Quantum Topol. 7 (2016), 435–503
DOI 10.4171/QT/79

Quantum Topology

Skein and cluster algebras of marked surfaces

Greg Muller[1]

Abstract. This paper considers several algebras associated to an oriented surface Σ with a finite set of marked points on its boundary. The first is the *skein algebra* $\mathsf{Sk}_q(\Sigma)$, which is spanned by links in the surface which are allowed to have endpoints at the marked points, modulo several locally defined relations. The product is given by superposition of links. A basis of this algebra is given, as well as several algebraic results.

When Σ is triangulable, a *quantum cluster algebra* $\mathcal{A}_q(\Sigma)$ and *quantum upper cluster algebra* $\mathcal{U}_q(\Sigma)$ can be defined. These are algebras coming from the triangulations of Σ and the elementary moves between them. Cluster algebras have been a subject of significant recent interest, due in part to their extraordinary positivity and Laurent properties.

Natural inclusions $\mathcal{A}_q(\Sigma) \subseteq \mathsf{Sk}_q^o(\Sigma) \subseteq \mathcal{U}_q(\Sigma)$ are shown, where $\mathsf{Sk}_q^o(\Sigma)$ is a certain Ore localization of $\mathsf{Sk}_q(\Sigma)$. When Σ has at least two marked points in each component, these inclusions are strengthened to equality, exhibiting a quantum cluster structure on $\mathsf{Sk}_q^o(\Sigma)$.

The method for proving these equalities has the potential to show $\mathcal{A}_q = \mathcal{U}_q$ for other classes of cluster algebras. As a demonstration of this fact, a new proof is given that $\mathcal{A}_q = \mathcal{U}_q$ for acyclic cluster algebras.

Mathematics Subject Classification (2010). Primary: 13F60, Secondary: 57M50, 16S99.

Keywords. Cluster algebra, quantum cluster algebra, skein algebra, quantum torus, triangulation of surfaces.

[1] This work was supported by the VIGRE program at LSU, National Science Foundation grant DMS-0739382.

Contents

1. Introduction

In this paper, we consider *marked surfaces*: compact, oriented surfaces, possibly with boundary, together with a finite set of marked points in the boundary.[1]

1.1. The skein module. Motivated by computing the Jones polynomial of a knot, Kauffman [18] introduced the *Kauffman bracket*, a (framed)[2] knot invariant defined by the two local relations in Figure 1.

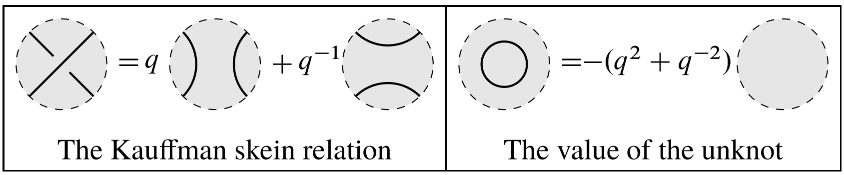


Figure 1. The skein relations (without marked points).

[1] This contrasts with some references, where 'marked surfaces' may have interior marked points.

[2] We suppress the details of framing a knot; all drawn knots will be given the blackboard framing.

These relations are defined as manipulations of a knot (or link) in an oriented 3-manifold, where the dashed circle represents a small sphere, and the links are understood to be kept identical outside this sphere. Using these relations, any link in $\mathbb{R}^3$ can be reduced to a Laurent polynomial in q (the Kauffman bracket of the link) times the empty link.

For a general oriented 3-manifold, these relations can be encoded in the 'skein module,' introduced independently by Turaev [35] and [36] and Przytycki [28]. Let $\mathbb{Z}_q := \mathbb{Z}[q^{\pm\frac{1}{2}}]$ denote the Laurent ring in the indeterminant[3] $q^{\frac{1}{2}}$, and let $\mathbb{Z}_q^{\mathsf{Links}}$ be the module of $\mathbb{Z}_q$-linear combinations of ambient isotopy classes of framed links. Imposing the skein relations defines a quotient $\mathbb{Z}_q$-module of $\mathbb{Z}_q^{\mathsf{Links}}$, called the *skein module* of the 3-manifold. The skein module of $\mathbb{R}^3$ is the free $\mathbb{Z}_q$-module spanned by the empty link.

1.2. The skein algebra $\mathsf{Sk}_q(\Sigma)$ (without marked points).

When the 3-manifold in question is $\Sigma \times [0,1]$ for an unmarked surface Σ, two extra structures appear. First, two links in $\Sigma\times[0,1]$ can be 'stacked' vertically to give a new link in $\Sigma\times[0,1]$ which contains the first link in $\Sigma \times [0, \frac{1}{2}]$ and the second in $\Sigma \times [\frac{1}{2}, 1]$. This gives a well-defined *superposition product* on the skein module of $\Sigma \times [0,1]$ and makes it into an associative $\mathbb{Z}_q$-algebra called the *skein algebra* of Σ.

Second, any link in $\Sigma \times [0,1]$ can be projected into Σ, with overcrossings and undercrossings used to keep track of the original link.[4] As an abuse of terminology, such a diagram will be called a *link* in Σ. The skein algebra of Σ can be computed directly from the set of links in Σ, as the quotient of $\mathbb{Z}_q^{\mathsf{Links}}$ by a submodule generated by the skein relations. In this way, the skein algebra can be associated directly to the surface Σ.

1.3. The skein algebra $\mathsf{Sk}_q(\Sigma)$ (with marked points).

Motivated by examples coming from the theory of cluster algebras and Teichmüller theory, we define a generalization of skein algebras to marked surfaces.

Let Σ be a surface with a finite set of 'marked points' $\mathcal{M}$. A *link* in Σ will be a collection of immersed curves in Σ, with transverse intersections and boundary contained in $\mathcal{M}$, together with a 'crossing data.' This is a choice, for each intersection, of the order in which the curves pass over each other (see Section 2.3). Links are considered up to homotopy through the set of links.

[3] The justification for including the half-power of q will come later.

[4] Ambient homotopy may be required to ensure the projection has simple transverse intersections.

Remark 1.1. Actually, we will extend this definition of link to allow simultaneous crossings at marked endpoints. Two curves can then arrive transversely at a marked point in three ways: over, under and simultaneous. This generalization does not affect the subsequent skein algebra (see Remark 3.1).

Let $\mathbb{Z}_q^{\mathsf{Links}}$ denote the free $\mathbb{Z}_q$-module spanned by (homotopy classes of) links in Σ, and define a quotient $\mathbb{Z}_q$-module $\mathsf{Sk}_q(\Sigma)$ by imposing the relations in Figure 2. The element in $\mathsf{Sk}_q(\Sigma)$ corresponding to a link $\mathbf{X}$ will be denoted $[\mathbf{X}]$.

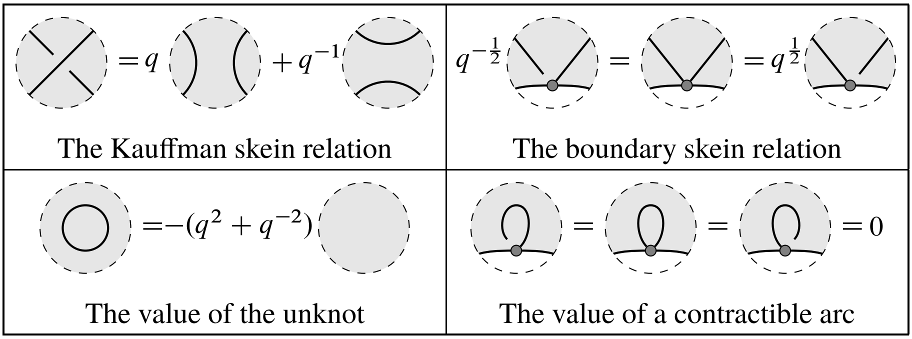


Figure 2. The skein relations with marked points.

Notes on the figure.

- A dashed circle denotes a small disc in Σ, and the links in each term of the equality are understood to be identical outside the circle.

- A solid curve between grey and white denotes the boundary of Σ, and a dark dot denotes a marked point.

- We also allow additional undrawn curves at the marked points which have the same order with respect to the drawn curves.

These relations imply several other relations (Proposition 3.2): the (framed) Reidemeister moves from knot theory, as well as an additional marked variation of the second Reidemeister move (Figure 5).

Given two transverse links $\mathbf{X}$ and $\mathbf{Y}$, their *superposition* $\mathbf{X} \cdot \mathbf{Y}$ is the union of the two links, with every curve in $\mathbf{X}$ passing over every curve in $\mathbf{Y}$. This extends to a well-defined product on $\mathsf{Sk}_q(\Sigma)$ and makes $\mathsf{Sk}_q(\Sigma)$ into an associative $\mathbb{Z}_q$-algebra (Proposition 3.7), which we call the *skein algebra* of Σ.

Many properties of $\mathsf{Sk}_q(\Sigma)$ are shown, which generalize known unmarked results.

- (Corollary 6.16) $\mathsf{Sk}_q(\Sigma)$ is a domain.
- (Theorem A.4) $\mathsf{Sk}_q(\Sigma)$ is finitely generated.
- (Lemma 4.1) $\mathsf{Sk}_q(\Sigma)$ has a $\mathbb{Z}_q$-basis parametrized by *simple multicurves*.

1.4. Triangulations. Curves in Σ come in two types,

- *loops*: immersed images of S^1, and
- *arcs*: immersed images of $[0, 1]$, with endpoints mapping to marked points.

A *triangulation* Δ of Σ is a simple multicurve consisting of arcs, such that the complement of the arcs is a disjoint union of discs with three marked points. As elements in $\mathsf{Sk}_q(\Sigma)$, the arcs in Δ *quasi-commute*; that is, for $\mathsf{x}_i, \mathsf{x}_j \in \Delta$, there is a $\Lambda^{\Delta}_{i,j} \in \mathbb{Z}$ such that

$$[\mathsf{x}_i][\mathsf{x}_j] = q^{\Lambda^{\Delta}_{i,j}}[\mathsf{x}_i][\mathsf{x}_j].$$

The numbers $\Lambda^{\Delta}_{i,j}$ correspond to entries in a *orientation matrix* (Section 6.2).

Triangulations of Σ give embeddings of the skein algebra into well-behaved algebras. Let the *quantum torus* $\mathbb{T}_\Delta$ associated to Δ be the $\mathbb{Z}_q$-algebra with a $\mathbb{Z}_q$-basis of elements of the form M^α, for all $\alpha \in \mathbb{Z}^\Delta$, and multiplication defined by[5]

$$M^\alpha M^\beta = q^{\frac{1}{2}\langle\alpha,\Lambda^\Delta\beta\rangle}M^{\alpha+\beta} = q^{\langle\alpha,\Lambda^\Delta\beta\rangle}M^\beta M^\alpha.$$

Theorem 6.14. *For each triangulation Δ of Σ, there is an injective Ore localization*

$$\mathsf{Sk}_q(\Sigma) \hookrightarrow \mathsf{Sk}_q(\Sigma)[\Delta^{-1}] \simeq \mathbb{T}_\Delta$$

which sends $[\mathsf{x}_i]$ to M^{e_i}.

The theorem says that $\mathsf{Sk}_q(\Sigma)$ embeds into its skew-field of fractions $\mathcal{F}$, and inside that skew-field, every element of $\mathsf{Sk}_q(\Sigma)$ can be written as a skew-Laurent polynomial in the arcs in Δ.

1.5. Three algebras. When Σ is triangulable, Theorem 6.14 leads to the definition of three related $\mathbb{Z}_q$-algebras.

[5] Here, $\langle -, - \rangle$ is the natural dot product on $\mathbb{Z}^\Delta$.

- THE LOCALIZED SKEIN ALGEBRA $\mathsf{Sk}_q^o(\Sigma)$ (Section 5).

  A *boundary arc* is a simple arc in Σ which is homotopic to an arc contained in the boundary. A triangulation Δ of Σ contains the set of boundary arcs, and so the localization $\mathsf{Sk}_q(\Sigma)[\Delta^{-1}]$ contains the inverse to each boundary arc. The *localized skein algebra* $\mathsf{Sk}_q^o(\Sigma)$ is the Ore localization of $\mathsf{Sk}_q(\Sigma)$ at the boundary arcs in Σ.

- THE (QUANTUM) CLUSTER ALGEBRA $\mathcal{A}_q(\Sigma)$ (Section 7.2).

  The skein algebra $\mathsf{Sk}_q(\Sigma)$ is generated by simple curves (Corollary 4.3), and so $\mathsf{Sk}_q^o(\Sigma)$ is generated by simple curves and the inverses to boundary curves. The (*quantum*) *cluster algebra* $\mathcal{A}_q(\Sigma)$ of Σ is the $\mathbb{Z}_q$-subalgebra of $\mathsf{Sk}_q^o(\Sigma)$ generated by simple arcs and the inverses to boundary arcs.

- THE (QUANTUM) UPPER CLUSTER ALGEBRA $\mathcal{U}_q(\Sigma)$ (Section 7.2).

  Since Σ may have many triangulations, Theorem 6.14 provides many distinct skew-Laurent expressions for an element in $\mathsf{Sk}_q(\Sigma)$. This property may be turned into a criterion for defining another algebra. The (*quantum*) *upper cluster algebra* $\mathcal{U}_q(\Sigma)$ of Σ is the $\mathbb{Z}_q$-algebra consisting of elements in the skew-field $\mathcal{F}$ which can be written as a skew-Laurent polynomial in each triangulation.

These algebras satisfy the following containments.

Theorem 7.15. *For any triangulable marked surface Σ,*

$$\mathcal{A}_q(\Sigma) \subseteq \mathsf{Sk}_q^o(\Sigma) \subseteq \mathcal{U}_q(\Sigma).$$

Our main result is that these are equalities for most marked surfaces.

Theorem 9.8. *For a triangulable marked surface Σ with at least two marked points in each connected component,*

$$\mathcal{A}_q(\Sigma) = \mathsf{Sk}_q^o(\Sigma) = \mathcal{U}_q(\Sigma).$$

Remark 1.2. The definitions given above for $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$ make Theorem 7.15 immediate, but it is not clear they are quantum cluster algebras in the sense of [3]. The body of the paper takes the opposite approach. In Section 7.2, $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$ are defined as quantum cluster algebras, and equivalence to the previous definitions will be a consequence of Theorem 7.15.

1.6. Quantum cluster algebras. Any two triangulations of Σ can be related by a sequence of *flips*, where a single arc is replaced by a distinct arc. The flip of an arc in Δ has a simple expression as a skew-Laurent polynomial in Δ, and by iterating these expressions, any arc in any triangulation can be obtained.

This process is a specific case of a more general framework: the theory of quantum cluster algebras (introduced in [11], quantized in [3]). We sketch this theory now, precise definitions are in Section 7.1. One starts with a *quantum seed*:

- a finite set of quasi-commuting *cluster variables* in a skew-field, which are designated either *exchangeable* or *frozen* and
- a rule (called *mutation*) for replacing any exchangeable cluster variable by a new exchangeable cluster variable, resulting in a new quantum seed.

The *quantum cluster algebra* $\mathcal{A}_q$ associated to a quantum seed is the $\mathbb{Z}_q$-algebra generated by all the cluster variables obtained by iterated mutations, and the inverses to the frozen cluster variables. A quantum seed also determines a *quantum upper cluster algebra* $\mathcal{U}_q$, which is an algebra containing $\mathcal{A}_q$ defined as an intersection of quantum tori.[6]

In case of marked surfaces, a triangulation Δ of Σ determines a quantum seed.

- The cluster variables are the arcs in Δ, as elements in $\mathcal{F}$, the skew-field of $\mathsf{Sk}_q(\Sigma)$. An arc is frozen if it is a boundary arc and exchangeable otherwise.
- The mutation rule is determined from the relative orientations of the arcs in Δ at the endpoints.[7]

The resulting algebras $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$ do not depend on the choice of triangulation (Definition 7.12), and coincide with the definitions of $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$ in the previous section (Theorem 7.15 and Remark 7.16).

The specialization $q^{\frac{1}{2}} = 1$ of $\mathcal{A}_q(\Sigma)$ becomes a commutative cluster algebra $\mathcal{A}_1(\Sigma)$. Commutative cluster algebras associated to marked surfaces have been introduced (in [15] and [7]) and extensively studied (in [9] and [10] and in [31],[33],[32], and [23]). The relation of $\mathcal{A}_1(\Sigma)$ to skein algebras was noticed in [7, Section 12.3]. The equality $\mathcal{A}_1(\Sigma) = \mathsf{Sk}_1^o(\Sigma)$ for triangulable surfaces with at least two marked points (the commutative specialization of Theorem 9.8) has been independently proven by Musiker, Schiffler, and Williams (in [24] and [26]) using more explicit methods than this paper.

[6] In this paper, 'cluster algebras' are quantum cluster algebras unless otherwise specified.

[7] Specifically, the exchange matrix is a restriction of the *skew-adjacency matrix* in Section 6.2.

Remark 1.3. The commutative cluster algebra of a marked surface (as defined in [9]) depends on a choice of coefficients. The commutative specialization $\mathcal{A}_1(\Sigma)$ has coefficients in the Laurent ring generated by the boundary arcs.

1.7. The structure of the paper. The first part of the paper focuses on skein algebras of general marked surfaces.

(2) Curves and links in marked surfaces. This section gives our definitions of 'curve,' 'multicurve' and 'link' for marked surfaces.

(3) The skein algebra $\mathsf{Sk}_q(\Sigma)$. The skein algebra is defined, first as a $\mathbb{Z}_q$-module, and then as a $\mathbb{Z}_q$-algebra under the superposition product. An anti-involution and a grading of $\mathsf{Sk}_q(\Sigma)$ are given.

(4) Simple multicurves. Lemma 4.1 proves that the simple multicurves define a $\mathbb{Z}_q$-basis of $\mathsf{Sk}_q(\Sigma)$. This is used to prove that simple curves are not zero-divisors (Lemma 4.10), and multiplication by a simple arc x reduces the 'crossing number' with x (Lemma 4.11).

(5) The localized skein algebra $\mathsf{Sk}_q^o(\Sigma)$. The localized skein algebra is defined, shown to be an Ore localization, and a $\mathbb{Z}_q$-basis by certain weighted simple multicurves is given.

The second part of the paper focuses on the case when Σ is triangulable, and the connection to cluster algebras.

(6) Triangulations. Triangulations and some of their basic properties are reviewed. A method is given for expressing an element of $\mathsf{Sk}_q(\Sigma)$ as a skew-Laurent polynomial in a given triangulation (Corollary 6.9). This is used to prove that the localization of $\mathsf{Sk}_q(\Sigma)$ at Δ is a quantum torus.

(7) Quantum cluster algebras of marked surfaces. Section 7.1 reviews the generalities of quantum cluster algebras. Section 7.2 defines the quantum seed associated to a triangulation of a marked surface (Proposition 7.8) and checks that the corresponding cluster algebras $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$ only depend on Σ (Corollary 7.11). These are related to the localized skein algebra by $\mathcal{A}_q(\Sigma) \subseteq \mathsf{Sk}_q^o(\Sigma) \subseteq \mathcal{U}_q(\Sigma)$ (Theorem 7.15).

(8) A general technique for $\mathcal{A}_q = \mathcal{U}_q$. This section develops an approach for showing $\mathcal{A}_q = \mathcal{U}_q$ for large classes of quantum cluster algebras. The final criterion is given in Lemma 8.13. This criterion is used to provide a new proof that $\mathcal{A}_q = \mathcal{U}_q$ for 'acyclic' cluster algebras (Proposition 8.17).

(9) $\mathcal{A}_q(\Sigma) = \mathsf{Sk}_q^o(\Sigma) = \mathcal{U}_q(\Sigma)$ for (most) marked surfaces. Theorem 9.8 is proven using the techniques of the preceding section.

The last part of the paper explores some cases and consequences of Theorem 9.8.

(10) Loop elements. The simple loops in $\mathsf{Sk}_q^o(\Sigma)$ define extra elements of $\mathcal{A}_q(\Sigma)$ which are not cluster variables. Considering these elements simplifies computations and provides a free $\mathbb{Z}_q$-basis of $\mathcal{A}_q(\Sigma)$.

(11) The commutative specialization $q^{\frac{1}{2}} = 1$. This section discusses the commutative specialization $\mathcal{A}_1(\Sigma) = \mathsf{Sk}_1^o(\Sigma) = \mathcal{U}_1(\Sigma)$. The commutative cluster algebra $\mathcal{A}_1(\Sigma)$ is *locally acyclic*, which implies additional results.

(12) Examples and non-examples. This section explores specific cases of Σ, such as discs and an annulus.

The paper concludes with an appendix showing that $\mathsf{Sk}_q(\Sigma)$ is finitely generated, by directly generalizing the original proof of Bullock in the unmarked case [2].

1.8. Acknowledgements. The author would like to thank a great many people for useful discussions and support; including A. Berenstein, M. Gekhtman, A. Knutson, G. Musiker, P. Plamondon, D. Thurston, M. Yakimov, and A. Zelevinsky. Helpful comments and edits were provided by J. Geiger, J. Matherne, G. Musiker, and H. Thomas.

2. Curves and links in marked surfaces

This section gives our definitions of ‘curve,’ ‘multicurve’ and ‘link’ for marked surfaces.

2.1. Curves. A (*framed*) *curve* x in Σ is an immersion $\mathsf{x}\colon C \to \Sigma$ of a compact, connected, 1-dimensional manifold into Σ, such that any boundary of C maps to $\mathcal{M}$ and the interior of C does not map to $\mathcal{M}$. There are two kinds of curves:

- Arcs: curves with endpoints in $\mathcal{M}$;
- Loops: closed loops without endpoints.

Homotopies between curves are always through the class of curves; that is, we only allow homotopies during which

- C remains immersed (*regular homotopy*),
- the endpoints remain in $\mathcal{M}$ (*endpoint-fixed*) and
- the interior remains disjoint from $\mathcal{M}$.

As an abuse of terminology, two curves will be called *homotopic* if they may be related by homotopy and orientation-reversal (so a homotopy class has no intrinsic orientation).

2.2. Multicurves. A *multicurve* $\mathtt{X}$ in $(\Sigma, \mathcal{M})$ will mean an unordered, finite set of curves in Σ which may contain duplicates (i.e., homotopic curves). Two multicurves are *homotopic* if there is a bijection between their constituent curves which takes a curve to a homotopic one. A curve can always be thought of as a single element multicurve.

We will often focus on multicurves locally, by restricting to arbitrarily small discs around a point in Σ. A *strand* in a multicurve $\mathtt{X}$ near a point $p \in \Sigma$ will be a component of the restriction of $\mathtt{X}$ to an arbitrarily small disc around p.

A multicurve $\mathtt{X}$ is *transverse* if

- at each intersection in $\mathtt{X}$, each strand has a different tangent direction and
- each interior intersection (called a *crossing*) is between only two strands.

Every multicurve is homotopic to a transverse multicurve.

A transverse multicurve is *simple* if it has no interior intersections, and no curves which are contractible. Contractible curves are either topologically trivial loops (called *unknots*) or arcs which cut out a disc (called *contractible arcs*).

Remark 2.1. A transverse multicurve will be drawn as the union of its curves. By the transverse condition, it is unambiguous what the constituent curves are.

2.3. Links. We now define links, by equipping a transverse multicurve with crossing data, about which strands are 'passing over' other strands. It will be convenient to allow strands at a marked point to either pass over each other, or to arrive simultaneously. This generalization is a convenience, not a necessity; see Remark 3.1.

A (*framed*) *link* $\mathbf{X}$ is a transverse multicurve $\mathtt{X}$, together with

- at each crossing, an ordering of the two strands and
- at each marked point, an equivalence relation on the strands and an ordering on the equivalence classes of the strands.

Intuitively, a strand at a crossing must pass over or under the other strand, and two strands at a marked point must pass over, under, or be simultaneous (the equivalence relation). This is drawn in the natural way (Figure 3).

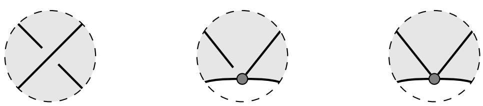


Figure 3. Crossings, ordered strands, and simultaneous strands.

A simple multicurve X can be regarded as a link with the simultaneous ordering at each endpoint; this will also be denoted by X.

Remark 2.2. Knot theory often considers 'links,' which would be links without arcs by the above definition, and 'virtual links,' which would be links without arcs, but where simultaneous crossings are allowed [19]. Thus, the above definition can be thought of as 'links with endpoints in $\mathcal{M}$, which can be virtual links at their endpoints.'

Links without arcs arise in knot theory, as projections of knots in 3-dimensional space onto 2-dimensional space. Similarly, our notion of links can be thought of as describing a multicurve in $\Sigma \times [0, 1]$, where $[0, 1]$ is the dimension coming 'out of the paper.'

Homotopies between links are through the class of transverse multicurves, where crossing data are not changed. This means the intersections are required to stay transverse, and so intersections can neither be created nor removed (in contrast with our definition of homotopy of multicurves). We will say two links are *homotopic* if they may be related by homotopy and orientation-reversal.

Remark 2.3. This notion of equivalence is weaker than the usual definition of equivalent links in knot theory, which uses Reidemeister moves and captures the notion of when two links describe ambient isotopic links in 3-dimensional space. This difference will become irrelevant later, as the skein relations will imply the Reidemeister moves (Proposition 3.2).

3. The skein algebra $\mathsf{Sk}_q(\Sigma)$

Inspired by knot theory, we now define an algebra associated to a marked surface, which consists of linear combinations of links modulo certain local relations, and whose product corresponds to superimposing links.

3.1. The skein relations. Let $\mathbb{Z}_q$ denote the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ of Laurent polynomials in the indeterminant $q^{\frac{1}{2}}$. For any marked surface Σ, let $\mathbb{Z}_q^{\mathsf{Links}}$ denote the free $\mathbb{Z}_q$-module with basis given by equivalence classes of links in Σ.

