Quantum Topol. 7 (2016), 553–638
DOI 10.4171/QT/81

Quantum Topology

Quantum shuffles and quantum supergroups of basic type

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Abstract. We initiate the study of several distinguished bases for the positive half of a quantum supergroup U_q associated to a general super Cartan datum $(\mathrm{I}, (\cdot,\cdot))$ of basic type inside a quantum shuffle superalgebra. The combinatorics of words for an arbitrary total ordering on I is developed in connection with the root system associated to I. The monomial, Lyndon, and PBW bases of U_q are constructed, and moreover, a direct proof of the orthogonality of the PBW basis is provided within the framework of quantum shuffles. Consequently, the canonical basis is constructed for U_q associated to the standard super Cartan datum of type $\mathfrak{gl}(n \mid 1)$, $\mathfrak{osp}(1 \mid 2n)$, or $\mathfrak{osp}(2 \mid 2n)$ or an arbitrary non-super Cartan datum. In the non-super case, this refines Leclerc's work and provides a new self-contained construction of canonical bases. The canonical bases of U_q, of its polynomial modules, as well as of Kac modules in the case of quantum $\mathfrak{gl}(2 \mid 1)$ are explicitly worked out.

Mathematics Subject Classification (2010). 17B37, 16T20.

Keywords. Shuffle algebras, Quantum supergroups, PBW bases, canonical bases.

Contents

1. Introduction

1.1. The Drinfeld–Jimbo quantum group associated to a simple Lie algebra admits extremely rich structures with a wide variety of applications in representation theory, low-dimensional topology, and mathematical physics. In particular, the positive half admits some remarkable bases with interesting geometric and categorical interpretations, including PBW bases and canonical bases introduced by Lusztig [26, 27, 28] (see also [14] for another approach to canonical bases from the viewpoint of crystals).

In contrast, the quantum supergroups associated to a simple Lie superalgebra are not well understood beyond the foundational work of Yamane [33, 34]. As Lie superalgebras form an important extension of Lie algebras, it is natural to ask which structural features carry over to the super setting.

Some reasons to hope such a structure exists are the recent categorification results for quantum supergroups in [16, 12, 15, 8, 17], following earlier pioneering works of Khovanov, Lauda, and Rouquier [18, 32]. However, due to various internal difficulties (e.g. lack of integral forms, isotropic odd roots, lack of positivity due to super signs), no construction of a canonical basis existed or was even conjectured in the super setting until recently the authors [6] constructed the canonical bases for the integrable modules and the positive half of quantum supergroups associated to the "anisotropic" super Cartan datum, meaning no isotropic odd simple roots occur. The anisotropic super Cartan datum is distinguished among all super Cartan datum in the sense that the corresponding Lie superalgebras and quantum supergroups admit a semisimple category of integrable modules in parallel to the usual Kac-Moody setting. The only anisotropic super Cartan datum of finite type corresponds to the Lie superalgebra $\mathfrak{osp}(1 \mid 2n)$.

There are many other finite-dimensional simple Lie superalgebras besides $\mathfrak{osp}(1 \mid 2n)$, among which the most important class are those of basic type. Similar to semisimple Lie algebras, the Lie superalgebras of basic type admit non-degenerate even bilinear forms, root systems, triangular decompositions, and so on (cf. [13, 7]). However, there is no reasonable semisimple category of finite-dimensional integrable modules for Lie superalgebras of basic type except for $\mathfrak{osp}(1 \mid 2n)$. Another phenomenon is the existence of non-conjugate simple systems for a general Lie superalgebra of basic type. The quantum supergroups studied in [33] are associated to these basic Lie superalgebras.

Let U_q denote the positive half of a quantum supergroup of basic type. Benkart, Kang, and Kashiwara [2] constructed the crystal (but not the global) bases for the polynomial representations of quantum $\mathfrak{gl}(m \mid n)$, and subsequently Kwon [20] constructed crystal bases for Kac modules of quantum $\mathfrak{gl}(m \mid n)$ (also cf. [21]

in the case of $\mathfrak{osp}(r \mid 2n)$ and [30] in the case of $\mathfrak{osp}(1 \mid 2n)$); none of these authors constructed crystal bases or canonical bases for U_q. As the works [6, 4] helped us to lift the mental block on the existence of canonical bases for a class of quantum supergroups, we are motivated to reexamine the possibilities for quantum supergroups of basic type.

Since the basic Lie superalgebras include simple Lie algebras as limiting cases, we require an approach toward canonical bases which would work equally well for the usual quantum group of finite type. However, Lusztig's geometric approach (via either perverse sheaves or quiver geometry) is not applicable for now, while Kashiwara's algebraic approach requires a semisimple category of integrable modules and hence works well only for the anisotropic quantum supergroups.

1.2. In this paper, we provide a first step toward the construction of canonical bases for quantum supergroups of basic type, and give a description of U_q which we believe will be useful for future studies on categorification (cf. [19, 10, 11, 29, 3]). Our approach through quantum shuffles is inspired by the work of Leclerc [23] which, in turn, builds on other foundational works of M. Lothaire [25], J. A. Green [9], P. Lalonde and A. Ram [22], and M. Rosso [31] on relations among combinatorics of words, root systems, quantum groups and quantum shuffles. In this paper, we systematically develop a super version of the aforementioned works, and almost always work in the most general setting of arbitrary (not merely the standard) simple systems of basic type. The passage from the classical to the super setting is highly nontrivial, due largely to the lack of positivity in the formula for the shuffle product. Moreover, our results go beyond those appearing in the literature, leading to new combinatorial proofs of classical results on quantized Lie algebras.

Among other results, we construct a family of monomial bases and orthogonal PBW bases of U_q, one for each total ordering of the index set I labeling the simple roots. We then construct an integral form in types $\mathfrak{gl}(m \mid n)$, $\mathfrak{osp}(1 \mid 2n)$ and $\mathfrak{osp}(2 \mid 2n)$, which yield a canonical basis for U_q when the Cartan data is of type $\mathfrak{gl}(m \mid 1)$, $\mathfrak{osp}(1 \mid 2n)$ and $\mathfrak{osp}(2 \mid 2n)$. We are also able to obtain a bar-invariant *psuedo-canonical* basis for $\mathfrak{gl}(m \mid n)$. However, this basis fails to be almost orthogonal with respect to the bilinear form and is not independent of the chosen ordering on I.

Unlike in the non-super setting, the PBW bases constructed here are not known to be orthogonal *a priori*. To obtain this result, we generalize a main result of Leclerc [23, Theorem 36] and prove it directly from the combinatorics of Lyndon words (Leclerc's proof used the orthogonality of PBW bases due to Lusztig); see Lemma 4.19 and Theorem 5.1. In the special case of the natural ordering on I given in Table 1, Yamane [33] constructs a PBW basis and proves that it is orthogonal through a case-by-case analysis. Our proof is type independent for almost all orderings on I. Our argument applies equally well to the Cartan-Killing root datum, yielding an independent proof of the orthogonality of the PBW bases and a new self-contained algebraic construction of the canonical basis of the positive half of a Drinfeld–Jimbo quantum group of finite type. After completion of this paper, we learned of a similar construction of orthogonal PBW-type bases for Nichols algebras appearing in [1].

1.3. We now provide a detailed description of the main results of the paper section by section. In the preliminary Section 2, we collect various basic results on quantum superalgebras of basic type, most of which can be found in Yamane's papers [33, 34].

In Section 3, generalizing the work of Rosso [31] and Green [9], we embed the positive half of a quantum supergroup U_q associated to a general Cartan datum $(\mathrm{I}, (\cdot,\cdot))$ of basic type in a quantum shuffle superalgebra. This should be viewed as a dual version to a construction of Lusztig who realized U_q as a quotient of a free algebra by the radical of a bilinear form. In the super setting we use (a variant of) a non-degenerate bilinear form on U_q constructed by Yamane [33].

The combinatorics of super words, such as dominant words (also known as good words) and Lyndon words, is then developed systematically in Section 4. Superizing the constructions of Leclerc [23], we construct monomial bases of U_q. More significantly, we develop a *highest word* theory for U_q and establish a bijection between the set of dominant Lyndon words and the reduced root system associated to I, generalizing a fundamental result of Lalonde and Ram [22]. Finally, we construct an auxiliary Lyndon basis for U_q and obtain Lemma 4.19.

In Section 5, we give a construction of PBW bases of U_q. From Lemma 4.19 we deduce Theorem 5.1, prove a Levendorskii–Soibelman type formula, and prove that these bases are orthogonal, see Theorem 5.5, Lemma 5.6 and Theorem 5.7. We note that Lemma 5.6 can be viewed as a combinatorial analog of [29, Lemma 3.2].

In Section 6, we compute the dominant Lyndon words and root vectors explicitly for quantum supergroups of type A-D. These PBW root vectors are very similar to those defined in [33], though we express them in the basis of words.

Additionally, we compute the inner product between any two root vectors. This information is also contained in [33, §10.3], but as our sign convention on the bilinear form differs from that in *loc. cit.* we derive the formulas directly. Theorem 5.7 explains how to compute the norm of any PBW basis vector.

In Section 7, we introduce the integral form of U_q, where we have to restrict ourselves to the standard simple systems, and to types $\mathfrak{gl}(m \mid n)$, $\mathfrak{osp}(1 \mid 2n)$ and $\mathfrak{osp}(2 \mid 2n)$, as well as any non-super type. In the non-super specialization, this allows us to give a new self-contained algebraic construction of a canonical basis of U_q; more importantly, we obtain a canonical basis of U_q in types $\mathfrak{gl}(m \mid 1)$, $\mathfrak{osp}(1 \mid 2n)$ and $\mathfrak{osp}(2 \mid 2n)$.

The case of $\mathfrak{gl}(2 \mid 1)$ is studied in detail in Section 8. Explicit formulas for the canonical basis of U_q were already given in [16]. We show that the canonical basis of U_q descends to a canonical basis of every polynomial representation and every Kac module of quantum $\mathfrak{gl}(2 \mid 1)$. On the other hand, we show that the canonical basis of U_q fails to descend to a canonical basis for certain finite-dimensional simple modules of quantum $\mathfrak{gl}(2 \mid 1)$. We conjecture these phenomena hold for general $\mathfrak{gl}(m \mid 1)$ case.

Acknowledgements. Weiqiang Wang is partially supported by the NSF grant DMS-1101268. The authors thank Institute of Mathematics, Academia Sinica, Taipei for providing an excellent working environment and support, which greatly facilitated this research. We also thank Bernard Leclerc for helpful discussions and clarifications regarding his paper.

2. Quantum supergroups of basic type

In this section, we review some fundamental properties of the positive half of a quantum supergroup of basic type, including the bilinear form and defining relations.

2.1. Root data. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a complex basic Lie superalgebra of rank $m + n + 1 = N$ of type A-G [13, 7]. Let $\widetilde{\Phi} = \widetilde{\Phi}_{\bar{0}} \sqcup \widetilde{\Phi}_{\bar{1}}$ be the root system for $\mathfrak{g}$, and let

$$\Phi = \Phi_{\bar{0}} \sqcup \Phi_{\bar{1}} = \left\{ \beta \in \widetilde{\Phi} \;\middle|\; \frac{1}{2}\beta \notin \widetilde{\Phi} \right\}$$

be the reduced root system for $\mathfrak{g}$, where $\Phi_s = \Phi \cap \widetilde{\Phi}_s$, for $s \in \{\bar{0}, \bar{1}\}$; as usual $\bar{0}$ and $\bar{1}$ here and below indicate the even and odd (roots) respectively. We will work with Φ and not $\widetilde{\Phi}$ until Section 7. Let

$$\Pi = \Pi_{\bar{0}} \sqcup \Pi_{\bar{1}} = \{\alpha_i \mid i \in \mathrm{I}\}$$

be a simple system for $\widetilde{\Phi}$ which is labelled by $\mathrm{I} = \mathrm{I}_{\bar{0}} \sqcup \mathrm{I}_{\bar{1}} = \{1, \ldots, N\}$, and let

$$\Phi^+ \subseteq \Phi$$

be the corresponding set of positive roots. We define the parity function $p(\cdot)$ on I by letting

$$p(i) = s \quad \text{for } i \in \mathrm{I}_s \text{ with } s \in \{\bar{0}, \bar{1}\}.$$

Let Q be the root lattice. The monoid

$$Q^+ := \bigoplus_{i \in \mathrm{I}} \mathbb{Z}_{\geq 0}\alpha_i$$

is $\mathbb{Z}_2$-graded by declaring

$$p(\alpha_i) = p(i)$$

and extending linearly. We further decompose

$$\Phi_{\bar{1}} = \Phi_{\mathsf{iso}} \sqcup \Phi_{\mathsf{n\text{-}iso}}$$

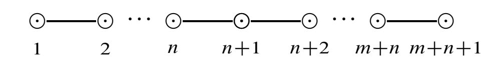
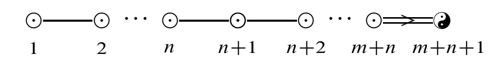
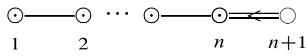
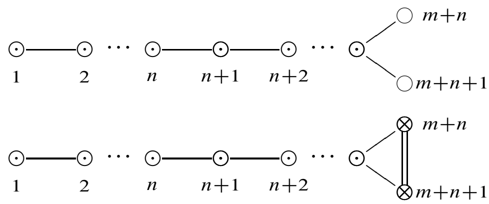
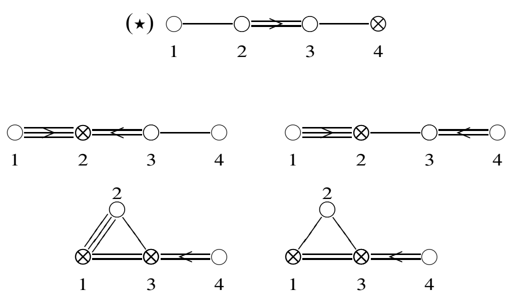
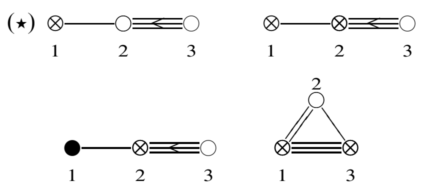
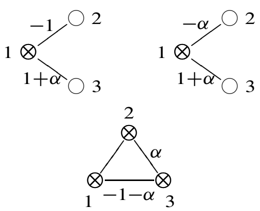
where Φ_{iso} (resp. $\Phi_{\mathsf{n\text{-}iso}}$) is the set of isotropic (resp. non-isotropic) odd roots. Decompose

$$\Pi_{\bar{1}} = \Pi_{\mathsf{iso}} \sqcup \Pi_{\mathsf{n\text{-}iso}}$$

(resp. $\mathrm{I}_{\bar{1}} = \mathrm{I}_{\mathsf{iso}} \sqcup \mathrm{I}_{\mathsf{n\text{-}iso}}$) accordingly.

In Table 1 below, we list the Dynkin diagrams which arise from an arbitrary choice of Φ^+ (for type A-D) and label the simple roots according to the labels on the nodes of the corresponding diagram. The diagrams labelled with ($\star$) in types $F(3 \mid 1)$ and $G(3)$ will be referred to as *distinguished diagrams* ($F(3 \mid 1)$ is often referred to as $F(4)$ in literature). The simple roots may be even, odd isotropic, or odd non-isotropic, and we will label the corresponding nodes ○, ⊗, and ●, respectively. We will use the notation ⊙ to denote a simple root which may be either odd isotropic or even, and ◕ for a simple root which may be either odd non-isotropic or even.

Table 1. Dynkin diagrams for general simple systems.

$A(m,n)$	
$B(m,n+1)$	
$C(n+1)$	
$D(m,n+1)$	
$F(3 \mid 1)$	
$G(3)$	
$D(2 \mid 1;\alpha)$ $(\alpha \in \mathbb{Z}_{>0})$	

The basic Lie superalgebras are examples of symmetrizable contragredient Lie superalgebras associated to (super generalized) Cartan matrices [13], which are endowed with a non-degenerate even supersymmetric bilinear form. Let $A = (a_{ij})_{i,j \in \mathrm{I}}$ be a symmetrizable Cartan matrix for $\mathfrak{g}$. Let d_i, $i \in \mathrm{I}$, be positive integers satisfying

$$d_i a_{ij} = d_j a_{ji}, \quad \text{and} \quad \gcd(d_i \mid i \in \mathrm{I}) = 1.$$

Define a symmetric bilinear form

$$(\cdot,\cdot)\colon Q \times Q \longrightarrow \mathbb{Z}$$

by letting

$$(\alpha_i, \alpha_j) = d_i a_{ij}, \quad i, j \in \mathrm{I}.$$

In particular, we have the following basic property.

Lemma 2.1. *The following are equivalent for $i \in \mathrm{I}$:*

(1) $a_{ii} = 0$;

(2) $i \in I_{\mathsf{iso}}$;

(3) $(\alpha_i, \alpha_i) = 0$.

We set the notation

$$\pi = -1, \tag{2.1}$$

which will be used to keep track of super-signs. Set

$$s_{ij} = \begin{cases} 1 & \text{if } (\alpha_i, \alpha_j) \geq 0, \\ \pi & \text{if } (\alpha_i, \alpha_j) < 0. \end{cases} \tag{2.2}$$

We call the triple $(\mathrm{I}, \Pi, (\cdot,\cdot)\,)$ a *Cartan datum of basic type.*

2.2. Quantum superalgebra U_q. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition of $\mathfrak{g}$. The quantized enveloping algebra $U_q(\mathfrak{g})$ with Chevalley generators $e_i, f_i, k_i^{\pm 1}$ $(i \in \mathrm{I})$ has been systematically defined and studied in [33] (here we choose to adopt a more standard version without an extra parity operator denoted by σ in *loc. cit.*). Let $U_q = U_q(\mathfrak{n}^+)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i $(i \in \mathrm{I})$. By definition, U_q is a quotient of a free superalgebra on the generators e_i by the radical of the bilinear form, just as defined by [28, Part I] in the non-super setting. We will use a rescaling of this bilinear form; see Proposition 2.4 below.

The algebra U_q is Q^+-graded by declaring that the degree of e_i is α_i:

$$U_q = \bigoplus_{\nu \in Q^+} U_{q,\nu}.$$

For homogeneous $u \in U_q$, we write $|u|$ for the degree of u in this grading. There is also a $\mathbb{Z}_2$-grading on U_q by setting

$$p(u) = p(\nu) \quad \text{if } |u| = \nu.$$

The next proposition is standard (see e.g. [33]); in the case of $B(0, n+1)$ the novel bar involution was introduced in [12].

Proposition 2.2. *The algebra U_q admits the following symmetries*:

(1) *a $\mathbb{Q}(q)$-linear anti-automorphism*

$$\tau : U_q \longrightarrow U_q$$

defined by

$$\tau(e_i) = e_i \text{ for all } i \in I \quad \text{and} \quad \tau(uv) = \tau(v)\tau(u). \tag{2.3}$$

(2) *A $\mathbb{Q}$-linear automorphism*

$$\bar{\ } : U_q \longrightarrow U_q$$

(called a ***bar involution****) defined by*

$$\bar{q} = \begin{cases} \pi q^{-1} & \text{if } U_q \text{ is of type } B(0, n+1), \\ q^{-1} & \text{otherwise,} \end{cases} \tag{2.4}$$

with

$$\overline{e_i} = e_i \quad \text{for all } i \in I,$$

and

$$\overline{uv} = \bar{u}\ \bar{v};$$

(3) *a $\mathbb{Q}$-linear anti-automorphism*

$$\sigma : U_q \longrightarrow U_q$$

defined by

$$\sigma(u) = \overline{\tau(u)}. \tag{2.5}$$

Proof. The existence of the anti-automorphism τ is proved in [34, Lemma 6.3.1]. The existence of the bar involution can be proved using similar arguments to those in [28, §1.2.12] (see also [5, Cor 1.4.4]). □

The algebra U_q has the structure of a twisted bi-superalgebra with coproduct defined on the generators by

$$\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i.$$

The coproduct is an algebra homomorphism

$$\Delta\colon U_q \longrightarrow U_q \otimes U_q$$

with respect to the twisted multiplication on $U_q \otimes U_q$:

$$(a \otimes b)(c \otimes d) = \pi^{p(b)p(c)} q^{-(|b|,|c|)} ac \otimes bd,$$

for $a, b, c, d \in U_q$ homogeneous in the $(Q^+ \times \mathbb{Z}_2)$-grading.

