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Quantum Topology

Lagrangian concordance is not a symmetric relation

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Abstract. We provide an explicit example of a non trivial Legendrian knot Λ such that there exists a Lagrangian concordance from Λ_0 to Λ where Λ_0 is the trivial Legendrian knot with maximal Thurston–Bennequin number. We then use the map induced in Legendrian contact homology by a concordance and the augmentation category of Λ to show that no Lagrangian concordance exists in the other direction. This proves that the relation of Lagrangian concordance is not symmetric.

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1. Introduction

In this paper we will only consider the standard contact $\mathbb{R}^3$ with the contact structure $\xi = \ker\alpha$ with $\alpha = dz - y\,dx$. A *Legendrian knot* is an embedding

$$i\colon S^1 \hookrightarrow \mathbb{R}^3$$

such that

$$i^*\alpha = 0.$$

The symplectisation of $(\mathbb{R}^3, \xi)$ is the symplectic manifold $(\mathbb{R}\times\mathbb{R}^3, d(e^t\alpha))$.

In [3] we introduced the notion of Lagrangian concordances and cobordisms between Legendrian knots and proved the basic properties of those relations. Roughly speaking a Lagrangian cobordism Σ from a knot Λ^- to a knot Λ^+ is a Lagrangian submanifold of the symplectisation which coincides at $-\infty$ with Λ^- and at $+\infty$ with Λ^+. When Σ is topologically a cylinder we say that Λ^- is Lagrangian concordant to Λ^+ (a relation we denote by $\Lambda^- \prec \Lambda^+$). Among the basic properties of oriented Lagrangian cobordisms we proved that

$$\mathrm{tb}(\Lambda^+) - \mathrm{tb}(\Lambda^-) = 2g(\Sigma)$$

where $\mathrm{tb}(\Lambda)$ is the Thurston–Bennequin number of Λ. This immediately implies that when a Lagrangian cobordism is not a cylinder then such a cobordism cannot be reversed. However we cannot apply such an argument to explicitly prove that the relation of concordance is not symmetric. In this paper we use more involved techniques, in particular recent results of T. Ekholm, K. Honda and T. Kálmán in [9] using pseudo-holomorphic curves and Legendrian contact homology, to give an example of a non reversible Lagrangian concordance. Namely we prove the following result.

Theorem 1.1. *Let Λ_0 be the Legendrian unknot with -1 Thurston–Bennequin invariant. There exists a Legendrian representative Λ of the knot $m(9_{46})$ of Rolfsen table of knots* (*see* [17]) *such that*

- $\Lambda_0 \prec \Lambda$,
- $\Lambda \not\prec \Lambda_0$.

The front and Lagrangian projections of Λ in the previous theorem are shown on Figure 1 (note that this Legendrian knot also appears in the end of [18] as an example of Lagrangian slice knot).

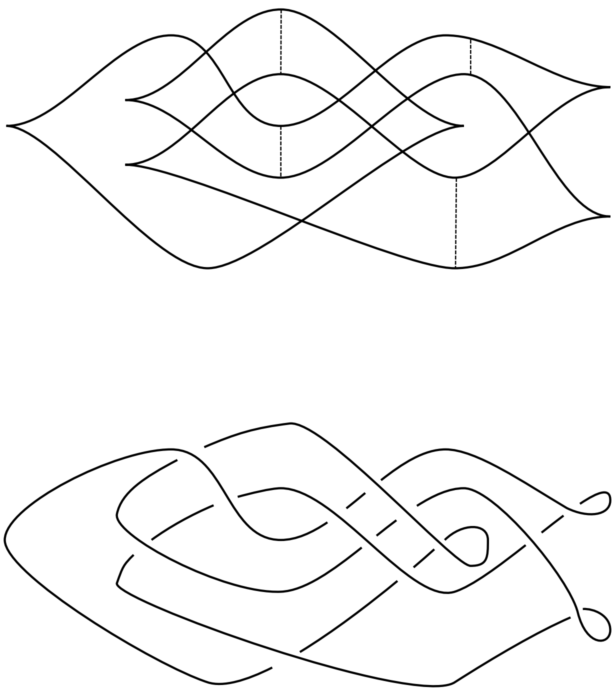

Figure 1. Front and Lagrangian projections of a Legendrian representative of $m(9_{46})$.

This example confirms the analogy of this relation with a partial order. Whether or not it is a genuine partial order (meaning that $\Lambda \prec \Lambda'$ and $\Lambda' \prec \Lambda$ would imply that Λ is Legendrian isotopic to Λ') is neither proved nor disproved; the author is unaware of any conjecture on how different the equivalence relation given by $\Lambda \prec \Lambda'$ and $\Lambda' \prec \Lambda$ is from the Legendrian isotopy relation.

The knot Λ is the "smallest" Lagrangianly slice Legendrian knot (as it is clear from the Legendrian knot atlas of [6]); it is therefore the first natural candidate fo an example a non-reversible concordance. Using connected sums it is possible to construct more examples of this kind. Another class of examples in dimension 3 will appear in forthcoming work by J. Baldwin and S. Sivek in [1] where they construct concordances where the negative ends are stabilisations and the positive ones have non-vanishing Legendrian contact homology. In higher dimensions recent results of Y. Eliashberg and E. Murphy [11] imply that if the negative end is loose (in the sense of [15]) then the Lagrangian concordance problem satisfies the h-principle. This can be used to prove further non reversible examples of Lagrangian concordances. Note that in both of those cases we still need pseudo-holomorphic curves techniques and the existence of maps in Legendrian contact homology to prove that the involved Lagrangian concordances cannot be reversed.

In order to prove the existence of the Lagrangian concordance claimed in Theorem 1.1 we use elementary Lagrangian cobordisms from [4] which we recall in Section 3. We also describe those elementary cobordisms in terms of Lagrangian projections as we will use those in Section 5 to compute maps between Legendrian contact homology algebras (LCH for short). As the negative end of the concordance is Λ_0 which has non-vanishing LCH the actual argument not only relies on the functoriality of Legendrian contact homology (as it is the case for the example of [11] and [1]) but also on a unknottedness result of Lagrangian concordances from Λ_0 to itself which follows from work of Y. Eliashberg and L. Polterovitch in [12] which we state in the following:

Theorem 1.2. *Consider the standard contact S^3 (seen as the compactification of the standard contact $\mathbb{R}^3$) and denote by K_0 the Legendrian unknot with -1 Thurston–Bennequin invariant (which corresponds to Λ_0 in $\mathbb{R}^3$).*

Let C be an oriented Lagrangian cobordism from K_0 to itself. Then there is a compactly supported symplectomorphism of $\mathbb{R} \times S^3$ such that $\phi(C) = \mathbb{R} \times K_0$.

Theorem 1.2 is proven in Section 6. Assuming then that a concordance C' from Λ to Λ_0 exists we could glue C to C' to get a concordance from Λ_0 to Λ_0 and applying Theorem 1.2 we deduce that the map induced in Legendrian contact homology is the identity (as stated in Theorem 6.1). We conclude the proof of Theorem 1.1 in Section 7. In order to do so, we use the augmentation categories of Λ and Λ_0 as defined in [2] and the functor between them induced by the concordance to find a contradiction to the existence of a concordance from Λ to Λ_0.

