' and B coincide on $A' \cap B$.

[5] Note that the special case $\dim(H) = 1$, $\dim(K) = \infty$ is straightforward, as fusing with an invertible bimodule certainly doesn't change the property of having a dual or not.

Let $\iota\colon A \to B$ be a subfactor. If the index $[B : A]$ is finite, we say that ι is a finite homomorphism. More generally, if A and B are von Neumann algebras with finite-dimensional centers, we say that a homomorphism $A \to B$ is finite if all the matrix entries of $\dim({}_AL^2B_B)$ are finite. Of course, this simply amounts to the following definition:

Definition 5.12. A homomorphism $A \to B$ between von Neumann algebras with finite-dimensional centers is *finite* if the associated bimodule ${}_AL^2B_B$ is dualizable.

When dealing with inclusions of von Neumann algebras with finite-dimensional center, the matrix $\dim({}_AL^2B_B)$ is much better behaved than the corresponding matrix of indices. We propose a new notation for it:

Definition 5.13. Given a finite homomorphism $f : A \to B$ between von Neumann algebras with finite-dimensional center, we let

$$[\![B : A]\!] \overset{\text{def}}{=} \dim({}_AL^2B_B)$$

denote the matrix of statistical dimensions of ${}_AL^2B_B$.

Following (5.6), the matrix of statistical dimensions satisfies

$$[\![B : A]\!][\![C : B]\!] = [\![C : A]\!]. \tag{5.14}$$

As an corollary of Lemma 4.10, we have the following result.

Lemma 5.15. *Let $f : A \to B$ be a finite homomorphism between von Neumann algebras with finite-dimensional center. Then the relative commutant of $f(A)$ in B is finite-dimensional.*

Proof. The relative commutant of $f(A)$ in B is the endomorphism algebra of the bimodule ${}_AL^2(B)_B$. Apply Lemma 4.10 to every summand in the decomposition (5.9) of that bimodule. □

Lemma 5.16. *Let ${}_AH_B$ be a bimodule between von Neumann algebras with finite-dimensional center. Assume B acts faithfully, and let $B' \supset A$ be the commutant of B on H. Then* $\dim({}_AH_B) = [\![B' : A]\!]$.

Proof. The bimodule ${}_{B'}H_B$ is a Morita equivalence, and its matrix of statistical dimensions is therefore an identity matrix. We have

$$\dim({}_AH_B) = \dim({}_AL^2B' \boxtimes_{B'} H_B) = \dim({}_AL^2B'_{B'})\dim({}_{B'}H_B) = \dim({}_AL^2B'_{B'}).$$

The last expression is the definition of the matrix $[\![B' : A]\!]$. □

Corollary 5.17. *If $A \subset B \subset \mathbf{B}(H)$ are von Neumann algebras with finite-dimensional centers, then $[\![B : A]\!] = [\![A' : B']\!]^T$. In particular, if A and B are factors, then $[B : A] = [A' : B']$.*

Proof. Let $\bar{H}$ denote the complex conjugate of H, with actions as in Proposition 3.10. Applying Lemma 5.16 twice, we have $[\![B : A]\!] = \dim({}_A H_{B'}) = \dim({}_{B'} \bar{H}_A)^T = [\![A' : B']\!]^T$. □

Lemma 5.18. *Let B be a factor, and let $A \subset B$ be a subalgebra with finite-dimensional center. Call its minimal central projections $p_1, \ldots, p_n$. Then*

$$\sum [p_i B p_i : p_i A] = \| [\![B : A]\!] \|^2,$$

where $\| \ \|$ stands for the ℓ^2-norm of a vector.

Proof. The ith entry in the vector $[\![B : A]\!] = \dim({}_A L^2 B_B)$ is by definition

$$\dim({}_{p_i A}(p_i L^2 B)_B) = \dim({}_{p_i A}(p_i L^2 B p_i)_{p_i B p_i}) = \dim({}_{p_i A} L^2 (p_i B p_i)_{p_i B p_i}),$$

where the first equality holds because ${}_B(L^2 B p_i)_{p_i B p_i}$ is an invertible bimodule, and the second one follows from Lemma 2.7. Therefore,

$$\dim({}_{p_i A}(p_i L^2 B)_B)^2 = [p_i B p_i : p_i A].$$

The results now follows by summing over all the indices i. □

For more results about statistical dimension and index, we refer the reader to [15, 17, 18, 19, 23, 24].

6. Functoriality of the L^2-space and of Connes fusion

The inner product on $L^2(A)$. We mentioned earlier that for a von Neumann algebra A, its L^2-space is a completion of the vector space $\bigoplus_{\phi \in L^1_+(A)} \mathbb{C}\sqrt{\phi}$ with respect to some pre-inner product. To define $\langle \sqrt{\phi}, \sqrt{\psi} \rangle$, one considers the function

$$f(t) \stackrel{\text{def}}{=} \phi([D\phi : D\psi]_t),$$

where $[D\phi : D\psi]_t \in A$ denotes Connes' Radon–Nikodym derivative.[6] The function f can be analytically continued from $\mathbb{R}$ to the strip $\Im\mathrm{m}(t) \in [0, 1]$, and the value of the inner product is then given by $f(i/2)$:

$$\langle \sqrt{\phi}, \sqrt{\psi} \rangle \overset{\text{def}}{=} \underset{t \to i/2}{\text{anal.cont.}} \phi([D\phi : D\psi]_t). \tag{6.1}$$

In particular, we have $\|\sqrt{\phi}\|^2 = \phi(1)$.

The cone of positive elements in $L^2 A$ is given by

$$L^2_+(A) \overset{\text{def}}{=} \{\sqrt{\phi} | \phi \in L^1_+(A)\},$$

and the two actions of A on $L^2 A$ are prescribed by

$$\langle a\sqrt{\phi}b, \sqrt{\psi} \rangle \overset{\text{def}}{=} \underset{t \to i/2}{\text{anal.cont.}} \phi([D\phi : D\psi]_t \sigma_t^{\psi}(b)a),$$

where σ_t^{ψ} is the modular flow.[7] The space $L^2 A$ is also equipped with the modular conjugation J_A, that sends $\lambda\sqrt{\phi}$ to $\bar{\lambda}\sqrt{\phi}$ for $\lambda \in \mathbb{C}$, and satisfies

$$J_A(a\xi b) = b^* J_A(\xi) a^*. \tag{6.2}$$

Altogether, the triple $(L^2(A), J_A, L^2_+(A))$ is a standard form for the von Neumann algebra A; compare [5, p.528].

The above constructions are compatible with spatial tensor product in the sense that there is a natural isomorphism $L^2(A \bar{\otimes} B) \cong L^2(A) \otimes L^2(B)$ that respects the left and right $A \bar{\otimes} B$-actions, and intertwines the modular involutions – see Example 2.10.

Remark 6.3 (The modular algebra). The construction of $L^2 A$ is best understood in the larger context of the modular algebra [29, 43] – recall Remark 2.4. The modular algebra is

$$L^* A \overset{\text{def}}{=} \bigoplus_{p \in \mathbb{C}^{\times}_{\Re\mathrm{e} \geq 0} \cup \{\infty\}} L^p A,$$

and can be represented as an algebra of unbounded operators on a Hilbert space. The product sends $L^p(A) \times L^q(A)$ to $L^{\frac{1}{1/p+1/q}}(A)$, and $L^{\infty}(A)$ is a synonym for A. Given $p \in \mathbb{C}^{\times}_{\Re\mathrm{e} \geq 0}$, then for every $\phi \in L^1_+ A$, its pth root $\phi^{1/p}$ (in the sense of functional calculus) belongs to $L^p A$. In particular, we have $\sqrt{\phi} \equiv \phi^{1/2} \in L^2(A)$.

[6] We work with a definition of the Radon–Nikodym derivative $[D\phi : D\psi]_t$ that does not require ϕ and ψ to be faithful; it satisfies $[D\phi : D\psi]_t \in \mathbf{s}_\phi A \mathbf{s}_\psi$ where $\mathbf{s}_\phi$ and $\mathbf{s}_\psi$ are the support projections of ϕ and ψ.

[7] We do not assume that ψ is faithful in defining the modular flow σ_t^{ψ}. For $a \in A$, we have $\sigma_t^{\psi}(a) \in \mathbf{s}_\psi A \mathbf{s}_\psi$.

The modular conjugation $J_A\colon L^2(A) \to L^2(A)$ is then simply the restriction of the $*$-operation on L^*A. There is also a faithful normal trace $\mathrm{Tr}\colon L^*A \to \mathbb{C}$ given by

$$\mathrm{Tr}(\phi) = \begin{cases} \phi(1) & \text{for } \phi \in L^1(A) \\ 0 & \text{for } \phi \in L^p(A),\, p \neq 1. \end{cases}$$

By definition, it satisfies $\mathrm{Tr}(\phi a) = \phi(a)$ for $\phi \in L^1 A$ and $a \in A$.

