Geometric filtrations of string links and homology cylinders

James Conant, Robert Schneiderman, and Peter Teichner[1]

Abstract. We show that the group of string links modulo order n twisted Whitney tower concordance is an extension of the image of the nilpotent Artin representation by a finite 2-group. Moreover, this 2-group is generated by band sums of iterated Bing-doubles of any string knot with nonzero Arf invariant. We also analyze the Goussarov–Habiro clasper filtration of the group of 3-dimensional homology cylinders modulo homology cobordism, importing techniques from our work on Whitney towers to improve on results of J. Levine. In particular, we classify the graded group associated to the Goussarov–Habiro filtration in all orders except $4n + 1$. In this last case, it is classified up to unknown 2-torsion with a precise upper bound. These calculations confirm conjectures of Levine in the even cases, and improve on his conjectures in the odd cases. In the last section of this paper we connect the settings of string links and homology cylinders by analyzing a geometric map, originally formulated by N. Habegger.

Contents

1. Introduction

This paper applies the techniques of our recent work on Whitney tower filtrations of classical link concordance [3, 4, 6] to the settings of string links and 3-dimensional homology cylinders. The goal is to understand certain geometric filtrations of links, string links and homology cylinders, respectively, and to describe relationships among them. We start by briefly summarizing the relevant filtrations; precise definitions are given in subsequent sections.

$\mathbb{L}$: The (framed) Whitney tower filtration $\mathbb{L} \supset \mathbb{W}_0 \supset \mathbb{W}_1 \supset \mathbb{W}_2 \supset \cdots$ and twisted Whitney tower filtration $\mathbb{W}_n^\infty \supset \mathbb{W}_n$ on the set $\mathbb{L} = \mathbb{L}(m)$ of concordance classes of framed m-component links in the 3–sphere. Here $\mathbb{W}_n$ (respectively $\mathbb{W}_n^\infty$) consists of those links that bound framed (respectively twisted) Whitney towers of order n in the 4–ball [4]. Note that the connected sum operation only gives a well defined group structure on $\mathbb{L}$ for $m = 1$. This changes when we go to string links.

$\mathbb{SL}$: – The analogous Whitney tower filtrations $\mathbb{SW}_n$ and $\mathbb{SW}_n^\infty$ on the group $\mathbb{SL} = \mathbb{SL}(m)$ of concordance classes of framed m-component string links (obtained from $\mathbb{W}_n$ and $\mathbb{W}_n^\infty$ via the usual closure operation from string links to links).

  – The Artin filtration $\mathbb{SA}_n$ on $\mathbb{SL}$, defined as kernels of nilpotent Artin representations $\mathrm{Artin}_n \colon \mathbb{SL} \to \mathrm{Aut}_0(F/F_{n+2})$, where $F = F(m)$ is a free group on m generators, F_n are the terms in its lower central series, and $\mathrm{Aut}_0(F/F_n)$ is the subgroup of automorphisms of F/F_n which are defined by conjugating each generator and which fix the product of generators [14].

  – The Goussarov–Habiro Y-filtration $\mathbb{SY}_n$ on $\mathbb{SL}$, consisting of string links obtained from the trivial string link via concordances and surgeries along claspers with n nodes [11, 16, 17].

$\mathbb{HC}$: The Artin and Goussarov–Habiro filtrations generalize to filtrations $\mathbb{J}_n$ and $\mathbb{Y}_n$ respectively on the group $\mathbb{HC} = \mathbb{HC}(g, b)$ of homology cobordism classes of homology cylinders over an oriented surface $\Sigma_{g,b}$ of genus g with b boundary circles [10, 21, 22]. $\mathbb{J}_n$ is called the Johnson filtration, as it also generalizes the Johnson filtration of the mapping class group [19].

Various graded groups associated to these filtrations will also be defined precisely below, and will be denoted by the sans-serif versions of the above letters, for example $\mathsf{SW}_n := \mathbb{SW}_n/\mathbb{SW}_{n+1}$.

Links and string links. To begin, we show that the graded groups associated to the Whitney tower filtrations on string links and links are isomorphic. One advantage of moving to string links is that the entire set of link concordance classes forms a group, and modding out by the Whitney tower filtration yields *nilpotent* groups:

Theorem 1. *The sets* $\$\mathbb{W}_n$ *and* $\$\mathbb{W}_n^{\infty}$ *are normal subgroups of* $\$\mathbb{L}$ *which are central modulo the next order. We obtain nilpotent groups* $\$\mathbb{L}/\$\mathbb{W}_n$ *and* $\$\mathbb{L}/\$\mathbb{W}_n^{\infty}$, *and associated graded groups*

$$\mathsf{SW}_n \cong \mathsf{W}_n \quad and \quad \mathsf{SW}_n^{\infty} \cong \mathsf{W}_n^{\infty}.$$

Here W_n and W_n^{∞} are the quotients of $\mathbb{W}_n$ and $\mathbb{W}_n^{\infty}$ by the equivalence relations of *order* $n + 1$ *Whitney tower concordance* and *order* $n + 1$ *twisted Whitney tower concordance*, respectively (see [4] and Section 2 below). Except for $m = 1$, the sets $\mathbb{W}_n$ and $\mathbb{W}_n^{\infty}$ don't have a natural group structure, however we showed in [4, Section 3] that a component-wise band connected sum operation makes W_n and W_n^{∞} into finitely generated abelian groups which determine graded groups associated to the filtrations in the sense that for any link L we have $L \in \mathbb{W}_{n+1}^{(\infty)}$ if and only if $L \in \mathbb{W}_n^{(\infty)}$ and $[L] = 0 \in \mathsf{W}_n^{(\infty)}$. Theorem 1 shows that considering string links removes this subtlety from the definition of the associated groups.

The groups W_n and W_n^{∞} are computed in [4, 5] by identifying the Whitney tower intersection/obstruction theory with the first non-vanishing Milnor invariants and certain higher-order Sato–Levine invariants and higher-order Arf invariants. As an application of these techniques to the setting of string links, Theorem 1 will be proven in Section 2.

Also proven in Section 2 are the next theorem and subsequent corollary, which follow from the classification of W_n^{∞}, together with the interpretation of the Artin representation as the "universal" Milnor invariant [14]. This gives a very satisfying algebraic interpretation of the quotients by the twisted Whitney tower filtration, up to a finite 2-group with well-understood generators.

Theorem 2. *We have* $\$\mathbb{W}_n^{\infty} \subset \$\mathbb{A}_n$, *and the Artin representation* Artin_n *induces an epimorphism*:

$$\mathrm{Artin}_n^{\infty} \colon \$\mathbb{L}/\$\mathbb{W}_n^{\infty} \longrightarrow \mathrm{Aut}_0(F/F_{n+2}).$$

The kernel is a finite 2-group, generated by (*internal band sums of*) *iterated Bing-doubles of the figure eight string knot* (*possibly with some additional trivial strands*). *In particular, for each n there is an upper bound on the size of this kernel.*

See Figures 4 and 5 for the definition of Bing-doubling and internal band sums in the setting of string links.

In fact, the kernel of Artin_n^∞ can be characterized geometrically in several other ways (the different geometric conditions are explained in Definition 22).

Corollary 3. *The following subsets of* $\mathbb{SL}/\mathbb{SW}_n^\infty$ *are equal to the kernel of* Artin_n^∞:

 (i) *the subgroup generated by* (*internal band sums of*) *iterated Bing-doubles of a fixed string knot with nontrivial Arf invariant* (*possibly with some additional trivial strands*);

 (ii) *the subgroup generated by* (*internal band sums of*) *iterated Bing-doubles of any string knots with non-trivial Arf invariant* (*possibly with some additional trivial strands*);

(iii) *the set of equivalence classes of boundary string links*;

(iv) *the set of equivalence classes of* π_1-*null string links*.

Regarding the Goussarov–Habiro Y-filtration $\mathbb{SY}_n$ on $\mathbb{SL}$, we will show in Proposition 47 that $\mathbb{SY}_n \subseteq \mathbb{SW}_n$. In other words, if σ is any string link obtained from the trivial string link by concordances and surgeries on claspers with n nodes, then the closure L_σ of σ bounds an order n Whitney tower. (In fact, in an upcoming paper we will prove that this is an if and only if statement: Whitney tower concordance is equivalent to the notion of C_n-concordance studied by Meilhan and Yasuhara [24].) Summarizing, we see that the filtrations on concordance classes $\mathbb{SL}$ of string links are ordered as follows:

$$\mathbb{SY}_n \subseteq \mathbb{SW}_n \subseteq \mathbb{SW}_n^\infty \subseteq \mathbb{SA}_n.$$

String links and homology cylinders.   By taking the complement of a string link in $D^2 \times [0, 1]$, one gets a well-known group homomorphism from string links to homology cylinders

$$C : \mathbb{SL}(m) \longrightarrow \mathbb{HC}(0, m + 1)$$

which takes $\mathbb{SA}_n$ to $\mathbb{J}_n$, and takes $\mathbb{SY}_n$ to $\mathbb{Y}_n$. Denote by $\mathbb{HC}_1(g, b)$ the homology bordism classes of homology cylinders inducing the identity map on integral first homology $H_1(\Sigma_{g,b}; \mathbb{Z})$. In unpublished work [12], Habegger used the fact that $\Sigma_{g,1} \times [0, 1] \cong \Sigma_{0,2g+1} \times [0, 1]$ to give a bijection

$$\mathbb{HC}_1(0, 2g + 1) \longleftrightarrow \mathbb{HC}_1(g, 1)$$

which is not a group homomorphism but identifies the filtrations $\mathbb{J}_n$ (respectively $\mathbb{Y}_n$) on the two different types of homology cylinders. Let $\mathbb{SL}_1(2g)$ denote the set of concordance classes of string links with $2g$ strands and trivial linking matrix. In Section 4 we generalize the map C to another geometric map

$$\mathrm{H} : \mathbb{SL}_1(2g) \longrightarrow \mathbb{HC}_1(g, 1) \quad \text{(see Figure 9)}$$

which is not a homomorphism but takes both $\mathbb{SY}_n$ to $\mathbb{Y}_n$ and $\mathbb{SA}_n$ to $\mathbb{J}_n$ for $n \geq 1$. We show that this map H agrees with Habegger's bijection pre-composed with C. Combining results from [10] and [14] it follows that the induced maps

$$\mathrm{C}_n : \mathsf{SA}_n \longrightarrow \mathsf{J}_n$$

are group isomorphisms for all $n \geq 1$, where

$$\mathsf{SA}_n := \mathbb{SA}_n/\mathbb{SA}_{n+1}$$

and

$$\mathsf{J}_n := \mathbb{J}_n/\mathbb{J}_{n+1}$$

are the quotient groups associated to the Artin and Johnson filtrations; and by composing with Habegger's bijection we see that the same is true for

$$\mathrm{H}_n : \mathsf{SA}_n \longrightarrow \mathsf{J}_n.$$

As will be explained in Section 4, the analogous induced maps for the Y-filtrations are not yet fully understood but again the statements for C_n and H_n are equivalent:

Theorem 4. *The induced maps*

$$\mathrm{C}_n, \mathrm{H}_n : \mathsf{SY}_n \longrightarrow \mathsf{Y}_n$$

are group isomorphisms for $n \equiv 0, 2, 3 \mod 4$. In the remaining cases, C_{4k+1}, H_{4k+1} are epimorphisms with finitely generated 2-torsion kernel.

The group SY_n is defined as the quotient of $\mathbb{SY}_n$ by the equivalence relation generated by surgeries along claspers with $n + 1$ nodes. (So two string links representing elements in $\mathbb{SY}_n$ are equal in SY_n if and only if they differ by a sequence of concordances and surgeries along claspers with $n + 1$ nodes.) Similarly Y_n is defined as the quotient of $\mathbb{Y}_n$ by the equivalence relation generated by surgeries along claspers with $n + 1$ nodes.

As an aside, we recall that by a theorem of Nielsen [25, 29], $\mathbb{HC}(g, b)$ contains the mapping class group of $\Sigma_{g,b}$. This is one source of interest in the filtrations $\mathbb{J}_n$ and $\mathbb{Y}_n$.

Levine had already observed that in $\mathbb{HC}(g, 1)$ there is an inclusion $\mathbb{Y}_n \subseteq \mathbb{J}_n$ and he started to study the difference in [21, 22]. He conjectured the statements of the next theorem which we proved in [6].

In the following, $\mathsf{L} = \mathsf{L}(2g)$ denotes the free Lie algebra (over the integers) on $2g$ generators (of degree 1), and L_n is the degree n part of L (additively generated by n-fold brackets of generators, modulo the Jacobi identity and the self-annihilation relations $[X, X] = 0$). Throughout this paper $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ denotes the additive group of integers mod 2.

Theorem 5 ([6]). *For $n \geq 1$, the inclusion of filtrations $\mathbb{Y}_n \subseteq \mathbb{J}_n$ of $\mathbb{HC}(g, 1)$ induces the following exact sequences*:

$$0 \longrightarrow \mathsf{Y}_{2k} \longrightarrow \mathsf{J}_{2k} \longrightarrow \mathsf{L}_{k+1} \otimes \mathbb{Z}_2 \longrightarrow 0,$$

$$\mathbb{Z}_2^m \otimes \mathsf{L}_{k+1} \longrightarrow \mathsf{Y}_{2k+1} \longrightarrow \mathsf{J}_{2k+1} \longrightarrow 0.$$

For all n, the groups J_n are free abelian of known rank (see discussion in Section 3.2), so this theorem provides a satisfactory classification of the even order groups Y_{2k} associated to the geometrically defined Goussarov–Habiro filtration of $\mathbb{HC}(g, 1)$ (and the same for SY_{2k} via Theorem 4). Levine did not conjecture that the map $\mathbb{Z}_2{}^m \otimes \mathsf{L}_{k+1} \to \mathsf{Y}_{2k+1}$ is injective, and in fact it is *not* because the *framing relations* introduced in [4] are also present in this context, as will be seen in Proposition 9 below. Unravelling this odd order case is one of the main results in this paper:

Theorem 6. *For $k \geq 1$, we have the following exact sequences*

$$0 \longrightarrow \mathsf{L}_{2k+1} \otimes \mathbb{Z}_2 \longrightarrow \mathsf{Y}_{4k-1} \longrightarrow \mathsf{J}_{4k-1} \longrightarrow 0,$$

$$0 \longrightarrow \mathsf{K}^{\mathsf{Y}}_{4k+1} \longrightarrow \mathsf{Y}_{4k+1} \longrightarrow \mathsf{J}_{4k+1} \longrightarrow 0,$$

and the kernel $\mathsf{K}^{\mathsf{Y}}_{4k+1}$ fits into the exact sequence

$$\mathsf{L}_{k+1} \otimes \mathbb{Z}_2 \xrightarrow{a_{k+1}} \mathsf{K}^{\mathsf{Y}}_{4k+1} \longrightarrow \mathsf{L}_{2k+2} \otimes \mathbb{Z}_2 \longrightarrow 0.$$

Theorem 6 will be proven in Section 3. The calculation of $\mathsf{K}^{\mathsf{Y}}_{4k+1}$ is thus reduced to the calculation of $\mathrm{Ker}(a_{k+1})$. As explained at the end of Section 4, this is the precise analog of the question *how nontrivial are the higher-order Arf invariants*? in the setting of Whitney tower filtrations of classical links (compare the a_{k+1} in Theorem 6 with the maps α_{k+1} defined in [3]).

