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A spectral sequence on lattice homology

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Abstract. Using the link surgery formula for Heegaard Floer homology we find a spectral sequence from the lattice homology of a plumbing tree to the Heegaard Floer homology of the corresponding 3-manifold. This spectral sequence shows that for graphs with at most two "bad" vertices, the lattice homology is isomorphic to the Heegaard Floer homology of the underlying 3-manifold.

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Contents

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1. Introduction

Heegaard Floer homologies were introduced in 2001 by the first and third authors as invariants of closed, oriented 3-manifolds [17, 18]. The construction of the invariants relies on a choice of a Heegaard decomposition of the 3-manifold at hand, and then applies Lagrangian Floer homology to a symplectic manifold (and two Lagrangian subspaces of it) associated to the Heegaard decomposition. The theory comes in many variants: the version $\mathrm{HF}^-(Y)$ is the most powerful in 3- and 4-dimensional applications, while the simpler $\widehat{\mathrm{HF}}(Y)$ turns out to be more accessible for computation. Since the introduction of the invariants, many results have been found towards their computability [3, 6, 13, 23], but a convenient computational scheme in general is still missing. For 3-manifolds which can be presented as the boundary of a negative definite plumbing with at most one *bad vertex* (in the sense of Definition 2.1), a relatively simple computational algorithm was described in [16].

Motivated by the result of [16], in [8] András Némethi introduced an algebraic object, the *lattice homology* for plumbed 3-manifolds, which – when considered for negative definite plumbings – provides a bridge between certain analytic properties of the singularity with resolution the given plumbing, and the differential topology of the boundary 3-manifold. Since lattice homology extends the combinatorial approach found in [16] to more general plumbings, it can be shown that for a negative definite plumbing tree G with at most one bad vertex, the lattice homology $\mathbb{HF}^-(G)$ and the Heegaard Floer homology group $\mathrm{HF}^-(Y_G)$ of the plumbed 3-manifold Y_G (obtained by plumbing circle bundles over spheres according to G) are isomorphic. Indeed, Némethi extended the isomorphism of [16] to a larger class of plumbing graphs which he called almost-rational [8]. (For the definition of these notions, see Section 2. See also [11] for related results.) His results can be viewed as evidence for a conjecture that, for a plumbing tree G, the lattice homology $\mathbb{HF}^-(G)$ is isomorphic to the Heegaard Floer homology $\mathrm{HF}^-(Y_G)$ of the corresponding 3-manifold Y_G. Further evidence to the validity of this conjecture is provided by the proof of a surgery exact triangle in lattice homology by Greene and (independently) by Némethi [2, 10], and by the introduction of knot lattice homology [14], cf. also [15].

In the present paper we show the existence of a spectral sequence from the lattice homology of a tree G to the Heegaard Floer homology of the corresponding plumbed 3-manifold Y_G. This spectral sequence is derived from the surgery presentation of Heegaard Floer homology from [5], compare also [20, 21]. In the statement below, the groups $\mathbb{HF}^-(G)$ and $\mathbf{HF}^-(Y_G)$ denote the regular lattice and Heegaard Floer homologies after completion (with respect to the U variable).

For a definition of $\mathbb{HF}^-(G)$ see Section 3. When the 3-manifold Y_G is a rational homology sphere then the completed versions of the homologies determine the ones defined over the polynomial ring, cf. [5]; moreover, the closed four-manifold invariants can be defined using only the completed theory. The main result of the paper is:

Theorem 1.1. *Suppose that G is a plumbing tree of spheres, and let Y_G be the corresponding 3-manifold. Then, there is a spectral sequence $\{\mathcal{E}_i\}_{i=1}^{\infty}$ with the following properties.*

- *The $\mathcal{E}_2$-term of the spectral sequence is isomorphic to the lattice homology $\mathbb{HF}^-(G)$.*
- *The spectral sequence converges to $\mathbf{HF}^-(Y_G)$.*
- *The lattice homology $\mathbb{HF}^-(G)$ naturally splits according to Spin^c structures over Y_G (see text preceding Definition 3.6); similarly, $\mathbf{HF}^-(Y_G)$ splits according to Spin^c structures. The spectral sequence respects these splittings.*
- *If $\mathbf{s} \in \mathrm{Spin}^c(Y_G)$ is a torsion Spin^c structure (e.g. if Y_G is a rational homology sphere, this holds for any $\mathbf{s} \in \mathrm{Spin}^c(Y_G)$), the isomorphism of the $\mathcal{E}_2$-term with $\mathbb{HF}^-(G)$ preserves the absolute Maslov grading.*
- *If $\mathbf{s} \in \mathrm{Spin}^c(Y_G)$ is a non-torsion Spin^c structure, the isomorphism of the $\mathcal{E}_2$-term with $\mathbb{HF}^-(G)$ preserves the relative Maslov grading.*

Remark 1.2. The $\mathcal{E}_\infty$-term of the above spectral sequence (as a sequence of modules over $\mathbb{F}[\![U]\!]$) recovers $\mathbf{HF}^-(Y_G)$ only as a vector space over $\mathbb{F}$. More information about the $\mathbb{F}[\![U]\!]$-module structure can be obtained by applying an analogous spectral sequence over $\mathbb{F}[U]/U^n$, see Theorem 4.11 below, and also the proof of Corollary 1.3.

As an application, we derive the following result. (For the definition of type n graphs, see Definition 2.1 in Section 2. Negative definite type n graphs include graphs with at most n bad vertices.) See [11, Section 8] for special cases of this result.

Corollary 1.3. *If a plumbing tree G is of type 2 then the lattice homology of G is isomorphic to the Heegaard Floer homology of the underlying 3-manifold Y_G.*

The paper is organized as follows. In Section 2 we fix notations and describe some necessary definitions, while in Section 3 we recall the basic concepts of lattice homology. Section 4 is devoted to the discussion of the spectral sequence, and finally in Section 5 we prove Corollary 1.3. In this proof we use the surgery exact sequence of Greene and Némethi [2, 10]. For completeness, in an Appendix we include a proof of this result adapted to the conventions used throughout our paper.

2. Background

Suppose that Γ is a tree on $n = n(\Gamma)$ vertices with vertex set $V = \mathrm{Vert}(\Gamma) = \{v_1, \dots, v_{n(\Gamma)}\}$, while G is the same graph together with an integer $m_v \in \mathbb{Z}$ (a *framing*) attached to each vertex v of Γ. Let M_G denote the associated incidence matrix (with framings in the diagonal). The plumbing 4-manifold defined by G (when we plumb disk bundles over spheres according to G) will be denoted by X_G, and its boundary 3-manifold is Y_G. It is not hard to see that M_G is the intersection matrix of the 4-manifold X_G in the basis $\{E_1, \dots, E_{n(\Gamma)}\} \subset H_2(X_G; \mathbb{Z})$ where E_i corresponds to the vertex v_i ($i = 1, \dots, n(\Gamma)$). Let d_v denote the number of neighbors of a vertex v in the tree G; this quantity is sometimes called the degree (or valency) of the vertex v_i. Although lattice homology can be defined for graphs containing cycles, in the present work we will restrict our attention to trees and forests (disjoint unions of trees).

Definition 2.1.
- Suppose that G is a negative definite plumbing tree (that is, the matrix M_G is negative definite). According to [1] there is a class $Z = \sum_i n_i E_i$ with $n_i \geq 0$ integers and $Z \neq 0$ which satisfies $Z \cdot E_i \leq 0$ for all i, and for any other class $Z' = \sum_i n_i' E_i$ with these properties $n_i \leq n_i'$ holds for all i. The plumbing tree G is called *rational* if for $Z = \sum_i n_i E_i$ we have

$$\Big(\sum_i n_i E_i\Big)^2 = 2\sum_i n_i + \sum_i n_i E_i^2 - 2.$$

(Let the canonical class $K_{\mathrm{can}} \in H^2(X_G; \mathbb{Z})$ defined by $K_{\mathrm{can}}(E_i) = -2 - E_i^2$ for each E_i with $i = 1, \dots, n(\Gamma)$. The above condition is then equivalent to requiring that the geometric genus $p(Z) = \frac{1}{2}(Z^2 + K_{\mathrm{can}} \cdot Z) + 1$ of the class Z vanishes.)

- The vertex v is a *bad vertex* of G if $d_v + m_v > 0$, i.e. the valency of the vertex is more than the negative of its framing.
- The plumbing tree G is of *type* k if it has k vertices $\{v_{i_1}, \dots, v_{i_k}\}$ on which we can change the framings $\{m_{i_1}, \dots, m_{i_k}\}$ in such a way that the result is rational.

Remark 2.2. The above definition differs from the definition of Némethi [7]: we use the term 'bad vertex' as it was used in [16]. For negative definite trees, the notion of almost-rational coincides with type 1. If a negative definite tree G has k bad vertices then it is of type k. The converse is false, cf. the example of Figure 1.

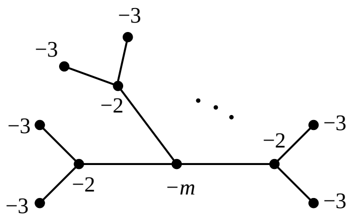

Figure 1. *The plumbing diagram of the figure has at least n bad vertices (where n is the valency of the central $(-m)$-framed vertex) and it is either type 1 or rational (depending on the actual value of m).* Note that all the (-2)-framed vertices are bad vertices (in the sense of Definition 2.1). It is easy to check that for $-m$ sufficiently negative the graph is rational, hence for any value of m the graph is of type 1.

Recall that a plumbing tree also provides a surgery diagram for the 3-manifold it represents: replace each vertex of the diagram with an unknot, and arrange them so that two unknots link if and only if the corresponding vertices are connected by an edge. The framings of the unknots are given by the integers attached to the vertices of the plumbing graph. Note that (viewing the resulting framed link $L = (L_1, \dots, L_\ell)$ as a Kirby diagram) this procedure actually gives the 4-manifold X_G with the given 3-dimensional boundary Y_G. In addition, if L', L'' are two sublinks of the resulting link L in such a way that $L' \subset L''$, then this surgery theoretic approach also provides a cobordism associated to the pair: attach the 4-dimensional 2-handles to $Y_{L'}$ along the components of $L'' - L'$ with the framings specified by G.

For later reference, let $\sigma(G)$ denote the signature of the intersection matrix M_G (or equivalently, the 4-manifold X_G), and define $\chi(G)$ as the cardinality $|V|$ of its vertex set.

3. Review of lattice homology

For the sake of completeness we review the basic notions of lattice homology. This notion was introduced by Némethi [8] (see also [9, 11]). The current presentation is similar to the one discussed in [14], with the difference that now we consider the completed version of the theory (cf. Remark 3.5) and we allow the plumbing graph to be not negative definite. Let G be a given plumbing tree/forest. Recall that G is specified by a graph Γ, together with a map m from the vertices $\mathrm{Vert}(G)$ to $\mathbb{Z}$, and the integer $m(v) = m_v$ is called the *framing* of v.