We will define a quotient $\mathbb{Z}_q$-module of $\mathbb{Z}_q^{\mathsf{Links}}$ by imposing several classes of relations (Figure 4), which are all defined in terms of local manipulations of a link. These relations are expressed in terms of small discs, where it is understood that they describe links identical to each other outside the disc. We also allow additional, undrawn curves at marked points, provided their order with respect to the drawn curves and each other does not change.

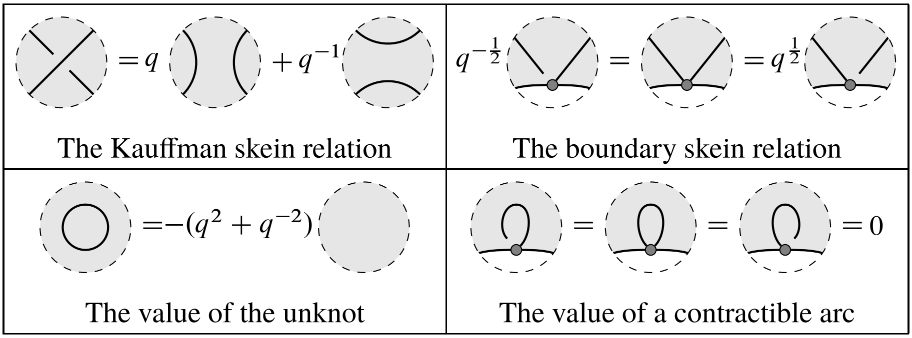


Figure 4. The defining relations of $\mathsf{Sk}_q(\Sigma)$.

Define the quotient $\mathbb{Z}_q$-module

$$\mathsf{Sk}_q(\Sigma) := \mathbb{Z}_q^{\mathsf{Links}}/I$$

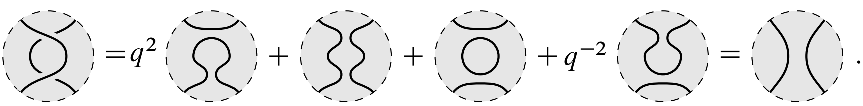
where I is the submodule generated by the set $\{l - r\}$, running over relations of the form $l = r$ in Figure 4. For a link $\mathbf{X}$, the class of $\mathbf{X}$ in $\mathsf{Sk}_q(\Sigma)$ will be denoted $[\mathbf{X}]$.

Remark 3.1. By the boundary skein relation, $\mathsf{Sk}_q(\Sigma)$ is spanned over $\mathbb{Z}_q$ by classes of links with no simultaneous endpoints. It would have been possible to define $\mathsf{Sk}_q(\Sigma)$ only in terms of those links; this would also eliminate the need for choosing a square root of q. However, allowing simultaneous endpoints gives topological realizations of the multicurve elements defined in Section 4.

3.2. The Reidemeister moves. The relations imposed in $\mathsf{Sk}_q(\Sigma)$ imply additional local relations which will be important (Figure 5). These are the (modified) Reidemeister moves from knot theory, together with an additional relation coming from the addition of marked endpoints.

Proposition 3.2. *The locally defined relations in Figure 5 hold in* $\mathsf{Sk}_q(\Sigma)$.

Proof. All four results follow from direct application of the relations in Figure 4. We show the computation for Reidemeister 2; the others are similar:



The four terms in the middle come from applying the Kauffman skein relation to the two crossings. The value of the unknot then cancels the first and last terms. □

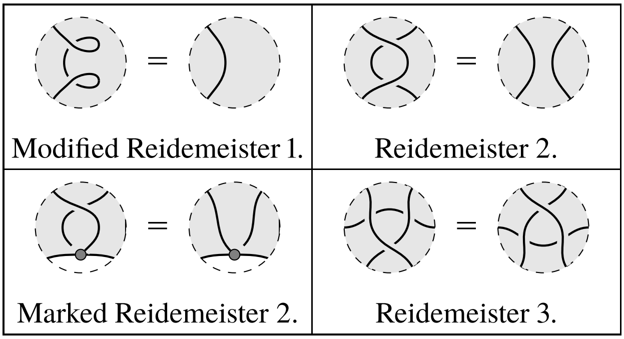


Figure 5. The Reidemeister moves for links with endpoints.

Remark 3.3. The fixed values of unknots and contractible arcs defined in Figure 4 are the only values for which the above Reidemeister moves hold (assuming the Kauffman and boundary skein relation).

Remark 3.4. The Reidemeister moves describe the minimal relations needed to relate two links (drawn in Σ with crossings) which describe the ambient isotopic framed links in $\Sigma \times [0,1]$. The endpoints of the framed link in $\Sigma \times [0,1]$ are required to stay in $\mathcal{M} \times [0,1]$, and the framing at the endpoints must stay tangent to $\mathcal{M} \times [0,1]$.

3.3. The superposition product. The $\mathbb{Z}_q$-module $\mathsf{Sk}_q(\Sigma)$ may be equipped with a $\mathbb{Z}_q$-bilinear, non-commutative product called the *superposition product*. If $\mathbf{X}$ and $\mathbf{Y}$ are two links such that the union of the underlying multicurves $\mathsf{X} \cup \mathsf{Y}$ is transverse, define the *superposition* $\mathbf{X} \cdot \mathbf{Y}$ to be the link which is $\mathsf{X} \cup \mathsf{Y}$ where each strand of $\mathbf{X}$ crosses over each strand of $\mathbf{Y}$ and all other crossings are ordered as in $\mathbf{X}$ and $\mathbf{Y}$.

Proposition 3.5. $[\mathbf{X} \cdot \mathbf{Y}]$ *only depends on the homotopy classes of* $\mathbf{X}$ *and* $\mathbf{Y}$.

Proof. Let $(\mathbf{X}', \mathbf{Y}')$ be a pair of links homotopic to $(\mathbf{X}, \mathbf{Y})$, such that the union of underlying multicurves $\mathsf{X}' \cup \mathsf{Y}'$ is transverse. There exists a family of pairs of links $(\mathbf{X}_t, \mathbf{Y}_t)$ for $t \in [0,1]$ such that

- $\mathbf{X}_t$ is a homotopy between $\mathbf{X}$ and $\mathbf{X}'$,
- $\mathbf{Y}_t$ is a homotopy between $\mathbf{Y}$ and $\mathbf{Y}'$,
- there is a finite subset $S \subset [0,1]$ such that, for $t \in [0,1] - S$, the union of underlying multicurves $\mathsf{X}_t \cup \mathsf{Y}_t$ is transverse, and

- for $t \in S$, $\mathsf{X}_t \cup \mathsf{Y}_t$ is transverse except for a single intersection, which is of one of the three types in Figure 6.

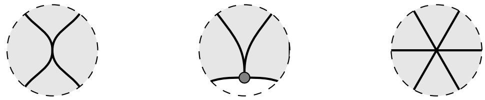


Figure 6. Elementary failures of transversality.

The superpositions $\mathbf{X}_{t_0} \cdot \mathbf{Y}_{t_0}$ and $\mathbf{X}_{t_1} \cdot \mathbf{Y}_{t_1}$ are homotopic if t_0 and t_1 are in the same component of $[0,1] - S$. If there is a single element of S between t_0 and t_1, then the two superpositions will be related by Reidemeister 2, Marked Reidemeister 2, or Reidemeister 3, depending on which of the three non-transverse intersections occurs. Then superpositions in adjacent components of $[0,1] - S$ are related by a single Reidemeister move, and so $[\mathbf{X} \cdot \mathbf{Y}]$ and $[\mathbf{X}' \cdot \mathbf{Y}']$ are related by a finite sequence of Reidemeister moves. □

Remark 3.6. The quotient of $\mathbb{Z}_q^{\mathsf{Links}}$ by the $\mathbb{Z}_q$-submodule generated by Reidemeister 2, Marked Reidemeister 2 and Reidemeister 3 also admits a well-defined superposition product, and $\mathsf{Sk}_q(\Sigma)$ can be defined as a quotient algebra of this algebra. Modified Reidemeister 1 is unnecessary for the product to be well-defined.

For general $\mathbf{X}$ and $\mathbf{Y}$, define the *superposition product* $[\mathbf{X}][\mathbf{Y}]$ by choosing homotopic links $\mathbf{X}'$ and $\mathbf{Y}'$ such that $\mathbf{X}' \cup \mathbf{Y}'$ is transverse, and letting

$$[\mathbf{X}][\mathbf{Y}] := [\mathbf{X}' \cdot \mathbf{Y}'].$$

By the proposition, this doesn't depend on the choice of $\mathbf{X}'$ and $\mathbf{Y}'$. Extend this product to all of $\mathsf{Sk}_q(\Sigma)$ by $\mathbb{Z}_q$-bilinearity.

Proposition 3.7. *The superposition product makes* $\mathsf{Sk}_q(\Sigma)$ *into an associative* $\mathbb{Z}_q$*-algebra with unit* $[\emptyset]$*, the class of the empty link.*

Proof. For links $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, find homotopic links $\mathbf{X}', \mathbf{Y}', \mathbf{Z}'$ such that the union of the underlying multicurves $\mathsf{X}' \cup \mathsf{Y}' \cup \mathsf{Z}'$ is transverse. Then

$$([\mathbf{X}][\mathbf{Y}])[\mathbf{Z}] = [\mathbf{X}' \cdot \mathbf{Y}' \cdot \mathbf{Z}'] = [\mathbf{X}]([\mathbf{Y}][\mathbf{Z}])$$

We also have $[\mathbf{X}][\emptyset] = [\mathbf{X} \cdot \emptyset] = [\mathbf{X}] = [\emptyset \cdot \mathbf{X}] = [\emptyset][\mathbf{X}]$. □

Definition 3.8. The algebra $\mathsf{Sk}_q(\Sigma)$ is the (*Kauffman*) *skein algebra* of Σ.

When Σ has no marked points, this definition coincides with the usual definition of the Kauffman skein algebra of an (unmarked) surface, defined in [28].

Remark 3.9. Some authors replace $\mathbb{Z}_q$ with a field k with a distinguished nonzero element λ, which plays the role of $q^{\frac{1}{2}}$. This setup can be recovered from ours as follows. The map $\mathbb{Z}_q \to \mathsf{k}$ with $q^{\frac{1}{2}} \mapsto \lambda$ makes k into a $\mathbb{Z}_q$-algebra. Then the k-algebra $\mathsf{k} \otimes_{\mathbb{Z}_q} \mathsf{Sk}_q(\Sigma)$ is the skein algebra defined over k. Since $\mathsf{Sk}_q(\Sigma)$ is a free $\mathbb{Z}_q$-module (Lemma 4.1), no torsion complications arise.

Remark 3.10. In [30, Definition 2.5], the authors also generalize skein algebras to 'marked surfaces.' However, they generalize skein algebras in an orthogonal direction, in that they require $\partial\Sigma = \emptyset$ but allow interior marked points. It is not clear if the two definitions can be combined in some 'best' way; see Remark 7.14.

3.4. The bar involution. For $\mathbf{X}$ any link, let $\mathbf{X}^\dagger$ be the link with the same underlying multicurve, but all crossing orders reversed.

Proposition 3.11. *The map* $[\mathbf{X}]^\dagger := [\mathbf{X}^\dagger]$ *and* $(q^{\frac{1}{2}})^\dagger := q^{-\frac{1}{2}}$ *extends to an involutive ring antiautomorphism of* $\mathsf{Sk}_q(\Sigma)$, *called the bar involution.*

Proof. Let $\dagger\colon \mathbb{Z}_q^{\mathsf{Links}(\Sigma)} \to \mathbb{Z}_q^{\mathsf{Links}(\Sigma)}$ send $[\mathbf{X}]$ to $[\mathbf{X}^\dagger]$ and $q^{\frac{1}{2}}$ to $q^{-\frac{1}{2}}$; this map is manifestly an involution. Each relation in Figure 4 goes to a relation of the same type, and so there is a quotient involution $\dagger\colon \mathsf{Sk}_q(\Sigma) \to \mathsf{Sk}_q(\Sigma)$.

For links $\mathbf{X}, \mathbf{Y}$, let $\mathbf{X}'$ and $\mathbf{Y}'$ be homotopic links with $\mathbf{X}' \cup \mathbf{Y}'$ transverse. Then

$$[\mathbf{X}]^\dagger[\mathbf{Y}]^\dagger = [\mathbf{X}^\dagger][\mathbf{Y}^\dagger] = [\mathbf{X}^\dagger \cdot \mathbf{Y}^\dagger] = [(\mathbf{Y}' \cdot \mathbf{X}')^\dagger] = [\mathbf{Y}' \cdot \mathbf{X}']^\dagger = ([\mathbf{Y}][\mathbf{X}])^\dagger.$$

Since $[\emptyset]^\dagger = [\emptyset]$, this is a ring homomorphism. □

The bar involution will be useful for two reasons. First, it shows $\mathsf{Sk}_q(\Sigma)$ is isomorphic to its opposite algebra, which cuts some proofs in half. Second, we are particularly interested in elements of $\mathsf{Sk}_q(\Sigma)$ which are fixed by the bar involution.

3.5. The endpoint E-grading. The skein algebra had an *endpoint grading*, where the degree of an arc is the formal sum of its endpoints, in the lattice $\mathbb{Z}^{\mathcal{M}}$ spanned by the marked points. This grading restricts to the following sublattice E in $\mathbb{Z}^{\mathcal{M}}$:

$$\mathsf{E} := \left\{ f\colon \mathcal{M} \longrightarrow \mathbb{Z} \;\middle|\; \text{for all connected components } \Sigma' \subseteq \Sigma, \sum_{m \in \mathcal{M} \cap \Sigma'} f(m) \text{ is even} \right\}.$$

Let E_+ be the subsemigroup whose image lands in $\mathbb{N} \subset \mathbb{Z}$.

For any $f : \mathcal{M} \to \mathbb{Z}$, let $(\mathsf{Sk}_q(\Sigma))_f$ be the $\mathbb{Z}_q$-submodule spanned by links with $f(m)$ strands at each marked point m; note that this is zero unless $f \in \mathsf{E}_+$.

Proposition 3.12. *This defines an E_+-grading on $\mathsf{Sk}_q(\Sigma)$.*

Proof. Two homotopic links have the same set of endpoints, so $\mathbb{Z}_q^{\mathsf{Links}}$ is naturally E_+-graded. The defining relations in $\mathsf{Sk}_q(\Sigma)$ are E_+-homogeneous by inspection. □

The degree zero part is the subalgebra $(\mathsf{Sk}_q(\Sigma))_0$ spanned by links without arcs; this is isomorphic to $\mathsf{Sk}_q(\Sigma_0)$, where Σ_0 is the unmarked version of Σ.

4. Simple multicurves

4.1. Simple multicurves in $\mathsf{Sk}_q(\Sigma)$. Recall that a *simple multicurve* X in Σ is a transverse multicurve with no crossings, no unknots and no contractible arcs. A simple multicurve can be regarded as a link with simultaneous endpoints; let $[\mathsf{X}]$ denote the corresponding element in $\mathsf{Sk}_q(\Sigma)$. No factor of q appears in the definition of $[\mathsf{X}]$, though it can be defined as a q-multiple of an ordered version of X.

This element is fixed by the bar involution, that is, $[\mathsf{X}]^\dagger = [\mathsf{X}]$. More so, the element $[\mathsf{X}]$ is the only $q^{\frac{1}{2}}$-multiple of itself or any other ordering of its endpoints which is fixed by the bar-involution. This gives an alternate definition of $[\mathsf{X}]$.

Let SMulti be the set of homotopy classes of simple multicurves.

Lemma 4.1. *Under $\mathsf{X} \mapsto [\mathsf{X}]$, the set SMulti maps to a $\mathbb{Z}_q$-basis of $\mathsf{Sk}_q(\Sigma)$.*

Proof. Let $\mathbb{Z}_q^{\mathsf{SMulti}}$ be the free $\mathbb{Z}_q$-module with basis SMulti. There is a map

$$s : \mathbb{Z}_q^{\mathsf{SMulti}} \longrightarrow \mathsf{Sk}_q(\Sigma)$$

which sends X to $[\mathsf{X}]$.

Define a map $\tilde{r} : \mathbb{Z}_q^{\mathsf{Links}} \to \mathbb{Z}_q^{\mathsf{SMulti}}$ as follows. Let $\mathbf{X}$ be a link.

(1) First, find $n \in \mathbb{Z}$ such that $[\mathbf{X}] = q^{\frac{n}{2}}[\mathbf{X}']$, where $\mathbf{X}'$ is identical to $\mathbf{X}$ except with the simultaneous ordering on endpoints.

(2) Then, by applying the Kauffman skein relation to each crossing in $\mathbf{X}'$, find links $\mathbf{X}_i$ and $m_i \in \mathbb{Z}$ (for an index set I) such that each $\mathbf{X}_i$ has no crossings, and

$$[\mathbf{X}'] = \sum_{i \in I} q^{m_i}[\mathbf{X}_i].$$

(3) Finally, remove contractible components of each $\mathbf{X}_i$ using the defined values. That is, let $\overline{\mathbf{X}}_i$ be $\mathbf{X}_i$ with the contractible components removed, and let $\lambda_i \in \mathbb{Z}_q$ be such that $[\mathbf{X}_i] = \lambda_i [\overline{\mathbf{X}}_i]$.

Since each $\overline{\mathbf{X}}_i$ is a simple multicurve, define

$$\tilde{r}(\mathbf{X}) = \sum_{i \in I} (q^{\frac{n}{2}+m_i} \lambda_i) \overline{\mathbf{X}}_i \in \mathbb{Z}_q^{\mathsf{SMulti}}.$$

Because the relations are local, the steps in the construction of $r(\mathbf{X})$ could have been taken in any order, with the exception of removing contractible components created by applying the Kauffman skein relation. It follows that r descends to a map

$$r\colon \mathsf{Sk}_q(\Sigma) \longrightarrow \mathbb{Z}_q^{\mathsf{SMulti}}.$$

By construction, $s(r([\mathbf{X}])) = [\mathbf{X}]$ as elements of $\mathsf{Sk}_q(\Sigma)$. If X is a simple multicurve, then the construction of $r(\mathsf{X})$ makes no changes, and so $r(s(\mathsf{X})) = \mathsf{X}$. Then s and r are inverses. □

For any element $x \in \mathsf{Sk}_q(\Sigma)$, there is a unique subset $\mathrm{Supp}(x) \subset \mathsf{SMulti}$ (called the *support* of x) and unique non-zero $\lambda_{\mathsf{Y}} \in \mathbb{Z}_q$ for each $\mathsf{Y} \in \mathrm{Supp}(x)$, such that

$$x = \sum_{\mathsf{Y} \in \mathrm{Supp}(x)} \lambda_{\mathsf{Y}} [\mathsf{Y}].$$

Remark 4.2. Because the skein relations are local, for any link $\mathbf{X}$, there are simple multicurves X_i which are each identical to $\mathbf{X}$ away from small neighborhoods of each crossing and marked point, such that $\mathrm{Supp}([\mathbf{X}]) = \{\mathsf{X}_i\}$.

Corollary 4.3. *The $\mathbb{Z}_q$-algebra $\mathsf{Sk}_q(\Sigma)$ is generated by the set of simple curves.*

Proof. If X is a simple multicurve consisting of simple curves $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$, then $[\mathsf{X}] = q^{\frac{\lambda}{2}} [\mathsf{x}_1][\mathsf{x}_2] \ldots [\mathsf{x}_n]$ for some $\lambda \in \mathbb{Z}$. Then the simple curves generate a $\mathbb{Z}_q$-subalgebra of $\mathsf{Sk}_q(\Sigma)$ which contains a basis, and so it coincides with all of $\mathsf{Sk}_q(\Sigma)$. □

4.2. Counting crossings. For any two simple multicurves X and Y, let $\mu(\mathsf{X}, \mathsf{Y})$ denote the minimum number of crossings between X' and Y', over all transverse pairs $(\mathsf{X}', \mathsf{Y}')$ homotopic to (X, Y). Note that intersections at marked points are not counted. We will say X and Y have *minimal crossings* if $\mathsf{X} \cdot \mathsf{Y}$ has $\mu(\mathsf{X}, \mathsf{Y})$ crossings.

Lemma 4.4. [8] *Let* $\mathsf{X}_1, \mathsf{X}_2, \ldots, \mathsf{X}_n$ *be a finite collection of simple multicurves. Then there are simple multicurves* $\mathsf{X}'_1, \mathsf{X}'_2, \ldots, \mathsf{X}'_n$ *such that*

- *for all* i, X'_i *is homotopic to* X_i *and*
- *for all* i *and* j, X'_i *and* X'_j *have minimal crossings.*

Idea of proof. This is done by choosing a hyperbolic metric on Σ. Then curve-shortening flow takes X_i to a geodesic X'_i, which also minimizes pairwise intersections. This may create intersections of higher order, but these can be resolved by a small perturbation. □

Corollary 4.5. *If* X *and* Y *are simple multicurves with components* $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_m$ *and* $\mathsf{y}_1, \mathsf{y}_2, \ldots, \mathsf{y}_n$*, then*

$$\mu(\mathsf{X}, \mathsf{Y}) = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \mu(\mathsf{x}_i, \mathsf{y}_j).$$

Proof. By Lemma 4.4, find X' and Y' homotopic to X and Y, respectively, so that $\mathsf{X}' \cup \mathsf{Y}'$ is transverse and each pair of components has minimal crossings. In particular, components in X' do not cross each other, and so X' is still a simple multicurve; likewise, Y' is still a simple multicurve. Since these are simple multicurves homotopic to X and Y with minimal total crossings, $\mu(\mathsf{X}, \mathsf{Y})$ is the number of crossings in $\mathsf{X}' \cup \mathsf{Y}'$, which can be counted by summing over all pairs. □

Extend μ to a map

$$\mu : \mathsf{Sk}_q(\Sigma) \times \mathsf{Sk}_q(\Sigma) \longrightarrow \mathbb{N},$$

$$\mu(x, y) := \max\{\mu\,(\mathsf{X}, \mathsf{Y}) \mid \mathsf{X} \in \mathrm{Supp}(x), \mathsf{Y} \in \mathrm{Supp}(y)\}.$$

Define $\mu(0, x) = 0$ for all x.

Remark 4.6. If $\mathbf{X}$ and $\mathbf{Y}$ are general links, then $\mu([\mathbf{X}], [\mathbf{Y}])$ is less than or equal to the minimum number of crossings between $\mathbf{X}'$ and $\mathbf{Y}'$, over all pairs $(\mathbf{X}', \mathbf{Y}')$ homotopic to $(\mathbf{X}, \mathbf{Y})$. Equality is not always true; see Lemma 4.11 for an example.

For a fixed element $x \in \mathsf{Sk}_q(\Sigma)$, $\mu(x, -)$ behaves like the degree of a polynomial.[8]

[8] If the reader enjoys complicated words, they may correctly call μ a *bisubtropical* map.

Lemma 4.7. *For $x, y, z \in \mathsf{Sk}_q(\Sigma)$,*

(1) $\mu(x, y) = \mu(y, x)$;

(2) $\mu(x, y + z) \leq \max(\mu(x, y), \mu(x, z))$;

(3) $\mu(x, yz) \leq \mu(x, y) + \mu(x, z)$;

(4) *if* $\mu(y, z) = 0$, *then* $\mu(x, yz) = \mu(x, y) + \mu(x, z)$.

Proof. The first two facts are clear from definitions, and the fourth follows from Corollary 4.5. The third fact is the only one which requires some work.

Let X, Y and Z be simple multicurves whose union is transverse, and such that each pair has minimal crossings. By Remark 4.2, there are simple multicurves T_i such that $\mathrm{Supp}(\mathsf{Y} \cdot \mathsf{Z}) = \{\mathsf{T}_i\}$ and each of the T_i are identical to $\mathsf{Y} \cdot \mathsf{Z}$ outside small neighborhoods of the intersections between Y and Z. Then the number of crossings between X and any T_i is $\mu(\mathsf{X}, \mathsf{Y}) + \mu(\mathsf{X}, \mathsf{Z})$, and so

$$\mu(\mathsf{X}, \mathsf{T}_i) \leq \mu(\mathsf{X}, \mathsf{Y}) + \mu(\mathsf{X}, \mathsf{Z}).$$

Then $\mu([\mathsf{X}], [\mathsf{Y}][\mathsf{Z}])$ is the maximum of $\mu(\mathsf{X}, \mathsf{T}_i)$ over all i, so it also satisfies the inequality.

Since any three simple multicurves are homotopic to such a triple by Lemma 4.4, the inequality is true for arbitrary simple multicurves. The general form of the inequality follows directly. □

Lemma 4.11 will give a non-trivial example where inequality (3) is strict.

Remark 4.8. Any element $x \in \mathsf{Sk}_q(\Sigma)$ gives an ascending filtration on $\mathsf{Sk}_q(\Sigma)$, by

$$\mathcal{F}_{x,i}(\mathsf{Sk}_q(\Sigma)) := \{y \in \mathsf{Sk}_q(\Sigma) \mid \mu(x, y) \leq i\}$$

4.3. Cancelling simple arcs. A useful algebraic lemma is the observation that simple arcs are not zero-divisors in $\mathsf{Sk}_q(\Sigma)$; this will eventually be used to show that all non-zero elements in $\mathsf{Sk}_1(\Sigma)$ are not zero-divisors (Corollary 6.16). This observation follows by associating an *initial multicurve* to every non-zero element of $\mathsf{Sk}_q(\Sigma)$, and then showing that multiplication by a simple curve induces an injective map on initial multicurves.

Fix an arbitrary total ordering $\preceq$ on the set of simple curves[9] in Σ. We may extend this to a total ordering $\preceq$ on the set $\mathsf{SMulti}(\Sigma)$ of simple multicurves in Σ with the following rules.[10]

[9] This will be used to break ties for defining initial terms, and will not meaningfully affect the outcome.