Conjecture 7. *The homomorphisms a_{k+1} are injective for all $k \geq 1$.*

Framing relations, the Levine Conjecture, and tree claspers. Underlying all of the above results are two key aspects of our recent classification of the Whitney tower filtrations on classical links (as surveyed in [3]): the first aspect is a strengthening of the tree-valued intersection theory for Whitney towers that incorporates the above-mentioned framing relations, which reflect geometric moves that affect obstructions to framing Whitney disks in a Whitney tower [4]. Second is the affirmation in [6] of a combinatorial conjecture of Levine [22] which allows for a complete computation of the resulting target groups for the Whitney tower intersection invariants. We briefly touch on the relevance of these points next, with details given in the body of the paper:

Definition 8. The abelian group $\mathcal{T} = \mathcal{T}(m)$ is additively generated by oriented unitrivalent trees, having each univalent vertex labeled by an element of the index set $\{1, \ldots, m\}$, modulo IHX relations and AS antisymmetry relations $t + (-t) = 0$, where $-t$ denotes the tree t with the opposite orientation (see e.g. [4, 6]). Here the *orientation* of a tree is given by cyclic orderings of the edges at all trivalent vertices, and trees are considered up to label-preserving isomorphisms. The *order* of a tree is the number of trivalent vertices. Since the AS and IHX relations are homogeneous with respect to order, $\mathcal{T}$ inherits a grading $\mathcal{T} = \oplus_n \mathcal{T}_n$, where $\mathcal{T}_n$ is the free abelian group on order n trees, modulo AS and IHX relations.

The resolution [6] of the Levine Conjecture allows for a complete computation of $\mathcal{T}_n$ for all n as free abelian groups of known rank in even orders, and having only 2-torsion of known rank in odd orders. Analogous groups with rational coefficients were well understood previously, and play a central role in finite type theories (e.g. [9, 12, 15, 16]). The move to integral coefficients is critical for extracting combinatorial/topological information from algebraic invariants such as Milnor's link invariants and the closely related Artin representation.

Using surgeries along tree claspers, Habiro [16] constructed surjective realization maps

$$\theta_n : \mathcal{T}_n \longrightarrow \mathsf{Y}_n$$

where Y_n filters either $\mathbb{HC}(0, m + 1)$ or $\mathbb{HC}(g, 1)$ (which are equivalent for $n \geq 1$ by the above discussion). As described in [4], the quotients $\widetilde{\mathcal{T}}_n$ of the $\mathcal{T}_n$ by the framing relations are natural targets for the intersection invariants associated to the obstruction theory of (framed) Whitney towers, and play a crucial role in the computation of W_n. (See Definition 40 below for the definition of $\widetilde{\mathcal{T}}_n$.) Most of the work in Section 3 goes into translating the geometry of the framing relations into the clasper language, culminating in the following result which is used in proving our main Theorem 6 above:

Proposition 9. *Habiro's realization map θ_{2k+1} factors through the quotient $\widetilde{\mathcal{T}}_{2k+1}$ for $k \geq 1$.*

In even orders $\widetilde{\mathcal{T}}_{2k} = \mathcal{T}_{2k}$ by definition (since the framing relations only occur in odd orders), and the main result of [6] also computes the groups $\widetilde{\mathcal{T}}_n$ for all n. These computations are essential to the proofs of Theorem 4 and Theorem 6. Further implications of [6] appear in the detailed description of the relationship between the string link and homology cylinder settings given in Section 4 ; as well as in the characterization of the kernel of $\mathrm{Artin}_n^{\mathrm{co}}$ given by Theorem 2 and Corollary 3 as explained in Section 2.

Organization. In Section 2, we recall basic Whitney tower techniques in the setting of string links and prove Theorems 1 and 2, as well as Corollary 3. Section 3 is devoted to clarifying and proving Theorem 6, and is mostly occupied with proving the just-stated Proposition 9. Finally, in section 4 we compare string links with homology cylinders using Habegger's map, proving Theorem 4 and explaining the connection between Conjecture 7 above and higher-order Arf invariants for links [3, 4].

2. The Artin Filtration and twisted Whitney towers

For the reader's convenience we sketch the relevant definitions of (twisted) Whitney towers and their intersection invariants, suppressing orientations and sign conventions. See [4] for a complete discussion. It will be convenient to use the bijective correspondence between formal non-associative bracketings of elements

from an index set $\{1, 2, 3, \ldots, m\}$ and *rooted* trees, which are oriented unitrivalent trees as in Definition 8 except that a single univalent vertex of each tree is left unlabeled. This unlabeled vertex is called the *root* of the tree, and all other univalent vertices are labeled from the index set. (The trees as in Definition 8 will occasionally be referred to as "unrooted" trees for emphasis.)

Definition 10. Let I and J be two rooted trees.

(i) The *rooted product* (I, J) is the rooted tree gotten by identifying the root vertices of I and J to a single vertex v and sprouting a new rooted edge at v. This operation corresponds to the formal bracket (Figure 1 upper right).

(ii) The *inner product* $\langle I, J \rangle$ is the unrooted tree gotten by identifying the roots of I and J to a single non-vertex point. (Figure 1 lower right.)

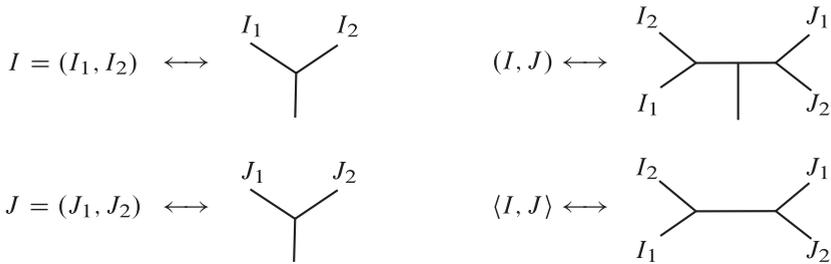(iii) The *order* of a tree, rooted or unrooted, is defined to be the number of trivalent vertices.

Figure 1. The *rooted product* (I, J) and *inner product* $\langle I, J \rangle$ of $I = (I_1, I_2)$ and $J = (J_1, J_2)$. The inner product is additive with respect to the orders of I and J, and the rooted product increases the sum of the orders by 1.

2.1. Whitney towers. Let L be an m-component framed link in $S^3 = \partial B^4$. A collection of oriented framed properly immersed 2-disks $A_1, \ldots, A_m \looparrowright (B^4, S^3)$ bounded by L is a *Whitney tower of order zero*. The framings of the immersed disks are required to induce the framing of L (framings are discussed below in 2.2), and being *properly immersed* means that the disks' interiors are generically immersed in $B^4 \smallsetminus S^3$.

To each *order zero disk* A_i in an order zero Whitney tower is associated the order zero rooted tree consisting of an edge with one vertex labeled by i, and to each transverse intersection $p \in A_i \cap A_j$ is associated the order zero tree

$$t_p := \langle i, j \rangle$$

consisting of an edge with vertices labeled by i and j. Note that for singleton brackets (rooted edges) we drop the bracket from notation, writing i instead of (i).

Suppose there exists a pair of oppositely signed intersection points between A_i and A_j. Joining these points by any embedded pair of arcs lying in (the interiors of) A_i and A_j (and avoiding all other intersection points) forms a loop, and any immersed disk in $B^4 \smallsetminus S^3$ bounded by such a loop is called a *Whitney disk* (for the pair of intersection points). Such a Whitney disk will, in general, also have interior intersections with other order zero disks and any other Whitney disks. To each Whitney disk $W_{(i,j)}$ pairing intersections between A_i and A_j is associated the order 1 rooted Y-tree (i, j), having a single trivalent vertex and two univalent labels i and j. This rooted tree can be thought of as being embedded in B^4, with its trivalent vertex and rooted edge sitting in $W_{(i,j)}$, and its two other edges descending into A_i and A_j as sheet-changing paths.

All Whitney disks will be assumed to be oriented, although orientation choices and conventions will be suppressed from the discussion (see [4, Section 2.3]).

Recursively, the rooted tree (I, J) is associated to any Whitney disk $W_{(I,J)}$ pairing intersections between W_I and W_J (see left-hand side of Figure 2); with the understanding that if, say, I is just a singleton i, then W_I denotes the order zero disk A_i.

To any transverse intersection $p \in W_{(I,J)} \cap W_K$ between $W_{(I,J)}$ and any W_K is associated the un-rooted tree

$$t_p := \langle (I, J), K \rangle$$

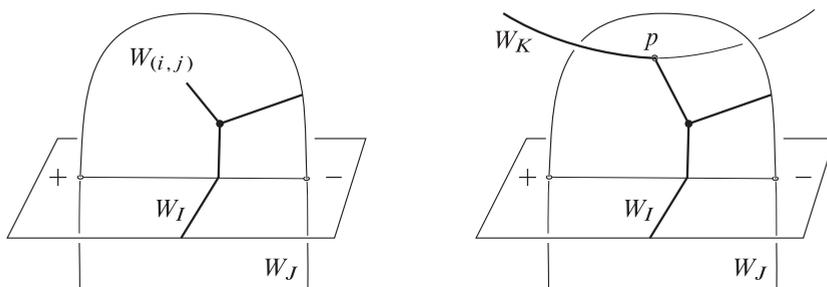(see right-hand side of Figure 2).

Figure 2. On the left, (part of) the rooted tree (I, J) associated to a Whitney disk $W_{(I,J)}$. On the right, (part of) the unrooted tree $t_p = \langle (I, J), K \rangle$ associated to an intersection $p \in W_{(I,J)} \cap W_K$. Note that p corresponds to where the roots of (I, J) and K are identified to a (non-vertex) point in $\langle (I, J), K \rangle$.

Definition 11. The *order of a Whitney disk W_I* is defined to be the order of the rooted tree I, and the *order of a transverse intersection p* is defined to be the order of the tree t_p.

Definition 12. A collection $\mathcal{W}$ of properly immersed disks bounding the link L, together with higher-order Whitney disks is an *order n Whitney tower* if $\mathcal{W}$ contains no unpaired intersections of order less than n.

Whitney disks in $\mathcal{W}$ must have disjointly embedded boundaries, and generically immersed interiors. Also, all Whitney disks and order zero disks must be *framed*, as discussed next.

2.2. Twisted Whitney disks and framings. The normal disk-bundle of a Whitney disk W in B^4 is isomorphic to $D^2 \times D^2$, and comes equipped with a canonical nowhere-vanishing *Whitney section* over the boundary ∂W given by pushing ∂W tangentially along one sheet and normally along the other, avoiding the tangential direction of W. Pulling back the orientation of B^4 with the requirement that the normal disks have $+1$ intersection with W means the Whitney section determines a well-defined (independent of the orientation of W) relative Euler number $\omega(W) \in \mathbb{Z}$ which represents the obstruction to extending the Whitney section to a nowhere-vanishing section over W. Following traditional terminology, when $\omega(W)$ vanishes W is said to be *framed*. (Since $D^2 \times D^2$ has a unique trivialization up to homotopy, this terminology is only mildly abusive.) In general when $\omega(W) = k$, we say that W is *k-twisted*, or just *twisted* if the (non-zero) value of $\omega(W)$ is not specified. So a 0-twisted Whitney disk is a framed Whitney disk. See [4, Section 2.2] for details.

Note that a *framing* of the link component ∂A_i (respectively the order zero disk A_i) is by definition a trivialization of the normal bundle of the immersion. Via a fixed orientation on (B^4, S^3), this is equivalent to an orientation and a non-vanishing normal vector field on ∂A_i (respectively A_i). The twisting $\omega(A_i) \in \mathbb{Z}$ of an order zero disk is also defined when a framing of ∂A_i is given, and A_i is said to be *framed* when $\omega(A_i) = 0$.

2.3. Twisted Whitney towers. In the definition of an order n Whitney tower given just above, all Whitney disks and order zero disks are required to be framed. It turns out that the natural generalization to twisted Whitney towers involves allowing twisted Whitney disks only in at least "half the order" as follows.

Definition 13. A *twisted Whitney tower of order* 0 bounded by a link $L \subset S^3$ is the same as a Whitney tower of order 0, except that the order 0 disks need not be framed.

For $k \geq 1$, a *twisted Whitney tower of order* $(2k-1)$ is just a (framed) Whitney tower of order $(2k-1)$ as in Definition 12 above.

For $k \geq 1$, a *twisted Whitney tower of order* $2k$ is a Whitney tower having all intersections of order less than $2k$ paired by Whitney disks, with all Whitney disks of order less than k required to be framed, but Whitney disks of order at least k allowed to be twisted.

For emphasis, we will sometimes refer to a Whitney tower as in Definition 12 as a "framed" Whitney tower. Note that any framed Whitney tower also satisfies the definition of a twisted Whitney tower.

Example 14. Any framed link $L \subset S^3$ bounds a twisted Whitney tower of order 0, and L bounds a framed Whitney tower of order 0 if and only if all components of L have even framings. Bounding an order 1 Whitney tower, which is the same as bounding an order 1 twisted Whitney tower, is equivalent to L having vanishing linking numbers and framings.

Example 15. An order 2 Whitney tower is illustrated in Figure 3.

Example 16. See [5, Intro and Section 6] for examples of links bounding twisted Whitney towers containing nontrivially-twisted Whitney disks.

Example 17. Suppose a link L is obtained from the unlink by surgery on a clasper with k nodes, where one of the leaves bounds a clean disk, but has a nonzero framing (like the twisted clasper leaf on the right of Figure 8). Then L bounds a twisted Whitney tower of order $2k$. To see this, one uses the clasper–grope correspondence of [7] and the twisted grope–twisted Whitney tower correspondence of [5]. The twisted leaf of the clasper corresponds to a twisted Whitney disk. See *twisted claspers* in Section 3 below.

2.4. Intersection invariants for (twisted) Whitney towers. As described above in 2.1 and illustrated in Figures 2 and 3, the trees associated to unpaired intersections in a Whitney tower can be considered to be embedded in the Whitney tower. A key idea in the theory of Whitney towers is that these same trees represent group elements which can be used to define intersection invariants giving obstructions to the existence of higher-order Whitney towers (details in [4, Section 2.4]).
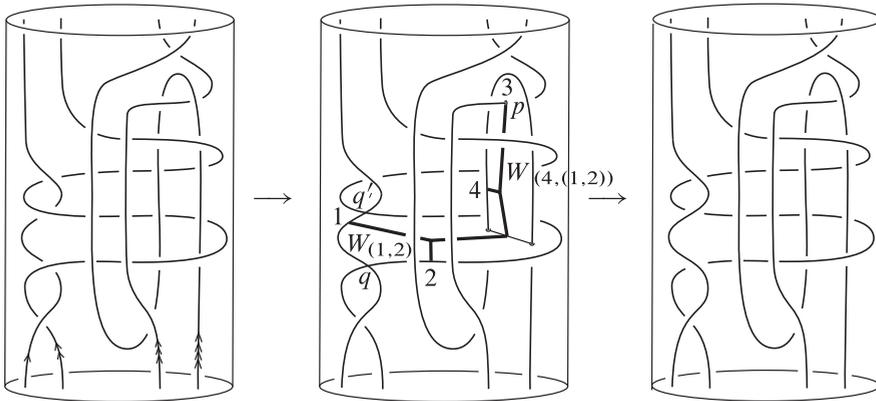
Figure 3. An order 2 Whitney tower $\mathcal{W}$ in $B^4 = D^2 \times I \times [0, 1]$, bounded by a link which is the union of a string link $\sigma \subset D^2 \times I \times \{0\}$ (left) and a trivial string link $\sigma_{\mathrm{id}} \subset D^2 \times I \times \{1\}$ (right) together with product arcs connecting $\partial\sigma$ and $\partial\sigma_{\mathrm{id}}$. The order zero disks A_i of $\mathcal{W}$ are described by the trace of the homotopy from σ to σ_{id} which has a single pair of crossing changes labeled q and q' (center). The order 1 Whitney disk $W_{(1,2)}$ pairs the corresponding intersections $q, q' \in A_1 \cap A_2$, and the order 2 Whitney disk $W_{(4,(1,2))}$ pairs the two intersections between A_4 and $W_{(1,2)}$. The single order 2 unpaired intersection point $p \in A_3 \cap W_{(4,(1,2))}$ of $\mathcal{W}$ has associated tree $t_p = \langle 3, (4, (1, 2)) \rangle$ which is embedded in $\mathcal{W}$.

