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Cyclic extensions of fusion categories via the Brauer–Picard groupoid

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Abstract. We construct a long exact sequence computing the obstruction space, $\pi_1 \underline{\mathrm{BrPic}}(\mathcal{C}_0)$, to G-graded extensions of a fusion category $\mathcal{C}_0$. The other terms in the sequence can be computed directly from the fusion ring of $\mathcal{C}_0$. We apply our result to several examples coming from small index subfactors, thereby constructing several new fusion categories as G-extensions. The most striking of these is a $\mathbb{Z}/2\mathbb{Z}$-extension of one of the Asaeda–Haagerup fusion categories, which is one of only two known 3-supertransitive fusion categories outside the ADE series.

In another direction, we show that our long exact sequence appears in exactly the way one expects: it is part of a long exact sequence of homotopy groups associated to a naturally occuring fibration. This motivates our constructions, and gives another example of the increasing interplay between fusion categories and algebraic topology.

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1. Introduction

In this paper we construct several new fusion categories related to the Asaeda–Haagerup subfactor [2] and the related $AH + 1$ and $AH + 2$ subfactors [1, 8]. These constructions require the calculation of certain obstruction groups which appear in the theory of G-extensions of fusion categories, and we construct a long exact sequence which allows us to do this calculation. This long exact sequence is in turn a consequence of a certain homotopy fibration of higher groups, as we explain.

Let us begin by specifying more precisely the concrete problem we wish to solve. Consider a finite index, finite depth subfactor pair $N \subset M$, which is *self-*

dual, i.e. is equipped with an equivalence of fusion categories between the *principal even part* $\mathcal{C}_0 = \langle {}_N M_N \rangle \subseteq N\text{-mod-}N$ and its dual $(\mathcal{C}_0)^*_{\mathcal{C}_1}$ with respect to the $\mathcal{C}_0$-module category $\mathcal{C}_1 = \langle {}_N M_M \rangle \subseteq N\text{-mod-}M$. Examples include the Izumi-Xu and $AH + 2$ subfactors. In this situation $\mathcal{C}_1$ is a $\mathcal{C}_0$-bimodule category, and it is natural to ask whether we can combine the common even part and the odd part to form a new fusion category $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$.

The theory of G-graded extensions of a fusion category, introduced in [6] and developed in [5], provides answers to precisely such questions. A fusion category $\mathcal{C}$ is graded by G if we have a decomposition $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$, compatible with tensor product. Thus the categories considered above are instances of $\mathbb{Z}/2\mathbb{Z}$-graded extensions of the even part $\mathcal{C}_0$.

The paper [5] constructs an equivalence between G-graded extensions of a fixed category $\mathcal{C}_0$, and homomorphisms $\rho\colon G \to \underline{\mathrm{BrPic}}(\mathcal{C}_0)$ to the categorical 2-group $\underline{\mathrm{BrPic}}(\mathcal{C}_0)$ of $\mathcal{C}_0$-bimodule categories. Standard arguments in algebraic topology then reduce the existence of such extensions to vanishing of obstruction classes $o_k \in H^k(G, \pi_{k-2}\underline{\mathrm{BrPic}}(\mathcal{C}_0))$, for $k = 3, 4$. When $G = \mathbb{Z}/2\mathbb{Z}$, or more generally when G is cyclic, o_4 is automatically trivial, so that we need only contend with $o_3 \in H^3(G, \pi_1\underline{\mathrm{BrPic}}(\mathcal{C}_0))$.

The main technical tool we develop in this paper is an exact sequence,

$$* \to \mathrm{Hom}(U(\mathcal{C}_0), \mathbb{C}^\times) \to \pi_1\underline{\mathrm{BrPic}}(\mathcal{C}_0) \to \mathrm{Inv}(\mathcal{C}_0), \tag{1}$$

which realizes $\pi_1\underline{\mathrm{BrPic}}(\mathcal{C}_0)$ as an extension of a subgroup of the group $\mathrm{Inv}(\mathcal{C}_0)$ of invertible objects of $\mathcal{C}_0$, by the group $\mathrm{Hom}(U(\mathcal{C}_0), \mathbb{C}^\times)$ of characters of the universal grading group of $\mathcal{C}_0$. The virtue of (1) is the other two groups can be read off directly from the fusion data of $\mathcal{C}_0$; we leverage this to show that $H^3(\mathbb{Z}/2\mathbb{Z}, \pi_1\underline{\mathrm{BrPic}}(\mathcal{C}_0))$ is trivial in our examples, so that the obstruction o_3 vanishes automatically.

In Section 3, we give an *ad hoc* construction of the sequence (1), which relies on an identification $\pi_1\underline{\mathrm{BrPic}}(\mathcal{C}_0) \cong \mathrm{Inv}(Z(\mathcal{C}_0))$, from [5]. However, (1) may be more properly understood a consequence of the following:

Theorem 3.4. *We have a homotopy fiber sequence,*

$$\underline{\mathrm{Inv}}(\mathcal{C}_0) \xrightarrow{F_-} \underline{\mathrm{Eq}}(\mathcal{C}_0) \xrightarrow{M_-} \underline{\mathrm{Out}}(\mathcal{C}_0).$$

Here $\underline{\mathrm{Inv}}$ and $\underline{\mathrm{Eq}}$ denote categorical 1-groups of invertible objects and tensor automorphisms, respectively, and $\underline{\mathrm{Out}}$ is a certain full subgroup of $\underline{\mathrm{BrPic}}$ with the

same π_1 and π_2. In Corollary 3.7, we deduce the sequence (1) as a fragment of the long exact sequence in homotopy groups induced by Theorem 3.4. The proof of Theorem 3.4 is delayed until Section 5.

In Section 4.1, we turn to applications of the sequence (1) to subfactors. Our primary application is the construction of a new fusion category $\mathcal{AH}+2$, built from the even and odd parts of the self-dual subfactor $AH+2$. This new fusion category $\mathcal{AH}+2$ is particularly notable because it is, along with Morrison-Penneys's 4442 fusion category [13], one of the first 3-supertransitive fusion category outside of the ADE families. It is generated by an object of dimension $\frac{1+\sqrt{17}}{2}$.