2.3. Bilinear forms on U_q. The goal of this section is to establish the existence of the bilinear form described in Proposition 2.4, a variant of which first appeared in [33]. Indeed, let $(\cdot,\cdot)_{\mathrm{sgn}}$ be the form appearing in *loc.cit.*. This form satisfies Conditions (B1)–(B3) in the statement of Proposition 2.4 below, but with the (q,π)-bialgebra structure on $U_q \otimes U_q$ replaced by a (q^{-1},π)-bialgebra structure and with the bilinear form satisfying

$$(x' \otimes x'', y' \otimes y'')_{\mathrm{sgn}} = \pi^{p(x'')p(y')}(x', y')_{\mathrm{sgn}}(x'', y'')_{\mathrm{sgn}}. \tag{2.6}$$

In order to deduce the proposition, we begin with some general comments about rescaling of bilinear forms. To this end, let

$$t\colon Q^+ \times Q^+ \longrightarrow \mathbb{Q}(q)^\times$$

be a function such that

$$t(\lambda,\nu) = t(\nu,\lambda),$$

$$t(\lambda+\nu,\eta) = t(\lambda,\eta)t(\nu,\eta),$$

$$t(\lambda,\nu+\eta) = t(\lambda,\nu)t(\lambda,\eta).$$

Lemma 2.3. *Assume we have a bilinear form* $\{\cdot,\cdot\}$ *on* U_q *such that*

(1) *for all* $\mu \neq \nu$ *in* Q^+,
$$\{U_{q,\mu}, U_{q,\nu}\} = 0;$$

(2) *for all* $i \in \mathrm{I}$,
$$\{1,1\} = 1 \quad \textit{and} \quad \{e_i, e_i\} \neq 0;$$

(3) *for all* $x, y, z \in U_q$,
$$\{xy, z\} = \{x \otimes y, \Delta(z)\},$$
where
$$\{x \otimes y, x' \otimes y'\} = t(|y|, |x'|)\{x, x'\}\{y, y'\}.$$

Then there is a symmetric bilinear form $(\cdot,\cdot)$ *on* U_q *such that*

(a) *for all* $\mu \neq \nu$,
$$(U_{q,\mu}, U_{q,\nu}) = 0;$$

(b) *for all* $i \in \mathrm{I}$,
$$(1,1) = 1 \quad \textit{and} \quad (e_i, e_i) \neq 0;$$

(c) *for all* $x, y, z \in U_q$,
$$(xy, z) = (x \otimes y, \Delta(z)),$$
where
$$(x \otimes y, x' \otimes y') = (x, x')(y, y').$$

Specifically, the bilinear form is given by
$$(x, y) = t(|x|)^{-1}\{x, y\},$$
where
$$t(\alpha_{i_1} + \ldots + \alpha_{i_n}) = \prod_{r<s} t(\alpha_{i_r}, \alpha_{i_s}).$$

Proof. Note that $t(\alpha_{i_1} + \ldots + \alpha_{i_n})$ defined above does not depend on the order because t is symmetric. Since this rescaling is well defined on each weight space, it suffices to show that the given bilinear form satisfies the required properties. (a) and (b) are trivially true, and the form $(\cdot,\cdot)$ is clearly symmetric. For (c), let x, y, z be homogeneous and
$$\Delta(z) = \sum z_1 \otimes z_2.$$

Then

$$\begin{aligned}(xy,z) &= t(|x|+|y|)^{-1}\{xy,z\}\\ &= t(|x|+|y|)^{-1}\{x\otimes y,\Delta(z)\}\\ &= t(|x|+|y|)^{-1}\sum t(|y|,|z_1|)\{x,z_1\}\otimes\{y,z_2\}\\ &= t(|x|+|y|)^{-1}\sum t(|y|,|x|)t(|x|)t(|y|)(x,z_1)\otimes(y,z_2).\end{aligned}$$

Observing that

$$t(|x|,|y|)t(|x|)t(|y|) = t(|x|+|y|)$$

finishes the proof. □

The following is a variant of a theorem due to Yamane [33, Section 2].

Proposition 2.4. *There exists a unique nondegenerate symmetric bilinear form*

$$(\cdot,\cdot)\colon U_q\times U_q\longrightarrow \mathbb{Q}(q)$$

satisfying

(B1) $(1,1)=1$;

(B2) $(e_i,e_j)=\delta_{ij}$, *for all* $i,j\in \mathrm{I}$;

(B3) $(x,yz)=(\Delta(x),y\otimes z)$, *for all* $x,y,z\in U_q$.

Here we have used

$$(x'\otimes x'',y'\otimes y'') := (x',y')(x'',y'').$$

Proof. Let $(\cdot,\cdot)_{\mathrm{sgn}}$ be the bilinear form appearing in [33, Section 2]. This bilinear form was shown to satisfy the 3 properties in the proposition with respect to (2.6). Take

$$t(\mu,\nu)=\pi^{p(\mu)p(\nu)}$$

and

$$\{x,y\}=\overline{(\bar{x},\bar{y})_{\mathrm{sgn}}},\quad x,y\in U_q.$$

Then the bilinear form $(\cdot,\cdot)$ obtained from $\{\cdot,\cdot\}$ satisfies the same properties, by Lemma 2.3. □

In [12, Proposition 3.3], the authors showed directly that the unsigned version of the bilinear form for U_q of type $B(0,n)$ (and other anisotropic Kac-Moody types) is well-defined. Our preference for this form is due to the fact that it agrees with a bilinear form arising from categorification.

Proposition 2.5. *Let*

$$e_i' : U_q \longrightarrow U_q$$

denote the adjoint of left multiplication by e_i with respect to the binear form:

$$(e_i u, v) = (u, e_i'(v)).$$

Then, e_i' satisfies

(1) $e_i'(e_j) = \delta_{ij}$;

(2) *for homogenous $u, v \in U_q$,*

$$e_i'(uv) = e_i'(u)v + \pi^{p(u)p(i)} q^{-(\alpha_i,|u|)} u e_i'(v);$$

(3) *for homogeneous $u \in U_q$,*

$$e_i'(u) = 0 \textit{ for all } i \in \mathrm{I} \iff |u| = 0.$$

Proof. Property (1) is obvious from the definition. To prove Property (2), let $x \in U_q$ and write

$$\Delta(x) = \sum x_1 \otimes x_2.$$

Then,

$$\begin{aligned}
(x, e_i'(uv)) &= (e_i x, uv) \\
&= ((e_i \otimes 1 + 1 \otimes e_i)\Delta(x), u \otimes v) \\
&= \sum (e_i x_1 \otimes x_2, u \otimes v) + \sum \pi^{p(x_1)p(i)} q^{-(\alpha_i,|x_1|)} (x_1 \otimes e_i x_2, u \otimes v) \\
&= \sum (e_i x_1, u)(x_2, v) + \sum \pi^{p(x_1)p(i)} q^{-(\alpha_i,|x_1|)} (x_1, u)(e_i x_2, v).
\end{aligned}$$

Note that if a summand of the second sum in the last line above is nonzero, then $|x_1| = |u|$ and $p(x_1) = p(u)$. Therefore,

$$\begin{aligned}
(x, e_i'(uv)) &= \sum (e_i x_1, u)(x_2, v) + \sum \pi^{p(u)p(i)} q^{-(\alpha_i,|u|)} (x_1, u)(e_i x_2, v) \\
&= \sum (x_1, e_i'(u))(x_2, v) + \sum \pi^{p(u)p(i)} q^{-(\alpha_i,|u|)} (x_1, u)(x_2, e_i'(v)) \\
&= \sum (x_1 \otimes x_2, e_i'(u) \otimes v + \pi^{p(u)p(i)} q^{-(\alpha_i,|u|)} u \otimes e_i'(v)) \\
&= (x, e_i'(u)v + \pi^{p(u)p(i)} q^{-(\alpha_i,|u|)} u e_i'(v)).
\end{aligned}$$

Since the form is nondegenerate, (2) follows.

Finally, to prove (3), note that if $|u| = \nu$, then $|e'_i(u)| = \nu - \alpha_i$. In particular, if $|u| = 0$, then $e'_i(u) = 0$ for all $i \in \mathrm{I}$. Conversely, if $e'_i(u) = 0$ for all i, then we have $(e_{i_1} \cdots e_{i_d}, u) = 0$ for all $i_1, \ldots, i_d \in \mathrm{I}$ and $d \geq 1$. As these monomials span $\bigoplus_{\nu \neq 0} U_{q,\nu}$, and the form is nondegenerate, we must have $|u| = 0$. □

Corollary 2.6. *The subalgebra $\mathcal{E}$ of $\mathrm{End}_{\mathbb{Q}(q)}(U_q)$ generated by the e'_i for $i \in I$ is isomorphic to U_q under the identification $e_i \mapsto e'_i$.*

Proof. Since the bilinear form is nondegenerate, the map $e_i \mapsto e'_i$ defines an anti-isomorphism between U_q and $\mathcal{E}$; Composing with the map τ defined in Proposition 2.2 yields the desired isomorphism. □

2.4. Defining relations for U_q. Define the q-commutator on homogeneous $u, v \in U_q$ by

$$\mathrm{ad}_q u(v) = [u, v]_q = uv - \pi^{p(u)p(v)} q^{(|u|,|v|)} vu.$$

Define the usual quantum integer and its super analogue for $n \in \mathbb{Z}_{\geq 0}$:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

and

$$\{n\} = \frac{\pi^n q^n - q^{-n}}{\pi q - q^{-1}}.$$

More generally, for $i \in I$, set

$$q_i = q^{d_i},$$
$$\pi_i = \pi^{p(i)},$$

and define

$$[n]_i = \begin{cases} \dfrac{\pi_i^n q_i^n - q_i^{-n}}{\pi_i q_i - q_i^{-1}} & \text{if } i \in \mathrm{I}_{\text{n-iso}}, \\[2ex] \dfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} & \text{otherwise,} \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i [n-1]_i \cdots [n-k+1]_i}{[k]_i!},$$

where $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$.

Proposition 2.7 ([33, 34]). *The algebra U_q satisfies the following relations whenever the given Dynkin subdiagram appears*:

(Iso) $e_i e_j = -e_j e_i$ *for* $i, j \in \mathrm{I}_{\bar{1}}$ *with* $a_{ij} = 0$;

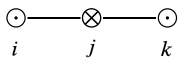
(N-Iso) *for* $i \in \mathrm{I}_{\bar{0}} \cup \mathrm{I}_{\text{n-iso}}$ *and* $i \neq j$,

$$\sum_{r+s=1+|a_{ij}|} (-1)^r \pi_i^{p(i,j;r)} \begin{bmatrix} 1+|a_{ij}| \\ r \end{bmatrix}_i e_i^r e_j e_i^s = 0,$$

where

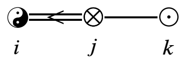
$$p(i,j;r) = \binom{r}{2} p(i) + r p(i) p(j);$$

(AB) *for*



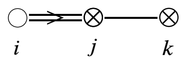
$(s_{ij} \neq s_{jk})$

or



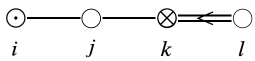
$$\mathrm{ad}_q e_j \circ \mathrm{ad}_q e_k \circ \mathrm{ad}_q e_j (e_i) = 0;$$

(CD1) *for*



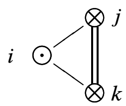
$$\mathrm{ad}_q e_j \circ \mathrm{ad}_q (\mathrm{ad}_q e_j (e_k)) \circ \mathrm{ad}_q e_i \circ \mathrm{ad}_q e_j (e_k) = 0;$$

(CD2) *for*



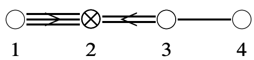
$$\mathrm{ad}_q e_k \circ \mathrm{ad}_q e_j \circ \mathrm{ad}_q e_k \circ \mathrm{ad}_q e_l \circ \mathrm{ad}_q e_k \circ \mathrm{ad}_q e_j (e_i) = 0;$$

(D) *for*



$$\mathrm{ad}_q e_k \circ \mathrm{ad}_q e_j (e_i) = \mathrm{ad}_q e_j \circ \mathrm{ad}_q e_k (e_i);$$

(F1) *for*

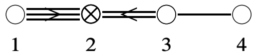


$$\mathrm{ad}_q E \circ \mathrm{ad}_q E \circ \mathrm{ad}_q e_4 \circ \mathrm{ad}_q e_3 \circ \mathrm{ad}_q e_2 = 0,$$

where

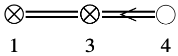
$$E = \mathrm{ad}_q (\mathrm{ad}_q e_1 (e_2)) \circ \mathrm{ad}_q e_3 (e_2);$$

(F2) *for*



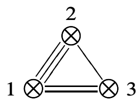
$$\mathrm{ad}_q(\mathrm{ad}_q e_1(e_2)) \circ \mathrm{ad}_q(\mathrm{ad}_q e_3(e_2)) \circ \mathrm{ad}_q e_3(e_4) \\ = \mathrm{ad}_q(\mathrm{ad}_q e_3(e_2)) \circ \mathrm{ad}_q(\mathrm{ad}_q e_1(e_2)) \circ \mathrm{ad}_q e_3(e_4);$$

(F3) *for*



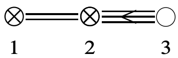
$$\mathrm{ad}_q e_3 \circ \mathrm{ad}_q e_1 \circ \mathrm{ad}_q e_3(e_4) = 0;$$

(F4) *for*



$$[3]\mathrm{ad}_q e_i \circ \mathrm{ad}_q e_j(e_k) + [2]\mathrm{ad}_q e_j \circ \mathrm{ad}_q e_i(e_k) = 0;$$

(G1) *for*

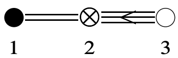


$$\mathrm{ad}_q E \circ \mathrm{ad}_q E \circ \mathrm{ad}_q E \circ \mathrm{ad}_q e_2(e_1) = 0,$$

where

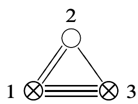
$$E = \mathrm{ad}_q e_2(e_3);$$

(G2) *for*



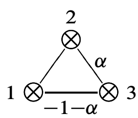
$$\mathrm{ad}_q e_2 \circ \mathrm{ad}_q e_3 \circ \mathrm{ad}_q e_3 \circ \mathrm{ad}_q e_2(e_1) = \mathrm{ad}_q e_3 \circ \mathrm{ad}_q e_2 \circ \mathrm{ad}_q e_3 \circ \mathrm{ad}_q e_2(e_1);$$

(G3) *for*



$$\mathrm{ad}_q e_1 \circ \mathrm{ad}_q e_2(e_3) - [2]\mathrm{ad}_q e_2 \circ \mathrm{ad}_q e_1(e_3) = 0;$$

(Dα) *for*



$$[\alpha + 1]\mathrm{ad}_q e_1 \circ \mathrm{ad}_q e_3(e_2) + [\alpha]\mathrm{ad}_q e_3 \circ \mathrm{ad}_q e_1(e_2) = 0.$$

Theorem 2.8 ([33, Proposition 10.4.1]). *If the Dynkin diagram for U_q is of type A-D, or the distinguished diagram in types F and G, then the relations given in Proposition 2.7 are defining relations for U_q.*

3. Quantum shuffle superalgebras

In this section, we formulate a quantum shuffle superalgebra associated to a Cartan datum of basic type, and construct an embedding of the half-quantum superalgebra U_q into a quantum shuffle superalgebra. These form super generalizations of constructions of Green [9] and Rosso [31].

3.1. The homomorphism Ψ, I. Let $(\mathrm{I}, \Pi, (\cdot,\cdot))$ be a Cartan datum of basic type. Let

$$\mathsf{F} = \mathsf{F}(\mathrm{I})$$

be the free associative superalgebra over $\mathbb{Q}(q)$ generated by I, with parity prescribed by $p(\cdot)$ on I. Let $\mathsf{W} = \sqcup_{d\geq 0}\mathrm{I}^d$ be the set of words in F, i.e., the monoid generated by I. The identity element is the empty word $\emptyset$, and a general word will be denoted by

$$\mathbf{i} = (i_1, i_2, \ldots, i_d) = i_1 i_2 \cdots i_d.$$

For $i \in I$ and $k \in \mathbb{N}$, we will use the notation

$$i^k = \underbrace{i\,i\ldots i}_{k}.$$

Note that F has a weight space decomposition

$$\mathsf{F} = \bigoplus_{\nu\in Q^+} \mathsf{F}_\nu$$

by setting

$$|(i_1,\ldots,i_d)| = \alpha_{i_1} + \ldots + \alpha_{i_d}$$

and extending linearly. We define

$$\mathsf{W}_\nu = \mathsf{W} \cap \mathsf{F}_\nu. \tag{3.1}$$

Finally, define the length function

$$\ell\colon \mathsf{W} \longrightarrow \mathbb{Z}_{\geq 0}$$

as

$$\ell(i_1, \ldots, i_d) = d. \tag{3.2}$$

Let $v \in \mathbb{Q}(q)$. We define the v-quantum shuffle product

$$\diamond_v\colon \mathsf{F} \times \mathsf{F} \longrightarrow \mathsf{F}$$

inductively by the formula

$$(xi)\diamond_v(yj) = (x\diamond_v(yj))i + \pi^{(p(x)+p(i))p(j)} v^{-(|x|+\alpha_i,\alpha_j)}((xi)\diamond_v y)j, \tag{3.3}$$

and $x\diamond_v\emptyset = \emptyset\diamond_v x = x$, for homogenous $x, y \in \mathsf{F}$ and $i, j \in \mathrm{I}$. The quantum shuffle products of interest will be those for $v = q$ or $v = q^{-1}$, so when there is no chance of confusion we will write

$$\diamond = \diamond_q \quad \text{and} \quad \bar{\diamond} = \diamond_{q^{-1}}.$$

Iterating (3.3) above, we obtain

$$(i_1, \ldots, i_a) \diamond (i_{a+1}, \ldots, i_{a+b}) = \sum_\sigma \pi^{\varepsilon(\sigma)} q^{-e(\sigma)} (i_{\sigma(1)}, \ldots, i_{\sigma(a+b)}), \tag{3.4}$$

where the sum is over minimal coset representatives in $S_{a+b}/S_a \times S_b$,

$$\varepsilon(\sigma) = \sum_{\substack{r\leq a<s \\ \sigma(r)<\sigma(s)}} p(i_{\sigma(r)})p(i_{\sigma(s)}), \quad \text{and} \quad e(\sigma) = \sum_{\substack{r\leq a<s \\ \sigma(r)<\sigma(s)}} (\alpha_{i_{\sigma(r)}}, \alpha_{i_{\sigma(s)}}). \tag{3.5}$$

We call each

$$(i_{\sigma(1)}, \ldots, i_{\sigma(a+b)})$$

in (3.4) a *shuffle* of $(i_1, \ldots, i_a)$ and $(i_{a+1}, \ldots, i_{a+b})$. More generally, given $x, y \in \mathsf{F}$ such that

$$x = \sum c_w w \quad \text{and} \quad y = \sum d_w w,$$

we say that a word $z \in \mathsf{W}$ *occurs as a shuffle* in $x \diamond y$ if z is a shuffle of words $w_1, w_2 \in \mathsf{W}$ such that

$$c_{w_1} d_{w_2} \neq 0.$$

Proposition 3.1. *The shuffle product is associative and satisfies*

$$x \diamond y = \pi^{p(x)p(y)} q^{-(|x|,|y|)} y \mathbin{\bar{\diamond}} x,$$

where we have used the notation

$$\bar{\diamond} = \diamond_{q^{-1}}.$$

Proof. The proof is straightforward using (3.4). □

We call $(\mathsf{F}, \diamond)$ the quantum shuffle (super)algebra associated to I.

