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Quantum Topology

Khovanov homology of a unicolored B-adequate link has a tail

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Abstract. C. Armond [3] and S. Garoufalidis and T. Le [6] have shown that a unicolored Jones polynomial of a B-adequate link has a stable tail at large colors. We categorify this tail by showing that Khovanov homology of a unicolored link also has a stable tail, whose graded Euler characteristic coincides with the tail of the Jones polynomial.

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Contents

1. Notations and basic facts

1.1. Links, adequate links and their diagrams. All diagrams of tangles and links in this paper are framed, we assume blackboard framing. Links are presumed unframed, unless specified otherwise.

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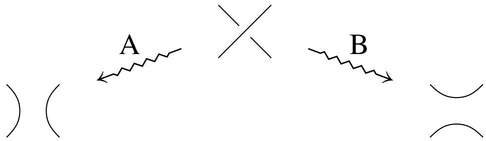
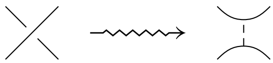
Each crossing of a link diagram D can be 'spliced' in two ways, we call them A-splicing and B-splicing:



Let $\mathrm{B}(D)$ denote the diagram which consists of two parts: the circles resulting from B-splicing of all crossings of D (B-circles) and segments connecting those circles at places where crossings were in D (struts). Schematically, one passes from D to $\mathrm{B}(D)$ in the following way:



where the arcs in the right diagram are parts of B-circles and the dashed segment is a strut.

Definition 1.1.
- A crossing of D and its strut in $\mathrm{B}(D)$ are called B-adequate, if the strut connects two different B-circles.
- A framed diagram D is B-adequate, if all of its crossings are B-adequate.
- A framed link is B-adequate, if it can be represented by a B-adequate framed diagram.
- An unframed link L is called B-adequate, if there is at least one framed B-adequate diagram which represents it.

Note that if an unframed link is B-adequate, then, generally, it can not be represented by a B-adequate framed diagram for all framings.

Adequate links were introduced by Raymond Lickorish and Morwen Thistlethwaite [9]. All alternating links are B-adequate, but not all B-adequate links are alternating: an example of this is a torus knot $\mathrm{T}_{m,-n}$, $n \geq m \geq 3$. More generally, a link constructed by closing a totally negative braid is B-adequate. Torus knots $\mathrm{T}_{m,n}$, $n \geq m \geq 3$ provide examples of links which are not B-adequate.

Here are some notations associated with a link diagram D throughout the paper:

$\mathfrak{V}$ – a set of crossings (struts) in D or in $\mathrm{B}(D)$,

χ_D – the number of crossings in $D2$,

χ_D^{in} – the number of B-inadequate crossings in D,

κ_D – the number of B-circles in $\mathrm{B}(D)$,

ϕ_D – the framing number of D.

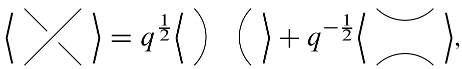
The following is an easy corollary of the results of section 7.7 of [8].

Theorem 1.2. *The numbers χ_L and κ_L are topological invariants of a B-adequate framed link L, because they do not depend on the choice of representative B-adequate diagram D for L. Moreover, if B-adequate framed links L and L′ differ only by framing, then*

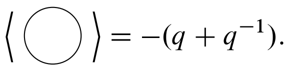
$$\chi_{L'} - \chi_L = \kappa_{L'} - \kappa_L = -(\phi_{L'} - \phi_L). \tag{1.1}$$

For an unframed B-adequate link L we define the minimal crossing number $\chi^!_L$ as the minimum among the numbers χ_D for B-adequate framed diagrams D representing L.

1.2. The Kauffman bracket and the Jones polynomial. The Kauffman bracket of a framed tangle diagram is defined by the splicing relation and the unknot normalization condition:

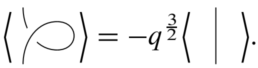


(1.2a)



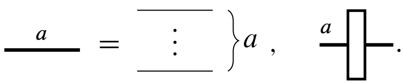
(1.2b)

Thus defined, the bracket is framing-dependent:



The Jones polynomial of a framed link L is the Kauffman bracket of its diagram: $\mathrm{J}_L(q) = \langle L \rangle$.

1.3. Cables and coloring. We introduce coloring of tangle and link components through cabling and Jones–Wenzl projectors. A cable of a strand is depicted by using a thicker line with the label indicating the number of strands, and the Jones–Wenzl projector is depicted by a box:



For a positive integer N let $\mathrm{J}_{N,L}(q)$ denote the unicolored Jones polynomial of L, that is, all components of L are colored by the same color N. A coloring of a link component by N means that we assign the $(N+1)$-dimensional irreducible representation of SU(2) to it. Equivalently, the color N means that the link component is N-cabled and we place the Jones–Wenzl projector on this cable.

In this paper we consider unicolored links, that is, links, all of whose components are colored by the same number N. Their colored Jones polynomial $\mathrm{J}_{N,L}(q)$ is a Laurent polynomial of q^2 up to an overall factor: if L is presented by a diagram D, then

$$q^{\frac{1}{2}\chi_D N^2+\kappa_D N}\mathrm{J}_{N,L}(q)\in\mathbb{Z}[q^{\pm 2}].$$

1.4. Homological notations. Let A be a finitely generated additive category: objects of A are finite sums of elements of a finite set $\mathfrak{A}$. Let $\mathbf{Kom}^+(\mathsf{A})$ denote the homotopy category of its complexes bounded from below: an object of $\mathbf{Kom}^+(\mathsf{A})$ is a chain

$$\mathbf{A}=(\cdots\longrightarrow A_{i+1}\longrightarrow A_i\longrightarrow\cdots\longrightarrow A_m),\tag{1.3}$$

where $A_i=\bigoplus_{\alpha\in\mathfrak{A}}m_{i,\alpha}\,\alpha$ and $m_{i,\alpha}\in\mathbb{Z}_{\geq 0}$ are the multiplicities of generators. The notation m for multiplicity is treated in this paper as an arbitrary constant, so the appearance of m in different expressions does not imply that there is a relation between the multiplicities, unless it is stated specifically. The special multiplicities appearing in a presentation of the categorified Jones–Wenzl projector are denoted by μ.

We use a non-standard notation for the translation functor: $\mathsf{h}\mathbf{A}=\mathbf{A}[1]$, which allows us to define a functor $p(\mathsf{h})$ for any polynomial $p(x)$ with integer non-negative coefficients. In particular, we use a functor $\left\{{i\atop j}\right\}_{\mathsf{h}}$ based on a combinatorial polynomial

$$\begin{Bmatrix} i\\ j\end{Bmatrix}_x=\frac{(1-x^{2i})(1-x^{2i-2})\cdots(1-x^{2i-2j+2})}{(1-x^2)(1-x^4)\cdots(1-x^{2j})}.\tag{1.4}$$

We also use a non-standard notation for the cone of two complexes:

$$\boxed{\mathsf{h}\mathbf{A}\xrightarrow{f}\mathbf{B}}=\mathrm{Cone}(\mathbf{A}\xrightarrow{f}\mathbf{B}).\tag{1.5}$$

in order to emphasize the fact that the cone $\mathrm{Cone}(\mathbf{A}\to\mathbf{B})$ can be presented as a sum $\mathsf{h}\mathbf{A}\oplus\mathbf{B}$ deformed by an extra differential $\mathbf{A}\xrightarrow{f}\mathbf{B}$. Moreover, when we work with bi-graded Khovanov complexes, there may be some confusion about which of two gradings is homological, but our non-standard notation (1.5) specifies all degree shifts explicitly.

The homological order $|O|_{\mathsf{h}}$ of an object $O\in\mathbf{Kom}^+(\mathsf{A})$ is the minimum number m, for which O can be presented by a complex (1.3).

Consider a direct system of complexes of $\mathbf{Kom}^+(\mathsf{A})$: $\mathbf{A}_0 \to \mathbf{A}_1 \to \cdots$. If this system is 'Cauchy', that is, if for the cones $\mathbf{B}_i = \mathrm{Cone}(\mathbf{A}_{i-1} \to \mathbf{A}_i)$ there is a limit $\lim_{i\to\infty} |\mathbf{B}_i|_{\mathsf{h}} = \infty$, then, according to Proposition 3.7 of [10], there exists a direct limit $\varinjlim \mathbf{A}_i$.

Since $\mathbf{A}_i \sim \mathrm{Cone}(\mathsf{h}^{-1}\mathbf{B}_i \to \mathbf{A}_{i-1})$, the direct limit $\varinjlim \mathbf{A}_i$ can be viewed as a result of attaching the complexes $\mathbf{B}_i$ one after another to the initial complex $\mathbf{B}_0 = \mathbf{A}_0$, hence we use the following notation for the complex $\varinjlim \mathbf{A}_i$:

$$\varinjlim \mathbf{A}_i \sim \boxed{\cdots \longrightarrow \mathbf{B}_i \longrightarrow \cdots}_{i=0}^{\infty} . \tag{1.6}$$

In fact, if all $\mathbf{B}_i$ are 'homologially minimal' representatives of their equivalence classes, then the sum $\bigoplus_{i=0}^{\infty} \mathbf{B}_i$ is well-defined (every chain object is finitely generated) and $\varinjlim \mathbf{A}_i$ is homotopy equivalent to $\bigoplus_{i=0}^{\infty} \mathbf{B}_i$ defomed by adding extra differentials $\mathbf{B}_i \xrightarrow{f_{ij}} \mathbf{B}_j$ for all pairs $i > j$.

We refer to the *r.h.s.* of eq. (1.6) as a *multi-cone*, and we also use a similar notation for the complex (1.3):

$$\mathbf{A} = \boxed{\cdots \longrightarrow \mathsf{h}^i A_i \longrightarrow \cdots}_{i=m}^{\infty} .$$

Note the use of the functor h to set explicitly the correct homological degree of the chain object A_i in the multi-cone.

If a multi-cone $\mathbf{A}$ is generated by complexes $\mathbf{B}_a$:

$$\mathbf{A} = \boxed{\cdots \longrightarrow \bigoplus_{j,a} m_{ij,a} \mathsf{h}^j\, \mathbf{B}_a \longrightarrow \cdots} \tag{1.7}$$

(where $m_{ij,a}$ are multiplicities) but we do not care how those complexes are arranged within the multi-cone, then we use a 'lump sum' notation

$$\mathbf{A} = \boxed{\bigoplus_{j,a} m_{j,a}^{\mathrm{tot}}\, \mathsf{h}^j\, \mathbf{B}_a}_{\circlearrowleft} , \qquad m_{j,a}^{\mathrm{tot}} = \sum_i m_{ij,a},$$

because, as a complex, $\mathbf{A}$ is a sum of $\mathbf{B}_a$ with total multiplicities $\sum_i m_{ij,a}$ deformed by an extra differential depicted as $\circlearrowleft$.

If the category A is abelian, then we can compute the homology of the multi-cone (1.6) with the help of the filtered complex spectral sequence. The E^1 term of this spectral sequence is the sum of homologies of $\mathbf{B}_i$: $E^1 = \bigoplus_{i=0}^{\infty} \mathrm{H}(\mathbf{B}_i)$ and it is determined by the lump sum form of the multi-cone.

Remark 1.3. Since subsequent terms in the spectral sequence get only smaller, there is a bound on the homological order of the homology of (1.6) in terms of homological orders of its constituent complexes:

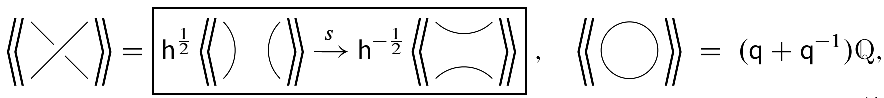
$$\left|\mathrm{H}\left(\boxed{\cdots \longrightarrow \mathbf{B}_i \longrightarrow \cdots}_{\,i=0}^{\,\infty}\right)\right|_{\mathrm{h}} \geq \min_i |\mathrm{H}(\mathbf{B}_i)|_{\mathrm{h}}.$$

In particular, for the lump sum multi-cone (1.7)

$$|\mathrm{H}(\mathbf{A})|_{\mathrm{h}} \geq \min\{|\mathrm{H}(\mathbf{B}_a)|_{\mathrm{h}} + j : m^{\mathrm{tot}}_{j,a} \neq 0\}.$$

1.5. Khovanov homology. In defining Khovanov complexes [8] for tangles we follow the cobordism based approach of D. Bar-Natan [2], albeit with a different grading convention. We still have two degrees: h-degree $\deg_{\mathrm{h}}$ and q-degree $\deg_{\mathrm{q}}$, and we use the notations h and q for their translation functors (these functors increase the corresponding degrees by 1). The q-degree is the genuine homological degree: it takes values in $\mathbb{Z}$ and its parity determines the sign factors. The h-degree is 'pseudo-homological', it takes values in $\frac{1}{2}\mathbb{Z}$ and it has no impact on signs, however it is the h-degree shift functor h which is present explicitly in the Khovanov bracket.