Next we recall the definition of the completed version of the lattice homology group of G. The group $\mathbb{HF}^-(G)$ is computed as the homology of the combinatorial chain complex $\mathbb{CF}^-(G)$, which is a module over the ring $\mathbb{F}[\![U]\!]$ of formal power series (where $\mathbb{F} \cong \mathbb{Z}/2\mathbb{Z}$). To define it, let $\mathrm{Char}(G) \subset H^2(X_G;\mathbb{Z})$ denote the set of characteristic cohomology classes on the 4-manifold X_G; i.e., it is the subset of those $K \in H^2(X_G;\mathbb{Z})$ which have the property that

$$K \cdot c \equiv c \cdot c \pmod 2$$

for all $c \in H_2(X_G;\mathbb{Z})$. Let $\mathbb{P}(V)$ be the power set of $V = \mathrm{Vert}(G)$, so that $E \in \mathbb{P}(V)$ simply means that $E \subset V$. Now, the $\mathbb{F}[\![U]\!]$-module underlying $\mathbb{CF}^-(G)$ is the direct product

$$\mathbb{CF}^-(G) = \prod_{[K,E]\in \mathrm{Char}(G)\times\mathbb{P}(V)} \mathbb{F}[\![U]\!]\langle [K,E]\rangle. \tag{3.1}$$

$\mathbb{CF}^-(G)$ naturally admits an integral grading, called the *δ-grading*. The δ-grading of an element $U^i \otimes [K,E]$ is given by the cardinality $|E|$ of the subset E. This grading naturally descends to a $\mathbb{Z}/2\mathbb{Z}$-grading by considering only the parity of $|E|$.

We define the boundary map

$$\partial\colon \mathbb{CF}^-(G) \longrightarrow \mathbb{CF}^-(G)$$

as follows. Given a subset $I \subset E$, we define the G-weight $f([K,I]) \in \mathbb{Z}$ of the pair $[K,I]$ by the formula

$$2f([K,I]) = \Big(\sum_{v\in I} K(v)\Big) + \Big(\sum_{v\in I} v\Big)\cdot\Big(\sum_{v\in I} v\Big). \tag{3.2}$$

Moreover, for a pair $[K,E]$, we define the *minimal G-weight* $g([K,E])$ by the formula

$$g([K,E]) = \min\{f([K,I]) \mid I \subset E\}.$$

Next, for the vertex $v \in E \subset V$ consider the quantities

$$A_v([K,E]) = g([K,E-v])$$

and

$$\begin{aligned} B_v([K,E]) &= \min\{f([K,I]) \mid v \in I \subset E\} \\ &= \Big(\frac{K(v)+v\cdot v}{2}\Big) + g([K+2v^*, E-v]), \end{aligned}$$

where v^* denotes the Poincaré dual of the vertex v (when v is regarded as an element of the second homology $H_2(X_G, Y_G; \mathbb{Z})$ of the plumbing 4-manifold). It follows trivially from the definition that $\min\{A_v([K,E]), B_v([K,E])\} = g([K,E])$. Let

$$a_v[K,E] = A_v([K,E]) - g([K,E])$$

and

$$b_v[K,E] = B_v([K,E]) - g([K,E]).$$

(The equality $\min\{a_v[K,E], b_v[K,E]\} = 0$ immediately follows.) We define the boundary map on $\mathbb{CF}^-(G)$ by the formula

$$\partial[K,E] = \sum_{v\in E} U^{a_v[K,E]} \otimes [K, E-v] + \sum_{v\in E} U^{b_v[K,E]} \otimes [K+2v^*, E-v] \quad (3.3)$$

on $[K,E]$ and extend it to $\mathbb{CF}^-(G)$ U-equivariantly and linearly. It is obvious that the boundary map drops the δ-grading by one. A simple calculation (cf. [14]) shows that

Lemma 3.1. *The pair* $(\mathbb{CF}^-(G), \partial)$ *is a chain complex, that is,* $\partial^2 = 0$.

Definition 3.2. The homology $H_*(\mathbb{CF}^-(G), \partial)$ of the chain complex $(\mathbb{CF}^-(G), \partial)$ is the *lattice homology* $\mathbb{HF}^-(G)$ of the plumbing graph G.

Lattice homology is the homology of an infinite direct product. Nonetheless, it enjoys the following finiteness property:

Proposition 3.3. *The lattice homology group* $\mathbb{HF}^-(G)$ *is a finitely generated* $\mathbb{F}[\![U]\!]$ *module.*

Proof. This can be easily seen by induction on the number of bad vertices in G, using the long exact sequence in lattice homology ([2, 10], see also Corollary 6.8), and using the result for graphs with no bad vertices as in [8]. □

Remark 3.4. A number of further variants can be introduced along the same lines: using the coefficient ring $\mathbb{F}[U^{-1}, U]\!]$ (the field of fractions for the ring of formal power series in U) we get $\mathbb{CF}^\infty(G)$ and the corresponding homology theory $\mathbb{HF}^\infty(G)$. Since $\mathbb{CF}^-(G)$ is a subcomplex of $\mathbb{CF}^\infty(G)$, we can consider the quotient complex $\mathbb{CF}^+(G)$, whose homology is $\mathbb{HF}^+(G)$. Setting $U = 0$ in $\mathbb{CF}^-(G)$ we get the homology theory $\widehat{\mathbb{HF}}(G)$, over the base ring $\mathbb{F}$. More generally, by setting $U^n = 0$ ($n \in \mathbb{N}$) we get the chain complex $\widehat{\mathbb{CF}}^{[n]}(G)$ and, as its homology, the version $\widehat{\mathbb{HF}}^{[n]}(G)$.

Remark 3.5. The conventional definition of lattice homology considers direct sum as opposed to direct product in the definition of $\mathbb{CF}^-(G)$ given in (3.1). Also, the usual coefficient ring is the polynomial ring $\mathbb{F}[U]$ rather than $\mathbb{F}[\![U]\!]$. With the changes in the present definition, in fact, we consider a completed version of the theory. If G is negative definite, then the usual definition (given for example, in [8, 14]) and the one given above determine each other. This principle is not true in general, cf. the second example in 3.11. We found the description adapted in this paper to be in accord with the corresponding Heegaard Floer homology theories.

The relation

$$K \sim K' \iff K - K' \in 2H^2(X_G, Y_G; \mathbb{Z})$$

splits the generators into equivalence classes: $U^i \otimes [K, E]$ and $U^j \otimes [K', E']$ are equivalent if $K \sim K'$. This relation then splits the chain complex $\mathbb{CF}^-(G)$ as well, and the definition of the boundary map in (3.3) shows that the boundary map respects this splitting. Since G is a tree, the 4-manifold X_G is simply connected, and hence an element of $K \in \mathrm{Char}(G)$ specifies a Spin^c structure $\mathbf{t}_K$ on X_G, therefore (by restricting $\mathbf{t}_K$ to the boundary Y_G) induces a Spin^c structure $\mathbf{s}_K$ on Y_G. It is not hard to see that $K \sim K'$ holds if and only if $\mathbf{s}_K$ and $\mathbf{s}_{K'}$ are isomorphic Spin^c structures on Y_G. Hence both the chain complexes and the homologies defined above split according to the Spin^c structures of Y_G. Recall that $\mathbb{CF}^-(G)$ admits a δ-grading (given for the generator $[K, E]$ by $|E|$), splitting the homologies further:

Definition 3.6. For $i \geq 0$ define $\mathbb{HF}^-_i(G, \mathbf{s})$ as the subgroup of $\mathbb{HF}^-(G)$ spanned by those pairs $[K, E]$ for which $\mathbf{s}_K = \mathbf{s}$ and $|E| = i$.

Lattice homology has a further grading, the (absolute or relative) *Maslov grading*. This structure is simplest to describe in the case where the underlying Spin^c structure is torsion (i.e. the first Chern class of that Spin^c structure is a torsion cohomology class). We give the grading in that case first.

Suppose that the Spin^c structure $\mathbf{s}_K$ associated to a generator $U^i \otimes [K, E]$ is torsion. In this case define the *Maslov grading* $\mathrm{gr}(U^i \otimes [K, E])$ of a generator $U^i \otimes [K, E]$ of $\mathbb{CF}^-(G)$ as

$$\mathrm{gr}(U^i \otimes [K, E]) = -2i + 2g(K, E) + |E| + \frac{1}{4}(K^2 - 3\sigma(G) - 2\chi(G)). \quad (3.4)$$

(Recall that K^2 is defined as the square of nK divided by n^2, where

$$nK \in H^2(X_G, Y_G; \mathbb{Z}) \cong H_2(X_G; \mathbb{Z}),$$

and therefore it admits a cup square. As a result we expect $\mathrm{gr}(U^i \otimes [K, E])$ to be a rational number rather than an integer.)

Lemma 3.7. (*cf.* [14]) *The boundary map drops the Maslov grading* gr *by one.*

Proof. Proceed separately for the two types of components of the boundary map. After obvious simplifications, according to the definition of $a_v[K, E]$ we have that

$$\begin{aligned}
&\mathrm{gr}(U^i \otimes [K, E]) - \mathrm{gr}(U^i \cdot U^{a_v[K,E]} \otimes [K, E - v]) \\
&\quad = 2g([K, E]) + |E| + 2a_v[K, E] - 2g([K, E - v]) - |E - v| \\
&\quad = 1.
\end{aligned}$$

Similarly,

$$\mathrm{gr}(U^i \otimes [K, E]) - \mathrm{gr}(U^i \cdot U^{b_v[K,E]} \otimes [K + 2v^*, E - v]) = 1$$

follows from the same simplifications and the definition of $B_v([K, E])$. □

We will find it convenient to use the following terminology:

Definition 3.8. A *Maslov graded chain complex* is a $\mathbb{Q}$-graded chain complex over $\mathbb{F}[\![U]\!]$ with the property that

- the differential drops grading by one and
- multiplication by U drops grading by two.

Lemma 3.7 and equation (3.4) together say that for a torsion Spin^c structure $\mathbf{s}$ the grading gr gives $\mathbb{CF}^-(G, \mathbf{s})$ a Maslov grading, in the sense of Definition 3.8.

Lemma 3.9. *Suppose that* $\mathbf{s}_K = \mathbf{s}_{K'}$ *is a torsion* Spin^c *structure. Then the difference*

$$\mathrm{gr}(U^i \otimes [K, E]) - \mathrm{gr}(U^j \otimes [K', E'])$$

is an integer, and it is congruent (mod 2) *to the difference* $|E| - |E'|$.