We now describe the bialgebra structure on F with respect to the concatenation product, and explain the relationship with the shuffle product. Equip $\mathsf{F} \otimes \mathsf{F}$ with the associative product

$$(w \otimes x)(y \otimes z) = \pi^{p(x)p(y)} q^{-(|x|,|y|)} (wy) \otimes (xz),$$

where we use the concatenation product on each tensor factor. Then,

$$\delta \colon \mathsf{F} \longrightarrow \mathsf{F} \otimes \mathsf{F}$$

given by

$$\delta(i) = i \otimes 1 + 1 \otimes i$$

is an algebra homomorphism with respect to the concatenation product on both sides.

Lemma 3.2. *The algebra* F *admits a symmetric bilinear form* $(\cdot, \cdot)$ *such that*

$$(1, 1) = 1,$$

and

$$(i, j) = \delta_{i,j}, \quad \text{for } i, j \in \mathrm{I},$$

$$(\mathbf{ij}, \mathbf{k}) = (\mathbf{i} \otimes \mathbf{j}, \delta(\mathbf{k})), \quad \text{for } \mathbf{i}, \mathbf{j} \in \mathsf{W},$$

where

$$(\mathbf{i}_1 \otimes \mathbf{i}_2, \mathbf{j}_1 \otimes \mathbf{j}_2) = (\mathbf{i}_1, \mathbf{j}_1)(\mathbf{i}_2, \mathbf{j}_2).$$

Proof. This can be proved by a standard argument; cf. [28, 5]. □

Note that there is an obvious surjective algebra homomorphism

$$\psi : \mathsf{F} \longrightarrow U_q$$

given by

$$i \longmapsto e_i;$$

moreover,

$$\Delta \circ \psi = (\psi \otimes \psi) \circ \delta,$$

and hence by Proposition 2.4,

$$(\mathbf{i}, \mathbf{j}) = (\psi(\mathbf{i}), \psi(\mathbf{j})).$$

Suppose that $\mathbf{i} = i_1 \cdots i_n$. For any $a < b \in \mathbb{N}$, set

$$[a.b] = \{a, a+1, \ldots, b-1, b\}.$$

Then for any subset $P = \{k_1 < \ldots < k_m\}$ of $[1.n]$, define

$$\mathbf{i}_P = i_{k_1} \cdots i_{k_m}$$

so that $\mathbf{i}_P$ is a word of length $m \le n$. We have

$$\delta(\mathbf{i}) = \prod_{k \in [1.n]} \delta(i_k) = \prod_{k \in [1.n]} (i_k \otimes 1 + 1 \otimes i_k),$$

where this non-commuting product is taken in the order $k = 1, \ldots, n$. The last product can be expanded as a sum

$$\sum_{P \subseteq [1.n]} z(P),$$

where

$$z(P) = z_1 \ldots z_n$$

with

$$z_k = i_k \otimes 1 \quad \text{if } k \in P$$

and

$$z_k = 1 \otimes i_k \quad \text{if } k \in P^c = [1.n] \setminus P.$$

Now expanding $z(P)$ using the tensor multiplication rule gives us

$$z(P) = \pi^{\varepsilon(\sigma_P)} q^{-e(\sigma_P)} \mathbf{i}_P \otimes \mathbf{i}_{P^c},$$

where σ_P is the minimal coset representative in $S_n / S_{n-m} \times S_m$ satisfying

$$\sigma_P([n-m+1.n]) = P$$

and $\varepsilon(\sigma_P)$ and $e(\sigma_P)$ are defined in (3.5). Hence, for a word $\mathbf{i} \in \mathsf{W}$ of length n, we have

$$\delta(\mathbf{i}) = \sum_{P \subseteq [1.n]} \pi^{\varepsilon(\sigma_P)} q^{-e(\sigma_P)} \mathbf{i}_P \otimes \mathbf{i}_{P^c}. \tag{3.6}$$

Let F^* be the graded dual of F. Then for any word $\mathbf{i}$ in F, we set $f_{\mathbf{i}}$ to be the dual basis element:

$$f_{\mathbf{i}}(\mathbf{j}) = \delta_{\mathbf{ij}}, \qquad \text{for all } \mathbf{i}, \mathbf{j} \in \mathsf{W}.$$

We endow F^* with an associative algebra structure with multiplication defined by

$$(fg)(x) = (g \otimes f)(\delta(x)), \quad \text{for } f, g \in \mathsf{F}^*, x \in \mathsf{F}.$$

Lemma 3.3. *The map*

$$\phi \colon \mathsf{F}^* \longrightarrow (\mathsf{F}, \diamond), \quad f_{\mathbf{i}} \longmapsto \mathbf{i}$$

is an isomorphism of algebras.

Proof. It is clear that the given map is a vector space isomorphism; it remains to show the products match. Let $\mathbf{i} = (i_1, \dots, i_n)$ and $\mathbf{j} = (j_1, \dots, j_m)$, and suppose that $\mathbf{k}$ has weight $|\mathbf{i}| + |\mathbf{j}|$. Then by (3.6) we have

$$\delta(\mathbf{k}) = \sum_{P \subseteq [1.n+m]} \pi^{\varepsilon(\sigma_P)} q^{-e(\sigma_P)} \mathbf{k}_P \otimes \mathbf{k}_{P^c}.$$

Then we see that

$$\lambda_{\mathbf{i},\mathbf{j}}^{\mathbf{k}} := (f_{\mathbf{j}} \otimes f_{\mathbf{i}})(\delta(\mathbf{k})) = \sum \pi^{\varepsilon(\sigma_P)} q^{-e(\sigma_P)},$$

where the sum is over $P \subset [1.n+m]$ such that

$$\mathbf{k}_P = \mathbf{j} \quad \text{and} \quad \mathbf{k}_{P^c} = \mathbf{i}.$$

Therefore,

$$f_{\mathbf{i}} f_{\mathbf{j}} = \sum \lambda_{\mathbf{i},\mathbf{j}}^{\mathbf{k}} f_{\mathbf{k}}. \tag{3.7}$$

On the other hand, by (3.4) that

$$\mathbf{i} \diamond \mathbf{j} = \sum_{\sigma} \pi^{\varepsilon(\sigma)} q^{-e(\sigma)} (l_{\sigma(1)}, \ldots, l_{\sigma(m+n)}),$$

where

$$\mathbf{i} \cdot \mathbf{j} = (l_1, \ldots, l_{m+n}),$$

$\sigma \in S_{n+m}/S_n \times S_m$ is a minimal coset representative, and

$$P = \{\sigma(n+1), \ldots, \sigma(n+m)\}.$$

Let $\mathbf{k} \in \mathsf{W}_{|\mathbf{i}|+|\mathbf{j}|}$. Then $\mathbf{k}$ appears as a summand of $\mathbf{i} \diamond \mathbf{j}$ if and only if $\mathbf{k} = (l_{\sigma(1)}, \ldots, l_{\sigma(m+n)})$ for some σ such that $\mathbf{k}_{\sigma([n+1.n+m]} = \mathbf{j}$ and $\mathbf{k}_{\sigma([1.n])} = \mathbf{i}$. In particular, σ satisfies

$$\sigma = \sigma_P \quad \text{for } P = \sigma([n+1.n+m]).$$

Therefore,

$$\mathbf{i} \diamond \mathbf{j} = \sum_{\mathbf{k}} \sum_{\substack{P \subset [1.n+m] \\ \mathbf{k}_P = \mathbf{j},\, \mathbf{k}_{P^c} = \mathbf{i}}} \pi^{\varepsilon(\sigma_P)} q^{-e(\sigma_P)} \mathbf{k} = \sum_{\mathbf{k}} \lambda_{\mathbf{i},\mathbf{j}}^{\mathbf{k}} \mathbf{k}.$$

Comparing this to (3.7) shows that ϕ is an algebra isomorphism. □

Corollary 3.4. *There exists an algebra embedding*

$$\Psi \colon U_q \longrightarrow (\mathsf{F}, \diamond_q)$$

such that

$$\Psi(e_i) = i.$$

Proof. The epimorphism

$$\psi \colon \mathsf{F} \longrightarrow U_q$$

induces an injective homomorphism of graded duals

$$\psi^* \colon U_q^* \longrightarrow \mathsf{F}^*.$$

But since $(\cdot, \cdot)$ on U_q is nondegenerate,

$$U_q^* \cong U_q;$$

on the other hand, we just proved that $\mathsf{F}^* \cong (\mathsf{F}, \diamond)$, and so the composition

$$\Psi \colon U_q \xrightarrow{\cong} U_q^* \xrightarrow{\psi^*} \mathsf{F}^* \xrightarrow{\cong} (\mathsf{F}, \diamond)$$

is the desired map. □

Define

$$\mathsf{U} = \Psi(U_q)$$

to be the subalgebra of $(\mathsf{F}, \diamond)$ generated by I.

3.2. The homomorphism Ψ, II. In the case where the diagram for U_q in Table 1 is of type A-D or the distinguished diagram in types F and G, we give an alternate description of the homomorphism Ψ above. This new description of Ψ and then U is suitable for computations later on.

For $x, y \in \mathsf{F}$, introduce the notation

$$x \diamond_{q,t} y = x \diamond_q y - x \diamond_t y. \tag{3.8}$$

Then Proposition 3.1 can be rephrased as

$$x \diamond y - \pi^{p(x)p(y)} q^{(|x|,|y|)} y \diamond x = x \diamond_{q,q^{-1}} y, \tag{3.9}$$

for $x, y \in \mathsf{F}$ homogeneous. We denote

$$i^{\diamond r} = \underbrace{i \diamond \cdots \diamond i}_{r \text{ times}}$$

below, and recall s_{ij} from (2.2).

Lemma 3.5. *The following identities hold in* F *whenever the indicated Dynkin subdiagram associated to* U_q *appears*:

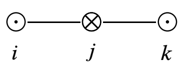
(Iso) *for* $i, j \in \mathrm{I}_{\bar{1}}$ *with* $a_{ij} = 0$,

$$i \diamond j + j \diamond i = 0;$$

(N-Iso) *if* $i \neq j$ *and* $i \in \mathrm{I}_{\bar{0}} \cup \mathrm{I}_{\text{n-iso}}$,

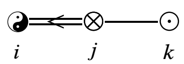
$$\sum_{r+s=1+|a_{ij}|} (-1)^r \pi_i^{p(i,j;r)} \begin{bmatrix} 1+|a_{ij}| \\ r \end{bmatrix}_i i^{\diamond r} \diamond j \diamond i^{\diamond s} = 0;$$

(A/B) *for*



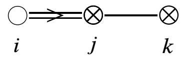
$(s_{ij} \neq s_{jk})$

or



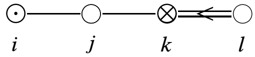
$$j \diamond_{q,q^{-1}} (k \diamond_{q,q^{-1}} (j \diamond_{q,q^{-1}} i)) = 0;$$

(CD1) *for*



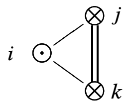
$$j \diamond_{q,q^{-1}} ((j \diamond_{q,q^{-1}} k) \diamond_{q,q^{-1}} (i \diamond_{q,q^{-1}} (j \diamond_{q,q^{-1}} k))) = 0;$$

(CD2) *for*



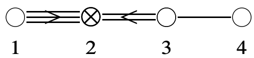
$$k \diamond_{q,q^{-1}} (j \diamond_{q,q^{-1}} (k \diamond_{q,q^{-1}} (l \diamond_{q,q^{-1}} (k \diamond_{q,q^{-1}} (j \diamond_{q,q^{-1}} i))))) = 0;$$

(D) *for*



$$k \diamond_{q,q^{-1}} (j \diamond_{q,q^{-1}} i) = j \diamond_{q,q^{-1}} (k \diamond_{q,q^{-1}} i);$$

(F1) *for*

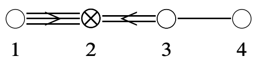


$$\mathcal{E} \diamond_{q,q^{-1}} (\mathcal{E} \diamond_{q,q^{-1}} (4 \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} 2))) = 0,$$

where

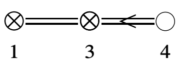
$$\begin{aligned} \mathcal{E} &= (1 \diamond_{q,q^{-1}} 2) \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} 2) \\ &= (q^5 + q^2 - q^{-2} - q^{-5})(3122 + 1322) + (q^2 - q^{-2})(1232); \end{aligned}$$

(F2) *for*



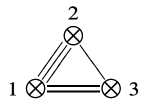
$$\begin{aligned} &(1 \diamond_{q,q^{-1}} 2) \diamond_{q,q^{-1}} ((3 \diamond_{q,q^{-1}} 2) \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} 4)) \\ &\quad = (3 \diamond_{q,q^{-1}} 2) \diamond_{q,q^{-1}} ((1 \diamond_{q,q^{-1}} 2) \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} 4)); \end{aligned}$$

(F3) *for*



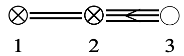
$$3 \diamond_{q,q^{-1}} (1 \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} 4)) = 0;$$

(F4) *for*



$$[3](i \diamond_{q,q^{-1}} (j \diamond_{q,q^{-1}} k)) + [2](j \diamond_{q,q^{-1}} (i \diamond_{q,q^{-1}} k)) = 0;$$

(G1) *for*

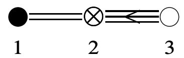


$$\mathcal{E} \diamond_{q,q^{-1}} (\mathcal{E} \diamond_{q,q^{-1}} (\mathcal{E} \diamond_{q,q^{-1}} (2 \diamond_{q,q^{-1}} 1))) = 0,$$

where

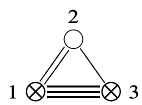
$$\mathcal{E} = (2 \diamond_{q,q^{-1}} 3) = -(q^3 - q^{-3})(23);$$

(G2) *for*



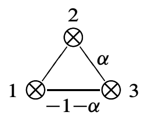
$$2 \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} (2 \diamond_{q,q^{-1}} 1))) \\ = 3 \diamond_{q,q^{-1}} (2 \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} (2 \diamond_{q,q^{-1}} 1)));$$

(G3) *for*



$$1 \diamond_{q,q^{-1}} (2 \diamond_{q,q^{-1}} 3) - [2](2 \diamond_{q,q^{-1}} (1 \diamond_{q,q^{-1}} 3)) = 0;$$

(Dα) *for*



$$[\alpha + 1](1 \diamond_{q,q^{-1}} (3 \diamond_{q,q^{-1}} 2)) + [\alpha](3 \diamond_{q,q^{-1}} (1 \diamond_{q,q^{-1}} 2)) = 0.$$

Proof. This follows from Corollary 3.4 and the corresponding relations for U_q given in Proposition 2.7. These can also be deduced directly by tedious (but straightforward) computer calculation, which we omit. □

Lemma 3.6. *For each $i \in \mathrm{I}$, define the $\mathbb{Q}(q)$-linear operator*

$$\varepsilon_i' : \mathsf{F} \longrightarrow \mathsf{F}$$

by

$$\varepsilon_i'(i_1, \ldots, i_d) = \delta_{i,i_d}(i_1, \ldots, i_{d-1})$$

and

$$\varepsilon_i'(\emptyset) = 0.$$

Then, the endomorphisms ε_i' satisfy

$$\varepsilon_i'(j) = \delta_{ij}$$

and

$$\varepsilon_i'(x \diamond y) = \varepsilon_i'(x) \diamond y + \pi^{p(x)p(y)} q^{-(\alpha_i, |x|)} x \diamond \varepsilon_i'(y).$$

Proof. This is immediate from the definition and (3.3). □

Given $\mathbf{i} = (i_1, \ldots, i_d) \in \mathrm{I}^d$, define

$$e_{\mathbf{i}}' = e_{i_1}' e_{i_2}' \cdots e_{i_d}' \quad \text{and} \quad \varepsilon_{\mathbf{i}}' = \varepsilon_{i_1}' \varepsilon_{i_2}' \cdots \varepsilon_{i_d}'. \tag{3.10}$$

Define a $\mathbb{Q}(q)$-linear map

$$\Psi : U_q \longrightarrow \mathsf{F}$$

by letting

$$\Psi(u) = \sum_{\mathbf{i} \in \mathsf{W}_\nu} e_{\mathbf{i}}'(u)\mathbf{i}, \quad \text{for } u \in U_{q,\nu}. \tag{3.11}$$

(Here we have abused the same notation Ψ as before, as it follows immediately by Proposition 3.7 below that they coincide.) Since

$$e_{\mathbf{i}}'(u) \in U_{q,0} = \mathbb{Q}(q),$$

this map is well defined. By Proposition 2.5, Ψ is injective and

$$\Psi(e_i) = i \quad \text{for } i \in \mathrm{I}.$$

Proposition 3.7. *When the diagram for U_q is of type A-D or the distinguished diagram in types F and G, the map*

$$\Psi\colon U_q \longrightarrow (\mathsf{F}, \diamond)$$

given by (3.11) *is an injective algebra homomorphism* (*and hence coincides with the* Ψ *given in Corollary* 3.4).

Proof. We have just seen the injectivity of Ψ above. In the cases we are considering, we have by Lemma 3.5 and Theorem 2.8 that there exists an algebra homomorphism

$$\Upsilon\colon U_q \longrightarrow (\mathsf{F}, \diamond)$$

such that

$$\Upsilon(e_i) = i \quad \text{for all } i \in \mathrm{I}.$$

Using Lemma 3.6, this map satisfies

$$\Upsilon \circ e_i'(u) = \varepsilon_i' \circ \Upsilon(u).$$

Let $u \in U_{q,\nu}$, and $\mathbf{i} \in \mathsf{W}_\nu$. Set $\gamma_{\mathbf{i}}(u)$ to be the coefficient of $\mathbf{i}$ in $\Upsilon(u)$. Then,

$$\gamma_{\mathbf{i}}(u) = \varepsilon_{\mathbf{i}}' \circ \Upsilon(u) = \Upsilon \circ e_{\mathbf{i}}'(u) = e_{\mathbf{i}}'(u)\Upsilon(1) = e_{\mathbf{i}}'(u),$$

where

$$\varepsilon_{\mathbf{i}}' = \varepsilon_{i_1}' \cdots \varepsilon_{i_d}'.$$

Hence

$$\Psi(u) = \Upsilon(u)$$

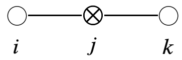
and so Ψ is an algebra homomorphism.