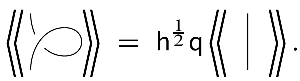
In our notations, Khovanov bracket of a crossing and of the unknot are



(1.8)

where h and q are degree shift functors, while s is the morphism corresponding to the saddle cobordism. Note that $\deg_{\mathrm{h}} s = -1$, while $\deg_{\mathrm{q}} s = 1$, so s is odd.

Thus defined, Khovanov bracket is invariant under the first Reidemeister move only up to a degree shift:



(1.9)

Relations (1.8) transform into the relations (1.2) after the substitution $\mathsf{h} \mapsto q$, $\mathsf{q} \mapsto -q$, hence in our notations the graded Euler characteristic of Khovanov homology of a framed link equals its Jones polynomial:

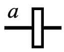
$$\mathrm{J}_L(q) = \sum_{i,j} (-1)^j q^{i+j} \dim \mathrm{H}^{\mathrm{Kh}}_{i,j}(L).$$

We will use the Khovanov bracket notation $\langle\!\langle - \rangle\!\rangle$ very sparingly, because it clutters the pictures, especially when the diagrams are big. Nevertheless, we hope that the distinction between diagrams and their Khovanov complexes will be clear.

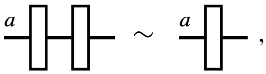
Actually, we blur this distinction further by allowing the presence of categorified Jones–Wenzl projectors within diagrams, since, strictly speaking, projectors are not diagrams but rather complexes within Bar-Natan's universal category.

1.6. A categorified Jones–Wenzl projector. An (a,b)-tangle is an embedding of circles and segments, the segment endpoints coinciding with initial a points or final b points. Imagine that the tangle goes from the bottom up. Depending on the position of its endpoints, the segment is either straight, or a cap, or a cup. If one of its endpoints is initial and the other is final (so the segment goes straight through the tangle), then the segment is *straight*, if both endpoints are initial, then the segment is a *cap*, and if both of its the segments are final, then the segment is a *cup*.

The *width* $|\tau|_{\mathrm{wd}}$ of a tangle τ is the number of its straight segments. An (a,a) tangle has an equal number of cups and caps, we call this number a *width deficit* and denote it as $|\tau|_{\mathrm{df}}$. Obviously, $|\tau|_{\mathrm{df}} = \frac{1}{2}(a - |\tau|_{\mathrm{wd}})$.

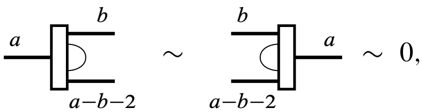
A *Temperley–Lieb* (TL) tangle is a flat tangle which contains no circles. Let $\mathfrak{T}_a$ be the set of all (a,a) TL tangles. The categorified Jones–Wenzl projector 

was constructed independently by Frenkel, Stroppel and Sussan [5], Cooper and Krushkal [3] and by the author [10]. It satisfies three essential properties: it is a projector:

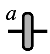


(1.10)

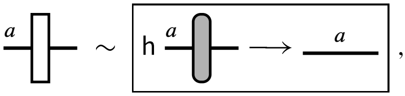
it annihilates cups and caps:



(1.11)

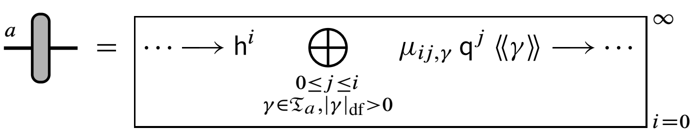
and it has a presentation as a cone of an identity braid and a complex 

generated by TL tangles with positive width deficit and with non-negative h-degree and q-degree shifts



(1.12)

where

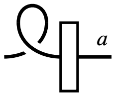


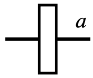
(1.13)

and $\mu_{ij,\gamma}$ are multiplicities.

1.7. Khovanov bracket of colored tangles. We define Khovanov bracket of colored tangles by cabling tangle components and adding at least one categorified Jones–Wenzl projector to each tangle component. This means that we allow semi-infinite complexes which may extend infinitely far into positive homological degree.

The colored Khovanov bracket is independent of the framing up to a degree shift:



$= \mathsf{h}^{\frac{1}{2}a^2}\mathsf{q}^a$


. (1.14)

2. Results

2.1. Overview

2.1.1. Bounds on colored Khovanov homology. Let L_N denote the N-unicolored version of the link L. For a B-adequate framed link L, a *shifted* Khovanov homology of L_N is defined by the formula

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(L_N) = \mathsf{h}^{\frac{1}{2}N^2\chi_L}\mathsf{q}^{\kappa_L N}\mathrm{H}^{\mathrm{Kh}}(L_N). \tag{2.1}$$

In view of eq. (1.1), if links L and L' differ only by framing, then their shifted Khovanov homologies are isomorphic: $\widetilde{\mathrm{H}}^{\mathrm{Kh}}(L'_N) = \widetilde{\mathrm{H}}^{\mathrm{Kh}}(L_N)$.

Theorem 2.1. *There are bounds on degrees of shifted homology of a B-adequate link L: $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,j}(L_N) = 0$, if one of the following conditions is met:*

$$i < 0, \tag{2.2}$$

$$j < -\tfrac{1}{2}i - \tfrac{1}{2}\chi^{!}_{L}, \tag{2.3}$$

$$j < -i, \tag{2.4}$$

$$j = -i \neq 0, \tag{2.5}$$

where $\chi^{!}_{L}$ is the minimum crossing number of a diagram representing L. Moreover,

$$\dim \widetilde{\mathrm{H}}^{\mathrm{Kh}}_{0,0} = 1. \tag{2.6}$$

2.1.2. Tail of Khovanov homology. The main result of this paper is a definition of special degree-preserving maps between shifted Khovanov homologies of unicolored B-adequate links, such that these maps are isomorphisms at low h-degrees.

Theorem 2.2. *A B-adequate diagram of a link L determines a sequence of degree preserving maps*

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(L_N) \xrightarrow{f_N} \widetilde{\mathrm{H}}^{\mathrm{Kh}}(L_{N+1}) \tag{2.7}$$

which are isomorphisms on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}(L_N)$ for $i \le N-1$.

Definition 2.3. The tail homology $\mathrm{H}^\infty(L)$ is the direct limit of the direct system determined by the sequence of maps f_N, $N \in \mathbb{Z}_+$:

$$\mathrm{H}^\infty(L) = \lim_{\rightarrow} \widetilde{\mathrm{H}}^{\mathrm{Kh}}(L_N). \tag{2.8}$$

Corollary 2.4. *The maps*

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(L_N) \longrightarrow \mathrm{H}^\infty(L) \tag{2.9}$$

associated with the direct limit (2.8) are isomorphisms on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}(L_N)$ for $i \le N-1$, hence the direct limit $\mathrm{H}^\infty(L)$ is finite-dimensional in every bi-degree, it satisfies the bounds $\mathrm{H}^\infty_{i,j}(L) = 0$ at the conditions (2.2)–(2.5) and

$$\dim \mathrm{H}^\infty_{0,0}(L_N) = 1. \tag{2.10}$$

Remark 2.5. To define the maps (2.7) we have to choose a diagram D representing L, however we expect that the maps can be defined canonically, that is, independently of that choice.

2.1.3. Relation to the tail of the Jones polynomial. The bounds (2.2) and (2.3) on $\mathrm{H}^\infty(L)$ mean that the graded Euler characteristic of the tail homology is well-defined, because in its presentation as an alternating sum of homology dimensions

$$\mathrm{J}_{L,\infty}(q) = \sum_{i,j} (-1)^j\, q^{i+j}\, \dim \mathrm{H}^\infty_{i,j}(D)$$

there is only a finite number of non-trivial terms for any given value of $i+j$.

The bound (2.3) indicates that $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,j}(L_N)$ and $\mathrm{H}^\infty_{i,j}(L)$ are trivial when $i+j < \frac{1}{2}i - \frac{1}{2}\chi^!_L$, hence their high h-degrees contribute only to coefficients at high powers of q in the graded Euler characteristic. Since the map (2.9) is an isomorphism at low h-degrees, we come to the following:

Theorem 2.6. *The graded Euler characteristic of the tail homology determines the lower powers of q in the unicolored Jones polynomial of a B-adequate link:*

$$\mathrm{J}_{N,L}(q) = (-1)^{\kappa_L N} q^{-\frac{1}{2}N^2\chi_L - \kappa_L N}\big(\mathrm{J}_{L,\infty}(q) + O(q^{\frac{1}{2}N - \frac{1}{2}\chi^!_L})\big).$$

This means that the tail homology categorifies the tail of the unicolored Jones polynomial of B-adequate links studied by C. Armond [1] and by S. Garoufalidis and T. Le [6].

2.1.4. Tail homology is determined by a reduced B-diagram of a link. A link diagram D' is a B-reduction of a link diagram D, if the diagram $\mathrm{B}(D')$ is constructed from $\mathrm{B}(D)$ in two stages: at first stage for each pair of distinct B-circles of $\mathrm{B}(D)$ connected by more that one strut we remove all connecting struts but one; at the second stage we remove all B-circles which have only one strut attached to them. Obviously, if D is B-adequate, then so is D'.

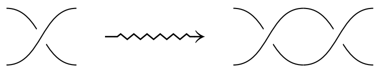
A link L' is a B-reduction of a B-adequate link L, if L' can be presented by a diagram which is a B-reduction of a B-adequate diagram presenting L.

Theorem 2.7. *If a link L' is a B-reduction of a B-adequate link L, then their tail homologies are isomorphic:* $\mathrm{H}^\infty(L') \cong \mathrm{H}^\infty(L)$.

Corollary 2.8. *If L_β is a circular closure of a connected negative braid β, then the tail homology of L_β is that of an unknot:* $\mathrm{H}^\infty(L_\beta) \cong \mathrm{H}^\infty(\bigcirc)$.

Proof. It is easy to see that L_β is B-adequate and a reduced B-diagram of L_β consists of a single circle without struts. □

Corollary 2.9 (Invariance under strut doubling). *If L is a B-adequate link and L' is constructed by performing a replacement*

 (2.11)

in a B-adequate diagram of L, then tail homologies of L and L' are isomorphic: $\mathrm{H}^\infty(L) \cong \mathrm{H}^\infty(L')$.

Proof. Obviously, L and L' have the same B-reduction. □

The latter corollary is a categorification of a similar property of the tail of the unicolored Jones polynomial observed by C. Armond and O. Dasbach [4, 1]. This property suggests that a single crossing plays the role of a categorified Jones–Wenzl projector in the tail homology. In order to make this statement precise, tail homology has to be defined for knotted graphs, which may include both finite and infinite colors, so that the essential property of contracting cups/caps can be formulated. We hope to address this issue in a subsequent paper. Meanwhile, we prove in Appendix that for large N a crossing of two N-cables is, indeed, homologically close to the Jones–Wenzl projector placed on two parallel N-cables.

2.2. Technicalities. We prove most statements of the previous subsection not just for B-adequate links, but for any link diagram. However, we conjecture that the results are trivial for B-inadequate diagrams, because their tail homology is trivial, if defined as a direct limit of the system that we construct. We expect that B-inadequate links also have tail homology, but the proof that the tail of their Khovanov homology stabilizes in the limit of large color requires new ideas.

2.2.1. Shifted Khovanov homology. Let D be a diagram of a tangle which may include single lines, cables and Jones–Wenzl projectors. We define $n_\times(D)$ to be the total number of single line crossings in D (that is, a crossing between an a-cable and a b-cable contributes ab to $n_\times(D)$). The following is an obvious corollary of eq. (1.8) and the fact that, according to (1.12) and (1.13), $|\!-\!\boxed{\,}\!-\!|_{\mathrm{h}} = 0$:

Theorem 2.10. *The complex $\langle\!\langle D \rangle\!\rangle$ has a lower homological bound: $|\langle\!\langle D \rangle\!\rangle|_{\mathrm{h}} \geq -\frac{1}{2} n_\times(D)$.*

Let D be a diagram of a link which may include single lines, cables and Jones–Wenzl projectors. Define $n_\circ(D)$ to be the total number of circles in the diagram constructed from D by replacing the Jones–Wenzl projectors with identity braids and performing B-splicings on all crossings. Now we define the shifted Khovanov homology of D:

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D) = \mathsf{h}^{\frac{1}{2} n_\times(D)} \mathsf{q}^{n_\circ(D)} \mathrm{H}^{\mathrm{Kh}}(D).$$

The following is a particular case of Theorem 2.10:

Theorem 2.11. *If $i < 0$, then $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}(D) = 0$.*

For a link diagram D let D_N denote the corresponding unicolored diagram (that is, every link components is N-cabled and contains at least one Jones–Wenzl projector). Then, obviously, $n_\times(D_N) = N^2 \chi_D$ and $n_\circ(D_N) = N\kappa_D$, so

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_N) = \mathsf{h}^{\frac{1}{2} N^2 \chi_D} \mathsf{q}^{\kappa_D N} \mathrm{H}^{\mathrm{Kh}}(D_N). \tag{2.12}$$

Hence, if D is B-adequate and represents a link L, then $\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_N)$ coincides with the shifted homology defined by eq. (2.1).