Definition 18. The *order n intersection invariant* $\tau_n(\mathcal{W})$ of an order n Whitney tower $\mathcal{W}$ is defined to be

$$\tau_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p \in \mathcal{T}_n$$

where the sum is over all order n intersections $p \in \mathcal{W}$, with $\epsilon_p = \pm 1$ the usual sign of a transverse intersection point.

The intersection invariants for Whitney towers are extended to twisted Whitney towers as follows.

Definition 19. The abelian group $\mathcal{T}^{\infty}_{2k-1}$ is the quotient of $\mathcal{T}_{2k-1}$ by the *boundary-twist relations*:

$$\langle (i, J), J \rangle = i \mathrel{-\!\!\prec^{J}_{J}} = 0.$$

Here J ranges over all order $k - 1$ rooted trees.

For any rooted tree J we define the corresponding ∞-*tree*, denoted by J^{∞}, by labeling the root univalent vertex with the symbol "∞":

$$J^{\infty} := \infty \!\!-\!\! J.$$

Definition 20. The abelian group $\mathcal{T}_{2k}^{\infty}$ is the free abelian group on order $2k$ trees and order k ∞-trees, modulo the following relations:

(i) the 'usual' AS and IHX relations on order $2k$ trees;

(ii) the *symmetry* relations: $(-J)^{\infty} = J^{\infty}$;

(iii) the *twisted IHX* relations: $I^{\infty} = H^{\infty} + X^{\infty} - \langle H, X \rangle$;

(iv) the *interior twist* relations: $2 \cdot J^{\infty} = \langle J, J \rangle$.

Here the AS and IHX relations are as in the "un-twisted" groups $\mathcal{T}_{2k}$, but they only apply to the order $2k$ trees, not to the order k ∞-trees.

Recall from Definition 13 that twisted Whitney disks only occur in even order twisted Whitney towers, and only those of half-order are relevant to the obstruction theory.

Definition 21. The *order n intersection intersection invariant* $\tau_n^{\infty}(\mathcal{W})$ of an order n twisted Whitney tower $\mathcal{W}$ is defined to be

$$\tau_n^{\infty}(\mathcal{W}) := \sum \epsilon_p \cdot t_p + \sum \omega(W_J) \cdot J^{\infty} \in \mathcal{T}_n^{\infty}$$

where the first sum is over all order n intersections p and the second sum is over all order $n/2$ Whitney disks W_J with twisting $\omega(W_J) \in \mathbb{Z}$. For $n = 0$, our notational convention is that W_j denotes A_j, and that $\omega(A_j) \in \mathbb{Z}$ is the relative Euler number of the normal bundle of A_j with respect to the given framing of the link component ∂A_j.

2.5. The obstruction theory. To summarize, we have defined what it means for a link $L \subset S^3$ to bound an order n (twisted) Whitney tower $\mathcal{W} \subset B^4$, and the intersection invariant $\tau_n^{(\infty)}(\mathcal{W}) \in \mathcal{T}_n^{(\infty)}$. One of the main results from [4] is that the vanishing of $\tau_n^{(\infty)}(\mathcal{W})$ is sufficient to conclude that L bounds a (twisted) Whitney tower of order $n + 1$, thus giving an obstruction theory for (twisted) Whitney towers. This obstruction theory was used in [4] to show that the sets $\mathbb{W}_n^{(\infty)}$ of links bounding order n (twisted) Whitney towers, modulo order $n + 1$ (twisted) Whitney tower concordance, form groups $\mathsf{W}_n^{(\infty)}$ under component-wise band sums, where the notion of (twisted) Whitney tower concordance involves (twisted) Whitney towers built on properly immersed annuli in $S^3 \times [0, 1]$ connecting the links in $S^3 \times \{0\}$ and $S^3 \times \{1\}$. As surveyed in [3], the sequence of papers [4, 5, 6] computes the groups $\mathsf{W}_n^{(\infty)}$ in terms of Milnor invariants together with *higher-order Arf invariants* taking values in finite 2-groups, with the caveat that the image of the higher-order Arf invariants is not known. (The framed case requires *higher-order Sato–Levine invariants* as well, which are certain mod 2 projections of Milnor invariants of higher order.)

2.6. The Artin filtration $\mathbb{A}_n$. Recall that the set of framed string links has a well-defined monoid structure given by stacking, which, modulo concordance, becomes a (nonabelian [20]) group $\mathbb{L}$. Let $\sigma \subset B = D^2 \times [0, 1]$ be a string link with m strands. The inclusions $(D^2 \setminus \{m \text{ points}\}) \times \{i\} \hookrightarrow B \setminus \sigma$ for $i = 0, 1$ induce isomorphisms on all lower central quotients of the fundamental groups by Stallings' Theorem [28]. In fact, the induced automorphisms of the lower central quotients F/F_n of the free group

$$F = \pi_1(D^2 \setminus \{m \text{ points}\})$$

are explicitly characterized by conjugating the meridional generators by the string link longitudes. Let $\mathrm{Aut}_0(F/F_n)$ consist of those automorphisms of F/F_n which are defined by conjugating each generator and which fix the product of generators. This leads to the (*nilpotent*) *Artin representation*

$$\mathrm{Artin}_n \colon \mathbb{L} \longrightarrow \mathrm{Aut}_0(F/F_{n+2}).$$

See [13, 14] for more details. (Note that our indexing of Artin_n is shifted by 1 compared to [13, 14], a result of our grading by order rather than by commutator length.)

The *Artin filtration* is defined as

$$\mathbb{A}_n := \mathrm{Ker}\{\mathrm{Artin}_n \colon \mathbb{L} \longrightarrow \mathrm{Aut}_0(F/F_{n+2})\}.$$

Observe that since conjugation acts trivially on the abelianization F/F_{n+2}, for the case $n = 0$ we have $\mathbb{A}_0 = \mathbb{L} = \mathbb{W}_0^{\infty}$ by Example 14.

2.7. The (twisted) Whitney tower filtration of $\mathbb{L}$. Given a string link σ, the standard closure $L_\sigma \subset S^3$ is obtained by connecting the endpoints by fixed arcs in the boundary of B and taking a standard embedding $B \subset S^3$ (i.e., thinking of $B = B \times \{0\} \subset B \times [0, 1]$, with $S^3 = \partial(B \times [0, 1])$; see Figure 5). Then a string link σ is by definition in $\mathbb{W}_n^{(\infty)}$ if L_σ is the boundary of a (twisted) Whitney tower of order n in $B \times [0, 1] \cong B^4$.

Proof of Theorem 1. String links σ and σ' are *order n* (*twisted*) *Whitney tower concordant* if the union of $\sigma \subset B \times \{0\}$ and $\sigma' \subset B \times \{1\}$ together with product arcs in $\partial B \times [0, 1]$ connecting the boundaries $\partial \sigma$ and $\partial \sigma'$ bounds an order n (twisted) Whitney tower $\mathcal{W}$ in $B \times [0, 1]$, and we say that "σ and σ' *cobound* $\mathcal{W}$." or that "$\mathcal{W}$ is a Whitney tower concordance *from* σ to σ'."

If $\mathcal{W}$ is a (twisted) Whitney tower concordance from σ to the trivial string link σ_{id}, then for brevity we will say that "σ bounds $\mathcal{W}$"; this streamlined terminology is justified by the easy observation that σ is order n (twisted) Whitney tower concordant to σ_{id} if and only if $\sigma \in \mathbb{SW}_n^{(\infty)}$.

The obstruction theory of [4] implies that if the closure L_σ of σ bounds an order n (twisted) Whitney tower $\mathcal{W}$ in $B \times [0, 1]$ with $\tau_n^{(\infty)}(\mathcal{W}) = 0 \in \mathcal{T}_n^{(\infty)}$, then $L_\sigma \in \mathbb{W}_{n+1}^{(\infty)}$, and hence $\sigma \in \mathbb{SW}_{(n+1)}^{(\infty)}$. Similarly, if σ bounds an order n (twisted) Whitney tower $\mathcal{W}$ (as in the previous paragraph) with $\tau_n^{(\infty)}(\mathcal{W}) = 0 \in \mathcal{T}_n^{(\infty)}$, then $\sigma \in \mathbb{SW}_{(n+1)}^{(\infty)}$.

Clearly $\mathbb{SW}_n$ and $\mathbb{SW}_n^{\infty}$ are closed under multiplication, since order n (twisted) Whitney towers bounded by σ and σ' can be stacked together to form an order n (twisted) Whitney tower bounded by the product $\sigma\sigma'$. Also, the reflection $\bar{\mathcal{W}}$ of any order n (twisted) Whitney tower $\mathcal{W}$ bounded by σ is an order n (twisted) Whitney tower bounded by the concordance inverse $\bar{\sigma}$ of σ. Note that Section 3 of [4] implies that $\tau_n^{(\infty)}(\mathcal{W}) = -\tau_n^{(\infty)}(\bar{\mathcal{W}})$ for any such $\mathcal{W}$ and $\bar{\mathcal{W}}$.

To see that $\mathbb{SW}_n$ is normal in $\mathbb{SL}$, observe that if σ_1 bounds an order n Whitney tower $\mathcal{W}_1$, then for any string link σ_2, the conjugate string link $\sigma_2\sigma_1\overline{\sigma_2}$ bounds an order n Whitney tower $\mathcal{W}_2$: Such a $\mathcal{W}_2$ is constructed by first stacking the Whitney tower concordance $\mathcal{W}_1$ from σ_1 to σ_{id} between product concordances of σ_2 and $\overline{\sigma_2}$ to get an order n Whitney tower concordance from $\sigma_2\sigma_1\overline{\sigma_2}$ to $\sigma_2\sigma_{\mathrm{id}}\overline{\sigma_2} = \sigma_2\overline{\sigma_2}$; then composing with any concordance $\mathcal{W}_0$ from $\sigma_2\overline{\sigma_2}$ to σ_{id}. (The concordance $\mathcal{W}_0$ is a Whitney tower of arbitrarily high order since the disjointly embedded order zero disks are free of singularities.)

To see that $\mathbb{SW}_n/\mathbb{SW}_{n+1}$ is central in $\mathbb{SL}/\mathbb{SW}_{n+1}$, first observe that the Whitney towers $\mathcal{W}_2$ and $\mathcal{W}_1$ described in the previous paragraph satisfy $\tau_n(\mathcal{W}_2) = \tau_n(\mathcal{W}_1) \in \mathcal{T}_n$, since $\tau_n(\mathcal{W}_2) = \tau_n(\mathcal{W}_1 \cup \mathcal{W}_0) = \tau_n(\mathcal{W}_1)$. (In fact $\mathcal{W}_2$ and $\mathcal{W}_1$ have identical *intersection forests* (disjoint unions of signed trees, see [4, Section 2.5]) since the concordance $\mathcal{W}_0$ has no intersection points.) An order n Whitney tower $\mathcal{W}$ bounded by the commutator $\sigma_2\sigma_1\overline{\sigma_2\sigma_1}$ can be formed by stacking $\mathcal{W}_2$ with the reflection $\overline{\mathcal{W}_1}$ of $\mathcal{W}_1$. We have

$$\tau_n(\mathcal{W}) = \tau_n(\mathcal{W}_2) + \tau_n(\overline{\mathcal{W}_1}) = \tau_n(\mathcal{W}_2) - \tau_n(\mathcal{W}_1) = 0 \in \mathcal{T}_n,$$

so $\sigma_2\sigma_1\overline{\sigma_2\sigma_1}$ bounds an order $n+1$ Whitney tower by the obstruction theory of [4].

The same arguments hold in the twisted setting, showing that $\mathbb{SW}_n^{\infty}$ is normal in $\mathbb{SL}$, and central modulo $\mathbb{SW}_{n+1}^{\infty}$.

In order to see that the string link quotient groups

$$\mathsf{SW}_n := \$\mathbb{W}_n^{(\infty)}/\$\mathbb{W}_{n+1}^{(\infty)}$$

are isomorphic to the corresponding link groups

$$\mathsf{W}_n^{(\infty)} := \mathbb{W}_n^{(\infty)}$$

modulo order $n+1$ (twisted) Whitney tower concordance, recall from [13] that the closure operation $\sigma \mapsto L_\sigma$ maps the set of string links onto the set of links in the 3-sphere: Given a link L, a string link in the preimage is constructed by cutting S^3 open along a choice of "disk-basing" for L. To see that this map induces an isomorphism

$$\mathsf{SW}_n \xrightarrow{\cong} \mathsf{W}_n$$

for each n, note that by definition if $\sigma \in \$\mathbb{W}_n^{(\infty)}$, then $L_\sigma \in \mathbb{W}_n^{(\infty)}$; and also by definition if $L_\sigma \in \mathbb{W}_{n+1}^{(\infty)}$, then $\sigma \in \$\mathbb{W}_{n+1}^{(\infty)}$. To see that the induced map preserves the group operations, note that $L_{\sigma\sigma'} \subset S^3 = \partial(B \times [0,1])$ can be expressed as the sum $L_\sigma \sharp L_{\sigma'}$ by cutting S^3 along the 2-sphere $\partial(D \times [0,1])$, where $D \subset B$ is any 2-disk separating σ and σ' in the product stacking. As explained in [4], the band sum operation in $\mathbb{W}_n^{(\infty)}$ is well-defined modulo order $n+1$ Whitney tower concordance. □

Proof of Theorem 2. We may assume that $n \geq 1$, by the observation at the end of Subsection 2.6 above. To see that $\$\mathbb{W}_n^{\infty} \subset \$\mathbb{A}_n$, i.e. that $\text{Artin}_n(\$\mathbb{W}_n^{\infty})$ is trivial, we note that by computations in [5, Theorem 5], the longitudes of any string link in $\$\mathbb{W}_n^{\infty}$ all lie in the $(n+1)$th term of the lower central series, so that conjugation by such an element is trivial modulo the $(n+2)$th term.

If Artin_n^{∞} vanishes on σ, then all Milnor invariants of σ up to order $n-1$ (length $n+1$) vanish. Since these Milnor invariants together with higher-order Arf invariants detect all nontrivial elements of $\$\mathbb{L}/\$\mathbb{W}_n^{\infty}$ (see [4, Corollary 1.16]), the kernel of Artin_n^{∞} is generated by links introduced in Lemma 12 of [5] which have trivial Milnor invariants (because they are boundary links) but which realize all values of the higher-order Arf invariants. As shown in [5], these are indeed (internal band sums of) iterated Bing-doubles of the figure-eight knot. See Figures 4 and 5 for the Bing-doubling and internal band sum construction in the setting of string links.