Using complex exponentiation in the algebra L^*A, the Radon–Nikodym derivative and the modular flow can be recovered[8] as

$$\begin{aligned} [D\phi : D\psi]_t &= \phi^{it}\psi^{-it} \\ \sigma_t^{\psi}(a) &= \psi^{it} a \psi^{-it} \quad (t \in \mathbb{R}). \end{aligned} \tag{6.4}$$

We can therefore rewrite the quantity that appears in the right hand side of (6.1) as

$$\phi([D\phi : D\psi]_t) = \mathrm{Tr}(\phi[D\phi : D\psi]_t) = \mathrm{Tr}(\phi\phi^{it}\psi^{-it}) = \mathrm{Tr}(\phi^{1+it}\psi^{-it}).$$

The last expression $\mathrm{Tr}(\phi^{1+it}\psi^{-it})$ can be evaluated for any t in the strip $\Im\mathrm{m}(t) \in [0,1]$, because $\Re\mathrm{e}(1+it)$ and $\Re\mathrm{e}(-it)$ are both non-negative there. Moreover, the dependence on t is analytic by [43, Corollary 2.6]. One can therefore rewrite the inner product on $L^2(A)$ as

$$\langle \sqrt{\phi}, \sqrt{\psi} \rangle = \mathrm{Tr}(\phi^{1+it}\psi^{-it})|_{t=i/2} = \mathrm{Tr}(\phi^{1/2}\psi^{1/2}),$$

and the fact that it is symmetric follows from the trace property. The inner product also admits the following alternative definition:

$$\langle \sqrt{\phi}, \sqrt{\psi} \rangle \overset{\text{def}}{=} \underset{t \to -i/2}{\text{anal.cont.}} \psi([D\phi : D\psi]_t).$$

This definition agrees with definition (6.1) because

$$\psi([D\phi : D\psi]_t) = \mathrm{Tr}(\psi\phi^{it}\psi^{-it}) = \mathrm{Tr}(\phi^{it}\psi^{1-it})$$

and

$$\mathrm{Tr}(\phi^{it}\psi^{1-it})|_{t=-i/2} = \mathrm{Tr}(\phi^{1/2}\psi^{1/2}).$$

We will need the following lemma later on in order to identify the dual of the bimodule ${}_A L^2 B_B$ associated to a finite homomorphism $A \to B$.

[8] Unfortunately, one cannot use (6.4) to *define* $[D\phi : D\psi]_t$ and σ_t^{ψ}, as the Radon–Nikodym derivative and the modular flow are needed for the construction of the modular algebra – see [43].

Lemma 6.5. *Let* $\{p_i \in A\}$ *be orthogonal projections adding up to* 1. *If* $\phi \in L^1_+(A)$ *satisfies* $p_i \phi p_j = 0$ *for all* i *and* j *with* $i \neq j$, *then* $p_i \sqrt{\phi} p_j = 0$ *for all* i *and* j *with* $i \neq j$.

Proof. Applying functional calculus to an (unbounded) operator in block diagonal form yields an operator in block diagonal form. The result follows since the modular algebra has a representation by unbounded operators [43], and $\sqrt{\phi}$ is obtained from ϕ by functional calculus. □

In our analysis of conditional expectations in section 7, we will use the following general fact relating Radon–Nikodym derivatives in different algebras – see [4, Lemma 1.4.4] and [8, Theorem 4.7]. Let $A \subset B$ be a subalgebra, and let $E: B \to A$ be a faithful completely positive normal map such that $E(axb) = aE(x)b$ for $x \in B$, $a, b \in A$; in this case,

$$[D(\phi \circ E) : D(\psi \circ E)]_t = [D\phi : D\psi]_t. \tag{6.6}$$

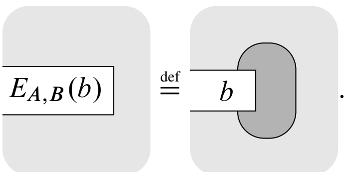
Functoriality of the L^2-space. The following theorem is closely related to some known results [15, 19]. Nevertheless, it appears to be new:

Theorem 6.7. *The assignment* $A \mapsto L^2(A)$ *defines a functor from the category of von Neumann algebras with finite-dimensional center and finite homomorphisms, to the category of Hilbert spaces and bounded linear maps.*

Proof. Given a finite homomorphism $A \to B$ between von Neumann algebras with finite-dimensional center, let $E_{A,B}: B \to A$ be the map given by

$$E_{A,B}(b)\xi \overset{\text{def}}{=} R^*(b \otimes 1)R\xi, \quad \xi \in L^2(A), \tag{6.8}$$

where $R\colon {}_AL^2(A)_A \to {}_AL^2(B) \boxtimes_B \overline{L^2(B)}_A$ is as in (4.1), and the b that appears in the right hand side of (6.8) acts by left multiplication on $L^2(B)$. Graphically, this is



As before, the two shades represent the algebras A and B, and the lines stand for the bimodule ${}_AL^2B_B$ and its dual. The fact that the box labeled $E_{A,B}(b)$ extends to the left of the picture refers to the fact that the map $E_{A,B}(b)\colon {}_AL^2A_A \to {}_AL^2A_A$ is only right A-linear.

The map (6.8) satisfies $E_{A,B}(aba') = aE_{A,B}(b)a'$ for any $a, a' \in A$ and $b \in B$. Moreover, for every sequence $A \to B \to C$ of composable arrows, we have

$$E_{A,B} \circ E_{B,C} = E_{A,C} \tag{6.9}$$

by Lemma 4.8. The map

$$L^2(f)\colon L^2(A) \longrightarrow L^2(B)$$

associated to the finite homomorphism $f\colon A \to B$ is then defined by

$$L^2(f)\colon \sqrt{\phi} \mapsto \sqrt{\phi \circ E_{A,B}}. \tag{6.10}$$

To see that this map is well defined and bounded, we exhibit a constant C such that

$$\Big\| \sum_j c_j \sqrt{\phi_j \circ E_{A,B}} \Big\|^2 \leq C \cdot \Big\| \sum_j c_j \sqrt{\phi_j} \Big\|^2, \quad c_j \in \mathbb{C}, \phi_j \in L^1_+(A).$$

Let $\{p_\alpha\}$ be the minimal central projections of A. Since $E_{A,B}(1)$ is central, we can write it as $E_{A,B}(1) = \sum_\alpha C_\alpha p_\alpha$ for some given constants C_α. Let

$$C \stackrel{\text{def}}{=} \max_\alpha C_\alpha = \|E_{A,B}(1)\|.$$

Using the shorthand notation

$$\phi_{j,\alpha} \stackrel{\text{def}}{=} \phi_j p_\alpha,$$

we then have

$$\begin{aligned}
&\| \textstyle\sum_j c_j \sqrt{\phi_j \circ E_{A,B}} \|^2 \\
&\quad = \textstyle\sum_{j,k} c_j \bar{c}_k \langle \sqrt{\phi_j \circ E_{A,B}}, \sqrt{\phi_k \circ E_{A,B}} \rangle \\
&\quad = \textstyle\sum_{j,k} c_j \bar{c}_k \underset{t \to i/2}{\text{anal.cont.}} \phi_j \circ E_{A,B}([D(\phi_j \circ E_{A,B}) : D(\phi_k \circ E_{A,B})]_t) \\
&\quad = \textstyle\sum_{j,k} c_j \bar{c}_k \underset{t \to i/2}{\text{anal.cont.}} \phi_j \circ E_{A,B}([D\phi_j : D\phi_k]_t) \\
&\quad = \textstyle\sum_{j,k} c_j \bar{c}_k \underset{t \to i/2}{\text{anal.cont.}} \phi_j (E_{A,B}(1)[D\phi_j : D\phi_k]_t) \\
&\quad = \textstyle\sum_{\alpha,j,k} c_j \bar{c}_k \underset{t \to i/2}{\text{anal.cont.}} \phi_{j,\alpha}(C_\alpha [D\phi_{j,\alpha} : D\phi_{k,\alpha}]_t) \\
&\quad = \textstyle\sum_\alpha C_\alpha \| \sum_j c_j \sqrt{\phi_{j,\alpha}} \|^2 \\
&\quad \leq C \cdot \textstyle\sum_\alpha \| \sum_j c_j \sqrt{\phi_{j,\alpha}} \|^2 \\
&\quad = C \cdot \Big\| \displaystyle\sum_j c_j \sqrt{\phi_j} \Big\|^2,
\end{aligned}$$

where the third equality follows from (6.6) and the fourth one follows from the A-linearity of $E_{A,B}$.

The compatibility of (6.10) with composition follows from (6.9). □

Remark 6.11. Given a finite homomorphism $f: A \to B$ between von Neumann algebras with finite-dimensional centers, one can also define

$$L^p(f): L^pA \longrightarrow L^pB, \quad \phi^{1/p} \longmapsto (\phi \circ E_{A,B})^{1/p}.$$

These assemble to a $*$-algebra homomorphism $\bigoplus L^pA \to \bigoplus L^pB$; see [43, Section 3].

Corollary 6.12. *Let ${}_AH_B$ be a bimodule between von Neumann algebras with finite-dimensional center. Then its dual bimodule, if it exists, is canonically isomorphic to the complex conjugate Hilbert space, with actions given by*

$$b\bar{\xi}a \stackrel{\text{def}}{=} \overline{a^*\xi b^*}.$$

Proof. Let ${}_AH_B$ be dualizable. By Lemma 4.10 and the decomposition (5.9), this bimodule is a finite direct sum of irreducible bimodules. Both duals and complex conjugates being compatible with the direct sum operation, it is enough to treat the irreducible case. We assume for simplicity that the action $\rho: A \to \mathbf{B}(H)$ is faithful. The general case follows.