[10] This is analogous to a *graded lexicographic order* among monomials in a polynomial ring.

- If two multicurves have a different number of curves, then the multicurve with more curves is larger.
- If two multicurves have the same number of curves, then the multicurve with the $\preceq$-largest curve not possessed by the other multicurve is larger.

By Lemma 4.1, any element x in $\mathsf{Sk}_q(\Sigma)$ can be uniquely expressed as

$$x = \sum_{i\in\mathbb{Z}/2} q^i x_i,$$

where x_i is a $\mathbb{Z}$-linear combination of simple multicurves, with all but finitely many x_i zero. Define the *initial* multicurve $\mathrm{in}(x)$ of x to be the $\preceq$-largest simple multicurve in $\mathrm{Supp}(x_i)$, where x_i is the non-zero term in x with the largest i.

Lemma 4.9. *Let* x *be a simple arcs in* Σ. *Then the map*

$$\mathsf{Y} \longmapsto \mathrm{in}([\mathsf{x}][\mathsf{Y}])$$

is injection from SMulti *to itself.*

In fact, this map is given by choosing the 'positive smoothing' of every crossing in $\mathsf{x}\cdot\mathsf{Y}$; consequently, the above map does not depend on $\preceq$. The proof of this lemma may be found in Appendix C.

The lemma has the following algebraic consequence.

Lemma 4.10. *If* x *is a simple arc, then* $[\mathsf{x}]$ *is a not a zero divisor in* $\mathsf{Sk}_q(\Sigma)$.

Proof. Let $y \in \mathsf{Sk}_q(\Sigma)$ be such that $[\mathsf{x}]y = 0$. Write

$$y = \sum_{\mathsf{Y}\in\mathrm{Supp}(y)} \lambda_{\mathsf{Y}}[\mathsf{Y}]$$

for λ_{Y} non-zero in $\mathbb{Z}_q$. Then

$$0 = [\mathsf{x}]y = \sum_{\mathsf{Y}\in\mathrm{Supp}(y)} \lambda_{\mathsf{Y}}[x][\mathsf{Y}] = \sum_{\mathsf{Y}\in I} \lambda_{\mathsf{Y}}(q^{i_{\mathsf{Y}}}[\gamma_{\mathsf{x}}(\mathsf{Y})] + \text{lower order terms in } q).$$

Let $i = \max_I(\deg_q(\lambda_{\mathsf{Y}}) + i_{\mathsf{Y}})$, the maximal power of q appearing above,

$$0 = \mathrm{in}_q([\mathsf{x}]y) = \sum_{\substack{\mathsf{Y}\in\mathrm{Supp}(y)\\ \deg(\lambda_{\mathsf{Y}})+i_{\mathsf{Y}}=i}} \mathrm{in}_q(\lambda_{\mathsf{Y}})[\gamma_{\mathsf{x}}(\mathsf{Y})].$$

Since the map γ_{x} is an injection and SMulti is a basis, the elements $[\gamma_{\mathsf{x}}(\mathsf{Y})]$ are independent over $\mathbb{Z}_q$. Since $\mathrm{in}_q(\lambda_{\mathsf{Y}})$ cannot be zero, the support $\mathrm{Supp}(y)$ must be empty, and so $y = 0$.

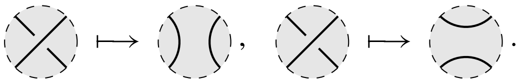
Then $[\mathsf{x}]$ is not a left zero divisor. By applying the bar involution, $[\mathsf{x}]^\dagger = [\mathsf{x}]$ is not a right zero divisor. □

4.4. Reducing crossings. Next, we consider how multiplication by a simple curve affects crossing number. We observe that multiplication by the class of a simple arc x reduces the crossing number with respect to that curve.

Lemma 4.11. *If* x *is a simple arc, then for all* $y \in \mathsf{Sk}_q(\Sigma)$ *such that* $\mu([\mathsf{x}], y) > 0$*,*

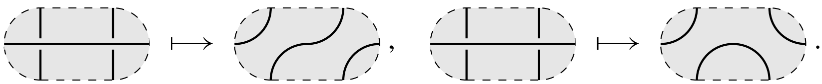
$$\mu([\mathsf{x}], [\mathsf{x}]y) \leq \mu([\mathsf{x}], y) - 1.$$

Proof. First consider the case when y is a simple multicurve Y so that $\mathsf{x} \cdot \mathsf{Y}$ is transverse, and $\mathsf{x} \cdot \mathsf{Y}$ has $\mu(\mathsf{x}, \mathsf{Y})$ crossings (the minimal number, up to homotopy). Consider the set I of multicurves which can be obtained by applying some combination of the following two local relations to each crossing in $\mathsf{x} \cdot \mathsf{Y}$.



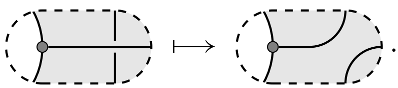
Since the simple multicurves in the support $\mathrm{Supp}([\mathsf{x}][\mathsf{Y}])$ come from applying the Kauffman skein relation to the crossings in $\mathsf{x} \cdot \mathsf{Y}$, we have $\mathrm{Supp}([\mathsf{x}][\mathsf{Y}]) \subset I$.

Consider a simple multicurve $\mathsf{Z} \in I$. For two adjacent crossings in $\mathsf{x} \cdot \mathsf{Y}$ along x, there are two local possibilities for Z, up to reflection across x.



In this local picture, the first case is homotopic to a multicurve with crossing x once, and the second is homotopic to a multicurve which does not cross x.

Between a crossing in $\mathsf{x} \cdot \mathsf{Y}$ and an end of x, there is one local possibility for Z, up to reflection across x.



This local picture is homotopic to one which does not cross x.

Then Z is homotopic to a simple multicurve Z', such that $\mathsf{x} \cdot \mathsf{Z}'$ is transverse and the crossings in $\mathsf{x} \cdot \mathsf{Z}'$ occur at most once between each pair of adjacent crossings in $\mathsf{x} \cdot \mathsf{Y}$ (and $\mathsf{x} \cdot \mathsf{Z}'$ has no other crossings). Therefore, $\mathsf{x} \cdot \mathsf{Z}'$ has strictly fewer crossings than $\mathsf{x} \cdot \mathsf{Y}$. Since the latter already has $\mu(\mathsf{x}, \mathsf{Y})$ crossings,

$$\mu([\mathsf{x}], [\mathsf{Z}']) \leq \mu(\mathsf{x}, \mathsf{Y}) - 1.$$

Because $\mathrm{Supp}([\mathsf{x}][\mathsf{Y}]) \subset I$,

$$\mu([\mathsf{x}], [\mathsf{x}][\mathsf{Y}]) \leq \mu([\mathsf{x}], [\mathsf{Y}]) - 1.$$

The general form of the lemma follows from this case. □

Remark 4.12. Multiplication by simple loops does not reduce crossing number.

Lemma 4.11 is useful, because multiplication by a sufficiently high power of $[\mathsf{x}]$ will make an element $y \in \mathsf{Sk}_q(\Sigma)$ have zero crossing number with $[\mathsf{x}]$.

Corollary 4.13. *If* x *is a simple arc, then for all* $y \in \mathsf{Sk}_q(\Sigma)$*,*

$$\mu([\mathsf{x}], [\mathsf{x}]^{\mu([x],y)} y) = 0.$$

Proof. By iterating Lemma 4.11, if $i \leq \mu([\mathsf{x}], y)$

$$\mu([\mathsf{x}], [\mathsf{x}]^i y) \leq \mu([\mathsf{x}], y) - i.$$

In particular, $\mu([\mathsf{x}], [\mathsf{x}]^{\mu([x],y)} y) \leq 0$, so it is zero. □

5. The localized skein algebra $\mathsf{Sk}_q^o(\Sigma)$

The connection from $\mathsf{Sk}_q(\Sigma)$ to cluster algebras will be through the localization $\mathsf{Sk}_q^o(\Sigma)$ of $\mathsf{Sk}_q(\Sigma)$ at the set of boundary curves.

5.1. The localized skein algebra. A *boundary curve* is a simple curve which is homotopic to a subset of the boundary $\partial\Sigma$. A boundary curve is either an arc connecting adjacent marked points on the same boundary component, or a loop homotopic to an unmarked boundary component. The set of boundary curves is finite; it is the number of marked points plus the number of unmarked boundary components.

Definition 5.1. The localization of $\mathsf{Sk}_q(\Sigma)$ at the set of boundary curves is the *localized skein algebra* of Σ, denoted $\mathsf{Sk}_q^o(\Sigma)$.

For the moment, $\mathsf{Sk}_q^o(\Sigma)$ is defined as an abstract localization; that is, the universal algebra with a map from $\mathsf{Sk}_q(\Sigma)$ such that every boundary curve is sent to a unit. This is improved with the following proposition.

Proposition 5.2. *The algebra* $\mathsf{Sk}_q^o(\Sigma)$ *is an injective Ore localization of* $\mathsf{Sk}_q(\Sigma)$.

Proof. Given a boundary curve x and a link $\mathbf{Y}$, there are homotopic links x' and $\mathbf{Y}'$ which only intersect at the boundary. Then, there is some $\lambda \in \mathbb{Z}$ such that $[\mathsf{x}][\mathbf{Y}] = q^{\frac{\lambda}{2}}[\mathbf{Y}][\mathsf{x}]$. Therefore, the set of all products of boundary curves is right and left permutable. By Ore's theorem, the localization is injective Ore. □

We will identify $\mathsf{Sk}_q(\Sigma)$ with its image in $\mathsf{Sk}_q^o(\Sigma)$.

The extra structures on $\mathsf{Sk}_q(\Sigma)$ extend to $\mathsf{Sk}_q^o(\Sigma)$. The bar involution $\dagger$ on $\mathsf{Sk}_q(\Sigma)$ (see Section 3.4) extends to an involutive ring antiautomorphism on $\mathsf{Sk}_q^o(\Sigma)$, by $(xy^{-1})^\dagger = (y^\dagger)^{-1}x^\dagger$. The endpoint E-grading on $\mathsf{Sk}_q^o(\Sigma)$ (see Section 3.5) extends to an endpoint E-grading on $\mathsf{Sk}_q^o(\Sigma)$, with $\deg(xy^{-1}) = \deg(x) - \deg(y)$.

5.2. The basis of weighted simple multicurves. The $\mathbb{Z}_q$-basis of $\mathsf{Sk}_q(\Sigma)$ by the set SMulti of simple multicurves can be extended to a $\mathbb{Z}_q$-basis of $\mathsf{Sk}_q^o(\Sigma)$ in the following (somewhat artificial) way.

Define a *weighted simple multicurve* X to be a simple multicurve X, together with an integer 'weight' w_{x} for each $\mathsf{x} \in \mathsf{X}$. Two weighted simple multicurves X and Y are *equivalent* if, for each simple curve x in Σ, the sum of the weights on curves in X homotopic to x is the same as the sum of the weights on curves in Y homotopic to x. Intuitively, a curve x of weight $w_{\mathsf{x}} \in \mathbb{N}$ is equivalent to w_{x}-many copies of x.

Let SMulti^o be the set of equivalence classes of weighted simple multicurves with positive weights on non-boundary curves (and arbitrary integral weights on boundary curves). Given $\mathsf{X} \in \mathsf{SMulti}^o$, define an element $[\mathsf{X}] \in \mathsf{Sk}_q^o(\Sigma)$ by

$$[\mathsf{X}] := q^{\frac{\lambda}{2}} \prod_{\mathsf{x} \in \mathsf{X}} [\mathsf{x}]^{w_{\mathsf{x}}},$$

where $q^{\frac{\lambda}{2}}$ is the unique q-power such that $[\mathsf{X}]^\dagger = [\mathsf{X}]$.

The $\mathbb{Z}_q$-basis of $\mathsf{Sk}_q(\Sigma)$ then extends to a $\mathbb{Z}_q$-basis of $\mathsf{Sk}_q^o(\Sigma)$.

Proposition 5.3. *Under* $\mathsf{X} \to [\mathsf{X}]$*, the set* SMulti^o *maps to a* $\mathbb{Z}_q$*-basis of* $\mathsf{Sk}_q^o(\Sigma)$.

Proof. Any element of $\mathsf{Sk}_q^o(\Sigma)$ can be written as xy^{-1}, with y a product of boundary curves. Then $y = q^j[\mathsf{Y}]$, where Y is some simple multicurve of boundary arcs. The element x can be written as

$$x = \sum_i \lambda_i [\mathsf{X}_i],$$

a $\mathbb{Z}_q$-linear combination of simple multicurves X_i. Then

$$xy^{-1} = \sum_i q^{-j}\lambda_i [\mathsf{X}_i][\mathsf{Y}]^{-1}.$$

It is always possible to add boundary curves to any simple multicurve, without violating simplicity. Let X_i' be the weighted simply multicurve which contains all the curves in X_i and Y, with each weight counting how many times a given curve appeared in X_i minus how many times it appeared in Y. Then there are λ_i' such that

$$xy^{-1} = \sum_i \lambda_i' [\mathsf{X}_i']$$

and so SMulti^o spans $\mathsf{Sk}_q^o(\Sigma)$ over $\mathbb{Z}_q$.

To show this is a $\mathbb{Z}_q$-basis, consider any relation between the weighted simple multicurves. Denominators may be cleared by multiplying by a sufficiently large multicurve $[\mathsf{Z}]$ in the boundary curves, giving a relation between weighted simple multicurves with positive weights. This gives a relation between simple multicurves in $\mathsf{Sk}_q(\Sigma)$, which must be the trivial relation (Lemma 4.1). Since $[\mathsf{Z}]$ is not a zero divisor (Lemma 4.10), the original relation was also trivial. □

This basis is fixed by the bar involution, and is homogeneous for the E-grading.

6. Triangulations

This section explores the extra structure on $\mathsf{Sk}_q(\Sigma)$ coming from a triangulation of Σ. Since triangulations only exist when there are enough marked points, this demonstrates an advantage over the unmarked case.

6.1. Triangulations. A marked surface Σ is *triangulable* if

- $\partial\Sigma$ is not empty,
- each boundary component contains a marked point, and
- no connected component of Σ is a disc with one or two marked points.[11]

A *triangulation*[12] of a triangulable Σ is a simple multicurve Δ such that

- no two curves in Δ are homotopic,
- Δ is maximal amongst simple multicurves with the first property, and
- Δ consists entirely of arcs.

A triangulation of Δ is a collection of arcs which cut Σ into a union of triangles.

[11] This last condition is unnecessary for subsequent results on skein and cluster algebras.

[12] This is sometimes called an *ideal triangulation*, to distinguish from triangulations which are allowed to have vertices away from marked points.

Remark 6.1. If only the first two conditions hold, Δ is called a *maximal multicurve*.

If $\mathsf{x} \in \Delta$ is a non-boundary arc, then is it an edge in two distinct triangles[13] in $\Sigma - \Delta$. There is a unique other curve x' such that $(\Delta - \mathsf{x}) \cup \mathsf{x}'$ is also a triangulation; both the curve and the resulting triangulation may be called the *flip* of x in Δ.

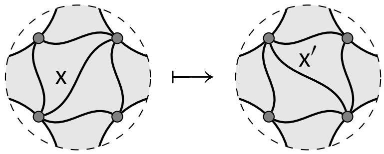


Figure 7. Flipping an arc.

Proposition 6.2. *Let Σ be a triangulable surface with marked points $\mathcal{M}$.*

(1) *Triangulations of Σ always exist.*[14]

(2) *Any simple multicurve of distinct arcs is contained in some triangulation.*

(3) *An arc is in every triangulation if and only if it is a boundary arc.*

(4) *Every triangulation has $|\Delta| = 6g + 3h + 2|\mathcal{M}| - 6$ arcs, where g is the genus and h is the number of boundary components of Σ.*

(5) *Every pair of triangulations are related by a sequence of flips.*

Proof. Our triangulations differ from those in [9], in that they forbid boundary arcs. However, their results can still be applied with appropriate modification.

(1) [9, Lemma 2.13].

(2) This follows from the given definition of 'triangulation.'

(3) A boundary arc is in every triangulation because it has no crossings with any other arcs, and so it can always be added to a simple multicurve without breaking simplicity. For any non-boundary arc x, find a triangulation containing x and flip x, to get a new triangulation which does not contain x.

(4) By [9, Proposition 2.10], there are $|\Delta| - |\mathcal{M}|$ non-boundary arcs in every triangulation. Since there are always $|\mathcal{M}|$ boundary arcs, the claim follows.

(5) [9, Proposition 3.8].

[13] This is a consequence of requiring that marked points are on the boundary. If there are interior marked points, there can be 'self-folded triangles': triangles without all edges distinct.

[14] That is, the definitions 'triangulable' and 'triangulation' behave as expected.

□

It will frequently be useful to index the arcs in a triangulation with numbers $1, 2, \ldots, |\Delta|$; this will often be done without comment. Then we can write

$$\Delta = \{\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_{|\Delta|}\}.$$

Let $\mathbb{Z}^\Delta$ denote the rank $|\Delta|$ lattice generated by the elements of Δ. For an indexed triangulation, $\mathbb{Z}^\Delta \simeq \mathbb{Z}^{|\Delta|}$, and we identify elements α of $\mathbb{Z}^\Delta$ with $|\Delta|$-tuples of integers $(\alpha_1, \alpha_2, \ldots, \alpha_{|\Delta|})$.

6.2. The orientation matrix and the signed adjacency matrix. An *end* of an arc x will be a strand of x in a small neighborhood of an endpoint. For an arc x, let $\partial_1(\mathsf{x})$ and $\partial_2(\mathsf{x})$ denote the two ends of x (for an arbitrary numbering).

For two simple curves x, y with $\mathsf{x} \cup \mathsf{y}$ simple, define

$$\Lambda_{\mathsf{x},\mathsf{y}} = \sum_{i,j \in \{1,2\}} \left\{ \begin{array}{c|c} 0 & \text{if } \partial_i(\mathsf{x}) \text{ and } \partial_j(\mathsf{y}) \text{ have different endpoints} \\ 1 & \text{if } \partial_i(\mathsf{x}) \text{ is clockwise to } \partial_j(\mathsf{y}) \\ -1 & \text{if } \partial_i(\mathsf{y}) \text{ is clockwise to } \partial_j(\mathsf{x}) \end{array} \right\}.$$

This measures the power of q which relates the superposition $[\mathsf{x}][\mathsf{y}] = [\mathsf{x} \cdot \mathsf{y}]$ to the (simultaneous) simple multicurve $[\mathsf{x} \cup \mathsf{y}]$.

Proposition 6.3. *Let* x *and* y *be simple curves with* $\mathsf{X} = \mathsf{x} \cup \mathsf{y}$ *a simple multicurve.*

$$[\mathsf{x}][\mathsf{y}] = q^{\frac{1}{2}\Lambda_{\mathsf{x},\mathsf{y}}}[\mathsf{X}] = q^{\Lambda_{\mathsf{x},\mathsf{y}}}[\mathsf{y}][\mathsf{x}].$$

Proof. This is a restatement of the boundary skein relation (Figure 4). □

Given an indexed triangulation $\Delta = \{\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_{|\Delta|}\}$, define a skew-symmetric $|\Delta| \times |\Delta|$-matrix Λ^Δ, called the *orientation matrix of* Δ, by

$$\Lambda^\Delta_{ij} := \Lambda_{\mathsf{x}_i, \mathsf{x}_j}.$$

Finally, extend Λ^Δ to a skew-symmetric bilinear form $\Lambda^\Delta \colon \mathbb{Z}^\Delta \times \mathbb{Z}^\Delta \to \mathbb{Z}$ by

$$\Lambda^\Delta(\alpha, \beta) := \langle \alpha, \Lambda^\Delta \beta \rangle = \sum_{1 \leq i,j \leq |\Delta|} \Lambda^\Delta_{ij} \alpha_i \beta_j.$$

Later on, we will also need a related matrix which measures when two ends are immediately clockwise in a triangulation. For two simple curves x, y in a indexed triangulation Δ, define

$$\mathsf{Q}^\Delta_{\mathsf{x},\mathsf{y}} = \sum_{i,j \in \{1,2\}} \left\{ \begin{array}{c|c} 0 & \text{if } \partial_i(\mathsf{x}) \text{ and } \partial_j(\mathsf{y}) \text{ have different endpoints} \\ -1 & \text{if } \partial_i(\mathsf{x}) \text{ is immediately clockwise to } \partial_j(\mathsf{y}) \text{ in } \Delta \\ 1 & \text{if } \partial_i(\mathsf{y}) \text{ is immediately clockwise to } \partial_j(\mathsf{x}) \text{ in } \Delta \end{array} \right\}.$$

Note the sign-reversal. Define a skew-symmetric $|\Delta| \times |\Delta|$ matrix Q^Δ, called the *skew-adjacency matrix of* Δ, by

$$\mathsf{Q}^\Delta_{ij} := \mathsf{Q}^\Delta_{\mathsf{x}_i,\mathsf{x}_j}.$$

Finally, extend Q^Δ to a skew-symmetric bilinear form $\mathsf{Q}^\Delta\colon \mathbb{Z}^\Delta \times \mathbb{Z}^\Delta \to \mathbb{Z}$ by

$$\mathsf{Q}^\Delta(\alpha,\beta) := \alpha^\dagger \mathsf{Q}^\Delta \beta = \sum_{1\leq i,j\leq|\Delta|} \mathsf{Q}^\Delta_{ij}\alpha_i\beta_j.$$

6.3. Monomials in Δ. Fix a triangulation Δ of Σ. For $\alpha \in \mathbb{N}^\Delta$, let Δ^α denote a simple multicurve which has α_i-many curves homotopic to x_i, for each i, and no other components. The corresponding class $[\Delta^\alpha] \in \mathsf{Sk}_q(\Sigma)$ does not depend on the choice of such a multicurve. Such an element is called a *monomial* in the triangulation Δ.

Multiplication of monomials can be computed using the following proposition.

Proposition 6.4. *We have*

$$[\Delta^\alpha] = q^{-\frac{1}{2}\sum_{i<j}\Lambda^\Delta_{ij}\alpha_i\alpha_j}[\mathsf{x}_1]^{\alpha_1}[\mathsf{x}_2]^{\alpha_2}\dots[\mathsf{x}_{|\Delta|}]^{\alpha_{|\Delta|}},$$

$$[\Delta^\alpha][\Delta^\beta] = q^{\frac{1}{2}\Lambda^\Delta(\alpha,\beta)}[\Delta^{\alpha+\beta}] = q^{\Lambda^\Delta(\alpha,\beta)}[\Delta^\beta][\Delta^\alpha].$$

Proof. The superposition product $[\mathsf{x}_1]^{\alpha_1}[\mathsf{x}_2]^{\alpha_2}\dots[\mathsf{x}_{|\Delta|}]^{\alpha_{|\Delta|}}$ corresponds to a link $\mathbf{X}$ which has the same underlying multicurve as Δ^α; however, the ordering on $\mathbf{X}$ is via superposition, and the ordering on Δ^α is simultaneous. By repeatedly applying the boundary skein relation (Figure 4), one obtains the first identity.

The second identity follows from the first identity, or by direct application of the boundary skein relation. □

Monomials can be characterized as follows. For any element $y \in \mathsf{Sk}_q(\Sigma)$, define the element $\mu_\Delta(y) \in \mathbb{N}^\Delta$ by

$$\mu_\Delta(y) := (\mu([\mathsf{x}_1],y), \mu([\mathsf{x}_2],y),\dots,\mu([\mathsf{x}_{|\Delta|}],y)).$$

Lemma 4.7 implies that

$$\mu([\Delta^\alpha],x) = \alpha\cdot\mu_\Delta(x),$$

where the dot product uses the standard basis in $\mathbb{N}^\Delta$.

Proposition 6.5. *For* X *a simple multicurve,* $[\mathsf{X}]$ *is a monomial in* Δ *if and only if* $\mu_\Delta([\mathsf{X}]) = 0$.

Proof. If $[\mathsf{X}] = [\Delta^\alpha]$, then

$$\mu([\mathsf{x}_i], [\Delta^\alpha]) = \sum_{1\leq j\leq|\Delta|} \alpha_j \mu([\mathsf{x}_i], [\mathsf{x}_j]) = 0$$

and so $\mu_\Delta([\Delta^\alpha]) = 0$.

Now, assume $\mu_\Delta([\mathsf{X}]) = 0$. Then there is a homotopic simple multicurve X' which does not cross any $\mathsf{x}_i \in \Delta$. Then each component of X' is a simple curve which does not cross any $\mathsf{x}_i \in \Delta$. Because Δ is maximal, each component of X' is homotopic to some arc in Δ. Then every component of X' is homotopic to an arc in Δ, and so $[\mathsf{X}'] = [\Delta^\alpha]$ for some α. □

A *polynomial*[15] in Δ is a $\mathbb{Z}_q$-linear combination of monomials, and the set of polynomials in Δ is a $\mathbb{Z}_q$-subalgebra of $\mathsf{Sk}_q(\Sigma)$ by the proposition. By the proposition, $x \in \mathsf{Sk}_q(\Sigma)$ is a polynomial in Δ if and only if $\mu_\Delta(x) = 0$.

Remark 6.6. A triangulation Δ gives $\mathsf{Sk}_q(\Sigma)$ an $\mathbb{N}^N$-filtration, where $\mathbb{N}^N$ has the partial order $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all i. The filtration is

$$\mathcal{F}_{\Delta,\alpha}(\mathsf{Sk}_q(\Sigma)) := \{x \in \mathsf{Sk}_q(\Sigma) \mid \mu_\Delta(x) \leq \alpha\}.$$

Then the subalgebra of polynomials in Δ is $\mathcal{F}_{\Delta,\mathbf{0}}(\mathsf{Sk}_q(\Sigma))$.