The fact that the kernel is a 2-group can be shown as follows. Let σ be a string link in the kernel of Artin_n^{∞}. For any order k twisted Whitney tower $\mathcal{W}$ bounded by σ with $k < n$, it follows from [4, Theorem 1.13 and Proposition 1.14] (or [5, Theorem 7 and Proposition 8]), that $\tau_k^{\infty}(\mathcal{W})$ vanishes if $k \equiv 0, 1, 3 \mod 4$

and $\tau_k^{\infty}(\mathcal{W})$ is 2-torsion if $k \equiv 2 \mod 4$. So if $\mathcal{W}_\sigma$ is any order 2 twisted Whitney tower bounded by σ, then σ^2 bounds $\mathcal{W}_{\sigma^2}$ formed by stacking two copies of $\mathcal{W}_\sigma$, and $\tau_2^{\infty}(\mathcal{W}_{\sigma^2}) = 2\tau_2^{\infty}(\mathcal{W}_\sigma) = 0$. Hence $\sigma^2 \in \mathbb{SW}_6^{\infty}$. But then $\tau_6^{\infty}(\mathcal{W}_{\sigma^4}) = 0$, where $\mathcal{W}_{\sigma^4}$ is the double of an order 6 twisted Whitney tower bounded by σ^2. Hence $\sigma^4 \in \mathbb{SW}_{10}^{\infty}$, etc. In particular, if $4k - 2 > n$, then $\sigma^{2^k} = 1 \in \mathbb{SL}/\mathbb{SW}_{n+1}^{\infty}$. $\square$
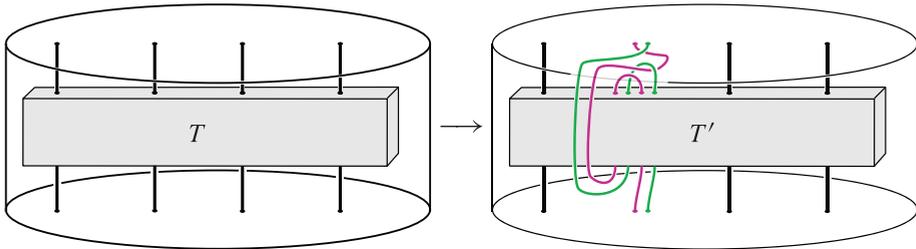

Figure 4. From left to right: *Bing-doubling* a strand of a framed string link represented by a tangle T. The four parallel strands in the tangle T' that replace the strand in T are assumed to follow the framing of the original strand. The string link σ in Figure 3 is (isotopic to) the result of Bing-doubling a pure braid commutator.
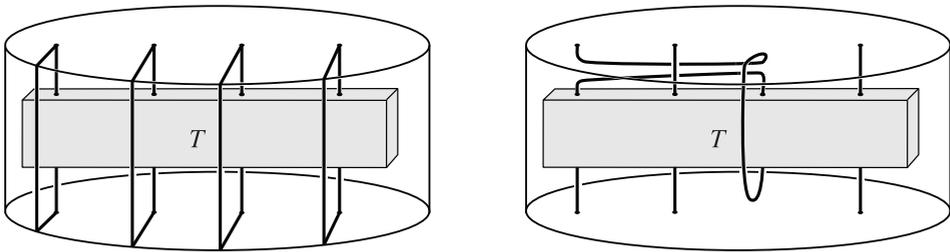

Figure 5. Left. The standard closure $L_\sigma \subset S^3$ of a string link $\sigma \subset B$ represented by a tangle T, using a standard embedding $B \subset S^3$. Right. An *internal band sum* between the first and third strands of σ (see Definition 22).

Proof of Corollary 3. The statement of Corollary 3 uses the following notions.

Definition 22. A string link σ is a *boundary string link* if the components of the standard closure L_σ bound disjoint surfaces in the 3-ball B; and a π_1-*null string link* is a string link σ whose standard closure L_σ bounds a surface Σ in the 4-ball $B^4 = B \times [0, 1]$ such that $\pi_0(L_\sigma) \to \pi_0(\Sigma)$ is a bijection and for which there is a push-off inducing the trivial homomorphism $\pi_1(\Sigma) \to \pi_1(B^4 \setminus \Sigma)$. Note that the

string link *Bing-doubling* operation shown in Figure 4 preserves boundary links: the standard closure of the string link in Figure 4 is designed to look like Figure 21 in [5], for which there are evident disjoint Seifert surfaces. Given a string link, we define an *internal band sum* between two of the components i, j as follows (see Figure 5). Add the boundary arc which makes the standard closure of component j, and push this closure into B slightly. This gives us a tangle where strand j is now a closed component. Choose an arc in the 3-ball B between the i and j strands, which will guide the band-sum, and do surgery along the chosen arc. In the case that the string link is a boundary string link, choose the arc to avoid the bounding surfaces as well as the strands. Then by construction the internal band sum will also be a boundary string link.

All of the four sets listed in the statement of Corollary 3 are certainly contained in the kernel of Artin_n^∞, since Milnor invariants vanish on all of these classes, and the order n Artin representation corresponds to the universal Milnor invariants of order n (length $n + 2$) as described in Section 3 of [14]. On the one hand, these sets are increasing in size, and on the other hand, the smallest set is equal to the kernel of Artin_n^∞ as in the proof of Theorem 2. □

3. Comparing filtrations of the group of homology cylinders

This section clarifies and proves Theorem 6 from the introduction, which significantly sharpens Levine's description of the relationship between the algebraically defined Johnson filtration and the geometrically defined filtration based on the Goussarov–Habiro theory of finite type 3-manifold invariants and clasper surgery. The central idea of the proof is to show that Habiro's realization map

$$\theta_{2k-1}\colon \mathcal{T}_{2k-1} \longrightarrow \mathsf{Y}_{2k-1}$$

in Theorem 23 below descends to a map $\tilde{\theta}_{2k-1}$ on the quotient $\widetilde{\mathcal{T}}_{2k-1}$ of $\mathcal{T}_{2k-1}$ by the framing relations, as stated in Proposition 9. We will recall the definition of $\widetilde{\mathcal{T}}_n$ in Definition 40, and prove Proposition 9 in Subsection 3.7. This also leads to the formulation of higher-order Arf invariants defined for certain homology cylinders (Remark 26) analogous to those defined in the Whitney tower filtrations on links.

3.1. Homology cylinders. We briefly introduce the group $\mathbb{HC}$ of homology cylinders modulo homology cobordism. For details, background and related developments the reader is referred to the recent survey [18] and its references.

Let $\Sigma_{g,1}$ denote the compact oriented surface of genus g with one boundary component. Following Garoufalidis and Levine [10, 21], we define a *homology cylinder* over $\Sigma_{g,1}$ to be a compact oriented 3-manifold M equipped with a homeomorphism

$$m\colon \partial(\Sigma_{g,1} \times [-1,1]) \longrightarrow \partial M$$

such that the two maps $m_{\pm} = m|_{\Sigma_{g,1} \times \pm 1}$, when composed with the inclusion $\partial M \subset M$, give an isomorphism on homology. This definition differs slightly from Habiro and Massuyeau [17], who reserve the term "homology cylinder" for those homology cylinders with a trivial action on $H_*(\Sigma_{g,1})$. Conveniently, the two definitions coincide for positive order in the Johnson filtration, which is all we will consider anyway. Two homology cylinders M_0 and M_1 are said to be *homology cobordant* if there is a compact oriented 4-manifold W with $\partial W = M_0 \cup_\partial (-M_1)$, such that the inclusions $M_i \hookrightarrow W$ are homology isomorphisms. This defines an equivalence relation on the set of homology cylinders. Let $\mathbb{HC}(g,1)$ be the set of homology cylinders up to homology cobordism over $\Sigma_{g,1}$. $\mathbb{HC}(g,1)$ is a group via the "stacking" operation ([10, 21]).

3.2. The Johnson filtration $\mathbb{J}_n$. Extending the Johnson homomorphisms on the mapping class group [19], Garoufalidis and Levine [10, 21] introduced an Artin-type representation $\mathbb{HC}(g,1) \to A_0(F/F_{n+1})$ where F is the free group on $2g$ generators, and $A_0(F/F_{n+1})$ is the group of automorphisms ϕ of F/F_{n+1} which lift to an endomorphism $\tilde{\phi}\colon F \to F$ which fixes the product $[x_1, y_1] \cdots [x_g, y_g]$ modulo F_{n+2}. Here $\{x_i, y_i\}_{i=1}^{g}$ is a standard symplectic basis for $\pi_1\Sigma_{g,1} \cong F$, and by benign abuse of notation we use the same letters for elements in F/F_{n+1}. The *Johnson filtration* (also known as the *relative weight* filtration) of $\mathbb{HC}(g,1)$ is defined by

$$\mathbb{J}_n := \mathrm{Ker}\{\mathbb{HC}(g,1) \longrightarrow A_0(F/F_{n+1})\},$$

and the associated graded is denoted $\mathsf{J}_n = \mathbb{J}_n/\mathbb{J}_{n+1}$. In [10] ([21, Corollary 2.1]), it is shown that $\mathsf{J}_n \cong \mathsf{D}_n$, where D_n is the kernel of the bracket map $\mathsf{L}_1 \otimes \mathsf{L}_{n+1} \to \mathsf{L}_{n+2}$ with L_k the degree k component of the free Lie algebra (over $\mathbb{Z}$) on the rank $2g$ free abelian group $F/F_2 \cong H_1(\Sigma_{g,1})$. (See also [12], Theorem 2.13 and Corollary 2.14.) The rank of the free abelian group D_n, which has appeared in quite a few different topological settings (see discussions in [6, 23]), is equal to the number of independent first non-vanishing Milnor invariants of order n (length $n+2$), as first shown by K. Orr [26].

3.3. The Goussarov–Habiro filtration $\mathbb{Y}_n$. On the other hand, there is a filtration of $\mathbb{HC}$ related to the Goussarov–Habiro theory of finite type 3-manifold invariants [9, 16]: Define the relation of n-equivalence (also known as A_n-equivalence or Y_n-equivalence) to be generated by the following move: $M \sim_n M'$ if M' is diffeomorphic to M_C, surgery on some connected graph clasper $C \subset M$ whose associated graph has at least n trivalent vertices. The *Y-filtration* of $\mathbb{HC}(g, 1)$ is defined by taking $\mathbb{Y}_n$ to be the subgroup of all homology cylinders n-equivalent to the trivial product $\Sigma_{g,1} \times I$, and we denote the associated graded by

$$\mathsf{Y}_n := \mathbb{Y}_n / \sim_{n+1} .$$

3.4. Levine's work. As Levine observes, $\mathbb{Y}_n \subset \mathbb{J}_n$, giving rise to a map of associated graded groups $\mathsf{Y}_n \to \mathsf{J}_n$. Rationally, Levine showed that this map is an isomorphism, and in fact both Y_n and J_n are rationally isomorphic with $\mathfrak{T}_n = \mathfrak{T}_n(2g)$. This is implied by the following theorem, proven in the pair of papers [21, 22], and stated in the introduction of [22]. It uses the surjective clasper surgery map $\theta_n \colon \mathfrak{T}_n \to \mathsf{Y}_n$ introduced by Habiro in [16], and further elucidated in [17] (Remarks 6.6 and 7.8).

Theorem 23 (Levine). *For $n > 1$, there is a commutative diagram*

$$\mathfrak{T}_n \xrightarrow[\eta_n]{\theta_n} \mathsf{Y}_n \longrightarrow \mathsf{J}_n \xrightarrow{\cong} \mathsf{D}_n .$$

Moreover, for all $n > 1$, η_n, and hence all of the maps, are rational isomorphisms.

Here, and throughout this section, the generators of $\mathfrak{T}_n$ have univalent vertices labeled by elements of $H_1(\Sigma_{g,1})$, with multi-linearity relations for labels included along with the IHX and antisymmetry relations. The realization map θ_n is defined by sending a tree to surgery on $\Sigma_{g,1} \times I$ along a clasper with the corresponding tree-type; and the combinatorially defined map η_n essentially sums over all choices of roots on order n trees, using the correspondence between rooted binary trees and non-associative bracketings. Note that terminology and notation varies throughout the literature. Translation from the notation of Levine's more recent work [21, 22, 23] to ours is as follows:

$$\mathcal{A}_n^t = \mathfrak{T}_n, \quad \mathcal{F}_n^w = \mathbb{J}_n, \quad \mathcal{G}_n^w = \mathsf{J}_n, \quad \mathcal{F}_n^Y = \mathbb{Y}_n, \quad \mathcal{G}_n^Y = \mathsf{Y}_n.$$

By linearity, the labels on univalent vertices of the generators of $\mathcal{T}_n$ can be assumed to be singletons from the set $\{x_i, y_i\}_{i=1}^g$; this corresponds to the geometric fact that clasper leaves can be "split" modulo higher-order equivalence.

The maps in Theorem 23 are not isomorphisms, and analyzing the above diagram over $\mathbb{Z}$ is the main subject of this section. With the same goal in mind, Levine was led to study [22, 23] the free *quasi*-Lie algebra L', in which the self-annihilation relation $[X, X] = 0$ in L is replaced by the weaker antisymmetry relation $[X, Y] = -[Y, X]$ in L'. In [22] Levine conjectures that a map

$$\eta_n' : \mathcal{T}_n \longrightarrow \mathsf{D}_n'$$

is an isomorphism, where D_n' is the kernel of the quasi-Lie bracket map $\mathsf{L}_1' \otimes \mathsf{L}_{n+1}' \to \mathsf{L}_{n+2}'$. This map is analogous to η_n in Theorem 23 above and we proved this "Levine conjecture" in [6]. Our analysis below is an application of this result.

Remark 24. It is interesting to observe that with this new notion of a quasi-Lie algebra, Conjecture 7 from the introduction implies an exact sequence for all $k \geq 1$:

$$0 \longrightarrow \mathbb{Z}_2 \otimes \mathsf{L}_{2k+2}' \longrightarrow \mathsf{Y}_{4k+1} \longrightarrow \mathsf{J}_{4k+1} \longrightarrow 0.$$

This uses the following split exact sequence from [23], where the left map sends X to $[X, X]$, and the right map is the natural projection

$$0 \to \mathbb{Z}_2 \otimes \mathsf{L}_n \longrightarrow \mathsf{L}_{2n}' \longrightarrow \mathsf{L}_{2n} \longrightarrow 0.$$

Notice that Conjecture 7 would not be correct in the $k = 0$ case: Habiro showed that there is a non-canonical isomorphism

$$\mathsf{K}_1^{\mathsf{Y}} \cong (\mathbb{Z}_2 \otimes \mathsf{L}_2') \oplus \mathbb{Z}_2,$$

which has an extra $\mathbb{Z}_2$-summand.

3.5. Proof of Theorem 6. Recall from the statement of Theorem 6 that $\mathsf{K}_n^{\mathsf{Y}}$ is the kernel of the map on the associated graded groups $\mathsf{Y}_n \to \mathsf{J}_n$ induced from the inclusion $\mathbb{Y}_n \subset \mathbb{J}_n$. We will need a homomorphism

$$\kappa_{2k-1} : \mathsf{K}_{2k-1}^{\mathsf{Y}} \longrightarrow \mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$$

defined as follows. Suppose $[M] \in \mathsf{K}_{2k-1}^{\mathsf{Y}}$ is an equivalence class of homology cylinders up to $\sim_{2k}$. Then $M \in \mathbb{J}_{2k}$. So M determines an element of $\mathsf{J}_{2k} \cong \mathsf{D}_{2k}$. Applying the right-hand map in the top sequence of Theorem 5 yields an element

of $\mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$. This does not depend on the choice of representative, for if $M' \sim_{2k} M$, then $M' \cdot M^{-1} \in \mathbb{Y}_{2k}$ and $[M' \cdot M^{-1}] \mapsto 0 \in \mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$ by the exactness of the top sequence in Theorem 5. So M and M' both map to the same element of $\mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$.

The following result will be proven in Section 3.8.

Proposition 25. *The homomorphism* $\kappa_{2k-1} \colon \mathsf{K}^{\mathsf{Y}}_{2k-1} \to \mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$ *is surjective.*

Remark 26. Regardless of whether or not Conjecture 7 is true, a_{k+1} induces an injection $\bar{a}_{k+1}$ on $(\mathbb{Z}_2 \otimes \mathsf{L}_{k+1})/\operatorname{Ker} a_{k+1}$, and in direct analogy with the higher-order Arf invariants defined for links bounding certain Whitney towers [3], one can define higher-order Arf invariants for homology cylinders representing elements of $\mathsf{K}^{\mathsf{Y}}_{4k+1}$ by inverting $\bar{a}_{k+1}$. This analogy with [3] can be further extended by defining higher-order Sato–Levine invariants for homology cylinders using the homomorphism κ_{2k-1}.