Proof. In the difference the terms coming from $\sigma(G)$ and $\chi(G)$ cancel, and the ones originating from the U-exponents or from the g-function are obviously even. We claim that the difference $\frac{1}{4}(K^2-(K')^2)$ is also even. Since $\mathbf{s}_K = \mathbf{s}_{K'}$, we have that $K' = K + 2x$ for some vector $x \in H^2(X_G, Y_G; \mathbb{Z})$, therefore

$$\frac{1}{4}(K^2 - (K')^2) = x \cdot (K + x),$$

which is even since K is characteristic. (Note that since x is in the relative cohomology, the above product always makes sense.) The only remaining terms are $|E| - |E'|$, verifying the statement. □

We now turn to the non-torsion case. In this case the term K^2 is not defined, since nK is not in $H^2(X_G, Y_G; \mathbb{Z})$ for any non-zero n. Nevertheless, if $\mathbf{s}_K = \mathbf{s}_{K'}$, we can still consider the difference $K^2-(K')^2$ by writing it as $(K-K')\cdot(K+K')$. The assumption $\mathbf{s}_K = \mathbf{s}_{K'}$ then ensures that $K-K'$ admits a lift from $H^2(X_G;\mathbb{Z})$ to $H^2(X_G, Y_G; \mathbb{Z})$, hence the above product makes sense. This provides a possibility of defining a relative Maslov grading. Note, however, that the lift of $K - K'$ is not unique in general: by the long exact sequence of the pair (X_G, Y_G) the ambiguity for choosing such a lift lies in the group $H^1(Y_G;\mathbb{Z}) \cong H_2(Y_G;\mathbb{Z})$. Suppose that x is a lift of $\frac{1}{2}(K - K')$ and $y \in H_2(Y_G;\mathbb{Z})$. Then the difference we get for $K^2 - (K')^2$ by using x or $x + y$ can be easily computed to be equal to $K|_{Y_G}(y)$. (If the restriction $K|_{Y_G}$ is torsion, then this evaluation is obviously zero, and we are in the previous situation of having absolute Maslov gradings in torsion Spin^c structures.) Therefore if d denotes the divisibility of $K|_{Y_G}$ (that is, this cohomology class equals d-times a primitive one), then the value $K^2-(K')^2$ is well-defined up to $4d$, hence the relative Maslov grading is well-defined modulo d only. (Note that for a characteristic cohomology class K the divisibility d of the restriction $K|_{Y_G}$ is always even.) In summary, we have:

Lemma 3.10. *Fix two generators $U^i \otimes [K, E]$ and $U^j \otimes [K', E']$ and suppose that $\mathbf{s}_K = \mathbf{s}_{K'}$ is a non-torsion* Spin^c *structure over Y_G. Then, the relative Maslov grading*

$$\begin{aligned}&\mathrm{gr}(U^i \otimes [K, E], U^j \otimes [K', E'])\\ &\qquad = -2(i-j) + 2g(K,E) - 2g(K,E') + |E| - |E'| + \frac{1}{4}(K^2 - (K')^2),\end{aligned}$$

gives a well-defined element of $\mathbb{Z}/d\mathbb{Z}$, where d denotes the divisibility of $c_1(\mathbf{s}_K)$.

The proof of Lemma 3.7 readily adapts to the non-torsion case: in this case, the lattice complex is a relatively $\mathbb{Z}/d\mathbb{Z}$-graded Maslov-graded complex.

Examples 3.11. • Consider the example of the graph G with a single vertex v, no edges and the decoration of the single vertex to be equal to $+1$. Then a characteristic cohomology class K can be identified with the odd number $K(v)$ it takes as a value on v. The generators of $\mathbb{CF}^-(G)$ are then $[2n+1, \{v\}]$ and $[2n+1]$. The boundary of $[2n+1]$ is 0, while

$$\partial[2n+1, \{v\}] = \begin{cases} [2n+1] + U^{n+1} \otimes [2n+3] & \text{if } n \geq -1, \\ U^{-(n+1)} \otimes [2n+1] + [2n+3] & \text{if } n < -1. \end{cases}$$

The map ∂ is then obviously injective on the subspace given by the finite sums of elements of the form $[2n+1, v]$. By allowing infinite sums (as we did), the element

$$\sum_{n=-\infty}^{-1} U^{\frac{1}{2}(n+1)(n+2)}[2n+1, v] + \sum_{n=0}^{\infty} U^{\frac{1}{2}n(n+1)}[2n+1, v]$$

generates $\mathbb{HF}^-(G)$ over $\mathbb{F}[\![U]\!]$. This shows that in this case

$$\mathbb{HF}^-(G) = \mathbb{HF}^-_1(G) = \mathbb{F}[\![U]\!].$$

A simple calculation shows that this element has zero Maslov grading, in accordance with the Heegaard Floer homological computation for the plumbing manifold Y_G given by G, which is diffeomorphic to S^3.

- In the next example we assume that G still has a single vertex v (and no edges) and the framing of the single vertex is zero. The underlying 3-manifold is now $S^1 \times S^2$. The generators are of the form $[2n]$ and $[2n, v]$, and two generators are in the same Spin^c structure if and only if the characteristic cohomology classes coincide. As always, $\partial[2n] = 0$. A simple calculation shows that

$$\partial[2n, v] = (1 + U^n)[2n].$$

Considering the theory over $\mathbb{F}[U]$ (and allowing only finite sums) the homology for the Spin^c structure $n \neq 0$ is $\mathbb{F}[U]/(U^n)$, while for $n = 0$ it is $\mathbb{F}[U] \oplus \mathbb{F}[U]$. Working with the completed groups (and hence using the coefficient ring $\mathbb{F}[\![U]\!]$), the term $(1 + U^n)$ is invertible for $n \neq 0$ (and vanishes if $n = 0$), hence according to the definition we adopted in the present paper we have that

$$\mathbb{HF}^-(G, \mathbf{s}_n) = 0, \quad \text{if } n \neq 0,$$

and

$$\mathbb{HF}^-(G, \mathbf{s}_0) = \mathbb{F}[\![U]\!] \oplus \mathbb{F}[\![U]\!].$$

(A simple computation shows that the Maslov gradings of the two generators are $\frac{1}{2}$ and $-\frac{1}{2}$.) Moreover,

$$\mathbb{HF}_0^-(G, \mathbf{s}_0) \cong \mathbb{F}[\![U]\!] \quad \text{and} \quad \mathbb{HF}_1^-(G, \mathbf{s}_0) \cong \mathbb{F}[\![U]\!].$$

This simple computation shows that for non-torsion Spin^c structures the completed theory (over the ring $\mathbb{F}[\![U]\!]$) loses some information. On the other hand, for torsion Spin^c structures the completed theory determines the one referred to in Remark 3.5 (which is defined over $\mathbb{F}[U]$). We just note here that the resulting homologies are again isomorphic to the corresponding completed Heegaard Floer homology groups.

4. The spectral sequence

Before turning to the proof of our main result, we need to recall some definitions and constructions from [5] (cf. also [19]). Recall that the plumbing graph determines a link $L = (L_1, \ldots, L_\ell)$ in S^3: each vertex of the plumbing tree gives rise to an unknot and these unknots are linked if and only if the corresponding vertices are connected in the graph by an edge.

4.1. Constructions from link Floer homology. Let H denote the homology group $H_1(S^3 - L; \mathbb{Z})$. By fixing an orientation on the component L_i, it gives rise to an oriented meridian μ_i, and these meridians generate H. Using these meridians we can identify the group ring $\mathbb{Z}[H]$ with the ring of Laurent polynomials on ℓ variables. Define $\mathbb{H}(L)$ as

$$\Big\{ \sum a_i \cdot [\mu_i] \mid a_i \in \mathbb{Q}, \text{ and } 2a_i + \ell k(L_i, L - L_i) \in 2\mathbb{Z} \Big\},$$

where $\ell k(L_i, L - L_i)$ is the linking number of the component L_i with the rest of the link. As it was discussed in [5, 19], the set $\mathbb{H}(L)$ parametrizes the relative Spin^c structures on $S^3 - L$.

Fix a multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ representing the link L, as in [19]. In this diagram $\mathbf{w} = (w_1, \ldots, w_\ell)$ and $\mathbf{z} = (z_1, \ldots, z_\ell)$ are basepoints with the property that the pair w_i and z_i represents the i^{th} component L_i of L. Recall that the multi-pointed diagram, in fact, specifies an orientation on the link. When we wish to underscore this structure, we write an oriented link as $\vec{L}$.

Given the Heegaard diagram and a choice of an element $\mathbf{s} \in \mathbb{H}(L)$, we define the chain complex $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$ as follows. Any intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ has a Maslov grading $M(\mathbf{x}) \in \mathbb{Z}$ (since the link L is in S^3) and an Alexander multi-grading $A(\mathbf{x}) \in \mathbb{H}(L)$, defined using the Heegaard diagram. This Alexander multi-grading is specified (up to an overall additive constant, i.e. up to a vector), as follows. If w_i and z_i are the pair of basepoints belonging to the i^{th} component of the link, and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is any homotopy class connecting $\mathbf{x}$ and $\mathbf{y}$, then the i^{th} component $A_i(\mathbf{x})$ of $A(\mathbf{x})$ satisfies

$$A_i(\mathbf{x}) - A_i(\mathbf{y}) = n_{z_i}(\phi) - n_{w_i}(\phi).$$

In an integral homology sphere (and specifically in S^3) such ϕ always exists and the difference above is independent of the choice of ϕ.

Given $\mathbf{s} = (s_1, \dots, s_\ell)$ and ϕ, we define the $\mathbf{s}$-modified multiplicity of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ by the formulas

$$E^i_{\mathbf{s}}(\phi) = \max\{s_i - A_i(\mathbf{x}), 0\} - \max\{s_i - A_i(\mathbf{y}), 0\} + n_{z_i}(\phi) \tag{4.1}$$

$$= \max\{A_i(\mathbf{x}) - s_i, 0\} - \max\{A_i(\mathbf{y}) - s_i, 0\} + n_{w_i}(\phi). \tag{4.2}$$

This quantity has the following two properties.

- $E^i_s(\phi) \geq 0$ if all the local multiplicities of ϕ are non-negative.
- If $\phi_1 \in \pi_2(\mathbf{p}, \mathbf{q})$ and $\phi_2 \in \pi_2(\mathbf{q}, \mathbf{r})$, then for $\phi_1 * \phi_2 \in \pi_2(\mathbf{p}, \mathbf{r})$

$$E^i_s(\phi_1 * \phi_2) = E^i_s(\phi_1) + E^i_s(\phi_2).$$

Given $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{H}(L)$, we define the corresponding chain complex $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$, which is a free module over the algebra $\mathbb{A} = \mathbb{F}[\![U_1, \dots, U_\ell]\!]$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and equipped with the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#\Big(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\Big) \cdot U_1^{E^1_{s_1}(\phi)} \cdots U_\ell^{E^\ell_{s_\ell}(\phi)} \cdot \mathbf{y}. \tag{4.3}$$

Note that this complex also depends on the choice of a suitable almost complex structure on the symmetric product. We suppress this almost complex structure from the notation for simplicity.