The Ψ here and the Ψ given in Corollary 3.4 coincide since both are algebra homomorphisms satisfying $\Psi(e_i) = i$ for $i \in \mathrm{I}$. □

Let Γ be the Dynkin diagram associated to U and let $\langle\Gamma\rangle$ be the set of subdiagrams inducing relations associated to (AB)–(D) in Lemma 3.5. Then using (3.9), we may rewrite the relation corresponding to $\Gamma' \in \langle\Gamma\rangle$ in the form

$$\sum_{\mathbf{i}=(i_1,\ldots,i_d)\in\mathsf{W}} \vartheta_{\Gamma'}(\mathbf{i})(i_1 \diamond i_2 \diamond \cdots \diamond i_d) = 0, \quad \text{for } \vartheta_{\Gamma'}(\mathbf{i}) \in \mathbb{Q}(q). \tag{3.12}$$

Example 3.8. Let U be associated to the diagram



$(s_{ij} = -1 \neq s_{jk} = 1)$.

The only subdiagram causing a relation of the form (AB)–(D) is the whole diagram (which corresponds to (AB)) so

$$\langle \Gamma \rangle = \{i, j, k\}$$

(where we identify the subdiagram with its set of labels). We have

$$\vartheta_{\{i,j,k\}}(\mathbf{i}) = \begin{cases} 1 & \text{if } \mathbf{i} \in \{jkji, jijk, kjij, ijkj\}, \\ -q & \text{if } \mathbf{i} \in \{jjik, jikj\}, \\ -q^{-1} & \text{if } \mathbf{i} \in \{kijj; jkij\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.9. *Let* U *be associated to a diagram of type* A-D*, or to the distinguished diagram of type* F *or* G*. The element*

$$x = \sum_{\mathbf{k} \in \mathsf{W}} \gamma_{\mathbf{k}}(x)\mathbf{k} \in \mathsf{F}$$

belongs to U *if and only if the following statements hold for all* $\mathbf{h}, \mathbf{h}' \in \mathsf{W}$.

(1) *For all* $i, j \in \mathrm{I}_{\mathsf{iso}}$ *with* $a_{ij} = 0$*,*

$$\gamma_{\mathbf{h}\cdot ij\cdot\mathbf{h}'}(x) + \gamma_{\mathbf{h}\cdot ji\cdot\mathbf{h}'}(x) = 0;$$

(2) *For all* $i \in \mathrm{I}_{\bar{0}} \cup \mathrm{I}_{\mathsf{n\text{-}iso}}$ *and* $j \in \mathrm{I}$ *with* $i \neq j$*,*

$$\sum_{r+s=1+|a_{ij}|} (-1)^r \pi_i^{p(i,j;k)} \begin{bmatrix} 1+|a_{ij}| \\ r \end{bmatrix}_i \gamma_{\mathbf{h}\cdot i^r\cdot j\cdot i^s\cdot\mathbf{h}'}(x) = 0;$$

(3) *For all* $\Gamma' \in \langle \Gamma \rangle$*, and with* $\vartheta_{\Gamma'}$ *defined as in* (3.12)*,*

$$\sum_{\mathbf{i} \in \mathsf{W}} \vartheta_{\Gamma'}(\mathbf{i})\gamma_{\mathbf{h}\cdot\mathbf{i}\cdot\mathbf{h}'}(x) = 0.$$

Proof. Let V be the subspace of F spanned by those elements that satisfy the statements (1)–(3). Let

$$x = \Psi(u) = \sum_{\mathbf{k}\in\mathsf{W},|\mathbf{k}|=\nu} \gamma(\mathbf{k})\mathbf{k} \in \mathsf{U}_\nu$$

be the image of some $u \in U_q$. Then, for $\mathbf{k} = (k_1, \ldots, k_d)$,

$$\gamma(\mathbf{k}) = e'_{\mathbf{k}}(u) = (e_{k_1} \cdots e_{k_d}, u)$$

by definition. Then by Corollary 2.6, $x \in \mathsf{V}$.

Conversely, note that by Lemma 3.5 $x \in F$ satisfies (1)-(3) exactly when x is orthogonal to a subspace of F^* isomorphic to the kernel of the algebra surjection

$$F \longrightarrow (F, \diamond).$$

Therefore, we see that

$$\mathsf{V}_\nu = \mathsf{F}_\nu \cap \mathsf{V}$$

has the same dimension as $U_{q,\nu}$. As Ψ is injective,

$$\dim \mathsf{U}_\nu = \dim \mathsf{V}_\nu,$$

and therefore

$$\mathsf{U} = \mathsf{V}. \qquad \square$$

3.3. Automorphisms of U. For

$$\nu = \sum_{i\in\mathrm{I}} c_i\alpha_i \in Q^+,$$

we set

$$N(\nu) = \frac{1}{2}\Big((\nu,\nu) - \sum_{i\in\mathrm{I}} c_i(\alpha_i,\alpha_i)\Big) \tag{3.13a}$$

and

$$P(\nu) = \frac{1}{2}\Big(p(\nu)^2 - \sum_{i\in\mathrm{I}} c_i\, p(\alpha_i)\Big), \tag{3.13b}$$

where here we interpret $p(\alpha_i) \in \{0,1\}$ and $p(\nu) = \sum_{i\in\mathrm{I}} c_i\, p(\alpha_i)$ as integers. Below we realize certain automorphisms of U, whose counterparts for U_q were given in Proposition 2.2, as restrictions of simple linear maps on F (compare [23, Proposition 6]).

Proposition 3.10. (1) *Let*

$$\tau\colon \mathsf{F} \longrightarrow \mathsf{F}$$

be the $\mathbb{Q}(q)$*-linear map defined by*

$$\tau(i_1,\dots,i_d) = (i_d,\dots,i_1).$$

Then,

$$\tau(x \diamond y) = \tau(y) \diamond \tau(x) \quad \textit{for all } x, y \in \mathsf{F}.$$

In particular,

$$\tau\Psi(u) = \Psi\tau(u) \quad \textit{for all } u \in U_q,$$

see (2.3).

(2) *Let*

$$x \longmapsto \bar{x}$$

be the $\mathbb{Q}$*-linear map* $\mathsf{F} \to \mathsf{F}$ *such that*

$$\bar{q} = \begin{cases} \pi q^{-1} & \textit{if } U_q \textit{ is of type } B(0, n+1), \\ q^{-1} & \textit{otherwise,} \end{cases}$$

and

$$\overline{(i_1,\dots,i_d)} = \pi^{\sum_{s<t} p(i_s)p(i_t)} q^{-\sum_{s<t}(\alpha_{i_s},\alpha_{i_t})}(i_d,\dots,i_1).$$

Then,

$$\overline{x \diamond y} = \bar{x} \diamond \bar{y}$$

and

$$\overline{\Psi(u)} = \Psi(\bar{u}) \quad \textit{for all } u \in U_q.$$

(3) *Let*

$$\sigma\colon \mathsf{F} \longrightarrow \mathsf{F}$$

be the $\mathbb{Q}$*-linear map defined by*

$$\sigma(x) = \overline{\tau(x)}.$$

Then,

$$\sigma\Psi(u) = \Psi\sigma(u) \quad \textit{for all } u \in U_q$$

and for

$$\nu = \sum_{i \in \mathrm{I}} c_i \alpha_i \in Q^+$$

and $\mathbf{i} \in \mathsf{W}_\nu$,

$$\sigma(\mathbf{i}) = \pi^{P(\nu)} q^{-N(\nu)} \mathbf{i}.$$

Proof. First note that

$$P(\alpha_{i_1} + \ldots + \alpha_{i_n}) = \sum_{s<t} p(i_s)p(i_t)$$

and

$$N(\alpha_{i_1} + \ldots + \alpha_{i_n}) = \sum_{s<t} (\alpha_{i_s}, \alpha_{i_t}),$$

so (3) follows from (1) and (2). We need only check (1) and (2) when $x, y \in \mathsf{W}$. Note that (1) is clear from (3.4). To prove (2), proceed by induction. Suppose (2) holds provided $\ell(x) + \ell(y) \le n$ (the case $n = 1$ being trivial). Applying τ to the expression for $(\tau(y)j) \diamond (\tau(x)i)$ given by (3.3), we have

$$(ix) \diamond (jy) = \pi^{p(i)(p(y)+p(j))} q^{-(\alpha_i, |y|+\alpha_j)} i(x \diamond (jy)) + j((ix) \diamond y).$$

Therefore, assuming

$$\ell(xi) + \ell(yj) = n + 1,$$

we have

$$\begin{aligned}
&\overline{(ix) \diamond (jy)} \\
&\quad = \overline{\pi^{p(i)(p(y)+p(j))} q^{-(\alpha_i, |y|+\alpha_j)} i(x \diamond jy) + j(ix \diamond y)} \\
&\quad = \pi^{p(i)p(x)} q^{-(\alpha_i, |x|)} \overline{(x \diamond jy)} i \\
&\qquad + \pi^{p(j)(p(x)+\alpha_i+p(y))} q^{-(\alpha_j, \alpha_i+|x|+|y|)} \overline{(ix \diamond y)} j \\
&\quad = \pi^{p(i)p(x)+p(j)p(y)} q^{-(\alpha_i,|x|)-(\alpha_j,|y|)} (\bar{x} \diamond \bar{y} j) i \\
&\qquad + \pi^{p(i)p(x)+p(j)p(y)+p(j)(p(x)+p(i))} q^{-(\alpha_i,|x|)-(\alpha_j,\alpha_i+|x|+|y|)} (\bar{x} i \diamond \bar{y}) j \\
&\quad = \pi^{p(i)p(x)+p(j)p(y)} q^{-(\alpha_i,|x|)-(\alpha_j,|y|)} (\bar{x} i \diamond \bar{y} j) \\
&\quad = (\overline{ix} \diamond \overline{jy}).
\end{aligned}$$

This proves (2). ∎

3.4. The bialgebra structure of U. We now transport the bilinear form from U_q to U via Ψ.

Proposition 3.11. *Let*

$$\Delta \colon \mathsf{F} \longrightarrow \mathsf{F} \otimes \mathsf{F}$$

be the map

$$\Delta(i_1, \ldots, i_d) = \sum_{0 \le k \le d} (i_{k+1}, \ldots, i_d) \otimes (i_1, \ldots, i_k).$$

Then,

$$\Delta(x \diamond y) = \Delta(x) \diamond \Delta(y),$$

where we define the shuffle product on $\mathsf{F} \otimes \mathsf{F}$ *by*

$$(w \otimes x) \diamond (y \otimes z) = \pi^{p(x)p(y)} q^{-(|x|,|y|)} (w \diamond y) \otimes (x \diamond z).$$

In particular,

$$\Delta\Psi = (\Psi \otimes \Psi)\Delta.$$

Proof. For $x \in \mathsf{W}$, we write

$$\Delta(x) = \sum x_2 \otimes x_1.$$

Then, for any $i \in \mathrm{I}$,

$$\Delta(xi) = \Delta(x) \cdot (i \otimes 1) + 1 \otimes xi = \sum x_2 i \otimes x_1 + 1 \otimes xi,$$

where we have used the associative multiplication

$$(w \otimes x) \cdot (y \otimes z) = wy \otimes xz.$$

Let $x, y \in \mathsf{W}$ and $i, j \in \mathrm{I}$. Assume the proposition is proved provided $\ell(x) + \ell(y) \le n$ (the case $n = 1$ being trivial). Suppose

$$\ell(xi) + \ell(yj) = n + 1.$$

Write

$$\Delta(x) = \sum x_2 \otimes x_1$$

and

$$\Delta(y) = \sum y_2 \otimes y_1.$$

We compute

$$\begin{aligned}\Delta(xi \diamond yj) &= \Delta((x \diamond yj)i + \pi^{(p(x)+p(i))p(j)} q^{-(|x|+\alpha_i,\alpha_j)}(xi \diamond y)j) \\ &= \Delta(x \diamond yj) \cdot (i \otimes 1) \\ &\qquad + 1 \otimes (x \diamond yj)i \\ &\qquad + \pi^{(p(x)+p(i))p(j)} q^{-(|x|+\alpha_i,\alpha_j)}(\Delta(xi \diamond y) \cdot (j \otimes 1) \\ &\qquad + 1 \otimes (xi \diamond y)j).\end{aligned}$$

By induction, this equals

$$\begin{aligned}&(\Delta(x) \diamond \Delta(yj)) \cdot (i \otimes 1) \\ &\qquad + \pi^{(p(x)+p(i))p(j)} q^{-(|x|+\alpha_i,\alpha_j)}(\Delta(xi) \diamond \Delta(y)) \cdot (j \otimes 1) \\ &\qquad + 1 \otimes (xi \diamond yj) \\ &= \Big[\Big(\sum x_2 \otimes x_1\Big) \diamond \Big(\sum y_2 j \otimes y_1 + 1 \otimes yj\Big)\Big] \cdot (i \otimes 1) \\ &\qquad + \pi^{(p(x)+p(i))p(j)} q^{-(|x|+\alpha_i,\alpha_j)}\Big[\Big(\sum x_2 i \otimes x_1 + 1 \otimes xi\Big) \\ &\qquad\qquad \diamond \Big(\sum y_2 \otimes y_1\Big)\Big] \cdot (j \otimes 1) \\ &\qquad + 1 \otimes (xi \diamond yj) \\ &= \sum \pi^{p(x_1)(p(y_2)+p(j))} q^{-(|x_1|,|y_2|+\alpha_j)}(x_2 \diamond y_2 j)i \otimes (x_1 \diamond y_1) \\ &\qquad + \sum x_2 i \otimes (x_1 \diamond yj) \\ &\qquad + \pi^{(p(x)+p(i))p(j)} q^{-(|x|+\alpha_i,\alpha_j)} \\ &\qquad\qquad \sum \pi^{p(x_1)p(y_2)} q^{-(|x_1|,|y_2|)}(x_2 i \diamond y_2)j \otimes (x_1 \diamond y_1) \\ &\qquad + \pi^{(p(x)+p(i))p(j)} q^{-(|x|+\alpha_i,\alpha_j)} \\ &\qquad\qquad \sum \pi^{(p(x)+p(i))p(y_2))} q^{-(|x|+\alpha_i,|y_2|)} y_2 j \otimes (xi \diamond y_1) \\ &\qquad + 1 \otimes (xi \diamond yj) \\ &= \sum \pi^{p(x_1)(p(y_2)+p(j))} q^{-(|x_1|,|y_2|+\alpha_j)}((x_2 \diamond y_2 j)i \\ &\qquad + \pi^{(p(x_2)+p(i))p(j)} q^{-(|x_2|+\alpha_i,\alpha_j)}(x_2 i \diamond y_2)j) \otimes (x_1 \diamond y_1) \\ &\qquad + \sum x_2 i \otimes (x_1 \diamond y_j) \\ &\qquad + \sum \pi^{(p(x)+p(i))(p(y_2)+p(j))} q^{-(|x|+\alpha_i,y_2+\alpha_j)} y_2 j \otimes (xi \diamond y_1) \\ &\qquad + 1 \otimes (xi \diamond yj) \\ &= \Delta(xi) \diamond \Delta(yj).\end{aligned}$$

This completes the proof. □

Remark 3.12. The formulas in this paper differ slightly from those appearing in [19], where multiplication and comultiplication correspond to induction and restriction at the categorified level. If we regard the shuffle product in this paper as a map

$$m_\diamond : \mathsf{U} \otimes \mathsf{U} \longrightarrow \mathsf{U},$$

then the precise relationship with induction and restriction in a categorification of U will be

$$[\mathrm{Ind}] = \tau \circ m_\diamond \circ (\tau \otimes \tau)$$

and

$$[\mathrm{Res}] = (\tau \otimes \tau) \circ \Delta \circ \tau.$$

As a consequence of Proposition 3.11, we obtain the following counterpart of Proposition 3.11 via the algebra isomorphism

$$\Psi : U_q \longrightarrow \mathsf{U}.$$

Proposition 3.13. *There exists a symmetric nondegenerate bilinear form*

$$(\cdot,\cdot) : \mathsf{U} \otimes \mathsf{U} \longrightarrow \mathbb{Q}(q)$$

satisfying

(1) $(1, 1) = 1$;

(2) *for* $i, j \in I$, $(i, j) = \delta_{ij}$;

(3) $(x, y \diamond z) = (\Delta(x), y \otimes z)$, *for* $x, y, z \in \mathsf{U}$.

4. Combinatorics of words

In this section, we will develop word combinatorics for the q-shuffle superalgebra following closely [23, Section 3] (which was in turn built on [25, 22]).

4.1. Dominant words and monomial bases. We now fix a total ordering $\leq$ on I. Let $\mathsf{W} = (\mathsf{W}, \leq)$ be the ordered set with respect to the corresponding lexicographic order:

$$\mathbf{i} = (i_1, \ldots, i_d) < (j_1, \ldots, j_k) = \mathbf{j}$$

if there exists an r such that $i_r < j_r$ and $i_s = j_s$ for $s < r$, or if $d < k$ and $i_s = j_s$ for $s = 1, \ldots d$ (i.e., $\mathbf{i}$ is a proper left factor of $\mathbf{j}$).

For $x \in \mathsf{F}$, we set

$$\max(x) = \mathbf{i}$$

if $\kappa_{\mathbf{i}} \neq 0$ in the expansion $x = \sum_{\mathbf{j} \in \mathsf{W}} \kappa_{\mathbf{j}} \mathbf{j}$ (where $\kappa_{\mathbf{j}} \in \mathbb{Q}(q)$) and $\kappa_{\mathbf{j}} = 0$ unless $\mathbf{i} \geq \mathbf{j}$. A word $\mathbf{i} \in \mathsf{W}$ is called *dominant* (also called *good* in [23]) if $\mathbf{i} = \max(u)$ for some $u \in \mathsf{U}$, and let W^+ denote the subset of dominant words of W.

The following proposition proves that the set W^+ labels bases of U_q and U. The proof proceeds exactly as in [23, Proposition 12].

Proposition 4.1. (1) *There exists a unique basis of homogeneous vectors*

$$\{m_{\mathbf{j}} \mid \mathbf{j} \in \mathsf{W}^+\}$$

in U *such that*

$$\varepsilon'_{\mathbf{i}}(m_{\mathbf{j}}) = \delta_{\mathbf{ij}} \qquad \text{if } |\mathbf{i}| = |\mathbf{j}|,$$

where $\varepsilon'_{\mathbf{i}}$ *is defined in Lemma* 3.6 *and* (3.10).

(2) *The set*

$$\{e_{\mathbf{i}} = e_{i_1} \cdots e_{i_d} \mid \mathbf{i} = (i_1, \ldots, i_d) \in \mathsf{W}^+\}$$

is a basis (*called* ***monomial basis***) *of* U_q.

For $\mathbf{i} = (i_1, \ldots, i_d) \in \mathsf{W}$, define

$$\varepsilon_{\mathbf{i}} = i_1 \diamond \cdots \diamond i_d = \Psi(e_{\mathbf{i}}).$$

Define the *monomial basis* for U to be

$$\{\varepsilon_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}. \tag{4.1}$$

The next lemma generalizes [22] (cf. [23]).

Lemma 4.2. *Every factor of a dominant word is dominant.*

Proof. This follows from the fact that U is stable under the action of ε'_i and $\varepsilon''_i = \tau\varepsilon'_i\tau$, $i \in \mathrm{I}$. See [23, Lemma 13]. □

4.2. Lyndon words

4.2.1. A word $\mathbf{i} = (i_1, \dots, i_d) \in \mathsf{W}$ is called *Lyndon* if it is smaller than any of its proper right factors:

$$\mathbf{i} < (i_r, \dots, i_d), \quad \text{for } 1 < r \leq d. \tag{4.2}$$

Let L denote the set of Lyndon words in W.