Proof of Theorem 6 (*order* $4k - 1$ *case*). We claim there is a commutative diagram of short exact sequences:

$$
\begin{array}{ccccc}
\mathbb{Z}_2 \otimes \mathsf{L}_{2k+1} & \rightarrowtail & \widetilde{\mathcal{T}}_{4k-1} & \xrightarrow{\tilde{\eta}} & \mathsf{D}_{4k-1} \\
\downarrow & & \downarrow {\scriptstyle \tilde{\theta}_{4k-1}} & & \downarrow {\scriptstyle \cong} \\
\mathsf{K}^{\mathsf{Y}}_{4k-1} & \rightarrowtail & \mathsf{Y}_{4k-1} & \twoheadrightarrow & \mathsf{J}_{4k-1}
\end{array}
$$

The existence of $\tilde{\theta}_{4k-1}$ follows from Proposition 9 (see Subsection 3.7 below). We already know from Theorem 23 that the right-hand square commutes when $\tilde{\theta}_{4k-1}$ and $\widetilde{\mathcal{T}}_{4k-1}$ are replaced by θ_{4k-1} and $\mathcal{T}_{4k-1}$, which implies the right-hand square commutes in the present context. The exactness of the top sequence follows by combining Theorems 5.1 and 6.5(ii) of [4], using that $\mathsf{L}_{2k+1} = \mathsf{L}'_{2k+1}$. The surjection $\tilde{\theta}_{4k-1}$ restricts to a surjection

$$\mathbb{Z}_2 \otimes \mathsf{L}_{2k+1} \twoheadrightarrow \mathsf{K}^{\mathsf{Y}}_{4k-1}.$$

By Proposition 25, there is also a surjection in the opposite direction. Since these are $\mathbb{Z}_2$-vector spaces, this is sufficient to conclude that $\mathbb{Z}_2 \otimes \mathsf{L}_{2k+1} \to \mathsf{K}^{\mathsf{Y}}_{4k-1}$ is an isomorphism. By the 5-lemma, we can also conclude that the middle map is an isomorphism. This completes the analysis of the order $4k - 1$ case. $\qquad\square$

Proof of Theorem 6 (*order* $4k + 1$ *case*). We again have a commutative diagram of exact sequences

$$
\begin{array}{ccccc}
\mathbb{Z}_2 \otimes \mathsf{L}'_{2k+2} & \rightarrowtail & \widetilde{\mathcal{T}}_{4k+1} & \xrightarrow{\ \tilde{\eta}\ } & \mathsf{D}_{4k+1} \\
\downarrow & & \downarrow{\scriptstyle \tilde{\theta}_{4k+1}} & & \downarrow{\scriptstyle \cong} \\
\mathsf{K}^{\mathsf{Y}}_{4k+1} & \rightarrowtail & \mathsf{Y}_{4k+1} & \twoheadrightarrow & \mathsf{J}_{4k+1}
\end{array}
$$

The $4k + 1$ case would follow by the same argument as for $4k - 1$, except that κ_{4k+1} maps to $\mathsf{L}_{2k+2} \neq \mathsf{L}'_{2k+2}$, so we can no longer conclude that the left-hand map is an isomorphism. However, from the following commutative diagram it is clear that $\mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$ surjects onto $\mathrm{Ker}(\kappa_{4k+1})$:

$$
\begin{array}{ccccc}
\mathbb{Z}_2 \otimes \mathsf{L}_{k+1} & \rightarrowtail & \mathbb{Z}_2 \otimes \mathsf{L}'_{2k+2} & \longrightarrow & \mathbb{Z}_2 \otimes \mathsf{L}_{2k+2} \\
 & & \downarrow & \nearrow{\scriptstyle \kappa_{4k+1}} & \\
 & & \mathsf{K}^{\mathsf{Y}}_{4k+1} & &
\end{array}
$$

The top line of this diagram comes from the second exact sequence in Remark 24 of Subsection 3.4 above. The fact that this diagram commutes will be explained in Remark 43 during the proof of Proposition 25 in Subsection 3.8 below. $\square$

3.6. Clasper conventions and notations.

For details on *claspers* (also called *clovers*) see [16, 9] as well as [7]. Given a tree clasper with n nodes embedded in a homology cylinder, there is an associated element of $\mathcal{T}_n$ defined by labeling the univalent vertices of the underlying vertex-oriented tree by the homology classes determined by the clasper leaves, which are canonically identified with elements of $H_1(\Sigma_{g,1}; \mathbb{Z})$. The *order* of a tree clasper is the order of its associated tree. All claspers will be assumed to be of order at least 1, unless otherwise specified. Surgery on a clasper $C \subset M$ is denoted by M_C, and for $C' \subset M \setminus C$, we denote by C'_C the image of C' in M_C.

Rooted trees will usually be denoted by capital letters, and unrooted trees will usually be denoted by t. Lower case letters labeling univalent vertices represent homology elements and capital letters represent sub-trees. Choosing a preferred leaf of a clasper determines an associated rooted tree.

We will need the *zip construction* as formulated in section 4.2 of [7]: Given a clasper C, one can cut one of its leaves along a framed arc into two framed embedded loops, yielding two daughter claspers C_1 and C_2, where C_1 is exactly C with one loop of the leaf, and C_2 has leaves parallel to those of C_1, except

where it uses the other loop of the leaf, and its edges and vertices lie in a regular neighborhood of C_1 avoiding any caps it might have (a *cap* is a framed embedded disk bounded by a leaf).

3.7. Proof of Proposition 9. Habiro's surjective realization map $\theta_n \colon \mathcal{T}_n \twoheadrightarrow \mathsf{Y}_n$ is defined on a tree t by choosing a clasper C in the identity homology cylinder $\Sigma_{g,1} \times I$ whose associated tree is t. One shows this is well-defined by proving that any two such embeddings of a clasper C are related by $(> n)$-equivalences and homology bordisms, and that the IHX and AS relations are realized geometrically (this argument will be presented and extended in this subsection).

To prove Proposition 9 (stating that θ_{2k-1} descends to $\tilde{\theta}_{2k-1} \colon \widetilde{\mathcal{T}}_{2k-1} \twoheadrightarrow \mathsf{Y}_{2k-1}$) we introduce the notion of a *twisted clasper* (Definition 31 below), which is required to have one leaf that is a $+1$-framed unknot, and show that it actually represents a $(2k-1)$-equivalence up to homology bordism, when it has k nodes, i.e. when its associated "twisted" ∞-tree is of order k (Definition 32). We will extend Habiro's result to show that the homology cylinders resulting from surgery on any two twisted claspers corresponding to the same ∞-tree are $2k$-equivalent up to homology bordism.

Lemma 27. *Suppose M is a 3-manifold, possibly with boundary, and C is a tree clasper embedded in M such that a leaf ℓ of C bounds an embedded disk, consistent with the framing of ℓ, which possibly intersects the rest of C. Then M is homology bordant to M_C.*

Proof. This is similar to Theorem 2 of [21] and the the proof given there works with very little modification. Surgery on the clasper C is defined as surgery on a corresponding link L, and by lemmata 2.5 and 2.6 of [21], to show that surgery on C is a homology bordism, it suffices to decompose L as a union $L' \cup L''$ where L' is a trivial 0-framed link, and the matrix of linking numbers $\lambda(L'_i, L''_j)$ between the components of L' and L'' is non-singular. Orient the edges of the clasper so that every leaf but ℓ is at the end of the arrow, ℓ is the source of an arrow, and no trivalent vertex is a source (as in Lemma 2.6 of [21]). Let L'' be the link components at the end of the arrows, and let L' be the link components at the beginning of arrows. Then the linking matrix between L' and L'' is the identity matrix with one row of possibly non-zero off-diagonal elements, so it is invertible. $\square$

It follows from Lemma 27 that the relation of n-equivalence on $\mathbb{HC}(g, 1)$ is generated by clasper surgery on *tree* claspers of order (at least) n, as observed in [21]. The relationship between the choice of embedding of a clasper representing a tree-type and the resulting surgery is described by the following lemmata.

Lemma 28. *Let $C_1 \cup C_2 \subset M$ be an embedding of two tree claspers, of orders n_1 and n_2 respectively, in a homology cylinder M. Suppose $C_1' \cup C_2' \subset M$ is an embedding that differs from $C_1 \cup C_2 \subset M$ by*

(i) *crossing a leaf of C_1 through a leaf of C_2, then*

$$M_{C_1 \cup C_2} \sim_{n_1 + n_2} M_{C_1' \cup C_2'};$$

(ii) *crossing a leaf of C_1 through an edge of C_2, then*

$$M_{C_1 \cup C_2} \sim_{n_1 + n_2 + 1} M_{C_1' \cup C_2'};$$

(iii) *crossing an edge of C_1 through an edge of C_2, then*

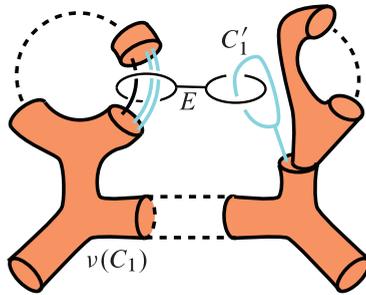$$M_{C_1 \cup C_2} \sim_{n_1 + n_2 + 2} M_{C_1' \cup C_2'}.$$

Proof. This is standard clasper calculus. It follows by the zip construction and Figure 31 of [16]. □

Lemma 29. *Suppose C_1 and C_2 are order k tree claspers embedded in a homology cylinder M, such that C_2 is formed from C_1 by a finger move that pushes an arc of a leaf of C_1 across a leaf of C_1. Then $M_{C_1} \sim_{2k} M_{C_2}$ up to homology bordism.*
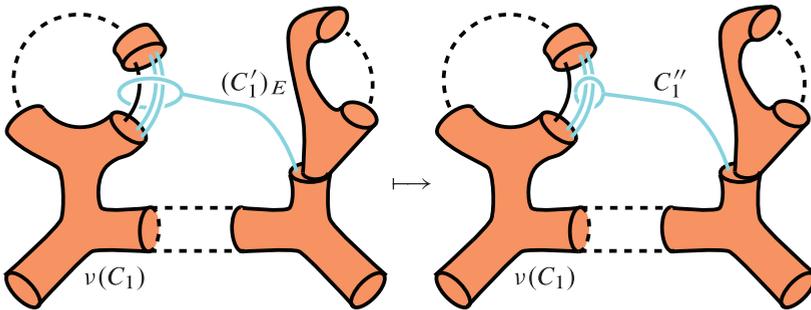

Proof. Assume that the crossing is between two different leaves. (The case when it is a self-crossing of a leaf is similar.) As pictured above, the finger move can be realized by surgery on a clasper E, with one edge and two leaves, where each leaf of E is a meridian to one of the two leaves of C_1 involved in the finger move. In other words, $C_2 = (C_1)_E$. Now use the zip construction of [7] to cut one of the

leaves of C_1 so that one piece forms a meridian to the E leaf. This gives us two claspers: the original C_1 together with C_1', where C_1' has edges that wind through a regular neighborhood $\nu(C_1)$ of C_1 in a complicated way, as in the following picture:

In the preceding picture, C_1' is depicted in blue and the regular neighborhood of C_1 is depicted in orange. Near the left-hand E-leaf, we have one (black) strand of a leaf of C_1, and several parallel (blue) strands which represent a leaf and possibly many edges of C_1'.

On the left-hand side of the following picture we depict what happens after surgering along the E-clasper:

The above-pictured operation of pulling the leaf of $(C_1')_E$ across the indicated leaf of C_1 corresponds to a $2k$-equivalence, by Lemma 28. And by Lemma 27, clasper surgery on $C_1 \cup C_1''$ in the above picture on the right is homology bordant to surgery on C_1, completing the proof. $\qquad\square$

Lemma 30. *Let C_1 and C_2 be order k tree claspers embedded in a homology cylinder M such that C_2 differs from C_1 by a finger move of an edge of C_1 through a leaf of C_1, or by a finger move of an edge of C_1 though an edge of C_1. Then $M_{C_1} \sim_{2k} M_{C_2}$ up to homology bordism.*

Proof. Any clasper homotopies of this form can be realized by leaf-leaf crossings, which are covered by Lemma 29. To see this note that a tree clasper is the union of a tree and some circles (the leaves). Fixing a handlebody neighborhood of the clasper, the tree can be contracted by an isotopy to be in an arbitrarily small ball, pulling parts of the leaves toward this ball, and fixing the complement of the handlebody. The homotopies in the lemma are realized by crossing the handlebody through itself. Since all of the edges are in a small ball, this translates into multiple crossings of the leaves.   □

In analogy to twisted Whitney towers we introduce twisted claspers. Notice that in a homology cylinder M, every (simple) closed curve has a canonical framing: the curve is homologous into ∂M, and this homology determines a parallel push-off of the curve.

Definition 31. A *twisted clasper of order* $2k - 1$ is an order k tree clasper C embedded in a homology cylinder M, such that one leaf of C is a $+1$-framed unknot bounding a disk whose interior is disjoint from C, and all other leaves have the canonical framing.

Here "$+1$-framed" means that the canonical parallel push-off of the unknotted leaf has $+1$ intersection with the disk bounded by the leaf.

Definition 32. The ∞-tree associated to a twisted order $2k - 1$ clasper C is gotten from the usual order k tree by labeling the univalent vertex which corresponds to the $+1$-framed leaf of C by the symbol ∞ (this ∞ label replaces the trivial homology element that would usually label such a vertex). Such a tree is called an ∞-tree, and for any rooted tree J the notation J^{∞} denotes the ∞-tree gotten by labeling the root of J by ∞.

In the next lemma we see that defining the order of a twisted clasper in this way is justified, as surgery along it is an equivalence of the correct order in $\mathbb{HC}(g, 1)$.

Lemma 33. *Let C be a twisted clasper of order $2k - 1$ embedded in a homology cylinder M. Then*

$$M \sim_{2k-1} M_C$$

up to homology bordism.

Proof. By geometric IHX relations [2, 8] (which preserve framings) one can write

$$M_C = M_{C_1 \cup \cdots \cup C_r},$$

where each C_i is a clasper of order k with a $+1$-framed unknotted leaf adjacent to a trivalent vertex which is adjacent to another leaf. The $+1$-framed unknotted leaf of each C_i bounds an embedded disk which intersects the other claspers and may even have interior intersections with C_i. Using Lemma 28, the intersections between this disk and the other claspers may be removed, up to $2k$-equivalence. Using Lemma 29 and Lemma 30, interior intersections of this disk with C_i can also be removed, up to $2k$-equivalence and homology bordism.

So it suffices to show the lemma is true for any twisted clasper where the $+1$-framed unknotted leaf is next to another leaf. By inserting a Hopf link into an edge (Figure 14 of [7]), any such clasper of the form $J \prec_h^\infty$ can be converted (without changing the result of surgery) into an order one Y-clasper with one twisted leaf, one leaf of homology class h, and one leaf linking a clasper C_J with rooted tree J, as depicted in Figure 6A. Sliding the twisted leaf in the Y-clasper over the leaf linking C_J results in two parallel leaves linking the C_J as depicted in Figure 6B.