Remark 1.3. The main result was announced in the addendum in the introduction of [3]. When it was written bilinearised LCH was not known to the author. The original proof of the non-symmetry followed however similar lines. The idea is to construct several other concordances C_i from Λ_0 to Λ (every dashed line in Figure 1 is a chord where we can apply move number 4 of Figure 2 to get such a concordance). For each of those we computed the associated map similarly to what is done in Section 5. We then used Theorem 6.1 to prove that for each of them the composite map in Legendrian contact homology is the identity and deduce after some effort a contradiction. The existence of the augmentation category allows us to give a more direct final argument and use only one explicit concordance from Λ_0 to Λ.

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2. Lagrangian concordances and Legendrian contact homology

We recall in this section the main definition from [3].

Definition 2.1. Let $\Lambda^-\colon S^1 \hookrightarrow \mathbb{R}^3$ and $\Lambda^+\colon S^1 \hookrightarrow \mathbb{R}^3$ be two Legendrian knots in $\mathbb{R}^3$. We say that Λ^- is *Lagrangian concordant* to Λ^+ if there exists a Lagrangian embedding $C : \mathbb{R} \times \Lambda \hookrightarrow \mathbb{R} \times \mathbb{R}^3$ such that

(1) $C|_{(-\infty,-T)\times\Lambda} = \mathrm{Id} \times \Lambda^-$,

(2) $C|_{(T,\infty)\times\Lambda} = \mathrm{Id} \times \Lambda^+$.

In this situation C is called a *Lagrangian concordance* from Λ^- to Λ^+.

It was proven in [10] that two Legendrian isotopic Legendrian knots are indeed Lagrangian concordant. Another proof is given in [3] where we also proved that under Lagrangian concordances the classical invariants tb and r are preserved.

A Lagrangian concordance C is always an exact Lagrangian submanifold of $\mathbb{R} \times \mathbb{R}^3$ in the sense of [9] and thus following [9] it defines a DGA-map

$$\varphi_C : \mathcal{A}(\Lambda^+) \longrightarrow \mathcal{A}(\Lambda^-),$$

where $\mathcal{A}(\Lambda^\pm)$ denote the Chekanov algebras of the Legendrian submanifolds $\Lambda^\pm$. The homology of $\mathcal{A}(\Lambda)$ (denoted by $\mathrm{LCH}(\Lambda)$) is called the Legendrian contact homology of Λ (see [5] and [8]). This map is defined by a count of pseudo-holomorphic curves with boundary on C.

If C_1 is a Lagrangian concordance from Λ_0 to Λ_1 and C_2 a Lagrangian concordance from Λ_1 to Λ_2. We denote by $C_1 \#_T C_2$ the Lagrangian concordance from Λ_0 to Λ_2 which is equal to a translation of C_1 for $t < -T$ and a translation of C_2 for $t > T$. Then [9, Theorem 1.2] implies that there exists a sufficiently big T such that $\varphi_{C_1 \#_T C_2} = \varphi_{C_1} \circ \varphi_{C_2}$, in particular the association $C \to \varphi_C$ is functorial on LCH.

3. Elementary Lagrangian cobordisms and their Lagrangian projections

For a Legendrian knot Λ in $\mathbb{R}^3$ we call the projection of Λ on the xz-plane along the y direction the *front projection* of Λ. The projection on the xy plane along the z direction is called the *Lagrangian projection* of Λ.

In order to produce an example of a non-trivial Lagrangian concordance we will use a sequence of elementary cobordisms as defined in [4] and [9]. A combination of results from [3], [4] and [9] implies that the local moves of Figure 2 can be realised by Lagrangian cobordisms (the arrows indicate the increasing $\mathbb{R}$ direction in $\mathbb{R} \times \mathbb{R}^3$).

The first three moves are Legendrian Reidemeister moves arising along generic Legendrian isotopies, in each case the associated cobordism is a concordance. The fourth move is a saddle cobordism which corresponds to a 1-handle attachment. The cobordism corresponding to the fifth move is a disk.

In Section 5 we will compute the induced map in Legendrian contact homology by a concordance. It will then be convenient to have a description of this concordance in terms of the Lagrangian projection. As it is easier in general to draw isotopy of front projections, we will use procedure of [16] to draw Lagrangian projections from front projections.

The idea is to write front projections in piecewise linear forms where the slope of a strand is always bigger than the one under it except before a crossing or a cusp. Such front diagrams are then easily translated into Lagrangian projections.

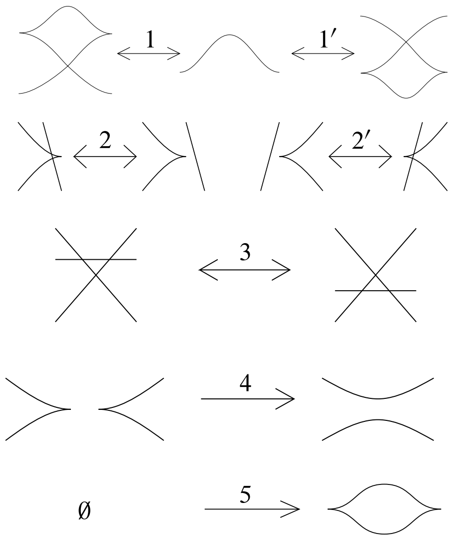

Figure 2. Local bifurcations of fronts along elementary Lagrangian cobordisms.

In Figure 3 we provide, on the left, the elementary moves in front diagrams of this form associated to elementary cobordisms which we translate then, on the right, in terms of Lagrangian projections. As in Figure 2 the arrows represent the increasing time direction.

We label an arrow according to the corresponding bifurcation of the Lagrangian projection where **II**, **III** and **III**′ correspond to the notation of [14]. However, as a cobordism from Λ^- to Λ^+ induce a map from $\mathcal{A}(\Lambda^+)$ to $\mathcal{A}(\Lambda^-)$ (i.e. following the decreasing time direction) we labelled a move in Figure 3 by the corresponding move from [14] following the arrow backward. As an example, if Λ^- differs from Λ^+ by a move number **II** from [14] we will label the arrow by a $\mathbf{II}^{-1}$ as it is this move we will use to compute the map from $\mathcal{A}(\Lambda^+)$ to $\mathcal{A}(\Lambda^-)$. We denote by **IV** the saddle cobordism denoted L_{sa} in [9] and by **V** the Lagrangian filling of Λ_0 denoted by L_{mi} in [9]. In move number 4, we also provide an intermediate step which corresponds to the creation of two Reeb chords one of which being then resolved by the cobordism (this procedure guaranties that the smallest newly created chord is contractible).

This language being understood we will be able to translate any bifurcation of fronts as a bifurcation of Lagrangian projections and we will keep drawing qualitative Lagrangian projections.