Now Theorem 2.1 is a corollary of Theorem 2.11 and the following:

Theorem 2.12. *The shifted homology of a unicolored diagram D_N has a bound: $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,j}(D_N) = 0$ if one of the following conditions is satisfied:*

$$j < -\tfrac{1}{2} i - \tfrac{1}{2}\chi_D - \tfrac{3}{2}\chi_D^{\mathrm{in}}, \tag{2.13}$$

$$j < -i - \chi_D^{\mathrm{in}} \tag{2.14}$$

Moreover, if D is B-adequate, then

$$\dim \widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,-i} = \begin{cases} 0, & \text{if } i > 0, \\ 1, & \text{if } i = 0. \end{cases} \tag{2.15}$$

Theorem 2.2 is a special case of the following:

Theorem 2.13. *For any link diagram D there is a sequence of degree preserving maps*

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_N) \xrightarrow{f_N} \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_{N+1}), \tag{2.16}$$

which are isomorphisms on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}$ for $i \leq N-1$.

This theorem implies that the direct system formed by maps (2.16) has a limit

$$\mathrm{H}^{\infty}(D) = \lim_{\to} \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_N). \tag{2.17}$$

which is finite-dimensional in every bi-degree.

Conjecture 2.14. *If the diagram D is not B-adequate, then the direct limit* (2.17) *is trivial:* $\mathrm{H}^{\infty}(D) = 0$.

2.3. Discussion. We conjecture that B-adequate links have a tri-graded homology $\mathrm{H}^{\sim}(L)$, which has an additional 'b-grading', whose zero-degree part coincides with the tail homology $\mathrm{H}^{\infty}(L)$:

$$\mathrm{H}^{\sim}(L) = \bigoplus_{\substack{i,j \\ k \geq 0}} \mathrm{H}^{\sim}_{i,j,k}(L), \quad \mathrm{H}^{\sim}_{i,j,0}(L) = \mathrm{H}^{\infty}_{i,j}(L).$$

This homology should have a family of mutually anti-commuting differentials d_N, $N \in \mathbb{Z}_+$, with degrees $\deg_{\mathrm{q}} d_N = -1$, $\deg_{\mathrm{h}} d_N = 1$ and $\deg_{\mathrm{b}} = -1$ such that homology of $\mathrm{H}^{\sim}(L)$ with respect to d_N matches the shifted Khovanov homology $\widetilde{\mathrm{H}}^{\mathrm{Kh}}$ up to a level proportional to N^2, after the b-degree is converted into h-degree:

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{\tilde{\imath},\bullet}(L) = \bigoplus_{i+Nk=\tilde{\imath}} \mathrm{H}^{d_N}_{i,\bullet,k}\big(\mathrm{H}^{\sim}(L)\big), \quad \text{if } \tilde{\imath} \geq a_L N^2, \tag{2.18}$$

where a_L is a constant determined by L.

There are three reasons to formulate this conjecture. The first reason is that the proof of Theorem 2.2 is based on numerous long exact sequences (3.2), in which the 'correction homology' starts at homological degree proportional to N.

The second reason is that, according to Garoufalidis and Le [6], the tail of the Jones polynomial of B-adequate links has a 'telescopic' structure. They show that if L is alternating, then there exists a family of Laurent series $\Phi_n(q) = \sum_m a_m q^m$ such that for any $k > 0$ the combined series $F_k(q) = \sum_{n=0}^{k} \Phi_n(q)$ is a better approximation for the tail of the colored Jones polynomial that just the first term $\Phi_0(q) = \mathrm{J}_{L,\infty}(q)$. With the help of the colored Kauffman bracket (*cf.* (3.10)) we can prove a similar result for all B-adequate links and we expect that the 2-variable series

$$\mathrm{J}_{L,\sim}(b,q) = \sum_{n=0}^{\infty} b^n \Phi_n(q) \tag{2.19}$$

is the bi-graded Euler characteristic of the tri-graded homology $\mathrm{H}^{\sim}(L)$.

The third reason for our conjecture comes from the paper by Gukov and Stošić; see [7]. Based on QFT models of Khovanov homology, they suggest that its dependence on color should be similar to the dependence of the $\mathrm{SU}(n)$ homology on n: this homology may be presented as a homology of a special differential acting on $\mathrm{SU}(N)$ homology if $N > n$. We suggest to go half step further. Ultimately, the $\mathrm{SU}(N)$ homology may be presented, at least, conjecturally, from the tri-graded HOMFLY-PT homology with the help of special differentials d_n and we expect that a similar process may work for the tail homology.

We expect that the formation of a stable tail of a unicolored B-adequate link is a general feature which originates in the tri-graded homology when the Young diagram describing the color has a very large value of one of the differences between the lengths of rows or columns. In particular, it could be easy to follow the tail formation in case when the diagram consists of a single very large column.

Witten suggested [11] that a series of the form (2.19) should represent the graded Euler characteristic of Khovanov homology in the background of a flat U(1)-reducible SU(2) connection in the link complement. We conjecture that if a link can be presented as a circular closure of a totally negative braid, then the tail homology coincides with the one related to the flat U(1)-reducible SU(2) connection.

2.4. Acknowledgements. The author thanks Eugene Gorsky for sharing the results of his unfinished research and, in particular, the conjecture about the structure of Khovanov homology of a colored unknot and its stabilization in the limit of high color.

3. Five tools

The proof of Theorems 2.12 and 2.13 requires five tools: local transformations, purging, braid straightening, colored Khovanov bracket and recurrence relations between categorified Jones–Wenzl projectors. Local transformations relate homologies of similar diagrams. Purging gets rid of redundant TL tangles in complexes, which contain Jones–Wenzl projectors, thus improving estimates of homological order. Straightening a braid is a simple observation that braiding within a cable attached to a projector results only in degree shifts. The colored Khovanov bracket is a special presentation of a crossing of two cables attached to Jones–Wenzl projectors. Finally, recurrence relations relate Jones–Wenzl projectors on N and $N+1$ strands.

3.1. Local replacements and local transformations. A *local replacement* is a pair of tangles τ_{i} and τ_{f}, which may contain single and cabled lines, as well as Jones–Wenzl projectors. Both tangles should have the same sets of incoming legs and the same sets of outgoing legs. Hence if an initial diagram D_{i} contains the tangle τ_{i} attached by its legs to the rest of the diagram, then we can construct a final diagram D_{f} by replacing τ_{i} with τ_{f}. If τ_{i} or τ_{f} is not an actual diagram, but rather a complex of diagrams within the universal category, the local replacement still makes sense as a construction of $\langle\!\langle D_{\mathrm{f}}\rangle\!\rangle$ from $\langle\!\langle D_{\mathrm{i}}\rangle\!\rangle$.

A *local transformation* is a local replacement together with a specified degree preserving morphism $\langle\!\langle \tau'_{\mathrm{f}}\rangle\!\rangle \xrightarrow{g} \langle\!\langle \tau_{\mathrm{i}}\rangle\!\rangle$, where we use a shortcut $\langle\!\langle \tau'_{\mathrm{f}}\rangle\!\rangle = \mathsf{q}^{m_{\mathrm{f}}}\mathsf{h}^{n_{\mathrm{f}}}\langle\!\langle \tau_{\mathrm{f}}\rangle\!\rangle$.[1] The morphism g determines the presentation of $\langle\!\langle \tau_{\mathrm{i}}\rangle\!\rangle$ as a cone

$$\langle\!\langle \tau_{\mathrm{i}}\rangle\!\rangle \sim \boxed{\langle\!\langle \tau_{\mathrm{c}}\rangle\!\rangle \longrightarrow \langle\!\langle \tau'_{\mathrm{f}}\rangle\!\rangle}\,, \quad \text{where } \langle\!\langle \tau_{\mathrm{c}}\rangle\!\rangle = \boxed{\mathsf{h}\langle\!\langle \tau'_{\mathrm{f}}\rangle\!\rangle \xrightarrow{g} \langle\!\langle \tau_{\mathrm{i}}\rangle\!\rangle}\,. \tag{3.1}$$

Up to a degree shift, the 'correction' complex $\langle\!\langle \tau_{\mathrm{c}}\rangle\!\rangle$ may be the categorification complex of an actual tangle τ_{c}, or it may be just a convenient shortcut.

Let D_{i} be a diagram of a link which contains τ_{i} and let D_{f} and D_{c} be the diagrams constructed by replacing τ_{i} with τ_{f} and τ_{c}. The relations (3.1) imply a long exact sequence

$$\mathsf{h}^{-1}\mathrm{H}^{\mathrm{Kh}}(D_{\mathrm{c}}) \longrightarrow \mathsf{q}^{m_{\mathrm{f}}}\mathsf{h}^{n_{\mathrm{f}}}\mathrm{H}^{\mathrm{Kh}}(D_{\mathrm{f}}) \xrightarrow{g} \mathrm{H}^{\mathrm{Kh}}(D_{\mathrm{i}}) \longrightarrow \mathrm{H}^{\mathrm{Kh}}(D_{\mathrm{c}}). \tag{3.2}$$

For all local transformations considered in this paper, there are relations

$$n_{\mathrm{f}} = \tfrac{1}{2}n_{\times}(D_{\mathrm{f}}) - \tfrac{1}{2}n_{\times}(D_{\mathrm{i}}), \quad m_{\mathrm{f}} = n_{\circ}(D_{\mathrm{f}}) - n_{\circ}(D_{\mathrm{i}}),$$

[1] The unnatural direction of morphism is chosen for future convenience.

hence eq. (3.2) turns into the following sequence of degree preserving maps between shifted homologies:

$$\mathsf{h}^{-1}\mathrm{H}^{\mathrm{Kh}}(D'_{\mathrm{c}}) \longrightarrow \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_{\mathrm{f}}) \xrightarrow{g} \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_{\mathrm{i}}) \longrightarrow \mathrm{H}^{\mathrm{Kh}}(D'_{\mathrm{c}}),$$

where $\mathrm{H}^{\mathrm{Kh}}(D'_{\mathrm{c}}) = \mathsf{h}^{\frac{1}{2}n_{\times}(D_{\mathrm{i}})}\mathsf{q}^{n_{\circ}(D_{\mathrm{i}})}\mathrm{H}^{\mathrm{Kh}}(D_{\mathrm{c}})$. This exact sequence implies the following:

Proposition 3.1. *If* $\mathrm{H}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{c}}) = 0$ *for* $i \leq M_{\mathrm{h}} - 1$, *then the degree preserving map*

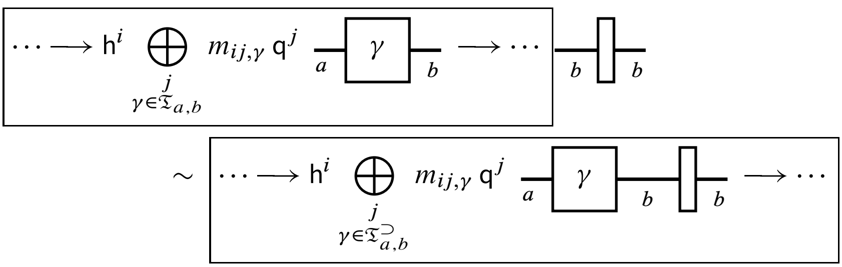
$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_{\mathrm{f}}) \xrightarrow{g} \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_{\mathrm{i}}), \tag{3.3}$$

is an isomorphism on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}$ *for*

$$i \leq M_{\mathrm{h}} + \tfrac{1}{2}n_{\times}(D_{\mathrm{i}}) - 2. \tag{3.4}$$

3.2. Purging. Purging is a process of using eq. (1.11) to remove constituent TL tangles of a complex, whose cups or caps are connected directly to a Jones–Wenzl projector. Let $\mathfrak{T}_{a,b}$ be the set of all (a,b) TL tangles and let $\mathfrak{T}^{\supset}_{a,b} = \{\gamma \in \mathfrak{T}^{\supset}_{a,b} \colon |\gamma|_{\mathrm{wd}} = b\}$. In other words, $\mathfrak{T}^{\supset}_{a,b}$ is a subset of (a,b) TL tangles, which contain no cups, but only caps and straight segments.

