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Quantum Topology

Comparing invariants of Legendrian knots

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Abstract. We prove the equivalence of the invariants $\mathrm{EH}(L)$ and $\mathfrak{L}^-(\pm L)$ for oriented Legendrian knots L in the 3-sphere equipped with the standard contact structure, partially extending a previous result by Stipsicz and Vértesi. In the course of the proof we relate the sutured Floer homology groups associated with a knot complement $S^3 \setminus K$ and the knot Floer homology of (S^3, K) and define intermediate Legendrian invariants.

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1. Introduction

In recent years, many Floer-theoretic invariants for Legendrian knots have been introduced: in 2008, Ozsváth, Szabó and Thurston [28] used grid diagrams to define two invariants $\hat{\lambda}_\pm(L)$ and $\lambda_\pm(L)$ of oriented Legendrian knots L in (S^3, ξ_{st}), taking values in a combinatorial version of knot Floer homology. Shortly afterwards, Lisca, Ozsváth, Stipsicz and Szabó [19] used open books to construct two other invariants of oriented nullhomologous Legendrian knots L, called $\widehat{\mathfrak{L}}(L)$ and $\mathfrak{L}^-(L)$, taking values in the original version knot Floer homology.

In 2006, Juhász defined a version of Heegaard Floer homology for manifolds with "marked" boundary, which he called sutured Floer homology [16]. Honda, Kazez and Matić soon constructed invariants for contact manifolds with convex boundary, taking values in a sutured Floer cohomology group [14]: the key feature of their invariant (and of sutured Floer homology) is its behaviour with respects to gluing manifolds along their (compatible) boundaries [15].

In this context, to every Legendrian knot L in a contact 3-manifold one can associate a contact manifold with convex boundary, and therefore a contact invariant $\mathrm{EH}(L)$ living in some sutured Floer homology group. Some natural questions arise at this point: is there any relation between $\mathrm{EH}(L)$ and the $\mathfrak{L}$ invariants? If so, what is this relation exactly?

Late in 2008, a first answer to these questions was given by Stipsicz and Vértesi, who explained how $\mathrm{EH}(L)$ determines $\widehat{\mathfrak{L}}(L)$ [31]; recently, Baldwin, Vela-Vick, and Vértesi were able to prove the equivalence of the combinatorial invariants λ and the LOSS invariants $\mathfrak{L}$ [1].

Our main result is the following ($-L$ means L with the reversed orientation).

Theorem 1.1. *For two oriented, topologically isotopic Legendrian knots L_0, L_1 in (S^3, ξ_{st}), the following are equivalent*:

(i) $\mathrm{EH}(L_0) = \mathrm{EH}(L_1)$;

(ii) $\mathfrak{L}^-(L_0) = \mathfrak{L}^-(L_1)$ *and* $\mathfrak{L}^-(-L_0) = \mathfrak{L}^-(-L_1)$.

The same result has been obtained, in greater generality, by Etnyre, Vela-Vick, and Zarev [7]. In fact, using the same techniques together with a generalisation of [18, Theorem 11.35], one can prove the generalisation of Theorem 1.1 to Legendrian knots in arbitrary contact 3-manifolds (Y, ξ) such that $c(Y, \xi) \neq 0$, and it is always the case that $\mathrm{EH}(L)$ determines $\mathfrak{L}^-(\pm L)$.

Organisation. This paper is organised as follows: we first review the setting we are working in, giving a brief introduction to sutured Floer homology in Section 2 and the EH invariants in Section 3. Then we analyse in some detail the groups and the maps we are dealing with, in Section 4. In Section 5 the relation between various sutured Floer homology associated to a knot complement and HFK^- are explained; this will lead to the proof of the equivalence of the two invariants EH and $\mathfrak{L}^-$ in the last section.

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2. Sutured Floer homology and gluing maps

2.1. Sutured manifolds. The definition of balanced sutured manifold is due to Juhász [16].

Definition 2.1. A **balanced sutured manifold**, is a pair (M, Γ) where M is an oriented 3-manifold with nonempty boundary ∂M, and Γ is a family of oriented curves in ∂M that satifies:

- Γ intersects each component of ∂M;
- Γ disconnects ∂M into R_+ and R_-, with $\pm\Gamma = \partial R_\pm$ (as oriented manifolds);
- $\chi(R_+) = \chi(R_-)$.

Remark 2.2. The condition $\chi(R_+) = \chi(R_-)$ is called the *balancing* condition. Since this is the only kind of sutured manifolds we are dealing with, we prefer to just drop the adjective 'balanced'.

Example 2.3. Any M oriented 3-manifold with S^2-boundary, can be turned into a sutured manifold $(M, \{\gamma\})$ by choosing any simple closed curve γ in ∂M. We will often write $M = Y(1)$, where $Y = M \cup_\partial D^3$ is the "simplest" closed 3-manifold containing M.

For every integer f, we have a sutured manifold $S^3_{K,f}$ given by pairs $(S^3 \setminus N(K), \{\gamma_f, -\gamma_f\})$, where γ_f is an oriented curve on the boundary torus $\partial N(K)$ of an open small neighbourhood $N(K)$ of K. The slope of γ_f is $\lambda_S + f \cdot \mu$, and $-\gamma_f$ is a parallel push-off of γ_f, with the opposite orientation. Here λ_S denotes the Seifert longitude of K. We will use the shorthand Γ_f for $\{\gamma_f, -\gamma_f\}$.

Example 2.4. To any Legendrian knot $L \subset (Y, \xi)$ in an arbitrary 3-manifold Y one can associate in a natural way a sutured manifold, that we will denote with Y_L, constructed as follows: there is a standard open Legendrian neighbourhood $\nu(L)$

for L, whose complement has convex boundary. The dividing set Γ_L on the boundary consists of two parallel oppositely oriented curves parallel to the contact framing of L. The manifold Y_L is then defined as the pair $(Y \setminus \nu(L), \Gamma_L)$. We are mainly interested in the case $Y = S^3$; if L is of topological type K and has Thurston–Bennequin number $\mathrm{tb}(L)$, then $S^3_L = S^3_{K,\mathrm{tb}(L)}$.

We will often use Y_L also to denote the contact manifold with convex boundary $(Y \setminus \nu(L), \xi|_{Y\setminus\nu(L)})$, without creating any confusion.

There is a decomposition/classification theorem for sutured manifolds, completely analogous to the Heegaard decomposition/Reidemeister–Singer theorem for closed 3-manifolds. Consider a compact surface Σ with boundary together with two collections of pairwise disjoint simple closed curves $\boldsymbol{\alpha}, \boldsymbol{\beta} \subset \Sigma$, such that each collection is a linearly independent set in $H_1(\Sigma; \mathbb{Z})$; suppose moreover that $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. We can build a balanced sutured manifold out of this data as follows: take $\Sigma \times [0, 1]$, glue a 2-handle on $\Sigma \times \{0\}$ for each α-curve, and a 2-handle on $\Sigma \times \{1\}$ for each β-curve, and let M be the manifold obtained after smoothing corners; declare $\Gamma = \partial\Sigma \times \{1/2\}$. The pair (M, Γ) is a balanced sutured manifold, and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is called a (*sutured*) *Heegaard diagram* of (M, Γ).

Theorem 2.5 ([16]). *Every balanced sutured manifold admits a Heegaard diagram, and every two such diagrams become diffeomorphic after a finite number of isotopies of the curves, handleslides and stabilisations taking place in the interior of the Heegaard surface.*

2.2. The Floer homology packages. This is meant to be just a recollection of facts about the Floer homology theories we will be working with. The standard references for the material in this subsection are [24, 25, 17] for the Heegaard Floer part, and [16] for the sutured Floer part.

In order to avoid sign issues, we will work with $\mathbb{F} = \mathbb{F}_2$ coefficients.

Consider a pointed Heegaard diagram $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ representing a 3-manifold Y, and form two Heegaard Floer complexes $\widehat{\mathrm{CF}}(Y)$ and $\mathrm{CF}^-(Y)$: the underlying modules are freely generated over $\mathbb{F}$ and $\mathbb{F}[U]$ by g-tuples of intersection points in $\bigcup_{i,j}(\alpha_i \cap \beta_j)$, so that there is exactly one point on each curve in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$.

The differentials $\hat\partial, \partial^-$ are harder to define, and count certain pseudo-holomorphic discs in a symmetric product $\mathrm{Sym}^g(\Sigma_g)$, or maps from Riemann surfaces with boundary in $\Sigma_g \times \mathbb{R} \times [0, 1]$, with the appropriate boundary conditions. The homology groups $\widehat{\mathrm{HF}}(Y) = H_*(\widehat{\mathrm{CF}}(Y), \hat\partial)$ and $\mathrm{HF}^-(Y) = H_*(\mathrm{CF}^-(Y), \partial^-)$ so defined are called *Heegaard Floer homologies* of Y, and are independent of the (many) choices made along the way [24].

Sutured Floer homology is a variant of this construction for sutured manifolds (M, Γ). The starting point is a sutured Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for (M, Γ). We form a complex $\mathrm{SFC}(M, \Gamma)$ in the same way, generated over $\mathbb{F}$ by d-tuples of intersection points as above, where $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. The differential ∂ is defined by counting pseudo-holomorphic discs in $\mathrm{Sym}^d(\Sigma)$ or maps from Riemann surfaces to $\Sigma \times \mathbb{R} \times [0, 1]$, again with the appropriate boundary conditions.

The homology $\mathrm{SFH}(M, \Gamma) = H_*(\mathrm{SFC}(M, \Gamma), \partial)$ is called the *sutured Floer homology* of (M, Γ), and is shown to be independent of all the choices made [16]. It naturally corresponds to a 'hat' theory.

Proposition 2.6 ([16]). *For a closed* 3*-manifold* Y*,*

$$\widehat{\mathrm{HF}}(Y) = \mathrm{SFH}(Y(1)).$$

For a knot K *in a closed* 3*-manifold* Y*,*

$$\widehat{\mathrm{HFK}}(Y, K) = \mathrm{SFH}(Y_{K,m}),$$

where m *is the meridian for* K *in* Y*.*

2.3. Floer-theoretic contact invariants. The first contact invariant to be defined in Heegaard Floer homology was Ozsváth and Szabó's c [26]. We sketch here the construction of the contact class EH [14], and we will relate it to c below.

Definition 2.7. A *partial open book* is a triple (S, P, h) where S is a compact open surface, P is a proper subsurface of S which is a union of 1-handles attached to $S \setminus P$ and $h \colon P \to S$ is an embedding that pointwise fixes a neighbourhood of $\partial P \cap \partial S$.

We can build a contact manifold with convex boundary out of these data in a fashion similar to the usual open books: instead of considering a mapping torus, though, we glue two asymmetric halves, quotienting the disjoint union $S \times [0, 1/2] \coprod P \times [1/2, 1]$ by the relations $(x, t) \sim (x, t')$ for $x \in \partial S$, $(y, 1/2) \sim (y, 1/2)$, $(h(y), 1/2) \sim (y, 1)$ for $y \in P$. The contact structure is uniquely determined if we require – as we do – tightness and prescribed sutures on each half $S \times [0, 1/2]/\!\sim$ and $P \times [1/2, 1]/\!\sim$ (see [13] for details). Moreover, to any contact manifold with convex boundary we can associate a partial open book, unique up to Giroux stabilisations.

We can build a balanced diagram out of a partial open book. The Heegaard surface Σ is obtained by gluing P to $-S$ along the common boundary.

Definition 2.8. A *basis* for (S, P) is a set $\mathbf{a} = \{a_1, \ldots, a_k\}$ of arcs properly embedded in $(P, \partial P \cap \partial S)$ whose homology classes generate $H_1(P, \partial P \cap \partial S)$.

Given a basis as above, we produce a set $\mathbf{b} = \{b_1, \ldots, b_k\}$ of curves using a Hamiltonian vector field on P: we require that under this perturbation the endpoints of a_i move in the direction of ∂P, and that each a_i intersects b_i in a single point x_i, and is disjoint from all the other b_j's.

Finally define the two sets of attaching curves: $\boldsymbol{\alpha} = \{\alpha_i\}$ and $\boldsymbol{\beta} = \{\beta_i\}$, where $\alpha_i = a_i \cup -a_i$ and $\beta_i = h(b_i) \cup -b_i$: the sutured manifold associated to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is (M, Γ). We call $\mathbf{x}(S, P, h)$ the generator $\{x_1, \ldots, x_k\}$ in $\mathrm{SFC}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ supported inside P.

Theorem 2.9 ([14]). *The chain* $\mathbf{x}(S, P, h) \in \mathrm{SFC}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ *is a cycle, and its class in* $\mathrm{SFH}(-M, -\Gamma)$ *is an invariant of the contact manifold* (M, ξ) *defined by the partial open book* (S, P, h).

Definition 2.10. $\mathrm{EH}(M, \xi)$ is the class $[\mathbf{x}(S, P, h)] \in \mathrm{SFH}(-M, -\Gamma)$ for some partial open book (S, P, h) supporting (M, ξ).

The type of invariants that we are going to deal with are either invariants of (complements of) Legendrian knots or invariants coming from contact structures on closed manifolds: this allows us to consider only sutured manifolds with sphere/torus boundary and one/two sutures, as described in Examples 2.3 and 2.4.