This twisted-leaf-sliding move is pictured in the proof of Lemma 4.9 of [9], and in the proof of Theorem 3.1 of [9] it is explained why the twisting gets killed (this is a 3-dimensional analog of the 4-dimensional *boundary-twist* maneuver on a Whitney disk). Next apply a zip construction [7, 16] to the leaf of C_J, splitting apart the parallel leaves of the Y-clasper, and turning C_J into claspers C_J and C_J' with parallel leaves and the same tree-type. The edges of C_J' are contained in a neighborhood of C_J (Figure 6C). Then combine C_J' and the Y-clasper by converting the right-hand Hopf link into an edge of an order k clasper C''. The left-hand leaf of C'' links the root leaf of C_J and also has strands of C'' passing through it (Figure 6D). By Lemma 30 these strands of C'' can be pushed out by a $2k$-equivalence up to homology cobordism, so that the leaf forms a clean Hopf link pair with the root leaf of C_J (Figure 6E). Finally, combining C_J and C'' by replacing their Hopf-linked leaves with an edge forms an order $2k - 1$ clasper of the form $h \prec_J^J$, which represents a $2k - 1$-equivalence (Figure 6F). □

Lemma 34. *Let C_1 be a clasper in a homology cylinder M which is either of order $2k - 1$ or is twisted of order $2k - 1$. Suppose C_2 differs from C_1 by band-summing an edge e into a simple closed curve γ in M. Then $M_{C_1} \sim_{2k} M_{C_2}$, up to homology bordism.*
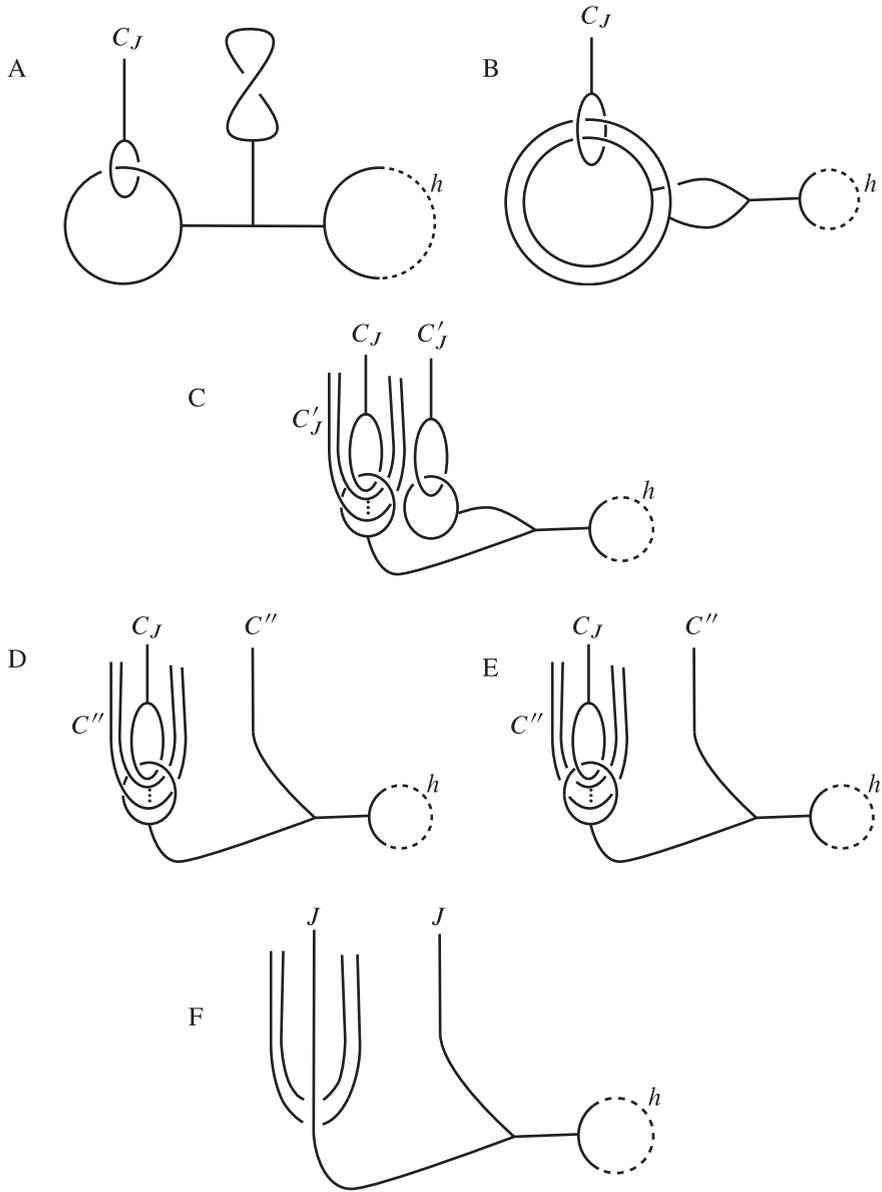
Figure 6. From the proof of Lemma 33.

Proof. This move is realized by surgery on an order 0 clasper one of whose leaves is a meridian to the edge, and the other is a copy of γ, as depicted in Figure 7A. By clasper calculus (see [16], Figure 30), we can write $M_{C_2} = M_{C_1 \cup C'}$, where C' has order $k + 1$ and the tree of C' is gotten from that of C_1 by attaching a new edge to an interior point in e (with the leaf of the new edge given by γ). See Figure 7B.
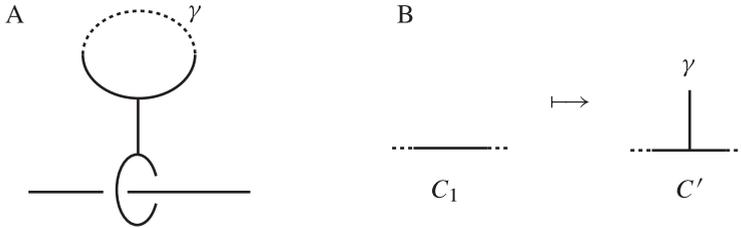

Figure 7. From the proof of Lemma 34.

We may arrange that C_1 sits in a regular neighborhood of C', which is the clean daughter clasper in the zip construction. In the case that C_1 is of order $2k - 1$, we are done, since C' is of order $2k$, so now assume that C_1 is twisted of order $2k - 1$. A leaf and edges of C' link the twisted leaf of C_1 nontrivially, but by Lemma 28 we can "disentangle" the twisted leaf of C' from the clasper C_1 by a $2k + 1$-equivalence so that C' has a clean twisted leaf. Thus C' has been modified to be twisted of order $2k+1$. So modulo $\sim_{2k}$ we can disregard C', since it determines a $2k + 1$-equivalence by Lemma 33, and we have $M_{C_2} \sim_{2k} M_{C_1}$. □

The following is a version of Lemma 28 for twisted claspers.

Lemma 35. *Let $C_1 \cup C_2$ be an embedding of two twisted tree claspers in a homology cylinder M, each of order $2k - 1$. Suppose $C_1' \cup C_2'$ is an embedding that differs from $C_1 \cup C_2$ by crossing a strand of a leaf of C_1 across a leaf of C_2. Then*

$$M_{C_1 \cup C_2} \sim_{2k} M_{C_1' \cup C_2'}.$$

Proof. Each of the twisted claspers C_1 and C_2 can be thought of as a clasper of order k. Then Lemma 28 proves that $M_{C_1 \cup C_2} \sim_{2k} M_{C_1' \cup C_2'}$. □

We also need a version of Lemma 29 for twisted claspers.

Lemma 36. *Let C_1 be a twisted tree clasper of order $2k - 1$ embedded in a homology cylinder M. If C_2 is any twisted tree clasper formed from C_1 by a finger move that pushes an arc of a leaf of C_1 across another leaf of C_1, then $M_{C_1} \sim_{2k} M_{C_2}$ up to homology bordism.*

Proof. We trace through the proof of Lemma 29. The only step of the proof that was not a diffeomorphism or a homology bordism is the step at the end of the proof where a leaf of $(C_1')_E$ is pushed across a leaf of C_1, which is a $2k$-equivalence by Lemma 35. $\square$

Lemma 37. *Let C_1 and C_2 be twisted order $2k - 1$ tree claspers embedded in a homology cylinder M which differ by a finger move of an edge through a leaf of C_1 or by a finger move of an edge though an edge of C_1. Then*

$$M_{C_1} \sim_{2k} M_{C_2}$$

up to homology bordism.

Proof. As in the proof of Lemma 30, this follows from the leaf-leaf case. $\square$

Lemma 38. *Let C_1 be a clasper in a homology cylinder M which is either of order $2k - 1$ or is twisted of order $2k - 1$. Suppose C_2 differs from C_1 by band-summing a leaf into a null-homologous simple closed curve in M. Then*

$$M_{C_1} \sim_{2k} M_{C_2}$$

up to homology bordism.

Proof. The curve we are banding into is null-homologous and so it bounds an immersed surface S, which may intersect C_2. This surface has a 1-dimensional spine, and by contracting S toward the spine we may take it to be embedded and disjoint from the rest of C_2. This contraction homotopy will introduce crossings of the part of the leaf of C_2 lying in ∂S with the rest of C_2 and with itself. These are $2k$-equivalences up to homology bordism, by Lemmata 28, 29, 35, and 36. A zip construction splitting the leaf yields the original clasper C_1 together with a clasper C', contained in a neighborhood of C_1 except for one leaf that bounds the embedded surface S. If C_1 is of order $2k - 1$ then S can be used to turn C' into a clasper of order $2k$ (using the grope-to-clasper construction). On the other hand if C_1 is twisted of order $2k - 1$, the twisted leaves of C_1 and C' will interlink, but can be separated by a $2k$-equivalence up to homology bordism by Lemma 28. Furthermore any edges of C_1 or C' that link with the twisted leaf of C' can be pushed out of that leaf by Lemmata 28 and 29, so that C' becomes twisted of order $2k + 1$, so can be ignored as it represents a $2k + 1$-equivalence up to homology bordism by Lemma 33. $\square$

Lemma 39. *If C and C' are two (twisted) claspers of order $2k - 1$ embedded in a homology cylinder M, such that C and C' have the same tree type, then $M_C \sim_{2k} M_{C'}$ up to homology bordism.*

Proof. First consider the untwisted case. Two such embeddings $C \subset M$ and $C' \subset M$ differ, up to framing, by a sequence of crossings, band-sums of edges into nontrivial curves and band-sums of leaves into null-homologous curves, all of which are $2k$-equivalences and homology bordisms by Lemmata 29, 30, 34, and 38. By introducing crossings of a leaf with itself (Lemma 29), one can change the framing by multiples of 2. We can change the leaf framings in this way to be either 0 or $+1$. In the latter case, by the zip construction, one can split the leaf into a 0-framed leaf and a $+1$-framed unknot. Any clasper of (untwisted) order $2k - 1$ with a $+1$-framed unknotted leaf represents a higher-order $(4k - 3)$-equivalence up to homology cobordism by Lemma 33 (which also holds if more than one leaf is twisted). Thus we may assume all leaves are canonically framed.

In the twisted case, all framings on the leaves of C and C' are assumed to agree, and we can again move the two claspers into the same position by a sequence of crossings, band-sums of edges into nontrivial curves and band-sums of leaves into null-homologous curves, all of which are $2k$-equivalences up to homology bordism. □

Definition 40 ([4]). The group $\widetilde{\mathcal{T}}_{2k-1}$ is defined as $\mathcal{T}_{2k-1}/ \operatorname{Im} \Delta_{2k-1}$, where the homomorphism
$$\Delta_{2k-1} \colon \mathbb{Z}_2 \otimes \mathcal{T}_{k-1} \longrightarrow \mathcal{T}_{2k-1}$$
is defined as follows. Suppose t is a generator of $\mathcal{T}_{k-1}$. For any univalent vertex v of t, denote by $\ell(v)$ the label of that vertex, and denote by $T_v(t)$ the rooted tree obtained from t by thinking of v as the (unlabeled) root. With these notations
$$\Delta_{2k-1}(t) = \sum_v \ell(v) \prec^{T_v(t)}_{T_v(t)} .$$

The following presentation of $\widetilde{\mathcal{T}}_{2k-1}$ will be used to complete the proof of Proposition 9.

Lemma 41. *The group $\widetilde{\mathcal{T}}_{2k-1}$ is isomorphic to the abelian group generated by order $2k - 1$ (non-∞) trees and order k ∞-trees, with the following relations*:

 (i) *IHX and AS relations among order $2k - 1$ trees*;

 (ii) *IHX relations among ∞-trees*;

(iii) *Boundary twist relations: $h \prec^\infty_J = h \prec^J_J$, where h is a tree of order 0.*

Proof. Let $\overline{\mathfrak{T}}_{2k-1}$ be the group defined by the above presentation. Define a map $\widetilde{\mathfrak{T}}_{2k-1} \to \overline{\mathfrak{T}}_{2k-1}$ by sending each order $2k-1$ generator to itself. Clearly this vanishes on IHX and AS relators, since these are also present in the target group. A framing relator $\Delta(t)$ is sent to the sum of attaching an ∞-labeled edge to the edge adjacent to every univalent vertex of t, using the boundary twist relations in the target group. This sum is zero as a consequence of IHX relations on ∞-trees, so the map is indeed well-defined. On the other hand, we define a map $\overline{\mathfrak{T}}_{2k-1} \to \widetilde{\mathfrak{T}}_{2k-1}$ back the other way as follows. Send each order $2k-1$ generator to itself, and send each ∞-tree, $\infty\!\!-\!\!<_K^J$, to the sum

$$\sum_{v \in J} \ell(v) \!-\!\!<_{T_v(J-K)}^{T_v(J-K)}$$

where we sum over all leaves of J in the tree $J - K$. This is well-defined, since doing this same construction with K differs by $\Delta(J - K)$ from doing it with J. This map clearly vanishes on IHX and AS relators for order $2k-1$ trees, as well as $2J^\infty$ relators. To see that it vanishes on IHX relators for ∞-trees, notice that these are generated by IHX relators on ∞-trees where the ∞ is adjacent to a leaf: $\infty\!\!-\!\!<_k^I - \infty\!\!-\!\!<_k^H + \infty\!\!-\!\!<_k^X$. This maps to $\sum_v k\!-\!\!<_{T_v(I-H+X)}^{T_v(I-H+X)}$, which vanishes by IHX relations for non-∞ trees. The IHX relators involving the ∞-edge also map to 0, as do the relators corresponding to the boundary twist relations. $\square$

Define the map

$$\tilde{\theta}_{2k-1} \colon \widetilde{\mathfrak{T}}_{2k-1} \longrightarrow \mathsf{Y}_{2k-1}$$

on generators by sending each tree to $(\Sigma_{g,1} \times I)_C$, where C is any clasper realizing that tree type, both in the twisted and untwisted cases.

Lemma 42. *The homomorphism $\tilde{\theta}_{2k-1} \colon \widetilde{\mathfrak{T}}_{2k-1} \to \mathsf{Y}_{2k-1}$ is well-defined.*

Proof. The definition of $\tilde{\theta}_{2k-1}$ does not depend on the choices of embeddings of claspers by Lemma 39. So it suffices to check that $\tilde{\theta}_{2k-1}$ respects the relations in Lemma 41. The fact that $\tilde{\theta}_{2k-1}$ vanishes on IHX relators follows because all IHX relations can be realized geometrically and the claspers can be pulled apart modulo order $2k$ clasper surgeries, as we mentioned in the proof of Lemma 33. The boundary twist relations were also shown in that proof. Antisymmetry relations follow from Figure 45 of [16] or Corollary 4.6 of [9]. This implies that if C is a clasper, then there is a clasper $\widetilde{C}$ in the complement of C of the same tree type, except that a half-twist has been introduced into an edge, such that $M_{C \cup \widetilde{C}} \cong M$. These two framings correspond to oppositely signed trees in $\widetilde{\mathfrak{T}}_{2k-1}$.

The boundary twist relation was proven explicitly in the argument for Lemma 33. $\square$

The proof of Proposition 9 is completed by the observation that the surjectivity of $\tilde{\theta}_{2k-1}$ stated in Proposition 9 follows since all relations are realized in target, and the original θ-map was onto.