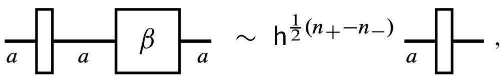
Proposition 3.2. *There is a homotopy equivalence*



(3.5)

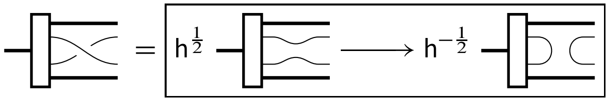
3.3. Straightening a braid attached to a Jones–Wenzl projector

Theorem 3.3. *The categorification complex of the tangle composition of the Jones–Wenzl projector with a braid* β *is homotopy equivalent to the shifted complex of the Jones–Wenzl projector:*



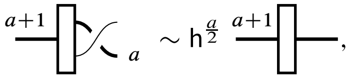
where n_+ (n_-) *is the number of positive (negative) elementary crossings in a presentation of* β.

Proof. It is sufficient to prove this equivalence in the case of $n_+ = 1$ and $n_- = 0$ (the case of $n_+ = 0$ and $n_- = 1$ is similar and other cases can be proved by consequent composition of elementary crossings). Thus we replace the positive crossing by its Khovanov complex

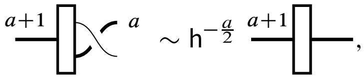


and observe that the second term in the resulting cone is contractible. □

This proposition has important special cases:



(3.6a)

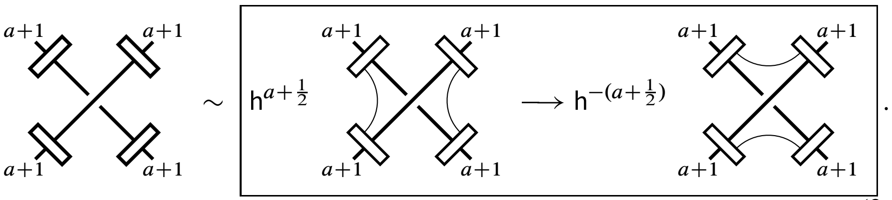


(3.6b)

and two similar cases with opposite powers of h when the cable runs over the single line.

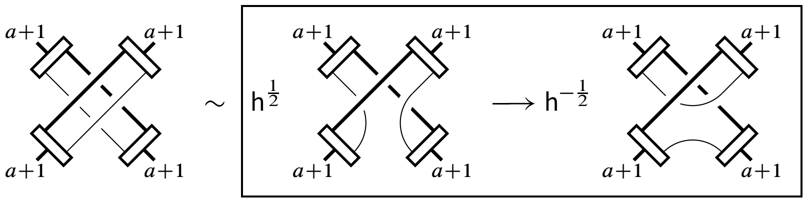
3.4. Colored Khovanov bracket

Theorem 3.4 (Single strand splicing). *A Khovanov bracket of the crossing of two equally colored strands can be presented as a cone:*



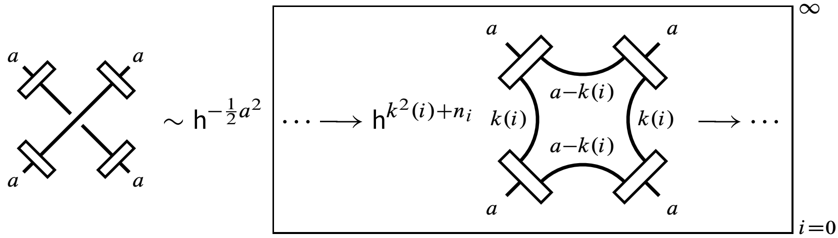
(3.7)

Proof. Split off a single strand from each crossing cable, apply the Khovanov bracket relation (1.8) to their crossing:



and then use the relations (3.6) to bring both tangles to the form of eq. (3.10). □

Theorem 3.5 (Colored Khovanov bracket). *A Khovanov bracket of the crossing of two equally colored strands can be presented as a multi-cone of crossingless colored tangles:*

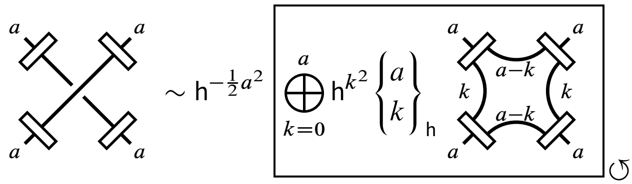


, (3.8)

such that

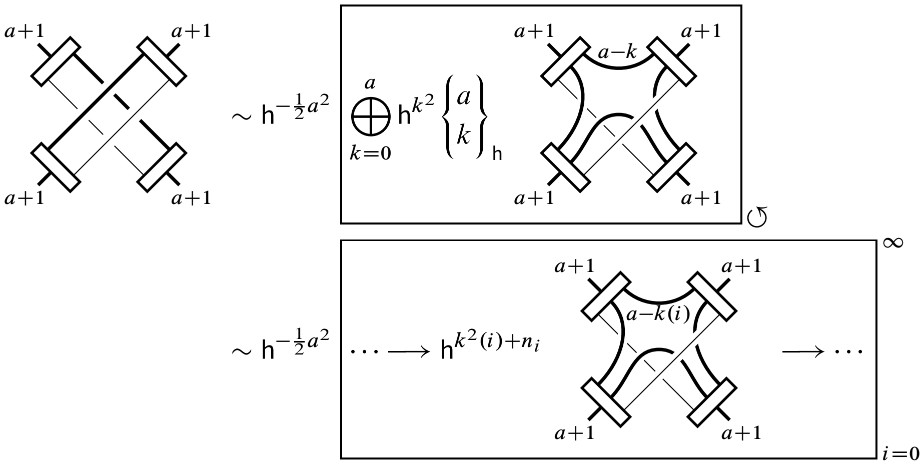
$$n_i \geq 0, \quad i \geq 2^k - 1 \tag{3.9}$$

and the lump sum form of this multi-cone is



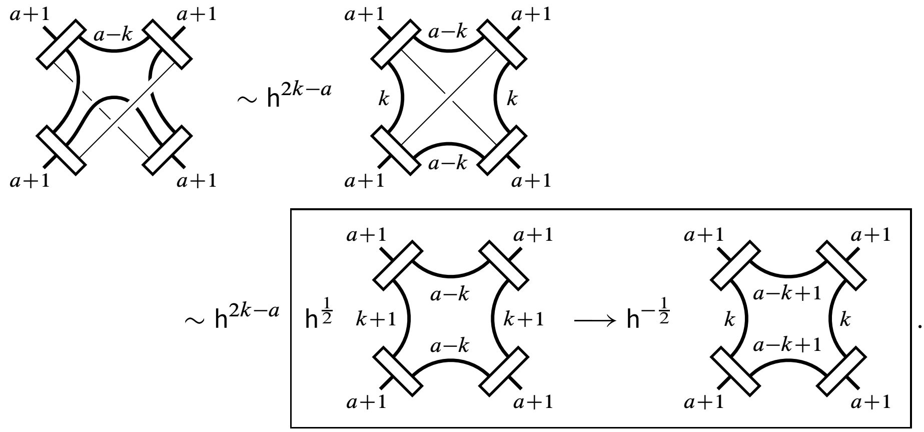
. (3.10)

Proof. We prove this theorem by induction over a. At $a = 1$ it amounts to Khovanov bracket (1.8). Suppose that it holds for some a and consider the crossing of two $(a + 1)$-cables. We split each cable into an a-cable and a single line and apply eqns. (3.8) and (3.10) to the crossing of a-cables:



. (3.11)

The categorification complex of a constituent tangle of the resulting multi-cone can be simplified:



(3.12)

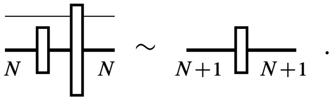
Here the first homotopy equivalence follows from eq. (3.6) and the second one is the application of Khovanov bracket (1.8) to the crossing of two single lines. We substitute eq. (3.12) for every constituent tangle in both multi-cones of eq. (3.11). The lump sum multi-cone transforms into the *r.h.s.* of eq. (3.10) for the intersection of two $(a+1)$-cables with the help of a simple identity

$$\begin{Bmatrix} a \\ k-1 \end{Bmatrix}_{\mathsf{h}} + \mathsf{h}^{2k} \begin{Bmatrix} a \\ k \end{Bmatrix}_{\mathsf{h}} = \begin{Bmatrix} a+1 \\ k \end{Bmatrix}_{\mathsf{h}}.$$

Associativity of the cone operation implies that the second multi-cone of eq. (3.11) can be brought to the linear form of the *r.h.s.* of eq. (3.8), so it remains to verify inequalities (3.9). The first inequality follows from the lump sum multi-cone formula (3.10) which we have just proved. Let us verify the second inequality for two tangles of the cone (3.12) after they appear through the substitution in the second multi-cone of eq. (3.11). Since every constituent tangle of (3.11) is replaced by a cone of two tangles, the second tangle of the cone (3.12) will appear at the multi-cone position $i' = 2i$ and the second inequality of (3.9) for it obviously holds. The first tangle of (3.11) appears at the position $i' = 2i + 1$ and it carries $k' = k + 1$. The inequality $2i' + 1 \geq 2^{k'} - 1$ follows easily from the assumed inequality $i \leq 2^k - 1$. □

3.5. Recurrence relations for categorified Jones–Wenzl projectors

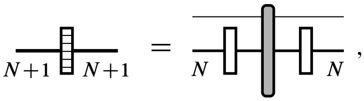
Proposition 3.6. *A larger Jones–Wenzl projector absorbs a smaller one:*



(3.13)

Proof. In view of eqns. (1.12) and (1.13) for $a = N$, this equivalence is a result of purging the smaller projector with the larger one. □

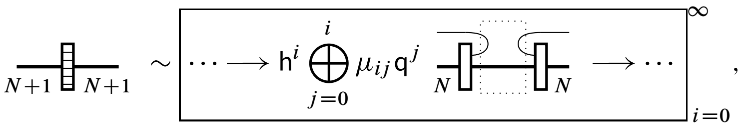
Let us introduce a shortcut notation:



(3.14)

where the complex is defined by eq. (1.12).

Proposition 3.7. *Thus defined, the complex has a multi-cone presentation*

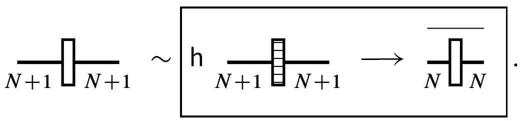


(3.15)

where μ_{ij} are the multiplicities of the Temperley–Lieb tangle inside the dotted box, with which it appears in the r.h.s. of eq. (1.13).