Consider a closed contact manifold (Y, ξ), and let $B \subset Y$ be a small, closed Darboux ball with convex boundary. Then consider the manifold $(Y(1), \xi(1))$ where $Y(1)$ is obtained from Y by removing the interior of B, and $\xi(1)$ is $\xi|_{Y(1)}$.

Proposition 2.11 ([14]). *There is an isomorphism of graded complexes from* $\widehat{\mathrm{HF}}(Y)$ *to* $\mathrm{SFH}(Y(1))$ *that maps the Ozsváth–Szabó contact invariant* $c(Y, \xi)$ *to the Honda–Kazez–Matić class* $\mathrm{EH}(Y(1), \xi(1))$.

Suppose now that $L \subset Y$ is a Legendrian knot with respect to a contact structure ξ: the contact manifold Y_L defined in Example 2.4 determines a contact invariant $\mathrm{EH}(Y_L) \in \mathrm{SFH}(-Y_L)$. We will denote this invariant by $\mathrm{EH}(L)$, considering it as an invariant of the Legendrian isotopy class of L rather than of its complement.

2.4. Gluing maps. In their paper [15], Honda, Kazez and Matić define maps associated to the gluing of a contact manifold to another one along some of the boundary components, and show that these maps preserve their EH invariant.

Consider two sutured manifolds $(M,\Gamma) \subset (M',\Gamma')$, where M is embedded in $\mathrm{Int}(M')$; let ξ be a contact structure on $N \stackrel{\mathrm{def}}{=} M' \setminus \mathrm{Int}(M)$ such that ∂N is ξ-convex and has dividing curves $\Gamma \cup \Gamma'$. For simplicity, and since this will be the only case we need, we will restrict to the case when each connected component of N intersects $\partial M'$ (*i.e.* gluing N to M doesn't kill any boundary component).

Theorem 2.12. *The contact structure ξ on N induces a gluing map Φ_ξ, that is a linear map*

$$\Phi_\xi \colon \mathrm{SFH}(-M,-\Gamma) \longrightarrow \mathrm{SFH}(-M',-\Gamma').$$

If ξ_M is a contact structure on M such that ∂M is ξ_M-convex with dividing curves Γ, then

$$\Phi_\xi(\mathrm{EH}(M,\xi_M)) = \mathrm{EH}(M',\xi_M \cup \xi).$$

This theorem has interesting consequences, even in simple cases:

Corollary 2.13. *If (M,Γ) embeds in a Stein fillable contact manifold (Y,ξ), and ∂M is ξ-convex, divided by Γ, then* $\mathrm{EH}(M,\xi|_M)$ *is not trivial.*

There is also a naturality statement, concerning the composition of two gluing maps: suppose that we have three sutured manifolds $(M,\Gamma) \subset (M',\Gamma') \subset (M'',\Gamma'')$ as at the beginning of the section, and suppose that ξ and ξ' are contact structures on $M' \setminus \mathrm{Int}(M)$ and $M'' \setminus \mathrm{Int}(M')$ respectively, that induce sutures Γ, Γ' and Γ'' on ∂M, $\partial M'$ and $\partial M''$ respectively.

Theorem 2.14. *If ξ and ξ' are as above, then*

$$\Phi_{\xi\cup\xi'} = \Phi_{\xi'} \circ \Phi_\xi.$$

Much of our interest will be devoted to stabilisations of Legendrian knots and associated maps, whose discussion will occupy Subsection 3.3: we give a brief summary of the contact side of their story here.

Let us start with a definition, due to Honda [13]:

Definition 2.15. Let η be a tight contact structure on $T^2 \times I$ with two dividing curves on each boundary component: call γ_i, $-\gamma_i$ the homology class of the two dividing curves on $T^2 \times \{i\}$, and let $s_i \in \mathbb{Q} \cup \{\infty\}$ be their slope. $(T^2 \times I, \eta)$ is a *basic slice* if it is of the form above, and also satisfies the following three conditions:

- $\{\gamma_0, \gamma_1\}$ is a basis for $H_1(T^2)$;

- ξ is *minimally twisting*, *i.e.* if $T_t = T \times \{t\}$ is convex, the slope of the dividing curves on T_t belongs to $[s_0, s_1]$ (where we assume that if $s_0 > s_1$ the interval $[s_0, s_1]$ is $[-\infty, s_1] \cup [s_0, \infty]$);

Honda proved the following:

Proposition 2.16 ([13]). *For every integer t there exist exactly two basic slices $(T^2 \times I, \xi_j)$ (for $j = 1, 2$) with boundary slopes $t/1$ and $(t-1)/1$. The sutured complement of a stabilisation L' of L is gotten by attaching one of the two basic slices to Y_L, where the trivialization of T^2 is given by identifying the slopes $0/1$ and $t/1$ with a meridian μ and the contact framing c for L, respectively.*

These two different layers correspond to the positive and negative stabilisation of L, once we've chosen an orientation for the knot; reversing the orientation swaps the labelling signs. Since we will be considering oriented Legendrian knots, we can label the two slices with a sign.

Definition 2.17. We call *stabilisation maps* the gluing maps associated to the attachment of a stabilisation basic slice: these will be denoted with $\sigma_\pm$.

Remark 2.18. As it happens for the Stipsicz–Vértesi map [31], these basic slice attachments correspond to single bypass attachments, too.

3. A few facts on SFH($S^3_{K,n}$) and $\sigma_\pm$

Given a topological knot K in S^3, denote with $S^3_m(K)$ the manifold obtained by (topological) m-surgery along K, and let $\widetilde{K}$ be the dual knot in $S^3_m(K)$, that is the core of the solid torus we glue back in. Notice that an orientation on K induces an orientation of $\widetilde{K}$, by imposing that the intersection of the meridian μ_K of K on the boundary of the knot complement has intersection number $+1$ with the meridian $\mu_{\widetilde{K}}$ of $\widetilde{K}$ on the same surface.

Fix a contact structure ξ on S^3 and a Legendrian representative L of K: we will write t for $\mathrm{tb}(L)$. Since t measures the difference between the contact and the Seifert framings of L, $S^3_t(K)_{\widetilde{K},\infty}$ and S^3_L are sutured diffeomorphic: in particular, $\mathrm{EH}(L)$ lives in $\mathrm{SFH}(-S^3_t(K)_{\widetilde{K},\infty}) = \widehat{\mathrm{HFK}}(-S^3_t(K), \widetilde{K})$, the identification depending on the choice of an orientation for K (or $\widetilde{K}$).

We will often write $\widehat{\mathrm{CFK}}(Y, K)$ to denote any chain complex computing $\widehat{\mathrm{HFK}}(Y, K)$ that comes from a Heegaard diagram, even though the complex itself depends on the choice of the diagram.

3.1. Gradings and concordance invariants. The groups $\widehat{\mathrm{HFK}}(S^3, K)$ and $\widehat{\mathrm{HFK}}(-S^3_m(K), \widetilde{K})$ come with a grading, that we call the *Alexander grading*. A Seifert surface $F \subset S^3$ for K gives a relative homology class

$$[F, \partial F] \in H_2(S^3 \setminus N(K), \partial N(K)) = H_2(S^3_m(K) \setminus N(\widetilde{K}), \partial N(\widetilde{K})).$$

Given a generator $\mathbf{x} \in \widehat{\mathrm{CFK}}(S^3, K)$, there is an induced relative Spin^c structure $\mathfrak{s}(\mathbf{x})$ in $\underline{\mathrm{Spin}}^c(S^3, K)$ [12, Equation 2], and the Alexander grading of $\mathbf{x}$ is defined as

$$A(\mathbf{x}) = \frac{1}{2}\langle c_1(\mathfrak{s}(\mathbf{x})) - PD([\mu_K]), [F, \partial F]\rangle,$$

where PD denotes Poincaré duality.

Likewise, given a generator $\mathbf{x} \in \widehat{\mathrm{CFK}}(-S^3_m(K), \widetilde{K})$, there is an induced relative Spin^c structure $\mathfrak{s}(\mathbf{x}) \in \underline{\mathrm{Spin}}^c(S^3_m(K), \widetilde{K})$, and we can define $A(\mathbf{x})$ as

$$A(\mathbf{x}) = \frac{1}{2}\langle c_1(\mathfrak{s}(\mathbf{x})) - PD([\mu_{\widetilde{K}}]), [F, \partial F]\rangle. \tag{3.1}$$

We now turn to recalling the definition of $\tau(K)$, due to Ozsváth and Szabó [22].

Recall that the Alexander grading induces a filtration on the knot Floer chain complex $(\widehat{\mathrm{CFK}}(S^3, K), \partial)$, where the differential ∂ ignores the presence of the second basepoint, that is $H_*(\widehat{\mathrm{CFK}}(S^3, K), \partial) = \widehat{\mathrm{HF}}(S^3)$. In particular, every sublevel $\widehat{\mathrm{CFK}}(S^3, K)_{A \leq s}$ is preserved by ∂, and we can take its homology.

Definition 3.1. $\tau(K)$ is the smallest integer s such that the inclusion of the s-th filtration sublevel induces a nontrivial map

$$H_*(\widehat{\mathrm{CFK}}(S^3, K)_{A \leq s}, \partial) \longrightarrow \widehat{\mathrm{HF}}(S^3) = \mathbb{F}.$$

This invariant turns out to provide a powerful lower bound for the slice genus of K, in the sense that $|\tau(K)| \leq g_*(K)$ [22]. One of the properties it enjoys, and that we will need, is that $\tau(\overline{K}) = -\tau(K)$ for every K.

3.2. Modules. We turn our attention back to $\widehat{\mathrm{HFK}}(-S^3_t(K), \widetilde{K}) \simeq \mathrm{SFH}(-S^3_{K,t})$. Recall that this is a $\mathbb{F}$-vector space on which the A defines a grading.

The group $\widehat{\mathrm{CFK}}(S^3, K)$ is a graded vector space that comes with two differentials, ∂_K and ∂, such that the complex $(\widehat{\mathrm{CFK}}(S^3, K), \partial)$ has homology $\widehat{\mathrm{HF}}(S^3) = \mathbb{F}$, while the complex $(\widehat{\mathrm{CFK}}(S^3, K), \partial_K)$ is the associated graded object with respect to the Alexander filtration. By definition $\widehat{\mathrm{HFK}}(S^3, K)$ is the homology of this latter complex; as such, it inherits an Alexander grading that we call A.

Let us set

$$d = \dim \widehat{\mathrm{HFK}}(S^3, K),$$

and fix a basis

$$\mathcal{B} = \{\eta_i, \eta'_j \mid 0 \le i < d\}$$

of $\widehat{\mathrm{CFK}}(S^3, K)$ such that the set $\{\eta_i^{\mathrm{top}}, (\eta'_j)^{\mathrm{top}}\}$ of the highest nontrivial Alexander-homogeneous components of the η_i's and η'_j's is still a basis for $\widehat{\mathrm{CFK}}(S^3, K)$, and the following relations hold (see [18, Section 11.5]):

$$\begin{aligned}
\partial\eta_0 &= 0, & \partial_K\eta_0 &= 0,\\
\partial\eta_{2i-1} &= \eta_{2i}, & \partial_K\eta_i &= 0,\\
\partial\eta'_{2j-1} &= \eta'_{2j}, & \partial_K\eta'_{2j-1} &= \eta'_{2j}.
\end{aligned}$$

Observe that the set of homology classes of the η_i's is a basis for $\widehat{\mathrm{HFK}}(S^3, K) = H_*(\widehat{\mathrm{CFK}}(S^3, K), \partial_K)$. We will write $A(\eta)$ for $A(\eta^{\mathrm{top}})$. Finally, call

$$\delta(i) = A(\eta_{2i-1}) - A(\eta_{2i});$$

let us remark that, by definition, $A(\eta_0) = \tau \overset{\mathrm{def}}{=} \tau(K)$.

Theorem 3.2 ([18]). *The* homology *group* $\widehat{\mathrm{HFK}}(-S^3_m(K), \widetilde{K})$ *is an* $\mathbb{F}$*-vector space with basis*

$$\{d_{i,j}, d^*_{i,j}, u_\ell \mid 1 \le i \le k, 1 \le j \le \delta(i), 1 \le \ell \le |2\tau - m|\},$$

where the generators satisfy

$$A(d_{i,j}) = A(\eta_{2i}) - (j-1) - (m-1)/2 = -A(d^*_{i,j})$$

and

$$A(u_\ell) = \tau - (\ell - 1) - (m-1)/2.$$

Generators with a $*$ are to be thought of as symmetric to the generators without it, and each family $\{d_{i,j}\}_j$ can be interpreted as representing the arrow

$$\eta_{2i-1} \overset{\partial}{\longmapsto} \eta_{2i}$$

(notice that i varies among *positive* integers), counted with a multiplicity equalling its length (*i.e.* the distance it covers in Alexander grading).

Remark 3.3. Not any basis of $\widehat{\mathrm{HFK}}(-S^3_m(K), \widetilde{K})$ with the same degree properties works for our purposes: we are actually choosing a basis that is compatible with stabilisation maps, as we are going to see in Theorem 3.7.

Definition 3.4. Call S_+ the subspace of $\widehat{\mathrm{HFK}}(-S^3_m(K), \widetilde{K})$ generated by $\{d_{i,j}\}$, and S_- the one generated by $\{d^*_{i,j}\}$: the subspace $S = S_+ \oplus S_-$ is the *stable complex*, and elements of S are called *stable elements*. The subspace spanned by $\{u_\ell\}$ is called the *unstable complex* and will be denoted with U_m (although the subscript will often be dropped), so that $\widehat{\mathrm{HFK}}(-S^3_m(K), \widetilde{K})$ decomposes as $S_+ \oplus U_m \oplus S_-$.