According to [5], the above complex is related to the Heegaard Floer homology of the 3-manifold obtained as sufficiently large surgeries on a link. (See also [20, 22] for the analogues for knots.) More formally, let $\Lambda = (\Lambda_1, \dots, \Lambda_\ell) \in \mathbb{Z}^\ell$ be a vector of framings, and let $Y_\Lambda(L)$ denote the 3-manifold we get by performing Λ_i-surgery on L_i for $i = 1, \dots, \ell$. Then, the following holds.

Theorem 4.1. ([5, Theorem 10.1]) *If Λ is sufficiently large (that is, for all i the coordinate $\Lambda_i \in \mathbb{Z}$ is sufficiently large) then the Heegaard Floer chain complex* $\mathbf{CF}^-(Y_\Lambda(L), \mathbf{s})$ *is quasi-isomorphic to* $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$.

Although for general links $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$ can be challenging to compute, in the case where L is the link diagram associated to a plumbing tree, the complex $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$ can be easily determined with the help of the above theorem. Recall that an *L-space* is a rational homology 3-sphere Y with the property that for each Spin^c structure $\mathbf{s}$ over Y, the Heegaard Floer homology $\mathbf{HF}^-(Y, \mathbf{s}) \cong \mathbb{F}[\![U]\!]$.

Lemma 4.2. *Let G be a plumbing tree and L be its corresponding link in S^3. Then, for each* $\mathbf{s} \in \mathbb{H}(L)$*, there is a homotopy equivalence* $\mathfrak{A}^-(\mathcal{H}, \mathbf{s}) \simeq \mathbb{F}[\![U]\!]$.

Proof. By Theorem 4.1 (which is identical to [5, Theorem 10.1]), $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$ computes the Heegaard Floer homology of a 3-manifold obtained by sufficiently positive surgeries on L. By [16, Lemma 2.6], this 3-manifold is an L-space, providing the desired isomorphism. □

The result given in [5, Theorem 7.7] (restated in Theorem 4.3 below) provides a chain complex, described in terms of the $\mathfrak{A}^-(\mathcal{H}, \mathbf{s})$ from above, which computes the Heegaard Floer homology of arbitrary surgeries on L. To describe this, we need a little more notation. Let us fix $\Lambda = (\Lambda_1, \dots, \Lambda_\ell)$. Let $M \subseteq L$ be a sublink with m components. The projection map

$$\psi^M : \mathbb{H}(L) \longrightarrow \mathbb{H}(L - M)$$

is defined as follows. Label the components of $L = L_1 \cup \dots \cup L_\ell$, and the components of $L - M = L_{j_1} \cup \dots \cup L_{j_{\ell - m}}$. We then define

$$\psi^M = (\psi^M_{j_1}, \dots, \psi^M_{j_{\ell-m}})$$

by

$$\psi^M_{j_i}(\mathbf{s}) = s_{j_i} - \frac{\ell k(L_{j_i}, M)}{2}.$$

Here, $\ell k(L_{j_i}, M)$ denotes the linking number of L_{j_i} with M; recall that both are oriented (via an orientation induced from the ambient link $\vec{L}$).

For each sublink $M \subset L$ fix a Heegaard diagram $\mathcal{H}^{L-M}$ for $L - M$. As a module over $\mathbb{A} = \mathbb{F}[\![U_1, \dots, U_\ell]\!]$, the surgery complex for the 3-manifold $Y_\Lambda(L)$ is defined by

$$(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-) = \bigoplus_{M \subseteq L} \prod_{\mathbf{s} \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})). \tag{4.4}$$

To define its differential, we need yet more notation. We need to give some algebraically defined maps, which are indexed by sublinks $M \subseteq L$, equipped with orientations (not necessarily agreeing with the induced orientation from $\vec{L}$). We write this data (sublink, together with a possibly different orientation) $\vec{M}$; and let $I_+(\vec{L}, \vec{M})$ resp. $I_-(\vec{L}, \vec{M})$ denote the sublink consisting of components of M whose orientation (in $\vec{M}$) agree resp. disagree with the orientation on the ambient link $\vec{L}$. For a sublink $M \subseteq L$, we let $\Omega(M)$ denote the set of orientations on M.

Let $\overline{\mathbb{H}}(L)$ denote the extension of $\mathbb{H}(L)$, where we allow some of the components to be $\pm\infty$. For $i \in \{1, \dots, \ell\}$, we define a projection map

$$p^{\vec{M}} : \mathbb{H}(L) \longrightarrow \overline{\mathbb{H}}(L)$$

so that the i^{th} component of $p^{\vec{M}}(\mathbf{s})$ is specified by

$$\begin{cases} +\infty & \text{if } i \in I_+(\vec{L}, \vec{M}), \\ -\infty & \text{if } i \in I_-(\vec{L}, \vec{M}), \\ s_i & \text{otherwise.} \end{cases}$$

There are algebraically defined maps

$$\mathcal{I}_{\mathbf{s}}^{\vec{M}} : \mathfrak{A}^-(\mathcal{H}, \mathbf{s}) \longrightarrow \mathfrak{A}^-(\mathcal{H}, p^{\vec{M}}(\mathbf{s}))$$

given by

$$\mathcal{I}_{\mathbf{s}}^{\vec{M}} \mathbf{x} = \prod_{i \in I_+(\vec{L}, \vec{M})} U_i^{\max(A_i(\mathbf{x}) - s_i, 0)} \cdot \prod_{i \in I_-(\vec{L}, \vec{M})} U_i^{\max(s_i - A_i(\mathbf{x}), 0)} \cdot \mathbf{x}.$$

Fix an orientation $\vec{M}$ on M and let $J(M) \subset \overline{\mathbb{H}}(L)$ denote the subspace $\mathbf{s} = (s_1, \dots, s_\ell)$, for which $s_i = +\infty$ if $L_i \in I_+(M)$, and $s_i = -\infty$ if $L_i \in I_-(M)$. Counting holomorphic curves induces a homotopy equivalence

$$\theta_{\mathbf{s}}^{\vec{M}} : \mathfrak{A}^-(\mathcal{H}, p^{\vec{M}}(\mathbf{s})) \longrightarrow \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^{\vec{M}}(\mathbf{s})).$$

(This homotopy equivalence was called $\hat{D}_{\mathbf{s}}^{\vec{M}}$ in [5]. We renamed it so that that it does not look like a differential.)

The differential on the surgery complex is given as a sum of components

$$\Phi_{\mathbf{s}}^{\vec{M}} : \mathfrak{A}^-(\mathcal{H}, \mathbf{s}) \longrightarrow \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^{\vec{M}}(\mathbf{s})),$$

defined by

$$\Phi_{\mathbf{s}}^{\vec{M}} = \theta_{p^{\vec{M}}(\mathbf{s})}^{\vec{M}} \circ \mathcal{I}_{\mathbf{s}}^{\vec{M}}.$$

We now define the boundary operator $\mathcal{D}^-$ on the surgery complex (4.4) as follows. For $\mathbf{s} \in \mathbb{H}(L)$ and $\mathbf{x} \in \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s}))$, we set

$$\mathcal{D}^-(\mathbf{s}, \mathbf{x}) = \sum_{N \subseteq L-M} \sum_{\vec{N} \in \Omega(N)} (\mathbf{s} + \Lambda_{\vec{L},\vec{N}}, \Phi^{\vec{N}}_{\psi^M(\mathbf{s})}(\mathbf{x}))$$
$$\in \bigoplus_{N \subseteq L-M} \bigoplus_{\vec{N} \in \Omega(N)} \mathfrak{A}^-(\mathcal{H}^{L-M-N}, \psi^{M \cup \vec{N}}(\mathbf{s})) \subseteq \mathfrak{A}^-(\mathcal{H}, \Lambda).$$

Of course, the homotopy equivalences $\theta^{\vec{M}}_{\mathbf{s}}$ appearing in the differential $\Phi^{\vec{M}}_{\mathbf{s}}$ are, in general, tricky to compute. For our present purposes, though, it turns out that a precise computation is unnecessary.

Recall that $(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-)$ is a module over $\mathbb{F}[\![U_1, \dots, U_n]\!]$. Choosing $U = U_1$, we can view it as a module over $\mathbb{F}[\![U]\!]$ (it will turn out that our results are independent of the numbering of the U_i).

The complex $(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-)$ admits a natural splitting into summands, as follows. Consider the subspace $H(L, \Lambda)$ of $H_1(Y - L)$ spanned by framings Λ_i of the components of L. The complex $(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-)$ naturally splits into summands indexed by the quotient space $\mathbb{H}(L)/H(L, \Lambda)$. In turn, this quotient space is naturally identified with $\mathrm{Spin}^c(Y, \Lambda)$, via for example, the filling construction from [19, Section 3.7].

One of the key results in [5] is the following:

Theorem 4.3. ([5, Theorem 7.7]) *The homology of the chain complex* $(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-)$ *is identified with* $\mathbf{HF}^-(Y_G)$. *Indeed, the identification respects the splitting of both spaces into summands indexed by* $\mathrm{Spin}^c(Y_G)$.

The surgery complex has a natural filtration S induced by the number of components in the sublink M. The differential $\mathcal{D}^-$ then splits as

$$\mathcal{D}^- = \sum_{k=0}^{\infty} \mathcal{D}^-_k,$$

where $\mathcal{D}^-_k$ is a term which drops the filtration level by exactly k. In particular, $\mathcal{D}^-_0$ is the differential on the associated graded complex.

By the $\mathcal{E}_1$*-term of the spectral sequence*, we mean the chain complex whose underlying $\mathbb{F}[\![U]\!]$-module is $H_*(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-_0)$, and whose differential is induced by $\mathcal{D}^-_1$.

Proposition 4.4. *The* $\mathcal{E}_1$*-term in the filtration on* $(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-)$ *is identified with* $\mathbb{CF}^-(G)$.

Proof. Let us first identify the $\mathbb{F}[\![U]\!]$-modules. Recall that $\mathrm{Vert}(G)$ can be used to index the components of L, therefore sublinks of L naturally correspond subsets of $V = \mathrm{Vert}(G)$. Furthermore, a characteristic element $K \in H^2(X_G; \mathbb{Z})$ specifies a Spin^c structure on X_G, and therefore an element $\mathbf{s} \in \mathbb{H}(L)$. By Lemma 4.2, we have that $H_*(\mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s}))) = \mathbb{F}[\![U]\!]$. Mapping the generator $U^i \otimes [K, E]$ of $\mathbb{CF}^-(G)$ to U^i in the factor $H_*(\mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})))$ of $H_*(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}_0^-)$ corresponding to the sublink M indexed by E and the Spin^c structure $\mathbf{s}$ corresponding to K, we get an isomorphism

$$\mathbb{CF}^-(G) \to H_*(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}_0^-)$$

of $\mathbb{F}[\![U]\!]$-modules.