Let $\mathbf{i} \in \mathsf{L}$. Call the decomposition $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ the *co-standard factorization* of $\mathbf{i}$ if $\mathbf{i}_1, \mathbf{i}_2 \neq \emptyset$, $\mathbf{i}_1 \in \mathsf{L}$, and the length of $\mathbf{i}_1$ is maximal among all such decompositions. In this case, it is known that $\mathbf{i}_2 \in \mathsf{L}$ as well, see [25, Chapter 5]. Call the decomposition $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ the *standard factorization* if $\mathbf{i}_1, \mathbf{i}_2 \neq \emptyset$, $\mathbf{i}_2 \in \mathsf{L}$, and the length of $\mathbf{i}_2$ is maximal among all such decompositions. As above, we have $\mathbf{i}_1 \in \mathsf{L}$ as well.

We will frequently use the following lemma.

Lemma 4.3 ([23, Lemma 14]). *Let $\mathbf{i} \in \mathsf{L}$, and let $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ be its co-standard factorization. Then,*

$$\mathbf{i}_2 = \mathbf{i}_1^r \mathbf{i}_1' i,$$

where $r \geq 0$, $\mathbf{i}_1'$ is a (possibly empty) proper left factor of $\mathbf{i}_1$, and $\mathbf{i}_1' i > \mathbf{i}_1$.

We also have the following converse to this lemma.

Lemma 4.4. *If $\mathbf{i} \in \mathsf{L}$ and $\mathbf{j} = \mathbf{i}^r \mathbf{i}' i$ where $r \geq 1$, $\mathbf{i}'$ is a (possibly empty) proper left factor of $\mathbf{i}$, and $\mathbf{i} < \mathbf{i}' i$, then*

$$\mathbf{j} \in \mathsf{L}.$$

Proof. It is enough to prove the statement when $r = 1$, the general case being similar. To this end, assume $\mathbf{i} = (i_1, \dots, i_d) \in \mathsf{L}$ and $\mathbf{j} = \mathbf{i}\mathbf{i}' i$ satisfies the conditions of the lemma. Then $\mathbf{j} = (i_1, \dots, i_d, i_1, \dots, i_k, i)$. If $\mathbf{j}''$ is a right factor of $\mathbf{j}$ then either

(1) $\mathbf{j}'' = (i_r, \dots, i_d, i_1, \dots, i_k, i)$, or

(2) $\mathbf{j}'' = (i_r, \dots, i_k, i)$.

In case (1), we have

$$\mathbf{i} = (i_1, \dots, i_d) < (i_r, \dots, i_d)$$

since $\mathbf{i} \in \mathsf{L}$. As $\ell(i_r, \dots, i_d) < \ell(\mathbf{i})$ we my conclude that $\mathbf{j} < \mathbf{j}''$. For case (2), we have

$$\mathbf{i} < (i_r, \dots, i_d) < (i_r, \dots, i_k, i),$$

so $\mathbf{j} < \mathbf{j}''$ as well. This completes the proof. ☐

Let L^+ be the set of dominant Lyndon words in W. Note that

$$\mathsf{L}^+ = \mathsf{L} \cap \mathsf{W}^+ \subset \mathsf{W}^+ \subset \mathsf{W}.$$

It is well known [25] that every word $\mathbf{i} \in \mathsf{W}$ has a *canonical factorization* as a product of non-increasing Lyndon words:

$$\mathbf{i} = \mathbf{i}_1 \cdots \mathbf{i}_d, \quad \mathbf{i}_1, \ldots \mathbf{i}_d \in \mathsf{L},\ \mathbf{i}_1 \geq \cdots \geq \mathbf{i}_d. \tag{4.3}$$

Lemma 4.5. *Let* $\mathbf{i} \in \mathsf{L}$ *and* $\mathbf{j} \in \mathsf{W}$. *Assume that* $\mathbf{i} \geq \mathbf{j}$, *and further assume* $\mathbf{i} \neq \mathbf{j}$ *if* $|\mathbf{i}| \in Q^+$ *is isotropic odd. Then*

$$\max(\mathbf{i} \diamond \mathbf{j}) = \mathbf{ij}.$$

Proof. We will prove a slightly stronger statement. Namely, we will prove that

$$\max(\mathbf{i} \diamond \mathbf{j}) \leq \mathbf{ij}$$

and

(1) if $\mathbf{i} > \mathbf{j}$, then the coefficient of $\mathbf{ij}$ in $\mathbf{i} \diamond \mathbf{j}$ is $\pi^{p(\mathbf{i})p(\mathbf{j})} q^{-(|\mathbf{i}|,|\mathbf{j}|)}$ and,

(2) if $\mathbf{i} = \mathbf{j}$, then the coefficient of $\mathbf{ii}$ in $\mathbf{i} \diamond \mathbf{i}$ is $1 + \pi^{p(\mathbf{i})} q^{-(|\mathbf{i}|,|\mathbf{i}|)}$.

Let $\mathbf{i} = (i_1, \ldots, i_d)$ and $\mathbf{j} = (j_1, \ldots, j_k)$. We prove this statement by a double induction on $\ell(\mathbf{i}) = d$ and $\ell(\mathbf{j}) = k$. To this end, suppose $\ell(\mathbf{i}) = 1$, i.e. $\mathbf{i} = i_1 = i \in \mathrm{I}$. If $i > \mathbf{j}$, then $i > j_1$, so clearly $\max(i \diamond \mathbf{j}) = i\mathbf{j}$ and $i\mathbf{j}$ occurs with the coefficient given in (1). If $\mathbf{j} = \mathbf{i}$, then $\mathbf{j} = j_1 = i$ and

$$i \diamond i = (1 + \pi^{p(i)} q^{-(\alpha_i,\alpha_i)})(i\,i).$$

Hence (2) follows.

Now, suppose that $\ell(\mathbf{j}) = 1$, so $\mathbf{j} = j_1 = j \in \mathrm{I}$. The case $\mathbf{i} = \mathbf{j}$ is treated above, so assume that $\mathbf{i} > j$. Then, $j < i_1$. Assume

$$\mathbf{k} = (k_1, \ldots, k_{d+1}) = (i_1, \ldots, i_{r-1}, j, i_r, \ldots, i_d)$$

is any word occurring as a nontrivial shuffle in $\mathbf{i} \diamond j$. Then, $k_r = j < i_1 \leq i_r$, so $\mathbf{k} < \mathbf{i}j$ and (1) holds.

We now proceed to the inductive step.

CASE 1: $\mathbf{i} > \mathbf{j}$. Let $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ be the co-standard factorization of $\mathbf{i}$ and recall that $\mathbf{i}_2$ is of the form $\mathbf{i}_2 = \mathbf{i}_1'^r\mathbf{i}_1' i$, see Lemma 4.3. Then, if $\mathbf{k}$ occurs as a nontrivial shuffle in $\mathbf{i} \diamond \mathbf{j}$, there exists a factorization $\mathbf{j} = \mathbf{j}_1\mathbf{j}_2$ such that $\mathbf{k}$ occurs in $(\mathbf{i}_1 \diamond \mathbf{j}_1)(\mathbf{i}_2 \diamond \mathbf{j}_2)$.

If $\mathbf{i}_1 \geq \mathbf{j}_1$, then by induction on $\ell(\mathbf{i})$, $\max(\mathbf{i}_1 \diamond \mathbf{j}_1) \leq \mathbf{i}_1\mathbf{j}_1$. It now follows that

$$\mathbf{k} \leq \mathbf{i}_1\mathbf{j}_1 \max(\mathbf{i}_2 \diamond \mathbf{j}_2).$$

Since $\mathbf{j} < \mathbf{i} < \mathbf{i}_2$, induction on $\ell(\mathbf{i})$ implies that $\max(\mathbf{i}_2 \diamond \mathbf{j}) = \mathbf{i}_2\mathbf{j}$ and any nontrivial shuffle is strictly smaller. Now, since any word occurring in $\mathbf{j}_1(\mathbf{i}_2 \diamond \mathbf{j}_2)$ is a proper shuffle in $\mathbf{i}_2 \diamond (\mathbf{j}_1\mathbf{j}_2) = \mathbf{i}_2 \diamond \mathbf{j}$, we have

$$\mathbf{k} \leq \mathbf{i}_1\mathbf{j}_1 \max(\mathbf{i}_2 \diamond \mathbf{j}_2) < \mathbf{i}_1 \max(\mathbf{i}_2 \diamond \mathbf{j}) = \mathbf{i}\mathbf{j}.$$

Assume $\mathbf{i}_1 < \mathbf{j}_1$. Since $\mathbf{i} > \mathbf{j}$, we must have $\mathbf{j}_1 = \mathbf{i}_1\mathbf{j}_1'$ with $\mathbf{j}_1'\mathbf{j}_2 < \mathbf{i}_2$. Note that any shuffle occurring in $\mathbf{i}_1 \diamond \mathbf{j}_1$ must occur in $(\mathbf{i}_{11} \diamond \mathbf{i}_1)(\mathbf{i}_{12} \diamond \mathbf{j}_1')$ for some factorization $\mathbf{i}_1 = \mathbf{i}_{11}\mathbf{i}_{12}$. By induction, $\max(\mathbf{i}_{11} \diamond \mathbf{i}_1) \leq \mathbf{i}_1\mathbf{i}_{11}$, so

$$\mathbf{k} \leq \mathbf{i}_1\mathbf{i}_{11} \max(\mathbf{i}_{12} \diamond \mathbf{j}_1') \max(\mathbf{i}_2 \diamond \mathbf{j}_2).$$

Any word occurring in $\mathbf{i}_{11}(\mathbf{i}_{12} \diamond \mathbf{j}_1')$ must also occur in $\mathbf{i}_1 \diamond \mathbf{j}_1'$, and any word occurring in $\mathbf{i}_1(\mathbf{i}_1 \diamond \mathbf{j}_1')(\mathbf{i}_2 \diamond \mathbf{j}_2)$ also occurs in $\mathbf{i}_1(\mathbf{i} \diamond (\mathbf{j}_1'\mathbf{j}_2))$.

Set $\mathbf{h} = \mathbf{j}_1'\mathbf{j}_2$. If $\mathbf{h} < \mathbf{i}$, then induction on $\ell(\mathbf{j})$ implies that $\max(\mathbf{i} \diamond \mathbf{h}) = \mathbf{i}\mathbf{h}$ and any proper shuffle is strictly smaller. Hence,

$$\mathbf{k} \leq \mathbf{i}_1\mathbf{i}_{11} \max(\mathbf{i}_{12} \diamond \mathbf{j}_1') \max(\mathbf{i}_2 \diamond \mathbf{j}_2) < \mathbf{i}_1 \max(\mathbf{i} \diamond \mathbf{h}) = \mathbf{i}_1\mathbf{i}_1'^{r+1}\mathbf{i}_1' i\mathbf{h} < \mathbf{i} < \mathbf{i}\mathbf{j}.$$

We may, therefore, assume that $\mathbf{h} \geq \mathbf{i}$.

Recall that $\mathbf{h} < \mathbf{i}_2 = \mathbf{i}_1'^r\mathbf{i}_1' i$. If $\mathbf{h} \leq \mathbf{i}_1'^r\mathbf{i}_1'$, then $\mathbf{h} < \mathbf{i}$ since $\mathbf{i}_1'^r\mathbf{i}_1'$ is a left factor of $\mathbf{i}$. This contradicts our assumption, leaving us to consider the case where $\mathbf{h} > \mathbf{i}_1'^r\mathbf{i}_1'$.

Since $\mathbf{h} < \mathbf{i}_2$, it follows that $\mathbf{h} = \mathbf{i}_1'^r\mathbf{i}_1'\mathbf{h}'$, where $\mathbf{h}' < i$. Suppose for the moment that $\mathbf{h}' = j \in I$, i.e. $\mathbf{h} = \mathbf{i}_1'^r\mathbf{i}_1' j$, $j < i$. Since $\mathbf{h} > \mathbf{i}$, $\mathbf{i}_1' j > \mathbf{i}_1$ and, therefore, $\mathbf{h} \in \mathsf{L}$ by Lemma 4.4. Since $\ell(\mathbf{h}) < \ell(\mathbf{i})$ we may apply induction to conclude that $\max(\mathbf{i}\mathbf{h}) \leq \mathbf{h}\mathbf{i}$. Hence,

$$\mathbf{k} \leq \mathbf{i}_1\mathbf{i}_1'^r\mathbf{i}_1' j\mathbf{i} < \mathbf{i} = \mathbf{i}_1'^{r+1}\mathbf{i}_1' i < \mathbf{i}\mathbf{j}.$$

More generally, when $\mathbf{h}' = j\mathbf{h}''$ is not a letter, any word in $\mathbf{i}_1(\mathbf{i} \diamond \mathbf{h})$ can be obtained by first shuffling $\mathbf{i}_1'^r\mathbf{i}_1' j$ into $\mathbf{i}$ to obtain a word $\mathbf{i}_1\mathbf{l} = \mathbf{i}_1(\mathbf{l}_1 j\mathbf{l}_2)$, and then shuffling $\mathbf{h}''$ into $\mathbf{l}_2$. Since we already have proved that the maximum of the $\mathbf{i}_1\mathbf{l}_1 j\mathbf{l}_2$ appearing this way is $\mathbf{i}_1'^{r+1}\mathbf{i}_1' j\mathbf{i}$, $\mathbf{i}_1'^{r+1}\mathbf{i}_1' j < \mathbf{i}$ and $\ell(\mathbf{i}_1'^{r+1}\mathbf{i}_1' j) = \ell(\mathbf{i})$, the same holds in general. This finishes Case 1 and proves (1).

CASE 2: $\mathbf{i} = \mathbf{j}$. This case is almost identical to Case 1 except in the last step where now $\mathbf{h} = \mathbf{i}_1^r \mathbf{i}_1' i$. From this we see that there are exactly two ways in which $\mathbf{ii}$ occurs in $\mathbf{i} \diamond \mathbf{i}$ and (2) follows. □

The next statement follows immediately from the proof above.

Corollary 4.6. *Assume that* $\mathbf{i} \in \mathsf{L}$ *and* $|\mathbf{i}| = \nu$ *is isotropic odd, then*

$$\max(\mathbf{i} \diamond \mathbf{i}) < \mathbf{ii}.$$

The next proposition now follows as in [23, Proposition 16].

Proposition 4.7. *Let* $\mathbf{i} \in \mathsf{L}^+$ *and* $\mathbf{j} \in \mathsf{W}^+$ *with* $\mathbf{i} \geq \mathbf{j}$*, and further assume* $\mathbf{i} \neq \mathbf{j}$ *if* $|\mathbf{i}| \in Q^+$ *is isotropic odd. Then,*

$$\mathbf{ij} \in \mathsf{W}^+.$$

Theorem 4.8. *The map*

$$\mathbf{i} \longmapsto |\mathbf{i}|$$

defines a bijection from L^+ *to* Φ^+*. Moreover,* $\mathbf{i} \in \mathsf{W}^+$ *if and only if its canonical factorization is of the form* $\mathbf{i} = \mathbf{i}_1 \cdots \mathbf{i}_r$*, where* $\mathbf{i}_1, \ldots, \mathbf{i}_r \in \mathsf{L}^+$ *satisfy* $\mathbf{i}_1 \geq \ldots \geq \mathbf{i}_r$ *and* $\mathbf{i}_s$ *appears only once whenever* $|\mathbf{i}_s|$ *is isotropic odd.*

Proof. We prove both statements simultaneously by induction. Let

$$\mathsf{L}_n^+ = \{\mathbf{i} \in \mathsf{L}^+ \mid \ell(\mathbf{i}) = n\},$$

$$\Phi_n^+ = \{\beta \in \Phi^+ \mid \mathrm{ht}(\beta) = n\}$$

and let $\mathsf{W}^\oplus$ be the set of words in W satisfying the conditions of the theorem. By Proposition 4.7,

$$\mathsf{W}^\oplus \subset \mathsf{W}^+.$$

Assume that for $r < n$ there is a bijection

$$\mathsf{L}_r^+ \longrightarrow \Phi_r^+,$$

and

$$\mathsf{W}_\nu^\oplus = \mathsf{W}_\nu^+ \quad \text{whenever } \mathrm{ht}(\nu) < n.$$

The base case is the bijection

$$\mathsf{L}_1^+ = \mathrm{I} \longleftrightarrow \Pi = \Phi_1^+.$$

We now proceed to the inductive step. Let $\preceq$ be an arbitrary total ordering on Φ^+. For $\nu \in Q^+$, let

$$d(\nu) = \dim U_{q,\nu},$$

and define

$$d'(\nu) = |\{(\beta_1, \ldots, \beta_d) \in (\Phi^+)^d \mid d \geq 2,\ \beta_1 \preceq \cdots \preceq \beta_d,\ \beta_1 + \ldots + \beta_d = \nu\}|.$$

Then, by the PBW theorem for U_q (cf. [33]),

$$d(\nu) = 1 + d'(\nu) \quad \text{if } \nu \in \Phi^+$$

and

$$d(\nu) = d'(\nu) \qquad \text{otherwise.}$$

Assume that $\mathbf{i} \in \mathsf{L}_n^+$, $|\mathbf{i}| = \nu \in Q^+$. By induction,

$$|\mathsf{W}_\nu^\oplus \backslash \{\mathbf{i}\}| \geq d'(\nu).$$

Since $\mathsf{W}_\nu^\oplus \subset \mathsf{W}_\nu^+$, and $|\mathsf{W}_\nu^+| = d(\nu)$,

$$d(\nu) = |\mathsf{W}_\nu^+| \geq |\mathsf{W}_\nu^\oplus| \geq 1 + d'(\nu) \geq d(\nu).$$

This forces

$$d(\nu) = 1 + d'(\nu)$$

and, therefore, $\nu \in \Phi_n^+$. Moreover, it follows that $\mathbf{i} \in \mathsf{W}_\nu^+$ is the unique Lyndon word of its degree. Hence, the map $\mathsf{L}_n^+ \to \Phi_n^+$ is injective and $\mathsf{W}_\nu^\oplus = \mathsf{W}_\nu^+$ whenever $\mathrm{ht}(\nu) = n$ and $\mathsf{L}_\nu^+ \neq \emptyset$.

We now prove this map is surjective. To this end, let $\beta \in \Phi_n^+$. By induction $|\mathsf{W}_\beta^\oplus| \geq d'(\beta)$ and $|\mathsf{W}_\beta^\oplus| > d'(\beta)$ if and only if $\mathsf{L}_\beta^+ \neq \emptyset$ (in which case there is a unique $\mathbf{i}(\beta) \in \mathsf{L}_\beta^+$). Suppose that the map is not surjective; that is, $|\mathsf{W}_\beta^\oplus| = d'(\beta)$. Then, there exists $\mathbf{j} \in \mathsf{W}_\beta^+ \backslash \mathsf{W}_\beta^\oplus$ with $\mathbf{j} = \mathbf{j}_1 \cdots \mathbf{j}_r$ with $\mathbf{i} = \mathbf{j}_s = \mathbf{j}_{s+1}$ odd isotropic for some s. If $\mathbf{j} \neq \mathbf{ii}$, then $\mathbf{ii} \in \mathsf{W}^+$ by Lemma 4.2. Since $\ell(\mathbf{ii}) < \ell(\mathbf{j})$, $\mathsf{W}_{2|\mathbf{i}|}^\oplus = \mathsf{W}_{2|\mathbf{i}|}^+$ and so $\mathbf{ii} \in W^\oplus$, contradicting the definition of $W^\oplus$. But, the only alternative is $\mathbf{j} = \mathbf{ii}$, which implies both $2|\mathbf{i}| = \beta$ and $|\mathbf{i}|$ are in Φ^+, contradicting the fact that Φ^+ is reduced. It now follows that

$$|\mathsf{W}_\nu^\oplus| = d(\nu) = |\mathsf{W}_\nu^+|$$

for all $\nu \in Q^+$, which completes the proof of both statements of the theorem. □

4.3. Bracketing and triangularity. For homogeneous $x, y \in \mathsf{F}$, define

$$[x, y]_q = xy - \pi^{p(x)p(y)} q^{(|x|,|y|)} yx. \tag{4.4}$$

When $\mathbf{i} \in \mathsf{L}^+$, we define $[\mathbf{i}]^+ \in \mathsf{F}$ inductively by

$$[\mathbf{i}]^+ = i \qquad \text{if } \mathbf{i} = i \in \mathrm{I}$$

and

$$[\mathbf{i}]^+ = [\mathbf{i}_1, \mathbf{i}_2]_q \quad \text{otherwise,}$$

where $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ is the co-standard factorization of $\mathbf{i}$.