It is worth remarking that the decomposition given in the definition above *does* depend on our choice of the basis: the three stable subspaces $S_\pm$ and S are independent on this choice, but the unstable complex is not; see also Remark 3.9 below.

There is a good and handy pictorial description when $|m|$ is sufficiently large; we will be mostly dealing with negative values of m, so let us call $m' = -m \gg 0$. Consider a direct sum $\widetilde{C} = \bigoplus_{i=1}^{m'} C_i$ of m' copies of $C = \widehat{\mathrm{CFK}}(S^3, K)$, and (temporarily) denote by $\mathbf{x}_i$ the copy of the element $\mathbf{x} \in C$ in C_i. Endow $\widetilde{C}$ with a shifted Alexander grading:

$$\widetilde{A}(\mathbf{x}_i) = \begin{cases} A(\mathbf{x}) - (i-1) - (m-1)/2 & \text{for } i \le m'/2, \\ -A(\mathbf{x}) - (i-1) - (m-1)/2 & \text{for } i > m'/2. \end{cases}$$

for each homogeneous $\mathbf{x}$ in $\widehat{\mathrm{CFK}}(S^3, K)$. We picture this situation by considering each copy of C as a vertical tile of $2g(K)+1$ boxes – each corresponding to a value for the Alexander grading, possibly containing no generators at all, or more than one generator – and stacking the m' copies of C in staircase fashion, with C_1 as the top block and $C_{m'}$ as the bottom block. Notice that, by our grading convention, the copies in the bottom part of the picture are turned upside down: for example, if $\mathbf{x}^{\max} \in C$ has maximal Alexander degree $A(\mathbf{x}) = g(K)$, then $\mathbf{x}_1^{\max}$ lies in the top box of C_1, while $\mathbf{x}_{m'}^{\max}$ lies in the bottom box of $C_{m'}$. Likewise, an element

$\mathbf{x}^\tau \in C$ has Alexander degree $A(\mathbf{x}) = \tau$, then $\mathbf{x}_1^\tau$ lies in the $(g(K) - \tau + 1)$-th box from the top in C_1, and $\mathbf{x}_{m'}^\tau$ lies in the $(g(K) - \tau + 1)$-th box from the bottom in $C_{m'}$.

Our construction is a variant of Hedden's construction: while in general our chain complex for $\widehat{\mathrm{HFK}}(S^3_m(K), \widetilde{K})$ differs in from his complex in the region with intermediate Alexander grading, the resulting homologies nevertheless agree.

The situation is depicted in Figure 1: in this concrete example we have $g(K) = 2$ and $\tau(K) = -1$; accordingly, there are $2g(K) + 1 = 5$ boxes in each vertical column and $\mathbf{x}_1^\tau$ lies in the fourth box from the top in C_1.

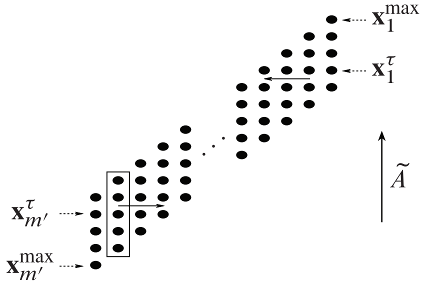

Figure 1. We represent here the top (on the right) and bottom (on the left) parts of $\widehat{\mathrm{HFK}}(S^3_m(K), \widetilde{K})$ for $m \ll 0$. Each vertical tile is a copy of $\widehat{\mathrm{CFK}}(S^3, K)$, and the arrows show the direction of the differentials.

Now define a differential $\tilde{\partial}$ on $\widetilde{C}$ in the following way:

$$\tilde{\partial}\colon \begin{cases} (\eta_0)_i \mapsto 0 & \text{for small and large } i, \\ (\eta_{2j-1})_i \mapsto (\eta_{2j})_{i+\delta(j)} \mapsto 0 & \text{for small } i, \\ (\eta_{2j-1})_i \mapsto (\eta_{2j})_{i-\delta(j)} \mapsto 0 & \text{for large } i, \\ (\eta'_{2j-1})_i \mapsto (\eta'_{2j})_i \mapsto 0 & \text{for every } i. \end{cases}$$

We extend the differential to be any map $\tilde{\partial}$ such that the level $\{A = j\}$ is a subcomplex for every j, whose homology is $\mathbb{F}$ for intermediate values of j (this is possible since $\{A = j\}$ has odd rank for every intermediate value of j).

We are now going to analyse what happens on the top and bottom part of the complex (*i.e.* when i is small or large, in what follows), when we take the homology.

Pairs $(\eta'_{2j-1})_i, (\eta'_{2j})_i$ cancel out in homology. The element $(\eta_{2j})_i$ is a cycle for each i, j, and it is a boundary only when $j > 0$ and either $i > \delta(j)$ or $i < m' - \delta(j)$: so there are $2\delta(j)$ surviving copies of η_{2j}, in degrees

$A(\eta_{2j}) - k - (m-1)/2$ and $-A(\eta_{2j}) + k + (m-1)/2$ for $k = 0, \dots, \delta(j) - 1$. We can declare $d_{i,j} = [(\eta_{2j}^{\mathrm{top}})_i]$ and $d_{i,j}^* = [(\eta_{2j}^{\mathrm{top}})_{m'-i}]$.

The element $(\eta_0)_i$ is a cycle for every i, and it is never cancelled out, so it survives when taking homology. Given our grading convention, for *small* values of i, $\widetilde{A}((\eta_0)_i) = A(\eta_0) - (i-1) - (m-1)/2 = \tau(K) - (i-1) - (m-1)/2$, and in particular we have a nonvanishing class $[(\eta_0^{\mathrm{top}})_i] = u_i$ in degrees $\tau(K) - (m-1)/2, \tau(K) - (m-1)/2 - 1 \dots$. On the other hand, when i is *large*, $[(\eta_0)_i]$ lies in degree $-\tau(K) - (i-1) - (m-1)/2$, and we get a nonvanishing class $[(\eta_0^{\mathrm{top}})_i] = u_{2\tau(K)+i+(m-1)/2}$ in degrees $-\tau(K) + (m-1)/2$, $-\tau(K) + (m-1)/2 + 1 \dots$.

We also have a string of $\mathbb{F}$ summands in between, giving us a strip of unstable elements of length $2\tau(K) - m$, as in Theorem 3.2.

3.3. Stabilisation maps. We are going to study the action of the two stabilisation maps $\sigma_\pm$ of Definition 2.17 on the sutured Floer homology groups $\mathrm{SFH}(-S^3_L)$. It is worth stressing that these maps do not depend on the particular Legendrian representative, but only on its Thurston–Bennequin number: in fact, the topological type of L determines the complement $S^3 \setminus \nu(L)$ and $\mathrm{tb}(L)$ determines the sutures on $\partial\nu(L)$, hence the sutured manifold S^3_L depends only on these data. A gluing map $\Phi_\xi \colon \mathrm{SFH}(M, \Gamma) \to \mathrm{SFH}(M', \Gamma')$ only depends on the contact structure ξ on the layer and not on the contact structure on (M, Γ) (in fact, no such contact structure is required in the definition of Φ_ξ).

Note that if L is a Legendrian knot in S^3 with $\mathrm{tb}(L) = n$, then, as a sutured manifold, S^3_L is just $S^3_{K,n}$. Moreover, if L' is a stabilisation of L, then $S^3_{L'}$ is isomorphic to $S^3_{K,n-1}$ as a sutured manifold.

Recall that we have two families (indexed by the integer n) of stabilisation maps, $\sigma_\pm \colon \mathrm{SFH}(-S^3_{K,n}) \to \mathrm{SFH}(-S^3_{K,n-1})$, corresponding to the gluing of the negative and positive stabilisation layer: if the knot K is oriented, these maps can be labelled as σ_- or σ_+. With a slight abuse of notation, we are going to ignore the dependence of these maps on the framing.

Remark 3.5. Notice that orientation reversal of L or K is not seen by the sutured groups nor by $\mathrm{EH}(L)$, but it swaps the rôles of σ_- and σ_+.

Remark 3.6. Let us recall that for an *oriented* Legendrian knot L of topological type K in S^3 the Bennequin inequality holds:

$$\mathrm{tb}(L) + r(L) \leq 2g(K) - 1.$$

In [29], Plamenevskaya proved a sharper result:

$$\mathrm{tb}(L) + r(L) \leq 2\tau(K) - 1. \tag{3.2}$$

This last form of the Bennequin inequality, together with Theorem 3.2, tells us that, whenever we are considering knots in the standard S^3, the unstable complex is never trivial in $\mathrm{SFH}(-S^3_{K,n})$: more precisely we are always (strictly) below the threshold $2\tau \overset{\text{def}}{=} 2\tau(K)$, so that $2\tau - m$ is always positive; in particular, the dimension of the unstable complex is always positive and *increases* under stabilisations. We will state the theorem in its full generality anyway, even though this remark tells us we need just half of it when working in (S^3, ξ_{st}).

The following theorem is proved in [9, Section 3.4].

Theorem 3.7. *The maps*

$$\sigma_-, \sigma_+ \colon \mathrm{SFH}(-S^3_{K,n}) \longrightarrow \mathrm{SFH}(-S^3_{K,n-1})$$

act as follows:

$$\sigma_- \colon \begin{cases} d_{i,j} \mapsto d_{i,j}, \\ u_\ell \quad \mapsto u_\ell, \\ d^*_{i,j} \mapsto d^*_{i,j+1}, \end{cases} \qquad \sigma_+ \colon \begin{cases} d_{i,j} \mapsto d_{i,j+1}, \\ u_\ell \quad \mapsto u_{\ell+1}, \\ d^*_{i,j} \mapsto d^*_{i,j}, \end{cases} \qquad \text{for } n \leq 2\tau;$$

$$\sigma_- \colon \begin{cases} d_{i,j} \quad \mapsto d_{i,j}, \\ u_\ell \quad\quad \mapsto u_\ell, \\ u_{n-2\tau} \mapsto 0, \\ d^*_{i,j} \quad \mapsto d^*_{i,j+1}, \end{cases} \qquad \sigma_+ \colon \begin{cases} d_{i,j} \mapsto d_{i,j+1}, \\ u_\ell \quad \mapsto u_{\ell-1}, \\ u_1 \quad \mapsto 0, \\ d^*_{i,j} \mapsto d^*_{i,j}, \end{cases} \qquad \text{for } n > 2\tau.$$

Notice that we are implicitly choosing an appropriate isomorphism between the group $\mathrm{SFH}(-S^3_{K,n})$ and the vector space generated by the $d_{i,j}$'s and the u_i's (see Theorem 3.2).

There is an interpretation of the maps $\sigma_\pm \colon \mathrm{SFH}(-S^3_{K,n}) \to \mathrm{SFH}(-S^3_{K,n-1})$ in terms of Figure 1, when $n \ll 0$: fix a chain complex C computing $\widehat{\mathrm{HFK}}(S^3, K)$ and call $(\widetilde{C}_n, \widetilde{\partial})$ and $(\widetilde{C}_{n-1}, \widetilde{\partial})$ the two complexes defined in the previous section, computing $\mathrm{SFH}(-S^3_{K,n})$ and $\mathrm{SFH}(-S^3_{K,n-1})$ starting from C. We have two "obvious" chain maps $s_\pm \colon \widetilde{C}_n \to \widetilde{C}_{n-1}$: s_- sends $\mathbf{x}_i \in \widetilde{C}_n$ to $\mathbf{x}_i \in \widetilde{C}_{n-1}$, while s_+ sends $\mathbf{x}_i \in \widetilde{C}_n$ to $\mathbf{x}_{i+1} \in \widetilde{C}_{n-1}$. The maps $s_\pm$ induce the two stabilisation maps $\sigma_\pm$ at the homology level.

The map s_- is the inclusion $\widetilde{C}_n \hookrightarrow \widetilde{C}_{n-1}$ that misses the leftmost vertical tile (that is, the copy C_{1-n} of C that is in lowest Alexander degree), while s_+ is the inclusion that misses the rightmost vertical tile (the copy C_1 of C that lies in highest Alexander degree).

As a corollary (of the proof), we obtain a graded version of the result:

Corollary 3.8. *The maps $\sigma_\pm$ are Alexander-homogeneous of degree $\mp 1/2$.*

Remark 3.9. Notice that the maps σ_- preserve S_+ and eventually kill S_-, whereas the maps σ_+ have the opposite behaviour. Moreover, σ_- and σ_+ are injective on the unstable complex for $n \leq 2\tau$, while they eventually kill it for $n > 2\tau$.

Namely, for $n \leq 2\tau$, the subcomplex $S_\pm = \bigcup_{m>0} \ker \sigma_\pm^m = \ker \sigma_\pm^N$ for some large N (depending on K, but not on the slope n: any $N > 2g(K)$ works), do not depend on the basis we've chosen. For $n < 2\tau$, though, the unstable subspace *does* depend on this choice: this reflects the fact that it is a section for the projection map $\mathrm{SFH}(-S^3_{K,n}) \to \mathrm{SFH}(-S^3_{K,n})/(S_+ + S_-)$.