Remark 6.7. If Δ is a maximal multicurve (possibly with loops), the results of this section remain true (where $\Lambda_{ij}^\Delta := 0$ if either x_i or x_j is a loop).

6.4. Laurent expressions. In Section 4.4, it was shown that multiplying y by a sufficiently high power of an arc $[\mathsf{x}]$ had zero crossing number with $[\mathsf{x}]$. This can be directly generalized to triangulations.

Lemma 6.8. *For all $y \in \mathsf{Sk}_q(\Sigma)$, $\mu_\Delta([\Delta^{\mu_\Delta(y)}]y) = 0$.*

Proof. By Proposition 6.4, there is some $n \in \mathbb{Z}$ such that

$$[\Delta^{\mu_\Delta(y)}] = q^{\frac{1}{2}n}[\Delta^\alpha][\mathsf{x}_i]^{\mu(\mathsf{x}_i,y)}.$$

By Lemma 4.7,

$$\mu([\mathsf{x}_i], [\Delta^{\mu_\Delta(y)}]y) \leq \mu([\mathsf{x}_i], [\Delta^\alpha]) + \mu([\mathsf{x}_i], [\mathsf{x}_i]^{\mu(\mathsf{x}_i,y)}y).$$

The first term on the right is zero by Proposition 6.5, and the second is zero by Corollary 4.13. Therefore, $\mu([\mathsf{x}_i], [\Delta^{\mu_\Delta(y)}]y) = 0$ and so every term in $\mu_\Delta([\Delta^{\mu_\Delta(y)}]y)$ is zero. □

[15] Some call these 'skew-polynomials,' to emphasize their monomials only quasi-commute.

Corollary 6.9. *For all $y \in \mathsf{Sk}_q(\Sigma)$, $[\Delta^{\mu_\Delta(y)}]y$ is a polynomial in Δ.*

Remark 6.10. If $[\Delta^{\mu_\Delta(y)}]$ had a left inverse in $\mathsf{Sk}_q(\Sigma)$, then we could write

$$y = \sum_\alpha \lambda_\alpha [\Delta^{\mu_\Delta(y)}]^{-1}[\Delta^\alpha].$$

This can be regarded as a (skew) Laurent polynomial in Δ; this will be made precise by introducing quantum tori. Such an inverse does not exist in $\mathsf{Sk}_q(\Sigma)$, but it will exist in an appropriate localization.

6.5. Quantum tori. Let Λ be a skew-symmetric $N \times N$ matrix with integral coefficients. Define the (*based*) *quantum torus* $\mathbb{T}_\Lambda$ of Λ to be the associative $\mathbb{Z}_q$-algebra such that

- as a $\mathbb{Z}_q$-module, $\mathbb{T}_\Lambda$ has a free $\mathbb{Z}_q$-basis denoted M^α as α runs over $\mathbb{Z}^N$ and
- the product of these basis elements is given by

$$M^\alpha \cdot M^\beta = q^{\frac{1}{2}\Lambda(\alpha,\beta)} M^{\alpha+\beta},$$

  and general products are determined by $\mathbb{Z}_q$-bilinearity.

These are 'based' quantum tori because the lattice $\mathbb{Z}^N$ comes with an explicit basis, denoted $\{e_1, e_2, \ldots, e_N\}$. There are then distinguished elements of the form M^{e_i}, which generate $\mathbb{T}_\Lambda$ together with M^{-e_i}. The basis $\{e_1, e_2, \ldots, e_N\}$ of $\mathbb{Z}^N$ gives elements $\{M^{e_1}, M^{e_2}, \ldots, M^{e_N}\}$ and $\{M^{-e_1}, M^{-e_2}, \ldots, M^{-e_N}\}$ which generate the algebra $\mathbb{T}_\Lambda$.

Remark 6.11. The ring $\mathbb{T}_\Lambda$ is also called a ring of 'skew-Laurent polynomials.' The name 'quantum torus' is motivated as follows. The ring $\mathbb{C} \otimes_\mathbb{Z} (\mathbb{T}_\Lambda / \langle q^{\frac{1}{2}} - 1\rangle)$ is a ring of complex Laurent polynomials in N variables (independent of Λ), which is the ring of regular functions on the variety $(\mathbb{C}^*)^N$, called the 'N-dimensional algebraic torus.' In this way, $\mathbb{C} \otimes_\mathbb{Z} \mathbb{T}_\Lambda$ defines a quantization of the algebraic torus with parameter $q^{\frac{1}{2}}$.

Proposition 6.12. [17] *The quantum torus $\mathbb{T}_\Lambda$ is a Noetherian Ore domain.*

As a consequence, $\mathbb{T}_\Lambda$ embeds into its skew-field of fractions $\mathcal{F}$.

When $\Lambda = \Lambda^\Delta$, the orientation matrix of a triangulation, we write $\mathbb{T}_\Delta$ for $\mathbb{T}_{(\Lambda^\Delta)}$.

6.6. Embeddings into quantum tori. We now have all the tools needed to show that a skein algebra embeds into a quantum torus for each triangulation.

Lemma 6.13. *The set of monomials in Δ generate an Ore set.*

Proof. Let $x \in \mathsf{Sk}_q(\Sigma)$ and $[\Delta^\beta]$ be a monomial in Δ. By Corollary 6.9,

$$[\Delta^{\mu_\Delta(x)}]x = \sum_{\alpha\in\mathbb{N}^N} \lambda_\alpha[\Delta^\alpha]$$

for finitely many non-zero $\lambda_\alpha \in \mathbb{Z}_q$, and so

$$\begin{aligned}[\Delta^{\beta+\mu_\Delta(x)}]x &= q^{-\frac{1}{2}\Lambda^\Delta(\beta,\mu_\Delta(x))}[\Delta^\beta] \sum_{\alpha\in\mathbb{N}^N} \lambda_\alpha[\Delta^\alpha] \\ &= \Big(q^{-\frac{1}{2}\Lambda^\Delta(\beta,\mu_\Delta(x))} \sum_{\alpha\in\mathbb{N}^N} q^{\Lambda^\Delta(\beta,\alpha)}\lambda_\alpha[\Delta^\alpha]\Big)[\Delta^\beta].\end{aligned}$$

Then the set of monomials in Δ satisfies the left Ore condition. Since the bar-involution sends monomials to themselves, they automatically satisfy the right Ore condition as well. □

Let $\mathsf{Sk}_q(\Sigma)[\Delta^{-1}]$ be the localization at the monomials[16] in Δ. For any $\alpha \in \mathbb{Z}^\Delta$, define the *Laurent monomial* $[\Delta^\alpha] \in \mathsf{Sk}_q(\Sigma)[\Delta^{-1}]$ by the rule

$$[\Delta^{\beta'-\beta}] := q^{\frac{1}{2}\Lambda^\Delta(\beta,\beta')}[\Delta^\beta]^{-1}[\Delta^{\beta'}].$$

One may check that this is independent of the representation $\alpha = \beta' - \beta$, and the multiplication rules of Proposition 6.4 hold for general $\alpha, \beta \in \mathbb{Z}^N$.

Theorem 6.14. *For each triangulation Δ of Σ, there is an injective Ore localization*

$$\mathsf{Sk}_q(\Sigma) \hookrightarrow \mathsf{Sk}_q(\Sigma)[\Delta^{-1}] \simeq \mathbb{T}_\Delta$$

which sends $[\Delta^\alpha]$ to M^α.

Proof. The injectivity of the Ore localization $\mathsf{Sk}_q(\Sigma) \to \mathsf{Sk}_q(\Sigma)[\Delta^{-1}]$ follows because the Ore set consists of non-zero-divisors.

Let $f\colon \mathbb{T}_\Delta \to \mathsf{Sk}_q(\Sigma)[\Delta^{-1}]$ be the $\mathbb{Z}_q$-linear map defined by $f(M^\alpha) = [\Delta^\alpha]$. This is an algebra homomorphism by Proposition 6.4.

[16] This notation is non-abusive, because $\mathsf{Sk}_q(\Sigma)$ and $[\Delta]^{-1}$ generate $\mathsf{Sk}_q(\Sigma)[\Delta^{-1}]$.

Let $[\Delta^\alpha]^{-1}x$ be an arbitrary element in $\mathsf{Sk}_q(\Sigma)[\Delta^{-1}]$, with $x \in \mathsf{Sk}_q(\Sigma)$ and $\alpha \in \mathbb{N}^N$. By Corollary 6.9, $y = [\Delta^{\mu_\Delta(x)}]x$ is a polynomial in Δ, so there is some $Y \in \mathbb{T}_\Delta$ with $f(Y) = y$. Then

$$\begin{aligned} f(q^{\Lambda(\alpha,\mu_\Delta(x))}M^{-(\alpha+\mu_\Delta(x))}Y) &= q^{\Lambda(\alpha,\mu_\Delta(x))}[\Delta^{-(\alpha+\mu_\Delta(x))}]y \\ &= [\Delta^\alpha]^{-1}[\Delta^{\mu_\Delta(x)}]^{-1}y \\ &= [\Delta^\alpha]^{-1}x. \end{aligned}$$

Therefore, f is surjective.

Let $\gamma = \sum_\alpha \lambda_\alpha M^\alpha$ be an element in the kernel of f. Let $\beta \in \mathbb{N}^N$ such that $\alpha + \beta \in \mathbb{N}^N$ for all α with $\lambda_\alpha \neq 0$,

$$0 = [\Delta^\beta] f\Big(\sum_\alpha \lambda_\alpha M^\alpha\Big) = \sum_\alpha \lambda_\alpha [\Delta^\beta][\Delta^\alpha] = \sum_\alpha \lambda_\alpha q^{\lambda^\Delta(\beta,\alpha)/2}[\Delta^{\alpha+\beta}].$$

Since $\alpha+\beta$ is in $\mathbb{N}^N$, the elements $[\Delta^{\alpha+\beta}]$ are simple multicurves. By Lemma 4.1, these are independent over $\mathbb{Z}_q$, and so $\lambda_\alpha = 0$ for all α. Then the kernel of f is 0, so f is an isomorphism. □

Corollary 6.15. *The Laurent monomials in Δ are a $\mathbb{Z}_q$-basis of $\mathsf{Sk}_q(\Sigma)[\Delta^{-1}]$.*

Proof. This is true for $\mathbb{T}_\Delta$ by construction. □

Corollary 6.16. *For any Σ, $\mathsf{Sk}_q(\Sigma)$ and $\mathsf{Sk}_q^o(\Sigma)$ are Ore domains.*

Proof. If $\partial\Sigma = \emptyset$, then this is [29, Theorem 4.7]. For any Σ with $\partial\Sigma \neq \emptyset$, it is possible to add marked points to Σ to get a marked surface Σ' with a triangulation Δ. By Theorem 6.14,

$$\mathsf{Sk}_q(\Sigma) \hookrightarrow \mathsf{Sk}_q(\Sigma') \hookrightarrow \mathsf{Sk}_q(\Sigma')[\Delta^{-1}] \simeq \mathbb{T}_\Delta.$$

Then $\mathsf{Sk}_q(\Sigma)$ includes into an Ore domain, so it is an Ore domain. Since $\mathsf{Sk}_q^o(\Sigma)$ is an injective Ore localization of an Ore domain, it is also an Ore domain. □

7. Quantum cluster algebras of marked surfaces

We now turn to cluster algebras of marked surfaces. Cluster algebras are defined in terms of a set of ‘seeds’; combinatorial objects with the property that the full set of seeds can be recovered from any individual seed by ‘mutation.’ In the case of triangulable marked surfaces, seeds will correspond to triangulations and mutation will correspond to flipping an arc inside a triangulation.

There are many variations on cluster algebras. We highlight one distinction.

- *Commutative cluster algebras* $\mathcal{A}$ are (as you would expect) commutative algebras, defined as subalgebras of $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ generated by a set of elements produced by an iterative mutation rule.
- *Quantum cluster algebras* $\mathcal{A}_q$ are $\mathbb{Z}_q$-subalgebras of a skew-field $\mathcal{F}$ generated by a set of elements produced by an iterative mutation rule.

A quantum cluster algebra $\mathcal{A}_q$ always becomes a commutative cluster algebra $\mathcal{A}_1$ under the specialization $q^{\frac{1}{2}} \to 1$. However, not every commutative cluster algebra can arise this way (see Remark 7.14 for a relevant example), and multiple quantum cluster algebras can have the same commutative specialization (see Section 12.1 for a relevant example).

We focus on the quantum case, and so 'cluster algebra' will refer to a quantum cluster algebra. Commutative cluster algebras will always be labeled as such.

7.1. Quantum cluster algebras. In [11], commutative cluster algebras were introduced to axiomatize structures occurring in the study of canonical bases, and it was rapidly discovered that these algebras occur in many areas of math. In [14], the authors introduced the idea of a 'compatible' Poisson structure on a commutative cluster algebra; and in [3], these Poisson structures were 'quantized' by quantum cluster algebras.

A *quantum seed* (of skew-symmetric type[17]) in a skew-field $\mathcal{F}$ is a triple $(\mathbf{B}, \Lambda, M)$, where

- the *exchange matrix* $\mathbf{B}$ is an $N \times \mathbf{ex}$ integer matrix (for a subset $\mathbf{ex} \subseteq \{1, \ldots, N\}$), such that $\pi\mathbf{B}$ is skew-symmetric, where π is the $\mathbf{ex} \times N$ matrix which projects $\mathbb{Z}^N$ onto $\mathbb{Z}^{\mathbf{ex}}$,
- the *compatibility matrix* Λ is an $N \times N$ skew-symmetric, integer matrix, such that $\Lambda\mathbf{B} = D\iota$, where ι is the $N \times \mathbf{ex}$ matrix which includes $\mathbb{Z}^{\mathbf{ex}}$ into $\mathbb{Z}^{\mathbb{N}}$, and D is an $N \times N$ diagonal matrix with entries $D_{ii} > 0$ (the identity $\Lambda\mathbf{B} = D\iota$ is called the *compatibility condition*), and
- $M : \mathbb{Z}^N \to \mathcal{F} - \{0\}$ is a function such that
$$M(\alpha)M(\beta) = q^{\frac{1}{2}\Lambda(\alpha,\beta)} M(\alpha + \beta).$$
We require that the $\mathbb{Z}_q$-span of $M(\mathbb{Z}^N) \subset \mathcal{F}$ is a based quantum torus of Λ whose skew-field of fractions is $\mathcal{F}$.

Note that Λ can be recovered from M by the quasi-commutation relations.

[17] This is to distinguish from more general 'skew-symmetrizable' quantum seeds.

Remark 7.1. The notation for a quantum seed here differs from [3], who would write $(M, \mathbf{B})$ where we write $(\mathbf{B}, \Lambda, M)$.

The following proposition is useful to know.

Proposition 7.2. [3, Proposition 3.3],[14] *For a quantum seed* $(\mathbf{B}, \Lambda, M)$*, the matrix* $\mathbf{B}$ *has rank* $|\mathbf{ex}|$ (*the largest possible*)*.*

A quantum seed $(\mathbf{B}', \Lambda', M')$ is the *mutation at* $i \in \mathbf{ex}$ of a quantum seed $(\mathbf{B}, \Lambda, M)$, both in $\mathcal{F}$, if

- the *exchange relation* holds:

$$\mathbf{B}'_{jk} = \begin{cases} -\mathbf{B}_{jk} & \text{if } i = j \text{ or } i = k, \\ \mathbf{B}_{jk} + \frac{1}{2}(|\mathbf{B}_{ji}|\mathbf{B}_{ik} + \mathbf{B}_{ji}|\mathbf{B}_{ik}|) & \text{otherwise,} \end{cases}$$

- for $\alpha \in \mathbb{Z}^N$ such that $\alpha_i = 0$, $M(\alpha) = M'(\alpha)$, and
- the *quantum cluster relation* holds:

$$M'(e_i) = M\Big(-e_i + \sum_{\mathbf{B}_{ji}>0} \mathbf{B}_{ji} e_j\Big) + M\Big(-e_i - \sum_{\mathbf{B}_{ji}<0} \mathbf{B}_{ji} e_j\Big).$$

For a given quantum seed $(\mathbf{B}, \Lambda, M)$ and i, there always exists a unique mutation at i (see [3, Section 4.4]). Mutating twice in a row at the same index returns to the original quantum seed, and if $\mathbf{B}_{ij} = 0$, then mutating at i and at j commutes. Two quantum seeds $(\mathbf{B}, \Lambda, M)$ and $(\mathbf{B}', \Lambda', M')$ in $\mathcal{F}$ are *mutation equivalent* if they can be related by an arbitrary sequence of mutations and reordering indices.

Definition 7.3. The *quantum cluster algebra* $\mathcal{A}_q(\mathbf{B}, \Lambda, M)$ of a quantum seed $(\mathbf{B}, \Lambda, M)$ is the $\mathbb{Z}_q$-subalgebra of $\mathcal{F}$ generated by all elements of the form $M'(\alpha)$, with $(\mathbf{B}', \Lambda', M')$ mutation equivalent to $(\mathbf{B}, \Lambda, M)$, $\alpha_i \in \mathbb{N}$ for $i \in \mathbf{ex}$ and $\alpha_i \in \mathbb{Z}$ for $i \in N - \mathbf{ex}$.

When the quantum seed is clear, the cluster algebra will be denoted $\mathcal{A}_q$. An element of the form $M'(e_i) \in \mathcal{F}$ is called a *cluster variable* in $\mathcal{A}_q(\mathbf{B}, \Lambda, M)$. If $i \in \mathbf{ex}$, then $M(e_i)$ is called a *mutable variable*; otherwise, it is a *frozen variable*. Then $\mathcal{A}_q(\mathbf{B}, \Lambda, M)$ is the subalgebra of $\mathcal{F}$ generated by the cluster variables, together with the inverses of the frozen variables.

Proposition 7.4. *Any element of* $\mathcal{A}_q$ *may be written as* $a^{-1}b$*, where* a *is a product of frozen variables and* b *is a polynomial in cluster variables.*

Proof. Since they are never mutated, frozen variables are represented in every quantum seed of $\mathcal{A}_q$. Then frozen variables and their inverses quasi-commute with every cluster variable, so they may be collected on the left of any expression in $\mathcal{A}_q$. □

Any quantum cluster algebra determines a quantum upper cluster algebra.

Definition 7.5. The *quantum upper cluster algebra* $\mathcal{U}_q(\mathbf{B},\Lambda,M)$ is defined as the intersection of the based quantum tori defined by M', for each quantum seed (B',Λ',M') equivalent to (B,Λ,M),

$$\mathcal{U}_q(\mathbf{B},\Lambda,M)=\bigcap_{(\mathbf{B}',\Lambda',M')\sim(\mathbf{B},\Lambda,M)}\mathbb{Z}_q\cdot M'(\mathbb{Z}^N).$$

Remark 7.6. By [3, Theorem 5.1], it suffices to only intersect the $|\mathbf{ex}|+1$ quantum tori corresponding to $(\mathbf{B},\Lambda,M)$ and its one-step mutations.

A main result in the theory of cluster algebras is the *Laurent phenomenon*.

Theorem 7.7. [3, Corollary 5.2] $\mathcal{A}_q(\mathbf{B},\Lambda,M)\subseteq\mathcal{U}_q(\mathbf{B},\Lambda,M)$.

While this inclusion is not always equality, there are many important examples where it is. Determining when $\mathcal{A}_q=\mathcal{U}_q$ is an active area of research in both the quantum and commutative settings. Techniques for attacking this problem will be developed in Section 8.

Quantum cluster algebras are quantizations of *commutative cluster algebras*, as defined in [11]. These are commutative algebras defined only by an exchange matrix $\mathbf{B}$.

Commutative cluster algebras may be recovered from their quantizations by specializing $q^{\frac{1}{2}}$ to 1; that is, quotienting out by the ideal generated by $q^{\frac{1}{2}}-1\in\mathbb{Z}_q$,

$$\mathcal{A}_1(\mathbf{B}):=\mathcal{A}_q(\mathbf{B},\Lambda,M)/\langle q^{\frac{1}{2}}-1\rangle,$$

$$\mathcal{U}_1(\mathbf{B}):=\mathcal{U}_q(\mathbf{B},\Lambda,M)/\langle q^{\frac{1}{2}}-1\rangle.$$

7.2. Quantum cluster algebras of marked surfaces. In [15], the authors observe that a triangulable marked surface Σ determines a commutative cluster algebra.We now extend their construction to a quantum cluster algebra.

Let Σ be a marked surface, and let $\mathcal{F}$ be the skew-field of fractions of the skein algebra $\mathsf{Sk}_q(\Sigma)$. For any triangulation Δ, construct a quantum seed in $\mathcal{F}$ as follows.

- **ex** $\subset \{1, 2, \ldots, N\} \simeq \Delta$ is the subset of non-boundary arcs [18] in Δ.
- $\mathbf{B}^\Delta = \mathsf{Q}^\Delta \circ \iota$, where $\iota : \mathbb{Z}^{\mathbf{ex}} \to \mathbb{Z}^N$ is the natural inclusion.
- Λ^Δ is the orientation matrix of Δ.
- $M^\Delta : \mathbb{Z}^N \to \mathcal{F}$ is given by $M_\Delta(\alpha) = [\Delta^\alpha]$.

Proposition 7.8. *The triple* $(\mathbf{B}^\Delta, \Lambda^\Delta, M^\Delta)$ *is a quantum seed.*

Proof. The only non-trivial fact to prove is that $\Lambda^\Delta \mathbf{B}^\Delta = 4\iota$ (the *compatibility condition*). Let $\mathsf{x}_j \in \Delta$ be a non-boundary arc. For $\mathsf{x}_i \in \Delta$, consider the matrix entry

$$(\Lambda^\Delta \mathsf{Q}^\Delta)_{ij} = \sum_{1 \le k \le N} \Lambda^\Delta_{ik} \mathsf{Q}^\Delta_{kj}.$$

The curve x_j is an edge in two distinct triangles in $\Sigma - \Delta$. Let $\mathsf{x}_{k_1}, \mathsf{x}_{k_2}, \mathsf{x}_{k_3}, \mathsf{x}_{k_4}$ be the other arcs around these triangles, ordered as in Figure 8. Note that these arcs need not be distinct nor have distinct endpoints, despite the figure.

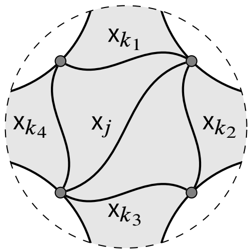


Figure 8. The adjacent arcs.

From the definition of Q^Δ, $\mathsf{Q}^\Delta_{kj} = (-1)^\ell$ if $k = k_\ell$, and 0 otherwise. Then

$$(\Lambda^\Delta \mathsf{Q}^\Delta)_{ij} = \sum_{1 \le \ell \le 4} (-1)^\ell \Lambda^\Delta_{ik_\ell}.$$

The arcs k_ℓ need not be distinct for the above sum to remain valid.

We consider x_i in three cases.

- Case 1: $i \notin \{j, k_1, k_2, k_3, k_4\}$. At each end of x_i, either there are no ends of the arcs x_{k_ℓ}, or there are two of the form x_{k_ℓ} and $\mathsf{x}_{k_{\ell+1}}$ for some ℓ. In the latter case, both x_{k_ℓ} and $\mathsf{x}_{k_{\ell+1}}$ are either clockwise or counter-clockwise to x_i, and so $\Lambda^\Delta_{ik_\ell} = \Lambda^\Delta_{ik_{\ell+1}}$. Therefore, $(\Lambda^\Delta \mathsf{Q}^\Delta)_{ij} = 0$.

[18] Recall that a *boundary arc* is an arc homotopic to an arc contained in the boundary $\partial\Sigma$.

- Case 2: $i = k_\ell$ for some ℓ. Then $\Lambda^\Delta_{ik_{\ell+1}} = -\Lambda^\Delta_{ik_{\ell-1}}$ and all others are zero, so $(\Lambda^\Delta \mathsf{Q}^\Delta)_{ij} = 0$.
- Case 3: $i = j$. In this case, $\Lambda^\Delta_{ik_\ell} = \mathsf{Q}^\Delta_{ik_\ell} = (-1)^\ell$, and so $(\Lambda^\Delta \mathsf{Q}^\Delta)_{ij} = 4$.

By definition, e_j is in the image of ι if and only if x_j is a non-boundary arc, so $\Lambda^\Delta \mathbf{B}^\Delta = \Lambda^\Delta \mathsf{Q}^\Delta \iota = 4\iota$. □

From the definitions, $M^\Delta(e_i) = [\Delta^{e_i}] = [\mathsf{x}_i] \in \mathsf{Sk}_q(\Sigma)$.

Theorem 7.9. *For any triangulation Δ, and any flip Δ' of Δ at a non-boundary arc x_j, $(\mathbf{B}^{\Delta'}, \Lambda^{\Delta'}, M^{\Delta'})$ is the mutation of $(\mathbf{B}^\Delta, \Lambda^\Delta, M^\Delta)$ at j.*

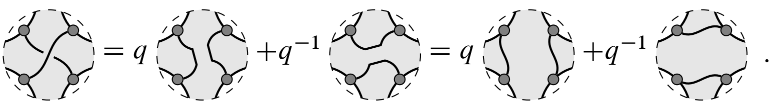
Proof. The exchange relation is unchanged from the commutative version of this theorem, which can be found in [9, Proposition 4.8]. It is also clear that $M^\Delta(\alpha) = M^{\Delta'}(\alpha)$ if $\alpha_j = 0$. The remaining work is the quantum cluster relation.