The next two propositions are proved exactly as in [23, Propositions 19 and 20].

Proposition 4.9. *For* $\mathbf{i} \in \mathsf{L}^+$,

$$[\mathbf{i}]^+ = \mathbf{i} + x,$$

where x *is a linear combination of words* $\mathbf{j} \in \mathsf{W}^+$ *satisfying*

$$\mathbf{j} > \mathbf{i}.$$

Now, for $\mathbf{i} \in \mathsf{W}$, let $\mathbf{i} = \mathbf{i}_1 \cdots \mathbf{i}_r$, where $\mathbf{i}_1, \ldots, \mathbf{i}_r \in \mathsf{L}^+$ and $\mathbf{i}_1 \geq \ldots \geq \mathbf{i}_r$, be its canonical factorization. Define

$$[\mathbf{i}]^+ = [\mathbf{i}_1]^+ \cdots [\mathbf{i}_r]^+.$$

Proposition 4.10. *The set* $\{[\mathbf{i}]^+ \mid \mathbf{i} \in \mathsf{W}\}$ *is a basis for* F.

Now, let

$$\Xi : (\mathsf{F}, \cdot) \longrightarrow (\mathsf{F}, \diamond)$$

be the algebra homomorphism defined by

$$\Xi(i_1, \ldots, i_d) = i_1 \diamond \cdots \diamond i_d.$$

Obviously, we have $\Xi(\mathsf{F}) = \mathsf{U}$. The next lemma generalizes [23, Lemma 21] with an identical proof.

Lemma 4.11. *A word* $\mathbf{i} \in \mathsf{W}$ *is dominant if and only if it cannot be expressed modulo* $\ker \Xi$ *as a linear combination of words* $\mathbf{j} > \mathbf{i}$.

4.4. Lyndon bases. For $\mathbf{i} \in \mathsf{W}^+$ we define

$$\mathrm{R}_{\mathbf{i}} = \Xi([\mathbf{i}]^+).$$

Proposition 4.12. *Let* $\mathbf{i} \in \mathsf{L}^+$ *and* $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ *be the co-standard factorization of* $\mathbf{i}$. *Then,*

$$\mathrm{R}_{\mathbf{i}} = \mathrm{R}_{\mathbf{i}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{i}_2}.$$

Proof. Observe that $\mathbf{i}_1, \mathbf{i}_2 \in \mathsf{L}^+$ by Lemma 4.2 and §4.3. Therefore, we compute that

$$\begin{aligned}\mathrm{R}_{\mathbf{i}} &= \Xi([[\mathbf{i}_1]^+, [\mathbf{i}_2]^+]_q) \\ &= \Xi([\mathbf{i}_1]^+) \diamond \Xi([\mathbf{i}_2]^+) - \pi^{p(\mathbf{i}_1)p(\mathbf{i}_2)} q^{-(|\mathbf{i}_1|,|\mathbf{i}_2|)} \Xi([\mathbf{i}_2]^+) \diamond \Xi([\mathbf{i}_1]^+) \\ &= \mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_2} - \pi^{p(\mathbf{i}_1)p(\mathbf{i}_2)} q^{-(|\mathbf{i}_1|,|\mathbf{i}_2|)} \mathrm{R}_{\mathbf{i}_2} \diamond \mathrm{R}_{\mathbf{i}_1}.\end{aligned}$$

The proposition now follows by applying Proposition 3.1. □

Recall the monomial basis from (4.1). The next result generalizes [23, Proposition 22].

Proposition 4.13. *For* $\mathbf{i} \in \mathsf{W}^+$, *we have*

$$\mathrm{R}_{\mathbf{i}} = \varepsilon_{\mathbf{i}} + \sum_{\mathbf{j} \in \mathsf{W}^+, \mathbf{j} > \mathbf{i}} \chi_{\mathbf{ij}}\, \varepsilon_{\mathbf{j}},$$

for some $\chi_{\mathbf{ij}} \in \mathbb{Q}(q)$. *In particular, the set* $\{\mathrm{R}_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}$ *is a basis for* U.

Proof. By Lemma 4.11 we have

$$[\mathbf{i}]^+ \in \mathbf{i} + \sum_{\mathbf{j} \in \mathsf{W}^+, \mathbf{j} > \mathbf{i}} \chi_{\mathbf{ij}}\, \mathbf{j} + \ker \Xi,$$

for some $\chi_{\mathbf{ij}} \in \mathbb{Z}[q, q^{-1}]$. Therefore, the first statement follows by applying Ξ. The second statement follows since the transition matrix from the monomial basis is triangular. □

Call the basis $\{\mathrm{R}_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}$ the *Lyndon basis* for U. The following theorem is an analogue of [23, Theorem 23] and is immediate from Theorem 4.8 and the definitions.

Proposition 4.14. *The Lyndon basis has the form*

$$\left\{ \mathrm{R}_{\mathbf{i}_1} \diamond \cdots \diamond \mathrm{R}_{\mathbf{i}_k} \,\middle|\, \begin{array}{l} \mathbf{i}_1, \ldots, \mathbf{i}_k \in \mathsf{L}^+, \\ \mathbf{i}_1 \geq \cdots \geq \mathbf{i}_k, \ \textit{and} \\ \mathbf{i}_{s-1} > \mathbf{i}_s > \mathbf{i}_{s+1} \ \textit{if} \ |\mathbf{i}_s| \in \Phi^+_{\bar{1}} \ \textit{is isotropic} \end{array} \right\}.$$

4.5. Computing dominant Lyndon words. Given $\mathbf{i} \in \mathsf{L}^+$, write

$$\mathbf{i} = \mathbf{i}^+(\beta)$$

if $\beta \in \Phi^+$ is the image of $\mathbf{i}$ under the bijection $\mathsf{L}^+ \to \Phi^+$ (i.e. $|\mathbf{i}| = \beta$).

Proposition 4.15. *Let $\beta_1, \beta_2 \in \Phi^+$ be such that $\beta_1 + \beta_2 = \beta \in \Phi^+$. Suppose that $\mathbf{i}^+(\beta_1) < \mathbf{i}^+(\beta_2)$. Then*

$$\mathbf{i}^+(\beta_1)\mathbf{i}^+(\beta_2) \leq \mathbf{i}^+(\beta).$$

Proof. This proof essentially proceeds as in [23, Proposition 24]. Indeed, write

$$\mathbf{i}_1 = \mathbf{i}^+(\beta_1),$$

$$\mathbf{i}_2 = \mathbf{i}^+(\beta_2)$$

and

$$\mathbf{i} = \mathbf{i}^+(\beta).$$

We have that

$$\mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_2} = \sum_{\mathbf{j} \in \mathsf{W}^+, \mathbf{j} \geq \mathbf{i}_1 \mathbf{i}_2} z_{\mathbf{j}} \, \mathrm{R}_{\mathbf{j}},$$

where $z_{\mathbf{j}} \in \mathbb{Z}[q, q^{-1}]$. It is therefore necessary to show that $z_{\mathbf{i}} \neq 0$.

For this, we appeal to [33, Theorem 10.5.8] which provides a specialization $x \mapsto \underline{x}$ from U_q to $U(\mathfrak{n})$. Write

$$s_{\mathbf{j}} = \Psi^{-1}(\mathrm{R}_{\mathbf{j}}) \quad \text{for } \mathbf{j} \in \mathsf{W}^+.$$

Then $\underline{s_{\mathbf{j}}} \in \mathfrak{n}$ being an iterated bracket of Chevalley generators. We have that $\underline{s_{\mathbf{i}}} = [\underline{s_{\mathbf{i}_1}}, \underline{s_{\mathbf{i}_2}}]$ belongs to the β-weight space of $\mathfrak{n}$, which is 1-dimensional and spanned by $\underline{s_{\mathbf{i}}}$. Therefore,

$$\underline{s_{\mathbf{i}_1}}\,\underline{s_{\mathbf{i}_2}} = \pi^{p(\mathbf{i}_1)p(\mathbf{i}_2)} \underline{s_{\mathbf{i}_2}}\,\underline{s_{\mathbf{i}_1}} + \lambda \underline{s_{\mathbf{i}}} \in U(\mathfrak{n})$$

for some nonzero $\lambda \in \mathbb{Z}$. It now follows that $z_{\mathbf{i}} \neq 0$ and hence $\mathbf{i} \geq \mathbf{i}_1 \mathbf{i}_2$. ☐

This yields an inductive method for computing dominant Lyndon words as described in [23, §4.3]. We recall it here. Let

$$C(\beta) = \{(\beta_1, \beta_2) \in \Phi^+ \times \Phi^+ \mid \beta_1 + \beta_2 = \beta \text{ and } \mathbf{i}^+(\beta_1) < \mathbf{i}^+(\beta_2)\}.$$

Then, the next proposition is a super-analogue of [23, Proposition 25].

Proposition 4.16. *For $\beta \in \Phi^+$,*

$$\mathbf{i}^+(\beta) = \max\{\mathbf{i}^+(\beta_1)\mathbf{i}^+(\beta_2) \mid (\beta_1, \beta_2) \in C(\beta)\}.$$

Moreover, if $(\beta_1, \beta_2) \in C(\beta)$ achieves the maximum, then

$$\mathbf{i}^+(\beta) = \mathbf{i}^+(\beta_1)\mathbf{i}^+(\beta_2)$$

is the co-standard factorization of $\mathbf{i}^+(\beta)$.

Corollary 4.17 ([23, Corollary 27]). *For $\beta \in \Phi^+$, $\mathbf{i}^+(\beta)$ is the smallest dominant word of its degree.*

4.6. Further properties of Lyndon bases

Lemma 4.18. *Let $\mathbf{i} = (i_1, \ldots, i_d) \in \mathsf{L}^+$. Then, i_1 is a left factor of every word appearing in the expansion of $\mathrm{R}_{\mathbf{i}}$.*

Proof. Proceed by induction on the length of $\mathbf{i}$, the case $\mathbf{i} = i_1 \in \mathrm{I}$ being trivial.

For the inductive step, let $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ be the costandard factorization of $\mathbf{i}$. By [23, Lemma 14], $\mathbf{i}_2 = \mathbf{i}_1^r\mathbf{i}_1' i$ where $r \geq 0$, $\mathbf{i}_1'$ is a (possibly empty) left factor of $\mathbf{i}_1$ and $i \in \mathrm{I}$ is such that $\mathbf{i}_1' i > \mathbf{i}_1$. By Proposition 4.12,

$$\mathrm{R}_{\mathbf{i}} = \mathrm{R}_{\mathbf{i}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{i}_2}.$$

By induction, i_1 is a left factor of every word in the expansion of $\mathrm{R}_{\mathbf{i}_1}$. If $\mathbf{i}_2 = \mathbf{i}_1^r\mathbf{i}_1' i$ with either $r > 0$ or $\mathbf{i}_1' \neq \emptyset$, then i_1 is a left factor of every word in the expansion of $\mathrm{R}_{\mathbf{i}_2}$ and therefore the same holds for $\mathrm{R}_{\mathbf{i}}$. Otherwise, $\mathbf{i}_2 = i$, and, if $\mathbf{k} = (i_1, k_2, \ldots, k_{d-1})$ is a word appearing in the expansion of $\mathrm{R}_{\mathbf{i}_1}$ then

$$\mathbf{k} \diamond i = \pi^{p(i_1)p(i)} q^{-(\alpha_{i_1}, \alpha_i)} i_1((k_2, \ldots, k_{d-1}) \diamond i) + i\mathbf{k}.$$

In particular, i_1 is a left factor of every word appearing in $\mathbf{k} \diamond_{q,q^{-1}} i$. This proves the lemma. □

Lemma 4.19. *For* $\mathbf{i} \in \mathsf{L}^+$, *we have* $\max(\mathrm{R}_{\mathbf{i}}) = \mathbf{i}$.

Proof. We proceed by induction on the length $\ell(\mathbf{i})$, the case $\mathbf{i} = i \in \mathrm{I}$ being clear. For the inductive step, let $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ be the co-standard factorization of $\mathbf{i} \in \mathsf{L}^+$. Induction applies to $\mathbf{i}_1$ and $\mathbf{i}_2$, so $\max(\mathrm{R}_{\mathbf{i}_1}) = \mathbf{i}_1$ and $\max(\mathrm{R}_{\mathbf{i}_2}) = \mathbf{i}_2$. In particular, $\max(\mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_2}) \leq \max(\mathbf{i}_1 \diamond \mathbf{i}_2)$. Since $\mathbf{i}_1 < \mathbf{i}_2$ and the words appearing as shuffles in $\mathbf{i}_1 \diamond \mathbf{i}_2$ are the same as the words appearing as shuffles in $\mathbf{i}_1 \bar{\diamond}\, \mathbf{i}_2$ and $\mathbf{i}_2 \diamond \mathbf{i}_1$, Lemma 4.5 implies that

$$\max(\mathrm{R}_{\mathbf{i}}) = \max(\mathrm{R}_{\mathbf{i}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{i}_2}) \leq \mathbf{i}_2\mathbf{i}_1.$$

Now $\mathbf{i}_2\mathbf{i}_1$ only appears in $\mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_2}$ as a summand of $\mathbf{i}_1 \diamond \mathbf{i}_2$, and using 3.4 we see that it appears with coefficient equal to 1, hence

$$\max(\mathrm{R}_{\mathbf{i}}) < \mathbf{i}_2\mathbf{i}_1.$$

We will prove that if $\mathbf{k} \in \mathsf{W}^+$ occurs as a shuffle in $\mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_2}$, and $\mathbf{i}_1\mathbf{i}_2 \leq \mathbf{k} < \mathbf{i}_2\mathbf{i}_1$, then $\mathbf{k} = \mathbf{i}_1\mathbf{i}_2$. To this end, we use Lemma 4.3, which says that $\mathbf{i}_2 = \mathbf{i}_1^r\mathbf{i}_1' i$ where $r \geq 0$, $\mathbf{i}_1'$ is a (possibly empty) left factor of $\mathbf{i}_1$ and $i \in \mathrm{I}$ is such that $\mathbf{i}_1' i > \mathbf{i}_1$.

Assume $\mathbf{k} = \mathbf{k}_1 \cdots \mathbf{k}_n$ is the canonical factorization of $\mathbf{k}$ into a nonincreasing product of dominant Lyndon words. Write $\mathbf{i}_1 = (i_1, \ldots, i_d)$ and $\mathbf{i}_2 = (i_1, \ldots, i_r)$. If $\mathbf{k}$ occurs in $\mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_1' i}$, then by Lemma 4.18, $\mathbf{k}_1 = (i_1, \ldots)$. As $\mathbf{i}_1$ is Lyndon, we have $i_1 \leq i_s$ for any $s \leq d$. In particular, the inequality $\mathbf{k}_1 \geq \mathbf{k}_t$ now implies that $\mathbf{k}_t = (i_1, \ldots)$ for all t.

Assume until the last paragraph of this proof that if U of type $F(3 \mid 1)$ in Table 1 we consider only its distinguished diagram and $3 \in \mathrm{I}$ is not minimal, or if U is of type $G(3)$ in Table 1 we consider only its distinguished diagram and $2 \in \mathrm{I}$ is not minimal. Here, $3 \in \mathrm{I}$ (resp. $2 \in \mathrm{I}$) refer to the labels appearing in Table 1 for the distinguished diagrams marked by $(\star)$.

An inspection of the root systems of basic Lie superalgebras implies that $n \leq 3$ since $|\mathbf{k}| \in \Phi^+$, and $n\alpha_{i_1}$ appears in its support. It follows that if i_1 occurs only once in $\mathbf{i}$, then $\mathbf{k} = \mathbf{k}_1$ is Lyndon. Since $|\mathbf{k}| = |\mathbf{i}|$ we must have $\mathbf{k} = \mathbf{i}$ as $\mathbf{i}$ is the unique dominant Lyndon word of its degree. The $n = 3$ case can only occur in type $G(3)$ (see [33, p.45]) and corresponds to $|\mathbf{i}|$ being a root of the Lie algebra of type G_2 where the result can be verified by inspection of [23, §5.5.4].

Let us now consider the case where i_1 appears twice in $|\mathbf{i}|$ and suppose $\mathbf{k} = \mathbf{k}_1\mathbf{k}_2$ is the canonical factorization of a word $\mathbf{k} \in \mathsf{W}^+$ appearing in $\mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_2}$. We want to show that $\mathbf{k}_2 = \emptyset$, so suppose otherwise. By the assumption in the cases of $F(3 \mid 1)$ and $G(3)$, we have $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$, where $\mathbf{i}_2 = \mathbf{i}_1' i$ and $\mathbf{i}'$ is a left factor of $\mathbf{i}_1$ (now, possibly empty or equal to $\mathbf{i}$).

Suppose first that $\mathbf{i}_1' \neq \emptyset$. Let $\mathbf{h}$ be any word occurring as a summand in $\mathrm{R}_{\mathbf{i}_1}$, let $\mathbf{l}$ be any word occurring as a summand in $\mathrm{R}_{\mathbf{i}_2}$, and assume that $\mathbf{k}$ occurs as a shuffle in $\mathbf{h} \diamond \mathbf{l}$. First observe that $\mathbf{h} = (i_1, h_2, \dots, h_d)$ and $\mathbf{l} = (i_1, l_1, \dots, l_e)$ with $i_1 < h_s$ and $i_1 < l_t$ for all s, t. Note $\mathbf{k}_1 \neq \mathbf{h}$ unless $\mathbf{h} = \mathbf{i}_1$ and, since $\mathbf{k}_2 \in \mathsf{L}^+$ is the unique dominant Lyndon word of weight $|\mathbf{i}| - |\mathbf{i}_1|$, $\mathbf{k}_2 = \mathbf{i}_2 = \mathbf{l}$. Similarly, $\mathbf{k}_1 \neq \mathbf{l}$ unless $\mathbf{l} = \mathbf{i}_2$ and $\mathbf{k}_2 = \mathbf{i}_1 = \mathbf{h}$. The case $\mathbf{k}_1 = \mathbf{l}$ contradicts the fact that $\mathbf{k} < \mathbf{i}_2\mathbf{i}_1$, and the case $\mathbf{k}_1 = \mathbf{l}$ contradicts $\mathbf{k}_1 > \mathbf{k}_2$. So in either case, we arrive at a contradiction.