On the other hand, for $m > 2\tau$ the situation is reversed: the unstable complex is the intersection of the kernels of $\sigma_\pm^N$, and $S_\pm$ is a section of the projection map $\ker \sigma_\mp^N \to (\ker \sigma_\mp^N)/(\ker \sigma_-^N \cap \ker \sigma_+^N)$.

The action of $\sigma_\pm$ on the unstable complex is just by degree shift, as in Theorem 3.7.

4. An apparently new Legendrian invariant

4.1. Some remarks on EH(L). Given an oriented Legendrian knot L, we define $L^{m,n}$ to be the Legendrian knot obtained from L via m negative and n positive stabilisations.

The main character of the subsection will be an *unoriented* Legendrian knot L in the 3-sphere S^3, equipped with some contact structure ξ.

Proposition 4.1. *If* $\mathrm{tb}(L) \leq 2\tau(K)$, *the pair* $\{\mathrm{EH}(L^{0,n}), \mathrm{EH}(L^{n,0})\}$ *determines* $\mathrm{EH}(L)$.

Strictly speaking, since L is not oriented, $\mathrm{EH}(L^{0,n}), \mathrm{EH}(L^{n,0})$ are not individually defined, but the pair $\{\mathrm{EH}(L^{0,n}), \mathrm{EH}(L^{n,0})\}$ is, as the unordered pair $\{\sigma_-^n(\mathrm{EH}(L)), \sigma_+^n(\mathrm{EH}(L))\}$ for either orientation of L.

Proof. Since σ_- preserves S_- and σ_+ preserves S_+, knowing the pair we know what the stable part of $\mathrm{EH}(L)$ is.

Let us consider now the unstable component of $\mathrm{EH}(L)$: since $\mathrm{EH}(L)$ is represented by a single generator in the chain complex, it is Alexander-homogeneous; moreover, since the stable and unstable complexes are generated by homogeneous elements, both the stable and unstable components of $\mathrm{EH}(L)$ are Alexander-homogeneous. We now state a proposition that will turn out to be useful later, and we will prove it below.

Proposition 4.2. *ξ is overtwisted if and only if* $\mathrm{EH}(L)$ *is stable.*

Now, if ξ is overtwisted, $\mathrm{EH}(L)$ is stable, so we are done.

On the other hand, if $\xi = \xi_{\mathrm{st}}$, the unstable component of $\mathrm{EH}(L)$ is nonvanishing, and – when fixing either orientation – has Alexander degree $2\widetilde{A}(\mathrm{EH}(L)) = \widetilde{A}(\mathrm{EH}(L^{n,0})) + \widetilde{A}(\mathrm{EH}(L^{0,n}))$, and this suffices to determine it. □

Remark 4.3. Proposition 4.2 is a analogue to Theorem 1.2 in [19], which tells us that $\mathcal{L}^-(L)$ is mapped to $c(\xi)$ by setting $U = 1$ in the complex $\mathrm{HFK}^-(-S^3, K)$. See also Proposition 4.14 below.

Proof of Proposition 4.2. We are first going to prove that if $\mathrm{EH}(L)$ is stable, ξ is overtwisted, via the following lemma (which will turn out to be useful also later). Let ψ_∞ denote the gluing map associated to the gluing of the standard neighbourhood of a Legendrian knot (*i.e.* the difference $\mathbb{T}_\infty = Y(1) \setminus \mathrm{Int}(Y_L)$).

Lemma 4.4. *A homogeneous element $x \in \mathrm{SFH}(-S^3_{K,n})$ is stable if and only if $\psi_\infty(x) = 0$.*

Proof. Consider the Legendrian unknot $L \subset (S^3, \xi_{st})$ with $\mathrm{tb}(K_0) = -1$, and stabilise it once (with either sign) to get L'. By gluing $\mathbb{T}_\infty$ to either S^3_L or $S^3_{L'}$ we obtain the contact structure ξ_{st} on S^3. Observe now that $S^3_{L'}$ is obtained from S^3_L by a stabilisation basic slice: it follows in particular that the union $\mathbb{T}'$ of this basic slice and $\mathbb{T}_\infty$ is a tight solid torus. Honda's classification of tight contact structures of solid tori tells us that $\mathbb{T}'$ is isotopic to $\mathbb{T}_\infty$.

Now the associativity of gluing maps (Theorem 2.14) tells us that, as $\mathbb{T}'$ is isotopic (as a contact manifold) to $\mathbb{T}_\infty$, $\psi_\infty \circ \sigma_\pm = \psi_\infty$.

Suppose that x is stable, then there exists a positive integer N such that $(\sigma_- \circ \sigma_+)^N(x) = 0$, and therefore

$$\psi_\infty(x) = \psi_\infty((\sigma_- \circ \sigma_+)^N(x)) = 0.$$

Suppose now that x is not stable. Then $(\sigma_- \circ \sigma_+)^N(x) \neq 0$ for all N. Note that $\sigma_- \circ \sigma_+$ carries homogenous elements to homogenous elements, and has degree 0. By Theorem 3.7, there is a sufficiently large integer N such that the image of x under $(\sigma_- \circ \sigma_+)^N$ lies in the middle part of the complex. More precisely, it lies in a homogenous component of dimension 1, and in particular $x_N = (\sigma_- \circ \sigma_+)^N(x)$ is the generator of the unstable complex in its Alexander-degree summand.

We claim that ψ_∞ doesn't kill x_N.

Now take a knot L' that is Legendrian *with respect to the standard contact structure*, and consider $\mathrm{EH}(L')$. From the first part of the proof, we know that, for all $k, \ell \geq 0$, $(\sigma_-^k \circ \sigma_+^\ell)(\mathrm{EH}(L')) \overset{\psi_\infty}{\longmapsto} c(\xi_{\mathrm{st}})$. But there are positive integers k, ℓ, m such that $(\sigma_-^k \circ \sigma_+^\ell)(\mathrm{EH}(L'))$ and x_{N+m} have the same Alexander degree, and are both nonzero. Since they live in the same 1-dimensional summand, they are equal, and in particular $\psi_\infty(x_{N+m}) = \psi_\infty(\mathrm{EH}(L')) = c(\xi_{\mathrm{st}}) \neq 0$. □

We can now conclude the proof of Proposition 4.2: recall that Eliashberg [2] proved that the only tight contact structure on S^3 is the standard one, and in particular a contact structure ξ on S^3 is overtwisted if and only if $c(\xi) = 0$. By the lemma above, though, $c(\xi) = 0$ if and only if $\mathrm{EH}(L)$ is stable. □

We can pin down the Alexander grading of $\mathrm{EH}(L)$ using an argument analogous to the one that Ozsváth and Stipsicz use for $\mathfrak{L}^-(L)$ [21].

Proposition 4.5. *Identifying* $\mathrm{SFH}(-S^3_L) = \widehat{\mathrm{HFK}}(S^3_{\mathrm{tb}(L)}(K), \widetilde{K})$ *as in Proposition* 2.6*,* $\mathrm{EH}(L)$ *is homogenous of Alexander degree* $-r(L)/2$*.*

Proof. In [21, Theorem 4.1], Ozsváth and Stipsicz compute the Alexander degree of $\mathfrak{L}^-(L)$ by a combinatorial argument on an open book compatible with L. They obtain that

$$A(\mathfrak{L}^-(L)) = \frac{1}{2}\langle c_1(\mathfrak{s}(\mathbf{x}_L)) - PD([\mu_L]), [F, \partial F]\rangle = \frac{\mathrm{tb}(L) - r(L) + 1}{2},$$

where $\mathbf{x}_L$ is a generator representing $\mathfrak{L}^-(L)$ in some Heegaard diagram for S^3.

Let us consider the following set up: let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z, w)$ be a triple Heegaard diagram, where $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ is obtained from an open book compatible with L as in [19], so that $\mathfrak{L}^-(L)$ is represented by a generator $\mathbf{x}$ in $\mathrm{CFK}^-(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$. Now define $\boldsymbol{\gamma}$ to be obtained from $\boldsymbol{\beta}$ by replacing β_0 with $L \subset F$ as sitting inside the page of the open book, and positioned with respect to z, w as in Figure 2.

Note that $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, z, w)$ represents an unknot in $\#^{g-1}(S^1 \times S^2)$, therefore we can choose a generator $\boldsymbol{\Theta}$ representing the top-dimensional class in $\widehat{\mathrm{HFK}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, z, w)$.

Ozsváth and Szabó proved in [23, Section 2] that, whenever we have a triangular domain $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \boldsymbol{\Theta})$, then

$$\mathfrak{s}(\mathbf{y}) - \mathfrak{s}(\mathbf{x}) = (n_w(\psi) - n_z(\psi))PD(\mu). \tag{4.1}$$

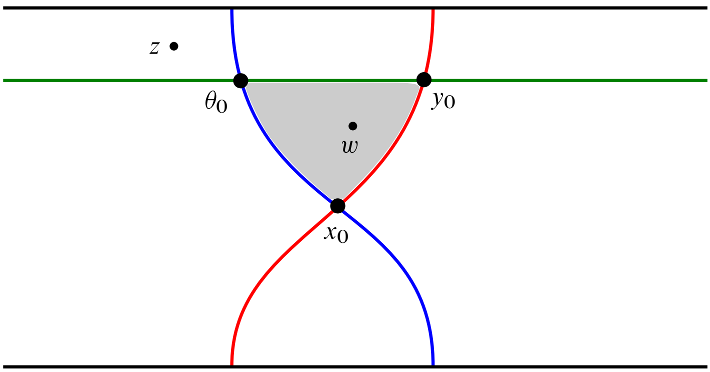

Figure 2. The triple Heegaard diagram used in the proof of Proposition 4.5

We exhibit in Figure 2 a Whitney triangle ψ in $\pi_2(\mathbf{x}, \mathbf{y}, \boldsymbol{\Theta})$ with $n_w(\psi) = 1$, $n_z(\psi) = 0$ connecting the generator $\mathbf{x}$ in $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$ representing $\mathcal{L}(L)$ and the generator $\mathbf{y}$ for in $(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\alpha}, D)$ representing $\mathrm{EH}(L)$, where D is a disc on γ_0 that touches the two regions of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\gamma})$ containing z and w. Notice that $\mathbf{x}$ and $\mathbf{y}$ live in the *cohomology* groups of $\mathrm{CFL}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ and

$$\mathrm{SFC}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, D) = \widehat{\mathrm{CFK}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, z, w),$$

so we need to be careful when using Equation 4.1.

More precisely, we want to consider a map

$$\mathrm{SFC}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}) \longrightarrow \mathrm{SFC}(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\alpha})$$

(we omit basepoint for the sake of clarity), that is dual to a map

$$\mathrm{SFC}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}) \longrightarrow \mathrm{SFC}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

so we should be looking at triangles in the triple Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})$ instead of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. In particular, the grading shifts are reversed: for every triangular domain D in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ we associate the domain $-D$ in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})$, so that n_z and n_w change signs.

Since $c_1(\mathfrak{s}+\alpha) = c_1(\mathfrak{s}) + 2\alpha$ for every $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ and every $\alpha \in H^2(Y)$, it follows from the computations in [21, Section 4] that

$$\begin{aligned} \langle c_1(\mathfrak{s}(\mathbf{y})), [F, \partial F]\rangle &= \langle c_1(\mathfrak{s}(\mathbf{x})), [F, \partial F]\rangle - 2 \\ &= 2A(\mathcal{L}^-(L)) + 1 - 2 \\ &= \mathrm{tb}(L) - r(L). \end{aligned}$$

If we now plug this in Equation 3.1 and we use $\langle PD([\mu_{\tilde{K}}]), [F, \partial F]\rangle = \mathrm{tb}(L)$, we get

$$A(\mathrm{EH}(L)) = \frac{\langle c_1(\mathfrak{s}(\mathbf{y})) - PD([\mu_{\tilde{K}}]), [F, \partial F]\rangle}{2} = -\frac{r(L)}{2},$$

since $\langle PD([\mu_{\tilde{K}}]), [F, \partial F]\rangle = \mathrm{tb}(L)$ by construction of S^3_L. □

We now prove that the hypothesis $\mathrm{tb}(L) \leq 2\tau(K) - 1$ above is necessary:

Proposition 4.6. *For every non-loose unknot L in S^3,* EH(L) *is nonvanishing and purely unstable.*

Proof. When K is the unknot, the stable complex of $S^3_{K,n}$ is trivial for all values of n. Also, $\tau(K) = 0$.

According to Eliashberg and Fraser [3], L has non-negative Thurston–Bennequin number $\mathrm{tb}(L) \geq 0 = 2\tau(K)$, and admits a tight Legendrian surgery (Y, ξ). Since L is topologically unknotted, Y is a lens space, and any tight contact structure on a lens space is Stein fillable: in particular $c(Y, \xi) \neq 0$. Then Lemma 2.13 applies, showing that also $\mathrm{EH}(L) \neq 0$. □

We conclude the section by giving an alternative proof of the following fact, due to Etnyre and Van Horn-Morris, and Hedden [6, 11]. If $K \subset S^3$ is a fibred knot, then it is the binding of an open book for S^3, and any fibre is a minimal genus Seifert surface for K: call ξ_K the contact structure on S^3 supported by this open book.