In fact, $\mathcal{AH}+2$ is just one of eighteen new examples of fusion categories we build as $\mathbb{Z}/2\mathbb{Z}$-extensions of Asaeda–Haagerup type categotries. By [8], the principal even parts $\mathcal{AH}_1, \mathcal{AH}_2, \mathcal{AH}_3$ of AH, $AH+1$, $AH+2$, respectively, each have three non-trivial bimodule categories up to equivalence. We have:

Theorem 4.4. *Each of the three non-trivial bimodule categories over each $\mathcal{AH}_i$, $i = 1, 2, 3$, is the odd component of exactly two $\mathbb{Z}/2\mathbb{Z}$-graded extensions of $\mathcal{AH}_i$.*

These techniques work well more generally when applied to any fusion category coming from a 2-supertransitive subfactor. In Section 4.2, we give one other source of such examples, the near group categories $\mathcal{C}_p$ associated to $\mathbb{Z}/p\mathbb{Z}$. These are fusion categories of the form $\mathrm{Vec}(\mathbb{Z}/p\mathbb{Z}) \oplus \mathrm{Vec}$, which are not necessarily $\mathbb{Z}/2\mathbb{Z}$-graded. The group algebra $\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]$ is an algebra in $\mathcal{C}_p$, and we we let $\mathcal{M}_p$ denote its category of modules. We say that $\mathcal{C}_p$ is self-dual if $\mathcal{C}_p \cong (\mathcal{C}_p)^*_{\mathcal{M}_p}$. We have:

Theorem 4.9. *Suppose that $p > 2$, that $\mathcal{C}_p$ is a self-dual $\mathbb{Z}/p\mathbb{Z}$ near-group category with trivial outer automorphism group. Then there exist exactly two $\mathbb{Z}/2\mathbb{Z}$-extensions of $\mathcal{C}_p$ by $\mathcal{M}_p$.*

Applying Theorem 4.9 and Han's thesis [10], we give a third proof of a result first proved by Ostrik [4, Appendix] and second by Morrison-Penneys [13], establishing the existence of a certain $\mathbb{Z}/2\mathbb{Z}$-graded extension of the Izumi-Xu category $\mathcal{IX}$. It is our hope that Theorem 4.9 will find application in more near-group examples; however this will require developing techniques to establish self-duality and to calculate outer automorphism groups.

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2. Preliminaries

2.1. Categorical *n*-groups. In order to state the results of [5] on extensions of fusion categories we will need to use the notions of higher groupoids and higher categorical groups. We will need only the notion of a categorical 0-group (i.e. a group), 1-group, and 2-group in this paper.

By an n-groupoid, we will mean an n-category $\mathcal{C}$, all of whose morphisms at all levels are invertible. Recall from [5] that a categorical n-group $\mathcal{G}$ is a monoidal n-groupoid, in which all objects are invertible, or equivalently, an $(n+1)$-groupoid with a single object. A homomorphism of categorical n-groups is a monoidal functor of n-groupoids or a functor of connected $(n+1)$-groupoids in that formulation.

For $m > k$, we can regard any categorical k-group as a categorical m-group, with trivial morphisms in degree $k+1, \ldots, m$. Thus we will often speak of a homorphism from, say, a categorical 0-group to a categorical 2-group, and that will mean a homomorphism regarding both as categorical 2-groups.

Remark 2.1. The classifying space construction $\mathcal{C} \mapsto B\mathcal{C}$ defines an equivalence between the category of n-groupoids and homotopy n-types; for this reason (more precisely, invertibility of morphisms at all levels), the well-known subtleties in the foundations of higher categories are largely absent from the theory of higher groupoids and higher categorical groups. In particular, a categorical n-group may be regarded as a connected homotopy $n+1$-type, just as a group may be identified with the connected homotopy 1-type of its classifying space. This identification shifts dimensions: we have $\pi_k \mathcal{C} = \pi_{k+1} B\mathcal{C}$, canonically.

Remark 2.2. Categorical n-groups are often called $(n+1)$-groups in the literature (e.g. [3]). Both indexings are reasonable, depending on whether you think of a group as a set with an operation or as a 1-category with only one object.

2.2. Fusion categories and their extensions. In this subsection we recall the extension theory of fusion categories developed in [5].

Definition 2.3. [6] A fusion category over $\mathbb{C}$ is a finite $\mathbb{C}$-linear semisimple rigid monoidal category with simple identity object.

For definitions of module categories, bimodule categories, tensor products of bimodule categories, and invertibility, see [5]; in this paper we assume all module categories are semisimple.

Definition 2.4. [5] The Brauer–Picard groupoid of a fusion category $\mathcal{C}$ is a 3-groupoid whose:

- objects are fusion categories which are Morita equivalent to $\mathcal{C}$;
- 1-morphisms are invertible bimodule categories between such fusion; categories
- 2-morphisms are equivalences of such bimodule categories;
- 3-morphisms are isomorphisms of such equivalences.

The Brauer–Picard categorical 2-group $\underline{\mathrm{BrPic}}(\mathcal{C})$ is the full subgroupoid of the Brauer–Picard groupoid whose only object is $\mathcal{C}$.

An extension of a fusion category $\mathcal{C}$ by a finite group G is a G-graded fusion category whose 0-graded part is equivalent to $\mathcal{C}$. There is a natural notion of equivalence of G-extensions.

Theorem 2.5 ([5]). (a) *Equivalence classes of G-extensions of $\mathcal{C}$ are given by categorical 2-group homomorphisms from G to $\underline{\mathrm{BrPic}}(\mathcal{C})$.*

(b) *Such homomorphisms (and hence, G-extensions) are parameterized by triples (c, M, α), where c is a group homomorphism*

$$c\colon G \to \mathrm{BrPic}(\mathcal{C}),$$

M belongs to a certain $H^2(G, \pi_1\underline{\mathrm{BrPic}}(\mathcal{C}))$-torsor T^2_c, and α belongs to a certain $H^3(G, \pi_1\underline{\mathrm{BrPic}}(\mathcal{C}))$-torsor $T^3_{c,M}$.