Let ${}_B\bar{H}^c{}_A$ denote the complex conjugate of ${}_AH_B$, and let B' be the commutant of B on H. By Proposition 3.10, we have ${}_{B'}H \boxtimes_B \bar{H}^c{}_{B'} \cong {}_{B'}L^2(B')_{B'}$, and so

$$\begin{aligned} {}_AH \boxtimes_B \bar{H}^c{}_A &\cong {}_AL^2(B') \boxtimes_{B'} H \boxtimes_B \bar{H}^c \boxtimes_{B'} L^2(B')_A \\ &\cong {}_AL^2(B') \boxtimes_{B'} L^2(B') \boxtimes_{B'} L^2(B')_A \\ &\cong {}_AL^2(B')_A. \end{aligned}$$

By Theorem 6.7, we therefore get a map ${}_AL^2(A)_A \to {}_AH \boxtimes_B \bar{H}^c{}_A$ which is nontrivial by construction – see for instance equation (6.20). The result now follows from Lemma 4.6. □

Remark 6.13. The isomorphism between any dual and the complex conjugate bimodule constructed in the proof of Corollary 6.12 is in fact unitary. We do not include a proof – see Proposition 6.16 for a related result.

In the special case of the bimodule ${}_AL^2(B)_B$ associated to a finite homomorphism $f: A \to B$, together with a chosen dual $({}_B\overline{L^2B}_A, R, S)$, the isomorphism $\overline{L^2B} \cong \overline{L^2B}^c$ is given by

$$\overline{L^2B} \cong \overline{L^2B} \boxtimes_A L^2A \xrightarrow{1 \otimes L^2(f)} \overline{L^2B} \boxtimes_A L^2B \cong \overline{L^2B} \boxtimes_A L^2B \boxtimes_B L^2B$$
$$\xrightarrow{S^* \otimes 1} L^2B \boxtimes_B L^2B \cong L^2B \xrightarrow{J} \overline{L^2B}^c,$$

where J is the modular conjugation. This isomorphism $\overline{L^2B} \cong \overline{L^2B}^c$ is chosen so as to make the composite

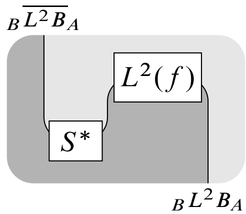
$${}_AL^2(A)_A \xrightarrow{R} {}_AL^2(B) \boxtimes_B \overline{L^2(B)}_A$$
$$\cong {}_AL^2(B) \boxtimes_B \overline{L^2(B)}^c{}_A \xrightarrow{1 \otimes J} {}_AL^2(B) \boxtimes_B L^2(B)_A \cong {}_AL^2(B)_A$$

equal to $L^2(f)$.

Instead of identifying the dual of ${}_AL^2B_B$ with ${}_B\overline{L^2B}^c{}_A$, we can identify it with ${}_BL^2B_A$, as follows. There is an isomorphism Φ between any dual of ${}_AL^2(B)_B$ and ${}_BL^2(B)_A$ given by

$$\begin{aligned} {}_B\overline{L^2B}_A &\cong {}_B\overline{L^2B} \boxtimes_A L^2A_A \xrightarrow{1 \otimes L^2(f)} {}_B\overline{L^2B} \boxtimes_A L^2B_A \\ &\cong {}_B\overline{L^2B} \boxtimes_A L^2B \boxtimes_B L^2B_A \xrightarrow{S^* \otimes 1} {}_BL^2B \boxtimes_B L^2B \cong {}_BL^2B_A. \end{aligned} \tag{6.14}$$

In graphical notation we have

$\Phi \stackrel{\text{def}}{=}$  .

The isomorphism Φ makes the following diagram commutative:

$$\begin{array}{ccc} {}_AL^2A_A & \xrightarrow{R} & {}_AL^2(B) \boxtimes_B \overline{L^2(B)}_A \\ {\scriptstyle L^2(f)}\downarrow & & \downarrow{\scriptstyle 1 \otimes \Phi} \\ {}_AL^2B_A & \xrightarrow{\cong} & {}_AL^2(B) \boxtimes_B L^2(B)_A\,. \end{array} \tag{6.15}$$

Proposition 6.16. *Let $f\colon A \to B$ be a finite homomorphism, and let $(\overline{L^2B}, R, S)$ be a chosen dual to the bimodule ${}_AL^2B_B$ associated to f. The isomorphism*

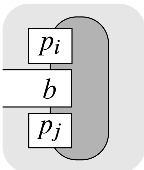
$$\Phi \overset{\text{def}}{=} (S^* \otimes 1)(1 \otimes L^2(f))$$

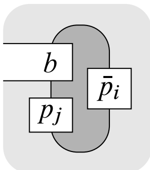
from ${}_B\overline{L^2B}_A$ to ${}_BL^2B_A$ is unitary.

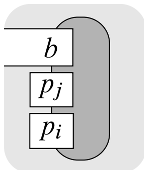
Proof. The algebra $B \cap A' = \mathrm{End}({}_AL^2B_B)$ is finite-dimensional by Lemma 4.10 and decomposition (5.9). Let $p_1, \dots, p_n \in B \cap A'$ be mutually orthogonal minimal projections adding up to 1, and let $\bar{p}_i \overset{\text{def}}{=} (S^* \otimes 1)(1 \otimes p_i \otimes 1)(1 \otimes R)$ be the dual projection defined in equation (4.21). Let $E\colon B \to A$ be as in (6.8). For every $i \neq j$ and $\phi \in L^1_+A$, the element $p_i(\phi \circ E)p_j \in L^1(B)$ is zero, as

$$(p_i(\phi \circ E)p_j)(b) = \phi \circ E(p_jbp_i)$$

and

$E(p_jbp_i) =$ 

$=$ 

$=$ 

$= 0.$

It follows from Lemma 6.5 that $p_i\sqrt{\phi \circ E}\,p_j = 0$ for $i \neq j$. The map

$$L^2(f)\colon \sqrt{\phi} \longmapsto \sqrt{\phi \circ E}$$

therefore factors as

$$\begin{array}{ccc} L^2(f)\colon L^2A & \longrightarrow & L^2(B) \cong \bigoplus_{ij} p_iL^2(B)p_j \\ & \overset{\bigoplus [L^2(f)]_i}{\searrow} & \uparrow \\ & & \bigoplus_i p_iL^2(B)p_i\,. \end{array}$$

We have a similar factorization of R by Lemma 4.7:

$$\begin{array}{ccc} R\colon L^2A & \longrightarrow & L^2(B) \underset{B}{\boxtimes} \overline{L^2(B)} \cong \bigoplus_{ij} p_iL^2(B) \underset{B}{\boxtimes} \bar{p}_j\overline{L^2(B)} \\ & \overset{\bigoplus R_i}{\searrow} & \uparrow \\ & & \bigoplus_i p_iL^2(B) \boxtimes_B \bar{p}_i\overline{L^2(B)}\,. \end{array}$$

Let us write

$$\Phi_{jk} : \bar{p}_j \overline{L^2(B)} \longrightarrow L^2(B)p_k$$

for the components of Φ. Given that ${}_B\bar{p}_j\overline{L^2(B)}_A$ and ${}_BL^2(B)p_{kA}$ are irreducible bimodules, the maps Φ_{jk} are either zero or a scalar multiple of some unitary. By the commutativity of (6.15) (and since $R_i \neq 0$), the subspace

$$\bigoplus_i \, p_i L^2(B) \boxtimes_B \bar{p}_i \overline{L^2(B)}$$

of $L^2(B) \boxtimes_B \overline{L^2(B)}$ goes to $\bigoplus_i p_i L^2(B) p_i$ under the map

$$\begin{aligned} 1 \otimes \Phi\colon \; & \textstyle\bigoplus_{ij} \, p_i L^2(B) \underset{B}{\boxtimes} \bar{p}_j \overline{L^2(B)} \\ & \longrightarrow \textstyle\bigoplus_{ik} \, p_i L^2(B) \boxtimes_B L^2(B) p_k \cong \bigoplus_{ik} \, p_i L^2 B p_k. \end{aligned}$$

It follows that $\Phi_{jk} = 0$ whenever $j \neq k$. We can therefore rewrite Φ as

$$\Phi = \bigoplus \Phi_i : \; \textstyle\bigoplus_i \, \bar{p}_i \overline{L^2(B)} \longrightarrow \bigoplus_i \, L^2(B) p_i,$$

where each Φ_i is a scalar multiple of some unitary.

To finish the argument, we show that each Φ_i has norm 1. Let

$$q_i \in Z(A') = Z(A)$$

be the central support projection of $p_i \in A'$. The maps

$$[L^2(f)]_i : {}_AL^2A_A \longrightarrow {}_Ap_i L^2(B) p_{iA}$$

and

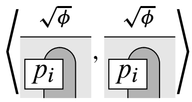
$$R_i : {}_AL^2A_A \longrightarrow {}_Ap_i L^2(B) \boxtimes_B \bar{p}_i \overline{L^2(B)}_A$$

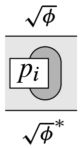
factor through ${}_Aq_i(L^2A)_A$, and are therefore scalar multiples of partial isometries. Given $\phi \in q_i(L^1_+A)$, we have

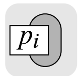
$$\begin{aligned} \left\| [L^2(f)]_i(\sqrt{\phi}) \right\|^2 &= \left\| p_i L^2(f)(\sqrt{\phi}) \right\|^2 \\ &= \left\| p_i \sqrt{\phi \circ E} \right\|^2 \\ &= \langle p_i \sqrt{\phi \circ E}, \sqrt{\phi \circ E} \rangle \\ &= \phi \circ E(p_i) = E(p_i) \cdot \phi(1), \end{aligned}$$

where $E(p_i) \in q_i Z(A) \cong \mathbb{C}$. Similarly, we have

$$\left\| R_i(\sqrt{\phi}) \right\|^2 = \left\| p_i R(\sqrt{\phi}) \right\|^2$$

$=$ 

$=$ 

$=$  $\cdot \langle \sqrt{\phi}, \sqrt{\phi} \rangle$

$$= E(p_i) \cdot \phi(1).$$

It follows that $\|R_i\|^2 = \|[L^2(f)]_i\|^2 = E(p_i)$. Since (6.15) is commutative, we thus get $\|\Phi_i\| = \|[L^2(f)]_i\| / \|R_i\| = 1$, and the map $\Phi = \bigoplus \Phi_i$ is therefore unitary. □

The reader may wonder whether the condition of finite-dimensional center was really needed in Theorem 6.7. We saw in Theorem 4.12 that a bimodule between von Neumann algebras with finite-dimensional centers is dualizable if and only if there exist maps R and S satisfying (4.2): though a priori dualizability requires both conditions (4.2) and (4.3), in fact it is detected by condition (4.2) only. If the centers of A and B are not atomic (that is, if one of them contains $L^\infty([0,1])$), then we do not know how to formulate (4.3). We therefore do not have a good notion of duality in that context; however, we may still define a homomorphism $f : A \to B$ between arbitrary von Neumann algebras to be finite if there exist maps R and S satisfying (4.2), that is giving a not-necessarily normalized dual for the bimodule ${}_A L^2 B_B$.