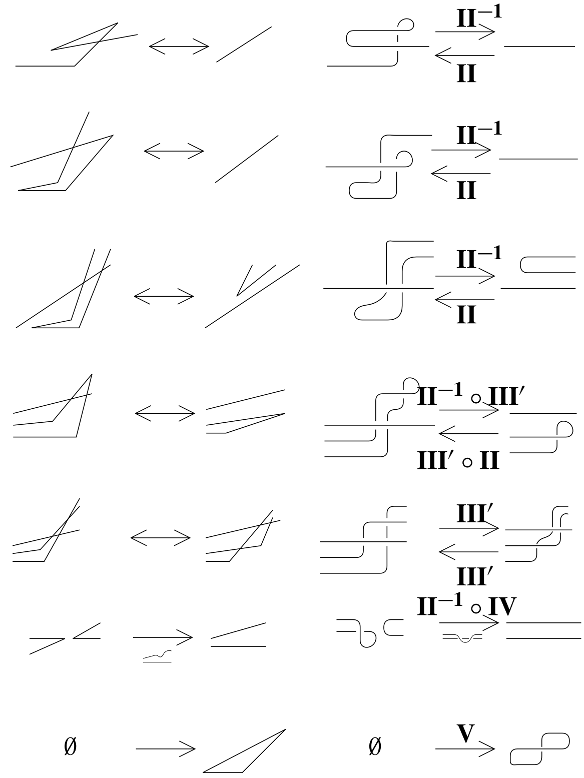

Figure 3. Lagrangian projections of elementary cobordisms.

4. Example of a non-trivial concordance

Using the moves of Figure 2 we are able to provide a non trivial Lagrangian concordance from Λ_0 to Λ. Note that the knot $m(9_{46})$ is the first Legendrian knot in the Legendrian knot atlas of [6] with

$$g_s(K) = 0 \text{ and } \max\{\text{tb}(\Lambda) | \Lambda \text{ Legendrian representative of } K\} = -1,$$

thus, following [3, Theorem 1.4], it is the simplest candidate for such an example. The bifurcations of the fronts along the non trivial concordance is given on Figure 4.

One can see that it is indeed a concordance either by using [3, Theorem 1.3] and deduce from $\text{tb}(\Lambda) - \text{tb}(\Lambda_0) = 0$ that the genus of the cobordism is 0 or by explictly seeing that the projection to $\mathbb{R}$ of C has only two critical points, one of index 1 and one of index 0 which implies that C is a cylinder.

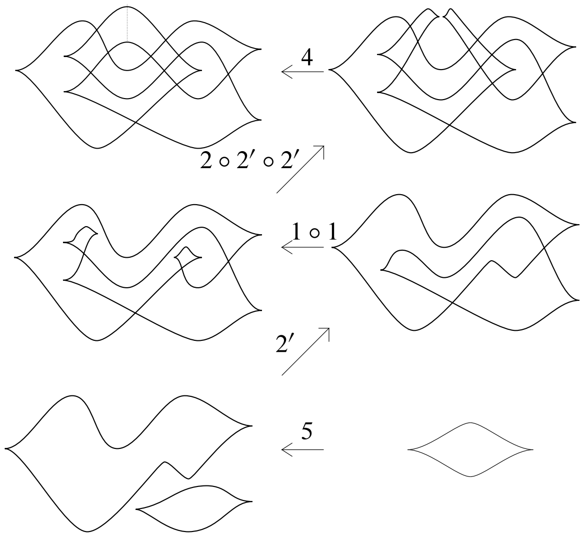

Figure 4. A non trivial Lagrangian concordance.

5. Legendrian contact homology of Λ and some geometrical maps

We compute now the boundary operator on the Chekanov algebra of Λ (see [5]). As $r(\Lambda) = 0$ it is a differential $\mathbb{Z}$-graded algebra over $\mathbb{Z}_2$ freely generated by the double points of the Lagrangian projection of Λ. The generators of $\mathcal{A}(\Lambda)$ are represented on Figure 5 where each a_i has degree 1, each b_i degree 0 and each c_i degree -1.

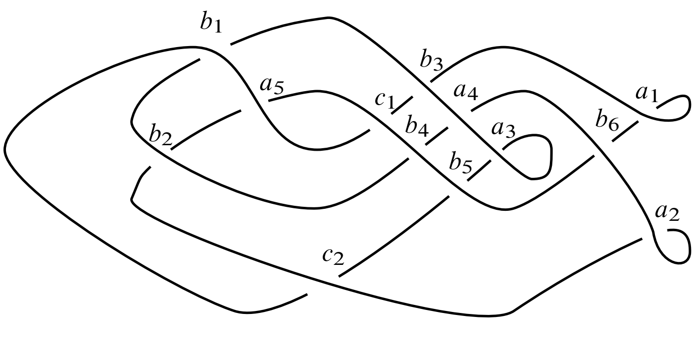

Figure 5. Generators of $\mathcal{A}(\Lambda)$.

The boundary operator on generators counts degree one immersed polygons with one positive corner and several negative corners and in our situation gives

$$\begin{aligned}
\partial a_1 &= 1 + a_5 c_2 b_2 + b_1 b_6 + b_2,\\
\partial a_2 &= 1 + b_2 c_2 a_4 b_2 + b_2 c_2 b_3 a_5 + b_6 b_4 b_2 + b_6 c_1 a_5 + b_6 + b_2,\\
\partial a_3 &= 1 + a_4 b_2 c_2 + b_3 a_5 c_2 + b_3 + b_2 b_5,\\
\partial a_4 &= 1 + b_3 b_1 + b_2 b_4,\\
\partial a_5 &= b_1 b_2,\\
\partial b_1 = \partial b_2 &= 0,\\
\partial b_3 &= b_2 c_1,\\
\partial b_4 &= c_1 b_1,\\
\partial b_5 &= b_4 b_2 c_2 + c_1 a_5 c_2 + c_2 + c_1,\\
\partial b_6 &= b_2 c_2 b_2,\\
\partial c_1 = \partial c_2 &= 0.
\end{aligned}$$

It is then extended to the whole algebra by Leibniz' rule: $\partial(ab) = \partial(a)b + a\partial(b)$.

We will now compute the map between Chekanov algebras associated to the concordance C of Figure 4. At each step we use the results of [9] which give a combinatorial description of the map associated to each elementary cobordism.

On Figure 6 we see the bifurcations of the Lagrangian projections along C using the correspondence between front moves and Lagrangian moves of Figure 3, for convenience we split the first two steps in two steps each. For a cobordism C_i we denote the differential of the DGA associated to the upper level by $\partial^+_{C_i}$ and the one corresponding to the lower level by $\partial^-_{C_i}$ (of course $\partial^+_{C_{i+1}} = \partial^-_{C_i}$). At each step we compute the map associated to these moves between the corresponding Chekanov algebras heavily using the results of [9, Section 6]. We provide the precise section of this paper we use for each of the corresponding move. We decorate the labels of the bifurcations of the Lagrangian projections with subscripts precising the chords involved by each move.

5.1. Map associated to C_1. The bifurcation associated to the cobordism C_1 is $\mathbf{II_{ab}}$ as in Figure 7. The computation of the map associated to this move is the most involved of all the DGA maps described in [14] and [9].