Therefore, in order to verify the lemma, we need to identify $\mathcal{D}_1^-$ with the boundary operator ∂ of $\mathbb{CF}^-(G)$ described in equation (3.3). Let $M' \subseteq M$ denote a sublink with $|M| = |M'| + 1$. The boundary map $\mathcal{D}_1^-$ applied to an element of $H_*(\mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})))$ has two components in $H_*(\mathfrak{A}^-(\mathcal{H}^{L-M'}, \psi^{M'}(\mathbf{s})))$, which correspond to the two orientations of the knot $M - M'$. Let us denote these two components by d_1^+ and d_1^-. Recall that $M - M'$ is a knot, and hence it corresponds to some vertex v of the plumbing graph G. Although $M' - M$ is the unknot in S^3, in Y_M it represents a possibly complicated knot, which we denote $K_v \subset Y_M$.

The components d_1^+ and d_1^- of the differential have an interpretation as a four-manifold invariant. Specifically, the following square commutes:

$$\begin{array}{ccc} H_*(\mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s}))) & \xrightarrow{d_1^{\pm}} & H_*(\mathfrak{A}^-(\mathcal{H}^{L-M'}, \psi^{M'}(\mathbf{s}))) \\ \downarrow & & \downarrow \\ \mathbf{HF}^-(Y_M) & \xrightarrow{\mathbf{t}_{\pm}} & \mathbf{HF}^-(Y_{M'}). \end{array}$$

Here, Y_M resp. $Y_{M'}$ denotes any sufficiently large positive surgery on M resp M', the vertical maps are the identifications from Lemma 4.2, the top horizontal map is either of the two maps $d_1^{\pm}$, and the bottom horizontal map is induced by the single two-handle cobordism W from Y_M to $Y_{M'}$, equipped with one of the two Spin^c structures $\mathbf{t}_+$ or $\mathbf{t}_-$ of maximal square. An orientation on $M - M'$ specifies which component $d_1^{\pm}$ we are using: when the orientation of $M - M'$ agrees with that on L, we denote the component by d_1^+, and the other by d_1^-.

The orientation on $M - M'$ also specifies which of the two maximal square Spin^c structures $\mathbf{t}_\pm$ we are using. Both $\mathbf{t}_+$ and $\mathbf{t}_-$ are Spin^c structures with maximal square, they have the same evaluation on Y_M, and

$$\mathbf{t}_+ = \mathbf{t}_- + \mathrm{PD}[F],$$

where here $F \in H_2(W, Y_M; \mathbb{Z}) \cong \mathbb{Z}$ is the generator with the property that $\partial F \in H_1(Y_M; \mathbb{Z})$ corresponds to our knot $M - M'$ with its given orientation. (Commutativity of the above square is verified in [5, Theorem 10.2].)

Both top horizontal maps are non-trivial (they are isomorphisms in all sufficiently large degrees), so they must both be multiplication by some power of U. We let α_v denote the U-power associated to d_1^+ and β_v denote the U-power associated to d_1^-. Before finishing the proof of Proposition 4.4, we need to verify a series of lemmas.

Lemma 4.5. *The exponents α_v and β_v are independent of the surgery coefficients Λ.*

Proof. This is clear: the maps d_1^+ and d_1^- make no reference to surgery coefficients. □

The same property holds on the lattice homology side.

Lemma 4.6. *Let G and G' be two plumbing graphs, whose underlying graphs Γ and Γ' coincide. Fix $K \in \mathrm{Char}(G)$, $E \subset \mathrm{Vert}(G) = \mathrm{Vert}(G')$ and $v \in E$. Let $K' \in \mathrm{Char}(G')$ be the characteristic vector with*

$$K'(v) + m'_v = K(v) + m_v,$$

$$K'(w) = K(w),$$

for all $w \neq v$. Then,

$$a_v[K, E] = a_v[K', E],$$

$$b_v[K, E] = b_v[K'E].$$

Proof. By equation (3.2) and the choice of K', $f[K, I] = f[K', I]$. Since f determines a_v and b_v, the claim follows. □

Lemma 4.7. *For sufficiently negative surgery coefficients along the sublink M, we have that*

$$a_v = \alpha_v \quad \text{and} \quad b_v = \beta_v.$$

Proof. If the surgery coefficients along the sublink M' are sufficiently negative, the 3-manifold $Y_{M'}$ is an L-space. Therefore the statement of the lemma is essentially [15, Proposition 4.1] (cf. [15, Remark 4.2]) applied to the graph M', where $v = M - M'$ is the distinguished vertex. □

Now we return to the conclusion of the proof of Proposition 4.4. The identification stated in the proposition is equivalent to the statement that $a_v = \alpha_v$ and $b_v = \beta_v$ for the given framing Λ. This statement, however, is an immediate consequence of Lemmas 4.5, 4.6, and 4.7. □

Proposition 4.8. *The identification of Proposition 4.4 respects the* (*relative or absolute, depending on the* Spin^c *structure*) *Maslov gradings.*

Proof. Recall that in the proof of Proposition 4.4 the generator $[K, E]$ of $\mathbb{CF}^-(G)$ has been identified with the pair $(\mathbf{s}, M)$, where M is a sublink of the link L defined by the plumbing graph G and $\mathbf{s}$ is a relative Spin^c structure. In particular, $|M| = |E|$. We claim that this identification respects Maslov gradings. Indeed, if K represents a torsion Spin^c structure, then the absolute Maslov grading of $[K, \emptyset]$ (thought of as an element of $\mathbb{CF}^-(G)$) coincides with that of $(\mathbf{s}, \emptyset)$ (thought of as an element of $(\mathcal{C}^-(\mathcal{H}, \Lambda), \mathcal{D}^-)$). Since the boundary map drops Maslov grading by one, the identification of Maslov gradings extends to all generators of the form $[K, E]$. The same argument applies in the relatively graded setting (when K restricts to a non-torsion class on $\partial X_G = Y_G$). □

We now turn to the proof of Theorem 1.1:

Proof of Theorem 1.1. Theorem 4.3 presents $\mathbf{HF}^-(Y_G)$ as the homology of a filtered chain complex. Theorem 1.1 now follows from this theorem, together with the interpretation of the $\mathcal{E}_1$-term on the filtration provided by Proposition 4.4. Proposition 4.8 then provides the proof of the claim about the identification of Maslov gradings. □

Certain higher differentials in the spectral sequence vanish for *a priori* reasons. This is most easily seen when one appeals to gradings.

Proposition 4.9. *The differential $\mathcal{D}^-_{2n}$ on the page E_{2n} vanishes.*

Proof. Note first that all differentials on E_r drop Maslov grading by 1, and in particular change the Maslov grading by $1 \pmod 2$ (see Lemma 3.7). By Lemma 3.9 the relative Maslov grading of any element $U^i \otimes [K, E]$ agrees with $|E| \pmod 2$. Moreover, $\mathcal{D}^-_k$ drops $|E|$ by k. It follows from these observations and the identification of the Maslov gradings on the two theories (given by Proposition 4.8) that D^-_{2n} vanishes. □

4.2. Module structures and the spectral sequence. After establishing Theorem 1.1, we need a slight further refinement in order to provide the proof of Corollary 1.3.

Suppose all the higher differentials on the spectral sequence appearing in Theorem 1.1 vanish. Even in this case we cannot necessarily conclude that $\mathbf{HF}^-(Y_G)$ is computed by lattice homology: this method allows us to identify the two theories only as vector spaces over $\mathbb{F}$, but not as $\mathbb{F}[\![U]\!]$-modules. In certain cases, this indeterminacy can be removed by working with coefficients in $\mathbb{F}[U]/U^n$ for all n. In the rest of the section we spell out the details of this observation.

The complex $(\mathcal{C}^{[n]}(\mathcal{H}, \Lambda), \mathcal{D}^-_{[n]})$ will denote the complex over $\mathbb{F}[U]/U^n$ defined by taking the complex defined in Proposition 4.4, $(\mathcal{C}(\mathcal{H}, \Lambda), \mathcal{D}^-)$, and setting $U^n = 0$. (Recall that we viewed $(\mathcal{C}(\mathcal{H}, \Lambda), \mathcal{D}^-)$ as a module over $\mathbb{F}[\![U]\!]$ by defining the action by U to be multiplication by U_1. To view it as a module over $\mathbb{F}[U]/U^n$, we must set $U_1^n = 0$.) The complex $(\mathcal{C}^{[n]}(\mathcal{H}, \Lambda), \mathcal{D}^-_{[n]})$ naturally inherits a filtration from $(\mathcal{C}(\mathcal{H}, \Lambda), \mathcal{D}^-)$.

Lemma 4.10. *Fix any positive integer n, and consider the spectral sequence on $(\mathcal{C}^{[n]}(\mathcal{H}, \Lambda), \mathcal{D}^-_{[n]})$ induced from its filtration. This spectral sequence has $\mathcal{E}_1$-term isomorphic to $\widehat{\mathbb{CF}}^{[n]}(G)$.*

Proof. This is true because (thanks to Lemma 4.2) the $\mathcal{E}_1$-term $H_*(\mathcal{C}(\mathcal{H}, \Lambda), \mathcal{D}^-_0)$ is torsion free, as an $\mathbb{F}[\![U]\!]$-module. More explicitly, consider the filtered chain complex

$$C = (\mathcal{C}(\mathcal{H}, \Lambda), \mathcal{D}^-).$$

The associated spectral sequence has

$$\mathcal{E}_1 = H_*(C, \mathcal{D}^-_0),$$

equipped with the differential induced by the $\mathcal{D}^-_0$-chain map $\mathcal{D}^-_1$.