(c) *Certain obstruction classes $o_3(c) \in H^3(G, \pi_1\underline{\mathrm{BrPic}}(\mathcal{C}))$ and $o_4(c, M) \in H^4(G, \pi_2\underline{\mathrm{BrPic}}(\mathcal{C}))$ must vanish for (c, M, α) to determine an extension.*

2.3. Subfactors. A *subfactor* is a unital inclusion $N \subseteq M$ of II_1 factors. A subfactor $N \subseteq M$ has finite index if M is a finitely-generated projective module over N, cf. [12, 16]. In this case, the $N - N$ bimodule ${}_N M_N$ tensor generates a

semisimple unitary rigid monoidal category of $N-N$ bimodules, called the *principal even part* of the subfactor. The $N-M$ bimodule ${}_N M_M$ generates a module category over the principal even part; the dual category of this module category, which is the category of $M-M$ bimodules tensor generated by ${}_M M_N \otimes_N {}_N M_M$, is called the *dual even part*. The subfactor is said to have *finite depth* if the even parts are fusion categories, i.e. if they each have finitely many simple objects, up to isomorphism.

If a subfactor has the same principal and dual principal parts, it is natural to ask whether there is a $\mathbb{Z}/2\mathbb{Z}$-extension whose 0-graded part is the even part of the subfactor, and whose 1-graded part is the odd part of this subfactor. In particularly nice situations (where the generator of the odd part $\mathcal{C}_1$ becomes self-dual in $\mathcal{C}$) this can be understood directly in terms of subfactors or planar algebras. In terms of the factors, you get such an extension when you can realize $N \subset M$ as coming from a self-dual bimodule over N. In terms of planar algebras, such an extension tells you that the shaded planar algebra comes from an unshaded planar algebra.

Note that although we are studying examples coming from subfactors, we do not explicitly address unitarity of the extension here. In particular, the $\mathbb{Z}/2\mathbb{Z}$-extensions come in pairs, and it does not seem reasonable to expect that they would both be unitary.

3. Computing π_1 of the Brauer–Picard group

The group $\pi_1\underline{\mathrm{BrPic}}(\mathcal{C}) \cong \mathrm{Inv}(Z(\mathcal{C}))$ houses the primary obstruction in the extension theory of $\mathcal{C}$. In this section, we use techniques from elementary homotopy theory to compute this group. In addition to the categorical 2-group $\underline{\mathrm{BrPic}}(\mathcal{C})$, the main examples we consider are as follows.

Definition 3.1. The categorical 2-group $\underline{\mathrm{Out}}(\mathcal{C})$, of outer equivalences of $\mathcal{C}$, is the full 2-subgroup of invertible bi-module categories that are equivalent to $\mathcal{C}$ as left $\mathcal{C}$-module categories.

Definition 3.2. The categorical 1-group $\underline{\mathrm{Inv}}(\mathcal{C})$ is the subcategory of invertible objects in $\mathcal{C}$ and their isomorphisms:

- objects are the invertible objects $(g, h, \ldots)$ in $\mathcal{C}$;
- $\mathrm{Hom}(g, h)$ are the isomorphisms $(\phi, \psi, \ldots)$ of such.

Definition 3.3. The categorical 1-group $\underline{\mathrm{Eq}}(\mathcal{C})$ is the category of tensor auto-equivalences of $\mathcal{C}$:

- Objects are tensor auto-equivalences $(F, G, \ldots)$ of $\mathcal{C}$,
- $\mathrm{Hom}(F, G)$ are monoidal natural isomorphisms of such.

The assignments $F_-: \underline{\mathrm{Inv}}(\mathcal{C}) \to \underline{\mathrm{Eq}}(\mathcal{C})$, sending g to the functor,

$$F_g(X) := g \otimes X \otimes g^*,$$

and $M_-: \underline{\mathrm{Eq}}(\mathcal{C}) \to \underline{\mathrm{Out}}(\mathcal{C})$, sending F to the outer bimodule category $M_F := \mathcal{C}$, with tensor product defined, for $X, Y \in \mathcal{C}$, and $m \in M_F$, by

$$X \otimes_{M_F} m \otimes_{M_F} Y := X \otimes m \otimes F(Y).$$

can each be upgraded to functors of categorical 2-groups (see §5). Our main result follows; its proof is deferred until the final section.

Theorem 3.4. *We have a homotopy fiber sequence,*

$$\underline{\mathrm{Inv}}(\mathcal{C}) \xrightarrow{F_-} \underline{\mathrm{Eq}}(\mathcal{C}) \xrightarrow{M_-} \underline{\mathrm{Out}}(\mathcal{C}).$$

Corollary 3.5. *We have the long exact sequence of homotopy groups,*

$$\begin{array}{rcl} & & 0 \longrightarrow \pi_2\underline{\mathrm{Out}}(\mathcal{C}) \\ \pi_1\underline{\mathrm{Inv}}(\mathcal{C}) & \longrightarrow \pi_1\underline{\mathrm{Eq}}(\mathcal{C}) & \longrightarrow \pi_1\underline{\mathrm{Out}}(\mathcal{C}) \\ \pi_0\underline{\mathrm{Inv}}(\mathcal{C}) & \longrightarrow \pi_0\underline{\mathrm{Eq}}(\mathcal{C}) & \longrightarrow \pi_0\underline{\mathrm{Out}}(\mathcal{C}) \longrightarrow 0. \end{array}$$

Proposition 3.6. *We have the following isomorphisms*:

- $\pi_2\underline{\mathrm{Out}}(\mathcal{C}) \cong \pi_1\underline{\mathrm{Inv}}(\mathcal{C}) \cong \mathbb{C}^\times$;
- $\pi_1\underline{\mathrm{Eq}}(\mathcal{C}) \cong \mathrm{Aut}(\mathrm{id}_\mathcal{C}) \cong \mathrm{Hom}(U(\mathcal{C}), \mathbb{C}^\times)$;
- $\pi_1\underline{\mathrm{Out}}(\mathcal{C}) \cong \pi_1(\underline{\mathrm{BrPic}}(\mathcal{C})) \cong \mathrm{Inv}(Z(\mathcal{C}))$, *the group of iso-classes of invertible objects in* $Z(\mathcal{C})$;
- $\pi_0\underline{\mathrm{Inv}}(\mathcal{C}) \cong \mathrm{Inv}(\mathcal{C})$, *the group of iso-classes of invertible objects in* $\mathcal{C}$;
- $\pi_0\underline{\mathrm{Eq}}(\mathcal{C}) \cong \mathrm{Eq}(\mathcal{C})$, *the group of tensor auto-equivalences of* $\mathcal{C}$;
- $\pi_0\underline{\mathrm{Out}}(\mathcal{C}) \cong \mathrm{Out}(\mathcal{C})$, *the group of tensor auto-equivalences, modulo inner equivalences.*