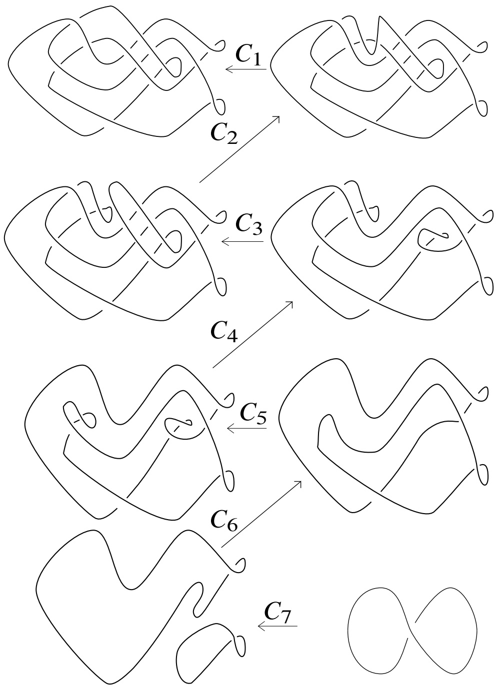

Figure 6. Bifurcations of Lagrangian projections along the non-trivial concordance.

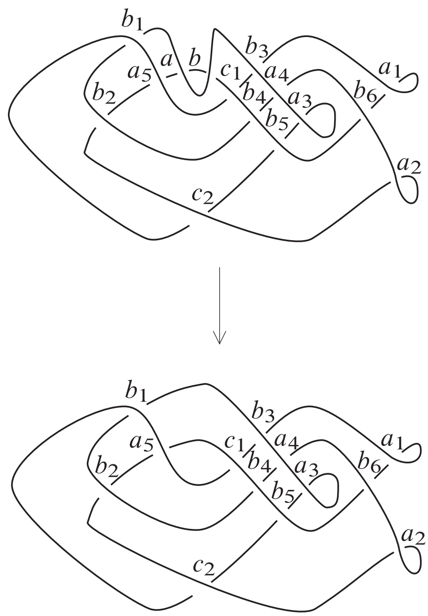

Figure 7. $\mathbf{II_{ab}}$.

Following [9, Section 6.3.4], in order to compute φ_{C_1} we need first to know $\partial^-_{C_1}$. We have

$$
\begin{aligned}
\partial^-_{C_1} a_1 &= 1 + a_5 c_2 b + b_1 b_6 + b,\\
\partial^-_{C_1} a_2 &= 1 + b_2 c_2 a_4 b_2 + b_2 c_2 b_3 a_5 + b_6 b_4 b_2 + b_6 c_1 a_5 + b_6 c_2 a,\\
&\quad + b_6 + b_2,\\
\partial^-_{C_1} a_3 &= 1 + a_4 b_2 c_2 + b_3 a_5 c_2 + b_3 + b b_5 + a c_2,\\
\partial^-_{C_1} a_4 &= 1 + b_3 b_1 + b b_4,\\
\partial^-_{C_1} a_5 &= b_1 b_2,\\
\partial^-_{C_1} b_1 = \partial^-_{C_1} b_2 &= 0,\\
\partial^-_{C_1} b_3 &= b c_1,\\
\partial^-_{C_1} b_4 &= c_1 b_1,\\
\partial^-_{C_1} b_5 &= b_4 b_2 c_2 + c_1 a_5 c_2 + c_2 + c_1,\\
\partial^-_{C_1} b_6 &= b_2 c_2 b\\
\partial^-_{C_1} c_1 = \partial^-_{C_1} c_2 &= 0,\\
\partial^-_{C_1} a &= b + b_2,\\
\partial^-_{C_1} b &= 0.
\end{aligned}
$$

Which we compare to $\partial^+_{C_1}$ computed above which gave

$$
\begin{aligned}
\partial^+_{C_1} a_1 &= 1 + a_5 c_2 b_2 + b_1 b_6 + b_2,\\
\partial^+_{C_1} a_2 &= 1 + b_2 c_2 a_4 b_2 + b_2 c_2 b_3 a_5 + b_6 b_4 b_2 + b_6 c_1 a_5 + b_6 + b_2,\\
\partial^+_{C_1} a_3 &= 1 + a_4 b_2 c_2 + b_3 a_5 c_2 + b_3 + b_2 b_5,\\
\partial^+_{C_1} a_4 &= 1 + b_3 b_1 + b_2 b_4,\\
\partial^+_{C_1} a_5 &= b_1 b_2,\\
\partial^+_{C_1} b_1 = \partial^+_{C_1} b_2 &= 0,\\
\partial^+_{C_1} b_3 &= b_2 c_1,\\
\partial^+_{C_1} b_4 &= c_1 b_1,\\
\partial^+_{C_1} b_5 &= b_4 b_2 c_2 + c_1 a_5 c_2 + c_2 + c_1,\\
\partial^+_{C_1} b_6 &= b_2 c_2 b_2,\\
\partial^+_{C_1} c_1 = \partial^+_{C_1} c_2 &= 0.
\end{aligned}
$$

A priori, in order to compute the associated map φ_{C_1} we need to order the Reeb chord according to the length filtration (see [14, Section 3.1] and [9, Section 6.3.4]). This ensure that when computing $\varphi_{C_1}(a)$ we already know the image by φ_{C_1} of any letter appearing in $\partial^-_{C_1}(a)$. But we actually do not need to understand the whole filtration in a concrete example. For this note that for any generator d of $\mathcal{A}(\Lambda^+)$ if b is not a letter appearing in $\partial^-_{C_1}(d)$ then $\varphi_{C_1}(d) = d$ regardless of its action. Thus in the end we need to understand the filtration on a_1, a_3, a_4, b_3 and b_6. One easily see that the action of a_1 can be made as big as we want without changing any other action. Then from the fact that $\partial^\pm$ decreases the action one get that $h(a_1) > h(a_3) > h(a_4) > h(b_3)$ and that $h(a_1) > h(b_6)$. This is enough to proceed with inductive process (as b_6 only appears in $\partial(a_1)$ we treat it as having action greater than a_3).

Also note that $\partial^-_{C_1}(a) = b = b + 0$ which give $v = 0$ (following the notation from [9]).

We start with b_3 following the notation of [9, Section 6.3.4] we need to write $\partial^-_{C_1} b_3 = \sum B_1 b B_2 b \dots B_k b A$ where all $B's$ are words with letters in the generator of $\mathcal{A}(\Lambda^+)$ (with lower action than b_3) and where every occurence of b in A follows an occurence of a. In our situation we have $\partial^-_{C_1} b_3 = bc_1 = bA$ with $A = c_1$ (and we have no word of type B_i). Thus b_3 is mapped to $b_3 + aA = b_3 + ac_1$.

We then proceed for a_4, we get $\partial^-_{C_1} a_4 = 1 + b_3 b_1 + bb_4 = A_1 + A_2 + bA_3$ with $A_1 = 1$, $A_2 = b_3 b_1$ and $A_3 = b_4$ (again no B's). Only A_3 is of interest here (as it belongs to a monomial containing b) and implies that a_4 is mapped to $a_4 + ab_4$.