Next, observe that $\mathbf{k}_1$ is not a proper left factor of $\mathbf{h}$. If it were, then $\mathbf{k}_1\mathbf{k}_2 < \mathbf{h} \leq \mathbf{i}_1 < \mathbf{i}_1\mathbf{i}_2$, since $\mathbf{k}_1\mathbf{k}_2 = (i_1, h_2, \dots, h_r, i_1, \dots)$ for some $r < d$ and $i_1 < h_{r+1}$, which is a contradiction with the choice of $\mathbf{k}$. Similarly, $\mathbf{k}_1$ is not a proper left factor of $\mathbf{l}$. If it were, then it would be less-than-or-equal-to the corresponding left factor of $\mathbf{i}_2$. As $\mathbf{i}_2 = \mathbf{i}_1' i$, any proper left factor of $\mathbf{i}_2$ is a left factor of $\mathbf{i}_1$. Hence, following the analysis of left factors of $\mathbf{h}$, we arrive at a contradiction. But then if $\mathbf{k}_1$ is not equal to a left factor of $\mathbf{h}$ or $\mathbf{l}$, it must contain both i_1's, contradicting the assumption that $\mathbf{k}_2 \neq \emptyset$.

We are, therefore, left to consider the case where $\mathbf{i}_1' = \emptyset$, so $\mathbf{i} = \mathbf{i}_1 i$. Then, $\mathbf{i}_1 = \mathbf{j}_1\mathbf{j}_2'$ where $\mathbf{i} = \mathbf{j}_1\mathbf{j}_2$ is the standard factorization of $\mathbf{i}$ and $\mathbf{j}_2 = \mathbf{j}_2' i$ (i.e. $\mathbf{j}_2$ is a Lyndon word of maximal length). We clearly have $\mathbf{j}_1$ and $\mathbf{j}_2$ of the form $\mathbf{j}_1 = (i_1, \dots)$ and $\mathbf{j}_2 = (i_1, \dots)$ and, since $\mathbf{i}$ is Lyndon, $\mathbf{j}_1 < \mathbf{j}_2$. In fact, since $\mathbf{j}_1\mathbf{j}_2' = \mathbf{i}_1$ is Lyndon,

$$\mathbf{j}_1 < \mathbf{j}_2'. \tag{4.5}$$

Claim ($\star$). $\mathrm{R}_{\mathbf{i}} = \mathrm{R}_{\mathbf{j}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{j}_2}$.

Assume the claim ($\star$) for the moment. Then, any $\mathbf{k} = \mathbf{k}_1\mathbf{k}_2 \in \mathsf{W}^+$ occurring in $\mathrm{R}_{\mathbf{i}}$ must occur as a shuffle $\mathbf{h} \diamond \mathbf{l}$ where $\mathbf{h} \leq \mathbf{j}_1$ occurs in $\mathrm{R}_{\mathbf{j}_1}$ and $\mathbf{l} \leq \mathbf{j}_2$ occurs in $\mathrm{R}_{\mathbf{j}_2}$. As before, $\mathbf{k}_1$ cannot be a left factor of $\mathbf{h}$ as this would imply $\mathbf{k} = \mathbf{k}_1\mathbf{k}_2 \leq \mathbf{j}_1\mathbf{j}_2 = \mathbf{i}$. We also cannot have $\mathbf{k}_1$ as a left factor of $\mathbf{l}$ unless $\mathbf{k}_1 \leq \mathbf{j}_1$ (in which case $\mathbf{k} < \mathbf{i}$). Otherwise, write $\mathbf{l} = \mathbf{k}_1\mathbf{l}''$. Then, $|\mathbf{k}_2| = |\mathbf{j}_1| + |\mathbf{l}''|$. While it is not necessarily true that $|\mathbf{l}''| \in \Phi^+$, there exists $\beta \in \Phi^+ \cup \{0\}$ and $\gamma \in \Phi^+$ such that $|\mathbf{j}_1| + \beta \in \Phi^+$ and $|\mathbf{j}_1| + \beta + \gamma = |\mathbf{k}_2|$ (choose $\alpha_r \in \Pi$ in the support of $|\mathbf{l}''|$ such that $|\mathbf{j}_1| + \alpha_r \in \Phi^+$ and continue this process one simple root at a time until arriving at β such $|\mathbf{l}''| - \beta \in \Phi^+$). Let $\mathbf{s} \in \mathsf{L}^+$ be the unique word of degree $|\mathbf{j}_1| + \beta$. Since i_1 is not in the support of $|\mathbf{l}''|$, it is not in the support of β. Consequently, $\mathbf{j}_1\mathbf{i}(\beta) > \mathbf{j}_1\mathbf{j}_2 = \mathbf{i}$. Therefore, by Proposition 4.16, it follows that $\mathbf{s} \geq \mathbf{j}_1 \cdot \mathbf{i}(\beta) > \mathbf{i}$. Hence,

$$\mathbf{k}_2 \geq \mathbf{s} \cdot \mathbf{i}(\gamma) > \mathbf{s} > \mathbf{i}.$$

Appealing to Proposition 4.16 again, we see that $(|\mathbf{k}_2|, |\mathbf{k}_1|) \in C(|\mathbf{i}|)$ and $\mathbf{k}_2\mathbf{k}_1 > \mathbf{i}$, contradicting the maximality of $\mathbf{i}$. But again, if $\mathbf{k}_1$ is not equal to a left factor of $\mathbf{h}$ or $\mathbf{l}$, it must contain both i_1's, contradicting the assumption that $\mathbf{k}_2 \neq \emptyset$. Then we see that $\mathbf{k}_2 = \emptyset$ and $\mathbf{k}$ is Lyndon, in which case the claim was already proven. Therefore, we see that $\max(\mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_2}) \leq \mathbf{i}$. On the other hand, $\mathrm{R}_{\mathbf{i}} = \mathrm{R}_{\mathbf{i}_1} \diamond \mathrm{R}_{\mathbf{i}_2}$ is a nonzero element in $\mathsf{U}_{|\mathbf{i}|}$, hence has a dominant word appearing with nonzero coefficient. Then by Corollary 4.17, this implies $\mathbf{i}$ appears with a nonzero coefficient and so the Lemma holds assuming $(\star)$.

Finally, we prove the claim $(\star)$ by induction on $\ell(\mathbf{j}_2)$. To begin induction, we note that $\mathbf{i} = \mathbf{i}_1 i$, where $\mathbf{i}_1 = \mathbf{j}_1\mathbf{j}_2{}'$, is the co-standard factorization and the computation below will eventually reduce to the case where the standard and co-standard factorization of $\mathbf{j}_1\mathbf{j}_2{}'$ coincide (i.e. $\mathbf{j}_2{}' = \mathbf{j}_1{}' j$ with $\mathbf{j}_1{}'$ a left factor of $\mathbf{j}_1$).

We now proceed to the inductive step. Observe that, by (3.9),

$$\mathrm{R}_{\mathbf{j}_1} \diamond i = \pi^{p(\mathbf{j}_1)p(i)} q^{-(|\mathbf{j}_1|, \alpha_i)} i \,\bar{\diamond}\, \mathrm{R}_{1_{\mathbf{i}}}, \tag{4.6}$$

since every word appearing in $\mathrm{R}_{\mathbf{j}_1}$ is homogeneous of degree $|\mathbf{j}_1|$.

Now, the co-standard factorization of $\mathbf{j}_2$ is

$$\mathbf{j}_2 = (\mathbf{j}_2{}')i,$$

so

$$\begin{aligned}
&\mathrm{R}_{\mathbf{j}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{j}_2}\\
&\quad = \mathrm{R}_{\mathbf{j}_1} \diamond_{q,q^{-1}} (\mathrm{R}_{\mathbf{j}_2{}'} \diamond_{q,q^{-1}} i)\\
&\quad = \mathrm{R}_{\mathbf{j}_1} \diamond (\mathrm{R}_{\mathbf{j}_2{}'} \diamond i) - \mathrm{R}_{\mathbf{j}_1} \diamond (\mathrm{R}_{\mathbf{j}_2{}'} \,\bar{\diamond}\, i) - \mathrm{R}_{\mathbf{j}_1} \,\bar{\diamond}\, (\mathrm{R}_{\mathbf{j}_2{}'} \diamond i) + \mathrm{R}_{\mathbf{j}_1} \,\bar{\diamond}\, (\mathrm{R}_{\mathbf{j}_2{}'} \,\bar{\diamond}\, i)\\
&\quad = \mathrm{R}_{\mathbf{j}_1} \diamond (\mathrm{R}_{\mathbf{j}_2{}'} \diamond i) - \pi^{p(\mathbf{j}_2{}')p(i)} q^{(|\mathbf{j}_2{}'|, \alpha_i)} \mathrm{R}_{\mathbf{j}_1} \diamond (i \diamond \mathrm{R}_{\mathbf{j}_2{}'})\\
&\qquad\quad - \pi^{p(\mathbf{j}_2)p(i)} q^{-(|\mathbf{j}_2{}'|, \alpha_i)} \mathrm{R}_{\mathbf{j}_1} \,\bar{\diamond}\, (i \,\bar{\diamond}\, \mathrm{R}_{\mathbf{j}_2{}'}) + \mathrm{R}_{\mathbf{j}_1} \,\bar{\diamond}\, (\mathrm{R}_{\mathbf{j}_2{}'} \,\bar{\diamond}\, i),
\end{aligned}$$

where we have used (3.9) for the last equality. On the other hand, the standard factorization of $\mathbf{i}_1$ is

$$\mathbf{i}_1 = \mathbf{j}_1\mathbf{j}_2{}'.$$

As $\ell(\mathbf{j}_2{}') < \ell(\mathbf{j}_2)$, induction applies and

$$\mathrm{R}_{\mathbf{i}_1} = \mathrm{R}_{\mathbf{j}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{j}_2'}.$$

Hence,

$$\begin{aligned}
\mathrm{R}_{\mathbf{i}} &= \mathrm{R}_{\mathbf{i}_1} \diamond_{q,q^{-1}} i \\
&= (\mathrm{R}_{\mathbf{j}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{j}_2{}'}) \diamond_{q,q^{-1}} i \\
&= (\mathrm{R}_{\mathbf{j}_1} \diamond \mathrm{R}_{\mathbf{j}_2{}'}) \diamond i - (\mathrm{R}_{\mathbf{j}_1} \diamond \mathrm{R}_{\mathbf{j}_2{}'}) \bar{\diamond}\, i - (\mathrm{R}_{\mathbf{j}_1} \bar{\diamond}\, \mathrm{R}_{\mathbf{j}_2{}'}) \diamond i + (\mathrm{R}_{\mathbf{j}_1} \bar{\diamond}\, \mathrm{R}_{\mathbf{j}_2{}'}) \bar{\diamond}\, i \\
&= (\mathrm{R}_{\mathbf{j}_1} \diamond \mathrm{R}_{\mathbf{j}_2{}'}) \diamond i - \pi^{p(\mathbf{j}_1)p(i)+p(\mathbf{j}_2{}')p(i)} q^{(|\mathbf{j}_1|+|\mathbf{j}_2{}'|,\alpha_i)} i \diamond (\mathrm{R}_{\mathbf{j}_1} \diamond \mathrm{R}_{\mathbf{j}_2{}'}) \\
&\quad - \pi^{p(\mathbf{j}_1)p(i)+p(\mathbf{j}_2{}')p(i)} q^{-(|\mathbf{j}_1|+|\mathbf{j}_2{}'|,\alpha_i)} i \,\bar{\diamond}\, (\mathrm{R}_{\mathbf{j}_1} \bar{\diamond}\, \mathrm{R}_{\mathbf{j}_2{}'}) + (\mathrm{R}_{\mathbf{j}_1} \bar{\diamond}\, \mathrm{R}_{\mathbf{j}_2{}'}) \bar{\diamond}\, i,
\end{aligned}$$

where we have used (3.9) to obtain the last equality. Finally, using Equation (4.6) and the associativity of $\diamond$ and $\bar{\diamond}$, The claim ($\star$) follows.

Finally, we consider the remaining diagrams and orderings when U of type $F(3 \mid 1)$ or $G(3)$. There are 6 orderings to consider in $F(3 \mid 1)$ and 2 orderings to consider in type $G(3)$. Inspection of the root systems shows that the argument above proves that $\max(\mathrm{R}_{\mathbf{i}}) = \mathbf{i}$ unless $|\mathbf{i}|$ is either $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$ or $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ in type $F(3 \mid 1)$, or $|\mathbf{i}|$ is $\alpha_1 + 3\alpha_2 + \alpha_3$, $\alpha_1 + 3\alpha_2 + 2\alpha_3$, or $\alpha_1 + 4\alpha_2 + 2\alpha_3$ in type $G(3)$. A direct computation of $\mathrm{R}_{\mathbf{i}}$ in these cases yields the theorem. □

5. Orthogonal PBW bases

In this section we will define a basis of PBW type for U and show it is orthogonal with respect to the bilinear form on U.

5.1. PBW bases. Let $\mathbf{i} = \mathbf{i}(\beta) \in \mathsf{L}^+$ for $\beta \in \Phi^+$. Set

$$d_\beta = \max\{|(\beta, \beta)|/2, 1\},$$

and define the quantum numbers

$$[n]_\beta = \begin{cases} [n]_i & \text{if } (\beta,\beta) = (\alpha_i, \alpha_i) \text{ and } \beta \in \Phi_{\mathsf{0}}^+ \cup \Phi_{\mathsf{iso}}^+, \\ \{n\}_i & \text{if } (\beta,\beta) = (\alpha_i, \alpha_i) \text{ and } \beta \in \Phi_{\mathsf{n\text{-}iso}}^+. \end{cases}$$

Let $\mathbf{i} = \mathbf{i}(\beta) = \mathbf{i}_1\mathbf{i}_2 = \mathbf{i}(\beta_1)\mathbf{i}(\beta_2)$ be the co-standard factorization and set

$$p_{\mathbf{i}} = \max\{p \in \mathbb{Z}_{\geq 0} \mid \beta_1 - p\beta_2 \in \Phi^+\}.$$

Define $\kappa_{\mathbf{i}}$ inductively by the formula

$$\kappa_{\mathbf{i}} = 1 \qquad \text{if } \mathbf{i} = i \in \mathrm{I}$$

and

$$\kappa_{\mathbf{i}} = [p_{\mathbf{i}} + 1]_{\beta_r} \kappa_{\mathbf{i}_1} \kappa_{\mathbf{i}_2} \quad \text{otherwise,}$$

where

$$(\beta_r, \beta_r) = \min\{(\beta_1, \beta_1), (\beta_2, \beta_2)\}$$

(note that there is no ambiguity in this definition since in all cases where $\kappa_{\mathbf{i}} \neq 1$ and $(\beta_1, \beta_1) = (\beta_2, \beta_2)$ we have $p(\beta_1) = p(\beta_2)$). Recalling the anti-automorphism σ on U from Proposition 3.10 and the Lyndon basis $\{\mathrm{R}_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}$ for U from Proposition 4.13, we define

$$\mathrm{E}_{\mathbf{i}} = \kappa_{\mathbf{i}}^{-1} \sigma(\mathrm{R}_{\mathbf{i}}), \quad \mathbf{i} \in \mathsf{L}^+. \tag{5.1}$$

We note that in the case of Lie algebras, this renormalization factor is the one computed in [3, Theorem 4.2].

More generally, if $\mathbf{i} = \mathbf{i}_1^{n_1} \cdots \mathbf{i}_d^{n_d}$ is the canonical factorization of $\mathbf{i}$ with $\mathbf{i}_1 > \cdots > \mathbf{i}_d$, set

$$E_{\mathbf{i}} = \mathrm{E}_{\mathbf{i}_d}^{(n_d)} \diamond \cdots \diamond \mathrm{E}_{\mathbf{i}_1}^{(n_1)} \tag{5.2}$$

where, for $\mathbf{j} \in \mathsf{L}^+$, we have denoted

$$\mathrm{E}_{\mathbf{j}}^{(n)} = \mathrm{E}_{\mathbf{j}}^{\diamond n} / [n]_{\mathbf{j}}!.$$

We first state the following theorem, which is a generalization of [23, Theorem 36] and follows from Lemma 4.19.

Theorem 5.1. *For all* $\mathbf{i} \in \mathsf{W}^+$*,*

$$\max(\mathrm{R}_{\mathbf{i}}) = \max(\mathrm{E}_{\mathbf{i}}) = \mathbf{i}.$$

Proof. It follows by Lemma 4.19 that $\max(\mathrm{E}_{\mathbf{i}}) = \mathbf{i}$, for $\mathbf{i} \in \mathsf{L}^+$, since $\mathrm{E}_{\mathbf{i}}$ is proportional to $\sigma(\mathrm{R}_{\mathbf{i}})$. Now the theorem follows by applying Lemma 4.5. $\square$

Corollary 5.2. *If* $\mathbf{i} \in \mathsf{L}^+_{\bar{1}}$, *then*

$$\mathrm{E}_{\mathbf{i}} \diamond \mathrm{E}_{\mathbf{i}} = 0.$$

Proof. By Theorem 5.1 and Corollary 4.6,

$$\max(\mathrm{E}_{\mathbf{i}} \diamond \mathrm{E}_{\mathbf{i}}) < \mathbf{ii}.$$

However, by [19, Lemma 5.9], $\mathbf{ii}$ is smaller than any dominant word of degree $2|\mathbf{i}|$. Hence, $\mathrm{E}_{\mathbf{i}} \diamond \mathrm{E}_{\mathbf{i}}$ must be 0. □

Proposition 5.3. *For each* $\mathbf{i} \in \mathsf{W}^+$, *there exists* $\kappa_{\mathbf{i}} \in \mathcal{A}$ *such that*

$$\overline{\kappa_{\mathbf{i}}} = \kappa_{\mathbf{i}} \quad \text{and} \quad \mathrm{E}_{\mathbf{i}} = \kappa_{\mathbf{i}}^{-1}\, \sigma(\mathrm{R}_{\mathbf{i}}).$$

Proof. This is by definition, taking

$$\kappa_{\mathbf{i}} = \prod_{s=1}^{d} \kappa_{\mathbf{i}_s} [n_s]_{\mathbf{i}_s}!. \tag{5.3}$$

See (5.1) and (5.2) above. □

It follows from Propositions 4.13 and 5.3 that $\{E_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}$ forms a basis for U, which will be called a *PBW basis*.

Proposition 5.4. *For* $\mathbf{i} \in \mathsf{W}^+$, *we have*

$$\mathrm{E}_{\mathbf{i}} = \kappa_{\mathbf{i}}^{-1} \varepsilon_{\tau(\mathbf{i})} + \sum_{\mathbf{j} > \mathbf{i}} \alpha_{\mathbf{ij}} \varepsilon_{\tau(\mathbf{j})}, \quad \textit{for } \alpha_{\mathbf{ij}} \in \mathbb{Q}(q).$$

Proof. This is immediate from Proposition 4.13 and Proposition 5.3. □

The next theorem is often referred to as the Levendorskii–Soibelman formula, see [24].