Conjecture 6.17. *The assignment $A \mapsto L^2 A$ extends to a functor from the category of all von Neumann algebras and finite homomorphisms to the category of Hilbert spaces and bounded linear maps.*

The following two lemmas describe how the functor L^2 interacts with the basic operations of taking corner and block-diagonal subalgebras. Recall from Lemma 2.7 that the L^2-space of the corner algebra

$$A_0 \stackrel{\text{def}}{=} pAp$$

is given by $L^2(A_0) = p(L^2 A)p$.

Lemma 6.18. *Let $f: A \to B$ be a finite homomorphism between von Neumann algebras with finite-dimensional centers. Given a projection $p \in A$, let*

$$A_0 \stackrel{\text{def}}{=} pAp,$$

$$B_0 \stackrel{\text{def}}{=} pBp,$$

and

$$f_0 \stackrel{\text{def}}{=} f|_{A_0}: A_0 \to B_0,$$

where we identify p with its image $f(p) \in B$. Then the homomorphism f_0 is finite, and

$$L^2(f_0) = L^2(f)|_{L^2(A_0)},$$

where we have identified $L^2(A_0)$ and $L^2(B_0)$ with the subspaces $pL^2(A)p$ and $pL^2(B)p$ of $L^2(A)$ and $L^2(B)$ respectively.

Proof. The structure maps (4.1) for the dual of ${}_A L^2(B)_B$ restrict to maps

$$\begin{aligned} R_0\colon {}_{A_0}L^2 A_{0A_0} &= {}_{A_0}pL^2Ap_{A_0} \\ &\longrightarrow {}_{A_0}pL^2B \boxtimes_B \overline{L^2B}p_{A_0} = {}_{A_0}pL^2Bp \boxtimes_{B_0} p\overline{L^2B}p_{A_0}, \end{aligned}$$

and

$$\begin{aligned} S_0\colon {}_{B_0}L^2 B_{0B_0} &= {}_{B_0}pL^2Bp_{B_0} \\ &\longrightarrow {}_{B_0}p\overline{L^2B} \boxtimes_A L^2Bp_{B_0} = {}_{B_0}p\overline{L^2B}p \boxtimes_{A_0} pL^2Bp_{B_0}. \end{aligned}$$

Here we use the invertibility of ${}_B L^2Bp_{B_0}$ to rewrite the targets of R_0 and S_0. These satisfy the duality equations (4.2) and the normalization (4.3), and therefore exhibit ${}_{B_0}(p\overline{L^2(B)}p)_{A_0}$ as the dual of ${}_{A_0}L^2(B_0)_{B_0}$. For every $b \in B_0$, we have

$$E_{A,B}(b)\xi = R^*(b \otimes 1)R\xi = R_0^*(b \otimes 1)R_0\xi = E_{A_0,B_0}(b)\xi \quad \text{for } \xi \in L^2(A_0),$$

and

$$E_{A,B}(b)\xi = R^*(b \otimes 1)R\xi = R^*(pb \otimes 1)R(1-p)\xi = 0 \qquad \text{for } \xi \in L^2(A_0)^\perp,$$

from which it follows that $E_{A_0,B_0} = E_{A,B}|_{B_0}$. Given a state $\phi\colon A_0 \to \mathbb{C}$, the image of $L^2(f_0)(\sqrt{\phi})$ in $L^2(B)$ is the square root of

$$b \longmapsto \phi(E_{A_0,B_0}(pbp)) = \phi(E_{A,B}(pbp)) = \phi(pE_{A,B}(b)p),$$

and is thus equal to the image of $\sqrt{a \mapsto \phi(pap)}$ under $L^2(f)$. ☐

Lemma 6.19. *Let A be a factor, and $p_1, \dots, p_n \in A$ be a collection of orthogonal projections that add up to 1. Let $\iota\colon \bigoplus p_i A p_i \to A$ denote the inclusion. Then $L^2(\iota)$ is the natural inclusion*

$$L^2(\textstyle\bigoplus p_i A p_i) \cong \bigoplus p_i L^2(A) p_i \hookrightarrow L^2(A),$$

where the first isomorphism is given by Lemma 2.7. In particular, $L^2(\iota)$ is an isometry.

Proof. We write A_i for $p_i A p_i$. The inclusions

$$R\colon {}_{\oplus A_i}L^2(\textstyle\bigoplus A_i)_{\oplus A_i} \cong \bigoplus p_i L^2(A) p_i \hookrightarrow L^2(A) \cong {}_{\oplus A_i}L^2(A) \boxtimes_A L^2(A)_{\oplus A_i}$$

and

$$S\colon {}_A L^2(A)_A \hookrightarrow \bigoplus_i L^2(A) \cong \bigoplus_i L^2(A) p_i \underset{\oplus A_i}{\boxtimes} p_i L^2(A) \cong {}_A L^2(A) \underset{\oplus A_i}{\boxtimes} L^2(A)_A$$

exhibit ${}_{\oplus A_i}L^2(A)_A$ as the dual of ${}_A L^2(A)_{\oplus A_i}$. For $\xi_i \in L^2(A_i)$ and $a \in A$, equation (6.8) reads

$$\bigoplus_i \xi_i \overset{R}{\longmapsto} \sum_i \xi_i \overset{a}{\longmapsto} \sum_i a\xi_i \overset{R^*}{\longmapsto} \bigoplus_j p_j \big(\sum_i a\xi_i\big) p_j = \bigoplus_i p_i a p_i \xi_i = \big(\bigoplus_i p_i a p_i\big)\big(\bigoplus_i \xi_i\big).$$

The map $E \overset{\text{def}}{=} E_{\oplus A_i, A}$ is therefore given by $E(a) = \bigoplus q_i(a)$, where

$$q_i(a) \overset{\text{def}}{=} p_i a p_i.$$

It follows that $L^2(\iota)(\sqrt{\oplus\phi_i}) = \sqrt{(\oplus\phi_i)\circ E} = \sqrt{\Sigma\phi_i \circ q_i} = \sum \sqrt{\phi_i \circ q_i}$. Since $\sqrt{\phi_i} \in L^2(A_i)$ maps to $\sqrt{\phi_i \circ q_i} \in L^2(A)$ under the map described in Lemma 2.7, this finishes the proof. ☐

One drawback of the construction presented in Theorem 6.7 is that the maps $L^2(f)\colon L^2(A) \to L^2(B)$ are not isometric. For example, if $\iota\colon A \to B$ is a finite map between factors, then $L^2(\iota)$ is $\sqrt[4]{[B : A]} = \sqrt{[\![B : A]\!]}$ times an isometry. This can be checked on positive vectors: since $\|\sqrt{\phi}\|^2 = \phi(1)$ and $\|\sqrt{\phi \circ E_{A,B}}\|^2 = \phi(E_{A,B}(1)) = E_{A,B}(1)\phi(1)$, it follows that

$$\|L^2(\iota)(\sqrt{\phi})\| / \|\sqrt{\phi}\| = \sqrt{E_{A,B}(1)} = \sqrt{R^*R} = \sqrt{\dim({}_A L^2 B_B)} \tag{6.20}$$

for any $\sqrt{\phi} \in L^2_+(A)$. In some sense, that is inevitable. Assuming that ι is injective, let $L^2(\iota)_{\text{iso}}$ denote the isometry in the polar decomposition of $L^2(\iota)$. The assignment

$$(\iota\colon A \to B) \longmapsto (L^2(\iota)_{\text{iso}}\colon L^2(A) \to L^2(B)) \tag{6.21}$$

is not a functor – this issue is already visible with finite-dimensional commutative von Neumann algebras. Nevertheless, we have the following result.