For a_3 we have $\partial^-_{C_1} a_3 = 1 + a_4 b_2 c_2 + b_3 a_5 c_2 + b_3 + bb_5 + ac_2$. The only relevant monomial is bb_5 implying that a_3 is mapped to $a_3 + ab_5$.

As for b_6 we have $\partial^-_{C_1} b_6 = b_2 c_2 b = Bb$. This implies that b_6 is mapped to $b_6 + \varphi_{C_1}(B) a = b_6 + b_2 c_2 a$.

Finally for a_1 we have $\partial^-_{C_1} a_1 = 1 + a_5 c_2 b + b_1 b_6 + b = A_1 + A_2 b + A_3 + A_4 b$ with the only relevant A_i being $A_2 = a_5 c_2$ and $A_4 = 1$ giving that a_1 is mapped to $a_1 + a_5 c_2 a + a$.

In summary we have that φ_{C_1} does the following:

$$
\begin{aligned}
a_1 &\to a_1 + a + a_5 c_2 a, \\
a_3 &\to a_3 + ab_5, \\
a_4 &\to a_4 + ab_4, \\
b_3 &\to b_3 + ac_1, \\
b_6 &\to b_6 + b_2 c_2 a,
\end{aligned}
$$

and all other generators are mapped to themselves.

5.2. Map associated to C_2. The bifurcation associated to C_2 is of type $\mathbf{IV_b}$ using the notations of Figure 8.

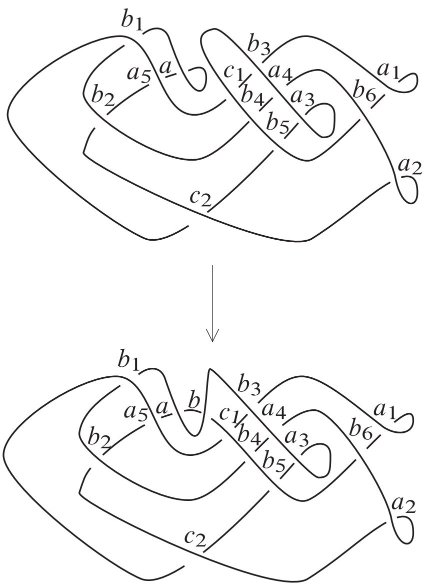

Figure 8. Saddle cobordism $\mathbf{IV_b}$.

An easy verification shows that the contractible Reeb chord b is simple (in the sense of [9]). We can thus apply [9, Proposition 6.17] and count immersed polygons with two positive corners (one on b). We get only three of those (the $\pm$ superscripts design postive and negative corners of the polygons):

$$a_2^+ b_6^- b^+ a^-,$$
$$b_4^+ b^+,$$
$$b_5^+ b^+ a^- c_2^-.$$

Which gives that the map φ_{C_2} does the following:

$$a_2 \to a_2 + b_6 a,$$
$$b_4 \to b_4 + 1,$$
$$b_5 \to b_5 + a c_2,$$
$$b \to 1,$$

all other generators being mapped to themselves.

This changes the differential as follows:

$$\begin{aligned}
\partial^-_{C_2} a_1 &= a_5 c_2 + b_1 b_6,\\
\partial^-_{C_2} a_2 &= 1 + b_2 c_2 a_4 b_2 + b_2 c_2 b_3 a_5 + b_6 b_4 b_2 + b_6 c_1 a_5 + b_2,\\
\partial^-_{C_2} a_3 &= 1 + a_4 b_2 c_2 + b_3 a_5 c_2 + b_3 + b_5,\\
\partial^-_{C_2} a_4 &= b_3 b_1 + b_4,\\
\partial^-_{C_2} a_5 &= b_1 b_2,\\
\partial^-_{C_2} b_1 = \partial^-_{C_2} b_2 &= 0,\\
\partial^-_{C_2} b_3 &= c_1,\\
\partial^-_{C_2} b_4 &= c_1 b_1,\\
\partial^-_{C_2} b_5 &= b_4 b_2 c_2 + c_1 a_5 c_2 + c_1,\\
\partial^-_{C_2} b_6 &= b_2 c_2,\\
\partial^-_{C_2} c_1 = \partial^-_{C_2} c_2 &= 0,\\
\partial^-_{C_2} a &= 1 + b_2,\\
\partial^-_{C_2} b &= 0.
\end{aligned}$$

5.3. Map associated to C_3. Using the notation of Figure 9, the bifurcations associated to C_3 are given by first $\mathbf{II}^{-1}_{\mathbf{b_3c_1}}$ then $\mathbf{II}^{-1}_{\mathbf{a_4b_4}}$ (going in the decreasing t direction).

From $\partial^+_{C_3}(b_3) = c_1 = c_1 + v$ with $v = 0$ we deduce (following [9, Section 6.3.3]) that at the first bifurcation b_3 maps to 0 and c_1 maps to v thus to 0. This implies that in the middle of the cobordism one has $\partial(a_4) = b_4$ implying that a_4 and b_4 maps to 0. Thus φ_{C_3} does the following:

$$\begin{aligned}
b_3 &\to 0,\\
c_1 &\to 0,\\
a_4 &\to 0,\\
b_4 &\to 0,
\end{aligned}$$

all other generators being mapped to themselves.

5.4. Map associated to C_4. Following the notation of Figure 10, the bifurcations associated to the cobordism C_4 are, again following the decreasing t direction, first $\mathbf{III}'_{\mathbf{b_1a_5a}}$ then $\mathbf{II}^{-1}_{\mathbf{a_5b_1}}$.

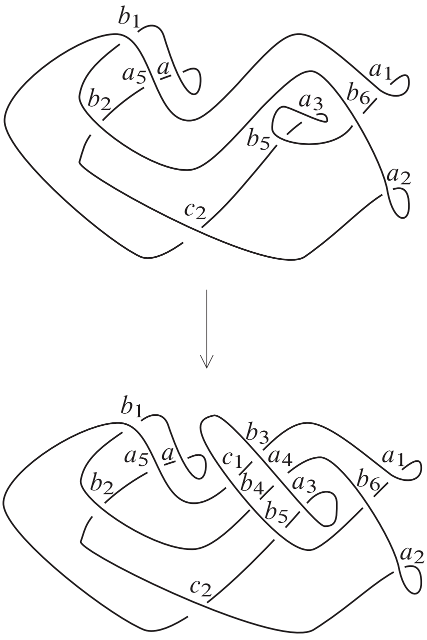

Figure 9. $\mathbf{II}^{-1}_{\mathbf{b_3c_1}} \circ \mathbf{II}^{-1}_{\mathbf{a_4b_4}}$.

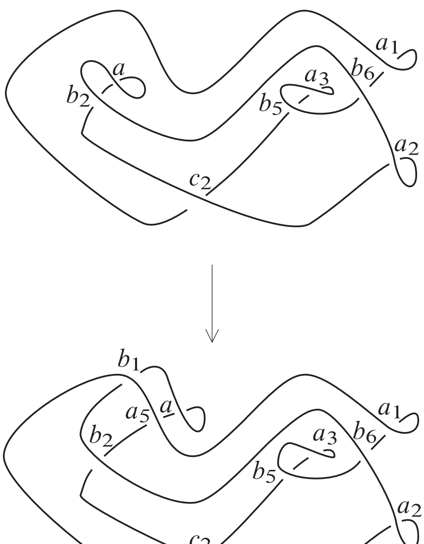

Figure 10. $\mathbf{III}'_{\mathbf{b_1a_5a}} \circ \mathbf{II}^{-1}_{\mathbf{a_5b_1}}$.