Theorem 4.7. *ξ_K is tight if and only if $\tau(K) = g(K)$.*

Proof. K sits in ξ_K as a transverse knot, and $sl(K) = 2g(K) - 1$. Let us consider a ξ_K-Legendrian approximation L of K such that $\mathrm{tb}(L) \ll 0$. Vela-Vick proved that $\widehat{\mathcal{L}}(L) \neq 0$, cf. [32], therefore $\mathrm{EH}(L) \neq 0$ [31]. Since K is fibred, $\widehat{\mathrm{HFK}}(S^3, K; g(K))$ is 1-dimensional [26]: using Proposition 4.5 above, together with Theorem 3.2 we see that $\mathrm{EH}(L)$ is the only nonzero element in the top degree component of $\mathrm{SFH}(-S^3_L)$.

If $\tau(K) = g(K)$, then $\mathrm{EH}(L)$ is also the generator in top degree of the unstable complex, and in particular $0 \neq \psi_\infty(\mathrm{EH}(L)) = c(\xi_K)$.

If $\tau(K) < g(K)$, on the other hand, the unstable complex is supported in degree strictly less than $A(\mathrm{EH}(L))$, so $0 = \psi_\infty(\mathrm{EH}(L)) = c(\xi_K)$.

Thus, applying Eliashberg's classification result [2] as above, ξ_K is tight if and only if $c(\xi_K) \neq 0$ if and only if $\tau(K) = g(K)$. $\square$

4.2. The group $\underrightarrow{\mathrm{SFH}}$. Let us step back for a second, and consider an oriented topological knot K in S^3.

Given a graded vector space $V = \bigoplus_d V_d$, we denote with $V\{s\}$ a graded vector space with graded components $(V\{s\})_d = V_{d-s}$. Consider the family of graded $\mathbb{F}$-vector spaces $(A_n \stackrel{\text{def}}{=} \widehat{\mathrm{HFK}}(-S^3_{-n}(K), \widetilde{K})\{(1-n)/2\})$, indexed by integers (notice the $-$ signs in the definition of A_n); for each n we have a degree 0 map $\sigma_-\colon A_n \to A_{n+1}$, the (negative) stabilisation map, induced by the negative basic slice attachment; these data can be conveniently summarized in a direct system $\mathbf{A}_- \stackrel{\text{def}}{=} ((A_n), (\psi_{m,n})_{n\geq m})$, where the map $\psi_{m,n}\colon A_m \to A_n$ is σ_-^{n-m}.

Definition 4.8. Let $\underrightarrow{\mathrm{SFH}}(-S^3, K)$ to be the direct limit $\varinjlim \mathbf{A}_\sigma$, and call ι_n the universal map

$$\iota_n\colon A_n \longrightarrow \underrightarrow{\mathrm{SFH}}(-S^3, K).$$

Remark 4.9. Since we are taking a direct limit, what counts is just what happens for sufficiently large indices. In particular, we just need to know what happens for $n \geq n_0 \stackrel{\text{def}}{=} -2\tau(K) + 1$: this also fits in the picture of contact topology, since this is the only interval where $\mathrm{EH}(L)$ can live for a Legendrian L in (S^3, ξ_{st}).

What happens for other indices is that, with respect to the maps $\psi_{m,n}$, the only component that survives is S: this is going to be more precise below, even though we discuss just the interval $n \geq n_0$.

As defined, $\underrightarrow{\mathrm{SFH}}$ is just a graded $\mathbb{F}$-vector space: using the other (*i.e.* the positive) stabilisation map σ_+, we can endow it with an $\mathbb{F}[U]$-module structure. One way to do it is to identify the projective limit with the quotient of the disjoint union $\coprod A_n$ by the relations $x_i \sim x_j$ whenever there exists N such that $\psi_{i,N}(x_i) = \psi_{j,N}(x_j)$ and defining $U \cdot [x] = [\sigma_+(x)]$: since σ_- and σ_+ commute, the map is well defined. Notice that the map σ_+ has now Alexander degree -1 (due to the degree shift introduced), and so does the map $U\cdot$ on $\mathrm{HFK}^-(S^3, K)$.

Alternatively, we can see the map induced by σ_+ in a more abstract (and universal) way, considering the following diagram:

$$\begin{tikzcd}
& \underrightarrow{\mathrm{SFH}}(-S^3, K) & . \\
& \underrightarrow{\mathrm{SFH}}(-S^3, K) \arrow[u, dashed, "U\cdot"] & \\
A_m \arrow[uur, bend left, "\iota_m \circ \sigma_+"] \arrow[ur, "\iota_m"] \arrow[rr, "\psi_{m,n}"'] & & A_n \arrow[uul, bend right, "\iota_n \circ \sigma_+"'] \arrow[ul, "\iota_n"']
\end{tikzcd}$$

Ignoring the dashed arrow, the diagram commutes, since σ_- and σ_+ commute, and by the universal property of the direct limit (and of the arrows ι_n!), there is a uniquely defined map $U\cdot$, that is the dashed arrow.

Remark 4.10. We have a dual direct system $\mathbf{A}_+$ defined using σ_+ rather than σ_-, and changing the sign of the degree shift.

Reversing the orientation of K induces, as expected, an isomorphism of $\mathbb{F}$-vector spaces $\underrightarrow{\mathrm{SFH}}(-S^3, K) \simeq \underrightarrow{\mathrm{SFH}}(-S^3, -K)$: this follows from the fact that the two direct systems $\mathbf{A}_-$ and $\mathbf{A}_+$ are isomorphic. Moreover, the universal isomorphism commutes with the U-action, and this U-equivariance gives the isomorphism in the category of $\mathbb{F}[U]$-modules.

This symmetry can also be seen as a choice for the labelling of positive *vs* negative stabilisation, which is in fact equivalent to the choice of an orientation.

Theorem 4.11. *The groups* $\underrightarrow{\mathrm{SFH}}(-S^3, K)$ *and* $\mathrm{HFK}^-(-S^3, K)$ *are isomorphic as* $\mathbb{F}[U]$*-modules.*

Before diving into the proof, recall Ozsváth and Szabó's description of HFK^- (see for example [27], especially Figures 1 and 2). The complex is a direct sum of countably many copies of $\widehat{\mathrm{HFK}}$, each thought of as $U^k \cdot \widehat{\mathrm{HFK}}$ for $k \in \mathbb{N}$: this gives the complex the $\mathbb{F}[U]$-structure; we think of each copy drawn as a vertical tile of Alexander-homogeneous components, and that all copies stacked in the plane like a staircase parallel to the $x = y$ diagonal; the differential comes from the complex $(\widehat{\mathrm{HFK}}, \partial)$ computing $\widehat{\mathrm{HF}}(S^3)$, and it can be depicted as a set of arrows pointing horizontally, each coming from a vertical arrow in $(\widehat{\mathrm{HFK}}, \partial)$ and corresponding to a domain crossing the auxiliary basepoint w. There is a quite striking similarity between the first chunks of this complex and the first chunks of the complexes computing A_n's, and this similarity is both the inspiration and the key of the proof of the theorem.

Proof. We will split the proof in two steps: first we will prove the isomorphism of the two as graded $\mathbb{F}$-vector spaces, and then as $\mathbb{F}[U]$-modules. As usual, we will call $g = g(K)$ and $\tau = \tau(K)$.

Step 1. We want to prove there are maps

$$j_n \colon A_n \longrightarrow H \overset{\text{def}}{=} \mathrm{HFK}^-(-S^3, K)$$

such that $(H, \{A_n\}, \{j_n\})$ satisfy the universal property for the direct limit of $\mathbf{A}_-$:

$$\begin{tikzcd}
& C & \\
& H \arrow[u, dashed, "\phi"] & \\
A_m \arrow[ur, "j_m"] \arrow[uur, bend left, "\phi_m"] \arrow[rr, "\psi_{m,n}"'] & & A_n \arrow[ul, "j_n"'] \arrow[uul, bend right, "\phi_n"']
\end{tikzcd}.$$

We need to define the maps j_n first, and then we need to prove that for every commutative diagram with maps ϕ_n to a module C there is a unique (dashed) map ϕ making the full diagram commute.

The maps j_n are easily defined: as described above, $\mathrm{HFK}^-(-S^3, K)$ is the direct sum of a copy of $S_- \subset A_n$ and a copy of $\mathbb{F}[U]$, with $A(U^k) = \tau - k$; imagining a superposition between the two pictures for the complexes computing A_n and H yields to the claim that j_n would like to be a fixed (*i.e.* not depending on n) graded isomorphism on S_-, zero on S_+ and the degree 0, injective map $U_n \to \mathbb{F}[U]$: the commutativity of the lower triangle of the diagram is clear by the description of the maps $\sigma_\pm$.

Now we can consider the full diagram, and show that ϕ is uniquely defined by $(\phi_n)_{n \geq n_0}$: consider an element $x_m = a_m + s_m \in A_m$, with $s_m \in S_+$ and $a_m \in S_- \oplus U_m$, and consider the diagram for $n = m + d \gg m$: since the lower triangle is commutative, we have that

$$\phi_m(x_m) = \phi_n(\sigma_-^d(x_m)) = \phi_n(\sigma_-^d(a_m)) = \phi_m(a_m),$$

so $\phi_m(S_+) = 0$: this implies that the map ϕ_m factors through j_m.

Now, define ϕ by $\phi|_{S_-} = \phi_m|_{S_-}$ for some m and $\phi|_{\mathbb{F}[U]/(U^m)} = \phi_m \circ j_m^{-1}$: notice how ϕ is well defined (since σ_- is an isomorphism on S_- and the injection of degree $+1/2$ on the unstable complex), and makes the diagram commute.

Since j_m is injective on $S_- \oplus U_m$ and $\mathbb{F}[U]$ is the direct limit of $\mathbb{F}[U]/(U^k)$, this is the only way we can define ϕ, and this concludes the first part of the proof.

Remark 4.12. It is worth remarking explicitly what we've proven: we've shown that the inclusion map $\iota_n \colon \widehat{\mathrm{HFK}}(-S^3_{-n}(K), \widetilde{K}) \to \underrightarrow{\mathrm{SFH}}(-S^3, K)$ is injective on $S_- \oplus U$, and that $S_+ = \ker \iota_n$ for each $n \geq n_0$. Moreover, for n sufficiently large, the map ι_n is an isomorphism between truncations of A_n and $\mathrm{HFK}^-(-S^3, K)$ that forgets of all elements of low Alexander degree.

Step 2. We now need to prove that the two $\mathbb{F}[U]$-module structure correspond under some map: we just need to show that the universal map Φ in the diagram

$$\begin{tikzcd}
& H & \\
& \underrightarrow{\mathrm{SFH}}(-S^3, K) \arrow[u, "\Phi"] & \\
A_m \arrow[rr, "\psi_{m,n}"'] \arrow[ur, "\iota_m"] \arrow[uur, bend left, "j_m"] & & A_n \arrow[ul, "\iota_n"'] \arrow[uul, bend right, "j_n"']
\end{tikzcd}$$

is U-equivariant, since the universal property for $(H, \{A_n\}, \{\iota_n\})$ already implies that it is an $\mathbb{F}$-isomorphism. For $x \in A_n$, the map Φ sends $\iota_n(x)$ to the class $[x] = j_n(x)$.

We have a good way to picture Φ when the framing is large: in this case, we just superpose the picture of the complex described in Section 3.2 above with Ozváth and Szabó's description, and identify generators pointwise. But we are working with the projective limit $\underrightarrow{\mathrm{SFH}}$, which is not the disjoint union $\coprod A_n$, but rather its quotient by the relation $x \sim \psi_{m,n}(x)$. Up to changing the choice of n and x, we can suppose that Theorem 3.2 above applies: in this case, the map σ_+ is just an injection of A_n *on the bottom* of A_{n+1}, which, in Ozsváth and Szabó's picture corresponds to shifting each copy $U^k \cdot \widehat{\mathrm{HFK}}(-S^3, K)$ to the next one, $U^{k+1} \cdot \widehat{\mathrm{HFK}}(-S^3, K)$, hence proving the U-equivariance of Φ. ☐

4.3. $\underrightarrow{\mathrm{EH}}$ invariants. Suppose now we have an oriented Legendrian knot L in (S^3, ξ), of topological type K: by construction, we have a naturally defined *oriented* contact class in $\underrightarrow{\mathrm{SFH}}(-S^3, K)$.

Definition 4.13. Define the class $\underrightarrow{\mathrm{EH}}(L) \in \underrightarrow{\mathrm{SFH}}(-S^3, K)$ as $[\mathrm{EH}(L)]$, in the identification $\underrightarrow{\mathrm{SFH}}(-S^3, K) = \coprod A_n / \sim$.