Proposition 6.22. *When restricted to the subcategory of von Neumann algebras with finite-dimensional center and injective finite homomorphisms $\iota\colon A \to B$ that satisfy $Z(B) \subset \iota(A)$, the assignment $\iota \mapsto L^2(\iota)_{\mathrm{iso}}$ is a functor.*

Proof. We can write $\iota\colon A \to B$ as a direct sum of maps $\iota_j : A_j \to B_j$, where each B_j is a factor. Let us decompose each A_j as a direct sum of factors $A_j = \bigoplus_i A_{ij}$, where $A_{ij} = p_{ij} A_j$, and p_{ij} are the minimal central projections of A_j. We can then factor ι as

$$\iota\colon A = \bigoplus_{ij} A_{ij} \longrightarrow \bigoplus_{ij} p_{ij} B_j p_{ij} \longrightarrow \bigoplus_j B_j = B.$$

Applying the functor L^2 (as defined in Theorem 6.7) to the above maps, we get

$$L^2(\iota)\colon L^2(A) = \bigoplus_{ij} L^2(A_{ij}) \longrightarrow \bigoplus_{ij} L^2(p_{ij} B_j p_{ij}) \xrightarrow{\star} \bigoplus_j L^2(B_j) = L^2(B).$$

The map $\star$ is an isometry by Lemma 6.19. The isometry $L^2(\iota)_{\mathrm{iso}}$ is therefore the composite of $L^2_{\mathrm{iso}}\colon \bigoplus L^2(A_{ij}) \to \bigoplus L^2(p_{ij} B_j p_{ij})$ with the natural inclusion $\bigoplus L^2(p_{ij} B_j p_{ij}) \hookrightarrow \bigoplus L^2(B_j)$ described in Lemma 2.7.

Given two composable inclusions $\iota\colon A \to B$ and $\kappa\colon B \to C$ with $Z(B) \subset \iota(A)$ and $Z(C) \subset \kappa(B)$, we now show that $L^2(\kappa\circ\iota)_{\mathrm{iso}} = L^2(\kappa)_{\mathrm{iso}} \circ L^2(\iota)_{\mathrm{iso}}$. Let us write $C = \bigoplus C_k$, $B = \bigoplus B_{jk}$, and $A = \bigoplus A_{ijk}$ as sums of factors, where $\iota(A_{ijk}) \subset B_{jk}$ and $\kappa(B_{jk}) \subset C_k$. The corresponding minimal central projections are denoted $p_{ijk} \in A_{ijk}$ and $q_{jk} \in B_{jk}$. To compare $L^2(\kappa \circ \iota)_{\mathrm{iso}}$ with $L^2(\kappa)_{\mathrm{iso}} \circ L^2(\iota)_{\mathrm{iso}}$, we consider the following diagram

$$\begin{array}{ccccc} \bigoplus_{jk} L^2(B_{jk}) & \xrightarrow{L^2_{\mathrm{iso}}} & \bigoplus_{jk} L^2(q_{jk} C q_{jk}) & \hookrightarrow & \bigoplus_k L^2(C_k) \\ \uparrow & & \uparrow & \nearrow & \\ \bigoplus_{ijk} L^2(p_{ijk} B p_{ijk}) & \xrightarrow{L^2_{\mathrm{iso}}} & \bigoplus_{ijk} L^2(p_{ijk} C p_{ijk}) & & \\ {\scriptstyle L^2_{\mathrm{iso}}}\uparrow & \nearrow{\scriptstyle L^2_{\mathrm{iso}}} & & & \\ \bigoplus_{ijk} L^2(A_{ijk}) & & & & \end{array}$$

The upper right triangle is a diagram of inclusions and commutes for obvious reasons. The upper left rectangle commutes by the functoriality of the L^2 construction (Theorem 6.7) and by the compatibility of polar decomposition with the operation of composing with an isometry. Finally, note that whenever we have a subfactor inclusion $f : N \to M$ then, by equation (6.20), the corresponding map $L^2(f)$ is a scalar multiple of an isometry. The commutativity of the bottom triangle thus holds because $A_{ijk} \hookrightarrow p_{ijk} B p_{ijk} \hookrightarrow p_{ijk} C p_{ijk}$ are subfactor inclusions. □

Functoriality of Connes fusion. By construction, the operation of Connes fusion $(H_A, {}_AK) \mapsto H \boxtimes_A K$ is a functor in H and K. We now investigate in what sense it is a functor of the *three* variables H, A, and K. Consider the following category. Its objects are triples (H, A, K) consisting of a von Neumann algebra A with finite-dimensional center, a right module H, and a left module K. A morphism from (H_1, A_1, K_1) to (H_2, A_2, K_2) is a triple $\alpha\colon A_1 \to A_2$, $h\colon H_1 \to H_2$, $k\colon K_1 \to K_2$, where α is a finite homomorphism, and h and k are A_1-linear maps.

Theorem 6.23. *The assignment*

$$(H, A, K) \longmapsto H \boxtimes_A K$$

extends to a functor from the category described above to the category of Hilbert spaces and bounded linear maps.

Proof. Given a morphism $(h, \alpha, k)\colon (H_1, A_1, K_1) \to (H_2, A_2, K_2)$ of the above category, we describe the induced map $h \boxtimes_\alpha k\colon H_1 \boxtimes_{A_1} K_1 \to H_2 \boxtimes_{A_2} K_2$. Recall that the composite (6.14) provides an isomorphism Φ between the dual of the bimodule ${}_{A_1}L^2(A_2)_{A_2}$ and the bimodule ${}_{A_2}L^2(A_2)_{A_1}$. Let

$$R\colon {}_{A_1}L^2(A_1)_{A_1} \longrightarrow {}_{A_1}L^2(A_2) \boxtimes_{A_2} \overline{L^2(A_2)}_{A_1} \xrightarrow{1\otimes\Phi} {}_{A_1}L^2(A_2) \boxtimes_{A_2} L^2(A_2)_{A_1},$$

and

$$S\colon {}_{A_2}L^2(A_2)_{A_2} \longrightarrow {}_{A_2}\overline{L^2(A_2)} \boxtimes_{A_1} L^2(A_2)_{A_2} \xrightarrow{\Phi\otimes 1} {}_{A_2}L^2(A_2) \boxtimes_{A_1} L^2(A_2)_{A_2}$$

denote the composition of the normalized duality maps (4.1) with the aforementioned isomorphism.

We define the image of an element

$$\phi_1 \otimes \xi_1 \otimes \psi_1 \in \hom_{A_1}\big(L^2A_1, H_1\big) \otimes L^2A_1 \otimes \hom_{A_1}\big(L^2A_1, K_1\big)$$

under the map $h \boxtimes_\alpha k$ to be $\phi_2 \otimes \xi_2 \otimes \psi_2$, where $\phi_2 \in \hom_{A_2}(L^2A_2, H_2)$ and $\psi_2 \in \hom_{A_2}(L^2A_2, K_2)$ are given by

$$\begin{aligned}\phi_2\colon L^2A_2 &\cong L^2A_1 \underset{A_1}{\boxtimes} L^2A_2 \xrightarrow{\phi_1\otimes 1} H_1 \underset{A_1}{\boxtimes} L^2A_2 \xrightarrow{h\otimes 1} H_2 \underset{A_1}{\boxtimes} L^2A_2 \\ &\cong H_2 \underset{A_2}{\boxtimes} L^2A_2 \underset{A_1}{\boxtimes} L^2A_2 \xrightarrow{1\otimes S^*} H_2 \underset{A_2}{\boxtimes} L^2A_2 \cong H_2,\end{aligned}$$

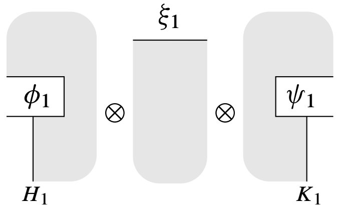
and

$$\begin{aligned}\psi_2\colon L^2A_2 &\cong L^2A_2 \underset{A_1}{\boxtimes} L^2A_1 \xrightarrow{1\otimes\psi_1} L^2A_2 \underset{A_1}{\boxtimes} K_1 \xrightarrow{1\otimes k} L^2A_2 \underset{A_1}{\boxtimes} K_2 \\ &\cong L^2A_2 \underset{A_1}{\boxtimes} L^2A_2 \underset{A_2}{\boxtimes} K_2 \xrightarrow{S^*\otimes 1} L^2A_2 \underset{A_2}{\boxtimes} K_2 \cong K_2,\end{aligned}$$

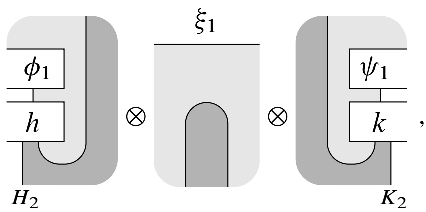
and

$$\xi_2 \stackrel{\text{def}}{=} R(\xi_1) \in L^2(A_2) \boxtimes_{A_2} L^2(A_2) \cong L^2(A_2).$$

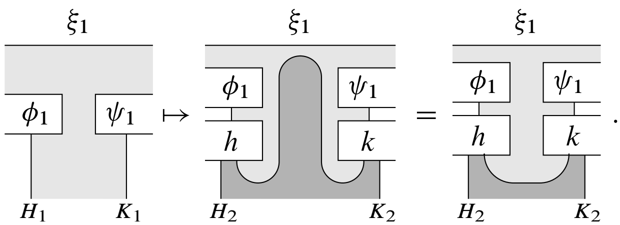
Note that $\xi_2 = L^2(\alpha)(\xi_1)$ by diagram (6.15). Graphically, the above map sends



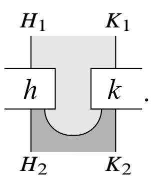
to



and is therefore given by

$h \boxtimes_\alpha k :$  (6.24)

Here, the two shades correspond to the algebras A_1 and A_2, the unlabeled line between those shades corresponds to the bimodule ${}_{A_1}L^2(A_2)_{A_2}$ and its dual bimodule ${}_{A_2}L^2(A_2)_{A_1}$, and the isomorphism (6.14) has been suppressed from the notation. Abstracting out ξ_1, ϕ_1, ψ_1 from (6.24), we can rewrite $h \boxtimes_\alpha k$ in a more concise form, as

$h \boxtimes_\alpha k =$ 

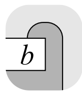
The latter description also makes it clear that $h \boxtimes_\alpha k$ is bounded. Compatibility with composition follows from Lemma 4.8. $\square$

We record the following lemma for future use. Once again, we make implicit use of the identification (6.14) and of its basic property (6.15).