To compute the map associated to $\mathbf{III}'_{\mathbf{b_1a_5a}}$ we apply [9, Section 6.3.2] and get that a_5 maps to $a_5 + b_1 a$ and all other generators are mapped to themselves.

One computes that in the middle

$$\partial(a_5) = \partial^+_{C_4}(a_5) + \partial(b_1 a) = b_1 b_2 + b_1 + b_1 b_2 = b_1.$$

Applying again [9, Section 6.3.3] we deduce that the bifurcation $\mathbf{II}^{-1}_{\mathbf{a_5b_1}}$ maps a_5 and b_1 to 0. This implies that φ_{C_4} does the following:

$$\begin{aligned} a_5 &\to 0, \\ b_1 &\to 0, \\ a &\to a \end{aligned}$$

all other generator being mapped to themselves.

The differential at this step is

$$\begin{aligned} \partial^-_{C_4} a_1 &= 0, \\ \partial^-_{C_4} a_2 &= 1 + b_2, \\ \partial^-_{C_4} a &= 1 + b_2, \\ \partial^-_{C_4} b_2 &= 0, \\ \partial^-_{C_4} b_6 &= b_2 c_2, \\ \partial^-_{C_4} a_3 &= 1 + b_5, \\ \partial^-_{C_4} b_5 &= 0. \end{aligned}$$

5.5. Map associated to C_5. Using the notation of Figure 11, the bifurcations corresponding to C_5 are $\mathbf{II}^{-1}_{\mathbf{a_3b_5}}$ and $\mathbf{II}^{-1}_{\mathbf{ab_2}}$ (these are commutative).

One easily see that φ_{C_5} does the following:

$$\begin{aligned} a &\to 0, \\ b_2 &\to 1, \\ a_3 &\to 0, \\ b_5 &\to 1, \end{aligned}$$

and all other generators are mapped to themselves.

The differential becomes

$$\begin{aligned} \partial^-_{C_5} a_1 &= 0, \\ \partial^-_{C_5} a_2 &= 0, \\ \partial^-_{C_5} b_6 &= c_2, \\ \partial^-_{C_5} c_2 &= 0. \end{aligned}$$

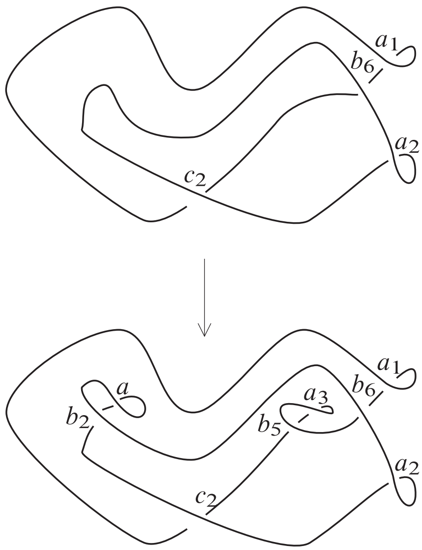

Figure 11. $\mathbf{II}^{-1}_{\mathbf{a_3b_5}} \circ \mathbf{II}^{-1}_{\mathbf{ab_2}}$.

5.6. Map associated to C_6. The bifurcation corresponding to C_6 is $\mathbf{II}^{-1}_{\mathbf{b_6c_2}}$.

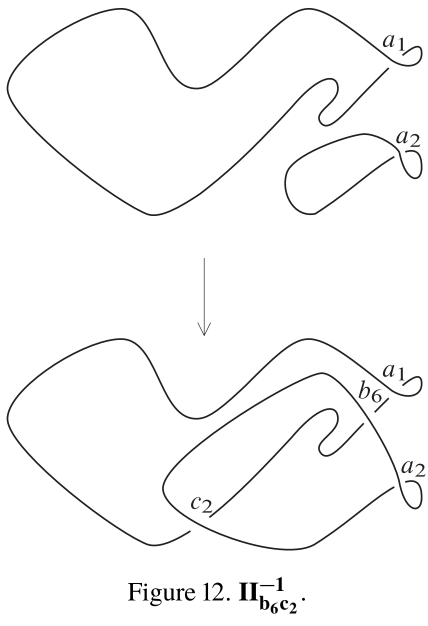

Figure 12. $\mathbf{II}^{-1}_{\mathbf{b_6c_2}}$.

We have that φ_{C_6} does

$$\begin{aligned}a_1 &\to a_1,\\ a_2 &\to a_2,\\ b_6 &\to 0,\\ c_2 &\to 0.\end{aligned}$$

5.7. Map associated to C_7 and the composition φ_C. The last part of C is filling one of the components of the link on Figure 12 with a Lagrangian disk (lets say the one with Reeb chord a_1). This has the effect of mapping the corresponding chord to 0, thus $\varphi_{C_7}(a_1) = 0$ and $\varphi_{C_7}(a_2) = a_0$ where a_0 is the unique Reeb chords of Λ_0.

Combining this to the previous paragraphs we get that the map

$$\varphi_C = \varphi_{C_7} \circ \varphi_{C_6} \circ \varphi_{C_5} \circ \varphi_{C_4} \circ \varphi_{C_3} \circ \varphi_{C_2} \circ \varphi_{C_1},$$

associated to the concordance of Figure 4 is

$$\begin{aligned}&a_2 \to a_0,\\ &a_1, a_3, a_4, a_5, b_1, b_3, b_6, c_1, c_2 \to 0,\\ &b_2, b_4, b_5 \to 1.\end{aligned}$$

6. Lagrangian concordances from Λ_0 to itself

The aim of this section is to prove the following:

Theorem 6.1. *Let C be a Lagrangian concordance from Λ_0 to Λ_0 then the map $\varphi_c : \mathcal{A}(\Lambda_0) \to \mathcal{A}(\Lambda_0)$ induced by C is the identity.*

This follows from Theorem 1.2 of which we give a proof now.

Proof of Theorem 1.2. This is actually a corollary of the main result of [12].

Let $C \subset \mathbb{R} \times S^3$ be an oriented Lagrangian cobordism from K_0 to itself. First note that since $\mathrm{tb}(K_0) - \mathrm{tb}(K_0) = 0$ it follows from [3] that C is topologically a cylinder.

The symplectisation of S^3 is symplectomorphic to $\mathbb{C}^2 \setminus 0$ with its standard symplectic form. Under this symplectomorphism the t-direction becomes the radial direction. A parametrisation of K_0 in S^3 is given by

$$\{(cos(\theta), \sin(\theta)) | \theta \in [0, 2\pi)\} \subset \mathbb{C}^2$$

i.e.

$$\Lambda_0 = \mathbb{R}^2 \cap S^3 \subset \mathbb{C}^2,$$

where $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\} \subset \mathbb{C}^2$. Thus C is a Lagrangian cylinder which coincides near 0 and outside a compact ball with the trivial Lagrangian plane, i.e. $C_1 = C \cup \{0\}$ is local Lagrangian knot (following the terminology of [12]). It follows from the main result of [12] that there exist a compactly supported Hamiltonian diffeomorphism ϕ_H such that $\phi_H(C_1) = \mathbb{R}^2 \subset \mathbb{C}^2$.