Let x'_j be the flip of x_j in Δ, so that $\Delta' = (\Delta - \mathsf{x}_j) \cup \mathsf{x}'_j$, and let $\mathsf{x}_{k_1}, \mathsf{x}_{k_2}, \mathsf{x}_{k_3}, \mathsf{x}_{k_4}$ be as in Figure 8. Because the endpoints of x_j and x'_j need not be distinct, the superposition $\mathsf{x}_j \cdot \mathsf{x}'_j$ may not have the simultaneous ordering on all of its ends. Let $\mathbf{X}$ be the link which is identical to $\mathsf{x}_j \cdot \mathsf{x}'_j$ except with the simultaneous ordering on the ends. There is then some $\lambda \in \mathbb{Z}$ such that

$$[\mathsf{x}_j][\mathsf{x}'_j] = q^{\frac{1}{2}\lambda}[\mathbf{X}].$$

If the endpoints of x_j and x'_j are all distinct, this correction is unneeded and $\lambda = 0$.

The link $\mathbf{X}$ has a single transverse crossing; by the Kauffman skein relation,



In the second equality, we have used homotopy to show that the resulting links have components corresponding to $\mathsf{x}_{k_2}, \mathsf{x}_{k_4}$ and $\mathsf{x}_{k_1}, \mathsf{x}_{k_3}$, respectively. Since $\mathbf{X}$ had simultaneous ends and a single transverse crossing, we can be assured that the right-hand side consists of simple multicurves. Therefore,

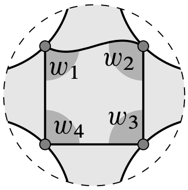
$$q^{-\frac{1}{2}\lambda}[\mathsf{x}_j][\mathsf{x}'_j] = q[\mathsf{x}_{k_2} \cup \mathsf{x}_{k_4}] + q^{-1}[\mathsf{x}_{k_1} \cup \mathsf{x}_{k_3}].$$

which we may rewrite as monomials in Δ, and divide by $q^{-\frac{1}{2}\lambda}[\Delta^{e_j}]$.

$$[(\Delta')^{e_j}] = q^{\frac{1}{2}(\lambda+2)}[\Delta^{-e_j}][\Delta^{e_{k_2}+e_{k_4}}] + q^{\frac{1}{2}(\lambda-2)}[\Delta^{-e_j}][\Delta^{e_{k_1}+e_{k_3}}]. \tag{7.1}$$

Lemma 7.10. $\lambda = \Lambda^\Delta_{jk_2} + \Lambda^\Delta_{jk_4} - 2 = \Lambda^\Delta_{jk_1} + \Lambda^\Delta_{jk_3} + 2$.

Proof. Let w_1, w_2, w_3, w_4 denote the four corners of the quadralateral cut out by the $\{\mathsf{x}_{k_\ell}\}$, thought of as wedges in small neighborhoods of the marked points,



.

For $m \neq n \in \{1, 2, 3, 4\}$, define

$$\Pi_{m,n} = \left\{ \begin{array}{c|c} 0 & \text{if } w_m \text{ and } w_n \text{ have disjoint marked points} \\ 1 & \text{if } w_m \text{ is clockwise to } w_n \text{ at a shared marked point} \\ -1 & \text{if } w_n \text{ is clockwise to } w_m \text{ at a shared marked point} \end{array} \right\}.$$

Since the interiors of the wedges are disjoint from each other, this is well-defined. Note that $\Pi_{m,n} = -\Pi_{n,m}$. From the definitions,

$$\lambda = \Pi_{2,1} + \Pi_{2,3} + \Pi_{4,1} + \Pi_{4,3},$$

$$\Lambda^\Delta_{jk_2} = 1 + \Pi_{2,3} + \Pi_{4,2} + \Pi_{4,3},$$

$$\Lambda^\Delta_{jk_4} = 1 + \Pi_{4,1} + \Pi_{2,4} + \Pi_{2,1}.$$

The first equality follows. The second equality is proved similarly. △

The lemma and Proposition 6.4 imply that

$$\begin{aligned} q^{\frac{1}{2}(\lambda+2)}[\Delta^{-e_j}][\Delta^{e_{k_2}+e_{k_4}}] &= q^{\frac{1}{2}(\lambda+2)} q^{\frac{1}{2}(-\Lambda^\Delta_{jk_2}-\Lambda^\Delta_{jk_4})}[\Delta^{e_{k_2}+e_{k_4}-e_j}] \\ &= [\Delta^{e_{k_2}+e_{k_4}-e_j}] \end{aligned}$$

and

$$\begin{aligned} q^{\frac{1}{2}(\lambda-2)}[\Delta^{-e_j}][\Delta^{e_{k_1}+e_{k_3}}] &= q^{\frac{1}{2}(\lambda-2)} q^{\frac{1}{2}(-\Lambda^\Delta_{jk_1}-\Lambda^\Delta_{jk_3})}[\Delta^{e_{k_1}+e_{k_3}-e_j}] \\ &= [\Delta^{e_{k_1}+e_{k_3}-e_j}]. \end{aligned}$$

Equation (7.1) then becomes

$$[(\Delta')^{e_j}] = [\Delta^{e_{k_2}+e_{k_4}-e_j}] + [\Delta^{e_{k_1}+e_{k_3}-e_j}].$$

Switching term on the right, this is the quantum cluster relation, as required.

$$M^{\Delta'}(e_j) = M^\Delta\Big(-e_j + \sum_{\mathbf{B}_{kj}>0} \mathbf{B}_{kj} e_k\Big) + M^\Delta\Big(-e_j - \sum_{\mathbf{B}_{kj}<0} \mathbf{B}_{kj} e_k\Big).$$

Then $(\mathbf{B}^{\Delta'}, \Lambda^{\Delta'}, M^{\Delta'})$ is the mutation of $(\mathbf{B}^\Delta, \Lambda^\Delta, M^\Delta)$ at j. □

Corollary 7.11. *For any two triangulations* Δ *and* Δ' *of* Σ*, the quantum seed* $(\mathbf{B}^{\Delta'}, \Lambda^{\Delta'}, M^{\Delta'})$ *is mutation equivalent to* $(\mathbf{B}^{\Delta}, \Lambda^{\Delta}, M^{\Delta})$*, and every seed mutation equivalent to* $(\mathbf{B}^{\Delta}, \Lambda^{\Delta}, M^{\Delta})$ *is of this form.*

Proof. Every mutation at $i \in \mathbf{ex}$ corresponds to a flipping a non-boundary arc in a triangulation, so any sequence of mutations corresponds to a sequence of flips. Since every triangulation Δ' is related to Δ by a sequence of flips, every quantum seed coming from a triangulation is mutation equivalent to $(\mathbf{B}^{\Delta}, \Lambda^{\Delta}, M^{\Delta})$. □

Thus, we can speak unambiguously about 'the' quantum cluster algebra $\mathcal{A}_q(\Sigma)$ and quantum upper cluster algebra $\mathcal{U}_q(\Sigma)$ of a triangulable marked surface Σ.

Definition 7.12. For any triangulation Δ of Σ, the subalgebras

$$\mathcal{A}_q(\Sigma) := \mathcal{A}_q(\mathbf{B}^{\Delta}, \Lambda^{\Delta}, M^{\Delta}) \subset \mathcal{F},$$

$$\mathcal{U}_q(\Sigma) := \mathcal{U}_q(\mathbf{B}^{\Delta}, \Lambda^{\Delta}, M^{\Delta}) \subset \mathcal{F}$$

are the *quantum cluster algebra of* Σ and the *quantum upper cluster algebra of* Σ, respectively.

Remark 7.13. These quantum cluster algebras are quantizations of the commutative cluster algebras of marked surfaces defined in [15] and [9], with *boundary coefficients*. This means the coefficients (in the sense of [11]) are the Laurent ring generated by the set of boundary arcs. The coefficient-free case may be recovered by quotienting $\mathcal{A}_q(\Sigma)$ or $\mathcal{U}_q(\Sigma)$ by the ideal generated by $q^{\frac{1}{2}} - 1$ and $\{\mathsf{x} - 1\}$ as x runs over the set of boundary arcs.

Remark 7.14. We can now justify requiring that the marked points are contained in the boundary. For a marked surface Σ with internal marked points, there is an associated commutative cluster algebra $\mathcal{A}(\Sigma)$ defined in [14] and [9] (where the coefficients are the Laurent ring generated by the boundary arcs). It is possible to use the 'tagged arcs' of [9] to define a commutative 'tagged skein algebra' $\mathsf{Sk}(\Sigma)$ (with $q^{\frac{1}{2}} = 1$) which has a localization $\mathsf{Sk}^o(\Sigma)$ which is naturally a cluster algebra.[19]

However, for any triangulation Δ of Σ, the corresponding exchange matrix $\mathbf{B}^{\Delta}$ will never be of full rank. Therefore, by Proposition 7.2, this commutative cluster algebra admits no quantization. It is possible there is a well-behaved generalization of Σ to the case of internal marked points for general q, but it cannot correspond to the quantum cluster algebra of Σ (with coefficients coming from boundary arcs).

[19] This is intended for a subsequent publication.

7.3. Relation to the skein algebra. The algebras $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$ were defined as subalgebras of $\mathcal{F}$, the skew-field of fractions of $\mathsf{Sk}_q^o(\Sigma)$, and so the three algebras can be compared as subalgebras.

Theorem 7.15. *For any triangulable marked surface* Σ,

$$\mathcal{A}_q(\Sigma) \subseteq \mathsf{Sk}_q^o(\Sigma) \subseteq \mathcal{U}_q(\Sigma).$$

It will be shown in Theorem 9.8 that these inclusions are equalities so long as Σ contains at least two marked points.

Proof of Theorem 7.15. Let Δ be a triangulation of Σ. By definition, $M^\Delta(\alpha) = [\Delta^\alpha]$. For any $\alpha \in \mathbb{Z}^N$ with $\alpha_i \geq 0$ for $i \in \mathbf{ex}$, write $\alpha = \beta - \beta'$, where $\beta, \beta' \in \mathbb{N}^N$ and $\beta'_i = 0$ for $i \in \mathbf{ex}$. Then

$$M^\Delta(\alpha) = [\Delta^\alpha] = q^{-\frac{1}{2}\Lambda^\Delta(\beta,\beta')}[\Delta^\beta][\Delta^{\beta'}]^{-1}.$$

Since $[\Delta^{\beta'}]$ is a monomial in the boundary arcs, $M^\Delta(\alpha) \in \mathsf{Sk}_q^o(\Sigma)$. Since this is true for any quantum seed and any α with $\alpha_i \geq 0$ for $i \in \mathbf{ex}$, $\mathcal{A}_q(\Sigma) \subseteq \mathsf{Sk}_q^o(\Sigma)$.

The quantum torus $\mathbb{Z}_q \cdot M^\Delta(\mathbb{Z}^N)$ is the same as $\mathbb{T}_\Delta$, because they are both the $\mathbb{Z}_q$-span of the $[\Delta^\alpha]$ for $\alpha \in \mathbb{Z}^N$. Then, Theorem 6.14 implies that $\mathsf{Sk}_q^o(\Sigma) \subseteq \mathbb{T}_\Delta$. Since this is true for any quantum seed, $\mathsf{Sk}_q^o(\Sigma) \subseteq \mathcal{U}_q(\Sigma)$. □

Remark 7.16. Under this inclusion, $\mathcal{A}_q(\Sigma)$ is the $\mathbb{Z}_q$-subalgebra of $\mathsf{Sk}_q^o(\Sigma)$ generated by arcs (ie, cluster variables) and inverses to boundary arcs. Hence, the definition given here for $\mathcal{A}_q(\Sigma)$ (Definition 7.12) agrees with the one give in the introduction.

Remark 7.17. Let $\mathcal{A}_q^\natural(\Sigma)$ be the $\mathbb{Z}_q$-subalgebra of $\mathcal{F}$ generated by the cluster variables, but not the inverses to 'frozen' variables. Then $\mathcal{A}_q^\natural(\Sigma) \subset \mathsf{Sk}_q(\Sigma)$ as the subalgebra generated by the arcs; however, this is only an equality when Σ is contractible. If there is a non-trivial loop $\ell \in \Sigma$, then $[\ell] \in \mathsf{Sk}_q(\Sigma)$ is not a scalar, but does have E-degree zero. This cannot happen in $\mathcal{A}_q^\natural(\Sigma)$, so $\mathcal{A}_q^\natural(\Sigma) \neq \mathsf{Sk}_q(\Sigma)$.

7.4. Laurent formulae and denominators. Given a link $\mathbf{X}$ and a triangulation Δ, the proof of Theorem 6.14 gives an explicit method to express $[\mathbf{X}]$ as an element of the quantum torus $\mathbb{T}_\Delta$. Specifically, applying the Kauffman skein relation repeatedly to $[\Delta^{\mu_\Delta([\mathbf{X}])}][\mathbf{X}]$ eventually gives a polynomial $\sum_\alpha \lambda_\alpha[\Delta^\alpha]$ in Δ, and so

$$[\mathbf{X}] = [\Delta^{\mu_\Delta([\mathbf{X}])}]^{-1}\sum_\alpha \lambda_\alpha[\Delta^\alpha] = \sum_\alpha \lambda'_\alpha[\Delta^{\alpha-\mu_\Delta([\mathbf{X}])}].$$

Since $\mathcal{A}_q(\Sigma) \subseteq \mathsf{Sk}_q^o(\Sigma)$, this can also be applied to any cluster variable in $\mathcal{A}_q(\Sigma)$. A cluster variable will correspond to a simple arc x, and so

$$[\mathsf{x}] = [\Delta^{\mu_\Delta([\mathsf{x}])}]^{-1} \sum_\alpha \lambda_\alpha [\Delta^\alpha] = \sum_\alpha \lambda'_\alpha [\Delta^{\alpha - \mu_\Delta([\mathsf{x}])}].$$

This gives an effective method for expressing a cluster variable as a skew-Laurent polynomial in the cluster variables of any other cluster. This approach is already well-known for commutative cluster algebras. For discs, explicit formulas appear in the work of Schiffler [31], and are more explicitly related to the skein relations in [16, Section 2.1.5]. For general marked surfaces, a related method for producing Laurent expansions in terms of *T-paths* and *snake graphs* has been developed in [32], [33], and [22].

One consequence of this formula is a denominator $[\Delta^{\mu_\Delta([\mathsf{x}])}]$ for the skew-Laurent expression. This is the smallest possible denominator, as this proposition shows.

Proposition 7.18 ([9, Theorem 8.6][20]). *If* x *is a simple arc in* Σ*,* Δ *is a triangulation of* Σ *and* $[\Delta^\alpha][\mathsf{x}] \in \mathsf{Sk}_q(\Sigma)$ *is a polynomial in* Δ*, then*

$$\alpha - \mu_\Delta([\mathsf{x}]) \in \mathbb{N}^\Delta.$$

7.5. Gradings on $\mathcal{A}_q(\Sigma)$.

In [14, Section 2.2], the authors define a grading[21] on any cluster algebra, which is the 'largest possible' compatible grading. For $\mathcal{A}_q(\Sigma)$, this is shown to coincide with the endpoint E-grading defined in Section 3.5.

For any abelian group L, an L-grading on a cluster algebra $\mathcal{A}_q$ is *compatible* if each cluster variable is homogeneous. Given a compatible L-grading on $\mathcal{A}_q$, a morphism $f : L \to L'$ induces a compatible L'-grading on $\mathcal{A}_q$, by $\deg_{L'}(x) := f(\deg_L(x))$. A compatible grading on $\mathcal{A}_q$ is *universal* if it is the initial object in the category of compatible gradings of $\mathcal{A}_q$ and induction maps between them.

In [14], the authors define compatible gradings, and characterize the universal compatible grading of any cluster algebra.

Lemma 7.19 ([14, Lemma 2.3][22]). *Let* $(\mathbf{B}, \Lambda, M)$ *be a quantum seed for* $\mathcal{A}_q$*. Then*

$$\deg(M(\alpha)) = \alpha + \mathbf{B}(\mathbb{Z}^{\mathbf{ex}}) \in (\mathbb{Z}^N / \mathbf{B}(\mathbb{Z}^{\mathbf{ex}}))$$

extends to a universal compatible grading of $\mathcal{A}_q$ *by* $\mathbb{Z}^N / \mathbf{B}(\mathbb{Z}^{\mathbf{ex}})$.

[20] Their result is for commutative cluster algebras, but it implies the quantum result.

[21] In truth, [14] define a torus action on $\mathcal{A}$, but semi-simple torus actions by $\mathbb{T}$ are equivalent to gradings by the character lattice of $\mathbb{T}$.

[22] The result in [14] is stated for torus actions, but is equivalent to the result stated here.

For the cluster algebras we consider, this coincides with the endpoint E-grading.

Proposition 7.20. *For any Δ, the map $\delta : \mathbb{Z}^\Delta \to \mathsf{E}$ which sends an arc in Δ to its endpoints induces an isomorphism $\mathbb{Z}^\Delta/\mathbf{B}(\mathbb{Z}^{\mathbf{ex}}) \xrightarrow{\sim} \mathsf{E}$.*

Proof. Simple arcs in Σ are E-homogeneous elements in $\mathsf{Sk}_q(\Sigma)$, so $\mathcal{A}_q(\Sigma)$ is generated by E-homogeneous elements (cluster variables and inverses to frozen variables); it follows that $\mathcal{A}_q(\Sigma)$ is compatibly E-graded.

Fix a triangulation Δ, and let $\delta\colon \mathbb{Z}^\Delta \to \mathsf{E}$ be the map which sends a monomial in Δ to its endpoint degree. The map δ kills the image $\mathbf{B}^\Delta(\mathbb{Z}^{\mathbf{ex}})$, and so it descends to a map $\delta'\colon \mathbb{Z}^\Delta/\mathbf{B}^\Delta(\mathbb{Z}^{\mathbf{ex}}) \to \mathsf{E}$ which is the map which induces the E-grading from the $\mathbb{Z}^\Delta/\mathbf{B}^\Delta(\mathbb{Z}^{\mathbf{ex}})$-grading.

For every pair of marked points in a connected component of Σ, there is an arc connecting them. The degrees of these arcs generated E, and so the map δ' is surjective.

The lattice E is a full-rank sublattice of $\mathbb{Z}^{\mathcal{M}}$, so it has rank $|\mathcal{M}|$. The lattice $\mathbb{Z}^\Delta/\mathbf{B}(\mathbb{Z}^{\mathbf{ex}})$ has rank equal to $|\Delta| - \mathrm{rank}(\mathbf{B}^\Delta)$. By Proposition 7.2, $\mathrm{rank}(\mathbf{B}^\Delta) = |\mathbf{ex}|$, and $|\Delta| - |\mathbf{ex}|$ is the number of boundary arcs $|\mathcal{M}|$. Then δ' is a surjective maps between lattices of the same rank, so it is an isomorphism. □

Corollary 7.21. *The endpoint E-grading on $\mathsf{Sk}_q^o(\Sigma)$ restricts to a universal compatible grading on $\mathcal{A}_q(\Sigma)$.*

8. A general technique for $\mathcal{A}_q = \mathcal{U}_q$

In this section, we develop a technique for simultaneously proving $\mathcal{A}_q = \mathcal{U}_q$ for classes of cluster algebras. Many of the ideas here are quantum analogs of commutative ideas which appeared in [25].

8.1. Exchange types. An $n \times n$ integral skew-symmetric matrix A may be *mutated* at an index $i \in \{1, \dots, n\}$ using the exchange relation as in Section 7. By construction, this notion of mutation is compatible with mutation of quantum seeds under the map which sends any quantum seed $(\mathbf{B}, \Lambda, M)$ to the matrix[23] $\pi\mathbf{B}$.

An *exchange type* $\mathcal{T}$ is an equivalence class of skew-symmetric matrices, under the relation generated by mutation and conjugation by a permutation matrix.

[23] Recall that π is the the $\mathbf{ex} \times N$-matrix which projects $\mathbb{Z}^N$ onto $\mathbb{Z}^{\mathbf{ex}}$, so that $\pi\mathbf{B}$ is the 'principal part' of $\mathbf{B}$ (see Section 7.1).

Given a quantum seed $(\mathbf{B}, \Lambda, M)$, the exchange type of $\pi\mathbf{B}$ consists of matrices of the form $\pi\mathbf{B}'$ for quantum seeds $(\mathbf{B}', \Lambda', M')$ mutation equivalent to $(\mathbf{B}, \Lambda, M)$. We say the *exchange type* of a quantum seed $(\mathbf{B}, \Lambda, M)$ is the exchange type of $\pi\mathbf{B}$, and the *exchange type* of a cluster algebra $\mathcal{A}_q$ is the exchange type of any of its quantum seeds.

The results which follow depend only on the exchange type of a cluster algebra.

Remark 8.1. An $n \times n$ integral skew-symmetric matrix A can be encoded in a *quiver* $Q(\mathsf{A})$, with vertex set $\{1, \ldots, n\}$ and A_{ij}-many arrows from j to i (where negative arrows are from i to j). Mutation can be encoded as an operation on a quiver [20, Section 2], and exchange types correspond to mutation-equivalence classes of quivers.

8.2. Isolated cluster algebras. A quantum seed or cluster algebra is called *isolated* if its exchange type is the zero matrix. Concretely, a cluster algebra is isolated if every quantum seed $(\mathbf{B}, \Lambda, M)$ has $\pi\mathbf{B} = 0$.

Proposition 8.2. *If $\mathcal{A}_q$ is isolated, $\mathcal{A}_q = \mathcal{U}_q$.*

Remark 8.3. In [3, Theorem 7.5], the authors show that $\mathcal{A}_q = \mathcal{U}_q$ whenever $\mathcal{A}_q$ has an *acyclic* exchange type, which immediately implies this proposition. We include a proof anyway, because a by-product of the techniques we develop will be a new proof of Berenstein and Zelevinsky's theorem (Proposition 8.17).

Proof of Proposition 8.2. Let $(\mathbf{B}, \Lambda, M)$ be a seed for $\mathcal{A}_q$, with corresponding quantum torus $\mathbb{T}_\Lambda$. Let R denote the subring of $\mathcal{A}_q$ generated by the frozen variables and their inverses. The ring R is naturally a quantum subtorus of $\mathbb{T}_\Lambda$, and so $\mathbb{T}_\Lambda$ is a free left R-module with basis $\{M(\alpha)\}$ as α runs over $\mathbb{Z}^{\mathbf{ex}}$.

Since $\mathbf{B}_{ij} = 0$ for all $i, j \in \mathbf{ex}$, all mutations commute with each other. Mutating once at each $i \in \mathbf{ex}$ in any order gives the quantum seed $((-1)^{|\mathbf{ex}|}\mathbf{B}, \Lambda', M')$, with[24]

$$P_i := M'(e_i)M(e_i) = q^\bullet M\Big(\sum_{\mathbf{B}_{ji}>0} \mathbf{B}_{ji} e_j\Big) + q^\bullet M\Big(-\sum_{\mathbf{B}_{ji}<0} \mathbf{B}_{ji} e_j\Big).$$

Since the expression on the right contains no indices in $\mathbf{ex}$, $P_i \in R$.

[24] Here, and throughout, $q^\bullet$ denotes a half-power of q not worth keeping careful track of.

Choose an element $x \in \mathcal{U}_q \subseteq \mathbb{T}_\Lambda$, write (for $\lambda_\alpha \in R$)

$$x = \sum_\alpha \lambda_\alpha M(\alpha) = \sum_{\alpha \in \mathbb{Z}^{\mathbf{ex}}} q^\bullet \lambda_\alpha M(e_1)^{\alpha_1} M(e_2)^{\alpha_2} \dots M(e_{|\mathbf{ex}|})^{\alpha_{|\mathbf{ex}|}}.$$

Let us rewrite x by replacing some of these cluster variables with their mutation. Choose any $I \subset \mathbf{ex}$.

$$\begin{aligned} x &= \sum_{\alpha \in \mathbb{Z}^{\mathbf{ex}}} q^\bullet \lambda_\alpha \Big(\prod_{i \in I} M(e_i)^{\alpha_i} \Big) \Big(\prod_{i \notin I} M(e_i)^{\alpha_i} \Big) \\ &= \sum_{\alpha \in \mathbb{Z}^{\mathbf{ex}}} q^\bullet \lambda_\alpha \Big(\prod_{i \in I} (M'(e_i)^{-1} P_i)^{\alpha_i} \Big) \Big(\prod_{i \notin I} M(e_i)^{\alpha_i} \Big) \\ &= \sum_{\alpha \in \mathbb{Z}^{\mathbf{ex}}} q^\bullet \lambda_\alpha \Big(\prod_{i \in I} P_i^{\alpha_i} \Big) \Big(\prod_{i \in I} M'(e_i)^{-\alpha_i} \Big) \Big(\prod_{i \notin I} M(e_i)^{\alpha_i} \Big). \end{aligned}$$

Let $\mathbb{T}_I$ be the quantum torus corresponding to the seed which is the mutation of $(\mathbf{B}, \Lambda, M)$ at the set I in any order. The cluster variables in this seed are $\{M'(e_i)\}_{i \in I} \cup \{M(e_i)\}_{i \notin I}$. Since the exponents α_i may be negative, it is not immediate that the coefficient $\lambda_\alpha \prod_{i \in I} P_i^{\alpha_i}$ is an element of R. However, because $x \in \mathcal{U}_q$, it is also in $\mathbb{T}_I$, and so this coefficient is in R.

Let $I_\alpha \subseteq \mathbf{ex}$ be the set on which α is negative. Then

$$x = \sum_{\alpha \in \mathbb{Z}^{\mathbf{ex}}} q^\bullet \Big(\lambda_\alpha \prod_{i \in I_\alpha} P_i^{\alpha_i} \Big) \Big(\prod_{i \in I_\alpha} M'(e_i)^{-\alpha_i} \Big) \Big(\prod_{i \notin I_\alpha} M(e_i)^{\alpha_i} \Big).$$

This expression is in $\mathcal{A}_q$, so $\mathcal{A}_q = \mathcal{U}_q$. □

Remark 8.4. This proof is essentially the same as that of [1, Lemma 4.1].