Lemma 6.25. *Let $f: A \to B$ be a finite map between von Neumann algebras with finite-dimensional center. Then the map*

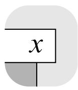
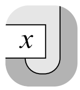
$$B \longrightarrow \hom(L^2 A_A, L^2 B_A)$$

given by

$$b \longmapsto (b \otimes 1) L^2(f) =$$ 

is an isomorphism.

Proof. The inverse map is

 $\longmapsto$  $\in \hom(L^2 B_B, L^2 B_B) \cong B.$ □

7. Index via conditional expectations

In this section, we recall the work of Pimsner and Popa on conditional expectations, and use it to establish the equivalence between the definition of index via statistical dimension (Definitions 5.1 and 5.10) and other notions of index that exist in the literature [9, 17, 18, 21, 30]. The basic inequality (7.4) was introduced in [30] for type *II* von Neumann algebras, and later in [19, 21, 22] for arbitrary von Neumann algebras. Further references include [32, section 1.1] and [18, section 3.4].

Given a subfactor [9] $N \subset M$, a completely positive normal map $E: M \to N$ is called a *conditional expectation* if $E(1) = 1$ and $E(axb) = aE(x)b$ for all $x \in M$ and $a, b \in N$. It may happen that, for some λ, the conditional expectation satisfies the *Pimsner–Popa inequality*

$$E(x) \geq \lambda^{-1} x, \quad x \in M_+.$$

Following [21], the index of the conditional expectation is the smallest possible such λ,

$$\mathrm{Ind}(E) \stackrel{\mathrm{def}}{=} \inf\{\lambda \mid E(x) \geq \lambda^{-1} x, x \in M_+\}. \tag{7.1}$$

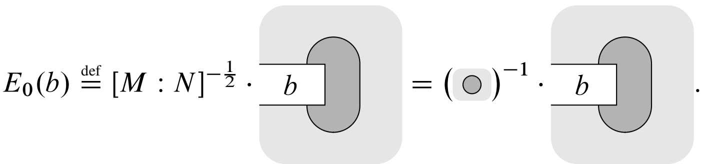
[9] Note that in this section, we usually (but not always) use the letters N and M to refer to factors, as is traditional, and use the letters A and B to refer to more general von Neumann algebras.

We call a conditional expectation finite if its index is finite. For subfactors admitting finite conditional expectations, Longo proves [21, Theorem 5.5] that there exists a unique conditional expectation minimizing $\mathrm{Ind}(E)$ – see also [9, 19]. For a general subfactor, he defines the *minimal index* to be

$$\mathrm{Ind}(N,M) \overset{\text{def}}{=} \inf_E \mathrm{Ind}(E) = \inf_E \inf\{\lambda \mid E(x) \geq \lambda^{-1}x, x \in M_+\}, \tag{7.2}$$

where the infimum runs over all conditional expectations $E\colon M \to N$. We will show later, in Corollary 7.14, that the index (Definition 5.10) agrees with the minimal index if N and M are infinite-dimensional – see Warning 7.16.

If the subfactor has finite index, then an example of a conditional expectation is given by $[M:N]^{-\frac{1}{2}}$ times the map (6.8),



We call E_0 the *minimal conditional expectation*. We will show later, in Proposition 7.10, that the minimal conditional expectation minimizes $\mathrm{Ind}(E)$, thus justifying its name.

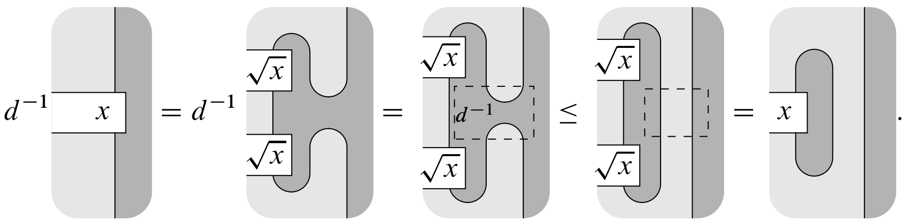
We begin by observing that the index of a subfactor provides an upper bound on the index of the minimal conditional expectation:

Proposition 7.3. *The minimal conditional expectation E_0 satisfies the inequality*

$$E_0(x) \geq [M:N]^{-1}x, \quad x \in M_+. \tag{7.4}$$

In other words, $\mathrm{Ind}(E_0) \leq [M:N]$.

Proof. Let x be a positive element of M, and let us write $d \overset{\text{def}}{=} [M:N]^{\frac{1}{2}}$ for the statistical dimension of ${}_N L^2 M_M$. Because the map d^{-1}  is a projection, we have d^{-1}  $\leq$ . As a consequence of the general fact $(a \leq b) \implies (yay^* \leq yby^*)$, it follows that



Now multiply both sides by d^{-1} to get the desired inequality. □

The following proposition establishes the connection between the Pimsner–Popa inequality and dualizability.

Proposition 7.5. *Let $A \subset B$ be von Neumann algebras with finite-dimensional centers, and let $E\colon B \to A$ be a conditional expectation. If there exists a constant $\mu > 0$ such that $E(x) \geq \mu x$ for all $x \in B_+$, then ${}_AL^2B_B$ is a dualizable bimodule.*

Proof. We show that ${}_BL^2B_A$ is the dual of ${}_AL^2B_B$. To do so, we construct maps

$R =$  $: {}_AL^2(A)_A \longrightarrow {}_AL^2B_A \cong {}_AL^2(B) \boxtimes_B L^2(B)_A$ (7.6a)

and

$S =$  $: {}_BL^2(B)_B \longrightarrow {}_BL^2(B) \boxtimes_A L^2(B)_B$ (7.6b)

satisfying the duality equations (4.2), and appeal to Theorem 4.12 in order to achieve the normalization (4.3).

Using equation (6.6) we see that the map R defined by $\sqrt{\phi} \mapsto \sqrt{\phi \circ E}$ is an isometry. Let

$e \overset{\text{def}}{=} RR^* =$ 

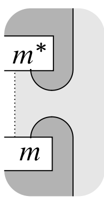
be the corresponding Jones projection. By [32, Theorem 1.1.6], there exists a set of elements $b_j \in B$ such that $\{b_j e b_j^*\}$ are mutually orthogonal projections forming a partition of unity, and such that $\sum b_j b_j^* \in B$ is a bounded operator. Here, both b_j and b_j^* refer to left multiplication operators on L^2B. It follows that the map $\sum b_j\colon \bigoplus_j L^2(B) \to L^2(B)$ is also bounded. Let K be the right A-module $\bigoplus_j L^2A$, and let m and $\bar{m}$ be the two maps

$$m\colon K \underset{A}{\boxtimes} L^2B \cong \bigoplus_j L^2(B) \xrightarrow{\sum(b_j\cdot)} L^2B,$$

and

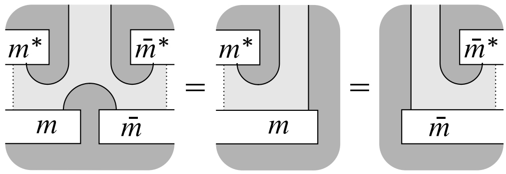
$$\bar{m}\colon L^2B \underset{A}{\boxtimes} \bar{K} \cong \bigoplus_j L^2(B) \xrightarrow{\sum(\cdot b_j)} L^2B.$$

Graphically, the equation $\sum b_j e b_j^* = 1$ means that the map



$: L^2(B) \longrightarrow L^2(B)$

is the identity, where the dotted line stands for K. It is then easy to check that, along with R, the map

$S \stackrel{\text{def}}{=}$ 

satisfies the duality equations (4.2). ☐

The above proof also shows that the following variant of Proposition 7.5 holds.

Proposition 7.7. *Let $f: A \to B$ be a map between arbitrary von Neumann algebras, and let $E: B \to A$ be a conditional expectation such that $E(x) \geq \mu x$ for all $x \in B_+$. Then f is a finite homomorphism in the sense (see the discussion before Conjecture 6.17) that ${}_AL^2B_B$ admits a not-necessarily normalized dual bimodule.* ☐

As a first application of the Pimsner–Popa inequality, we have the following result.

Lemma 7.8. *Let $N \subset P \subset M$ be factors. Then $[M : N] < \infty \implies [P : N] < \infty$.*

Proof. Let $E: M \to N$ be the minimal conditional expectation. Then $E|_P$ is a conditional expectation subject to the same bound: $E|_P(x) \geq [M : N]^{-1}x$, with $x \in P_+$. The subfactor $N \subset P$ satisfies the condition of Proposition 7.5, and so ${}_NL^2P_P$ is dualizable. ☐

Corollary 7.9. *Let N, P, M be factors, and let ${}_NH_P$ and ${}_PK_M$ be non-zero bimodules. If their fusion ${}_NH \boxtimes_P K_M$ is a dualizable N-M-bimodule, then ${}_NH_P$ and ${}_PK_M$ are dualizable.*

Proof. We show that ${}_NH_P$ is dualizable. Let P' be the commutant of P on H, and let M' be the commutant of M on $H \boxtimes_P K$. We have $N \subset P' \subset M'$. By Lemma 5.16, we have $[M' : N] < \infty$, which implies $[P' : N] < \infty$ by the above lemma. By a second application of Lemma 5.16, we deduce that ${}_NH_P$ is dualizable.