For $\epsilon > 0$ we denote by D_ϵ the ball of radius ϵ in $\mathbb{C}^2$. Take ϵ sufficiently small so that $C_\epsilon := C_1 \cap D_\epsilon = \mathbb{R}^2 \cap D_\epsilon$. Since ϕ_H maps C_1 to $\mathbb{R}^2$ then $\phi_H(C_\epsilon) \subset \mathbb{R}^2$ and there exists a compactly supported diffeomorphism isotopic to the identity f of $\mathbb{R}^2$ such that $f(\phi_H(C_\epsilon)) = C_\epsilon$. Using standard construction one can extend f to a compactly supported Hamiltonian diffeomorphism $\tilde{f}$ of $\mathbb{C}^2$ (which by assumption preserves $\mathbb{R}^2$). Thus $\phi_1 = \tilde{f} \circ \phi_H$ is a compactly supported Hamiltonian diffeomorphism mapping C_1 to $\mathbb{R}^2$ such that $\phi_1|_{C_\epsilon} = \mathrm{Id}$. Now standard application of Moser's path method leads to an Hamiltonian diffeomorphism ϕ' supported in D_ϵ such that ϕ' preserves $\mathbb{R}^2$ and $\phi' \circ \phi_1|_{D_{\epsilon'}} = \mathrm{Id}$ for $\epsilon' << \epsilon$. Restricting $\phi' \circ \phi_1$ to $\mathbb{C}^2 \setminus \{0\}$ proves the theorem. □

We are now able to prove Theorem 6.1.

Proof of Theorem 6.1. Take a contact embedding of $(\mathbb{R}^3, \xi_0) \to (S^3, \xi_0)$ as in [13, Proposition 2.1.8] such that Λ_0 is mapped to K_0. This embedding induces a symplectic embedding of $\mathbb{R} \times \mathbb{R}^3$ in $\mathbb{R} \times S^3 \simeq \mathbb{C}^2 \setminus \{0\}$. Under this identification the concordance C maps to a concordance from K_0 to itself. Theorem 1.2 implies that there exist a compactly supported symplectomorphism ϕ mapping C to the trivial cylinder of K_0.

Since ϕ is the identity near $\pm\infty$, for any cylindrical almost complex structure J on $\mathbb{R} \times S^3$ admissible (in the sense of [7]) for the trivial concordance we get that $(\phi^{-1})^* J$ is admissible for the original concordance C. This implies the induced map by C is the same map as the one induced by $\mathbb{R} \times K_0$ which is the identity (because the only degree 0 pseudo-holomorphic curve on the trivial concordance is the trivial one). Since $H(\mathcal{A}(\Lambda_0)) = \mathcal{A}(\Lambda_0)$ and the induced map in homology by φ_C do not depends on auxiliary choices, we get that the map do not depend on the choice of the almost complex structure cylindrical at infinities. This conclude the proof. □

7. Non symmetry of Lagrangian concordances

In order to prove Theorem 1.1 we use the augmentation category of Λ denoted by $Aug(\Lambda)$. This is an $\mathcal{A}_\infty$-category defined in [2] whose objects are augmentations of the Chekanov algebra and morphisms in the homological category are bilinearised Legendrian contact cohomology groups.

Recall that an augmentation ε of a DGA $(\mathcal{A}, \partial)$ over $\mathbb{Z}_2$ is simply a DGA map from $(\mathcal{A}, \partial)$ to $(\mathbb{Z}_2, 0)$.

Bilinearised cohomology groups are generalisations of linearised Legendrian contact cohomology groups (as defined in [5]) introduced in [2] using two augmentations instead of one and keeping track of the non-commutativity of $\mathcal{A}(\Lambda)$. Basically for two augmentations ε_1 and ε_2 and a word $b_1 \dots b_k$ in ∂a the expression

$$\sum_j \varepsilon_1(b_1)\varepsilon_1(b_2)\dots\varepsilon_1(b_{j-1}) \cdot b_j \cdot \varepsilon_2(b_{j+1})\dots\varepsilon_2(b_k)$$

contributes to $d^{\varepsilon_1,\varepsilon_2}a$.

Dualising $d^{\varepsilon_1,\varepsilon_2}$ leads to bilinearised Legendrian contact cohomology differential

$$\mu^1_{\varepsilon_1,\varepsilon_2} \colon C_{\varepsilon_1,\varepsilon_2}(\Lambda) \longrightarrow C_{\varepsilon_1,\varepsilon_2}(\Lambda)$$

(where $C_{\varepsilon_1,\varepsilon_2}(\Lambda)$ is the vector space generated by Reeb chords of Λ) whose homology forms morphisms space in the homological category of the augmentation category. Higher order compositions are defined using similar considerations with more than 2 augmentations. For instance the composition of morphisms $\mu^2_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is defined as the dual of the map $d_2^{\varepsilon_3,\varepsilon_2,\varepsilon_1}$ which to a word $b_1 \dots b_k$ in ∂a associates

$$\sum_{i,j} \varepsilon_3(b_1)\dots\varepsilon_3(b_{i-1}) \cdot b_i \cdot \varepsilon_2(b_{i+1})\dots\varepsilon_2(b_{j-1}) \cdot b_j \cdot \varepsilon_1(b_{j+1})\dots\varepsilon_1(b_k).$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The first part on the existence of the concordance has been proved in Section 4. It remains to prove that no concordance from Λ to Λ_0 exists.

Assume that such a concordance C' exists and denote by

$$\varphi_{C'} \colon \mathcal{A}(\Lambda_0) \longrightarrow \mathcal{A}(\Lambda)$$

the induced map. Let C be the concordance of Section 5 which induced the map φ_C.

The concatenation of C' with C leads to a concordance from Λ_0 to itself. Theorem 6.1 implies that the map induced by this concatenation is

$$\mathrm{Id} \colon \mathcal{A}(\Lambda_0) \longrightarrow \mathcal{A}(\Lambda_0).$$

Hence by [9, Theorem 1.2] we get that $\varphi_C \circ \varphi_{C'} = \mathrm{Id}$.

Now following [2, Section 2.4] we get that $\varphi_{C'}$ induces an $\mathcal{A}_\infty$-functor

$$\mathcal{F}_{C'}\colon Aug(\Lambda) \longrightarrow Aug(\Lambda_0)$$

(obtained by dualising the components of the map $\varphi_{C'}$). Similarly φ_C induces an $\mathcal{A}_\infty$-functor

$$\mathcal{F}_C : Aug(\Lambda_0) \longrightarrow Aug(\Lambda).$$

From $\varphi_C \circ \varphi_{C'} = \mathrm{Id}$ we get that

$$\mathcal{F}_{C'} \circ \mathcal{F}_C = \mathrm{Id}\,.$$

Note that $\mathcal{A}(\Lambda_0)$ has only one augmentation ε_0 (which maps a_0 to 0). By definition of $\mathcal{F}_C$ its action on the object of the augmentation category is given by $\varepsilon \to \varepsilon \circ \varphi_C$, thus the explicit computation of Section 5 shows that $\mathcal{F}_C(\varepsilon_0) = \varphi_C \circ \varepsilon_0 = \varepsilon_1$ where ε_1 is the first augmentation of Table 1. Table 1 also shows another augmentation of $\mathcal{A}(\Lambda)$ we will use to compute bilinearised cohomology groups.