8.3. Freezing and cluster localization. Let $\mathcal{A}_q$ be a quantum cluster algebra, with skew-field of fractions $\mathcal{F}$. Fix a quantum seed $(\mathbf{B}, \Lambda, M)$ of $\mathcal{A}_q$, and choose a set $s \subset \mathbf{ex}$ of exchangeable indices. If we let $\mathbf{ex}^{(s)} = \mathbf{ex} - s$ and $\mathbf{B}^{(s)}$ be the restriction of $\mathbf{B}$ to $\mathbf{ex}^{(s)}$, then $(\mathbf{B}^{(s)}, \Lambda, M)$ defines a new quantum seed, called the *freezing* of $(\mathbf{B}, \Lambda, M)$ at s. Let $\mathcal{A}_q^{(s)}$ and $\mathcal{U}_q^{(s)}$ be the corresponding cluster algebras of this new seed. By construction, these new algebras are subalgebras of $\mathcal{F}$.

On the level of the principal part $\pi\mathbb{B}$, freezing a set s is the square submatrix on the indices $\mathbf{ex} - s$. As a slight abuse of notation, for any skew-symmetric matrix A, we write $\mathsf{A}^{(s)}$ for the square submatrix of A after deleting the columns and rows in s (whether or not we think of A as the skew-symmetric part of an exchange matrix).

Denote by $S := \{M(e_i) \mid i \in s\}$ the initial cluster variables in $\mathcal{A}_q$. Let $\mathcal{A}_q[S^{-1}]$ (resp. $\mathcal{U}_q[S^{-1}]$) denote the subalgebra of $\mathcal{F}$ generated by $\mathcal{A}_q$ and S^{-1} (resp. $\mathcal{U}_q$ and S^{-1}).

These four algebras can be compared by the following proposition.[25]

Proposition 8.5. *There are inclusions in* $\mathcal{F}$

$$\mathcal{A}_q^{(s)} \subseteq \mathcal{A}_q[S^{-1}] \subseteq \mathcal{U}_q[S^{-1}] \subseteq \mathcal{U}_q^{(s)}.$$

Proof. The cluster variables of $\mathcal{A}_q^{(s)}$ are a subset of the cluster variables of $\mathcal{A}_q$. The only new generators are the inverses of the newly-frozen variables, but those are in the localization by construction. This gives the first inclusion. Similarly, $\mathcal{A}_q^{(s)}$ has fewer clusters than $\mathcal{A}_q$, so the intersection defining $\mathcal{U}_q^{(s)}$ has strictly fewer terms than $\mathcal{U}_q$; so $\mathcal{U}_q \subseteq \mathcal{U}_q^{(s)}$. Since $\mathcal{U}_q^{(s)}$ also contains the inverses of S, this gives the last inclusion. The middle inclusion follows from the inclusion $\mathcal{A}_q \subseteq \mathcal{U}_q$. □

If $\mathcal{A}_q^{(s)} = \mathcal{A}_q[S^{-1}]$, then $\mathcal{A}_q^{(s)}$ is a localization of $\mathcal{A}_q$ which is naturally a cluster algebra; in this case, we call $\mathcal{A}_q^{(s)}$ a *cluster localization* of $\mathcal{A}_q$. Determining which freezings give cluster localizations seems to be an interesting problem.

One nice aspect of cluster localizations is that they are Ore localizations.

Proposition 8.6. *If $\mathcal{A}_q^{(s)} = \mathcal{A}_q[S^{-1}]$ is a cluster localization, then it is an Ore localization of $\mathcal{A}_q$ at the multiplicative set generated by S.*

Proof. Any $x \in \mathcal{A}_q$ is in $\mathcal{A}_q^{(s)}$, and so by Proposition 7.4, $x = a^{-1}b$ for a a product of frozen variables of $\mathcal{A}_q^{(s)}$ and b a polynomial in the cluster variables of $\mathcal{A}_q^{(s)}$. The cluster variables of $\mathcal{A}_q^{(s)}$ are a subset of the cluster variables of $\mathcal{A}_q$, so $b \in \mathcal{A}_q$.

Frozen variables in $\mathcal{A}'_q$ are either frozen in $\mathcal{A}_q$ or in S, so we can write $a = q^{\lambda}cd$, where c is a product of frozen variables in $\mathcal{A}_q$ and d is a product of elements in S. Then $x = d^{-1}(q^{-\lambda}c^{-1}b)$, where d is a product of elements in S, and $q^{-\lambda}c^{-1}b \in \mathcal{A}_q$. Then $\mathcal{A}_q[S^{-1}]$ is a left Ore localization. Since the elements of S are fixed by the bar involution, it is also a right Ore localization. □

Remark 8.7. If $s = \mathbf{ex}$, then $\mathcal{A}_q^{(s)} = \mathcal{A}_q[S^{-1}]$ is the quantum torus $\mathbb{T}_\Lambda \subset \mathcal{F}$ corresponding to the quantum seed $(\mathbf{B}, \Lambda, M)$. In this way, cluster localizations generalize these embeddings.

Remark 8.8. In terms of the quiver $Q(\pi\mathbf{B})$, freezing deletes the vertices in s.

[25] This is the quantum analog of [25, Proposition 3.1].

8.4. Relatively prime elements. We give a technique for producing localizations of a cluster algebra whose collective intersection is the original cluster algebra. This algorithm will only depend on the skew-symmetric submatrix $\pi\mathbf{B}$.

Given an $n \times n$ skew-symmetric matrix A, $i \in \{1, \ldots, n\}$ is a *sink* if $\mathsf{A}_{ji} \geq 0$ for all j. Similarly, a *source* is an index $i \in \{1, \ldots, n\}$ such that $\mathsf{A}_{ji} \leq 0$ for all $j \in \mathbf{ex}$.

Remark 8.9. In terms of the quiver $Q(\mathsf{A})$, a source is a vertex without outgoing arrows, and a sink is a vertex without incoming arrows.

Sources and sinks are a source of pairs of cluster variables which generate $\mathcal{A}_q$.

Lemma 8.10. *Let $(\mathbf{B}, \Lambda, M)$ be a quantum seed, with $i, j \in \mathbf{ex}$ such that $\mathbf{B}_{ij} \neq 0$ and i is a sink or a source in $\pi\mathbf{B}$. Then the cluster variables $M(e_i)$ and $M(e_j)$ generate the trivial left ideal in $\mathcal{A}_q(\mathbf{B}, \Lambda, M)$.*

Proof. Assume i is a sink (the other case is similar); this implies $\mathbf{B}_{ji} > 0$. If $(\mathbf{B}', \Lambda', M')$ is the mutation of the original seed at i, then

$$M'(e_i)M(e_i) = q^\bullet M\Big(\sum_{\mathbf{B}_{ki}>0} \mathbf{B}_{ki}e_k\Big) + q^\bullet M\Big(-\sum_{\mathbf{B}_{ki}<0} \mathbf{B}_{ki}e_k\Big).$$

Since we may always factor a monomial $M(\alpha)$ into $q^\bullet M(\alpha-\beta)M(\beta)$ for any α and β, we move the first term on the right hand side to the left, and pull out an $M(e_j)$.

$$q^\bullet M'(e_i)M(e_i) - q^\bullet M\Big(-e_j + \sum_{\mathbf{B}_{ki}>0} \mathbf{B}_{ki}e_k\Big)M(e_j) = M\Big(-\sum_{\mathbf{B}_{ki}>0} \mathbf{B}_{ki}e_k\Big).$$

Since $\mathbf{B}_{ji} > 0$, the left-hand side is in any left $\mathcal{A}_q$-ideal containing $M(e_i)$ and $M(e_j)$. Since i is a sink, $\mathbf{B}_{ki} < 0$ implies that that $k \notin \mathbf{ex}$, and so the right hand side is a monomial in non-exchangeable indices, which are invertible by construction. It follows that the left $\mathcal{A}_q$-ideal generated by $M(e_i)$ and $M(e_j)$ is trivial. □

Remark 8.11. This lemma is weaker than its commutative analog, [25, Lemma 5.3], which applies to any 'covering pair,' which generalizes the condition on i and j. There is no obvious quantum analog of the argument for this more general condition.

Lemma 8.12. *If $M(e_i)$ and $M(e_j)$ generate $\mathcal{A}_q$ as a left ideal, then*

$$\mathcal{A}_q[M(e_i)^{-1}] \cap \mathcal{A}_q[M(e_j)^{-1}] = \mathcal{A}_q.$$

Proof. For any $x \in \mathcal{A}_q[M(e_i)^{-1}] \cap \mathcal{A}_q[M(e_j)^{-1}]$, let $n_x \in \mathbb{N}$ be the smallest positive integer such that, for all $a, b \in N$ such that $a + b \geq n_x$, $M(ae_i + be_j)x \in \mathcal{A}_q$. Such an n_x exists; to see this, write $x = M(ce_i)^{-1}y = M(de_j)^{-1}z$ for $y, z \in \mathcal{A}_q$ and note that $n_x \leq c + d$. Clearly, $n_x = 0$ if and only if $x \in \mathcal{A}_q$.

For contradiction, assume there exists $x \notin \mathcal{A}_q$ with n_x minimal among elements of $(\mathcal{A}_q[M(e_i)^{-1}] \cap \mathcal{A}_q[M(e_j)^{-1}]) - \mathcal{A}_q$. For any $a, b \in \mathbb{N}$ with $a+b \geq n_x - 1$,

$$M(ae_i + be_i)[M(e_i)x] = M((a+1)e_i + be_j)x \longrightarrow M(ae_i + be_i)[M(e_i)x] \in \mathcal{A}_q.$$

This implies that $n_{M(e_i)x} \leq n_x - 1$. Since n_x was minimal, $M(e_i)x \in \mathcal{A}_q$. By a symmetric computation, $M(e_j)x \in \mathcal{A}_q$.

Define the *left denominator ideal* I of x by

$$I := \{y \in \mathcal{A}_q \mid yx \in \mathcal{A}_q\}$$

This is a left $\mathcal{A}_q$-ideal. As has been observed, $M(e_i)$ and $M(e_j)$ are in I. By Lemma 8.10, $I = \mathcal{A}_q$. In particular, $1 \in I$ and $1 \cdot x \in \mathcal{A}_q$. This contradicts $x \notin \mathcal{A}_q$. □

8.5. A lemma for proving $\mathcal{A}_q = \mathcal{U}_q$. These techniques can be combined to give the following criterion for showing large classes of cluster algebras have $\mathcal{A}_q = \mathcal{U}_q$.

Lemma 8.13. *Let $\mathcal{P}$ be a set of exchange types. Assume that, for every non-isolated exchange type $\mathcal{T} \in \mathcal{P}$, there is a skew-symmetric matrix $\mathsf{A} \in \mathcal{T}$, and indices i, j such that*

(1) $\mathsf{A}_{ij} \neq 0$ *and i is either a source or a sink in A and*

(2) *the exchange types of the freezings $\mathsf{A}^{(i)}$ and $\mathsf{A}^{(j)}$ are both in $\mathcal{P}$.*

Then $\mathcal{A}_q = \mathcal{U}_q$ for all $\mathcal{A}_q$ with exchange type in $\mathcal{P}$.

Proof. Assume $\mathcal{P}$ non-empty; the alternative case is immediate.

We proceed by induction on the size of $\mathcal{T}$; this is the size of any matrix in $\mathcal{T}$. Let $\mathcal{T} \in \mathcal{P}$ have minimal size. If it is not isolated, then there is some $\mathsf{A} \in \mathcal{T}$ with a freezing $\mathsf{A}^{(i)}$ with exchange type in $\mathcal{P}$. Since the size of $\mathsf{A}^{(i)}$ is less than the size of A, this contracts minimality; so $\mathcal{T}$ is isolated. Then $\mathcal{A}_q = \mathcal{U}_q$ for any $\mathcal{A}_q$ of type $\mathcal{T}$ by Proposition 8.2.

Assume that $\mathcal{A}_q = \mathcal{U}_q$ for every $\mathcal{A}_q$ of type $\mathcal{T} \in \mathcal{P}$ with size $< n$. Let $\mathcal{T} \in \mathcal{P}$ be an exchange type of size n, and let $\mathcal{A}_q$ be a cluster algebra of type $\mathcal{T}$. If $\mathcal{A}_q$ is isolated, then $\mathcal{A}_q = \mathcal{U}_q$.

Else, let $\mathsf{A} \in \mathcal{T}$ be the matrix and i, j be the indices guarenteed by the hypothesis. Since $\mathcal{A}_q$ has type $\mathcal{T}$, there is a quantum seed $(\mathbf{B}, \Lambda, M)$ of $\mathcal{A}_q$ such that $\pi\mathbf{B} = \mathsf{A}$, and we identify i, j with indices in **ex**. Then the freezings $\mathcal{A}_q^{(i)}$ and $\mathcal{A}_q^{(j)}$ are of type $\mathsf{A}^{(i)}$ and $\mathsf{A}^{(j)}$ respectively. These exchange types are in $\mathcal{P}$ and so by the inductive hypothesis, $\mathcal{A}_q^{(i)} = \mathcal{U}_q^{(i)}$ and $\mathcal{A}_q^{(j)} = U_q^{(j)}$. Then the inclusions in Proposition 8.5 are equalities; in particular,

$$\mathcal{A}_q^{(i)} = \mathcal{A}_q[M(e_i)^{-1}] \quad \text{and} \quad \mathcal{A}_q^{(j)} = \mathcal{A}_q[M(e_j)^{-1}].$$

By Lemma 8.10, $M(e_i)$ and $M(e_j)$ generate $\mathcal{A}_q$ as a left ideal, so by Lemma 8.12,

$$\mathcal{U}_q \subseteq \mathcal{U}_q^{(i)} \cap \mathcal{U}_q^{(j)} = \mathcal{A}_q^{(i)} \cap \mathcal{A}_q^{(j)} = \mathcal{A}_q[M(e_i)^{-1}] \cap \mathcal{A}_q[M(e_j)^{-1}] = \mathcal{A}_q.$$

But $\mathcal{A}_q \subseteq \mathcal{U}_q$, so $\mathcal{A}_q = \mathcal{U}_q$. By induction, this is true for all $\mathcal{T} \in \mathcal{P}$. □

Remark 8.14. The above lemma is a weaker version of the Banff algorithm which appeared in [25, Section 5], reformulated as an criterion rather than an algorithm. Specifically, if the condition (2) in the lemma was replaced by the weaker condition '(i, j) is a covering pair in A,' then a set $\mathcal{P}$ satisfies the hypothesis of the new version of the lemma if and only if the Banff algorithm produces an acyclic cover for every commutative cluster algebra $\mathcal{A}_1$ with exchange type in $\mathcal{P}$. As a consequence, if $\mathcal{P}$ satisfies the lemma as it is stated above, every commutative cluster algebra $\mathcal{A}_1$ with exchange type in $\mathcal{P}$ is locally acyclic (see Section 11).

Remark 8.15. The union $\overline{\mathcal{P}}$ of all sets $\mathcal{P}$ which satisfy the lemma also satisfies the lemma, so $\overline{\mathcal{P}}$ is the unique maximal set of exchange types satisfying the lemma. Are there any cluster algebras $\mathcal{A}_q$ with $\mathcal{A}_q = \mathcal{U}_q$ and exchange type not in $\overline{\mathcal{P}}$?

8.6. Digression: acyclic cluster algebras. A $n \times n$ skew-symmetric matrix A is called *acyclic* if there is no sequence of indices $i_1, i_2, \ldots, i_n = i_1$ such that $\mathsf{A}_{i_{j+1} i_j} > 0$. An exchange type is acyclic if any matrix in it is.

Remark 8.16. The matrix A is acyclic if and only if $Q(\mathsf{A})$ has no directed cycles.

Cluster algebras of acyclic type are an important class of examples, for which many general results are known. A byproduct of Lemma 8.13 is a new proof that $\mathcal{A}_q = \mathcal{U}_q$ for acyclic cluster algebras, which first appeared in [3, Theorem 7.5].

Proposition 8.17. *If $\mathcal{A}_q$ has acyclic exchange type, then $\mathcal{A}_q = \mathcal{U}_q$.*

Proof. If A is acyclic and not zero, then there is some i which is a sink, and j with $\mathsf{A}_{ji} < 0$. This can be shown by starting at a non-isolated vertex in $Q(\mathsf{A})$ and moving along arrows; eventually a dead-end is reached because the index set is finite and cycles are forbidden. The freezings $\mathsf{A}^{(i)}$ and $\mathsf{A}^{(j)}$ are also acyclic. Therefore, the class of acyclic exchange types satisfies the hypothesis of Lemma 8.13. □

Remark 8.18. This is of limited usefulness for cluster algebras of marked surfaces, because $\mathcal{A}_q(\Sigma)$ has acyclic exchange type only for certain simple surfaces (see [9, Remark 10.11]).

9. $\mathcal{A}_q(\Sigma) = \mathcal{U}_q(\Sigma)$ for (most) marked surfaces

The techniques of the previous section can now be applied to the class of triangulable marked surfaces with at least two marked points on each connected component.

9.1. Marked surfaces with isolated cluster algebras.

The first step is to characterize which cluster algebras of marked surfaces have isolated exchange type.

Proposition 9.1. *If Σ is a union of topological discs, each with 3 or 4 marked points, then $\mathcal{A}_q(\Sigma)$ has isolated exchange type.*

Proof. Let Δ be a triangulation of Σ. The only non-boundary curves in Δ will be diagonals across connected components with 4 marked points. Since any two such curves x, y are in different components, $\mathsf{Q}^{\Delta}_{\mathsf{x},\mathsf{y}} = 0$, and so $\pi \mathbf{B}^{\Delta} = 0$. □

Remark 9.2. These are the only triangulable marked surfaces whose cluster algebras have isolated exchange type.[26]

9.2. Cutting a marked surface.

Freezing a quantum seed $(\mathbf{B}^{\Delta}, \Lambda^{\Delta}, M^{\Delta})$ can be interpreted as the topological action of 'cutting,' at least on the level of the skew-symmetric matrix $\pi \mathbf{B}^{\Delta}$.

Let x be a simple non-boundary arc[27] in Σ. The *cutting* $\chi_{\mathsf{x}}(\Sigma)$ of Σ along α is the marked surface obtained by cutting Σ along x, compactifying Σ by adding

[26] We emphasize that this is not true in the larger generality of marked surfaces with non-boundary marked points (not considered in this paper, but cover in [9].

[27] Cutting at any simple curve may be defined, but the resulting marked surface will only be triangulable for simple non-boundary arcs, so we do not consider the more general case.

boundary along the two sides of x, and adding marked points where the endpoints of x were. There is a natural map

$$\chi_{\mathsf{x}}(\Sigma) \longrightarrow \Sigma$$

which is a bijection away from $\mathsf{x} \subset \Sigma$, a 2-to-1 map over the interior of x, and such that the preimage of marked points are all marked. The two types of cut are pictured in Figure 9.

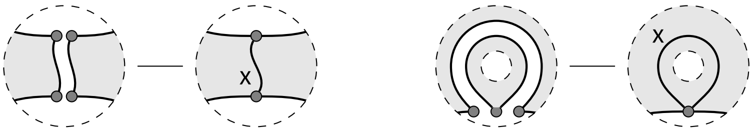


Figure 9. Types of cuts.

The map $\chi_{\mathsf{x}}(\Sigma) \to \Sigma$ takes a triangulation of $\chi_{\mathsf{x}}(\Sigma)$ to a triangulation of Σ which contains x. This induces a bijection between triangulations of $\chi_{\mathsf{x}}(\Sigma)$ and triangulations of Σ which contain x.

Proposition 9.3. *Let x be a simple non-boundary arc in Σ. Let Δ be a triangulation of Σ containing x, and Δ' be the corresponding triangulation of $\chi_{\mathsf{x}}(\Sigma)$. Then the skew-symmetric matrix $\pi\mathbf{B}^{\Delta'}$ is the submatrix $(\pi\mathbf{B}^{\Delta})^{(\mathsf{x})}$ of $\pi\mathbf{B}^{\Delta}$ where the row and column corresponding to x has been removed.*

Proof. Let $\mathsf{y},\mathsf{z} \in \Delta'$. Then

$$(\pi\mathbf{B}^{\Delta'})_{\mathsf{y},\mathsf{z}} = \mathsf{Q}^{\Delta'}_{\mathsf{y},\mathsf{z}} = \mathsf{Q}^{\Delta}_{\mathsf{y},\mathsf{z}} = (\pi\mathbf{B}^{\Delta})_{\mathsf{y},\mathsf{z}}.$$

The set $\mathbf{ex}' \subset \Delta'$ of non-boundary arcs is $\mathbf{ex} - \{\mathsf{x}\}$, so $\pi\mathbf{B}^{\Delta'}$ is the restriction of $\pi\mathbf{B}^{\Delta}$ away from x. □

Corollary 9.4. *Let x be a simple non-boundary arc in Σ, and let Δ be a triangulation of Σ containing x. Then $\mathcal{A}_q(\chi_{\mathsf{x}}(\Sigma))$ has the same exchange type as $\mathcal{A}_q(\Sigma)^{(\mathsf{x})}$, the freezing of x in the quantum seed corresponding to Δ.*

Remark 9.5. It is not true that $\mathcal{A}_q(\chi_{\mathsf{x}}(\Sigma)) = \mathcal{A}_q(\Sigma)^{(\mathsf{x})}$. The induced triangulation Δ' of $\chi_{\mathsf{x}}(\Sigma)$ has one more element than Δ, and so the cluster algebras in question do not have isomorphic skew-fields of fractions.

9.3. Finding relatively prime elements. We now topologically characterize pairs of cluster variables (ie, simple arcs) which satisfy Lemma 8.10.

Lemma 9.6. *Let* $\mathsf{x}, \mathsf{y}, \mathsf{z}$ *be non-crossing, simple arcs in* $(\Sigma, \mathcal{M})$ *as in Figure* 10*, with the endpoints of* y *distinct and* x, y *non-boundary.*[28] *Then, for any triangulation* $\{\mathsf{x}, \mathsf{y}, \mathsf{z}\} \subset \Delta$ *of* Σ*,* y *is a sink of the matrix* $(\pi \mathbf{B}^{\Delta})$ *with* $\mathbf{B}^{\Delta}_{\mathsf{yx}} > 0$*.*

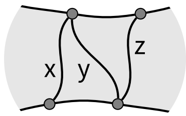


Figure 10. A configuration of arcs.

Proof. In any triangulation Δ of $(\Sigma, \mathcal{M})$ which contains x, y and z, there can be no non-boundary arcs immediate clockwise or counterclockwise to y other than x and z. Therefore, $\mathsf{Q}^{\Delta}_{\mathsf{x}}$ At each end of y, there will be no arcs in Δ which are counter-clockwise to y, and so there are no arrows out of y in Q_{Δ}. □

9.4. Proving $\mathcal{A}_q = \mathcal{U}_q$ for most marked surfaces. We are now in a position to prove that $\mathcal{A}_q(\Sigma) = \mathcal{U}_q(\Sigma)$ for a many marked surfaces.

Theorem 9.7. *If* $\mathcal{A}_q$ *is a cluster algebra with the same exchange type as* $\mathcal{A}_q(\Sigma)$ *for* Σ *a triangulable marked surfaces with at least two marked points in each connected component, then* $\mathcal{A}_q = \mathcal{U}_q$*.*

Proof. Let $\mathcal{P}$ be the set of exchange types comings from such marked surfaces. We show $\mathcal{P}$ satisfies the hypothesis of Lemma 8.13. Let $\mathcal{T}$ be an exchange type in $\mathcal{P}$, and let Σ be such that $\mathcal{A}_q(\Sigma)$ has exchange type $\mathcal{T}$. If every connected component of Σ is a disc 3 or 4 marked points, then $\mathcal{T}$ is isolated (Proposition 9.1).

Otherwise, choose a connected component Σ_0 of Σ which is not a disc with 3 or 4 marked points. Choose a simple non-boundary arc y with distinct endpoints (by hypothesis, Σ_0 has at least two marked points). There exists non-crossing simple arcs x and z (which may coincide) so that $\mathsf{x}, \mathsf{y}, \mathsf{z}$ are as in Figure 10. The curves x and z cannot both be boundary arcs, since that would force Σ_0 to be a disc with 4 marked points. Assume x is a non-boundary arc (the other case is identical).

[28] Other pairs of marked points may coincide, and x and z may coincide.

Choose a triangulation Δ containing $\mathsf{x}, \mathsf{y}, \mathsf{z}$. By Proposition 9.3, the freezing $(\pi\mathbf{B}^{\Delta})^{(\mathsf{x})} = (\pi\mathbf{B}^{\Delta'})$ where Δ' is the induced triangulation on the cutting $\chi_{\mathsf{x}}(\Sigma)$. Since the cutting $\chi_{\mathsf{x}}(\Sigma)$ is still a marked surface with at least two marked points in each connected component, the exchange type of $(\pi\mathbf{B}^{\Delta})^{(\mathsf{x})}$ is in $\mathcal{P}$. By an identical argument, the exchange type of $(\pi\mathbf{B}^{\Delta})^{(\mathsf{y})}$ is in $\mathcal{P}$.

Then $\pi\mathbf{B}^{\Delta} \in \mathcal{T}$ is non-isolated, with indices x, y such that

(1) $(\pi\mathbf{B}^{\Delta})_{\mathsf{x},\mathsf{y}} > 0$ and y is a sink of $\pi\mathbf{B}^{\Delta}$ (by Lemma 9.6) and

(2) $(\pi\mathbf{B}^{\Delta})^{(\mathsf{x})}$ and $(\pi\mathbf{B}^{\Delta})^{(\mathsf{x})}$ have exchange type in $\mathcal{P}$.