We can immediately read off some facts about this new invariant, that follow straight away from the definition:

Proposition 4.14. *Consider an oriented Legendrian L in (S^3,ξ) of topological type K; then*

(i) *for a* negative *stabilisation L' of L, $\underrightarrow{\mathrm{EH}}(L') = \underrightarrow{\mathrm{EH}}(L)$;*

(ii) *for a* positive *stabilisation L'' of L, $\underrightarrow{\mathrm{EH}}(L') = U \cdot \underrightarrow{\mathrm{EH}}(L)$;*

(iii) *$\underrightarrow{\mathrm{EH}}(L)$ is an element of U-torsion if and only if ξ is overtwisted;*

(iv) *$\underrightarrow{\mathrm{EH}}(L)$ sits in Alexander grading $\dfrac{\mathrm{tb}(L)-r(l)+1}{2}$.*

Proof. (i) L' is a negative stabilisation of L, so $\mathrm{EH}(L') = \sigma_-(\mathrm{EH}(L))$, and

$$\underrightarrow{\mathrm{EH}}(L') = [\mathrm{EH}(L')] = [\sigma_-(\mathrm{EH}(L))] = [\mathrm{EH}(L)] = \underrightarrow{\mathrm{EH}}(L).$$

(ii) L'' is a positive stabilisation of $-L$, so $\mathrm{EH}(L'') = \sigma_+(\mathrm{EH}(L))$, and

$$\underrightarrow{\mathrm{EH}}(L'') = [\mathrm{EH}(L'')] = [\sigma_+(\mathrm{EH}(L))] = U \cdot [\mathrm{EH}(L)] = U \cdot \underrightarrow{\mathrm{EH}}(L).$$

(iii) By definition, an element $[x]$ of $\underrightarrow{\mathrm{SFH}}(-S^3,K)$ vanishes if and only if $\sigma_-^k(x) = 0$ for some k, and is of U-torsion if and only if $[\sigma_+^h(x)] = 0$ for some h: in particular, since σ_- and σ_+ commute, $[x]$ is of U-torsion if and only if $(\sigma_- \circ \sigma_+)^\ell(x) = 0$ for some ℓ. If $\mathrm{tb}(L) > 2\tau(K)$ (and therefore ξ is overtwisted), we know that $\mathrm{SFH}(-S^3_L) = \ker(\sigma_- \circ \sigma_+)^\ell$, so in particular $\mathrm{EH}(L)$ is U-torsion. On the other hand, if $\mathrm{tb}(L) < 2\tau(L)$, Lemma 4.2 tells us that $(\sigma_- \circ \sigma_+)^\ell(\mathrm{EH}(L))$ vanishes if and only if ξ is overtwisted.

(iv) $\mathrm{EH}(L)$ lives in the group $\widehat{\mathrm{HFK}}(-S^3_{\mathrm{tb}(L)}(K),\widetilde{K})$, and by Proposition 4.5, its Alexander degree is $-r(L)/2$. Therefore, it lives in degree $\frac{\mathrm{tb}(L)-r(L)+1}{2}$ in $A_{-\mathrm{tb}(L)}$ and in $\underrightarrow{\mathrm{SFH}}(-S^3,K)$. □

Remark 4.15. $\mathrm{EH}(L)$ is an *unoriented* invariant, *i.e.* doesn't see orientation reversal, whereas the sign of the stabilisation does (see Remark 3.5), so one apparently can find a contradiction in Proposition 4.14. What happens is that when we reverse the orientation of L, we also reverse the orientation of K and we swap the rôles the two maps σ_- and σ_+ play. The two resulting groups, associated to $\mathbf{A}_-$ and $\mathbf{A}_+$ are – as already noticed – isomorphic, but in the first one σ_- acts trivially and σ_+ acts as U (as seen in the proof of Proposition 4.14.(i,ii)), while in the second one we'd have to write

$$\underrightarrow{\mathrm{EH}}(L') = [\mathrm{EH}(L')] = [\sigma_-(\mathrm{EH}(L))] = U \cdot [\mathrm{EH}(L)],$$
$$\underrightarrow{\mathrm{EH}}(L'') = [\mathrm{EH}(L'')] = [\sigma_+(\mathrm{EH}(L)] = [\mathrm{EH}(L))].$$

4.4. Transverse invariants. Let us just recall the classical theorem relating transverse and Legendrian knots: it will be the key fact throughout this subsection.

Theorem 4.16 ([5]). *Two transverse knots are transverse isotopic if and only if any two of their Legendrian approximations are Legendrian isotopic up to negative stabilisations.*

As it happens for $\mathfrak{L}^-$, also $\underrightarrow{\mathrm{EH}}$ descends to a transverse isotopy invariant of transverse knots:

Definition 4.17. Given a transverse knot T in (S^3, ξ) of topological type K, we can define $\underrightarrow{\mathrm{EH}}(T) = \underrightarrow{\mathrm{EH}}(L)$ for a Legendrian approximation L of T.

The transverse element is well-defined, in light of Proposition 4.14 and Theorem 4.16. A stronger statement holds, the natural counterpart of Proposition 4.1, that reveals a transverse nature of EH:

Theorem 4.18. *Suppose L, L' are two oriented Legendrian knots in S^3 that have the same classical invariants. Suppose also that both the transverse pushoffs of L, L' and the ones of $-L, -L'$ are transversely isotopic. Then* $\mathrm{EH}(L) = \mathrm{EH}(L')$.

Proof. Since the pushoffs of L and L' (respectively, of $-L$ and $-L'$) are transverse isotopic, $\underrightarrow{\mathrm{EH}}(L) = \underrightarrow{\mathrm{EH}}(L')$ (resp. $\underrightarrow{\mathrm{EH}}(-L) = \underrightarrow{\mathrm{EH}}(-L')$). By Remark 4.12, and by the behaviour of $\sigma_\pm$ on the unstable complex, we can reconstruct all three components (that is, along $S_\pm$ and U) of $\mathrm{EH}(L)$ from $\underrightarrow{\mathrm{EH}}(L)$ and $\underrightarrow{\mathrm{EH}}(-L)$, and this concludes the proof. □

5. $\underrightarrow{\mathrm{EH}}$ *vs* $\mathfrak{L}^-$

Fix an oriented Legendrian knot L in (S^3, ξ), of topological type K: the LOSS invariant $\mathfrak{L}^-(L)$ is an element of $\mathrm{HFK}^-(-S^3, K)$, which has just been proven isomorphic to $\underrightarrow{\mathrm{SFH}}(-S^3, K)$, where $\underrightarrow{\mathrm{EH}}(L)$ lives. Let us also recall the following theorem:

Theorem 5.1. [19, Theorems 1.2 and 1.6] *For L as before,*

(i) *for a* negative *stabilisation L' of L, $\mathfrak{L}^-(L') = \mathfrak{L}^-(L)$;*

(ii) *for a* positive *stabilisation L'' of L, $\mathfrak{L}^-(L'') = U \cdot \mathfrak{L}^-(L)$;*

(iii) *$\mathfrak{L}^-(L)$ is an element of U-torsion if and only if ξ is overtwisted*;

(iv) *$\mathfrak{L}^-(L)$ sits in Alexander degree* $\frac{\mathrm{tb}(L)-r(L)+1}{2}$.

Notice how the theorem above is formally identical to our Proposition 4.14: it is therefore natural to compare the two invariants $\underrightarrow{\mathrm{EH}}$ and $\mathcal{L}^-$.

Theorem 5.2. *Given L as before, there is an isomorphism of bigraded $\mathbb{F}[U]$-modules $\underrightarrow{\mathrm{SFH}}(-S^3, K) \to \mathrm{HFK}^-(-S^3, K)$ taking $\underrightarrow{\mathrm{EH}}(L)$ to $\mathcal{L}^-(L)$.*

We postpone the proof of the main theorem to the last subsection, and draw some conclusions from the theorem, first.

It is now worth stressing and making precise what we've announced in the introduction, that $\mathrm{EH}(L)$ (but *not* $\underrightarrow{\mathrm{EH}}(L)$!) contains at least as much information as $\mathcal{L}^-(L)$ and $\mathcal{L}^-(-L)$ together. We can prove the following refinement of Theorem 1.1:

Theorem 5.3. *For two oriented Legendrian knots L_0, L_1 in (S^3, ξ) of topological type K, with $\mathrm{tb}(L_0), \mathrm{tb}(L_1) \leq 2\tau(K)$, the following are equivalent*:

(i) $\mathrm{EH}(L_0) = \mathrm{EH}(L_1)$;

(ii) $\mathcal{L}^-(L_0) = \mathcal{L}^-(L_1)$ *and* $\mathcal{L}^-(-L_0) = \mathcal{L}^-(-L_1)$.

In general, withouth any restriction on the Thurston–Bennequin numbers of L_0 and L_1, (i) *implies* (ii).

Proof. $\mathrm{EH}(L_i)$ determines both $\mathcal{L}^-(L_i)$ and $\mathcal{L}^-(L_i)$ by Theorem 5.2, so (ii) follows from (i).

Let us now suppose that the constraint on the Thurston–Bennequin invariants holds. As already observed (see Remarks 3.5 and 4.15), $\underrightarrow{\mathrm{EH}}$ is an oriented invariant of Legendrian knots with the following property: $\underrightarrow{\mathrm{EH}}(L_1) = \underrightarrow{\mathrm{EH}}(L_2)$ if and only if the components of $\mathrm{EH}(L_1)$ and $\mathrm{EH}(L_2)$ along S_- and U agree. In particular, if $\mathcal{L}^-(L_0) = \mathcal{L}^-(L_1)$ the components of $\mathrm{EH}(L_i)$ along S_- and along U are equal; if $\mathcal{L}^-(-L_0) = \mathcal{L}^-(-L_1)$, then the S_+ components of $\mathrm{EH}(L_1)$ and $\mathrm{EH}(L_2)$ agree, too, thus showing that $\mathrm{EH}(L_1) = \mathrm{EH}(L_2)$. □

5.1. Triangle counts. The proof of Theorem 5.2 relies on bypass attachments on contact sutured knot complements and the induced gluing maps, henceforth called simply *bypass maps*.

There is another description of a sutured manifold with torus boundary and annular R_+ we are going to need: an *arc diagram* $\mathcal{H}^a$ is a quintuple $(\Sigma, \boldsymbol{\alpha}, \beta^a, \boldsymbol{\beta}^c, D)$, where Σ is a closed surface, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}^c$ are sets of non-disconnecting, simple closed curves in Σ, D is a closed disc disjoint from $\boldsymbol{\alpha} \cup \boldsymbol{\beta}^c$ and β^a is an arc properly embedded $\Sigma \setminus (\mathrm{Int}(D) \cup \boldsymbol{\beta}^c)$. We further ask that $|\boldsymbol{\alpha}| = g = g(\Sigma)$, and $|\boldsymbol{\beta}^c| = g - 1$.

We will often drop D from the notation and write $\boldsymbol{\beta}$ for $\boldsymbol{\beta}^c \cup \{\beta^a\}$, for sake of brevity.

We build a sutured manifold (M, Γ) with torus boundary and two parallel sutures out of $\mathcal{H}^a$ as follows: the set of α-curves determines how to attach g upside-down 2-handles on $\Sigma \times \{0\} \subset \Sigma \times [0, 1]$; we attach a 0-handle (a ball) to fill up the remaining component of the lower boundary; the set $\boldsymbol{\beta}^c$ of β-*curves* determines the attaching circles of 2-handles on $\Sigma \times \{1\}$. We define M to be the manifold obtained by smoothing corners after these handle attachments; notice that D is an embedded disc in ∂M, and β^a is an embedded arc in ∂M. Let R_+ be a small regular neighbourhood of $D \cup \beta^a$ and Γ be its boundary.

We can now consider the chain complex $\mathrm{SFC}(\mathcal{H}^a)$ as usual, by taking g-tuples of intersection points of α-curves and β-curves and arcs, so that no two points lie on the same curve or arc, and the differential counts holomorphic discs whose associated domains do not touch the disc D. It is clear that $\mathrm{SFC}(\mathcal{H}^a)$ is isomorphic as a chain complex to the complex associated to a doubly-pointed Heegaard diagram representing the dual knot $\widetilde{K}$ inside $S^3_\gamma(K)$; in particular, it is also chain homotopic to a complex computing $\mathrm{SFH}(M, \Gamma)$.

Remark 5.4. The construction above is related to Zarev's bordered sutured manifolds and their bordered sutured diagrams [33], and in fact generalises to sutured manifolds with connected R_+. What we called arc diagrams are in fact similar to bordered sutured diagrams (but not to what he calls arc diagrams).

In order to obtain the bypass maps we need to count holomorphic triangles in triple arc diagrams. At the level of arc diagrams, attaching a bypass to (M, Γ) corresponds to choosing another arc γ^a on Σ, which intersects β^a transversely in a single point θ^a. Every γ-curve is a small perturbation of a β-curve in $\mathcal{H}_\beta$, and therefore there is a preferred choice among the two intersection points (see [24] and Section 5.1.2 below), giving an element $\boldsymbol{\Theta}$. We then have:

Theorem 5.5 ([30]). *The bypass map is induced by the triangle count map* $F(\cdot \otimes \boldsymbol{\Theta})$ *associated to the triple diagram described above.*

Somewhat confusingly, the rôles of α- and β-curves are reversed when talking about contact invariants, since we are looking at elements in $\mathrm{SFH}(-M, -\Gamma)$ rather than in $\mathrm{SFH}(M, \Gamma)$: we will be very explicit and careful about the issue of triangle counts in this setting, as we discuss below.

Recall that, given a Legendrian knot L in any contact 3-manifold, $\mathrm{EH}(L)$ is the class of a generator $\mathbf{x}_{\mathrm{EH}} \in \mathrm{SFC}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$, where $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an arc diagram representing S^3_L (notice the order of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$).