Table 1. Two augmentations of $\mathcal{A}(\Lambda)$.

	b_1	b_2	b_3	b_4	b_5	b_6
ε_1	0	1	0	1	1	0
ε_2	1	0	1	0	0	1

We will now show that the two augmentations ε_1 and ε_2 are not equivalent.

Table 2 gives the bilinearised differential for all possible pairs out of those two augmentations (as b_1 and b_2 are always mapped to 0 we omit them from the table).

Table 2. Bilinearised differentials for Λ.

	a_1	a_2	a_3	a_4	a_5
$d^{\varepsilon_1,\varepsilon_1}$	b_2	b_2	$b_3+b_2+b_5$	b_2+b_4	b_1
$d^{\varepsilon_2,\varepsilon_2}$	$b_1+b_2+b_6$	b_2+b_6	b_3	b_3+b_1	b_2
$d^{\varepsilon_1,\varepsilon_2}$	b_1+b_2	b_6+b_2	b_3+b_5	b_3+b_4	0
$d^{\varepsilon_2,\varepsilon_1}$	b_6+b_2	b_6+b_4	b_3+b_2	b_1+b_2	b_1+b_2

	b_3	b_4	b_5	b_6
$d^{\varepsilon_1,\varepsilon_1}$	c_1	0	c_1	c_2
$d^{\varepsilon_2,\varepsilon_2}$	0	c_1	c_2+c_1	0
$d^{\varepsilon_1,\varepsilon_2}$	0	c_1	c_1	c_1
$d^{\varepsilon_2,\varepsilon_1}$	0	0	c_2+c_1	0

Notice that for linearised LCH (the first two lines) there are no non-trivial homology in degree -1 whereas for the mixed augmentation there is always a generator of degree -1. It follows then from [2, Theorem 1.4] that the two augmentations ε_1 and ε_2 are not equivalent.

In order to conclude, one must study the compositions in the augmentation category and its homological category, thus we need to consider the bilinearised cohomology groups. From Table 2 we get that the bilinearised differentials in cohomology are those given in Table 3.

Table 3. $\mu^1_{\varepsilon_i,\varepsilon_j}$ on Λ.

	b_1	b_2	b_3	b_4	b_5	b_6
$\mu^1_{\varepsilon_1,\varepsilon_1}$	a_5	$a_1+a_2+a_3+a_4$	a_3	a_4	a_3	0
$\mu^1_{\varepsilon_2,\varepsilon_2}$	a_1+a_4	$a_1+a_2+a_5$	a_3+a_4	0	0	a_1+a_2
$\mu^1_{\varepsilon_1,\varepsilon_2}$	a_4+a_5	$a_1+a_3+a_4+a_5$	a_3	a_2	0	a_1+a_2
$\mu^1_{\varepsilon_2,\varepsilon_1}$	a_1	a_1+a_2	a_3+a_4	a_4	a_3	0

	c_1	c_2
$\mu^1_{\varepsilon_1,\varepsilon_1}$	b_3+b_5	b_6
$\mu^1_{\varepsilon_2,\varepsilon_2}$	b_4+b_5	b_5
$\mu^1_{\varepsilon_1,\varepsilon_2}$	b_5	b_5
$\mu^1_{\varepsilon_2,\varepsilon_1}$	$b_3+b_4+b_5$	0

From Table 3 we can see that $\mathrm{LCH}^1_{\varepsilon_1}$ has one generator $[a_1]=[a_2]$ (since $a_1+a_2=\mu^1_{\varepsilon_1}(b_2+b_3+b_4)$) and that $\mathrm{LCH}^0_{\varepsilon_1}$ has dimension 0 (since $b_6=\mu^1_{\varepsilon_1}(c_2)$). As $\mathcal{F}_{C'}\circ\mathcal{F}_C$ is the identity we get that

$$H(\mathcal{F}^1_{C'})\circ H(\mathcal{F}^1_C)\colon\ \mathrm{LCH}_{\varepsilon_0}(\Lambda_0)\longrightarrow \mathrm{LCH}_{\varepsilon_0}(\Lambda_0)$$

is the identity. This implies that in the homological category

$$H(\mathcal{F}^1_C)\colon\ \mathrm{LCH}_{\varepsilon_1}(\Lambda)\longrightarrow \mathrm{LCH}_{\varepsilon_0}(\Lambda_0)$$

is surjective in particular the only generator $[a_2]$ of $\mathrm{LCH}^1_{\varepsilon_1}(\Lambda)$ is mapped to $[a_0]$ the generator $\mathrm{LCH}^1_{\varepsilon_0}(\Lambda_0)$.

In order to understand the compositions in the category, we need to compute

$\partial_2^{\varepsilon_1,\varepsilon_2,\varepsilon_1}$ which gives

$$
\begin{aligned}
a_1 &\to b_1 b_6,\\
a_2 &\to c_2 a_4 + c_2 a_5 + b_6 b_4,\\
a_3 &\to b_2 b_5,\\
a_4 &\to b_3 b_1 + b_2 b_4,\\
a_5 &\to b_1 b_2,\\
b_3 &\to b_2 c_1,\\
b_4 &\to c_1 b_1,\\
b_6 &\to c_2 b_2 + b_2 c_2.
\end{aligned} \tag{1}
$$

From the second line (1) of the preceding formula we see that $\mu^2_{\varepsilon_1,\varepsilon_2,\varepsilon_1}(a_5, c_2) = a_2 \in C_{\varepsilon_1,\varepsilon_1}(\Lambda)$. As the composition $[x] \circ [y]$ in the homological category is given by $[\mu^2(x, y)]$ we get that $[a_5] \circ [c_2] = [a_2]$. Since $\mathcal{F}_{C'}$ is an $\mathcal{A}_\infty$-functor we get that $H({\mathcal{F}_{\mathcal{C}'}}^1)$ preserves this composition (see [2, Section 2.3]) thus we have that $0 \neq [a_0] = H(\mathcal{F}^1_{C'})([a_2]) = H(\mathcal{F}^1_{C'})([a_5]) \circ H(\mathcal{F}^1_{C'})([c_2])$. However $H(\mathcal{F}^1_{C'})([c_2]) \in \mathrm{LCH}^{-1}_{\varepsilon_0}(\Lambda_0) \simeq \{0\}$. Thus $[a_0] = H(\mathcal{F}^1_{C'})([a_5]) \circ 0 = 0$, this contradicts the existence of $\mathcal{F}_{C'}$ and hence the existence of C'. Thus $\Lambda \not\prec \Lambda_0$. □

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