This argument might looks circular at first glance, as Lemma 5.16 depends on (5.6). However, Lemma 5.16 only depends on the special case of (5.6) mentioned in footnote 5, and is thus independent of the result of this corollary. ☐

Unless the factors are finite-dimensional, the Pimsner–Popa inequality also provides a characterization of the minimal conditional expectation and of the index. For a subfactor $N \subset M$ of finite index, let

$$E_0(m) \overset{\text{def}}{=} [M : N]^{-\frac{1}{2}} R^*(m \otimes 1) R,$$

as before.

Proposition 7.10. *Assume the factors N and M are infinite-dimensional, and $N \subset M$ is of finite index.*

(a) *If $0 < \lambda < [M : N]$, there exists $x \in M_+$ such that*

$$E_0(x) \not\geq \lambda^{-1} x. \tag{7.11}$$

In other words, $\mathrm{Ind}(E_0) \geq [M : N]$, *and therefore, by* (7.4), $\mathrm{Ind}(E_0) = [M : N]$.

(b) *If $E : M \to N$ is a conditional expectation and $E \neq E_0$, then there exists $x \in M_+$ such that*

$$E(x) \not\geq [M : N]^{-1} x. \tag{7.12}$$

In other words, $\mathrm{Ind}(E) > [M : N]$.

Proof. (a) We let

 $: L^2 N \longrightarrow L^2 M$

and

 $: L^2 M \longrightarrow L^2 M \boxtimes_N L^2 M$

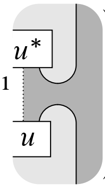
be normalized duality maps for the bimodule

$${}_N H_M \overset{\text{def}}{=} {}_N L^2 M_M.$$

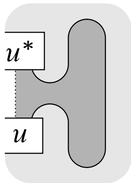
Let $d = [M : N]^{\frac{1}{2}}$ be the statistical dimension of H, and let $e = d^{-1}$  be the Jones projection. Since $\dim(N) = \infty$, one can find a right M-module K_M such that $K \boxtimes_M L^2(M)_N$ is isomorphic to $L^2 N_N$ – use the classification of modules over factors of type different from I_n. Pick a unitary isomorphism $u : K \boxtimes_M L^2(M)_N \to L^2 N_N$ and set

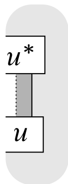
$$x \overset{\text{def}}{=} (u \otimes 1)(1 \otimes e)(u^* \otimes 1).$$

We then have

$$E_0(x) = E_0\Bigg(d^{-1}$$


$$\Bigg)$$

$$= d^{-2}$$


$$= [M : N]^{-1}$$


$$= [M : N]^{-1},$$

where the dotted line stands for K. Since x is a non-zero projection and $[M : N]^{-1}$ is a scalar, it follows that $E_0(x) \not\geq \mu x$ for any $\mu > [M : N]^{-1}$.

b. We need to check that E_0 minimizes $\mathrm{Ind}(E)$. Let p_i be the minimal central projections of $N' \cap M = \mathrm{End}({}_N L^2 M_M)$, let $d = [M : N]^{\frac{1}{2}}$, and let $d_i = [p_i M p_i : p_i N]^{\frac{1}{2}}$. Note that $p_i N \subset p_i M p_i$ is an irreducible subfactor, that is $(p_i N)' \cap p_i M p_i = \mathbb{C}$. Thus by [21, Proposition 5.3], there exits only one conditional expectation $p_i M p_i \to p_i N$. Using part (a) and Proposition 7.5 we conclude that for $p_i N \subset p_i M p_i$, the index coincides with the minimal index. Thus $d_i = (\mathrm{Ind}(p_i N, p_i M p_i))^{\frac{1}{2}}$. According to [21, Theorem 5.5], it suffices to check that $E_0|_{N' \cap M}$ is a trace and that

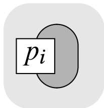
$$E_0(p_i) = \frac{d_i}{\sum d_i}. \tag{7.13}$$

The first condition was proven in Lemma 4.15. To check the latter, let

$$H_i \overset{\text{def}}{=} p_i(L^2(M)).$$

We then have

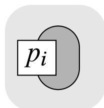
$$
\begin{aligned}
d_i &= \dim({}_{p_i N} L^2(p_i M p_i)_{p_i M p_i}) \\
&= \dim({}_{p_i N}\, p_i (L^2 M)\, p_i{}_{p_i M p_i}) \\
&= \dim({}_{p_i N}\, p_i L^2 M \underset{M}{\boxtimes} L^2 M p_i{}_{p_i M p_i}) \\
&= \dim({}_{p_i N}\, p_i L^2 M_M) \\
&= \dim(H_i) \\
&=
\end{aligned}
$$



,

where the second equality is Lemma 2.7, the fourth one holds by equations (5.4) and (5.6), and the last one is given by Lemma 4.20. Note that $\sum d_i = d$ now follows by equation (5.5). By the definition of E_0, we therefore have

$$\left(\sum d_i\right) \cdot E_0(p_i) = dE_0(p_i) =$$



$$= d_i. \qquad \square$$

Corollary 7.14. *Let $N \subset M$ be infinite-dimensional factors, let $[M : N]$ be the index, as in Definition 5.10, and let* $\mathrm{Ind}(N, M)$ *be the minimal index, as in equation (7.2). Then*

$$[M : N] = \mathrm{Ind}(N, M). \tag{7.15}$$

Warning 7.16. As noted in [21], the equality (7.15) fails to be true, for example, for the subfactors $\mathbb{C} \hookrightarrow M_n(\mathbb{C})$. The minimal index $\mathrm{Ind}(N, M)$ is not a good notion of index in the case of finite dimensional factors.

Now is an appropriate moment to pay our debt to Remark 4.14, by giving a particularly mild condition that ensures that a bimodule is dualizable – compare [22, Theorem 4.1].

Proposition 7.17. *Let ${}_A H_B$ and ${}_B K_A$ be irreducible bimodules between von Neumann algebras with finite-dimensional centers. If there exist non-zero maps*

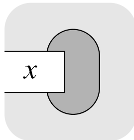
$$\tilde{R} \colon {}_A L^2(A)_A \longrightarrow {}_A H \boxtimes_B K_A$$

and

$$\tilde{S} \colon {}_B L^2(B)_B \longrightarrow {}_B K \boxtimes_A H_B,$$

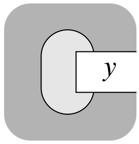
then ${}_A H_B$ and ${}_B K_A$ are dualizable.

Proof. We denote $\tilde{R}$ by  and $\tilde{S}$ by . We may assume without loss of generality that A and B are factors, and that $\tilde{R}$ and $\tilde{S}$ are isometries. Define conditional expectations $E\colon B' \to A$ and $F\colon A' \to B$ by

$E\colon x \longmapsto$ 

,

and

$F\colon y \longmapsto$ 

,

where the commutants are taken on H.

Denote by $U(A)$ the group of unitary elements of A. For any non-zero projection $p \in B'$, the least upper bound $\bigvee_{u \in U(A)} upu^*$ belongs to $A' \cap B' = \mathrm{End}({}_AH_B) = \mathbb{C}$ and is therefore equal to 1. If $E(p)$ were zero, we would have

$$1 = E(1) = E(\bigvee upu^*) = \bigvee E(upu^*) = \bigvee uE(p)u^* = 0.$$

Thus the conditional expectation E is faithful, and similarly F is faithful. It follows from [22, Proposition 4.4] that the inclusion $A \subset B'$ has finite index. By Lemma 5.16, we then have $\dim({}_AH_B) = [\![B' : A]\!] < \infty$, and so ${}_AH_B$ is dualizable. The bimodule ${}_BK_A$ is dualizable for similar reasons. $\square$

We finish this section by establishing some useful inequalities for the matrix of statistical dimensions $[\![B : A]\!]$ – recall Definition 5.13 – associated to a finite homomorphism $A \to B$ of von Neumann algebras with finite-dimensional centers. Our proofs are all based on the Pimsner–Popa inequality.

Let $A_1, B_1 \subset \mathbf{B}(H_1)$ and $A_2, B_2 \subset \mathbf{B}(H_2)$ be von Neumann algebras such that A_i commutes with B_i. The algebras $A_1 \vee B_1 \subset \mathbf{B}(H_1)$ and $A_2 \vee B_2 \subset \mathbf{B}(H_2)$ are therefore completions of the corresponding algebraic tensor products $A_1 \otimes_{\mathrm{alg}} B_1$ and $A_2 \otimes_{\mathrm{alg}} B_2$. Given homomorphisms $\alpha\colon A_1 \to A_2$ and $\beta\colon B_1 \to B_2$, the induced map $\alpha \otimes \beta\colon A_1 \otimes_{\mathrm{alg}} B_1 \to A_2 \otimes_{\mathrm{alg}} B_2$ does not always extend to a map $A_1 \vee B_1 \to A_2 \vee B_2$. This will however be the case in the presence of an $\alpha \otimes \beta$-equivariant surjective homomorphisms $h\colon H_1 \to H_2$.