3.8. Proof of Proposition 25. Recall that Proposition 25 asserts that the homomorphism

$$\kappa_{2k-1} \colon \mathsf{K}^{\mathsf{Y}}_{2k-1} \longrightarrow \mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$$

is surjective, where for

$$[M] \in \mathsf{K}^{\mathsf{Y}}_{2k-1} := \mathrm{Ker}\{\mathsf{Y}_{2k-1} \to \mathsf{J}_{2k-1}\}, \kappa_{2k-1}([M])$$

is defined by mapping $[M] \in \mathsf{J}_{2k} \cong \mathsf{D}_{2k}$ into $\mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$ via the right-hand map in the top sequence of Theorem 5. This latter map

$$\mathrm{sl}_{2k} \colon \mathsf{D}_{2k} \longrightarrow \mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$$

is defined by applying the snake lemma to the following diagram (see [4, Definition 5.6]):

$$
\begin{array}{ccccc}
 & & & & \mathbb{Z}_2 \otimes \mathsf{L}_{k+1} \\
 & & & & \downarrow {\scriptstyle \mathrm{sq}} \\
\mathsf{D}'_{2k} & \longrightarrow & \mathsf{L}_1 \otimes \mathsf{L}'_{2k+1} & \longrightarrow & \mathsf{L}'_{2k+2} \\
\downarrow & & \downarrow {\scriptstyle \cong} & & \downarrow \\
\mathsf{D}_{2k} & \longrightarrow & \mathsf{L}_1 \otimes \mathsf{L}_{2k+1} & \longrightarrow & \mathsf{L}_{2k+2} \\
\downarrow {\scriptstyle \mathrm{sl}_{2k}} & & & & \\
\mathbb{Z}_2 \otimes \mathsf{L}_{k+1} & & & &
\end{array}
$$

The two horizontal sequences are exact by definition and the vertical sequence on the right is exact by Theorem 2.2 of [23]. The squaring map on the upper right is

$$\mathrm{sq}(1 \otimes X) := [X, X].$$

Let J be any rooted tree of order k. Then J represents an element in $\mathbb{Z}_2 \otimes \mathsf{L}_{k+1}$, and J^∞ represents a generator in $\widetilde{\mathcal{T}}_{2k-1}$. To prove Proposition 25 we will first show that $\tilde{\theta}_{2k-1}(J^\infty) \in \mathsf{K}^{\mathsf{Y}}_{2k-1}$, and then check that that $\kappa_{2k-1}(\tilde{\theta}_{2k-1}(J^\infty)) = J$.

First we describe $\tilde{\theta}_{2k-1}(J^\infty)$ geometrically. Consider an embedded twisted clasper $C \subset \Sigma_{g,1} \times I$ which has the tree type of J^∞, and assume that all the non-twisted leaves are parallel copies of elements from a symplectic basis $\{x_i, y_i\}_{i=1}^g$ for $H_1(\Sigma_{g,1}; \mathbb{Z})$, with the twisted leaf being a clean $+1$ framed unknot (so each univalent label on J^∞ is an x_j or y_j except for the ∞-labeled vertex).

Then $\tilde{\theta}_{2k-1}(J^\infty) = [(\Sigma_{g,1} \times I)_C]$, and we want to geometrically understand the element of D_{2k} determined by $(\Sigma_{g,1} \times I)_C$.

Recall from [10, 21, 22] that $\mathbb{J}_n$ is defined to be the kernel of the map

$$\mathbb{HC}(g, 1) \longrightarrow A_0(F/F_{n+1})$$

induced by the composition $(\iota^-)^{-1} \circ \iota^+$, with F the free group on $\{x_i, y_i\}_{i=1}^g$. There is a short exact sequence

$$0 \longrightarrow \mathsf{D}_n \longrightarrow A_0(F/F_{n+2}) \longrightarrow A_0(F/F_{n+1}) \longrightarrow 0,$$

which implies that $\mathbb{J}_n \cong \mathsf{D}_n$. To read off an element in D_n for a given homology cylinder representing an element in $\mathbb{J}_n$, consider the corresponding automorphism f of F/F_{n+2} which by hypothesis acts trivially modulo F_{n+1}. For each $x \in F/F_{n+2}$, we have $f(x) = x\psi(x)$, where $\psi(x) \in F_{n+1}/F_{n+2} \cong \mathsf{L}_{n+1}$. Via the isomorphism $\mathrm{Hom}(H, \mathsf{L}_{n+1}) \cong H \otimes \mathsf{L}_{n+1}$ with $\{x_i, y_i\}_{i=1}^g$ as a basis for $H := F/F_2$, f is described by $\sum_i x_i \otimes \psi(y_i) - y_i \otimes \psi(x_i)$ which is an element of D_n (see the proof of Proposition 2.5 in [10]).

We want to calculate this element in the case of $(\Sigma_{g,1} \times I)_C$. The value $f(x_i)$ can be calculated by homotoping (a circle representing) x_i from $\Sigma_{g,1} \times \{0\}$ to $\Sigma_{g,1} \times \{1\}$, and such a homotopy can be arranged so that x_i intersects once with each leaf of C which is labeled by the dual element y_i, and misses all other leaves of C. Each time x_i pushes through such a y_i-labeled leaf ℓ_{y_i} of C the element $\psi(x_i)$ picks up a meridian of the clasper, which by Lemma 44 and Lemma 45 below corresponds to the iterated commutator determined by putting a root on the doubled tree $J \!-\! J$ at the univalent vertex corresponding to ℓ_{y_i} in one of the J sub-trees. So $\psi(x_i)$ is a product of such commutators (which we write as a sum of brackets in the abelian group $F_{2k+1}/F_{2k+2} \cong \mathsf{L}_{2k+1}$), with the terms of the product corresponding to the iterated commutators determined by putting roots at all the y_i-labeled vertices in one J-subtree of $J \!-\! J$. A similar discussion holds for $\psi(y_i)$ except for a minus sign coming from the identification of H with its dual H^* via the symplectic form ($x_i \mapsto -y_i^*$ and $y_i \mapsto x_i^*$). The upshot is that the element $\sum_i x_i \otimes \psi(y_i) - y_i \otimes \psi(x_i)$ is equal to

$$\sum_{v \in J} \ell(v) \otimes (J \!-\! J)_v$$

where the sum is over all univalent vertices v in one J-subtree of $J \!-\! J$, and $(J \!-\! J)_v$ denotes the element of L_{2k+1} determined by putting a root at v. This sum is equal to $\frac{1}{2}\eta_{2k}(J \!-\! J)$, where $\eta_{2k} : \mathcal{T}_{2k} \to \mathsf{D}_{2k}$ is the map in equation (1) of [22]. Since the coefficients of $\eta_{2k}(J \!-\! J)$ are even and D_{2k} is torsion-free, we conclude that $\sum_i x_i \otimes \psi(y_i) - y_i \otimes \psi(x_i)$ lies in D_{2k}.

So $[(\Sigma_{g,1} \times I)_C]$ maps to $\frac{1}{2}\eta(J \!-\! J) \in \mathsf{D}_{2k}$, and in particular is an element of $\mathsf{K}^{\mathsf{Y}}_{2k-1}$.

Tracing through the application of the snake lemma in the definition of sl_{2k} above shows that $\mathrm{sl}_{2k}(\frac{1}{2}\eta(J \!-\! J)) = J$, so κ_{2k-1} is surjective. $\qquad\square$

Remark 43. Note that the commutativity of the last diagram in Subsection 3.5 used in the proof of Theorem 6 comes from the fact that if $J \in \mathbb{Z}_2 \otimes \mathsf{L}'_{2k+2}$ and C is a clasper with this tree-type, then the element in D_{4k+2} represented by $(\Sigma_{g,1} \times I)_C$ is $\frac{1}{2}\eta_{4k+2}(J \!-\! J)$, which maps by sl_{4k+2} to $J \in \mathbb{Z}_2 \otimes \mathsf{L}_{2k+2}$ as required.

Let C be a tree clasper of order n embedded in an oriented 3-manifold M with fundamental group $G := \pi_1 M$. Orient C as a surface and orient the leaves consistent with this orientation. Consider a curve γ which is an oriented positive meridian to a leaf of C, connected by a whisker to the basepoint. Connect the cores of each leaf of C by whiskers to the basepoint, so that each leaf represents an element of G. Thinking of the leaf which γ links as a root, the tree associated to C determines an iterated commutator (bracketing expression) $c \in G_n$ of the group elements of the other leaves. For example, in Figure 8, $c = [\alpha_1, [\alpha_2, \alpha_3]]$.

Lemma 44. $\gamma = (-1)^{n-1}c \in G_n/G_{n+1}$.

Proof. This is proven by induction on n. Note that modulo higher order commutators, the choice of whiskers to the basepoint doesn't matter. So assume that the whiskers actually follow the clasper's body. Now form an iterated commutator $\tilde{c}$ defined like c, except that instead of $[u, v]$, we form $[u, v^{-1}]$. With these specific choices of whisker, we claim that $\gamma = \tilde{c}$. The base case is when $n = 1$. In this case, it's not hard to see that $\gamma = [\alpha_1, \alpha_2^{-1}] = -[\alpha_1, \alpha_2] \in G/G_2$ as desired (see e.g. Figure 23 of [7]). Given a clasper of order n, break it into a clasper C' of order $n - 1$ and a Y-clasper which links a leaf of C', and the other two leaves are the group elements α and β. Then by the base case, the leaf linking C' is the group element $[\alpha, \beta^{-1}]$, feeding this into the inductive computation of $\tilde{c}'$, yields the desired $\tilde{c}$. $\qquad\square$

Suppose in the setting of the previous lemma C is a twisted clasper of order n, where γ does not link the twisted leaf. Thinking of the γ-leaf as a root, the ∞-tree associated to C determines a formal commutator c_∞, where the twisted leaf corresponds to an ∞-symbol. Thinking of the ∞-vertex as the root, we get another iterated commutator of group elements ω. Let c be the commutator obtained by replacing the ∞-symbol in c_∞ by ω. For example, in Figure 8, $c_\infty = [\infty, [\alpha_2, \alpha_3]]$, $\omega = [[\alpha_2, \alpha_3], \alpha_1]$, and $c = [[[\alpha_2, \alpha_3], \alpha_1], [\alpha_2, \alpha_3]]$.
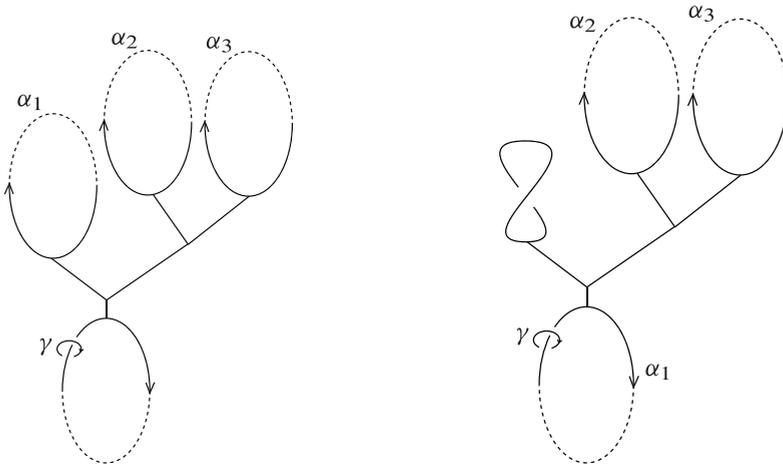
Figure 8. Left. An order 2 tree clasper in a 3-manifold. According to Lemma 44, $\gamma = -[\alpha_1, [\alpha_2, \alpha_3]] \in G_3/G_4$. Right. An order 4 twisted tree clasper. By Lemma 45, $\gamma = [[[\alpha_2, \alpha_3], \alpha_1], [\alpha_2, \alpha_3]] \in G_5/G_6$.

Lemma 45. $\gamma = c \in G_{2n-1}/G_{2n}$.

Proof. The ∞-symbol in the commutator c_{∞} geometrically corresponds to a longitude of the twisted leaf, which is homotopic to a meridian, so by the previous lemma, represents the commutator ω. Furthermore, there are two factors of $(-1)^n$ which cancel each other out. $\square$

4. From string links to homology cylinders

In this section we use a grope-to-Whitney tower construction from [2] together with algebraic results from [4] and Levine's Theorem 23 mentioned earlier to describe the relationship between the graded groups associated to the Whitney tower filtration $\mathbb{SW}_n$ of $\mathbb{SL}$ and the Y-filtration $\mathbb{Y}_n$ of $\mathbb{HC}$ via a geometric map from string links to homology cylinders. As mentioned in the introduction, Habegger defined a bijection between the set $\mathbb{HC}_1(g, 1)$ (of homology bordism classes of homology cylinders inducing the identity map on first homology $H_1(\Sigma_{g,1})$) and $\mathbb{HC}_1(0, 2g + 1)$ (which can be identified with the set $\mathbb{SL}_1(2g)$ of concordance classes of string links with $2g$ strands and trivial linking matrix in a homology ball), and used this bijection to equate the Milnor and Johnson invariants [12]. By Theorem 6 above and the Milnor invariant-Whitney tower relationship (e.g. [3]),

the results of this section can be thought of as a geometric analogue of the Milnor–Johnson correspondence described in [12].

Figure 9 illustrates a construction which sends a $2g$ component string link with trivial linking numbers to a homology cylinder over $\Sigma_{g,1}$: the arcs in the right-hand side have endpoints on a piece of a plane, which is the bottom of a topological ball containing the tangle T which defines the string link pictured on the left. We one-point compactify the bottom plane to S^2 and think of everything as taking place in a 3–ball B^3 bounded by that 2–sphere. Remove from B^3 a regular neighborhood of T in the right-hand picture to get a 3-manifold with boundary that decomposes into two surfaces $\Sigma^{\pm}$ as pictured. Since the pairwise linking numbers of the original string link are zero, the result is a homology cylinder over $\Sigma_{g,1}$. Moreover, the trivial string link gets sent to the identity product $\Sigma_{g,1} \times I$. This construction is not multiplicative on the monoids of string links and homology cylinders, but does give a well-defined map

$$\text{H}: \mathbb{SL}_1(2g) \longrightarrow \mathbb{HC}_1(g,1)$$

The map H takes concordant string links to homology cobordant homology cylinders, and is therefore well-defined.
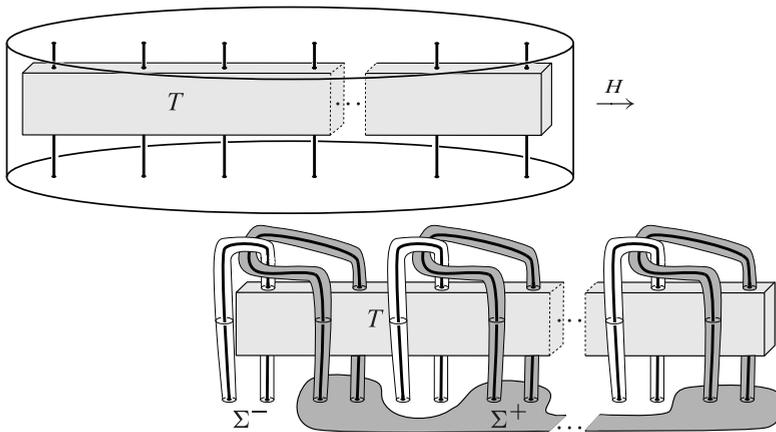

Figure 9. Closing a string link (defined by a tangle T) to a homology cylinder.

Proposition 46. *The map* $\text{H}: \mathbb{SL}_1(2g) \to \mathbb{HC}_1(g,1)$ *is the composition of the homomorphism*

$$C: \mathbb{SL}_1(2g) \longrightarrow \mathbb{HC}_1(0, 2g+1)$$

with the Habegger bijection.

Proof. Habegger's bijection is based on the fact that $\Sigma_{g,1} \times I$ is diffeomorphic to $\Sigma_{0,2g+1} \times I$. Refer to Figure 10, which shows a copy of $\Sigma_{1,1} \times I$ on the left, immersed into $\mathbb{R}^3$. The special surface $\Sigma^+ = \Sigma_{1,1} \times 1$ is shaded more darkly. To get a diffeomorphism to $\Sigma_{0,3} \times I$, take the left-most attaching rectangle of the right-hand handle, slide it up onto the top of the block, move it to the right and then back down onto the side of the block facing the viewer. The surface Σ^+ is moved to the position indicated on the right in Figure 10. Habegger maps string links in homology balls to homology cylinders by reparameterizing the boundary as indicated.

Figure 10. A diffeomorphism from $\Sigma_{1,1} \times I$ to $\Sigma_{0,3} \times I$.