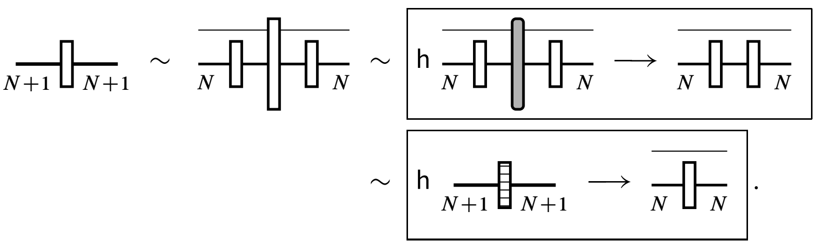
Proof. We purge the complex with the help of two N-strand Jones–Wenzl projectors. The tangle in the dotted box is the only $(N+1, N+1)$ TL tangle which is not contracted when sandwiched between them. □

Theorem 3.8. *The* $(N+1)$*-strand categorified Jones–Wenzl projector is homotopy equivalent to a cone*



(3.16)

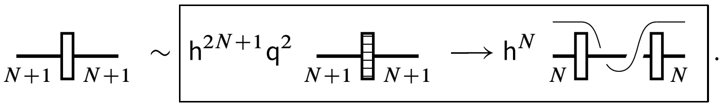
Proof. Consider a sequence of homotopy equivalences



(3.17)

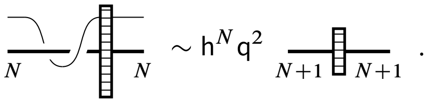
The first homotopy equivalence comes from eq. (3.13), the second follows from eq. (1.12) and the last one follows from eqns. (3.14) and (1.10). □

Theorem 3.9. *The* $(N+1)$*-strand categorified Jones–Wenzl projector can be presented as the following cone:*

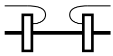


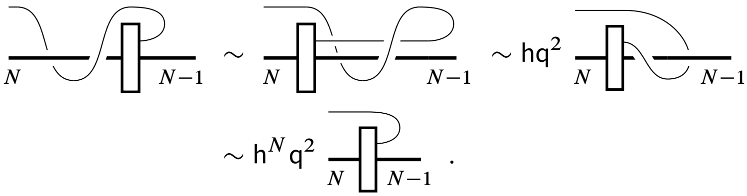
(3.18)

Lemma 3.10. *There is a homotopy equivalence*



(3.19)

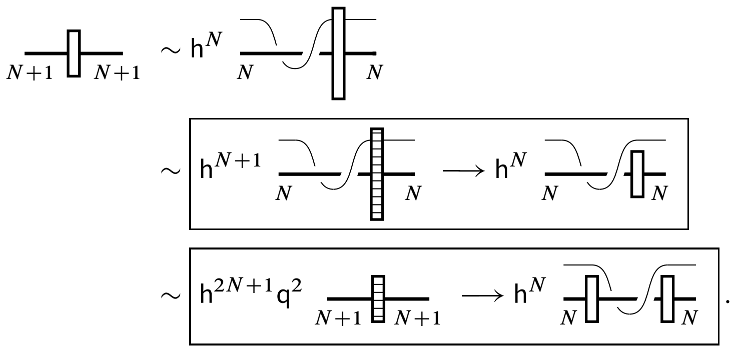
Proof. Consider the composition of the line winding around the N-cable with the left portion of the complex  which generates the multi-cone (3.15):



(3.20)

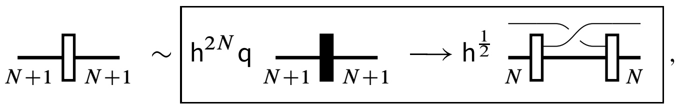
Here the first equivalence is purely topological: the projector is moved left along the cable, the second equivalence uses eq. (1.9) to remove two framing kinks on the single line and the third equivalence follows from eq. (3.6). The equivalence (3.19) comes from applying equivalence (3.20) to every constituent complex in the multi-cone (3.15). □

Proof of Theorem 3.9. Eq. (3.18) follows from a sequence of homotopy equivalences:



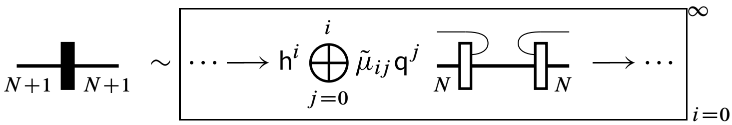
Here the first equivalence follows from eq. (3.6), the second equivalence follows from eq. (3.16), the third equivalence follows from eqns. (3.19) and (1.10). □

Theorem 3.11. *The* $(N+1)$*-strand categorified Jones–Wenzl projector is homotopy equivalent to a cone*



(3.21)

in which the complex  *has the following multi-cone presentation:*

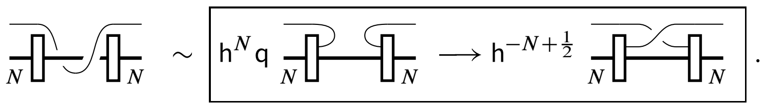


(3.22)

where

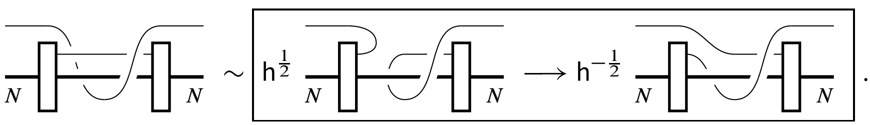
$$\tilde{\mu}_{ij} = \begin{cases} \mu_{i-1,j-1} & \textit{if } i \geq 1, \\ 1 & \textit{if } i = 0. \end{cases}$$

Lemma 3.12. *There is a homotopy equivalence*



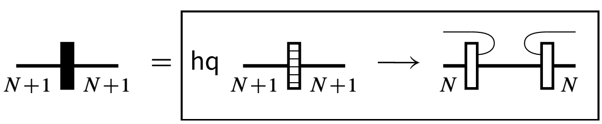
(3.23)

Proof. The lemma is proved by applying Khovanov bracket formula (1.8) to one of the elementary crossings in the *l.h.s.* diagram:



The diagrams in the *r.h.s.* cone are reduced to those of eq. (3.23) with the help of eqns. (1.9) and (3.6). □

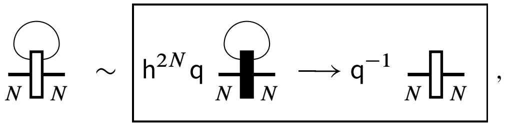
Proof of Theorem 3.11*.* A substitution of eq. (3.23) into eq. (3.18) yields the cone presentation (3.21) with



and eq. (3.22) follows from eq. (3.15). □

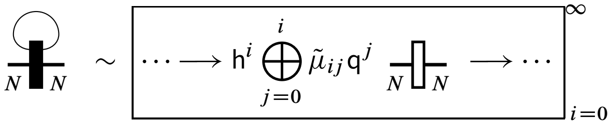
If we connect the endpoints of the upper single line in eq. (3.21) and apply the framing relation (1.9) to the last diagram in that relation, then we come to the following corollary

Corollary 3.13. *There is a homotopy equivalence*



(3.24)

where



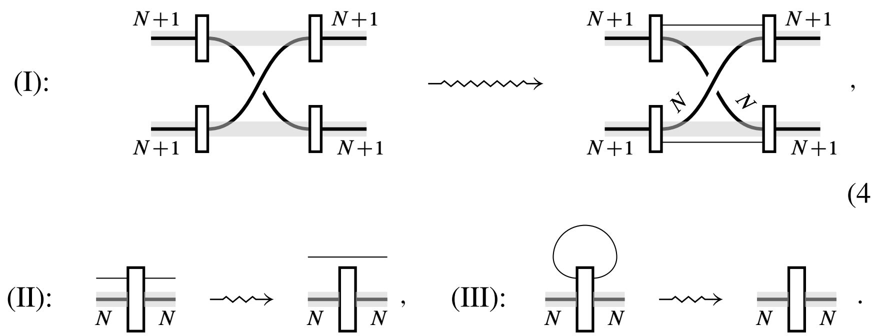
(3.25)

4. The morphisms f_N and the proof of Theorem 2.13

4.1. General setup. For a link diagram D we give a precise definition of a diagram (a complex) D_N. D_N is constructed by first N-cabling all components of D and then placing a categorified Jones–Wenzl projector at every edge of D, an edge being a piece of N-cabled strand between two crossings.

The map f_N of eq. (2.16) is a composition of many maps between Khovanov homologies of a sequence of diagrams related by local transformations, the first diagram in that sequence being D_{N+1} and the last being D_N (recall that maps go backwards).

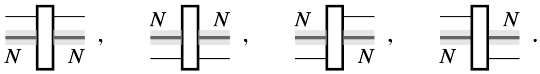
We use three types of local transformations, which are based on the following local replacements:



(4.1)

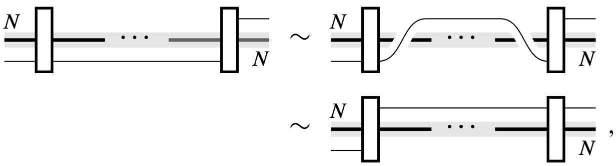
The thick gray lines in these pictures mark the B-circles of the diagram $B(D)$.

The transition from the diagram D_{N+1} to D_N is performed in two stages. At the first stage we apply the first replacement of (4.1) to every crossing of D_{N+1}. The result is the diagram $\widetilde{D}_N$, which consists of two parts connected at Jones–Wenzl projectors. The first part is the N-cabled diagram D_N and the second part consists of non-intersecting circles formed by single lines appearing in the final diagrams of replacements (I) of (4.1). These single line circles go along the B-circles. We orient them clockwise and assume that in our pictures the clockwise orientation corresponds to the direction from the left to the right. The circles are attached to D_N at the Jones–Wenzl projectors and those junctions have four possible forms:



(4.2)

At the second stage of the transition from D_{N+1} to D_N we remove the single circle lines of $\widetilde{D}_N$ one-by-one. In order to remove a particular circle we select an 'initial' Jones–Wenzl projector on it and then detach the single lines from other projectors going clockwise. During this process, the single line between the initial and current Jones–Wenzl projectors are kept on the same side of the B-circles. If the current projector has the incoming and outgoing single lines on the opposite sides of the B-circle (third and fourth type of (4.2)) then, prior to detachment, we perform the following transformation for the junction of the third type (and a similar transformation for the fourth type):



(4.3)

In these pictures the left projector is initial, the right projector is current, the first homotopy equivalence comes from the Reidemeister moves, while the second equivalence comes from eq. (3.6). Note that the single line between the initial and current projectors is kept always above the rest of the diagram.

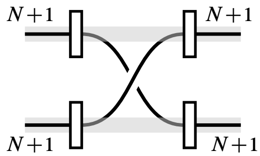
After the single lines attached to the current projector are brought to the same side of the B-circle, we detach the single line from that projector by the local replacement (II) of (4.1) and pass to the next projector on the single line.

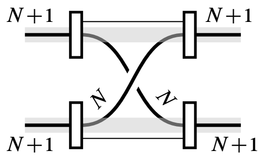
The single line is kept above the rest of the diagram, so once it is detached from all projectors except the initial one, it can be contracted to a small loop attached to that initial projector with the help of Reidemeister moves. The final step is the removal of that loop by the replacement (III) of (4.1). After all single line circles are removed, the diagram $\widetilde{D}_N$ becomes D_N.

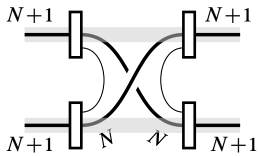
Our transition from D_{N+1} to D_N is generally similar to that used by C. Armond[1], but the details are different. In particular, we do not replace $(N+1)$-cable crossings by projectors, but rather apply replacements (I) of (4.1) directly to the crossings.

4.2. Local transformations generate isomorphisms at low *h*-degrees. We describe the local transformations related to replacements (4.1) and show that the corresponding maps (3.3) between shifted homologies are isomorphisms at low h-degrees, thus proving Theorem 2.13.

4.2.1. Local transformation I. Set

$\tau_{\mathrm{i}} =$  , (4.4a)

$\tau_{\mathrm{f}} =$  , (4.4b)

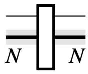
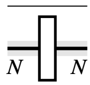
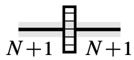
$\tau_{\mathrm{c}} = \mathsf{h}^{N+\frac{1}{2}}$  , (4.4c)

while $\langle\!\langle \tau_{\mathrm{f}}' \rangle\!\rangle = \mathsf{h}^{-N+\frac{1}{2}} \langle\!\langle \tau_{\mathrm{f}} \rangle\!\rangle$. Theorem 3.4 provides the exact triangle relation (3.1).

Proposition 4.1. *Let D_{i} be the diagram constructed by performing local replacements I of* (4.1) *on some vertices of D_{N+1} and let D_{f} be the diagram constructed by performing the local replacement I on the 'current' vertex in D_{i}. Then the degree preserving map* (3.3) *is an isomorphism on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}$ for $i \leq 2N-1$.*

Proof. Let D_{c} be the diagram constructed by performing the local replacement $\tau_{\mathrm{i}} \rightsquigarrow \tau_{\mathrm{c}}$ on the current vertex. We estimate the homological order of $\mathrm{H}^{\mathrm{Kh}}(D_{\mathrm{c}})$ with the help of Theorem 2.10: since $n_\times(D_{\mathrm{c}}) = n_\times(D_{\mathrm{i}}) - 2N - 1$, then $\mathrm{H}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{c}}) = 0$ for $i \leq -\frac{1}{2} n_\times(D_{\mathrm{i}}) + 2N$ (we took into account the shift $\mathsf{h}^{N+\frac{1}{2}}$ of τ_{c} in eq. (4.4)) and the claim of the theorem follows from Proposition 3.1. □

4.2.2. Local transformation II. Set

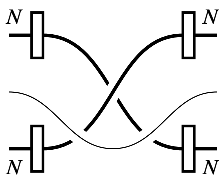
$\tau_{\mathrm{i}} =$  , $\tau_{\mathrm{f}} =$  , $\tau_{\mathrm{c}} = \mathsf{h}$  , (4.5)

while $\langle\!\langle \tau_{\mathrm{f}}' \rangle\!\rangle = \langle\!\langle \tau_{\mathrm{f}} \rangle\!\rangle$. The exact triangle relation (3.1) is provided by Theorem 3.8.