The filtered chain complex

$$C' = (\mathcal{C}^{[n]}(\mathcal{H}, \Lambda), \mathcal{D}^-_{[n]})$$

is gotten from C by $C \otimes \mathbb{F}[U]/U^n$. In general, its $\mathcal{E}_1$-term is computed by

$$\mathcal{E}_1(C') = H_*((C, \mathcal{D}^-_0) \otimes \mathbb{F}[U]/U^n) \oplus \mathrm{Tor}(H_*(C, \mathcal{D}^-_0) \otimes \mathbb{F}[U]/U^n),$$

converging to $H_*(C')$. In the case at hand, though, $H_*(C, \mathcal{D}^-_0)$ is a direct product of Heegaard Floer homology groups of 3-manifolds obtained as large surgeries on various components of our link, each of which, according to Lemma 4.2, contributing a factor of $\mathbb{F}[\![U]\!]$. Since

$$\mathrm{Tor}(\mathbb{F}[\![U]\!], \mathbb{F}[U]/U^n) = 0,$$

we have that

$$\mathrm{Tor}(H_*(C, \mathcal{D}^-_0) \otimes \mathbb{F}[U]/U^n) = 0.$$

It follows that

$$\mathcal{E}_1(C') = \mathcal{E}_1(C) \otimes \mathbb{F}[U]/U^n,$$

equipped with the differential induced from $\mathcal{D}^-_1$. But this $\mathcal{E}_1$-term is precisely $\widehat{\mathbb{CF}}^{[n]}(G)$. □

Now the version of Theorem 1.1 for the $U^n = 0$ truncated theory has the following shape.

Theorem 4.11. *Suppose that G is a plumbing tree of spheres, and let Y_G be the corresponding 3-manifold. Then there is a spectral sequence $\{\mathcal{E}_i\}_{i=1}^{\infty}$ with the property that*

- *the $\mathcal{E}_2$-term of the spectral sequence is isomorphic to the $U^n = 0$-specialized lattice homology $\widehat{\mathbb{HF}}^{[n]}(G)$ and*
- *the spectral sequence converges to the $U^n = 0$-specialized Heegaard Floer homology group $\widehat{\mathbf{HF}}^{[n]}(Y_G)$.*

Theorem 4.11 can be used to gain a little more information about the $\mathbb{F}[\![U]\!]$-module structure on $\mathbf{HF}^-(Y_G)$ (in terms of lattice homology). This improvement rests on the following algebraic result.

Lemma 4.12. *Suppose that C and C' are two Maslov-graded chain complexes over $\mathbb{F}[\![U]\!]$ whose homologies are finitely generated (as $\mathbb{F}[\![U]\!]$-modules). If for all $n \geq 1$,*

$$H_*(C \otimes \mathbb{F}[U]/U^n) \cong H_*(C' \otimes \mathbb{F}[U]/U^n)$$

as $\mathbb{F}$-vector spaces, then it follows that $H_*(C) \cong H_*(C')$ as $\mathbb{F}[\![U]\!]$-modules.

Proof. Fix a rational number d and an integer $k > 0$. Let $M(d,k)$ denote the Maslov-graded $\mathbb{F}[\![U]\!]$-module with the following two properties:

- $M(d,k) \cong \mathbb{F}[U]/U^k$ as an $\mathbb{F}[\![U]\!]$-module, and
- the generator of $M(d,k)$ has Maslov grading d (i.e. the whole module is supported in Maslov gradings between d and $d-2k$).

We extend the definition of $M(d,k)$ to $k = 0$ to be the Maslov-graded $\mathbb{F}[\![U]\!]$-module with the following two properties:

- $M(d,0) \cong \mathbb{F}[\![U]\!]$ as an $\mathbb{F}[\![U]\!]$-module, and
- the generator of $M(d,0)$ has Maslov grading d (i.e. the whole module is supported in Maslov gradings $\leq d$).

Since $\mathbb{F}[\![U]\!]$ is a principal ideal domain, the finitely-generated, Maslov-graded $\mathbb{F}[\![U]\!]$-module $H_*(C)$ splits as the direct sum of modules of the form $M(d,k)$; i.e.

$$H_*(C) \cong \bigoplus_{d\in\mathbb{Q}, k\in\{0,\dots\}} M(d,k)^{c_{d,k}},$$

where $c_{d,k}$ is a collection of non-negative integers, only finitely many of which are positive.

Our goal is to show that $\{H_*(C \otimes \mathbb{F}[U]/U^n)\}_{n=1}^{\infty}$ uniquely determines the isomorphism type of $H_*(C)$ as an $\mathbb{F}[\![U]\!]$-module, i.e. it uniquely determines the coefficients $\{c_{d,k}\}_{d,k}$.

This statement follows from an application of the universal coefficients theorem, stating that

$$\begin{aligned} &H_*(C \otimes \mathbb{F}[U]/U^n) \\ &\quad \cong (H_*(C) \otimes \mathbb{F}[U]/U^n) \oplus \mathrm{Tor}_{*-2k-1}(H_*(C), \mathbb{F}[U]/U^n), \end{aligned} \tag{4.5}$$

where the perhaps unfamiliar shift in grading (of $2k+1$, rather than simply 1) on the Tor results from the fact that the action by U shifts Maslov grading by 2.

We find it convenient to encode the input data in terms of a two-variable generating function

$$P_C(s,t) = \sum_{n\geq 0, m} \dim_{\mathbb{F}} H_m(C \otimes \mathbb{F}[U]/U^n) s^n t^{-m}.$$

By (4.5),

$$P_{H_*(C)} = \sum_{d,k} c_{d,k} \cdot P_{M(d,k)},$$

where

$$P_{M(d,k)} = \sum_{n\geq 0,m} \dim_{\mathbb{F}}(M_m(d,k)\otimes \mathbb{F}[U]/U^n)s^n t^{-m}$$
$$+ t^{2k+1} \sum_{n\geq 0,m} \dim_{\mathbb{F}}(M_m(d,k)\otimes \mathbb{F}[U]/U^n)s^n t^{-m}.$$

The lemma is proved once we show that the functions $P_{M(d,k)}$ are linearly independent (over $\mathbb{Z}$). This, in turn, follows form a straightforward calculation:

$$P_{M(d,k)} = t^{-d}\left(\frac{\sum_{i=0}^{k-1}(st^2)^i}{1-s} + t^{2k+1}\frac{\sum_{i=0}^{k-1} s^i}{1-st^2}\right)$$
$$= t^{-d}\left(\frac{1-(st^2)^k + t^{2k+1}(1-s^k)}{(1-s)(1-st^2)}\right),$$

thought of as a rational function in s; and also

$$P_{M(d,0)} = \frac{t^{-d}}{(1-s)(1-st^2)}.$$

Note that $(1-s)(1-st^2)P_{M(d,k)}$ is a degree-k polynomial in s. When $k>0$, the coefficient of s^k is $-t^{-d}(t^{2k}+t^{2k+1})$, while at $k=0$, we get the constant (in s) polynomial t^{-d}. The linear independence of the $P_{M(d,k)}$ follows immediately. □

Remark 4.13. A version of Lemma 4.12 applies when C and C' are two relatively $\mathbb{Z}/d\mathbb{Z}$ Maslov-graded chain complexes, as well. In that case, the generating function $P_M(s,t)$ is defined over $\mathbb{Z}[\mathbb{Z}/d\mathbb{Z}][\![s]\!]$; i.e. t is a primitive d^{th} root of unity.

Corollary 4.14. *Suppose that all higher differentials D_i^- vanish for $i\geq 2$ in the spectral sequence associated to $(\mathcal{C}^-(\mathcal{H},\Lambda),\mathcal{D}^-)$. Suppose that the same holds for all the truncated spectral sequences $(\mathcal{C}^{[n]}(\mathcal{H},\Lambda),\mathcal{D}^-_{[n]})$. Then, $\mathbb{HF}^-(G)$ and* $\mathbf{HF}^-(Y_G)$ *are isomorphic as $\mathbb{F}[\![U]\!]$-modules.*

Proof. This follows quickly from Lemma 4.12. □

5. Graphs of type 2

The proof of Corollary 1.3 relies on the following simple corollary of the existence of the surgery triangle for lattice homology. (The exact sequence we will use in the proof has been described by Greene [2, Theorem 3.1], and independently by Némethi [10]; see also the Appendix for a version adapted to the present notational conventions.)

Theorem 5.1. *Suppose that the plumbing tree G is of type k. Then*

$$\mathbb{HF}^-_q(G) = 0 \quad \textit{for } q > k.$$

Proof. The proof of the theorem proceeds by induction on k. For $k = 0$ (i.e. if G is rational), the claim follows from [8, Proposition 4.1.4.]. Suppose now that G is of type $(k+1)$ and assume that the claim of the theorem holds for graphs of type at most k. Let v be a vertex of G from the set $\{v_{i_1}, \ldots, v_{i_{k+1}}\}$ appearing in Definition 2.1 of the type of G. Then, by the same definition, $G - v$ is of type k. Let G_{-n} denote the graph we get from G by decreasing the framing of the chosen v by $n \in \mathbb{N}$. If n is sufficiently large, then (again by Definition 2.1) the graph G_{-n} is of type k. Fix now $q > k+1$ and consider the following portion of the long exact sequence associated to (G_{-n}, v) (cf. Corollary 6.8):

$$\cdots \longrightarrow \mathbb{HF}^-_q(G_{-n}) \longrightarrow \mathbb{HF}^-_q(G_{-n+1}) \longrightarrow \mathbb{HF}^-_{q-1}(G-v) \longrightarrow \cdots.$$

By the inductive assumption, the first and the third terms vanish, hence by exactness so does the middle term. Iterating this argument until we get the given framing on v, the result follows and shows that $\mathbb{HF}^-_q(G) = 0$ for $q > k+1$. $\square$

In a similar manner, we get

Theorem 5.2. *If the plumbing tree G is of type k then*

$$\widehat{\mathbb{HF}}^{[n]}_q(G) = 0 \quad \textit{for all } q > k \textit{ and all } n \in \mathbb{N}.$$

Remark 5.3. For G negative definite, Theorem 5.1 can be sharpened to $q \geq k$. This strengthening, however, does not hold for the truncated theories $\widehat{\mathbb{HF}}^{[n]}(G)$, hence the negative definite assumption does not improve Theorem 5.2 in this sense.

From these results the proof of the corollary is a simple exercise.

Proof of Corollary 1.3. Suppose that G is a plumbing tree (or forest) of type 2 and consider the spectral sequence provided by Theorem 1.1. By Proposition 4.9 we have that $\mathcal{E}_2 = \mathcal{E}_3$, and since by Theorem 5.1 the homology (and so the $\mathcal{E}_2$-table of the spectral sequence) concentrates on the rows with $|E|$-gradings $0, 1, 2$, the higher differentials point from or to vanishing groups, implying that $\mathcal{D}^-_i = 0$ for all $i \geq 3$. This means that $\mathcal{E}_2 = \mathcal{E}_\infty$, hence by Theorem 1.1 the lattice and Heegaard Floer homologies coincide, as vector spaces over $\mathbb{F}$. To get the corresponding isomorphism as $\mathbb{F}[\![U]\!]$-modules, we use the version of the spectral sequence over $\mathbb{F}[U]/U^n$, Theorem 4.11 cf. Corollary 4.14. For torsion Spin^c structures, the isomorphism of Maslov-graded $\mathbb{F}[\![U]\!]$-modules follows now from Lemma 4.12. For non-torsion Spin^c structures, we appeal to the modification of the proof of Lemma 4.12 described in Remark 4.13. □

6. Appendix: the exact sequence

For completeness, in this final section we prove the exact sequences in lattice homology used above. These results could be derived from [2, 10], but we find it convenient to include this proof here, as it follows the conventions and formalism introduced in Section 3.