To make this more explicit, we redraw $\Sigma_{0,3} \times I$ as in Figure 11(A), keeping track of the surface Σ^+, which we isotope to a more convenient position in Figure 11(B), (C) and (D). Now Habegger's construction, applied to a string link, can be seen as plugging in an arbitrary string link T (with trivial linking matrix) into the trivial string link represented in Figure 11(D). This is depicted in Figure 12(A). Finally, an isotopy of the surface Σ^+ leads us to the picture in Figure 12(B). Comparing this with Figure 9, it is clear that the map H coincides with Habegger's construction.
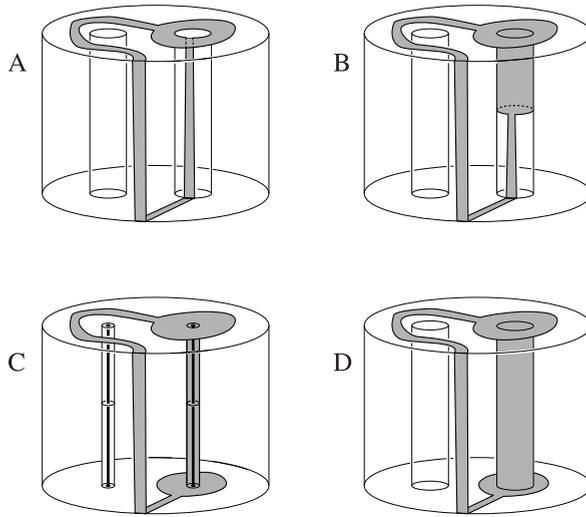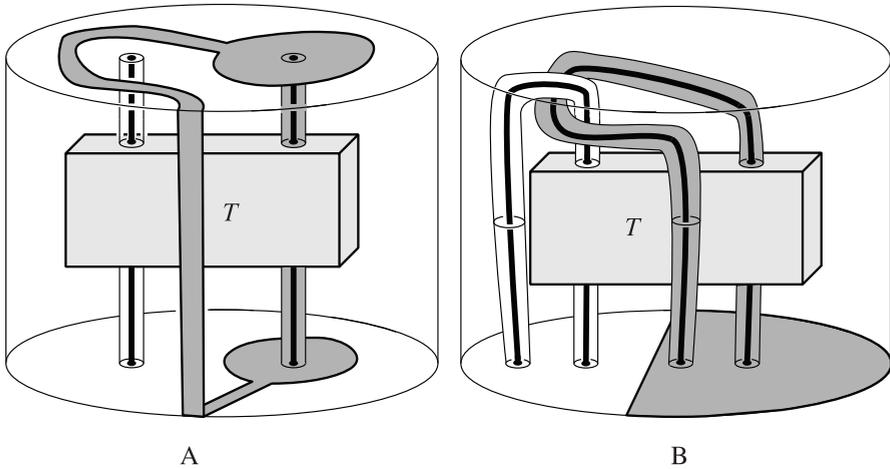
$\square$

Figure 11. Isotoping Σ^+.

Figure 12. Habegger's map.

Returning to the main discussion, it is not immediately obvious whether $H(\$W_n) \subset \mathbb{Y}_n$. However, recall the filtration $\$\mathbb{Y}_n$ on $\$\mathbb{L}$, where $\$\mathbb{Y}_n$ is defined to be the set of concordance classes of string links which differ from the trivial string link by a sequence of concordances and surgeries along claspers of order n.

Proposition 47. *For all $n \geq 1$, we have both the inclusion $\$\mathbb{Y}_n \subset \$W_n$ and the inclusion $H(\$\mathbb{Y}_n) \subset \mathbb{Y}_n$.*

Proof. By techniques of [7] (see Theorem 35 and the proof of Theorem 5 in [7]), the set $\$\mathbb{Y}_n$ can be defined using only concordances and surgeries along *simple* claspers of order $\geq n$, where by definition the leaves of a simple clasper bound disjointly embedded meridional disks to the string link components. A sequence of surgeries on simple claspers of order $\geq n$ is the same as a 3-dimensional capped grope cobordism of class $n + 1$ by [2, Theorem 23] (using [7]), which gives rise to a Whitney tower concordance of order n, as described in [2, Section 3.5] (using [27]). Thus $\$\mathbb{Y}_n \subset \$W_n$. On the other hand $H(\$\mathbb{Y}_n) \subset \mathbb{Y}_n$ follows because both filtrations are defined in terms of surgery on claspers (and clasper surgery obviously commutes with the map H). $\square$

Recall from Theorem 1 that the standard closure operation gives an isomorphism

$$\mathsf{SW}_n \cong \mathsf{W}_n$$

between the graded groups associated to the Whitney tower filtrations on string links and classical links. The rest of this section uses algebraic results from [4] to compare the Johnson and Y-filtrations (working with W_n rather than SW_n for ease of reference to [4]).

Recall Habiro's surjective realization map

$$\theta_n \colon \mathfrak{T}_n \longrightarrow \mathsf{Y}_n \quad (n > 1)$$
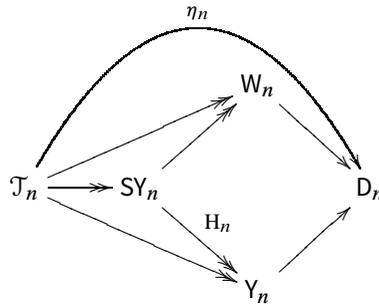
from Section 3 which can just as easily be defined using only simple clasper surgeries, thinking of the identity element in the group of homology cylinders as the image of the trivial string link under H. There is a surjective realization map $\mathfrak{T}_n \twoheadrightarrow \mathsf{SY}_n$ defined in an analogous way by embedding simple claspers in the complement of the identity string link, where SY_n is the quotient of $\$\mathbb{Y}_n$ by the equivalence relation generated by order $n + 1$ simple clasper surgery and concordance. There is also a realization surjection

$$R_n \colon \mathfrak{T}_n \longrightarrow \mathsf{W}_n$$

defined in [4] based on Cochran's Bing-doubling construction [1].

The following proposition combines Levine's Theorem 23 with the basic set-up of the Whitney tower theory surveyed in [3], and serves as a starting point for this section's results.

Proposition 48. *The following diagram is commutative for all $n > 1$:*

$$\eta_n$$

$$\mathcal{T}_n \longrightarrow SY_n \quad W_n \quad D_n$$

$$H_n \quad Y_n$$

Hence all maps are rational isomorphisms.

Proof. The bottom row factorization of η_n is Theorem 23 from Section 3. The map $W_n \to D_n$ is given by order n Milnor invariants μ_n, and the factorization $\eta_n = \mu_n \circ R_n$ is fundamental to the theory developed in [3, 4, 5, 6]. Commutativity of the bottom left triangle follows because the realization maps both are defined by embedding claspers. Commutativity of the triangle containing $SY_n \to W_n$ corresponds to the fact that the conversion from simple claspers to capped gropes to Whitney towers preserves tree type [2, 27]. Now, because $\mathcal{T}_n \to SY_n$ is surjective, diagram chasing is enough to show the right diamond commutes, and that the two maps emanating from SY_n are epimorphisms. $\qquad\square$

Remark 49. As mentioned in the introduction, in a future paper we will give a geometric argument that the map $SY_n \to SW_n \cong W_n$ is an isomorphism. This gives another proof that Habiro's surgery map induces a map $\widetilde{\mathcal{T}}_n \to Y_n$. However, we have presented the direct clasper arguments in Section 3 of this paper because they appear to give another fruitful line of approach to the theory.

To make progress understanding the analogous setting with $\mathbb{Z}$-coefficients, we appeal to the resolution of the Levine Conjecture in [6] and the resulting implications described in [4].

As mentioned in Section 3 above, in [22, 23] Levine studied a map

$$\eta'_n \colon \mathcal{T}_n \longrightarrow D'_n,$$
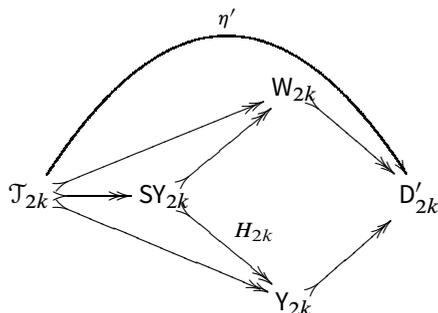
where D'_n is the kernel of the quasi-Lie bracket map

$$L'_1 \otimes L'_{n+1} \to L'_{n+2},$$

which is defined by the same formula as η_n. In [23] it is shown that $\mathsf{D}'_{2k} < \mathsf{D}_{2k}$, and in [6] it is shown that in all even orders the maps

$$\eta'_{2k} = \eta_{2k} \colon \mathfrak{T}_{2k} \to \mathsf{D}'_{2k}$$

are isomorphisms. This implies the following corollary to Proposition 48.

Corollary 50. *For all $k \geq 1$, the following diagram is commutative and all maps are $\mathbb{Z}$-module isomorphisms*:

In odd orders the situation depends on the order's parity modulo 4: Recall from [4, Section 6] that $\widetilde{\mathsf{D}}_{2k-1}$ is defined to be the quotient of D'_{2k-1} by the image under

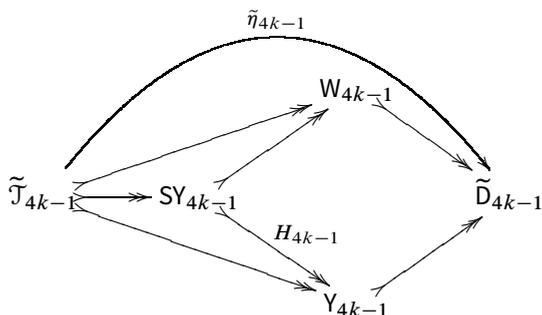$$\eta'_{2k-1} \colon \mathfrak{T}_{2k-1} \longrightarrow \mathsf{D}'_{2k-1}$$

of the framing relations (Definition 40 above) which define $\widetilde{\mathfrak{T}}_{2k-1}$, and the induced map

$$\tilde{\eta}_{2k-1} \colon \widetilde{\mathfrak{T}}_{2k-1} \longrightarrow \widetilde{\mathsf{D}}_{2k-1}$$

is an isomorphism for all $k \geq 1$.

In odd orders congruent to 3 mod 4, we get an analogous diagram of abelian groups and isomorphisms.

Corollary 51. *For all $k \geq 1$, the following diagram is commutative and all maps are $\mathbb{Z}$-module isomorphisms*:
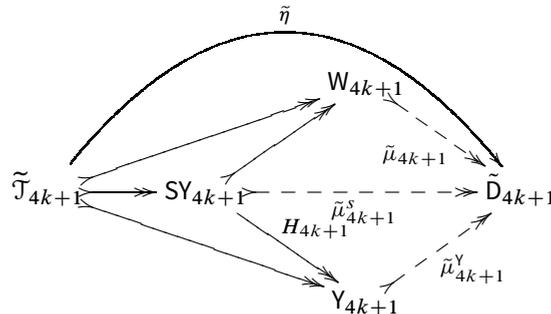
Proof. Consider the commutative diagram in Proposition 48. The realization maps emanating from $\mathcal{T}_{4k-1}$ are well-defined on the quotient $\widetilde{\mathcal{T}}_{4k-1}$. This follows from Theorem 1.5 of [4] for the upper map, Proposition 42 for the bottom map, and by an analogous argument applied to simple claspers on string links modulo concordance for the middle map. We also know, combining Theorems 5.1 and 6.5(ii) of [4], that there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \otimes \mathsf{L}_{2k+1} \longrightarrow \widetilde{\mathcal{T}}_{4k-1} \longrightarrow \mathsf{D}_{4k-1} \longrightarrow 0.$$

It follows that $\mathbb{Z}_2 \otimes \mathsf{L}_{2k+1}$ surjects onto the kernels of both $\mathsf{Y}_{4k-1} \to \mathsf{D}_{4k-1}$ and $\mathsf{W}_{4k-1} \to \mathsf{D}_{4k-1}$. Furthermore, by Theorem 6 and [4, Theorem 6.1], the kernels of both $\mathsf{Y}_{4k-1} \to \mathsf{D}_{4k-1}$ and $\mathsf{W}_{4k-1} \to \mathsf{D}_{4k-1}$ are isomorphic to $\mathbb{Z}_2 \otimes \mathsf{L}_{2k+1}$. It follows by the 5-lemma that the epimorphisms $\widetilde{\mathcal{T}}_{4k-1} \to \mathsf{Y}_{4k-1}$ and $\widetilde{\mathcal{T}}_{4k-1} \to \mathsf{W}_{4k-1}$ are actually isomorphisms. Furthermore, the fact that η_{4k-1} can be lifted to an isomorphism to $\widetilde{\mathsf{D}}_{4k-1}$ implies the maps $\mathsf{Y}_{4k-1}, \mathsf{W}_{4k-1} \to \mathsf{D}_{4k-1}$ can be lifted to maps, hence isomorphisms, to $\widetilde{\mathsf{D}}_{4k-1}$ as well. $\qquad\square$

Proof of Theorem 4. The last two corollaries prove the first sentence of Theorem 4. The statement for $4k + 1$ follows from Proposition 48 and the fact that the kernel of the map $\widetilde{\mathcal{T}}_{4k+1} \to \mathsf{D}_{4k+1}$ is the 2-torsion group $\mathbb{Z}_2 \otimes \mathsf{L}'_{2k+2}$, which follows from Theorems 1.13 and 5.1 of [4]. $\qquad\square$

4.1. Higher-order Arf invariants for homology cylinders.
Finally we consider the conjectural set-up in the order $4k + 1$ case, where the higher-order Arf invariants live (for framed filtrations). Recall from [4, Section 6.2] that the nontriviality of the higher-order Arf invariants for links is equivalent to the existence of a map $\tilde{\mu}_{4k+1} \colon \mathsf{W}_{4k+1} \to \widetilde{\mathsf{D}}_{4k+1}$ such that the composition $\widetilde{\mathcal{T}}_{4k+1} \to \mathsf{W}_{4k+1} \to \widetilde{\mathsf{D}}_{4k+1}$ is $\tilde{\eta}_{4k+1}$. By a similar argument, Conjecture 7 is true if and only if there is a map $\tilde{\mu}^{\mathsf{Y}}_{4k+1} \colon \mathsf{Y}_{4k+1} \to \widetilde{\mathsf{D}}_{4k+1}$ so that $\tilde{\theta}_{4k+1} \circ \tilde{\mu}^{\mathsf{Y}}_{4k+1} = \tilde{\eta}_{4k+1}$. So we get a commutative diagram of conjectured isomorphisms for $k > 0$:

Assuming our announced isomorphism $\mathsf{SY}_n \cong \mathsf{W}_n$, the existence of $\tilde{\mu}^s_{4k+1}$ is equivalent to the existence of $\tilde{\mu}_{4k+1}$. On the other hand, if $\tilde{\mu}^{\mathsf{Y}}_{4k+1}$ exists, then so does $\tilde{\mu}_{4k+1}$. So if Conjecture 7 is true, it would imply the existence of higher-order Arf invariants for links.

When $k = 0$, we almost have a diagram of isomorphisms, but there is an extra $\mathbb{Z}_2$ in Y_1 coming from the Rochlin invariant. That is, we have $\mathsf{Y}_1 \cong \mathsf{W}_1 \oplus \mathbb{Z}_2$ and $\widetilde{\mathcal{J}}_1 \cong \widetilde{\mathsf{D}}_1 \cong \mathsf{W}_1 \cong \mathsf{SY}_1$.

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James Conant, Department of Mathematics, University of Tennessee, Knoxville, 1403 Circle Dr, Knoxville, TN 37916, USA

e-mail: jim.conant@gmail.com

Robert Schneiderman, Department of Mathematics and Computer Science, Lehman College, City University of New York, 250 Bedford Park Blvd W, Bronx, NY 10468, USA

e-mail: robert.schneiderman@lehman.cuny.edu

Peter Teichner, Max Planck Institut für Mathematik, Bonn, Vivatsgasse 7, 53111 Bonn, Germany

Department of Mathematics, University of California in Berkeley, 970 Evans Hall, Berkeley, CA 94720, USA

e-mail: teichner@mac.com