Combining these identities with Theorem 3.4, and computing the connecting homomorphisms, yields:

Corollary 3.7. *We have a long exact sequence*:

$$* \to \mathrm{Hom}(U(\mathcal{C}), \mathbb{C}^\times) \to \pi_1 \underline{\mathrm{BrPic}}(\mathcal{C}) \xrightarrow{\delta} \mathrm{Inv}(\mathcal{C}) \xrightarrow{F_-} \mathrm{Eq}(\mathcal{C}) \xrightarrow{M_-} \mathrm{Out}(\mathcal{C}) \to *.$$

The corollary allows us to compute $\pi_1 \underline{\mathrm{BrPic}}(\mathcal{C}) \cong \mathrm{Inv}(Z(\mathcal{C}))$ in terms of its neighbors $\mathrm{Inv}(\mathcal{C})$ and $\mathrm{Hom}(U(\mathcal{C}), \mathbb{C}^\times)$, which may be read from the fusion rules of $\mathcal{C}$, and do not require us to comprehend the entire center $Z(\mathcal{C})$.

Since Corollary 3.7 is what we will use in applications, we give an independent, and elementary proof of it below. While Theorem 3.4 gives a more conceptual explanation, it is not strictly necessary for applications, and so we postpone its proof, and the consequent derivation of Corollary 3.7.

3.1. Independent proof of Corollary 3.7. The homomorphism M_- assigns to a tensor automorphism ϕ of $\mathcal{C}$ the bimodule $M_\phi = \mathcal{C}$, with regular left action, and with right action twisted by ϕ (see Section 5 for a precise definition). The homomorphism F_- assigns to an invertible object $g \in \mathcal{C}$ the tensor automorphism

$$F_g : X \mapsto g \otimes X \otimes g^*.$$

The homomorphism δ is induced by the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$.

Exactness at $\mathrm{Out}(\mathcal{C})$ and $\mathrm{Eq}(\mathcal{C})$ are the facts that all outer bimodules come from equivalences and that such a bimodule is trivial if, and only if, the equivalence is inner [5, §4.3]. Exactness at $\mathrm{Inv}(\mathcal{C})$ is the fact that an isomorphism $F_g \cong \mathrm{id}_\mathcal{C}$ yields a half-braiding on g.

Thus it remains only to identify $\mathrm{Hom}(U(\mathcal{C}), \mathbb{C}^\times))$ with the kernel of δ. Thus, we consider half-braidings $\sigma_{\mathbf{1},-}$ of the tensor unit. Since we have canonical isomorphisms $\mathbf{1} \otimes X \cong X \cong X \otimes \mathbf{1}$, the data of such a half-braiding is a scalar $c_X \in \mathbb{C}^\times$ for each simple object $X \in \mathcal{C}$. The Yang–Baxter equation and naturality condition for $\sigma_{\mathbf{1},-}$ imply that, for every simple object Z in the decomposition of $X \otimes Y$, we have $c_Z = c_X c_Y$. Thus the level sets of c determine a grading of $\mathcal{C}$ by a subgroup of $\mathbb{C}^\times$, yielding the required element of $\mathrm{Hom}(U(\mathcal{C}), \mathbb{C}^\times)$ via the universal property of $U(\mathcal{C})$. Finally, only a trivial such homomorphism can give rise to the trivial half-braiding.

4. Applications to subfactors

The results of the last section, combined with the obstruction theory of [5], allow us to construct several new fusion categories, starting with a fusion category and its invertible bimodule category.

A finite-index subfactor $N \subset M$ is called 2-*supertransitive* if $M \cong \mathbf{1} + \mathbf{X}$ as $N - N$ bimodules, where $\mathbf{1}$ is the trivial $N - N$ bimodule (N itself) and $\mathbf{X}$ is a simple object in the principal even part of $N \subset M$.

Lemma 4.1. *Let $N \subset M$ be a 2-supertransitive finite depth subfactor. Then the grading group of each of the even parts of $N \subset M$ is trivial.*

Proof. Let $\mathbf{A} = \mathbf{1}+\mathbf{X}$ be the algebra object in the principal even part $\mathcal{N}$ of $N \subset M$. Since the subfactor is 2-supertransitive, $\mathbf{X}$ is simple. Since $\mathbf{1} + \mathbf{X}$ is an algebra, we have that $\mathbf{X} \cong \mathbf{X}^*$ and that $\mathbf{X} \subset \mathbf{X} \otimes \mathbf{X} \cong \mathbf{X} \otimes \mathbf{X}^*$. Thus $\mathbf{X}$ lies in the 0-graded part of the principal even part $\mathcal{N}$ for any grading of $\mathcal{N}$; since $\mathbf{X}$ tensor generates the principal even part this means that the even principal part has no non-trivial gradings. Applying the same argument to the dual subfactor yields that the dual principal part has trivial grading group as well. □

Corollary 4.2. *Let $N \subset M$ be a 2-supertransitive finite depth subfactor such that the principal even part $\mathcal{N}$ has no invertible objects except for $\mathbf{1}$. Then*

$$\pi_1(\underline{\mathrm{BrPic}}(\mathcal{C})) \cong \mathrm{Inv}(Z(\mathcal{C}))$$

is trivial.

4.1. The Asaeda–Haagerup categories. We now turn to our main application, constructing several fusion categories which are extensions of fusion categories related to the Asaeda–Haagerup subfactor [2].

The Asaeda–Haagerup subfactor, which we will call AH, is a finite depth subfactor with index $\frac{5+\sqrt{17}}{2}$. The subfactors AH+1 and AH+2, constructed in [1, 8], have indices $\frac{7+\sqrt{17}}{2}$ and $\frac{9+\sqrt{17}}{2}$, respectively. The three subfactors AH, AH+1, and AH+2 all have the same dual even part but the principal even parts are three distinct fusion categories, which we call $\mathcal{AH}_1$, $\mathcal{AH}_2$, and $\mathcal{AH}_3$, respectively.

The subfactors AH, AH+1, and AH+2 are all 2-supertransitive. The categories $\mathcal{AH}_2$ and $\mathcal{AH}_3$ have non-trivial invertible objects but $\mathcal{AH}_1$ does not. Therefore by Corollary 4.2 we have that $\mathrm{Inv}(Z(\mathcal{AH}_i))$ is trivial for $i = 1, 2, 3$ (since the Drinfeld center is a Morita invariant [15, Cor. 2.1]).