Let G be a plumbing graph, and $v \in \mathrm{Vert}(G)$ be a distinguished vertex with framing m_v. $G - v$ will denote the graph obtained by omitting the vertex v. We define the *extension map*

$$\Phi_v : \mathbb{CF}^-(G - v) \longrightarrow \mathbb{CF}^-(G)$$

by the formula

$$\Phi_v([K, E]) = \sum_{p \equiv m_v \pmod 2} [(K, p), E]. \tag{6.1}$$

On the right-hand-side we write characteristic vectors for G as pairs (K, p), where K is a characteristic vector for $G - v$, and p is the evaluation of the characteristic vector on the distinguished vertex v. Since any component of $\Phi_v([K, E])$ determines $[K, E]$, it is easy to see that the above formula indeed provides a function on $\mathbb{CF}^-$ (meaning that any component of $\Phi_v(x)$ for a possibly infinite sum x has coefficient in $\mathbb{F}[\![U]\!]$). In fact, the above principle also shows that Φ_v is injective.

Lemma 6.1. *For each vertex $v \in \mathrm{Vert}(G)$, the map Φ_v is a chain map.*

Proof. This follows immediately from the fact that for any $E \subset G - v$, the $(G - v)$-weight $f_{G-v}[K, E]$ of the pair $[K, E]$ agrees with the G-weight $f_G[(K, p), E]$ of the pair $[(K, p), E]$ where p is any integer with the allowed parity. (Here f_{G-v} and f_G refer to the function defined in equation (3.2) with the respective graphs $G - v$ and G.) This implies that the corresponding functions g_{G-v} and g_G of minimal weights also coincide, and since the boundary maps are determined by these minimal weight functions, the result follows at once. □

Let $G_{+1}(v)$ denote the graph G with the same framings, except on the vertex v we consider $m_v + 1$ instead of m_v. Define the map

$$\Psi_v : \mathbb{CF}^-(G) \longrightarrow \mathbb{CF}^-(G_{+1}(v))$$

by the formula

$$\Psi_v[(K,p),E] = \sum_{m=-\infty}^{\infty} U^{s_m} \otimes [(K, p+2m-1), E], \tag{6.2}$$

where

$$s_m = g_{G_{+1}(v)}[(K, p+2m-1), E] - g_G[(K,p),E] + \frac{m(m-1)}{2}.$$

It is easy to see that when $v \notin I \subset E$ the equality

$$f_{G_{+1}(v)}[(K, p+2m-1), I] = f_G[(K,p), I]$$

holds, hence

$$s_m = \frac{m(m-1)}{2} \geq 0$$

in this case. If $v \in I \subset E$ then $f_{G_{+1}(v)}[(K, p+2m-1), I] - f_G[(K,p), I]$ is at most $|m|$ in absolute value, hence after adding $\frac{m(m-1)}{2}$ to it, the result will be nonnegative. In conclusion, s_m is nonnegative for any (K,p) and m.

Once again, a short argument is needed to confirm that the above formula defines a function on $\mathbb{CF}^-$, that is, for an infinite sum $\sum_{i\in\mathbb{Z}} U^{m_i}[K_i, E_i]$ all coordinates of the image admit a coefficient in $\mathbb{F}[\![U]\!]$. This property follows from the fact that if $p + 2m$ is fixed then the value s_m converges to infinity as $m \to \pm\infty$, implying that at most finitely many terms $[(K, p+2m-1), E]$ with $p+2m$ fixed can have a given U-power in the image.

Lemma 6.2. *The map Ψ_v is a chain map.*

Before starting the proof of this lemma, we need to define one further map. Suppose that the graph G_e is constructed from G by adding a new vertex e with framing (-1) and an edge connecting e and v. Consider the map

$$P : \mathbb{CF}^-(G_e) \longrightarrow \mathbb{CF}^-(G_{+1}(v))$$

given by the formula

$$P[(K, p, 2m-1), E] = \begin{cases} U^s \otimes [(K, p+2m-1), E] & \text{if } e \notin E, \\ 0 & \text{if } e \in E, \end{cases}$$

where

$$s_m = g_{G_{+1}(v)}[(K, p+2m-1), E] - g_{G_e}[(K, p, 2m-1), E] + \frac{m(m-1)}{2}.$$

(Once again, $(K, p, 2m-1)$ denotes the cohomology class on G_e which is K on $G - v$, takes the value p on v and the value $2m-1$ on e.) As above, it can be verified that P extends to a well-defined function on $\mathbb{CF}^-(G_e)$.

Lemma 6.3. *The map P is a chain map.*

Proof. We wish to prove that

$$\partial \circ P[(K, p, 2m-1), E] = P \circ \partial[(K, p, 2m-1), E].$$

First, we consider the case where $e \in E$. In this case the left hand side is zero. Moreover,

$$\begin{aligned} &P \circ \partial[(K, p, 2m-1), E] \\ &\quad = P(U^{a_e[(K,p,2m-1),E]} \otimes [(K, p, 2m-1), E-e]) \\ &\qquad + P(U^{b_e[(K,p,2m-1),E]} \otimes [(K, p+2, 2m-3), E-e]) \\ &\quad = U^{d_1} \otimes [(K, p+2m-1), E-e] + U^{d_2} \otimes [(K, p+2m-1), E-e], \end{aligned}$$

where

$$\begin{aligned} d_1 = a_e[(K, p, 2m-1), E] &+ g[(K, p+2m-1), E-e] \\ &- g[(K, p, 2m-1), E-e] + \frac{m(m-1)}{2} \end{aligned}$$

and

$$\begin{aligned} d_2 = b_e[(K, p, 2m-1), E] &+ g[(K, p+2m-1), E-e] \\ &- g[(K, p+2, 2m-3), E-e] + 2m^2 - 6m + 4. \end{aligned}$$

In fact, it is easy to see that

$$d_1 = g[(K, p+2m-1), E-e] - g[(K, p, 2m-1), E] + \frac{m(m-1)}{2} = d_2,$$

so the two terms cancel.

Next, suppose that $e \notin E$. Observe that

$$\begin{aligned} P \circ \partial[(K, p, 2m-1), E] = \sum_{w \in E} &U^{c_1(w)} \otimes [(K, p+2m-1), E-w] \\ &+ U^{d_1(w)} \otimes [(K, p+2m-1) + 2w^*, E-w], \end{aligned}$$

and

$$\partial \circ P[(K,p,2m-1),E] = \sum_{w\in E} U^{c_2(w)} \otimes [(K,p+2m-1),E-w] \\ + U^{d_2(w)} \otimes [(K,p+2m-1)+2w^*,E-w],$$

In fact, it is easy to see that

$$c_1(w) = g[(K,p+2m-1),E-w] - g[(K,p,2m-1),E] + \frac{m(m-1)}{2} \\ = c_2(w)$$

and

$$d_1(w) = g[(K,p+2m-1),E-w] - g[(K,p,2m-1),E] \\ + \frac{m(m-1)}{2} + \frac{L(w)+w\cdot w}{2} \\ = d_2(w),$$

where

$$L = (K,p+2m-1)$$

and $w\cdot w$ is taken in $G_{+1}(v)$. This completes the verification of the statement of the lemma. □

Proof of Lemma 6.2. Consider now the map

$$\Phi_e\colon \mathbb{CF}^-(G) \longrightarrow \mathbb{CF}^-(G_e).$$

The map Ψ_v is simply the composition $P\circ\Phi_e$, and since both maps are chain maps, so is Ψ_v, concluding the proof of the lemma. □

Theorem 6.4. *For any $v\in G$, the U-equivariant maps Ψ_v and Φ_v fit into a short exact sequence of chain complexes*

$$0 \longrightarrow \mathbb{CF}^-(G-v) \xrightarrow{\Phi_v} \mathbb{CF}^-(G) \xrightarrow{\Psi_v} \mathbb{CF}^-(G_{+1}(v)) \longrightarrow 0. \qquad (6.3)$$

The theorem could be proved by a direct check of exactness at each term – we rather choose an alternative way of first dealing with the $U=0$ theory (and the corresponding result there) and then apply abstract reasoning to verify the theorem. Define the map

$$\widehat{\Phi}_v\colon \widehat{\mathbb{CF}}(G-v) \longrightarrow \widehat{\mathbb{CF}}(G)$$

corresponding to Φ_v by the same formula as given by (6.1). Next define

$$\widehat{\Psi}_v : \widehat{\mathbb{CF}}(G) \longrightarrow \widehat{\mathbb{CF}}(G_{+1}(v))$$

(corresponding to the map Ψ_v) by the same formula as given in equation (6.2), after setting $U = 0$.

Lemma 6.5. *The map*

$$\widehat{\Psi}_v : \widehat{\mathbb{CF}}(G) \longrightarrow \widehat{\mathbb{CF}}(G_{+1}(v))$$

corresponding to Ψ_v in the $U = 0$ theory is given by the formula

$$\widehat{\Psi}_v([(K, p), E]) = [(K, p + 1), E] + [(K, p - 1), E] \tag{6.4}$$

if $v \notin E$ and by

$$\widehat{\Psi}_v([(K, p), E]) = \begin{cases} [(K, p+1), E] + [(K, p-1), E] & \text{if } A_v([(K, p), E]) < B_v([K, p], E), \\ [(K, p+1), E] + [(K, p-1), E] + [(K, p-3), E] & \text{if } A_v([(K, p), E]) = B_v([K, p], E), \\ [(K, p-1), E] + [(K, p-3), E] & \text{if } A_v([(K, p), E]) > B_v([K, p], E), \end{cases}$$

for $v \in E$.

Proof. Indeed, if $v \notin E$ we have

$$g_{G_1}[(K, p + 2m - 1), E] - g_G([(K, p), E] = 0,$$

hence $s_m = \frac{m(m-1)}{2}$, which is positive unless $m = 0, 1$, hence provides only the two terms of Formula (6.4) in the $U = 0$ theory.

The case of $v \in E$ requires a little more care. Suppose first that

$$A_v([(K, p), E]) < B_v([K, p], E),$$

meaning that the value $g[(K, p), E]$ is taken on a subset $I \subset E$ which does not contain v. Therefore for nonnegative m the difference of the g-functions is zero, hence $s_m = 0$ implies $\frac{m(m-1)}{2} = 0$, which holds exactly when $m = 0, 1$, providing the two terms in the expression. For $m < 0$ and

$$B_v([K, p], E) - A_v([(K, p), E]) = k > 0$$

the value of s_m is $m + k + \frac{m(m-1)}{2}$, which is strictly positive for any $m < 0$ (since $k \geq 1$).