Proposition 4.2. *Let D_{i} be a diagram constructed by removing some single line circles from $\widetilde{D}_N$ and by detaching the 'current' single line circle from the projectors which lie between the initial one and the current one and let D_{f} be the diagram constructed from D_{i} by detaching the single line from the current projector. Then the degree preserving map (3.3) is an isomorphism on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}$ for $i \le N-1$.*

The proof uses the following

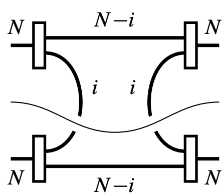
Lemma 4.3. *The tangle*

$\tau =$  (4.6)

has a homological bound $|\tau|_{\mathrm{h}} \ge -\frac{1}{2}N^2$.

Remark 4.4. This bound is better than the crude bound of Theorem 2.10. In fact, it coincides with that bound, if we neglect the intersections between the single line and the N-cables.

Proof of Lemma 4.3. Applying eq. (3.10) to the N-cable crossing in τ we get the presentation

$$\langle\!\langle \tau \rangle\!\rangle \sim \mathsf{h}^{-\frac{1}{2}N^2} \boxed{\bigoplus_{i=0}^{N} \mathsf{h}^{i^2} \begin{Bmatrix} N \\ i \end{Bmatrix}_{\mathsf{h}} \tau_i}_{\circlearrowleft}, \quad \langle\!\langle \tau_i \rangle\!\rangle =$$  .

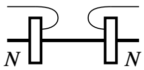
The homological order of τ_i can be estimated with the help of Theorem 2.10: $|\langle\!\langle \tau_i \rangle\!\rangle|_{\mathrm{h}} \ge -i$. Since the polynomial $\left\{ {N \atop i} \right\}_{\mathsf{h}}$ has only non-negative powers of h and $i^2 - i \ge 0$ for all integer i, we come to the estimate of Lemma 4.3. □

Proof of Proposition 4.2. Let D_{c} be the diagram constructed from D_{i} by replacing the current projector (τ_{i} of eq. (4.5)) with the tangle complex τ_{c} of eq. (4.5). By Proposition 3.1, we have to prove the bound:

$$\mathrm{H}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{c}}) = 0 \quad \text{for } i \le -\tfrac{1}{2} n_{\times}(D_{\mathrm{i}}) + N.$$

Since the complex τ_{c} of eq. (4.5) is a multi-cone (3.15) generated by an 'elementary' tangle

$$\tau_{\mathrm{e}} \; = \tag{4.7}$$

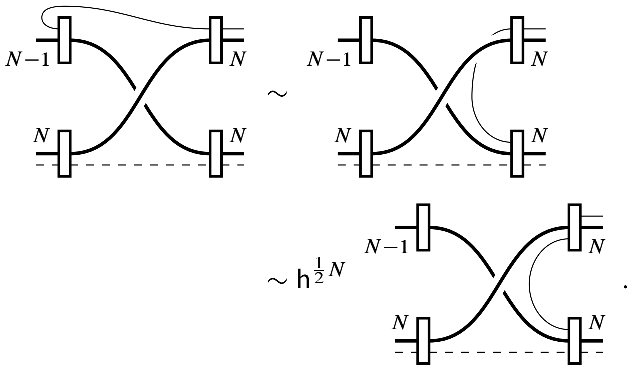


then, according to Remark 1.3, it is sufficient to prove

$$\mathrm{H}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{e}}) = 0 \quad \text{for } i \le -\tfrac{1}{2} n_{\times}(D_{\mathrm{i}}) + N - 1, \tag{4.8}$$

where D_{e} is the diagram constructed by replacing τ_{i} in D_{i} with τ_{e}.

Consider a tangle within D_{e} which consists of the right half of τ_{e} and the cable crossing which follows the current projector and transform its complex with the help of two homotopy equivalences:



(4.9)

The first equivalence comes from sliding the upper left projector down right along its N-cable, and the second equivalence comes from eq. (3.6). The dashed line indicates the possible presence of another single line which has not been removed yet, however, it plays no role in these calculations.

Let D'_{e} denote the diagram D_{e} in which the left tangle of eq. (4.9) has been replaced by the right tangle, then

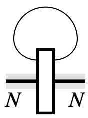
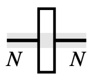
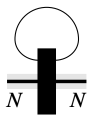
$$D_{\mathrm{e}} \sim \mathsf{h}^{\frac{1}{2}N} D'_{\mathrm{e}}. \tag{4.10}$$

We would like to estimate $|D'_{\mathrm{e}}|_{\mathrm{h}}$ with the help of Theorem 2.10. In doing so we would have to take into account possible crossings coming from the stretch of the single line between the initial projector and the left projector of the tangle τ_{e} of eq. (4.7) and N-cables participating in the crossings attached to the current single line circle. These new crossings are generated by the Reidemeister moves involved in the first homotopy equivalence of eq. (4.3): when a single line is flipped to the other side of the circle, it may come across the N-cable crossings, from which parts of this line originate through replacements I of (4.1) (see the picture (4.6) of the tangle τ). However, Remark 4.4 indicates that these crossings between the single line and the N-cables may be ignored when applying the estimate of Theorem 2.10, so $|D'_{\mathrm{e}}|_{\mathrm{h}} \geq -\frac{1}{2}n'_{\times}(D'_{\mathrm{e}})$, where $n'_{\times}(D'_{\mathrm{e}})$ is the number of single line intersections within D'_{e}, except those which we can ignore.

The cable intersection of the left tangle of eq. (4.9) involves two N-cables, while the same intersection in the right tangle involves both a N-cable and a $(N-1)$-cable, hence $n'_{\times}(D'_{\mathrm{e}}) = n_{\times}(D_{\mathrm{i}}) - N$ and the inequality (4.8) follows from eq. (4.10). □

4.2.3. Local transformation III.

Set

$\tau_{\mathrm{i}} =$  , $\tau_{\mathrm{f}} =$  , $\tau_{\mathrm{c}} = \mathsf{h}^{2N}\mathsf{q}$  , (4.11)

while $\langle\!\langle \tau'_{\mathrm{f}} \rangle\!\rangle = \mathsf{q}^{-1}\langle\!\langle \tau_{\mathrm{f}} \rangle\!\rangle$ and the cone relation (3.1) is eq. (3.24).

Proposition 4.5. *Let D_{i} be a diagram constructed by removing some single line circles from $\widetilde{D}_N$ and by detaching the 'current' single line circle from the all of its projectors, except the initial one, to which it is attached as in the picture (4.11) of tangle τ_{i}. Let D_{f} be the diagram D_{i} from which this circle is completely removed. Then the degree preserving map (3.3) is an isomorphism on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}$ for $i \leq 2N-2$.*

Proof. Since the complex τ_{c} of eq. (4.11) is a multi-cone (3.25) generated by the elementary tangle τ_{f} of eq. (4.11), then, according to Remark 1.3, the claim of this proposition would follow from the bound

$$\mathrm{H}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{f}}) = 0 \quad \text{for } i \leq -\tfrac{1}{2}n_{\times}(D_{\mathrm{i}}) - 1.$$

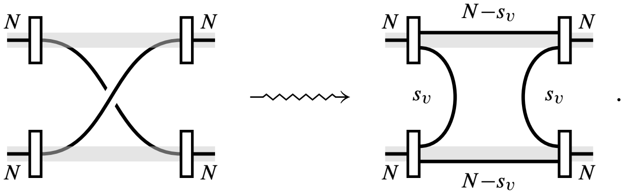
The latter follows from Theorem 2.10 coupled with an obvious relation $n_{\times}(D_{\mathrm{i}}) = n_{\times}(D_{\mathrm{f}})$. □

4.3. Proof of Theorem 2.13.

Of all three types of local transformations considered in Propositions 4.1, 4.2 and 4.5, it is the local transformation (4.5) which yields the weakest estimate of the homological degrees at which the map (2.16) is an isomorphism, and this is the estimate of Theorem 2.13 □

5. Proof of Theorem 2.12

In order to compute the shifted homology $\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_N)$, we apply the colored Khovanov bracket (3.10) to all crossings of D_N. As a result, this diagram turns into a multi-cone of flat diagrams of a special kind. Let $\mathfrak{V}$ be the set of crossings of D. A *state* of D_N is a map $s\colon \mathfrak{V} \to \{0, 1, \ldots, N\}$, $v \mapsto s_v$. It determines a diagram D_s constructed by performing the following local transformations at each crossing $v \in \mathfrak{V}$:



The gray strips in D_s combine into B-circles in the background of this diagram.

The diagrams D_s for all states s generate a multi-cone presentation of D_N, hence $\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_N)$ can be computed by spectral sequence, and its E_1 term is a sum of appropriately shifted homologies $\mathrm{H}^{\mathrm{Kh}}(D_s)$:

$$E^1 = \mathsf{q}^{N\kappa_D} \bigoplus_{\substack{s\\k\geq 0}} m_{s,k}\, \mathsf{h}^{\|s\|+k}\, \mathrm{H}^{\mathrm{Kh}}(D_s),$$

where $\|s\| = \sum_{v\in\mathfrak{V}} s_v^2$. Hence, a component $E^1_{i,j}$ of bi-degree i, j (both are homological and have nothing to do with filtration!) has the form

$$E^1_{i,j} = \bigoplus_{\substack{s\\k\geq 0}} m_{s,k} \mathrm{H}^{\mathrm{Kh}}_{i-\|s\|-k,\,j-N\kappa_D}(D_s). \tag{5.1}$$

As we already noted in Remark 1.3, further steps of spectral sequence may only reduce homology, hence $E^1_{i,j} = 0$ implies $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,j}(D_N) = 0$. Moreover, all differentials have bi-grading (-1,1), hence $E^1_{i+1,j-1} = E^1_{i-1,j+1} = 0$ implies $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,j}(D_N) = E^1_{i,j}$. These arguments imply that Theorem 2.12 follows from the proposition

Proposition 5.1. $E^1_{i,j} = 0$ *if one of the following conditions is satisfied:*

$$i < 0, \tag{5.2}$$

$$j < -\tfrac{1}{2}i - \tfrac{1}{2}\chi_D - \tfrac{3}{2}\chi_D^{\mathrm{in}}, \tag{5.3}$$

$$j < -i - \chi_D^{\mathrm{in}} - N\kappa_D. \tag{5.4}$$

Moreover, if D is B-adequate, then

$$E^1_{i,-i} = \begin{cases} 0, & \text{if } i \neq 0, \\ \mathbb{Q}, & \text{if } i = 0. \end{cases} \tag{5.5}$$

In view of eq. (5.1), Proposition 5.1 follows from the next one:

Proposition 5.2. $\mathrm{H}^{\mathrm{Kh}}_{i,j}(D_s) = 0$ *if one of the following three conditions is satisfied:*

$$i < 0, \tag{5.6}$$

$$j < -\tfrac{1}{2}\|s\| - \tfrac{1}{2}\chi_D - \tfrac{3}{2}\chi_D^{in} - N\kappa_D, \tag{5.7}$$

$$j < -\|s\| - \chi_D^{in} - N\kappa_D. \tag{5.8}$$

Furthermore, if a diagram D is B-adequate, then $\mathrm{H}^{\mathrm{Kh}}_{i,j}(D_s) = 0$ *for*

$$\mathrm{H}^{\mathrm{Kh}}_{0,-\|s\|-N\kappa_D}(D_s) = \begin{cases} 0, & \text{if } \|s\| > 0, \\ \mathbb{Q}, & \text{if } \|s\| = 0. \end{cases} \tag{5.9}$$

Proof of Proposition 5.1*.* Conditions (5.2)–(5.4) follow easily from (5.6)–(5.8). In order to prove eq. (5.5), observe that according to eq. (5.5), $E^1_{i,-i}$ is a sum of homologies $\mathrm{H}^{\mathrm{Kh}}_{i',j'}(D_s)$ with $i' = i - \|s\| - k$, $j' = -i - N\kappa_D$, hence

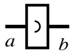
$$j' = -\|s\| - N\kappa_D - i' - k.$$

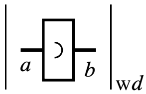
Since $i' \geq 0$ by eq. (5.6) and $k \geq 0$ by eq. (5.1), then in view of the bound (5.8) with $\chi_D^{in} = 0$ we conclude that non-trivial contributing homology exists only for $i' = k = 0$, so $j' = -\|s\| - N\kappa_D$ and $i = \|s\|$. Thus we proved that $E^1_{i,-i}$ is a sum of homologies $\mathrm{H}^{\mathrm{Kh}}_{0,-\|s\|-N\kappa_D}(D_s)$ with $\|s\| = i$, hence eq. (5.5) follows from eq. (5.9) and from the fact that the state s with $\|s\| = 0$ is unique (it corresponds to B-splicing all crossings in D_N) and its multiplicity in the presentation of $E^1_{i,-i}$ is one, because complete B-splicing has multiplicity one in eq. (3.10). □

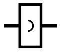
Proof of Proposition 5.2*.* First of all, we observe that the bound (5.6) follows from the fact that D_s has no crossings, while the formulas (1.12), (1.13) for the categorified Jones–Wenzl projector contain only non-negative shifts of h-degree.