Thus, $\mathcal{P}$ satisfies Lemma 8.13. □

The localized skein algebra $\mathsf{Sk}_q^o(\Sigma)$ is between $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$, so they coincide.

Theorem 9.8. *If Σ is triangulable and has at least two marked points in each connected component, then*

$$\mathcal{A}_q(\Sigma) = \mathsf{Sk}_q^o(\Sigma) = \mathcal{U}_q(\Sigma).$$

Proof. This is an immediate consequence of Theorems 7.15 and 9.7. □

Remark 9.9. One immediate advantage of this theorem is computational. Computations in cluster algebras can be quite difficult, for several reasons. Working with expressions in $\mathcal{A}_q$ in different seeds requires choosing an explicit sequence of mutations relating the seeds, and the complexity grows rapidly with the number of mutations. The upper cluster algebra $\mathcal{U}_q$ does not come with a generating set, and so working with general elements can be daunting.

The localized skein algebra is much easier to work with. Elements are expressed in terms of topological objects which fit on a piece of paper. The skein relations are local, and links may be freely homotoped; both of which keep complexity low.

10. Loop elements

10.1. Loop elements in $\mathcal{A}_q(\Sigma)$. By definition, the subalgebra $\mathcal{A}_q(\Sigma) \subset \mathsf{Sk}_q^o(\Sigma)$ contains arcs and inverses to boundary arcs. Therefore, the equality $\mathcal{A}_q(\Sigma) = \mathsf{Sk}_q^o(\Sigma)$ in Theorem 9.8 is equivalent to the following proposition.

Proposition 10.1. *Let Σ be a triangulable marked surface with at least two marked points in each connected component. For each simple loop $\ell \in \Sigma$,*

$$[\ell] = [\mathbf{Y}]^{-1} \sum_i \lambda_i [\mathsf{x}_{i,1}][\mathsf{x}_{i,2}] \dots [\mathsf{x}_{i,n_i}],$$

where $\mathbf{Y}$ is a link of boundary arcs, each $\mathsf{x}_{i,j}$ is an arc, and $\lambda_i \in \mathbb{Z}_q$.

Proof. Cluster variables in $\mathcal{A}_q(\Sigma)$ correspond to arcs, and so products of cluster variables correspond to general links. Frozen variables correspond to boundary arcs, and so a general element of $\mathcal{A}_q(\Sigma)$ can be written in the above form. By Theorem 9.8, this is equally true of all elements of $\mathsf{Sk}_q^o(\Sigma)$. □

These expressions are distinct from the skew-Laurent expressions from Corollary 6.9; the arcs $\mathsf{x}_{i,j}$ are allowed to cross each other, but there are no negative powers of non-boundary arcs.

Finding such an expression for a simple loop is typically very different from writing it as a skew-Laurent polynomial of the arcs in a triangulation. First, while every curve has a unique expression as a skew-Laurent polynomial in the arcs of a triangulation, there will be many ways to write a simple loop as a polynomial in arcs divided by a monomial in boundary arcs. Second, the author does not know of any direct algorithm to produce any such expression, analogous to Remark 6.10 or the *band graph* techniques found in [23].

From a cluster algebraic perspective, these loop elements are compelling. They are not an ingredient in the cluster structure on $\mathcal{A}_q(\Sigma)$, but they are a useful tool in computations. For example, a product of two simple arcs can have many crossings, and applying the Kauffman skein relation to each crossing may produce loops.

10.2. The $\mathbb{Z}_q$-basis of weighted simple multicurves. The localized skein algebra has a natural $\mathbb{Z}_q$-basis, given by the set SMulti^o of weighted simple multicurves with positive weights on non-boundary curves (Proposition 5.3). Theorem 9.8 implies this is also a basis for $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$.

Proposition 10.2. *Let Σ be a triangulable marked surface with at least two marked points in each connected component. Then SMulti^o maps to a $\mathbb{Z}_q$-basis of $\mathcal{A}_q(\Sigma)$ and $\mathcal{U}_q(\Sigma)$ under the map $\mathsf{X} \to [\mathsf{X}]$.*

The problem of finding natural bases for cluster algebras goes back to the origins of their study. Commutative cluster algebras were discovered in the study of Lusztig's dual canonical basis for $\mathbb{C}[G]$ of a reductive group, as a conjectural method of explicitly producing classical limits of elements of the dual canonical basis [11, Introduction]. A good history and bibliography of recent work on bases of cluster algebras can be found in the Introduction to [23].

Some of this basis comes directly from the cluster structure. If X is a weighted simple multicurve without loops, then there is some triangulation Δ which contains every arc in X. Then X is a monomial in Δ; in the language of cluster algebras, this is called a *cluster monomial*[29] in the seed corresponding to Δ.

The remaining basis elements contain loop elements. As has been mentioned, loop elements are difficult to express as explicit elements of $\mathcal{A}_q(\Sigma)$, and so these basis elements of $\mathcal{A}_q(\Sigma)$ do not follow naively from the cluster structure.

Remark 10.3. In the specialization $q^{\frac{1}{2}} = 1$, this basis automatically goes to a $\mathbb{Z}$-basis of the commutative cluster algebras $\mathcal{A}_1(\Sigma)$ and $\mathcal{U}_1(\Sigma)$. However, this basis is *not* a 'canonically positive' (or 'atomic') basis. That is, an element $x \in \mathcal{A}_1(\Sigma)$ can have a positive Laurent expression for each seed Δ, without being a positive combination of the basis elements SMulti^o.

An alternative basis for $\mathcal{A}_1(\Sigma)$ which may be canonically positive has been put forward in [7], [4], and [23]. This basis is related to the basis of weighted simple multicurves, by replacing simple loops with multiplicity by a single loop with self-crossings. Proofs of canonically positivity for some S can be found in [34] and [5].

11. The commutative specialization $q^{\frac{1}{2}} = 1$

In the specialization $q^{\frac{1}{2}} = 1$, Theorem 9.8 becomes equalities

$$\mathcal{A}_1(\Sigma) = \mathsf{Sk}_1^o(\Sigma) = \mathcal{U}_1(\Sigma)$$

This endows $\mathsf{Sk}_1^o(\Sigma)$ with the structure of a commutative cluster algebra.

11.1. Geometry of commutative cluster algebras. The equality $\mathcal{A}_1(\Sigma) = \mathcal{U}_1(\Sigma)$ was already shown in a previous work by the author [25, Theorem 10.6], using the idea of *local acyclicity*. This is a geometric notion which does not directly generalize to the quantum setting.[30]

Given a cluster algebra $\mathcal{A}_q$, the specialization $\mathcal{A}_1$ is commutative and so it can be studied geometrically, by considering the scheme $\mathrm{Spec}(\mathcal{A}_1)$. If $\mathcal{A}_q^{(s)}$ is a cluster

[29] Some references regard the Laurent ring of frozen variables as coefficients, rather than as cluster variables (as we are). In the former case, a cluster monomial would be a weighted simple multicurve without loops or boundary arcs, but the coefficient ring would be much larger.

[30] However, the techniques of Section 8 are based on this geometric approach (Remark 8.14).

localization of $\mathcal{A}_q$, then $\mathcal{A}_1^{(s)}$ is localization of $\mathcal{A}_1$, and so

$$\mathrm{Spec}(\mathcal{A}_1^{(s)}) \subseteq \mathrm{Spec}(\mathcal{A}_1)$$

is an open subscheme.

A collection $\{\mathcal{A}_1^{(s_i)}\}$ of cluster localizations of $\mathcal{A}_1$ is a *cover* if the corresponding open subschemes cover $\mathrm{Spec}(\mathcal{A}_1)$. If $\{\mathcal{A}_1^{(s_i)}\}$ is a cover of $\mathcal{A}_1$, then

$$\mathcal{A}_1 = \bigcap_i \mathcal{A}_1^{(s_i)}$$

though the converse is not true in general.

11.2. Local acyclicity. Recall that an exchange type $\mathcal{T}$ is *acyclic* if there is a skew-symmetric matrix $\mathsf{A} \in \mathcal{T}$ with no cycles.[31] If $\mathcal{A}$ has acyclic exchange type, then $\mathcal{A} = \mathcal{U}$ ([1, Corollary 1.19] or Proposition 8.17 and Remark 8.14).

This can be generalized, by checking acyclicity locally.

Definition 11.1 ([25, Definition 3.9]). A commutative cluster algebra $\mathcal{A}$ is *locally acyclic* if it has a cover $\{\mathcal{A}^{(s_i)}\}$ by acyclic cluster localizations.

Marked surfaces Σ such that $\mathcal{A}_1(\Sigma)$ is locally acyclic have been characterized.

Theorem 11.2 ([25, Theorems 10.6 and 10.10]). *The cluster algebra $\mathcal{A}_1(\Sigma)$ is locally acyclic if and only if Σ has at least two marked points in each connected component of Σ.*

Remark 11.3. Marked surfaces in [25] are allowed to have interior marked points, so the statements there are more general.

11.3. Consequences. Local acyclicity has several consequences.

Proposition 11.4. *Let $\mathcal{A}$ be a locally acyclic commutative cluster algebra. Then*

(1) [25, Theorem 4.1] $\mathcal{A} = \mathcal{U}$,

(2) [25, Theorem 4.2] $\mathcal{A}$ *is finitely generated, integrally closed and locally a complete intersection, and*

(3) [25, Theorem 7.7] $\mathbb{Q} \otimes \mathcal{A}$ *is a regular domain.*

These results can then be applied to commutative cluster algebras of marked surfaces, and the $q^{\frac{1}{2}} = 1$ localized skein algebra.

[31] A cycle is a list of indices $i_1, i_2, \ldots, i_{n-1}, i_n = i_1 \in \mathbf{ex}$ such that $\mathbf{B}_{i_j i_{j+1}} > 0$ for all j.

Corollary 11.5. *Let Σ be a triangulable marked surface with at least two marked points in each connected component.*

(1) $\mathcal{A}_1(\Sigma) = \mathsf{Sk}_1^o(\Sigma) = \mathcal{U}_1(\Sigma)$,[32]

(2) Sk_1^o *is finitely generated, integrally closed, and locally a complete intersection, and*

(3) $\mathbb{Q} \otimes \mathsf{Sk}_1^o(\Sigma)$ *is a regular domain.*

As a consequence of the last fact (see [25, Corollary 7.9]),

- $\mathrm{Spec}(\mathbb{Q} \otimes \mathsf{Sk}_1^o(\Sigma))$ is a smooth scheme,
- $\mathrm{Hom}(\mathsf{Sk}_1^o(\Sigma), \mathbb{C})$ is a smooth complex manifold, and
- $\mathrm{Hom}(\mathsf{Sk}_1^o(\Sigma), \mathbb{R})$ is a smooth real manifold.

Here, both Homs are as rings.

Remark 11.6. There is an open inclusion

$$\mathrm{Spec}(\mathsf{Sk}_1^o(\Sigma)) \subset \mathrm{Spec}(\mathsf{Sk}_1(\Sigma))$$

By analogy with the cluster structure on double Bruhat cells in semisimple Lie groups, it seems possible that $\mathrm{Spec}(\mathsf{Sk}_1^o(\Sigma))$ is the 'big cell' in some natural stratification of $\mathrm{Spec}(\mathsf{Sk}_1(\Sigma))$. Ideally, this is a finite stratification by smooth affine schemes, whose coordinate rings are commutative cluster algebras.

12. Examples and non-examples

12.1. Marked discs. Let Σ_n be the disc with n marked points on the boundary. A simple curve in Σ_n will always be homotopic to a chord $\mathsf{x}_{a,b}$ connecting distinct marked points a, b, and so $\mathsf{Sk}_q(\Sigma)$ is generated by the $\binom{n}{2}$-elements of the form $[\mathsf{x}_{a,b}]$ (Corollary 4.3). The relations are

$$[\mathsf{x}_{a,b}][\mathsf{x}_{b,c}] = q[\mathsf{x}_{b,c}][\mathsf{x}_{a,b}], \quad [\mathsf{x}_{a,b}][\mathsf{x}_{c,d}] = [\mathsf{x}_{c,d}][\mathsf{x}_{a,b}],$$

$$[\mathsf{x}_{a,c}][\mathsf{x}_{b,d}] = q[\mathsf{x}_{a,b}][\mathsf{x}_{c,d}] + q^{-1}[\mathsf{x}_{a,d}][\mathsf{x}_{b,c}].$$

as a, b, c, d run over distinct marked points in clockwise order around $\partial\Sigma_n$. The boundary arcs are the elements $[\mathsf{x}_{a,b}]$ for a, b adjacent on the boundary, and $\mathsf{Sk}_q^o(\Sigma_n)$ is the Ore localization at this set.

[32] This is to say; locally acyclic provides an alternative (though fundamentally the same) proof.

The surface is triangulable when $n \geq 3$, and so $\mathcal{A}_q(\Sigma_n) = \mathsf{Sk}_q^o(\Sigma_n) = \mathcal{U}_q(\Sigma_n)$ (Theorem 9.8). The cluster variables coincide with the set of chords $[\mathsf{x}_{a,b}]$, with clusters corresponding to triangulations.

The commutative cluster algebra $\mathcal{A}_1(\Sigma)$ is a basic example in cluster algebras; thorough investigations can be found in [16, Section 2.1] and [12, Section 3]. In our language, the main observation is that $\mathsf{Sk}_1(\Sigma_n)$ coincides with the homogeneous coordinate ring $\mathcal{O}[Gr_{\mathbb{C}}(2,n)]$ of the Grassmannian $Gr_{\mathbb{C}}(2,n)$. This isomorphism depends on an identification of the marked points with a basis of $\mathbb{C}^n$; a cluster variable $[\mathsf{x}_{a,b}]$ then corresponds to the Plücker coordinate $p_{a,b}$.

In [13], Grabowski and Launois exhibit a quantum cluster algebra structure on the *quantum Grassmannian* $\mathcal{O}_q[Gr(2,n)]$, a specific quantization of $\mathcal{O}[Gr_{\mathbb{C}}(2,n)]$. One might hope that the quantum Grassmannian would coincide with $\mathsf{Sk}_q(\Sigma_n)$. However, this is impossible; the quantum Grassmannian depends on an identification of the basis elements with the set $\{1,2,\ldots,n\}$; a cyclic permutation does not induce an automorphism of $\mathcal{O}_q[Gr(2,n)]$ [21] (cf. [37]). The skein algebra $\mathsf{Sk}_q(\Sigma_n)$ has no such dependency. Inspecting the quantum seeds in [13, Section 3.1] confirms that these are different quantizations of the same commutative cluster algebra.

12.2. A marked annulus. Let Σ be the annulus with a single marked point on each boundary component. Let a and b denote the two boundary arcs, and let ℓ denote the unique simple loop (Figure 11). The remaining simple curves are arcs connecting the two marked points; they may be parametrized by $\mathbb{Z}$ as follows. Choose such an arc to be x_0, and define the rest by the conditions that x_i and x_{i+1} do not intersect, and both ends of x_{i+1} are clockwise to both ends of x_i.

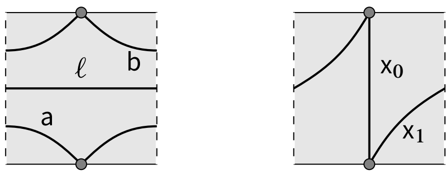


Figure 11. Simple curves in Σ (The two dashed edges are identified).

The simple curves $\mathsf{a}, \mathsf{b}, \ell$ and $\{\mathsf{x}_i\}_{i\in\mathbb{Z}}$ generate $\mathsf{Sk}_q(\Sigma)$ as a $\mathbb{Z}_q$-algebra. The elements $[\mathsf{a}]$ and $[\mathsf{b}]$ are central. Some relations among these generators are

$$[\ell][\mathsf{x}_i] = q[\mathsf{x}_{i+1}] + q^{-1}[\mathsf{x}_{i-1}],$$

$$[\mathsf{x}_i][\mathsf{x}_{i+1}] = q^{-1}[\mathsf{x}_{i+1}][\mathsf{x}_i],$$

$$[\mathsf{x}_i][\mathsf{x}_{i+2}] = [\mathsf{a}][\mathsf{b}] + q^{-2}[\mathsf{x}_{i+1}]^2,$$

$$[\mathsf{x}_i][\mathsf{x}_{i+3}] = q[\ell][\mathsf{a}][\mathsf{b}] + q^{-2}[\mathsf{x}_{i+1}][\mathsf{x}_{i+2}].$$

Since $[\mathsf{x}_{i+1}] = q[\ell][\mathsf{x}_i] - q^2[\mathsf{x}_{i-1}]$, the five elements $\mathsf{a}, \mathsf{b}, \ell, \mathsf{x}_0, \mathsf{x}_1$ generate $\mathsf{Sk}_q(\Sigma)$.

The triangulations of Σ are the sets $\{\mathsf{a}, \mathsf{b}, \mathsf{x}_i, \mathsf{x}_{i+1}\}$ for some i. Since Σ has two marked points, $\mathcal{A}_q(\Sigma) = \mathsf{Sk}_q(\Sigma) = \mathcal{U}_q(\Sigma)$ (Theorem 9.8).

The loop element $[\ell]$ can be written as a skew-Laurent polynomial in any triangulation (Theorem 6.14),

$$[\ell] = ([\mathsf{x}_i][\mathsf{x}_{i+1}])^{-1}(q[\mathsf{x}_i]^2 + q^{-1}[\mathsf{a}][\mathsf{b}] + q^{-3}[\mathsf{x}_{i+1}]^2).$$

and as a product of cluster variables divided by frozen variables (Proposition 10.1),

$$[\ell] = ([\mathsf{a}][\mathsf{b}])^{-1}(q^{-1}[\mathsf{x}_i][\mathsf{x}_{i+3}] - q^{-3}[\mathsf{x}_{i+1}][\mathsf{x}_{i+2}]).$$

Appendices

A. Finite generation of $\mathsf{Sk}_q(\Sigma)$

It has been shown by Bullock that $\mathsf{Sk}_q(\Sigma)$ is finitely generated when Σ is unmarked [2, Theorem 1]. The idea of his proof still works in the marked case, with the necessary modifications.

Remark A.1. What follows is a simplified version of Bullock's proof, since we will not explicitly bound the number of generators.

We assume $\partial\Sigma \neq \emptyset$ (otherwise, Bullock's result applies directly). The marked surface Σ has a handle decomposition (Figure 12); observe that every marked point can be placed on the boundary of the 0-handle.

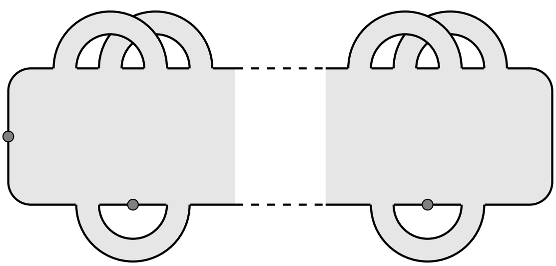


Figure 12. The handle decomposition of Σ. There are g-many pairs of 1-handles along the top, h-many 1-handles along the bottom, and any marked point may denote multiple close marked points (or none).

A link is in *standard position* (with respect to the handle decomposition) if its intersection with any 1-handle is a union of strands homotopic to the core, and the number of strands is minimal with respect to homotopy. Every link is homotopic to one in standard position; for the remainder of the section we assume all links are in standard position.

The *complexity* of a link is the total number of strands in the intersection with the 1-handles, minus the number of 1-handles it intersects. So, a link has complexity zero if its intersection with any 1-handle contains at most one strand.

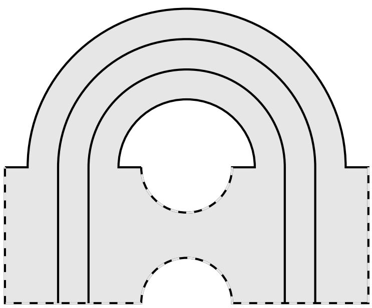
Proposition A.2. *The set of simple curves of complexity zero is finite.*

Proof. Fix a subset S of the 1-handles. If x is a simple curve of complexity zero which intersects exactly the 1-handles in S, then x is determined by its intersection with the 0-handle. The intersection of x with the 0-handle is a non-crossing matching between the attaching points of the 1-handles in S, and either 2 or 0 marked points. There are finitely many such non-crossing matchings, and finitely many subsets S of the 1-handles, so the set of zero complexity simple curves is finite. □

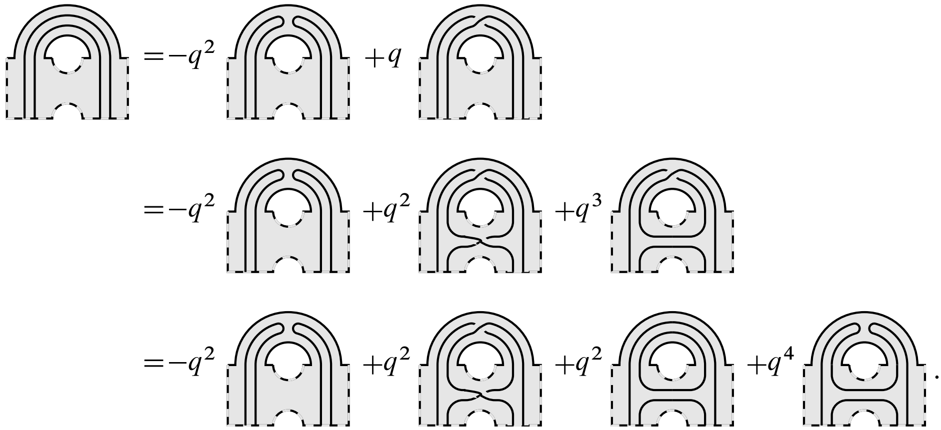
Lemma A.3. *The set of simple curves of complexity zero generates* $\mathsf{Sk}_q(\Sigma)$.

Proof. We claim every simple curve x in Σ is in the $\mathbb{Z}_q$-subalgebra of $\mathsf{Sk}_q(\Sigma)$ generated by simple curves of complexity zero. The proof is by induction on complexity κ. The case $\kappa = 0$ is trivial.

Assume $\kappa \geq 1$. Then there is some 1-handle which x intersects in multiple strands. Choose the two innermost strands, and consider the following picture, where there may be additional components in the 1-handle.



By repeated application of the Kauffman skein relation,



The four links on the right-hand side are products of simple curves with complexity $< \kappa$. By induction, [x] is in the subalgebra generated by the simple curves of complexity zero, and so every simple curve is. By Corollary 4.3, this set generates all of $\mathsf{Sk}_q(\Sigma)$. $\square$

Finite generation follows immediately.

Theorem A.4. $\mathsf{Sk}_q(\Sigma)$ *and* $\mathsf{Sk}_q^o(\Sigma)$ *are finitely generated.*

Proof. $\mathsf{Sk}_q(\Sigma)$ is generated by the simple curves of complexity zero, which is finite. The localized skein algebra $\mathsf{Sk}_q^o(\Sigma)$ is generated by the simple curves of complexity zero and the inverses to boundary curves, which is again finite. $\square$

B. Relation with Teichmüller space and quantum Teichmüller space

Here, we briefly describe the relation of the skein algebra of a marked surface to certain geometric and algebraic objects in Teichmüller theory.

B.1. Teichmüller spaces and moduli space of local systems. As before, let Σ be a marked surface; that is, a compact, oriented surface with a finite set of marked points $\mathcal{M}$ on the boundary. Let Σ^o be the corresponding *opened surface*, where a small ball around each point of $\mathcal{M}$ has been removed. Let $\partial\mathcal{M} \subset \Sigma^o$ be the boundary of these removed neighborhoods. Note that the boundary of Σ^o alternates between restrictions of components of $\partial\Sigma$ and components of $\partial\mathcal{M}$.

- The *Teichmüller space* $\mathcal{T}(\Sigma)$ is the moduli space of hyperbolic metrics (up to isotopy) on $\Sigma - \mathcal{M}$ such that each component of $\partial\Sigma - \mathcal{M}$ is a geodesic, and each point in $\mathcal{M}$ is a 'cusp.'
- The *decorated Teichmüller space* $\widehat{\mathcal{T}}(\Sigma)$ is the moduli space of hyperbolic metrics (up to isotopy) on Σ^o such that each component of the restriction of $\partial\Sigma$ is a geodesic, and each component of $\partial\mathcal{M}$ is a horocycle (that is, an arc of constant curvature).

There is a natural projection

$$\widehat{\mathcal{T}}(\Sigma, \mathcal{M}) \longrightarrow \mathcal{T}(\Sigma, \mathcal{M}).$$

which extends the metric to the small balls around points in $\mathcal{M}$. Both of these are real manifolds with a canonical *Weil-Petersson* Poisson structure [27].

This projection map was realized in [7] as the *positive part*[33] of a projection of complex varieties which parametrize certain local systems.

$$\begin{array}{ccc} \widehat{\mathcal{T}}(\Sigma) & \longrightarrow\!\!\!\!\!\rightarrow & \mathcal{T}(\Sigma) \\ \downarrow & & \downarrow \\ A(\Sigma) & \longrightarrow\!\!\!\!\!\rightarrow & X(\Sigma). \end{array}$$

Here, $A(\Sigma)$ is the moduli space of *decorated* SL_2*-local systems on* $(\Sigma, \mathcal{M})$,[34] and $X(\Sigma)$ is the moduli space of *framed* PGL_2*-local systems on* $(\Sigma, \mathcal{M})$.

Via Fock and Goncharov's theory of *cluster ensembles*, the moduli space $X(\Sigma)$ has a canonical Poisson structure, which extends the *Weil-Petersson form* on $\mathcal{T}(\Sigma)$. They also describe a *quantization* of $X(\Sigma)$: each triangulation Δ of Σ defines a quantum torus $QTS_{q,\Delta}(\Sigma)$ inside a common skew-field $QTS_q(\Sigma)$. This skew-field $QTS_q(\Sigma)$ had been previously introduced by Chekhov and Fock, and called the *quantum Teichmüller space* of Σ [6]. The $q = 1$ specialization produces a field $QTS_1(\Sigma)$ which is canonically isomorphic to the field of rational functions on the variety $X(\Sigma)$, and every element of $QTS_1(\Sigma)$ restricts to a well-defined function on $\mathcal{T}(X)$.