We recall the following result from [8]:

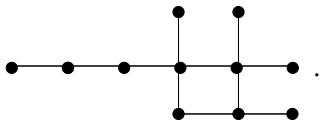
Theorem 4.3. *The Brauer–Picard group of each of the Asaeda–Haagerup fusion categories is* $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Therefore we have three order 2 invertible bimodule categories over each $\mathcal{AH}_i$, for $i = 1, 2, 3$. Full fusion rules for each of these bimodule categories were given in the ArXiv data supplement to [8]. We now show that each of these 9 bimodule categories admits two fusion category structures.

Theorem 4.4. *Each of the three non-trivial bimodule categories over each* $\mathcal{AH}_i$, $i = 1, 2, 3$, *is the odd component of exactly two* $\mathbb{Z}/2\mathbb{Z}$*-graded extensions of* $\mathcal{AH}_i$.

Proof. Each non-trivial bimodule over $\mathcal{AH}_i$ gives a map from $\mathbb{Z}/2\mathbb{Z}$ to the Brauer–Picard group of $\mathcal{AH}_i$. We want to show that this extends to a map of categorical 2-groups; that is, we want to show that the obstructions o_3 and o_4 from 2.5 vanish. It is enough to show that the groups that these obstructions lie in, namely $H^3(\mathbb{Z}/2\mathbb{Z}, \mathrm{Inv}(Z(\mathcal{C})))$ and $H^4(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$, vanish. As seen above, $\underline{\mathrm{Inv}}(Z(\mathcal{AH}_i))$ is trivial, so the first obstruction group vanishes. Finally $H^4(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$ vanishes since $\mathbb{Z}/2\mathbb{Z}$ is cyclic. There are two categories extending each bimodule, owing to the choice of α in Theorem 2.5. □

Among these new fusion categories, there is one with an object of dimension $\sqrt{\frac{9+\sqrt{17}}{2}}$. The fusion graph of this small object is (cf. [8])



This graph is 3-supertransitive (that is, it begins with a string of three edges). Outside the ADE series this is one of only two known fusion categories which is more than 2-supertransitive. The current record is the 4-supertransitive fusion category 4442 announced in [13].

Remark 4.5. It is easy to see that the Grothendieck ring of a $\mathbb{Z}/2\mathbb{Z}$-extension of a fusion category is determined by the following data: (a) The Grothendieck ring of the 0-graded part; (b) the bimodule fusion rules for the 1-graded part considered as a bimodule category over the 0-graded part; (c) the dual data of the 1-graded part.

The full multiplication rules for each of the three non-trivial bimodule categories over each $\mathcal{AH}_i$, $i = 1, 2, 3$ were given in the data supplement to [8], available on the arxiv, so (a) and (b) are known for all of the corresponding extensions.

Thus the only unknown information in the Grothendieck rings are the dual data of the 1-graded parts, which is given by an involution on the set of simple objects in the bimodule category which preserves Frobenius-Perron dimension. There are not very many possibilities, but we do not know how to compute these involutions at this time.

Remark 4.6. It is natural to wonder what extensions there are of $\mathcal{AH}_i$ by the Klein 4-group. This is somewhat more subtle as the obstruction group containing o_4 does not vanish.

4.2. The Izumi 2^p1 subfactors. Let p be a prime, and let R_p be the fusion ring generated by g and X with the relations $gX = X = Xg$, $g^p = 1$, and $X^2 = \sum_i g^i + pX$ (all sums here go from 0 to $p-1$). There is a left fusion module for R_p, which we will call M_p^{left}, with basis $a, b, gb, g^2b, \ldots, g^{p-1}b$ where $ga = a$, $Xa = \sum_i g^i b$, and $Xg^i b = a + \sum_i g^i b$. Since R_p is commutative, there is also a right fusion module M_p^{right} with the analogous fusion rules, and an R_p bimodule M_p^{bim}.

We will call $\mathcal{I}_p$ a $\mathbb{Z}/p\mathbb{Z}$ Izumi near-group category if its fusion ring is R_p. The study of these fusion categories was initiated by Izumi [11] and has been continued recently by Evans and Gannon [7].

Since $g^i X \cong X$, the subcategory of $\mathcal{I}_p$ generated by g has Vec as a module category, so this subcategory is $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$. In particular, there is an algebra object $\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]$ in $\mathcal{I}_p$.

The category of left modules for this algebra is a right module category over $\mathcal{I}_p$ which we will call $\mathcal{M}_p^{\text{right}}$. It is easy to see that the fusion rules for $\mathcal{M}_p^{\text{right}}$ give the fusion module M_p^{right}. Similarly we have a left module category $\mathcal{M}_p^{\text{left}}$. Furthermore, any (left or right) module category with fusion rules M_p must be equivalent to $\mathcal{M}_p$, because the internal endomorphisms of a must be $\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]$.

Picking an object in a module category gives a subfactor. The subfactor corresponding to $g^i b$ has principal and dual-principal graphs which are 2^p1 spoke graphs (that is, they have p spokes of length 2 and one spoke of length 1). Using [8, §3.3] it is not difficult to see that the resulting subfactor is independent of i, since the objects agree up to tensoring with an invertible in the dual graph. For general p, these 2^p1 subfactors were first studied by Izumi. When $p = 2$ this is just the E_6 subfactor, and when $p = 3$ there's an independent unpublished conformal inclusion construction due to Xu. We will call this series the 2^p1 subfactors,

rather than Izumi subfactors, to avoid confusion with another series of subfactors generalizing the Haagerup subfactor also constructed by Izumi in the same paper (sometimes called Izumi-Haagerup subfactors).

From looking at the dual graph of the 2^p1 subfactor, we see that the dual category to $\mathcal{I}_p$ over $\mathcal{M}_p$ is another $\mathbb{Z}/p\mathbb{Z}$ near-group category $\mathcal{I}'_p$. The fusion rules for $\mathcal{M}_p$ as a $\mathcal{I}_p$–$\mathcal{I}'_p$ bimodule are M_p^{bim}.