The proof of other bounds requires a simplification of the complex, whose homology yields Khovanov homology $\mathrm{H}^{\mathrm{Kh}}(D_s)$. We cut the diagram D_s into pieces (tangles), simplify their Khovanov complexes and then glue those complexes back together.

Consider a neighborhood of a B-circle c within a diagram D_s and cut in the middle all strut lines which are attached to it. We are going to simplify the complex of the resulting colored tangle $\tau_{s,c}$ by inserting two extra Jones–Wenzl projectors in it and then purging all other (preexisting) projectors.

For $a \geq b$ let the box  denote any TL tangle with the property

 $= b.$

In other words, a tangle  contains no cups, but only caps and straight segments.

Lemma 5.3. *The Khovanov categorification complex of the colored tangle diagram $\tau_{s,c}$ can be presented in the form*

$$\langle\!\langle \tau_{s,c} \rangle\!\rangle \sim \boxed{\cdots \longrightarrow \mathsf{h}^i \bigoplus_{0\leq j\leq i} \mathsf{q}^j \left(\bigoplus_{\tau} m_{ij,\tau} \langle\!\langle \tau \rangle\!\rangle \right) \longrightarrow \cdots}_{\,i=0}^{\,\infty}, \qquad (5.10)$$

where the diagrams τ are of one of two types depicted in Figure 1, in which $N_1, N_2 \leq N$.

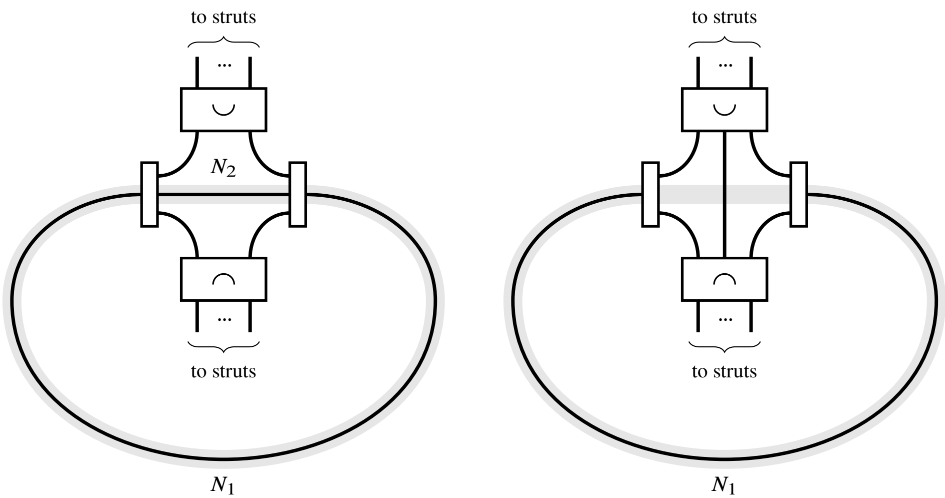


Figure 1. A purged vicinity of a B-circle

We will prove the lemma after we finish the proof of Proposition 5.2.

By gluing the complexes (5.10) back together we get a complex

$$\langle\!\langle D_s \rangle\!\rangle \sim \boxed{\cdots \longrightarrow \mathsf{h}^i \bigoplus_{0\le j\le i} \mathsf{q}^j \left(\bigoplus_{D_\circ} m_{ij,\tau} \langle\!\langle D_\circ \rangle\!\rangle \right) \longrightarrow \cdots}_{i=0}^{\infty}$$

such that $\mathrm{H}^{\mathrm{Kh}}(D_s) = \mathrm{H}(\langle\!\langle D_s \rangle\!\rangle)$. The 'circle diagrams' $D_\circ$ which result from gluing the diagrams of Figure 1 at the strut line cutting points and replacing projectors with complexes (1.12) and (1.13), consist of multiple single line circles. In view of the second formula of eq. (1.8), the lowest q-degree in the homology $\mathrm{H}^{\mathrm{Kh}}(D_s)$ may be bounded by the highest number of circles in those circle diagrams.

The circles in circle diagrams are of three types. The first type is *jumping* circles: they contain at least one strut line. The circles of the second and third type stay within the same B-circle. A *straight* circle goes along its B-circle, passing straight through each Jones–Wenzl projector on its way. A *winding* circle changes its direction at least twice, because it contains at least one cup and one cap of a constituent TL tangle coming from one of projectors.

Let us prove the inequalities (5.7) and (5.8) by finding upper bounds for the numbers n_{j}, n_{s} and n_{w} of jumping, straight and winding circles respectively in a circle diagram.

We begin with n_{j}. A jumping circle must contain at least two strut lines of an adequate crossing or at least one strut line of an inadequate crossing, so the number of jumping circles n_{j} has a bound:

$$n_{\mathrm{j}} \le \sum_{v\in\mathfrak{V}_{\mathrm{ad}}} s_v + 2 \sum_{v\in\mathfrak{V}_{\mathrm{in}}} s_v, \tag{5.11}$$

where $\mathfrak{V}_{\mathrm{ad}}, \mathfrak{V}_{\mathrm{in}} \subset \mathfrak{V}$ are the subsets of B-adequate and B-inadequate crossings. The obvious inequalities

$$s_v \le \tfrac{1}{2} s_v^2 + \tfrac{1}{2}, \quad 2s_v \le \tfrac{1}{2} s_v^2 + 2, \quad s_v \le s_v^2, \quad 2s_v \le s_v^2 + 1,$$

(the third inequality uses the fact that s_v is integer) indicate that the bound (5.11) implies to other bounds:

$$n_{\mathrm{j}} \le \tfrac{1}{2}\|s\| + \tfrac{1}{2}\chi_D + \tfrac{3}{2}\chi_D^{in}, \quad n_{\mathrm{j}} \le \|s\| + \chi_D^{in}, \tag{5.12}$$

which means that the first three terms in the *r.h.s.* of the inequality (5.7) and the first two terms in the inequality (5.8) bound the negative contribution of jumping circles to the q-degree of $\mathrm{H}^{\mathrm{Kh}}(D_s)$.

Next we prove the bound

$$n_{\mathrm{s},c} + n_{\mathrm{w},c} \le N, \tag{5.13}$$

where $n_{\mathrm{s},c}$ and $n_{\mathrm{w},c}$ are the numbers of straight and winding circles within any given B-circle c. It implies the bound $n_{\mathrm{s}} + n_{\mathrm{w}} \le N\kappa_D$ and combined with the bounds (5.12) they imply the bounds (5.7) and (5.8). In order to prove the bound (5.13), we observe that in the first diagram a straight circle contains one strand from the N_1-cable and one strand from the N_2-cable, while a winding circle contains at least two strands from one of these cables, hence there is a bound

$$n_{\mathrm{s},c} + n_{\mathrm{w},c} \le \tfrac{1}{2}(N_1 + N_2) \le N. \tag{5.14}$$

The second diagram is treated similarly, if we set $N_2 = 0$ in the previous argument. Thus we proved the bounds (5.7) and (5.8).

It remains to prove eq. (5.9). Since this time D is B-adequate, the second inequality of (5.12) becomes $n_{\mathrm{j}} \le \|s\|$. Since we consider only homology of zeroth h-degree, then according to eqns. (1.12) and (1.13), we may replace Jones–Wenzl projectors with identity braids, so there is only one circle diagram $D_\circ$ contributing to $\mathrm{H}^{\mathrm{Kh}}_{0,-\|s\|-N\kappa_D}(D_s)$, and this circle diagram has no winding circles: $n_{\mathrm{w}} = 0$. Furthermore, $n_{\mathrm{s}} \le N\kappa_D$, but if $\|s\| \ne 0$, then there is at least one pair of strut lines in D_s, so $n_{\mathrm{s}} < N$ and $n_{\mathrm{j}} + n_{\mathrm{s}} < \|s\| + N\kappa_D$, hence $\mathrm{H}^{\mathrm{Kh}}_{0,-\|s\|-N\kappa_D}(D_s) = 0$. If $\|s\| = 0$, then D_s has no strut lines and consists of disjoint N-cabled circles, so the relevant circle diagram $D_\circ$ consists of $N\kappa_D$ single-line circles, and $\mathrm{H}^{\mathrm{Kh}}_{0,-N\kappa_D}(D_s) = \mathbb{Q}$ follows from the second equation of (1.8). □

Proof of Lemma 5.3. We prove the lemma by 'purging' categorified Jones–Wenzl projectors appearing in the tangle $\tau_{s,c}$. In order to bring the complex $\langle\!\langle \tau_{s,c} \rangle\!\rangle$ to the form (5.10) with diagrams τ depicted in Figure 1, we insert two extra Jones–Wenzl projectors side-by-side at any place on the cable which runs along the B-circle. Then we go from the front one (relative to the clockwise orientation) to the back one in the clockwise direction, purging each preexisting projector that appears on our way. It is easy to prove by induction that after every projector purge we get a multi-cone presentation

$$\langle\!\langle \tau_{s,c} \rangle\!\rangle \sim \boxed{\cdots \longrightarrow \mathsf{h}^i \bigoplus_{0\le j\le i} \mathsf{q}^j \left(\bigoplus_{\tau} m'_{ij,\tau} \langle\!\langle \tau \rangle\!\rangle \right) \longrightarrow \cdots}_{\,i=0}^{\,\infty}$$

whose constituent diagrams τ have one of two possible forms between the front projector and the first unpurged projector (which lies in the pictures to the left of the dashed line) depicted in Figure 2.

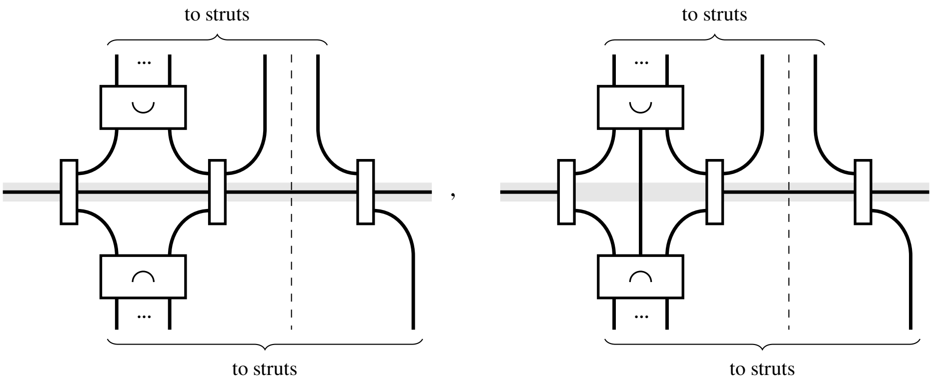


Figure 2. Purging Jones–Wenzl projectors along a B-circle

In both diagrams the left projector on the grey strip is the front one, the middle projector is the first unpurged one and the right projector is the second unpurged one. It is not hard to see that if we purge the middleprojector, then we get similar diagrams with the third projector becoming the first unpurged one (the left diagram may become of either left or right type after the purge, while the right diagram remains of the same type). The q-degree shifts remain non-negative, because the purging does not produce any circles: it just makes explicit various line connections that were hidden inside the constituent TL tangles of the purged projector. □

6. Proof of invariance of the tail homology under B-reduction

Proof of Theorem 2.7. A removal of a B-circle connected to the rest of the B-diagram by a single strut corresponds to the first Reidemeister move, hence the invariance of the tail homology under this removal follows from the fact that tail homology of a B-adequate link is determined by shifted Khovanov homologies of its unicolored diagrams and the latter are invariant under this type of first Reidemeister moves.

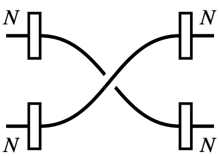
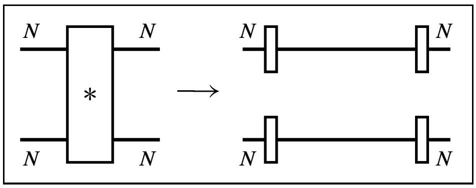
In view of Corollary 2.4, the invariance of the tail homology under the removal of 'extra' struts follows from the next lemma. □

Lemma 6.1. *Suppose that two distinct B-circles of a link diagram D are connected by multiple struts and the diagram D' is constructed by removal of one of those struts. Then there exists a degree preserving map* $\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D'_N) \xrightarrow{g} \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_N)$ *which is an isomorphism on* $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}$ *for* $i \leq N-1$.