Remark B.1. Unlike the quantum tori defining a quantum cluster algebra, the intersection of the quantum tori $QTS_{q,\Delta}(\Sigma)$ may be too small to generate $QTS_q(\Sigma)$ as a skew-field.

[33] Here, 'positive part' means the subset on which a certain system of distinguished coordinates has positive real values.

[34] In fact, these are local systems twisted by a 'spin structure'; specifically, they are SL_2-local systems on the unit tangent bundle to Σ with monodromy $-Id_2$ around any fiber.

B.2. Relation to skein algebra. Fock and Goncharov also define a (commutative) cluster structure on the variety $A(\Sigma)$. This cluster structure has a cluster variable for each arc in Σ, and the clusters correspond to triangulations.[35] This can be extended to a canonical isomorphism

$$\mathsf{Sk}_1(\Sigma) \xrightarrow{\sim} \mathcal{O}(A(\Sigma)).$$

Under this isomorphism, elements in $\mathsf{Sk}_1(\Sigma)$ restrict to functions on $\widehat{\mathcal{T}}(\Sigma)$ which take positive real values. Specifically, arcs and loops in $\mathsf{Sk}_1(\Sigma)$ restrict to the corresponding *Penner coordinates*[36] on $\widehat{\mathcal{T}}(\Sigma)$.

This story may be quantized as follows. For each triangulation Δ of Σ, there is a map

$$\rho_\Delta : \mathrm{QTS}_{q,\Delta}(\Sigma) \longrightarrow \mathsf{Sk}_q(\Sigma)[\Delta^{-1}] \simeq \mathbb{T}_\Delta.$$

A non-boundary arc $\mathsf{x} \in \Delta$ defines an element $X_\mathsf{x} \in \mathrm{QTS}(\Sigma)_{q,\Delta}$. Then ρ_Δ is defined by

$$\rho_\Delta(X_\mathsf{x}) = [\Delta^{\mathsf{Q}^\Delta \mathsf{x}}].$$

That is, $\rho_\Delta(X_\mathsf{x})$ is the 'cross-ratio' of the four arcs in $\mathsf{Sk}_q(\Sigma)$ immediately adjacent to x (normalized by a power of q so that $\rho_\Delta(X_\mathsf{x})$ is invariant under the bar involution). The map ρ_Δ extends to an inclusion of fraction fields

$$\rho\colon \mathrm{QTS}_q(\Sigma) \longrightarrow \mathcal{F}(\mathsf{Sk}_q(\Sigma)).$$

which does not depend on a choice of triangulation Δ. Hence, Chekhov and Fock's quantum Teichmüller space can be realized as a sub-skew-field of the fraction field [37] of $\mathsf{Sk}_q(\Sigma)$. Under the $q = 1$ specialization, the map

$$\rho : \mathrm{QTS}_1(\Sigma) \longrightarrow \mathcal{F}(\mathsf{Sk}_1(\Sigma)).$$

is the same as the map induced on fraction fields by the cluster ensemble map

$$A(\Sigma) \longrightarrow X(\Sigma).$$

Remark B.2. In many ways, the algebra $\mathsf{Sk}_q(\Sigma)$ is the 'decorated' analog of the quantum Teichmüller space $\mathrm{QTS}_q(\Sigma)$. In each case, each triangulation determines a quantum torus inside a fixed skew-field. The main difference is that the intersection of the quantum tori in $\mathrm{QTS}_q(\Sigma)$ is too small, and so one must keep track of the whole skew-field QTS $y_q(\Sigma)$ to have a reasonable invariant. By contrast, the intersection of quantum tori containing $\mathsf{Sk}_q(\Sigma)$ is $\mathsf{Sk}_q(\Sigma)$, which is large enough for every quantum torus to be recoverable as an Ore localization.

[35] This cluster algebra associated to Σ was independently introduced in [14], who also highlighted its realization as functions on $\mathcal{T}(\Sigma)$.

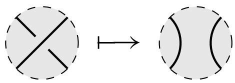
[36] An excellent reference for the connection between $\mathsf{Sk}_1(\Sigma)$ and Penner coordinates is [9].

[37] Note that ρ lands in the sub-skew-field of degree $\mathbf{0}$ for the endpoint grading of $\mathsf{Sk}_1(\Sigma)$.

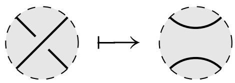
C. Proof of Lemma 4.9

C.1. The initial multicurve has positive smoothings. Let x be a simple arc in Σ. For a given Y, choose a homotopy representative of Y so that $\mathsf{x}\cdot\mathsf{Y}$ is transverse with minimal crossings.

Let $\mathsf{x}\cap\mathsf{Y}$ denote the set of crossings (that is, non-boundary intersections) in the superposition $\mathsf{x}\cdot\mathsf{Y}$. For any function $\sigma\colon \mathsf{x}\cap\mathsf{Y}\to\{-,+\}$, let R_σ be the multicurve obtained by applying the local relation (called a *positive smoothing*)



to each crossing sent to $+$ by σ, and by applying the local relation (called a *negative smoothing*)



to each crossing sent to $-$ by σ. The purpose if this is that

$$[\mathsf{x}][\mathsf{Y}] = q^a \sum_{\sigma:\mathsf{x}\cap\mathsf{Y}\longrightarrow\{-,+\}} q^{|\sigma^{-1}(+)|-|\sigma^{-1}(-)|}[\mathsf{R}_\sigma]$$

where $a\in\mathbb{Z}/2$ is the exponent produced by making the endpoints in $\mathsf{x}\cdot\mathsf{Y}$ simultaneous.

Lemma C.1. *In any* R_σ*,*

$$|\{\textit{contractible loops}\}| + \frac{1}{2}|\{\textit{contractible arcs}\}| \leq |\{\textit{negative smoothings}\}|$$

If equality holds, then each strand in each negative smoothing is in a contractible curve.

Proof. Choose a tubular neighborhood T of x small enough that Y intersects T a minimal number of times, but large enough to contain the chosen neighborhoods of each crossing in $\mathsf{x}\cdot\mathsf{Y}$.

Let z be a contractible curve in R_σ, and let $D\subset\Sigma$ denote the disc with boundary z. Construct a graph Γ whose vertices are connected components of $D\smallsetminus\partial T$, and with an edge between two components if they have common boundary in $D\cap\partial T$. The graph Γ is then a retract of the disc D, which implies that Γ is a tree. Since z intersects T but is not contained in T, there must be at least one component of $D\cap\partial T$ contained in T, and one component of $D\cap\partial T$ disjoint from T. It follows that T is a tree with at least two vertices, and so it has at least two vertices of degree 1.

A degree 1 vertex of Γ corresponds to a component of $D \smallsetminus \partial T$ with a single boundary component in $D \cap \partial T$. Let D_0 be such a component of $D \smallsetminus \partial T$; we split into three cases.

- ∂D_0 contains a marked point. This implies that z is a contractible arc; and so the marked point in ∂D_0 is the unique marked point in z. It follows that there can be at most one component of $D \smallsetminus \partial T$ of this type.
- $D_0 \subset D \cap T$ and ∂D_0 contains no marked points. The boundary of D_0 must contain a component of $\mathsf{x} \smallsetminus \mathsf{Y}$, whose endpoints are crossings in $\mathsf{x} \cap \mathsf{Y}$. In R_σ, one must be a negative smoothing and one must be a positive smoothing in R_σ; otherwise, D_0 could cross over x and have at least two boundary components in $D \cap \partial T$.
- $D_0 \subset D \smallsetminus T$ and ∂D_0 contains no marked points. This implies that the boundary of D_0 in z can be deformed to the interior T, contracting the assumption that T and Y intersect a minimum number of times. There are no components of this type.

Hence, if z is a contractible arc, then it must pass through a negative smoothing at least once, and if z is a contractible loop, then it must pass through a negative smoothing at least twice. Since each negative smoothing has two strands, this implies the stated lemma. □

Using the skein relations, we can write

$$[\mathsf{R}_\sigma] = q^{|\sigma^{-1}(+)|-|\sigma^{-1}(-)|}(-q^2-q^{-2})^{L_\sigma}\delta_\sigma[\widehat{\mathsf{R}}_\sigma].$$

where $\widehat{\mathsf{R}}_\sigma$ be the simple multicurve obtained by deleting contractible curves in R_σ, L_σ is the number of contractible loops in R_σ, and δ_σ is 0 if R_σ has a contractible arc and 1 if it does not.

$$\begin{aligned}[\mathsf{x}][\mathsf{Y}] &= q^a \sum_{\sigma:\mathsf{x}\cap\mathsf{Y}} q^{|\sigma^{-1}(+)|-|\sigma^{-1}(-)|}(-q^2-q^{-2})^{L_\sigma}\delta_\sigma[\widehat{\mathsf{R}}_\sigma] \\ &= q^a \sum_{\sigma:\mathsf{x}\cap\mathsf{Y}} \sum_{i=0}^{L_\sigma} \binom{L_\sigma}{i}(-1)^i q^{|\sigma^{-1}(+)|-|\sigma^{-1}(-)|+2L_\sigma-4i}\delta_\sigma[\widehat{\mathsf{R}}_\sigma].\end{aligned}$$

Using the fact that $|\mathsf{x}\cap\mathsf{Y}| = \sigma^{-1}(+)+\sigma^{-1}(-)$, we deduce that

$$[\mathsf{x}][\mathsf{Y}] = q^{a+|\mathsf{x}\cap\mathsf{Y}|} \sum_{\sigma:\mathsf{x}\cap\mathsf{Y}} \delta_\sigma[\widehat{\mathsf{R}}_\sigma] \sum_{i=0}^{L_\sigma} \binom{L_\sigma}{i}(-1)^i q^{2(L_\sigma-|\sigma^{-1}(-)|-2i)}.$$

Since $|\sigma^{-1}(-)|$ is the number of negative smoothings, $L_\sigma - |\sigma^{-1}(-)| \leq 0$ and equality only holds when R_σ has an equal number of contractible loops and negative smoothings (which implies that $\delta_\sigma = 1$).

Lemma C.2. *Let σ_+ denote the function which sends every crossing to $+$. If σ is another function such that R_σ has an equal number of contractible loops and negative smoothings, then $\widehat{\mathsf{R}}_\sigma \prec \widehat{\mathsf{R}}_{\sigma_+}$. Consequently,*

$$\mathrm{in}([\mathsf{x}][\mathsf{Y}]) = [\widehat{\mathsf{R}}_{\sigma_+}].$$

Proof. By Lemma C.1, neither R_{σ_+} nor R_σ have contractible arcs. Consequently, both $[\mathsf{R}_{\sigma_+}]$ and $[\mathsf{R}_\sigma]$ are non-zero.

Choose any negative smoothing in R_σ, and let $\mathsf{R}_{\sigma'}$ denote the multicurve in which it has been replaced by a positive smoothing. This alteration involves at most two curves in R_σ which must become at least one curve in $\mathsf{R}_{\sigma'}$, and so the total number of curves can decrease by at most 1. However, since the number of negative smoothings has decreased by 1, the number of contractible loops must have decreased by exactly 1, and so $\mathsf{R}_{\sigma'}$ has an equal number of contractible loops and negative smoothings.

The two strands in the chosen negative smoothing in R_σ must both be in contractible loops, by the preceding lemma. Since the number of contractible loops decreases by 1, there are two possibilities.

- The two strands were in distinct contractible loops in R_σ. They become one contractible loop in $\mathsf{R}_{\sigma'}$.
- The two strands are in the same contractible loop in R_σ. They become two loops in $\mathsf{R}_{\sigma'}$, which must be non-contractible.

In the first case, $\widehat{\mathsf{R}}_\sigma \preceq \widehat{\mathsf{R}}_{\sigma'}$, and in the second, $\widehat{\mathsf{R}}_\sigma \prec \widehat{\mathsf{R}}_{\sigma'}$.

By switching negative smoothings to positive smoothings one at a time, we may construct a sequence

$$\mathsf{R}_\sigma = \mathsf{R}_{\sigma_0}, \mathsf{R}_{\sigma_1}, \ldots, \mathsf{R}_{\sigma_n} = \mathsf{R}_{\sigma+}$$

such that at each step, $\widehat{\mathsf{R}}_{\sigma_i} \preceq \widehat{\mathsf{R}}_{\sigma_{i+1}}$. Furthermore, since $\mathsf{R}_{\sigma+}$ has no contractible loops, the last step in this sequence must be of the second type above; that is, $\widehat{\mathsf{R}}_{\sigma_{n-1}} \prec \widehat{\mathsf{R}}_{\sigma_+}$. By transitivity, $\widehat{\mathsf{R}}_\sigma \prec \widehat{\mathsf{R}}_{\sigma+}$. □

C.2. The map is an injection. Choose a tubular neighborhood T of x. For a given Y, choose a homotopy representative of Y so that Y intersects both x and T a minimal number of times.

Let γ_{x} denote the map $\mathsf{Y} \mapsto \mathrm{in}([\mathsf{x}][\mathsf{Y}]) = [\widehat{\mathsf{R}}_{\sigma+}]$. The multicurve $\gamma_{\mathsf{x}}(\mathsf{Y})$ has the following concrete construction inside the tubular neighborhood T: cut Y along each crossing in $\mathsf{x} \cdot \mathsf{Y}$ and reconnect the strands by shifting to the right along x. Any spare ends on either side are attached to the endpoints of x. The two cases (distinct versus identical endpoints of x) are illustrated in the Figure 13.

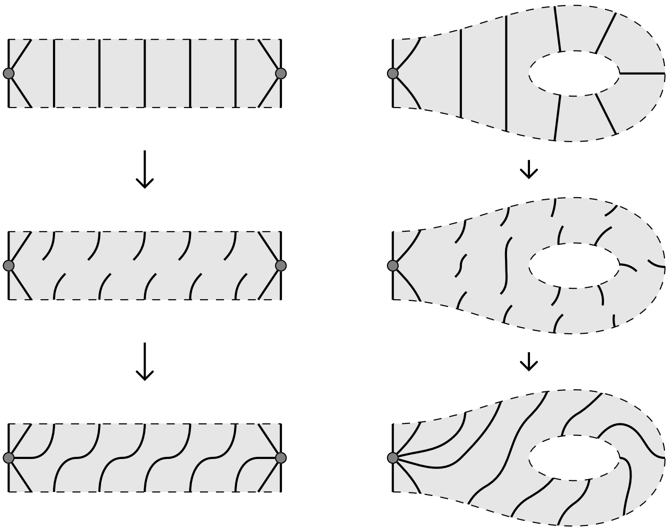


Figure 13. Explicit construction of $\gamma_{\mathsf{x}}(\mathsf{Y})$.

Lemma C.3. *For any simple curve, the map* $\gamma_{\mathsf{x}}\colon \mathsf{SMulti} \to \mathsf{SMulti}$ *is injective.*

Proof. Let Y be a simple multicurve transverse to x with minimal crossings. Define a new multicurve $\nu_{\mathsf{x}}(\mathsf{Y})$ as follows.

- If x has one end at a marked point p, and there are no strands of Y counter-clockwise to x at p, then $\nu_{\mathsf{x}}(\mathsf{Y})$ is the empty multicurve $\emptyset$.
- If x has both ends at a marked point p, and there are fewer than two strands of Y counter-clockwise to x, then $\nu_{\mathsf{x}}(\mathsf{Y})$ is the empty multicurve $\emptyset$.
- Otherwise, construct $\nu_{\mathsf{x}}(\mathsf{Y})$ as follows. Cut Y along x, and at each end of x, disconnect the first strand of Y counter-clockwise to x. Reconnect these ends by shifting to the left along x.

The composition $\nu_{\mathsf{x}}(\gamma_{\mathsf{x}}(\mathsf{Y})) = \mathsf{Y}$, therefore γ_{x} is injective. □

Lemma 4.9 is an immediate consequence of Lemmas C.2 and C.3.

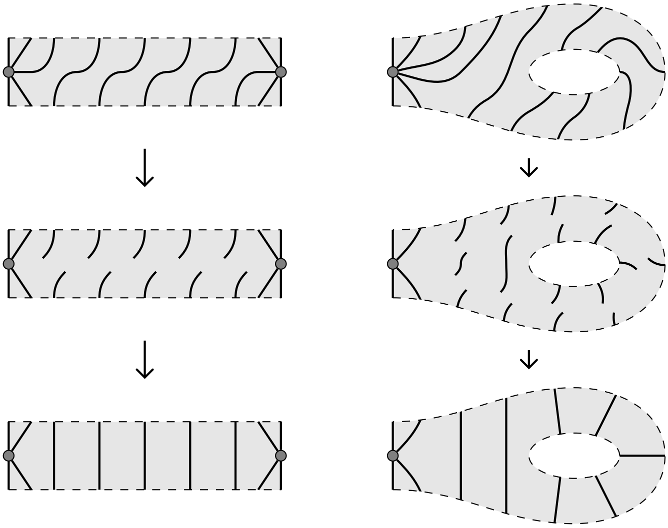


Figure 14. Explicit construction of $\nu_{\mathsf{x}}(\mathsf{Y})$.

References

[1] A. Berenstein, S. Fomin, and A. Zelevinsky, Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.* **126** (2005), no. 1, 1–52. MR 2110627 Zbl 1135.16013

[2] D. Bullock, A finite set of generators for the Kauffman bracket skein algebra. *Math. Z.* **231** (1999), no. 1, 91–101. MR 1696758 Zbl 0932.57016

[3] A. Berenstein and A. Zelevinsky, Quantum cluster algebras. *Adv. Math.* **195** (2005), no. 2, 405–455. MR 2146350 Zbl 1124.20028

[4] G. Dupont, Transverse quiver Grassmannians and bases in affine cluster algebras. *Algebra Number Theory* **4** (2010), no. 5, 599–624. MR 2679100 Zbl 1268.16019

[5] G. Dupont and H. Thomas, Atomic bases of cluster algebras of types A and $\widetilde{A}$. *Proc. Lond. Math. Soc.* (3) **107** (2013), no. 4, 825–850. MR 3108832 Zbl 1283.13019

[6] V. V. Fock and L. O. Chekhov, Quantum Teichmüller spaces. *Teoret. Mat. Fiz.* **120** (1999), no. 3, 511–528. In Russian. English translation, *Theoret. and Math. Phys.* **120** (1999), no. 3, 1245–1259 MR 1737362

[7] V. Fock and A. Goncharov, Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.* **103** (2006), 1–211. MR 2233852 Zbl 1099.14025

[8] M. Freedman, J. Hass, and P. Scott, Closed geodesics on surfaces. *Bull. London Math. Soc.* **14** (1982), no. 5, 385–391. MR 671777 Zbl 0476.53026

[9] S. Fomin, M. Shapiro, and D. Thurston, Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.* **201** (2008), no. 1, 83–146. MR 2448067 Zbl 1263.13023

[10] S. Fomin and D. Thurston, Cluster algebras and triangulated surfaces. II. Lambda lengths. Preprint 2012. arXiv:1210.5569 [math.GT]

[11] S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations. *J. Amer. Math. Soc.* **15** (2002), no. 2, 497–529 (electronic). MR 1887642 Zbl 1021.16017

[12] S. Fomin and A. Zelevinsky, Cluster algebras: notes for the CDM-03 conference. In B. Mazur, W. Schmid, S. T. Yau, A. J. de Jong, D. Jerison and G. Lustig (eds.), *Current developments in mathematics, 2003.* International Press, Somerville, MA, 2003, 1–34. MR 2132323 Zbl 1119.05108

[13] J. E. Grabowski and S. Launois, Quantum cluster algebra structures on quantum Grassmannians and their quantum Schubert cells: the finite-type cases. *Int. Math. Res. Not. IMRN* **2011**, no 10, 2230–2262. MR 2806564 Zbl 1223.14065

[14] M. Gekhtman, M. Shapiro, and A. Vainshtein, Cluster algebras and Poisson geometry. *Mosc. Math. J.* **3** (2003), no. 3, 899–934, 1199. Dedicated to V. I. Arnold on the occasion of his 65$^{\text{th}}$ birthday. MR 2078567 Zbl 1057.53064

[15] M. Gekhtman, M. Shapiro, and A. Vainshtein, Cluster algebras and Weil–Petersson forms. *Duke Math. J.* **127** (2005), no. 2, 291–311. MR 2130414 Zbl 1079.53124

[16] M. Gekhtman, M. Shapiro, and A. Vainshtein, *Cluster algebras and Poisson geometry.* Mathematical Surveys and Monographs, 167. American Mathematical Society, Providence, R.I., 2010. MR 2683456 Zbl 1217.13001

[17] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings.* London Mathematical Society Student Texts, 16. Cambridge University Press, Cambridge, 1989. MR 1020298 Zbl 0679.16001

[18] L. H. Kauffman, State models and the Jones polynomial. *Topology* **26** (1987), no. 3, 395–407. MR 899057 Zbl 0622.57004

[19] L. H. Kauffman, Virtual knot theory. *European J. Combin.* **20** (1999), no. 7, 663–690. MR 1721925 Zbl 0938.57006

[20] B. Keller, Categorification of acyclic cluster algebras: an introduction. In A. S. Cattaneo, A. Giaquinto, and P. Xu (ed.), *Higher structures in geometry and physics.* In honor of M. Gerstenhaber and J. Stasheff. Progress in Mathematics, 287. Birkhäuser/Springer, New York, 2011, 227–241. MR 2762547 Zbl 1220.18010

[21] S. Launois and T. H. Lenagan, Twisting the quantum Grassmannian. *Proc. Amer. Math. Soc.* **139** (2011), no. 1, 99–110. MR 2729074 Zbl 1219.16036

[22] G.+ Musiker and R. Schiffler, Cluster expansion formulas and perfect matchings. *J. Algebraic Combin.* **32** (2010), no. 2, 187–209. MR 2661414 Zbl 1246.13035

[23] G. Musiker, R. Schiffler, and L. Williams, Positivity for cluster algebras from surfaces. *Adv. Math.* **227** (2011), no. 6, 2241–2308. MR 2807089 Zbl 1331.13017

[24] G. Musiker, R. Schiffler, and L. Williams, Bases for cluster algebras from surfaces. *Compos. Math.* **149** (2013), no. 2, 217–263. MR 3020308 Zbl 1263.13024

[25] G. Muller, Locally acyclic cluster algebras. *Adv. Math.* **233** (2013), 207–247. MR 2995670 Zbl 1279.13032

[26] G. Musiker and L. Williams, Matrix formulae and skein relations for cluster algebras from surfaces. *Int. Math. Res. Not. IMRN* **2013**, no. 13, 2891–2944. MR 3072996 Zbl 1320.13028

[27] R. C. Penner, Decorated Teichmüller theory. With a foreword by Y. I. Manin. QGM Master Class Series. European Mathematical Society (EMS), Zürich, 2012. MR 3052157 Zbl 1243.30003

[28] J. H. Przytycki, Skein modules of 3-manifolds. *Bull. Polish Acad. Sci. Math.* **39** (1991), no. 1-2, 91–100. MR 1194712 Zbl 762.57013

[29] J. H. Przytycki and A. S. Sikora, On skein algebras and $\mathrm{Sl}_2(\mathbf{C})$-character varieties. *Topology* **39** (2000), no. 1, 115–148. MR 1710996 Zbl 0958.57011

[30] J. Rogers and T.+ Yang, The skein algebra of arcs and links and the decorated Teichmüller space. *J. Differential Geom.* **96** (2014), no. 1, 95–140. MR 3161387 Zbl 1290.53080

[31] R. Schiffler, A cluster expansion formula (A_n case). *Electron. J. Combin.* **15** (2008), no. 1, Research paper 64, 9 pp. MR 2398856 Zbl 1184.13064

[32] R. Schiffler, On cluster algebras arising from unpunctured surfaces. II. *Adv. Math.* **223** (2010), no. 6, 1885–1923. MR 2601004 Zbl 1238.13029

[33] R. Schiffler and H. Thomas, On cluster algebras arising from unpunctured surfaces. *Int. Math. Res. Not. IMRN* **2009**, no. 17, 3160–3189. MR 2534994 Zbl 1238.13029

[34] P. Sherman and A. Zelevinsky, Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. *Mosc. Math. J.* **4** (2004), no. 4, 947–974, 982. MR 2124174 Zbl 1103.16018

[35] V. G. Turaev, The Conway and Kauffman modules of a solid torus. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **167** (1988), Issled. Topol. 6, 79–89, 190. In Russian. English translation, *J. Soviet Math.* **52** (1990), no. 1, 2799–2805. MR 964255 Zbl 0706.57004

[36] V. G. Turaev, Algebras of loops on surfaces, algebras of knots, and quantization. In C. N. Yang and M. L. Ge (eds.), *Braid group, knot theory and statistical mechanics.* Advanced Series in Mathematical Physics, 9. World Scientific Publishing Co., Teaneck, N.J., 1989, 59–95. MR 1062423 Zbl 0732.57008

[37] M. Yakimov, Cyclicity of Lusztig's stratification of Grassmannians and Poisson geometry. In S. Caenepeel, J. Fuchs, S. Gutt, Ch. Schweigert, A. Stolin and F. Van Oystaeyen (eds.), *Noncommutative structures in mathematics and physics.* Papers from the congress held in Brussels, July 22–26, 2008. Koninklijke Vlaamse Academie van Belgie voor Wetenschappen en Kunsten (KVAB), Brussels, 2010, 259–263. MR 2742745 Zbl 1197.53112

Received May 9, 2013

Greg Muller, Department of Mathematics, University of Michigan, Ann Arbor, MI 48108, USA

e-mail: morilac@umich.edu