5.1.1. $\boldsymbol{\alpha}$-slides. If we want to do a handle-slide among the α-curves in $\mathcal{H}$, say changing $\boldsymbol{\alpha} = \{\alpha_i\}$ to $\boldsymbol{\alpha}' = \{\alpha_i'\}$, what we are doing is replacing the *second* set of curves in a (doubly-pointed) Heegaard diagram. A triangle count in $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\alpha}')$ gives a map

$$\mathrm{CF}(\boldsymbol{\beta}, \boldsymbol{\alpha}) \otimes \mathrm{CF}(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \otimes \mathrm{CF}(\boldsymbol{\alpha}', \boldsymbol{\beta}) \longrightarrow \mathbb{F},$$

which in turn gives a map

$$F_{\alpha\alpha'} \colon \mathrm{CF}(\boldsymbol{\beta}, \boldsymbol{\alpha}) \otimes \mathrm{CF}(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \longrightarrow \mathrm{CF}(\boldsymbol{\beta}, \boldsymbol{\alpha}').$$

(Here we've been dropping Σ from the notation, and we will do it again later.)

In all cases we are going to meet, the top-dimensional generator in $\mathrm{HF}(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$ will be represented by a single generator, that we call $\boldsymbol{\Theta}_{\alpha\alpha'}$, and the map we will be looking at is $\Psi_{\alpha\alpha'} \colon F_{\alpha\alpha'}(\cdot \otimes \boldsymbol{\Theta}_{\alpha\alpha'})$.

Consider a holomorphic triangle ψ connecting $\mathbf{x}$ to $\mathbf{y}$ giving a nontrivial contribution to the $\Psi_{\alpha\alpha'}$; that is, the Maslov index of ψ is 0 and the moduli space contains an odd number of points. The boundary of the domain $\mathcal{D}(\psi)$ associated to ψ has the following behaviour along its boundary:

A1. $\partial\partial_\alpha \mathcal{D}(\psi) = \mathbf{x} - \boldsymbol{\Theta}_{\alpha\alpha'}$;

A2. $\partial\partial_{\alpha'} \mathcal{D}(\psi) = \boldsymbol{\Theta}_{\alpha\alpha'} - \mathbf{y}$;

A3. $\partial\partial_\beta \mathcal{D}(\psi) = \mathbf{y} - \mathbf{x}$.

This amounts to saying that if we travel along $\partial\mathcal{D}(\psi)$ following the orientation induced by $\mathcal{D}(\psi)$ we cyclicly run along curves in the order β, α', α.

5.1.2. $\boldsymbol{\beta}$-slides. On the contrary, if we are doing some triangle count that changes the β-curves or arcs instead (as we will see below), we are going to face the opposite behaviour. More precisely, consider a set of curves $\boldsymbol{\beta}'$. A triangle count in $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\beta}')$ gives a map

$$\mathrm{CF}(\boldsymbol{\beta}, \boldsymbol{\alpha}) \otimes \mathrm{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}') \otimes \mathrm{CF}(\boldsymbol{\beta}', \boldsymbol{\beta}) \to \mathbb{F},$$

which in turn gives a map

$$F_{\beta\alpha\beta'} \colon \mathrm{CF}(\boldsymbol{\beta}, \boldsymbol{\alpha}) \otimes \mathrm{CF}(\boldsymbol{\beta}', \boldsymbol{\beta}) \to \mathrm{CF}(\boldsymbol{\beta}', \boldsymbol{\alpha}).$$

In all cases we are going to meet, the top-dimensional generator in $\mathrm{HF}(\boldsymbol{\beta}', \boldsymbol{\beta})$ will be represented by a single generator, that we call $\boldsymbol{\Theta}_{\beta'\beta}$, and the map we will be looking at is $\Psi_{\beta\beta'} \colon F_{\alpha\alpha'}(\cdot \otimes \boldsymbol{\Theta}_{\beta\beta'})$. Notice that $\boldsymbol{\Theta}_{\beta'\beta}$ represents the bottom-dimensional generator of $\mathrm{HF}(\boldsymbol{\beta}, \boldsymbol{\beta}')$.

Therefore, if we call ψ a triangle as above, giving a nontrivial summand $\mathbf{y}$ in $F_{\beta\beta'}(\mathbf{x})$ connecting generators $\mathbf{x}$ and $\mathbf{y}$, we get the following conditions on $\mathcal{D}(\psi)$:

B1. $\partial\partial_\alpha \mathcal{D}(\psi) = \mathbf{x} - \mathbf{y}$;

B2. $\partial\partial_\beta \mathcal{D}(\psi) = \mathbf{\Theta}_{\beta'\beta} - \mathbf{x}$;

B3. $\partial\partial_{\beta'} \mathcal{D}(\psi) = \mathbf{y} - \mathbf{\Theta}_{\beta'\beta}$.

That is to say that moving along $\partial\mathcal{D}(\psi)$ following the orientation induced by $\mathcal{D}(\psi)$ we meet the curves β, β', α.

5.2. Proof of Theorem 5.2. The idea underlying the proof is to find explicit representatives for the two contact invariants $\underrightarrow{\mathrm{EH}}(L)$ and $\mathfrak{L}^-(L)$ that live in suitable Heegaard diagram, and compare them.

The proof will be divided in three steps:

(1) We construct an open book (S^3, ξ, L') for a single negative stabilisation L' of L, together with an associated arc diagram $\mathcal{H}^{\mathrm{sut}}$ and an associated doubly-pointed Heegaard diagram $\mathcal{H}^{\mathrm{knot}}$, representing $\mathrm{SFH}(-S^3_{L'})$ and $\mathrm{HFK}^-(-S^3, L')$ respectively.

(2) We consider a large negative stabilisation L^{stab} of L'. Stabilisations corresponds to bypass attachments on $\mathcal{H}^{\mathrm{sut}}$: we compute the associated triangle counts, obtaining a generator in a diagram $\mathcal{H}^{\mathrm{stab}}$, representing $\mathrm{EH}(L^{\mathrm{stab}})$. Moreover, $\mathcal{H}^{\mathrm{stab}}$ has a handle that is very similar to the *winding region* (see Figure 3).

(3) Finally, we handle-slide a single α-curve and compare $\mathcal{H}^{\mathrm{stab}}$ with $\mathcal{H}^{\mathrm{knot}}$ using a refinement of a result of Hedden [10].

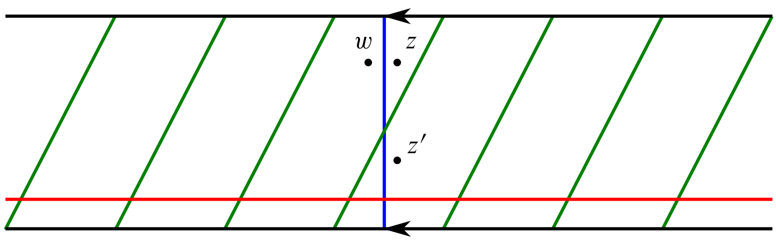

Figure 3. The winding region: the picture represents a handle, with the top and the bottom sides of the rectangle identified according to arrows. The horizontal curve (in red) is an α-curve, the vertical curve (in blue) is a β-curve, representing the meridian for the knot in S^3, whereas the curve that winds along the handle (in green) is the γ-curve representing the given framing on the boundary of $S^3 \setminus \nu(K)$: basepoints are placed such that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ represents (S^3, K), while $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, z', w)$ represents $(S^3_n(K), \widetilde{K})$.

Proof. **Step 1.** Recall the definition of $\mathcal{L}^-$: given $L \subset (S^3, \xi)$, using an idea of Giroux ([8], see also [4]) we can construct an open book (F, ϕ) with L sitting on one of the pages (identified with F, so that $L \subset F$) as a homologically nontrivial curve. We then choose a basis for F (in the sense of Definition 2.8) with only one arc, say a_1, intersecting L. We can construct a doubly-pointed Heegaard diagram as we did for the EH-diagram, the only thing to take care of being placing the two basepoints (see [19]). A representative for $\mathcal{L}^-$ is now given by the only intersection point entirely supported in $F \subset \Sigma$.

The following lemma is implicitly used by Stipsicz and Vértesi [31].

Lemma 5.6. *The partial open book* $(S, P, h) \stackrel{\text{def}}{=} (F, F \setminus \nu(L), \phi|_P)$ *represents the manifold* $(S^3_L, \xi|_{S^3_L})$*, where* $\nu(L)$ *is a small neighbourhood of* L *in* S^3.

Proof. The contact sutured manifold (M, Γ) associated to (S, P, h) embeds in S^3 as the complement of a small neighbourhood of L, since we can embed the two halves of M inside the two halves of S^3 given by (F, ϕ), respecting the foliation: this shows that (M, Γ) is contactomorphic to S^3_L. □

Remark 5.7. We can read off a sutured Heegaard diagram associated to (S, P, h) directly from the doubly-pointed Heegaard diagram for (S^3, L): we just need to remove the basepoints, together with a (small, open) neighbourhood of L in the Heegaard surface, and erase the two curves corresponding to a_1 and b_1. The remaining a_i's form a basis for the (S, P, h), so the EH invariant is already on the picture.

If we also want to have an arc diagram for S^3_L, we can to do the following. We start with the doubly-pointed Heegaard diagram, and replace the curve β_1 with a curve λ parallel to L. Then we add a disc D along it this curve, that is disjoint from α_1 and lies in the two regions that are occupied by the basepoints. Finally, we just forget about the basepoints and let β_1^c be the arc with endpoints in D that runs along λ. Notice that this arc arc intersects a single α-curve (namely, α_1) exactly once.

Notice that in this case the chain complexes associated to the sutured Heegaard diagram and the arc diagram are trivially isomorphic (as chain complexes), since they have exactly the same generators and count precisely the same curves (since the arc intersects only α_1, and in a single point).

We now want to know what happens to this picture when we stabilise L negatively to get L'. If L sits on a page F of the open book (F, ϕ), L' sits on a page of the open book $(F', \phi') = (F \cup H, \phi \circ \delta_c)$, where H is a 1-handle attached to

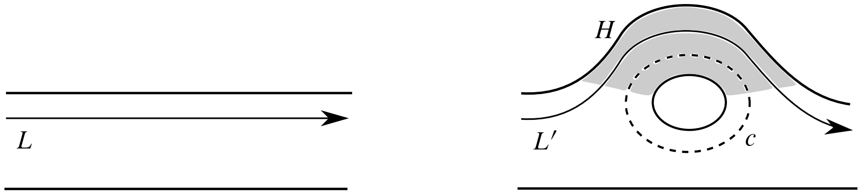

Figure 4. On the left we have a 1-handle of the page of an open book for (Y, ξ, L), where the arrow represents L. On the right we have the page with the additional handle H (shaded), the curve c along which we perform a positive Dehn twist; the arrow represents L' (which otherwise agrees with L).

the boundary of F as in Figure 4 and δ is a positive Dehn twist along the curve c, dashed in the figure. L' is isotopic to L inside F, except that it runs once along the handle [20].

Let us see what happens at the level of arc diagrams: recall that the invariant $\mathrm{EH}(L')$ is represented by a chain $\mathbf{x}_{\mathrm{EH}}$ in the arc diagram $\mathcal{H}^a = (\Sigma, \beta^a, \boldsymbol{\beta}^c, \boldsymbol{\alpha})$ coming from the open book (F', ϕ') together with the embedding $L' \subset F'$. In particular we have that $\Sigma = F' \cup -F'$ and $D \cup \beta^a \subset F' \subset \Sigma$. Call $g+1$ the genus of Σ; the α-curves are obtained after choosing a basis $\{a_0, a_1, \dots, a_g\}$ for F. We choose this basis so that a_0 is the co-core of the handle $H \subset F'$, and is the only arc intersecting L' inside the page, and a_1 is the only other arc intersecting the curve c above (this is always possible). Finally, we let $\beta_0 = \beta^a$ be the arc that runs parallel to L' inside F', $\alpha_0 = a_0 \cup -a_0$, which is the only curve that intersects β_0, $\alpha_1 = a_1 \cup -a_1$, and we number the remaining curves so that α_i and β_i intersect once inside F'. Recall that $\mathbf{x}_{\mathrm{EH}}$ is the generator consisting of all the intersection points $x_i = \alpha_i \cap \beta_i$ inside F'.

Step 2. We now want to attach bypasses to the sutured knot complement $S^3_{L'}$ and compute the associated gluing maps, as indicated in Theorem 5.5.

When we stabilise L' we attach a bypass to the sutured knot complement, and the framing of the sutures decreases by 1. We are going to obtain an arc diagram $(\Sigma, \gamma^a, \boldsymbol{\gamma}^c, \boldsymbol{\alpha})$ for a stabilisation L'' of L' by attaching a bypass to the sutured knot complement $S^3_{L'}$ (see [31]): the Heegaard surface Σ and the curves $\boldsymbol{\alpha}$ are the same as in $\mathcal{H}^a$; also, $\boldsymbol{\gamma}^c = \boldsymbol{\beta}^c$. The arc γ_0 is obtained by juxtaposing β_0 and μ as in Figure 5, where μ is the meridian of $L \subset S^3$. Notice that μ can be obtained by taking $a_0 = \alpha_0 \cap F'$ and letting $\mu = a_0 \cup -\phi'(a_0)$; in other words, μ is the curve β_0 in the doubly-pointed Heegaard diagram of representing $\mathrm{HFK}^-(-S^3, L)$. Observe also that the arc γ_0 intersects β_0 transversely in a single point, θ_0.