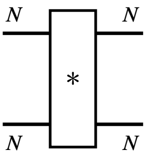
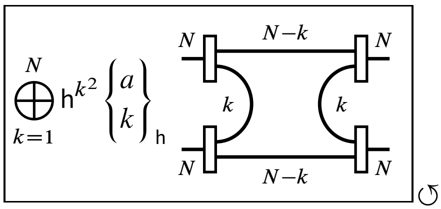
The proof of this lemma is similar to proofs of Section 4: we show that D_N can be constructed from D'_N with the help of a local transformation and prove the homological smallness of the correction diagram.

We need a simple corollary of Theorem 3.5

Corollary 6.2. *The Khovanov bracket formula* (3.8) *for the colored crossing can be recast in the form*

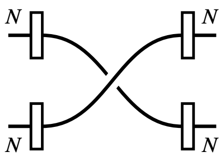
 $\sim \mathsf{h}^{-\frac{1}{2}N^2}$  , (6.1)

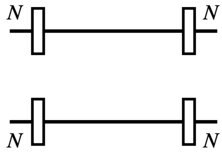
where

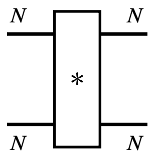
 $\sim$  . (6.2)

Proof. According to the second inequality of (3.9), $k(0) = 0$, hence the the right tangle of the cone (6.1) is the one that appears at $i = 0$ in the multi-cone (3.8). According to eq. (3.10), this tangle has multiplicity one, so this is the only place where it may appear in that multi-cone, and it has a zero shift of h-degree. □

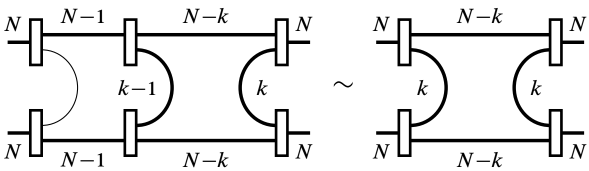
Proof of Lemma 6.1*.* Since D' is constructed from D by a removal of a single crossing (strut of $\mathrm{B}(D)$), we set

$\tau_{\mathrm{i}} =$  ,

$\tau_{\mathrm{f}} =$  ,

$\tau_{\mathrm{c}} = \mathsf{h}^{-\frac{1}{2}N^2}$  ,

so that $D_{\mathrm{i}} = D$ and $D_{\mathrm{f}} = D'$, while the relation (3.1) comes from (6.1) if we set $m_{\mathrm{f}} = 0$ and $n_{\mathrm{f}} = -\frac{1}{2}N^2$. Since $n_{\times}(\tau_{\mathrm{i}}) = N^2$, in view of Proposition 3.1 it remains to establish the bound $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{c}}) = 0$, if $i \leq N$. Let $D_{\mathrm{c},k}$ be the diagram D_{c} in which τ_{c} is replaced by a constituent tangle $\tau_{\mathrm{c},k}$ from the *r.h.s.* of eq. (6.2), in which one strand of a k-cable is separated from the others:

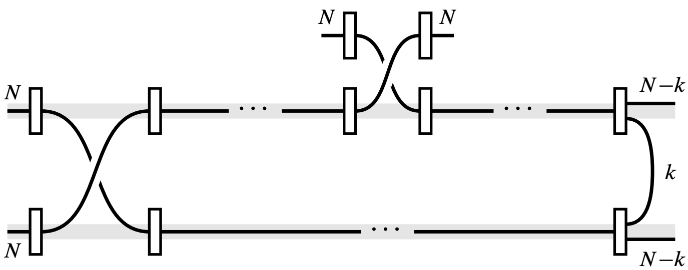
$\tau_{\mathrm{c},k} =$  .

The lump sum presentation (6.2) of τ_{c} allows us to use Remark 1.3: it is sufficient to establish a bound

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{c},k}) = 0, \quad \text{if } i \leq N - k, \tag{6.3}$$

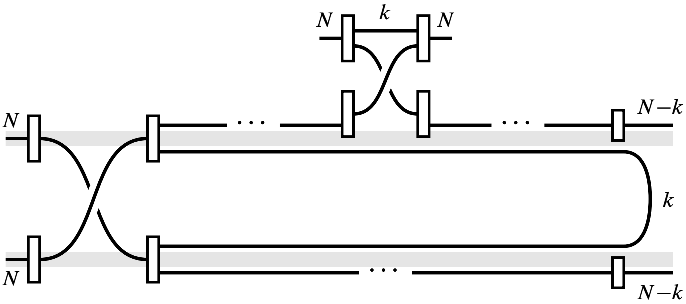
because the tangle $\tau_{\mathrm{c},k}$ has an extra h-degree shift h^{k^2} in the lump sum formula, see eq. (6.2), and $k^2 - k \geq 0$.

Consider the portion of $D_{\mathrm{c},k}$ between the left k-cable of $\tau_{\mathrm{c},k}$ and another crossing which connects the same B-circles:



(6.4)

As usual, gray strips indicate B-circles of the B-diagram. We showed explicitly one of the crossings attached to a B-circles. Consider a modification of this diagram which results from a repeated application of of Propositions 4.1 and 4.2 to these crossings:



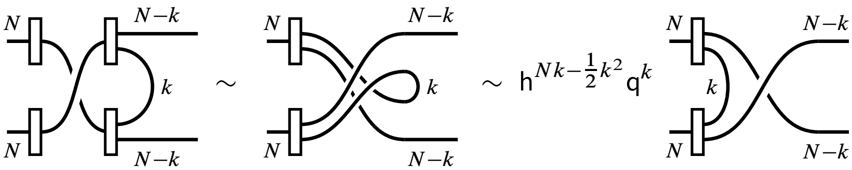
. (6.5)

Let $D'_{\mathrm{c},k}$ be the diagram constructed from $D_{\mathrm{c},k}$ by replacing the subdiagram (6.4) with the diagram (6.5). According to Propositions 4.1 and 4.2 there is a map of shifted Khovanov homologies $\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D'_{\mathrm{c},k}) \to \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D_{\mathrm{c},k})$, which is an isomorphism on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{c},k})$ for $i \leq N-k$. We are going to show that

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}(D'_{\mathrm{c},k}) = 0 \quad \text{if } i \leq N-k, \tag{6.6}$$

hence this will imply the bound (6.3).

Consider a sequence of homotopy equivalences:



. (6.7)

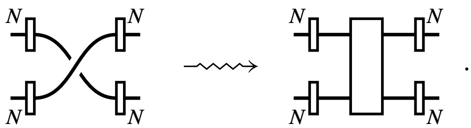
Here the first homotopy equivalence comes from sliding k-cable projectors to the left along N-cables and then contracting double projectors into single ones, while the second homotopy comes from eqns. (1.14) and (3.6). Let $D''_{\mathrm{c},k}$ be the diagram constructed from $D'_{\mathrm{c},k}$ by replacing the left tangle of eq. (6.7) with the right tangle. Since $n_\times(D''_{\mathrm{c},k}) = n_\times(D'_{\mathrm{c},k}) + (N-k)^2 - N^2$, homotopy equivalence (6.7) implies the isomorphism of shifted Khovanov homologies

$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D'_{\mathrm{c},k}) = \mathsf{h}^{k(2N-k)}\mathsf{q}^{k}\, \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D''_{\mathrm{c},k})$$

and the bound (6.6) follows from Theorem 2.11. □

A. A single crossing of colored strands approximates a Jones–Wenzl projector

Let D be a diagram which may include both single and cabled lines as well as Jones–Wenzl projectors. Suppose that D has a crossing of two N-cables with a projector on each. Let D' be a diagram, in which the crossing is replaced with a Jones–Wenzl projector:

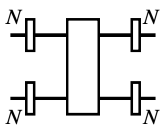


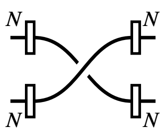
Theorem A.1. *There exists a map*

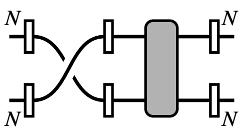
$$\widetilde{\mathrm{H}}^{\mathrm{Kh}}(D) \xrightarrow{g} \widetilde{\mathrm{H}}^{\mathrm{Kh}}(D')$$

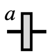
which is an isomorphism on $\widetilde{\mathrm{H}}^{\mathrm{Kh}}_{i,\bullet}$ *for* $i \le 2N-2$.

Proof. Consider three tangles

$\tau_{\mathrm{i}} =$  ,

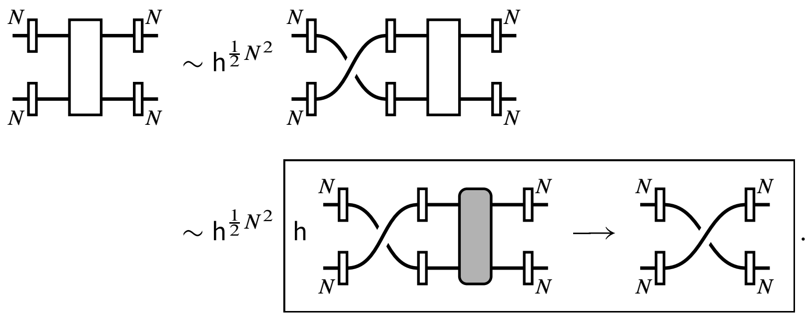
$\tau_{\mathrm{f}} =$  ,

$\tau_{\mathrm{c}} = \mathsf{h}^{\frac{1}{2}N^2}$  ,

where the complex  is defined by eq. (1.13), and set

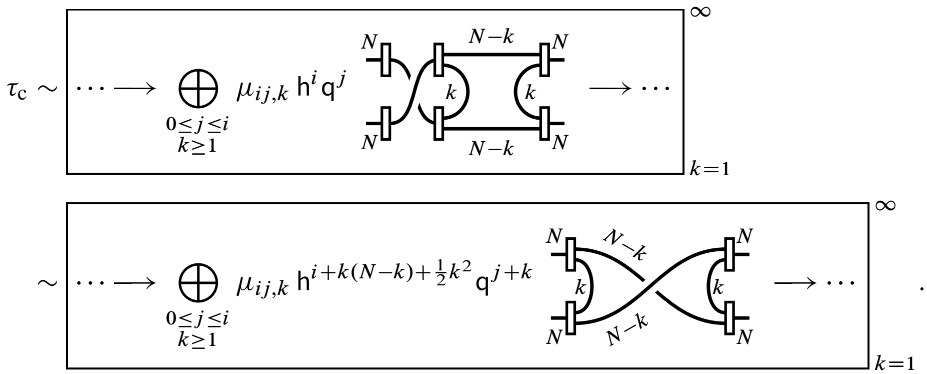
$$\langle\!\langle \tau'_{\mathrm{f}} \rangle\!\rangle = \mathsf{h}^{\frac{1}{2}N^2} \langle\!\langle \tau_{\mathrm{f}} \rangle\!\rangle.$$

These tangles have a relation (3.1) which comes from a sequence of homotopy equivalences:



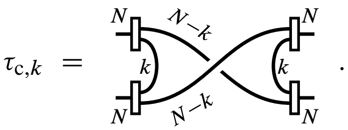
Here the first homotopy equivalence comes from eq. (3.6), while the second equivalence comes from eq. (1.12).

In order to put a bound on the homological order of D_{c}, we purge the gray box in τ_{c}, that is, we contract all constituent TL tangles, whose cup or cap is connected directly to Jones–Wenzl projectors sitting on N-cables. After the purge, the complex of τ_{c} takes the form



We used homotopy equivalence (6.7). Note that there are no tangles with $k = 0$, because the complex (1.13) does not contain identity braids.

Let $D_{\mathrm{c},k}$ be the diagram D_{c} in which the complex τ_{c} is replaced by the tangle diagram



According to Theorem 2.10, $\mathrm{H}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{c},k}) = 0$ for $i \le -\frac{1}{2}(N-k)^2 - \frac{1}{2}n_\times(D_{\mathrm{i}}) - 1$, so, by Remark 1.3, $\mathrm{H}^{\mathrm{Kh}}_{i,\bullet}(D_{\mathrm{c}}) = 0$ for $i \le -\frac{1}{2}n_\times(D_{\mathrm{i}}) + 2N - 2$ (here we used inequality $2Nk - k^2 \ge 2N - 1$ for $k \ge 1$). Now the claim of Theorem A.1 follows from eq. (3.4). $\square$

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