Lemma 7.18. *Let A_i, B_i, H_i, and h be as above. If the algebras A_i, B_i, and $A_i \vee B_i$ have finite-dimensional centers and the homomorphisms $\alpha\colon A_1 \to A_2$ and $\beta\colon B_1 \to B_2$ are finite, then the induced map*

$$\alpha \otimes \beta\colon A_1 \vee B_1 \longrightarrow A_2 \vee B_2 \tag{7.19}$$

is a finite homomorphism.

Proof. Let us write $\vee_{H_1}$ and $\vee_{H_2}$ for the completions inside $\mathbf{B}(H_1)$ and $\mathbf{B}(H_2)$, respectively. We can then factor the map (7.19) as

$$A_1 \vee_{H_1} B_1 \longrightarrow A_1 \vee_{H_2} B_1 \longrightarrow A_2 \vee_{H_2} B_1 \longrightarrow A_2 \vee_{H_2} B_2.$$

The first map is a projection, and therefore finite. We analyze the second map – the third one is similar. >From now on let $\vee$ mean $\vee_{H_2}$. The restriction to $A_1' \cap B_1' = (A_1 \vee B_1)'$ of the minimal conditional expectation $E_0\colon A_1' \to A_2'$ satisfies the same Pimsner–Popa bound as E_0. The homomorphism $(A_1 \vee B_1)' \to (A_2 \vee B_1)'$ is therefore finite by Proposition 7.5. Corollary 5.17 finishes the argument. □

Proposition 7.20. *Let A be an infinite-dimensional factor sitting in a von Neumann algebra B. If there exists a conditional expectation $E\colon B \to A$ satisfying the Pimsner–Popa bound*

$$E(x) \geq \mu^{-1} x, \quad x \in B_+, \tag{7.21}$$

then B has finite-dimensional center. Furthermore, letting p_i be the minimal central projections of B, we then have $\sum [p_i B : A] \leq \mu$. In other words, we have the inequality

$$\big\| [\![B : A]\!] \big\| \leq \sqrt{\mu},$$

where $\| \|$ stands for the ℓ^2 norm of a vector.

Proof. Let $q_i \in B$ be non-zero central projections adding up to 1. Since

$$aE(q_i) = E(aq_i) = E(q_i a) = E(q_i)a$$

for all $a \in A$, the element $E(q_i)$ is central in A, and hence a scalar. >From the bound (7.21), we conclude that $E(q_i) \geq \mu^{-1}$. Summing up over all indices i, we deduce

$$1 = E(1) = E\big(\textstyle\sum q_i\big) = \sum_i E(q_i) \geq \sum_i \mu^{-1},$$

from which it follows that the number of q_i's is at most μ. The center of B is therefore finite-dimensional.

Now let p_i be the minimal central projections of B, and let $B_i \stackrel{\text{def}}{=} p_i B$. The restriction $F_i \stackrel{\text{def}}{=} E|_{B_i} : B_i \to A$ satisfies all the properties for being a conditional expectation, except that it does not send the unit p_i of B_i to 1. The map $E_i \stackrel{\text{def}}{=} F_i(p_i)^{-1} F_i$ is therefore a conditional expectation. It satisfies the bound

$$E_i(x) \geq F_i(p_i)^{-1}\mu^{-1}x, \quad x \in B_{i+},$$

from which it follows that $[B_i : A] \leq F_i(p_i)\mu$. Adding up over indices, we get that $\sum[B_i : A] \leq \sum F_i(p_i)\mu = E(\sum p_i)\mu = E(1)\mu = \mu$. □

The following result is, in some sense, dual to Proposition 7.20.

Proposition 7.22. *Let $A = \oplus A_i$ be a sum of finitely many infinite-dimensional factors A_i, and suppose that A is a subalgebra of some factor B. Let $E : B \to A$ be a conditional expectation satisfying*

$$E(x) \geq \mu^{-1}x, \quad x \in B_+. \tag{7.23}$$

Letting p_i be the minimal central projections of A, we have $\sum[p_i B p_i : A_i] \leq \mu$. In other words, we have the inequality

$$\big\| [\![B : A]\!] \big\| \leq \sqrt{\mu},$$

where $\|\,\|$ stands for the ℓ^2 norm of a vector.

Proof. Under our assumption on A the optimal μ satisfying (7.23) can be identified with the Kosaki index $\|E^{-1}(1)\|$ of the conditional expectation E, see [32, Theorem 1.1.6]. By its definition [17, 18], the Kosaki index does not change under tensor product with another factor. In particular, given a type III factor R, we conclude that the conditional expectation $E \otimes R : B \bar{\otimes} R \to A \bar{\otimes} R$ satisfies the same bound (7.23) as E. The index of $A_i \otimes R$ in $p_i(B \otimes R)p_i$ being equal to that of A_i in $p_i B p_i$, we may assume without loss of generality that B is a type III factor.

Let us define $B_{ij} \stackrel{\text{def}}{=} p_i B p_j$. If B is a type III factor, then the projections p_i are all Murray-von Neumann equivalent; we can therefore identify each matrix block B_{ij} with a given algebra, say C, and get an isomorphism

$$B = \bigoplus_{ij} B_{ij} \cong M_n(C).$$

Taking the composite $B_{ii} \hookrightarrow B \xrightarrow{E} A \twoheadrightarrow A_i$, we get a conditional expectation $E_i : B_{ii} \to A_i$. Let λ_i be the smallest number for which the Pimsner–Popa inequality

$$E_i(x) \geq \lambda_i^{-1}x, \quad x \in B_{ii+}$$

holds, and note that there exist projections $e_i \in B_{ii}$ such that $E_i(e_i) = \lambda_i^{-1} p_i$; for example, we can take e_i to be a Jones projection as in the proof of Proposition 7.10 (a).

Let $u_{ij} \in C$ be partial isometries with $u_{ij}u_{ij}^* = e_i$, $u_{ij}^* = u_{ji}$, and $u_{ij}u_{jk} = u_{ik}$. In particular, we have $u_{ii} = e_i$. Consider now the projection $Q \in M_n(C)$ given by

$$Q_{ij} \overset{\text{def}}{=} \frac{\sqrt{\lambda_i \lambda_j}}{\sum_k \lambda_k} u_{ij}.$$

We then have

$$E(Q) = \bigoplus E_i(Q_{ii}) = \bigoplus E_i\Big(\frac{\lambda_i}{\sum_k \lambda_k} e_i\Big) = \bigoplus \frac{\lambda_i}{\sum_k \lambda_k} E_i(e_i) = \bigoplus \frac{1}{\sum_k \lambda_k} p_i$$
$$= \frac{1}{\sum_k \lambda_k}.$$

Combined with the bound (7.23), the above estimate shows that $\mu \geq \sum \lambda_k$. To finish the proof, we use the inequality $\lambda_i \geq [p_i B p_i : p_i A]$, which follows from (7.11) and (7.12). □

Remark 7.24. We expect that, analogously to Proposition 7.20, when $A \subset B$ with B a factor, the existence of a conditional expectation $B \to A$ satisfying a Pimsner–Popa bound actually implies that A has finite-dimensional center.

Given the results of Propositions 7.20 and 7.22 it is natural to ask the following:

Question 7.25. Let $A \subset B$ be von Neumann algebras with finite-dimensional center, and let $E: B \to A$ be a conditional expectation satisfying the Pimsner–Popa bound $E(x) \geq \mu^{-1}x$, for all $x \in B_+$. For which norm $\| \|$ on matrices do we then get the inequality $\|[\![B : A]\!]\| \leq \sqrt{\mu}$?

Finally, we use the previous two propositions to explain the relationship between index and the operations of relative commutant and of completed tensor product.

Corollary 7.26. *Let $N \subset M \subset A \subset \mathbf{B}(H)$ be subalgebras with N and M factors and $[M : N] < \infty$. Suppose that one of the two relative commutants $N' \cap A$ or $M' \cap A$ is a factor, and that the other one has finite-dimensional center. In this case,*

$$\big\|[\![N' \cap A : M' \cap A]\!]\big\| \leq [\![M : N]\!].$$

Proof. By Corollary 5.17, we know that $[N' : M'] = [M : N]$. Let $E' \colon N' \to M'$ be the minimal conditional expectation from N' to M'. If $a \in A' \subset N'$ and $x \in N' \cap A$, then we have $aE'(x) = E'(ax) = E'(xa) = E'(x)a$, showing that $E'(x) \in M' \cap A$. The restriction $E \stackrel{\text{def}}{=} E'|_{N' \cap A}$ is therefore a conditional expectation from $N' \cap A$ to $M' \cap A$. By the Pimsner–Popa inequality for E', we know that

$$E(x) \geq [N' : M']^{-1} x = [M : N]^{-1} x, \quad x \in N' \cap A.$$

By Proposition 7.20 or Proposition 7.22, it follows that $\big\| [\![N' \cap A : M' \cap A]\!] \big\| \leq [\![M : N]\!]$. □

Corollary 7.27. *Let $N \subset M \subset \mathbf{B}(H)$ be factors with $[M : N] < \infty$, and let $A \subset \mathbf{B}(H)$ be an algebra that commutes with M. Suppose that one of the algebras $N \vee A$ and $M \vee A$ is a factor, and that the other one has finite-dimensional center. In this case,*

$$\big\| [\![M \vee A : N \vee A]\!] \big\| \leq [\![M : N]\!].$$

Proof. By the previous corollary, we have $\big\| [\![N' \cap A' : M' \cap A']\!] \big\| \leq [\![M : N]\!]$. The result now follows from Corollary 5.17, because $(M \vee A)' = M' \cap A'$ and $(N \vee A)' = N' \cap A'$. □

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