Suppose now that

$$A_v([(K,p),E]) = B_v([K,p],E).$$

In this case for $m \geq 0$ the difference of the g-functions is zero, hence $s_m = 0$ is equivalent with $\frac{m(m-1)}{2} = 0$, providing the two terms corresponding to $m = 0, 1$. For negative m the term s_m is equal to $m + \frac{m(m-1)}{2}$, and this is zero exactly when $m = -1$, giving the third term in the expression.

Finally if

$$A_v([(K,p),E]) > B_v([K,p],E),$$

then for $m > 0$ the difference of the g-functions is positive (and $\frac{m(m-1)}{2}$ is non-negative), while for $m \leq 0$ the value of s_m is equal to $m + \frac{m(m-1)}{2}$, which is zero exactly when $m = 0, -1$, giving the claimed two terms in this case. $\square$

Having these formulae, now it is easy to see that the short sequence of (6.4) given by the maps on the $U = 0$ theory is exact, providing the long exact sequence on homologies:

Proposition 6.6. *For any $v \in G$, the maps $\widehat{\Phi}_v$ and $\widehat{\Psi}_v$ fit into the short exact sequence*

$$0 \longrightarrow \widehat{\mathbb{CF}}(G-v) \xrightarrow{\widehat{\Phi}_v} \widehat{\mathbb{CF}}(G) \xrightarrow{\widehat{\Psi}_v} \widehat{\mathbb{CF}}(G_{+1}(v)) \longrightarrow 0 \qquad (6.5)$$

of chain complexes.

Proof. Each group $\widehat{\mathbb{CF}}(G-v)$, $\widehat{\mathbb{CF}}(G)$, and $\widehat{\mathbb{CF}}(G_{+1})$ splits into a direct product indexed by pairs $K \in \mathrm{Char}(G-v)$, $E \subset \mathrm{Vert}(G)$. The maps $\widehat{\Phi}_v$ and $\widehat{\Psi}_v$ obviously respect this splitting. We claim that these maps fit into short exact sequences for each summand.

More precisely, in the case where $v \notin E$, the corresponding summand of $\widehat{\mathbb{CF}}(G-v)$ is one-dimensional, generated by the element $[K,E]$, and the desired short exact sequence is

$$0 \longrightarrow \mathbb{F}[K,E] \xrightarrow{\phi_v} \prod_{i\in\mathbb{Z}} \mathbb{F}[(K,k+2i),E] \xrightarrow{\psi_v} \prod_{i\in\mathbb{Z}} \mathbb{F}[(K,k+2i+1),E] \longrightarrow 0,$$

where

$$\phi_v([K,E]) = \sum_{i\in\mathbb{Z}}[(K,k+2i),E]$$

and

$$\psi_v([(K,p),E]) = [(K,p+1),E] + [(K,p-1),E].$$

A right inverse for ψ_v is determined by

$$r[(K,p-1),E] = \sum_{i=0}^{\infty}[(K,p+2i),E],$$

and it is easy to see that $\ker\psi_v = \mathrm{Im}\phi_v$.

In the case where $v \in E$, we declare the corresponding summand of $\widehat{\mathbb{CF}}(G-v)$ to be trivial, so we claim that the corresponding sequence

$$0 \longrightarrow 0 \xrightarrow{\phi_v} \prod_{i\in\mathbb{Z}} \mathbb{F}[(K,k+2i),E] \xrightarrow{\psi_v} \prod_{i\in\mathbb{Z}} \mathbb{F}[(K,k+2i+1),E] \longrightarrow 0$$

is short exact, i.e. ψ_v is an isomorphism. Indeed, the map

$$q\colon \prod_{i\in\mathbb{Z}} \mathbb{F}[(K,k+2i+1),E] \longrightarrow \prod_{i\in\mathbb{Z}} \mathbb{F}[(K,k+2i),E],$$

which is uniquely determined by

$$q([(K,p-1),E]) = \begin{cases} \displaystyle\sum_{i=0}^{\infty}[(K,p+2i),E] & \text{if } A_v([(K,p),E]) < B_v([(K,p),E]), \\ \displaystyle\sum_{i=-\infty}^{0}[(K,p+2i),E] & \text{if } A_v([(K,p),E]) > B_v([(K,p),E]), \\ \displaystyle\sum_{i=-\infty}^{\infty}[(K,p+2i),E] & \text{if } A_v([(K,p),E]) = B_v([(K,p),E]), \end{cases}$$

provides an inverse for ψ_v. Indeed, the fact that $\psi_v \circ q$ and $q \circ \psi_v$ are both equal to the (respective) identities follows from the principle, that for a given $[K,E]$ there is exactly one value of p for which

$$A_v[(K,p),E] = B_v[(K,p),E]. \qquad \square$$

The short exact sequences then induce a long exact sequence on homologies, and since both $\widehat{\Phi}_v$ and $\widehat{\Psi}_v$ respect the grading of $[K,E]$ induced by $|E|$, we get the following result.

Corollary 6.7. *The short exact sequence of Proposition 6.6 induces a long exact sequence*

$$\cdots \longrightarrow \widehat{\mathbb{HF}}_{i+1}(G_{+1}(v)) \longrightarrow \widehat{\mathbb{HF}}_i(G-v) \longrightarrow \widehat{\mathbb{HF}}_i(G) \\ \longrightarrow \widehat{\mathbb{HF}}_i(G_{+1}(v)) \longrightarrow \widehat{\mathbb{HF}}_{i-1}(G-v) \longrightarrow \cdots$$

on δ-graded lattice homology.

With the above result at hand we return to the theory over $\mathbb{F}[\![U]\!]$.

Proof of Theorem 6.4. First we claim that $\Psi_v \circ \Phi_v = 0$. This follows from the fact that

$$(\Psi_v \circ \Phi_v)[K, E] = \sum_p \sum_m U^{\frac{m(m-1)}{2}} \otimes [(K, p + 2m - 1), E].$$

(Note that since $v \notin E$, we have that

$$g_{G_{+1}(v)}[(K, p + 2m - 1), E] = g_G[(K, p), E],$$

hence $s_m = \frac{m(m-1)}{2}$.) Observe that each term in the above sum appears exactly twice: the term corresponding to (p, m) agrees with the term corresponding to $(p + 4m - 2, -m + 1)$. Indeed, the system

$$\begin{cases} p + 2m - 1 = p' + 2m' - 1, \\ m(m-1) = m'(m'-1) \end{cases}$$

has exactly the two solutions for (p', m') given above. This cancellation then shows that $(\Psi_v \circ \Phi_v)[K, E] = 0$, verifying the claim.

We define two homology theories associated to the pair (G, v): let $\widehat{\mathrm{H}}_{\mathrm{SES}}(G, v)$ denote the homology of the short exact sequence (6.5) (viewed as a chain complex with underlying group the sum of the terms in the sequence and boundary map equal to the maps in the sequence). Similarly, $\mathrm{H}^-_{\mathrm{SES}}(G, v)$ will denote the homology of the sequence (6.3). (Since the compositions of consecutive maps in these sequences are zero, these homologies are defined.) The content of Proposition 6.6 is that $\widehat{\mathrm{H}}_{\mathrm{SES}}(G, v) = 0$, while in Theorem 6.4 we want to show that $\mathrm{H}^-_{\mathrm{SES}}(G, v) = 0$. The two homologies are, however, connected by the Universal Coefficient Theorem. Indeed, $\mathrm{H}^-_{\mathrm{SES}}(G, v)$ is defined over the ring $\mathbb{F}[\![U]\!]$, while the chain complex defining $\widehat{\mathrm{H}}_{\mathrm{SES}}(G, v)$ can be given from (6.3) by considering the tensor product of the $\mathbb{CF}^-$-modules with $\mathbb{F}$ over $\mathbb{F}[\![U]\!]$, where a power series in $\mathbb{F}[\![U]\!]$ acts through its constant term on $\mathbb{F}$. By the Universal Coefficient Theorem [24] (and by the fact that $\mathbb{F}$ is a field) we get that $\mathrm{H}^-_{\mathrm{SES}}(G, v) \otimes_{\mathbb{F}[\![U]\!]} \mathbb{F} = \widehat{\mathrm{H}}_{\mathrm{SES}}(G, v) = 0$.

Since $\mathbb{F}[\![U]\!]$ is a principal ideal domain, the tensor product of any nontrivial module with $\mathbb{F}$ (over $\mathbb{F}[\![U]\!]$) is nontrivial: consider a nontrivial element $x \in \mathrm{H}^-_{\mathrm{SES}}(G, v)$ and observe that the submodule generated by it is isomorphic to $\mathbb{F}[\![U]\!]/(f(U))$ with $f(0) = 0$ (since $\mathbb{F}[\![U]\!]$ is a PID), and $\mathbb{F}[\![U]\!]/(f(U)) \otimes_{\mathbb{F}[\![U]\!]} \mathbb{F} = \mathbb{F} \neq 0$. Since we showed that $\widehat{\mathrm{H}}_{\mathrm{SES}}(G, v) = 0$, this last observation then implies that $\mathrm{H}^-_{\mathrm{SES}}(G, v) = 0$, concluding the proof of the Theorem. □

Corollary 6.8. *The short exact sequence of Theorem 6.4 induces a long exact sequence*

$$\cdots \longrightarrow \mathbb{HF}^-_{i+1}(G_{+1}(v)) \longrightarrow \mathbb{HF}^-_i(G - v) \longrightarrow \mathbb{HF}^-_i(G) \longrightarrow \mathbb{HF}^-_i(G_{+1}(v)) \longrightarrow \mathbb{HF}^-_{i-1}(G - v) \longrightarrow \cdots$$

on δ-graded lattice homology.

Proof. The short exact sequence of Theorem 6.4 induces a long exact sequence on the homologies, and it is easy to see that both Ψ_v and Φ_v respects the grading of a generator $[K, E]$ given by the cardinality of E, hence the long exact sequence admits the form stated in the corollary. □

Theorem 6.4 also gives a long exact sequence

$$\cdots \longrightarrow \widehat{\mathbb{HF}}^{[n]}_{i+1}(G_{+1}(v)) \longrightarrow \widehat{\mathbb{HF}}^{[n]}_i(G - v) \longrightarrow \widehat{\mathbb{HF}}^{[n]}_i(G) \longrightarrow \widehat{\mathbb{HF}}^{[n]}_i(G_{+1}(v)) \longrightarrow \widehat{\mathbb{HF}}^{[n]}_{i-1}(G - v) \longrightarrow \cdots .$$

This is gotten by tensoring the short exact sequence from equation (6.3) with $\mathbb{F}[U]/U^n$, and then taking the associated long exact sequence in homology.

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