It is natural to wonder whether $\mathcal{I}_p \cong \mathcal{I}'_p$, and if so whether there's a $\mathbb{Z}/2\mathbb{Z}$-extension of $\mathcal{I}_p$ by $\mathcal{M}_p$. To that end, we call $\mathcal{I}_p$ self-dual if the dual even part is equivalent to the principal even part. In such a circumstance, choosing an isomorphism between the two even parts endows $\mathcal{M}_p$ with the structure of a bimodule over $\mathcal{I}_p$. In general there may be several such choices, and there is no guarantee that any of them have order 2 in the Brauer–Picard group of $\mathcal{I}_p$. However, we have the following lemma.

Lemma 4.7. *Suppose that $\mathcal{I}_p$ is self-dual and that its outer automorphism group is trivial. Then the unique $\mathcal{I}_p$ bimodule structure on $\mathcal{M}_p$ has order 2 in the Brauer–Picard group.*

Proof. Consider $\mathcal{M}_p^{-1}$ as a $\mathcal{I}_p$-bimodule. Since this has fusion rules M_p, it must be equivalent to $\mathcal{M}_p$ as a left (or as a right) module category. Since both bimodules are invertible, this implies that $\mathcal{M}_p \cong \mathcal{M}_p^{-1} \boxtimes_{\mathcal{I}_p} \mathcal{F}$ as bimodules where $\mathcal{F}$ is an outer automorphism. Since the outer automorphism group is trivial, we have $\mathcal{M}_p^2 \cong \text{id}$. □

Remark 4.8. In general, this argument shows that for any choice of identification of $\mathcal{I}_p$ with $\mathcal{I}'_p$ the resulting bimodule $\mathcal{M}_p$ squares to an outer automorphism.

Theorem 4.9. *Suppose that $p > 2$ and that $\mathcal{I}_p$ is a self-dual $\mathbb{Z}/p\mathbb{Z}$ near-group category with trivial outer automorphism group. Then there exist exactly two $\mathbb{Z}/2\mathbb{Z}$-extensions of $\mathcal{I}_p$ by $\mathcal{M}_p$.*

Proof. We have only to show that the homomorphism, $c\colon \mathbb{Z}/2\mathbb{Z} \to \text{BrPic}(\mathcal{I}_p)$, corresponding to $\mathcal{M}_p$ is unobstructed, i.e. that o_3 and o_4 in Theorem 2.5 vanish. The obstruction o_4 vanishes because $H^4(G, \mathbb{C}^\times) = 0$ whenever G is cyclic.

By Corollary 3.7, we have the exact sequence:

$$* \to Hom(U(\mathcal{I}_p), \mathbb{C}^\times) \to \text{Inv}(Z(\mathcal{I}_p)) \to \text{Inv}(\mathcal{I}_p).$$

However, the categories $\mathcal{I}_p$ admit no non-trivial gradings, as is evident from the fusion rules. So we have that $\text{Inv}(Z(\mathcal{I}_p))$ includes into $\text{Inv}(\mathcal{I}_p) = \mathbb{Z}/p\mathbb{Z}$. Thus

the only remaining obstruction o_3 lies in $H^3(\mathbb{Z}/2\mathbb{Z}, \mathrm{Inv}(Z(\mathcal{J}_p))$. This obstruction group vanishes since $\mathrm{Inv}(Z(\mathcal{J}))$ is either $\mathbb{Z}/p\mathbb{Z}$ or trivial (here we use that p is odd, since $H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is nontrivial). Thus there exists a $\mathbb{Z}/2\mathbb{Z}$-extension of $\mathcal{J}_p$ by $\mathcal{M}_p$.

There are exactly two distinct such extensions, owing to the choice of α in Theorem 2.5. □

Remark 4.10. It is not difficult to work out the fusion rules for this extension of $\mathcal{J}_p$. Since p is odd, at least one of the $g^i b$ is self-dual, so without loss of generality b is self-dual. The rest is easy to work out.

The simplest near group categories with p odd are the Izumi-Xu examples where $p = 3$; there are two inequivalent such fusion categories corresponding to a choice of complex conjugation in the structure constants for the monoidal structure. In the following we will fix a choice of conjugation and refer to "the" Izumi-Xu category for $p = 3$, but everything holds equally true for either choice.

In this case there are already two constructions of $\mathbb{Z}/2\mathbb{Z}$-extensions, first by Ostrik in the appendix to [4], using constructions from affine Lie algebras and conformal embeddings, and second by Morrison and Penneys [13] using planar algebras. In order to recover their results we need to know that the Izumi-Xu category is self-dual and has no outer automorphisms. Both of these facts follow from Han's thesis [10]. Self-duality follows from uniqueness of the 2221 subfactor up to complex conjugacy, and no outer automorphisms follows from the explicit quadratic relations satisfied by Han's generators (as in [9, Lemma 5.3] and [8, Thm. 4.9]).

Remark 4.11. In fact, the Brauer–Picard 1-groupoid of the Izumi-Xu fusion category $\mathcal{J}_3$ is a single point with automorphism group $\mathbb{Z}/2\mathbb{Z}$. Following the approach in [9], the only possible minimal algebra objects in the Izumi-Xu fusion category are 1 and $1 + g + g^2$, and those each have unique algebra structures. Thus, the only simple module categories are $\mathcal{J}_3$ and $\mathcal{M}_3$. Since $\mathcal{J}_3$ is self-dual and has no outer automorphisms, the only nontrivial $\mathcal{J}_3$–$\mathcal{D}$ bimodule is $\mathcal{M}_3$ where $\mathcal{D} \cong \mathcal{J}_3$.

We are optimistic that the other near group categories coming from Izumi and Evans-Gannon also give $\mathbb{Z}/2\mathbb{Z}$-extensions. In theory, it should be possible to work out whether these categories are self-dual and what their outer automorphism groups are from the detailed descriptions given by Izumi and Evans-Gannon, but in practice this may be somewhat difficult to extract.

5. The homotopy fiber sequence

We begin by recalling the definitions of tensor functor and natural transformation of tensor functors, primarily to fix notation.