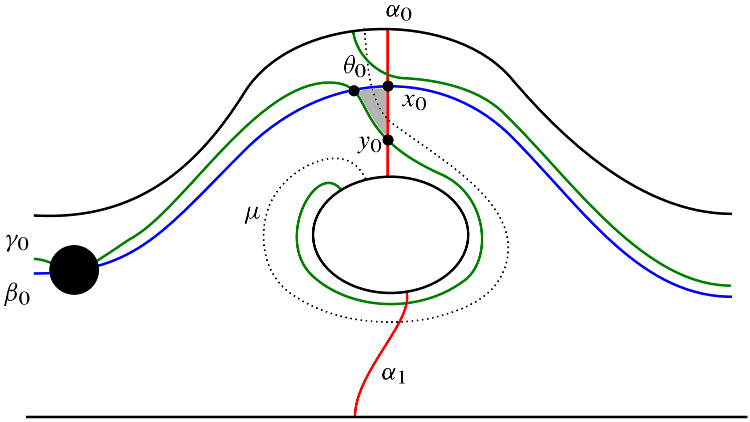

Figure 5. In this picture, we represent F' together with a small neighbourhood of c inside $-F'$; the dashed curve represents the meridian μ for $L' \subset S^3$, and γ_0 is obtained from β_0 through a right-handed Dehn twist along μ. We omit the curves β_1 and γ_1 to avoid cluttering the picture.

We are now ready to compute the action of the bypass map on $\mathbf{x}_{\mathrm{EH}}$; we are going to denote the bypass maps induced by negative stabilisations σ_-. In order to be able to do a triangle count, we need to perturb the β-curves to obtain curves $\gamma_1, \dots, \gamma_g$. We choose the perturbations so that γ_i has the following two properties:

- it intersects β_i transversely in two points, both inside F' and separated along $\beta_i \cap F'$ by α_i (see Figure 6);
- it intersects $\alpha_i \cap F'$ transversely in a single point y_i.

The two intersection points of β_i and γ_i are connected by a bigon B inside F. We label them θ_i and θ_i' so that B connects θ_i' *to* θ_i. Notice that this is the opposite of the usual convention for triangle counts (see 5.1.2 above). We let $\mathbf{\Theta} = \{\theta_i\}$.

We are going to do a triangle count in $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\gamma}, D)$; in the notation of 5.1.2 above, the map σ_- is induced by $F_- = F_{\beta\alpha\gamma}(\cdot \otimes \mathbf{\Theta})$. Let $F_-(\mathbf{x}_{\mathrm{EH}}) = \sum_{k=1}^n \mathbf{y}_k$, where all summands are distinct (such a representation exists and is unique up to permutations, since we are working with coefficients in $\mathbb{F}$).

Lemma 5.8. *For each $k = 1, \dots, n$, the intersection point of $\mathbf{y}_k$ along α_i for $i > 0$ is y_i.*

Proof. Recall that $\mathbf{x}_{\mathrm{EH}}$ is the generator which is entirely supported in F'. Let ψ be a holomorphic triangle contributing to the coefficient of the summand $\mathbf{y}_j$ in $F_-(\mathbf{x}_{\mathrm{EH}}) = \sum \mathbf{y}_k$.

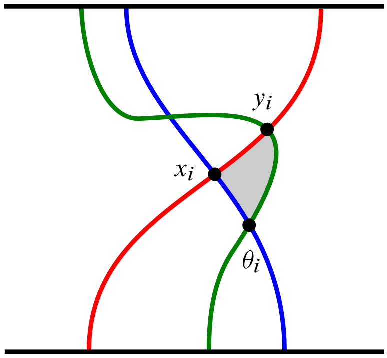

Figure 6. This picture represents a neighbourhood of α_i in F', for some positive i.

Let us consider $F' \subset \Sigma$ in a neighbourhood of α_i containing also β_i and γ_i. The arcs $a_i = \alpha_i \cap F'$ for $i \geq 0$ don't disconnect F' by construction; moreover, the arc β_0 is entirely contained in F' and does not meet any α_i for $i > 0$, while $\gamma_0 \cap F'$ is made of two arcs that run along β except near α_0. It follows that the two unbounded regions to the left and right of Figure 6 are in fact the same region, which touches the disc D. Therefore, the multiplicity of $\mathcal{D}(\psi)$ in these regions is 0.

Since the multiplicities at the left of x_i and at the right of θ_i both vanish, the β-boundary of $\mathcal{D}(\psi)$ is the arc from x_i to θ_i contained in F'. Also, since the multiplicity at the left of x_i is zero, the corner of $\mathcal{D}(\psi)$ at x_i is acute and is contained in the small triangle, shaded in the picture. Since the multiplicity at the right of y_i vanishes, there has to be a corner at y_i, too, and in particular the α_i-component of $\mathbf{y}_j$ has to be y_i. It also follows that the domain $\mathcal{D}(\psi)$ has to be the small triangle shaded in Figure 6. □

We now look at $\alpha_0 \cap \gamma_0$. Let us call y_0 the first intersection point of γ_0 and α_0 we meet when we travel along γ_0 starting from D and going in the direction of θ_0.

Lemma 5.9. *For each $k = 1, \ldots, n$, the intersection point of $\mathbf{y}_k$ along α_0 is y_0.*

Proof. The remaining intersection point of $\mathbf{y}_k$ lies on α_0 and β_0, since all other curves already have an intersection point on them. Notice also that there is a small triangle connecting θ_0, x_0 and y_0, so that $\bar{\mathbf{y}} = \{y_0, \ldots, y_g\}$ does in fact appear in the sum. See Figure 5.

Consider now a holomorphic triangle ψ and its domain $\mathcal{D} = \mathcal{D}(\psi)$. $\partial_\beta \mathcal{D}$ has to be the short arc connecting x_0 and y_0 in the handle H, since the complement of this arc touches the base-disc D. Consider a small push-off a of α_0 disjoint from this arc and from α_0 itself. Observe that $\partial\mathcal{D}$ is nullhomologous and $\partial\mathcal{D} \cap a = \partial_\gamma \mathcal{D} \cap a$. Therefore, a has to have trivial algebraic intersection with the γ-boundary of $\mathcal{D}$. Suppose that $\mathcal{D}$ connects x_0 with another intersection of γ_0 with α_0: its γ-boundary $\partial_\gamma \mathcal{D}$ is homologous to a linear combination of α_0 and the meridian μ, where μ appears with nonzero multiplicity. In particular, this contradicts the fact that a intersects $\partial_\gamma \mathcal{D}$ trivially, since $|a \cap \mu| = 1$. □

In particular, all $\mathbf{y}_k$s are equal, and thus $\mathrm{EH}(L'') = \sigma_-(\mathrm{EH}(L')) = [\bar{\mathbf{y}}]$.

We want to iterate the procedure, and stabilise L''. The bypass we need to attach only modifies γ_0 by juxtaposition with μ, and in particular Lemma 5.8 holds in this case as well. Notice also that the only thing we used in proving Lemma 5.9 is that x_0 and y_0 were the first intersection points of α_0 with the arcs β_0 and γ_0 respectively, so – up to notational modifications – Lemma 5.9 holds for iterations of bypass attachments.

In particular, we've computed the action of σ_-^n on $\mathrm{EH}(L')$ for every $n \geq 0$.

Step 3. We now slide α_1 over α_0 to obtain α_1'. Recall that $a_i = \alpha_i \cap F'$ intersects the curve c that we used to stabilise the open book (F, ϕ) only if $i = 0, 1$. In particular, α_1' is disjoint from μ and the only α-curve that intersects μ is α_0. Call $\mathcal{H}^{\mathrm{final}}$ the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}', D)$.

We are going to compute the action of the map HS induced by this handleslide on the contact invariant. Let $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\alpha}')$ be the triple Heegaard diagram associated to the handleslide, and $\mathbf{x}_{\mathrm{EH}}$ be the contact invariant as computed in the previous step and $\mathbf{y}_c$ be the intersection point in $\mathcal{H}^{\mathrm{final}}$ that is closest to $\mathbf{x}_{\mathrm{EH}}$ (see below for a more precise description).

Lemma 5.10. *The handleslide map HS sends $\mathbf{x}_{\mathrm{EH}}$ to $\mathbf{y}_c$.*

Proof. As above, let $HS(\mathbf{x}_{\mathrm{EH}}) = \sum \mathbf{y}_k$ where all summands are distinct.

On all α-curves other than α_0 and α_1 the same argument as in Lemma 5.8 applies with no modification (see 5.1.1 for the orientation issues): Figure 7 represents what happens locally around x_i and is obtained from Figure 6 through a rotation by 180 degrees.

In fact, the same argument applies to the triple $\beta_1, \alpha_1, \alpha_1'$: looking at Figure 8, we see that for every $\mathbf{y}_k$ in the sums the intersection point on α_1' is the intersection of α_1' and β_1 on F'. First of all, there is a small triangle T_1 connecting x_1 to y_1 inside F'. To prove that there can be no other domain, we observe that the

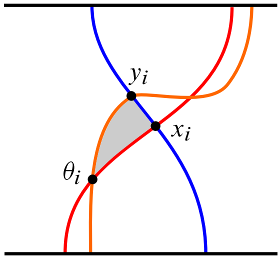

Figure 7. This picture represents a neighbourhood of α_i in F', for some positive i.

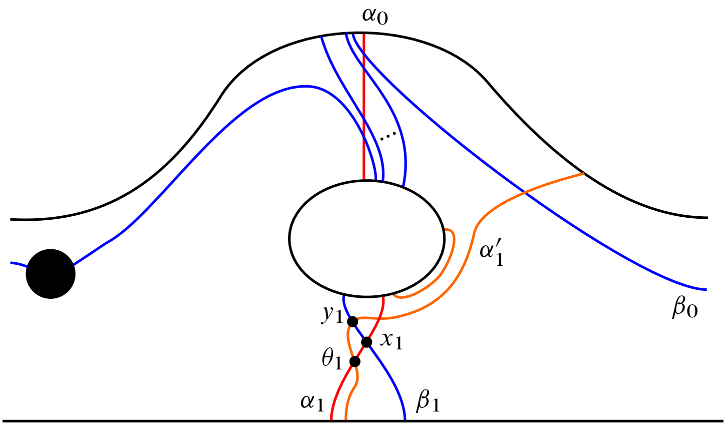

Figure 8. This picture represents a neighbourhood of the sliding region between α_0 and α_1 in F'

multiplicity has to vanish in the corner at x_1 across from T_1, since this region touches the disc D. On the other hand, this is enough for the proof of Lemma 5.8 to work.

Finally, we take care of the intersection point of $\beta_0 \cap \alpha'_0$, that is the first intersection point when moving from the D along β_0, traversing the handle H first. This is similar to the proof of Lemma 5.9 above, and it follows from the same homological considerations. $\square$

Observe that a neighbourhood of the meridian μ_L of $L \subset S^3$ in the diagram looks like *half* of the winding region, as in Figure 9. Call x_0 the intersection point of μ_L with α'_0, and number the intersection points of α'_0 with β_0 as $x_1, x_2, \ldots$ according to the order in which we meet them when travelling along α'_0 (so that x_0 comes first). An easy adaptation of the proof of Theorem 4.1 in [10] shows the following:

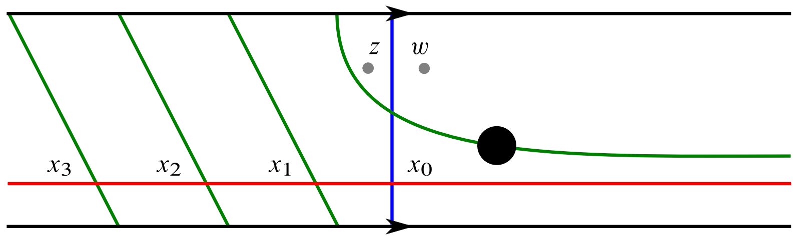

Figure 9. The neighbourhood of μ_L in $\mathcal{H}^{\text{final}}$. The twisting is all on one side of μ_L (the vertical curve). The intersection points on α_0' (the horizontal curve) are labelled $x_0, x_1, \ldots$ from right to left. We also put the basepoints z and w to represent the knot $K \subset S^3$.

Proposition 5.11. *All generators in $\mathcal{H}^{\text{final}}$ with sufficiently large Alexander degree have an intersection point in the winding region.*

Moreover, the map

$$\Phi\colon \text{SFC}_{A \geq N}(-S^3_{L^{\text{stab}}}) \longrightarrow \text{CFK}^-_{A \geq N'}(S^3, K)$$

defined by

$$\Phi(\{x_n\} \cup \mathbf{x}) = U^n \cdot (\{x_0\} \cup \mathbf{x})$$

induces an isomorphism of chain complexes when N is sufficiently large and

$$N' = N + \frac{\text{tb}(L^{\text{stab}}) + 1}{2}.$$

In particular, the generator $\mathbf{y}_0$ we've shown to represent $\text{EH}(L^{\text{stab}})$ is of the form $\{x_1\} \cup \overline{\mathbf{x}}$, where the generator $\{x_0\} \cup \overline{\mathbf{x}} \in \text{CFK}^-(S^3, K)$ represents $\mathcal{L}^-(L)$. It follows that under map induced at the chain level by Φ maps $\text{EH}(L^{\text{stab}})$ to $\mathcal{L}^-(L)$, therefore concluding the proof of Theorem 5.2. □

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