A tensor functor $(F, J_F)\colon \mathcal{C} \to \mathcal{D}$ is a functor F of abelian categories, together with a natural isomorphism,

$$J_F\colon F \circ \otimes_{\mathcal{C}} \xrightarrow{\sim} \otimes_{\mathcal{D}} \circ F \boxtimes F,$$

satisfying a certain cocycle condition[1]. More precisely, J_F consists of a family of isomorphisms,

$$J_F : F(X \otimes Y) \xrightarrow{\sim} F(X) \otimes F(Y),$$

natural in $X, Y \in \mathcal{C}$, and such that the following diagram commutes:

$$\begin{array}{ccc}
F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha_{\mathcal{C}})} & F(X \otimes (Y \otimes Z)) \\
{\scriptstyle J_F}\downarrow & & \downarrow{\scriptstyle J_F} \\
F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\
{\scriptstyle J_F}\downarrow & & \downarrow{\scriptstyle J_F} \\
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\alpha_{\mathcal{D}}} & F(X) \otimes (F(Y) \otimes F(Z))
\end{array}$$

where $\alpha_{\mathcal{C}}$ and $\alpha_{\mathcal{D}}$ are the associators of $\mathcal{C}$ and $\mathcal{D}$ respectively. In particular, we note that the tensor structure J_F is additional data packaged with F.

A tensor functor (F, J_F) is an equivalence if F is an equivalence of abelian categories.

Definition 5.1. A natural transformation $\theta\colon F \to G$ between tensor functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is *monoidal* if for all $X, Y \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
F(X \otimes Y) & \xrightarrow{\theta_{X\otimes Y}} & G(X \otimes Y) \\
{\scriptstyle J_F}\downarrow & & \downarrow{\scriptstyle J_G} \\
F(X) \otimes F(Y) & \xrightarrow{\theta_X \otimes \theta_Y} & G(X) \otimes G(Y)
\end{array}$$

[1] To ease notation, we adopt the usual convention of refering to tuples $(F, \cdots)$, consisting of a functor with structural isomorphisms (such as tensor or module functor structure) simply by "F".

Morphisms between module categories and bimodule categories are defined in an analagous way; namely, they are functors of abelian categories along with certain natural transformations making certain diagrams commute. An equivalence of module categories is a module functor whose underlying functor is an equivalence of abelian categories. See [14] for details.

We now construct the functors $F_{\_}$ and $M_{\_}$ from Section 3. For clarity of exposition, we suppress associators for the remainder of this section.

5.1. Construction of $F_{\_}$. Given $g \in \underline{\mathrm{Inv}}(\mathcal{C})$, we define $F_g \in \underline{\mathrm{Eq}}(\mathcal{C})$ as follows. For $X, Y \in \mathcal{C}$, and $\rho\colon X \to Y$, we set

$$F_g(X) := g \otimes X \otimes g^{-1}, \quad F_g(\rho) := \mathrm{id}_g \otimes \rho \otimes \mathrm{id}_{g^{-1}}.$$

We equip F_g with the tensor structure:

$$F_g(X) \otimes F_g(Y) = g \otimes X \otimes g^{-1} \otimes g \otimes Y \otimes g^{-1} \xrightarrow[\mathrm{ev}_g]{\sim} g \otimes X \otimes Y \otimes g^{-1} = F_g(X \otimes Y).$$

The assignment $g \mapsto F_g$, extends to a functor,

$$F_{\_}\colon \underline{\mathrm{Inv}}(\mathcal{C}) \to \underline{\mathrm{Eq}}(\mathcal{C}),$$

by assigning to any $\phi\colon g \xrightarrow{\sim} h$ the natural isomorphism,

$$F_g = g \otimes - \otimes g^{-1} \xrightarrow[\phi \otimes \mathrm{id} \otimes (\phi^{-1})^*]{\sim} h \otimes - \otimes h^{-1} = F_h.$$

The obvious natural isomorphisms, $F_{g \otimes h} \cong F_g \circ F_h$, endow $F_{\_}$ with a monoidal structure.

5.2. Construction of $M_{\_}$. For every $F \in \underline{\mathrm{Eq}}(\mathcal{C})$, we define an invertible $\mathcal{C}$-$\mathcal{C}$-bimodule category $M = M_F$, by letting $M = \mathcal{C}$ as an abelian category, and defining, for $X, Y \in \mathcal{C}$, $m \in M$:

$$X \otimes_{M_F} m \otimes_{M_F} Y := X \otimes m \otimes F(Y).$$

Here, unadorned tensor products denote the tensor product in $\mathcal{C}$. For every natural isomorphism $\theta\colon F \to G$, we have a $\mathcal{C}$-$\mathcal{C}$-bimodule auto-equivalence

$$(\mathrm{id}_{\mathcal{C}}, J_\theta)\colon M_F \to M_G,$$

which is the identity functor of $\mathcal{C}$, and where $J_\theta\colon F(Y) \to G(Y)$ is applied before tensoring on the right.

This gives a functor,

$$M_- : \underline{\mathrm{Eq}}(\mathcal{C}) \to \underline{\underline{\mathrm{Out}}}(\mathcal{C}),$$
$$F \mapsto M_F,$$

of 2-categories, where we regard $\underline{\mathrm{Eq}}(\mathcal{C})$ as a categorical 2-group with strict associativity of 1-morphisms.

We have a bi-additive functor of $\mathcal{C}$-$\mathcal{C}$-bimodules,

$$\hat{\otimes} : M_F \boxtimes M_G \to M_{F\circ G},$$
$$m \boxtimes n \mapsto m \otimes F(n),$$

and isomorhpisms,

$$M\hat{\otimes}(A \otimes N) = M \otimes \phi(A \otimes N) \xrightarrow{J_F} M \otimes \phi(A) \otimes \phi(N) = (M \otimes A)\hat{\otimes}N.$$

natural in M, A and N. Thus $\hat{\otimes}$ defines a functor,

$$\iota_{F,G} : M_F \otimes_{\mathcal{C}} M_G \to M_{F\circ G},$$

of abelian categories, which is clearly an equivalence. Moreover, we have an isomorphism of functors,

$$J_{F,G} : \otimes_{M_{F\circ G}} \circ \iota_{F,G} \to \iota_{F,G} \circ \otimes_{M_F \otimes_{\mathcal{C}} M_G},$$

$$X \otimes m \otimes F(n) \otimes F(G(Y)) \xrightarrow{J_F^{-1}} X \otimes m \otimes F(n \otimes G(Y)).$$

Thus, M_- induces a homomorphism of categorical 2-groups, which we also denote M_-.

5.3. The homotopy fiber of M. Let $p : \mathcal{G} \to \mathcal{H}$ be a homomorphism of categorical n-groups. The homotopy fiber, $p^{-1}(X)$, of $X \in \mathcal{H}$ has as its objects pairs $(Y \in \mathcal{G}, \phi : X \xrightarrow{\sim} p(Y))$. Morphisms are those inherited from $\mathcal{H}$, that is

$$\mathrm{Hom}_{p^{-1}(X)}((Y_1, \phi_1), (Y_2, \phi_2) := \mathrm{Hom}_{\mathcal{H}}(\phi_1, \phi_2).$$

Because all objects and morphisms in $\mathcal{H}$ are invertible, Quillen's Theorem B [17] asserts $p^{-1}(X) \to \mathcal{G} \to \mathcal{H}$ is a homotopy fiber sequence, for any object $X \in \mathcal{H}$.

In this section, we construct an equivalence between the full subcategory $\underline{\mathrm{Inv}}(\mathcal{C})$ of invertible objects in $\mathcal{C}$, and the homotopy fiber $M^{-1}(\mathcal{C})$ over the trivial $\mathcal{C}$-$\mathcal{C}$-bimodule $\mathcal{C}$.

We have equivalences of bimodule categories,

$$\Phi_g \colon \mathcal{C} \to M_{F_g}$$

$$X \mapsto X \otimes g^{-1},$$

equipped with tensor structure,

$$X \otimes \Phi_g(Y) \otimes F_g(Z) = X \otimes Y \otimes g^{-1} \otimes g \otimes Z \otimes g^{-1}$$
$$\xrightarrow[\mathrm{ev}_g]{\sim} X \otimes Y \otimes Z \otimes g^{-1} = \Phi_g(X \otimes Y \otimes Z).$$

We thus construct a functor

$$M_{F_-} \colon \underline{\mathrm{Inv}}(\mathcal{C}) \to M^{-1}(\mathcal{C}).$$

$$M_{F_-}(g) := (M_{F_g}, \Phi_g).$$

Theorem 5.2. *The functor M_{F_-} is an equivalence.*

Proof. Let $(F, J_F) \in \underline{\mathrm{Eq}}(\mathcal{C})$, and suppose we have an equivalence,

$$(\phi, J_\phi) \colon \mathcal{C} \to M_F,$$

of bimodule categories. Clearly, ϕ induces an auto-equivalence of $\mathcal{C}$ as a left-module category (recall that $M_F = \mathcal{C}$ canonically, as a left $\mathcal{C}$-module category). It is then routine to see that $\phi = - \otimes g^{-1}$, where $g^{-1} = \phi(\mathbf{1}) \in \underline{\mathrm{Inv}}(\mathcal{C})$.

The tensor data J_ϕ therefore consists of a family of isomorphisms,

$$X \otimes Y \otimes Z \otimes g = \phi(X \otimes Y \otimes Z) \xrightarrow[J_\phi]{\sim} X \otimes \phi(Y) \otimes F(Z) = X \otimes Y \otimes g \otimes F(Z),$$

which is natural in $X, Y, Z \in \mathcal{C}$. In particular, taking $X, Y = \mathbf{1}$, we obtain a natural isomorphism,

$$F \cong g \otimes - \otimes g^{-1} = F_g.$$

We thus have a canonical isomorphism $(M_F, \phi) \cong (M_{F_g}, \Phi_g)$ in the homotopy fiber, so that the induced functor $F_- \colon \underline{\mathrm{Inv}}(\mathcal{C}) \to M_{\mathcal{C}}$ is essentially surjective. It remains to show that it is fully faithful. We need to show that every equivalence, $\theta \colon M_{F_g} \xrightarrow{\sim} M_{F_h}$, of bimodule categories making the diagram

$$\begin{array}{ccc} M_{F_g} & \xrightarrow{\ \theta\ } & M_{F_h} \\ & {\scriptstyle \Phi_g^{-1}} \searrow \quad \swarrow {\scriptstyle \Phi_h^{-1}} & \\ & \mathcal{C} & \end{array}$$

commutative is in fact induced by a unique isomorphism $g \cong h$. The isomorphism required for fullness is

$$g = F_g(\mathbf{1}) \xrightarrow{\theta_{\mathbf{1}}} F_h(\mathbf{1}) = h.$$

The bimodule compatibility condition for θ implies that all the θ_X are determined by $\theta_{\mathbf{1}}$, thus implying faithfulness. □

Theorem 5.2 now implies 3.4, as it allows us to identify $\underline{\mathrm{Inv}}(\mathcal{C})$ with a homotopy fiber of the functor M_-. Corollary 3.5 follows from the long exact sequence in homotopy groups for a fibration.

Proof of Corollary 3.7. All that remains is to compute the connecting homomorphisms $\delta_2 \colon \pi_2\underline{\underline{\mathrm{Out}}}(\mathcal{C}) \to \pi_1\underline{\mathrm{Inv}}(\mathcal{C})$, and $\delta_1 \colon \pi_1\underline{\underline{\mathrm{Out}}}(\mathcal{C}) \to \pi_0\underline{\mathrm{Inv}}(\mathcal{C})$.

To this end, we identify $\pi_1\underline{\underline{\mathrm{Out}}}(\mathcal{C})$ with the automorphisms of the unit object of $\underline{\underline{\mathrm{Out}}}(\mathcal{C})$, namely the $\mathcal{C}$-$\mathcal{C}$-bimodule auto-equivalences of the regular bimodule $\mathcal{C}$. Every such auto-equivalence, ϕ, is isomorphic to the functor of tensoring by a central object g. Thus in $\underline{\mathrm{Eq}}(\mathcal{C})$, we have an isomorphism $\phi \cong - \otimes g$, and thus the connecting homomorphism $\delta_1 \colon \pi_1\underline{\underline{\mathrm{Out}}}(\mathcal{C}) \to \pi_0\underline{\mathrm{Inv}}(\mathcal{C})$ is simply the forgetful functor $\mathrm{For}\colon \underline{\mathrm{Inv}}(Z(\mathcal{C})) \to \underline{\mathrm{Inv}}(\mathcal{C})$. Similarly, δ_2 is the induced homomorphism $\mathrm{For}\colon \mathrm{Aut}_{Z(\mathcal{C})}(\mathbf{1},\mathbf{1}) \to \mathrm{Aut}(\mathbf{1},\mathbf{1})$. In particular, δ_2 is an isomorphism, so that the first two terms of the sequence split off, yielding Corollary 3.7. □

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