Theorem 5.5. *Suppose* $\mathbf{i}, \mathbf{j} \in \mathsf{L}^+$ *with* $\mathbf{i} < \mathbf{j}$. *Then,*

$$\mathrm{E}_{\mathbf{j}} \diamond \mathrm{E}_{\mathbf{i}} = \sum_{\substack{\mathbf{k} \in \mathsf{W}^+ \\ \mathbf{ij} \le \mathbf{k} \le \mathbf{ji}}} c^{\mathbf{k}}_{\mathbf{i},\mathbf{j}} \mathrm{E}_{\mathbf{k}}.$$

Proof. By Proposition 5.4,

$$\mathrm{E}_{\mathbf{j}} \diamond \mathrm{E}_{\mathbf{i}} = \Big(\kappa_{\mathbf{j}}^{-1}\varepsilon_{\tau(\mathbf{j})} + \sum_{\mathbf{k}>\mathbf{j}} \alpha_{\mathbf{j},\mathbf{k}}\varepsilon_{\tau(\mathbf{k})}\Big)\Big(\kappa_{\mathbf{i}}^{-1}\varepsilon_{\tau(\mathbf{i})} + \sum_{\mathbf{k}>\mathbf{i}} \alpha_{\mathbf{i},\mathbf{k}}\varepsilon_{\tau(\mathbf{k})}\Big)$$

$$= \sum_{\mathbf{k}\in\mathsf{W},\mathbf{k}>\mathbf{ij}} \beta_{\mathbf{ij}}^{\mathbf{k}}\varepsilon_{\tau(\mathbf{k})}$$

By Lemma 4.11, if $\mathbf{k} \notin \mathsf{W}^+$, then

$$\varepsilon_{\tau(\mathbf{k})} = \sum_{\mathbf{h}\in\mathsf{W}^+,\mathbf{h}>\mathbf{k}} \gamma_{\mathbf{k},\mathbf{h}}\varepsilon_{\tau(\mathbf{h})}.$$

Therefore,

$$\mathrm{E}_{\mathbf{i}} \diamond \mathrm{E}_{\mathbf{j}} = \sum_{\substack{\mathbf{k}\in\mathsf{W}^+\\ \mathbf{ij}\leq\mathbf{k}}} c_{\mathbf{i},\mathbf{j}}^{\mathbf{k}}\mathrm{E}_{\mathbf{k}}.$$

On the other hand, by Theorem 5.1, it follows that $c_{\mathbf{i},\mathbf{j}}^{\mathbf{k}} \neq 0$ only if $\mathbf{k} < \mathbf{ji}$. □

5.2. Orthogonality of PBW basis. We will prove that the PBW basis defined in the previous section is orthogonal with respect to the bilinear form on U.

Lemma 5.6. *For* $\mathbf{i} \in \mathsf{L}^+$, *we have*

$$\Delta(\mathrm{E}_{\mathbf{i}}) = \sum_{\mathbf{i}_1,\mathbf{i}_2\in\mathsf{W}^+} \vartheta_{\mathbf{i}_1\mathbf{i}_2}^{\mathbf{i}}\mathrm{E}_{\mathbf{i}_2}\otimes\mathrm{E}_{\mathbf{i}_1}, \quad \textit{for}\ \vartheta_{\mathbf{i}_1,\mathbf{i}_2}^{\mathbf{i}} \in \mathbb{Q}(q),$$

where $\vartheta_{\mathbf{i}_1,\mathbf{i}_2}^{\mathbf{i}} = 0$ *unless* $|\mathbf{i}_1| + |\mathbf{i}_2| = |\mathbf{i}|$ *and*

(1) $\mathbf{i}_1 \leq \mathbf{i}$, *and*

(2) $\mathbf{i} \leq \mathbf{i}_2$ *whenever* $\mathbf{i}_2 \neq \emptyset$.

Proof. Observe by Theorem 5.1 that

$$\mathrm{E}_{\mathbf{i}} = \sum_{\mathbf{j}\leq\mathbf{i}} \phi_{\mathbf{ij}}\mathbf{j},$$

for some $\phi_{\mathbf{ij}} \in \mathbb{Q}(q)$, so

$$\Delta(\mathrm{E}_{\mathbf{i}}) = \sum_{\substack{\mathbf{j}_1,\mathbf{j}_2;\\ \mathbf{j}_1\mathbf{j}_2=\mathbf{j}\leq\mathbf{i}}} \phi_{\mathbf{ij}}(\mathbf{j}_2\otimes\mathbf{j}_1).$$

Since $\mathbf{j}_1 \leq \mathbf{j} \leq \mathbf{i}$, part (1) follows.

We now prove (2) by induction on the length of $\mathbf{i}$, the case $\mathbf{i} = i \in \mathrm{I}$ being obvious.

To proceed to the inductive step, we need to make a few observations. First, given $\mathbf{i} \in \mathsf{L}^+$, $\mathrm{E}_{\mathbf{i}}$ is proportional to

$$\sigma(\mathrm{R}_{\mathbf{i}}) = \sigma(\mathrm{R}_{\mathbf{i}_2}) \diamond \sigma(\mathrm{R}_{\mathbf{i}_1}) - \pi^{p(\mathbf{i}_1)p(\mathbf{i}_2)} q^{-(|\mathbf{i}_1|,|\mathbf{i}_2|)} \sigma(\mathrm{R}_{\mathbf{i}_1}) \diamond \sigma(\mathrm{R}_{\mathbf{i}_1}),$$

where $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ is the costandard factorization of $\mathbf{i}$. In turn, the right hand side of the equation above is proportional to

$$\begin{aligned}\mathrm{E}_{\mathbf{i}_2} \diamond \mathrm{E}_{\mathbf{i}_1} &- \pi^{p(\mathbf{i}_1)p(\mathbf{i}_2)} q^{-(|\mathbf{i}_1|,|\mathbf{i}_2|)} \mathrm{E}_{\mathbf{i}_1} \diamond \mathrm{E}_{\mathbf{i}_2} \\ &= -\pi^{p(\mathbf{i}_1)p(\mathbf{i}_2)} q^{-(|\mathbf{i}_1|,|\mathbf{i}_2|)} (\mathrm{E}_{\mathbf{i}_1} \diamond_{q,q^{-1}} \mathrm{E}_{\mathbf{i}_2}).\end{aligned}$$

Therefore, it is sufficient to prove the lemma for $\mathrm{E}_{\mathbf{i}_1} \diamond_{q,q^{-1}} \mathrm{E}_{\mathbf{i}_2}$.

To this end, write

$$\mathbf{i}_1 = \mathbf{j} \quad \text{and} \quad \mathbf{i}_2 = \mathbf{k}$$

and note that induction applies to $\mathrm{E}_{\mathbf{j}}$ and $\mathrm{E}_{\mathbf{k}}$. Observe that if

$$\Delta(\mathrm{E}_{\mathbf{j}} \diamond \mathrm{E}_{\mathbf{k}}) = \sum_{\mathbf{h},\mathbf{l} \in \mathsf{W}^+} z_{\mathbf{h},\mathbf{l}} (\mathrm{E}_{\mathbf{h}} \otimes \mathrm{E}_{\mathbf{l}}),$$

then

$$\Delta(\mathrm{E}_{\mathbf{j}} \diamond_{q,q^{-1}} \mathrm{E}_{\mathbf{k}}) = \sum_{\mathbf{h},\mathbf{l} \in \mathsf{W}^+} (z_{\mathbf{h},\mathbf{l}} - \overline{z_{\mathbf{h},\mathbf{l}}})(\mathrm{E}_{\mathbf{h}} \otimes \mathrm{E}_{\mathbf{l}}) \tag{5.4}$$

since, replacing q with q^{-1} in Proposition 3.11 shows that Δ is an algebra homomorphism with respect to the (q^{-1}, π)-bialgebra structure on $\mathsf{U} \otimes \mathsf{U}$:

$$(w \otimes x) \mathbin{\bar{\diamond}} (y \otimes z) = \pi^{p(x)p(y)} q^{(|x|,|y|)} (w \mathbin{\bar{\diamond}} y) \otimes (x \mathbin{\bar{\diamond}} z).$$

On the other hand,

$$\Delta(\mathrm{E}_{\mathbf{j}} \diamond_{q,q^{-1}} \mathrm{E}_{\mathbf{k}}) = \Delta(\mathrm{E}_{\mathbf{k}} \diamond \mathrm{E}_{\mathbf{j}} - \pi^{p(\mathbf{j})p(\mathbf{k})} q^{-(|\mathbf{j}|,|\mathbf{k}|)} \mathrm{E}_{\mathbf{j}} \diamond \mathrm{E}_{\mathbf{k}}).$$

By Proposition 5.4, the transition matrix from the PBW basis to the basis $\{\varepsilon_{\tau(\mathbf{j})} \mid \mathbf{j} \in \mathsf{W}^+\}$ is triangular. Therefore, applying our inductive hypothesis, we have

$$\begin{aligned}\Delta(\mathrm{E}_{\mathbf{j}} \diamond \mathrm{E}_{\mathbf{k}}) &= \sum_{\substack{\mathbf{j}_1 \leq \mathbf{j} \leq \mathbf{j}_2 \\ \mathbf{k}_1 \leq \mathbf{k} \leq \mathbf{k}_2}} \vartheta^{\mathbf{j}}_{\mathbf{j}_1\mathbf{j}_2} \vartheta^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} (\mathrm{E}_{\mathbf{j}_2} \otimes \mathrm{E}_{\mathbf{j}_1}) \diamond (\mathrm{E}_{\mathbf{k}_2} \otimes \mathrm{E}_{\mathbf{k}_1}) \\ &= \sum_{\substack{\mathbf{h} \geq \mathbf{k}_2\mathbf{j}_2; \\ \mathbf{l} \geq \mathbf{k}_1\mathbf{j}_1}} \Theta_{\mathbf{h},\mathbf{l}} \mathrm{E}_{\mathbf{h}} \otimes \mathrm{E}_{\mathbf{l}}\end{aligned}$$

and similarly

$$\begin{aligned}\Delta(\mathrm{E}_{\mathbf{k}} \diamond \mathrm{E}_{\mathbf{j}}) &= \sum_{\substack{\mathbf{k}_1 \le \mathbf{k} \le \mathbf{k}_2 \\ \mathbf{j}_1 \le \mathbf{j} \le \mathbf{j}_2}} \vartheta^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} \vartheta^{\mathbf{j}}_{\mathbf{j}_1\mathbf{j}_2} (\mathrm{E}_{\mathbf{k}_2} \otimes \mathrm{E}_{\mathbf{k}_1}) \diamond (\mathrm{E}_{\mathbf{j}_2} \otimes \mathrm{E}_{\mathbf{j}_1}) \\ &= \sum_{\substack{\mathbf{h} \ge \mathbf{j}_2\mathbf{k}_2; \\ \mathbf{l} \ge \mathbf{j}_1\mathbf{k}_1}} \Theta'_{\mathbf{h},\mathbf{l}} \mathrm{E}_{\mathbf{h}} \otimes \mathrm{E}_{\mathbf{l}}.\end{aligned}$$

Comparing these equations to (5.4) we deduce that $\Theta_{\mathbf{h},\mathbf{l}} \neq 0$ if and only if $\Theta'_{\mathbf{h},\mathbf{l}} \neq 0$.

Now, assume $z_{\mathbf{hl}} - \overline{z_{\mathbf{hl}}} \neq 0$. The previous paragraph implies that

$$\mathbf{h} \ge \max\{\mathbf{j}_2\mathbf{k}_2, \mathbf{k}_2\mathbf{j}_2\}.$$

If $\mathbf{j}_2 \neq \emptyset$, then $\mathbf{j} \neq \emptyset$ and we obtain the inequality $\mathbf{h} \ge \mathbf{j}_2\mathbf{k}_2 \ge \mathbf{jk} = \mathbf{i}$ since $\mathbf{j}_2 \ge \mathbf{j}$, $\mathbf{k}_2 \ge \mathbf{k}$ and these are right factors of $\mathbf{j}$ and $\mathbf{k}$ respectively (note that if $\mathbf{j}_2$ is a proper right factor, we don't need to consider $\mathbf{k}$ and $\mathbf{k}_2$ at all). If $\mathbf{j}_2 = \emptyset$ and $\mathbf{k}_2 \neq \emptyset$, we have $\mathbf{h} \ge \mathbf{k}_2 \ge \mathbf{k} > \mathbf{jk}$ since, by Lemma 4.3, $\mathbf{k} = \mathbf{j}^r\mathbf{j}'j$ where $r \ge 0$, $\mathbf{j}'$ is a (possibly empty) left factor of $\mathbf{j}$ and $j \in \mathrm{I}$ satisfies $\mathbf{j}'j > \mathbf{j}$. If both $\mathbf{j}_2 = \mathbf{k}_2 = \emptyset$, the equality $|\mathbf{h}| = |\mathbf{j}_2| + |\mathbf{k}_2|$ forces $\mathbf{h} = \emptyset$. This proves part (2) and hence the lemma. □

Theorem 5.7. *Let* $\mathbf{i}, \mathbf{j} \in \mathsf{W}^+$. *Then,*

$$(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{j}}) = 0 \quad \textit{unless } \mathbf{i} = \mathbf{j}.$$

Moreover, if $\mathbf{i} = \mathbf{i}_1^{n_1} \cdots \mathbf{i}_d^{n_d}$, $\mathbf{i}_1 > \cdots > \mathbf{i}_d$ *is the canonical factorization of* $\mathbf{i}$ *into dominant Lyndon words, then,*

$$(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}}) = \pi^{\xi_{\mathbf{i}}} q^{-c_{\mathbf{i}}} \prod_{l=1}^{d} \frac{(\mathrm{E}_{\mathbf{i}_l}, \mathrm{E}_{\mathbf{i}_l})^{n_l}}{[n_l]_{\mathbf{i}_l}!},$$

where

$$\xi_{\mathbf{i}} = \sum_{l=1}^{d} \binom{n_l - 1}{2} p(\mathbf{i}_l) \quad \textit{and} \quad c_{\mathbf{i}} = \sum_{l=1}^{d} \binom{n_l}{2} \frac{(|\mathbf{i}_l|, |\mathbf{i}_l|)}{2}. \tag{5.5}$$

Proof. We proceed by induction on the length of $\mathbf{i}$, the case $\mathbf{i} = i \in \mathrm{I}$ being trivial. We first show that the theorem holds when $\mathbf{i} \in \mathsf{L}^+$. Indeed, suppose $\mathbf{j} \neq \mathbf{i}$ and let $\mathbf{j} = \mathbf{j}_1 \cdots \mathbf{j}_r$, where $\mathbf{j}_1 \ge \mathbf{j}_2 \cdots \ge \mathbf{j}_r$, be the canonical factorization of $\mathbf{j}$.

Then, $(E_{\mathbf{i}}, E_{\mathbf{j}})$ is proportional to

$$\begin{aligned}&\sum \vartheta^{\mathbf{i}}_{\mathbf{i}_1,\mathbf{i}_2}(E_{\mathbf{i}_2}\otimes E_{\mathbf{i}_1}, E_{\mathbf{j}_r}\otimes(E_{\mathbf{j}_{r-1}}\diamond\cdots\diamond E_{\mathbf{j}_1}))\\ &\quad=\sum \vartheta^{\mathbf{i}}_{\mathbf{i}_1,\mathbf{i}_2}(E_{\mathbf{i}_2}, E_{\mathbf{j}_r})(E_{\mathbf{i}_1},(E_{\mathbf{j}_{r-1}}\diamond\cdots\diamond E_{\mathbf{j}_1}))\end{aligned}\tag{5.6}$$

where the sum is over $\mathbf{i}_1 \le \mathbf{i} \le \mathbf{i}_2$ by Lemma 5.6. By assumption $|\mathbf{j}_r| \neq |\mathbf{i}|$, so we may take the sum to be over $\mathbf{i}_1 < \mathbf{i} < \mathbf{i}_2$. Therefore, since $\mathbf{j}_r \in \mathsf{L}^+$ has shorter length than $\mathbf{i}$, we may apply induction to conclude that the nonzero terms in the sum above satisfy $\mathbf{i}_2 = \mathbf{j}_r \in \mathsf{L}^+$ and $\mathbf{j}_1\cdots\mathbf{j}_{r-1} = \mathbf{i}_1$. But, now we have the inequalities

$$\mathbf{j}_1 \le \mathbf{j}_1\cdots\mathbf{j}_{r-1} = \mathbf{i}_1 < \mathbf{i}_2 = \mathbf{j}_r \le \mathbf{j}_1,$$

which is never satisfied. Hence, $(E_{\mathbf{i}}, E_{\mathbf{j}}) = 0$.

Now, let $\mathbf{i},\mathbf{j} \in \mathsf{W}^+_\nu$ be arbitrary and assume we have shown that $\{E_{\mathbf{k}} \mid \mathbf{k} \in \mathsf{W}^+_\mu\}$ is an orthogonal basis for U_μ whenever $\mu < \nu$ in the dominance ordering on Q^+ (the base case $\nu \in \Pi$ being trivial). Let $\mathbf{i} = \mathbf{i}_1\cdots\mathbf{i}_s$ and $\mathbf{j} = \mathbf{j}_1\cdots\mathbf{j}_r$ be the canonical factorizations of $\mathbf{i}$ and $\mathbf{j}$ into a nonincreasing product of dominant Lyndon words, and assume, without loss of generality, that $\mathbf{i}_1 \le \mathbf{j}_1$. If $\mathbf{i} \in \mathsf{L}^+$ or $\mathbf{j} \in \mathsf{L}^+$, then we are done, so assume that both $r, s > 1$. Then, $(E_{\mathbf{i}}, E_{\mathbf{j}})$ is proportional to (up to some suitable product of quantum factorials)

$$\begin{aligned}&(E_{\mathbf{i}_s}\diamond\cdots\diamond E_{\mathbf{i}_1}, E_{\mathbf{j}_r}\diamond\cdots\diamond E_{\mathbf{j}_1})\\ &\quad=(\Delta(E_{\mathbf{i}_s})\diamond\cdots\diamond\Delta(E_{\mathbf{i}_1}), (E_{\mathbf{j}_r}\diamond\cdots\diamond E_{\mathbf{j}_2})\otimes E_{\mathbf{j}_1})\\ &\quad=\sum \vartheta_{\mathbf{i}_{1,2},\ldots,\mathbf{i}_{s,2}}(E_{\mathbf{i}_{s,2}}\diamond\cdots\diamond E_{\mathbf{i}_{1,2}}, E_{\mathbf{j}_r}\diamond\cdots\diamond E_{\mathbf{j}_2})(E_{\mathbf{i}_{s,1}}\diamond\cdots\diamond E_{\mathbf{i}_{1,1}}, E_{\mathbf{j}_1}),\end{aligned}\tag{5.7}$$

where this sum is as in Lemma 5.6; in particular, $\mathbf{i}_{t,1} \le \mathbf{i}_t$, $\mathbf{i}_{t,1} \in \mathsf{W}^+$, for all $1 \le t \le s$ (note that $\mathbf{i}_{t,1}$ may be $\emptyset$).

Claim $(\star\star)$**.** *We have* $(E_{\mathbf{i}_{s,1}}\diamond\cdots\diamond E_{\mathbf{i}_{1,1}}, E_{\mathbf{j}_1}) = 0$ *unless there is a unique* k *such that* $\mathbf{i}_{k,1} = \mathbf{j}_1$ *and* $\mathbf{i}_{t,1} = \emptyset$ *for* $t \neq k$.

It is not necessarily the case that $E_{\mathbf{i}_{s,1}}\diamond\cdots\diamond E_{\mathbf{i}_{1,1}}$ belongs to the PBW basis, so we cannot apply earlier arguments. Therefore, suppose that k is maximal such that $\mathbf{i}_{k,1} \neq \emptyset$. Then,

$$(E_{\mathbf{i}_{k,1}}\diamond\cdots\diamond E_{\mathbf{i}_{1,1}}, E_{\mathbf{j}_1}) = \sum \vartheta^{\mathbf{j}_1}_{\mathbf{j}_{1,1},\mathbf{j}_{1,2}}(E_{\mathbf{i}_{k,1}}, E_{\mathbf{j}_{1,2}})(E_{\mathbf{i}_{k-1,1}}\diamond\cdots E_{\mathbf{i}_{1,1}}, E_{\mathbf{j}_{1,1}})$$

where the sum is as in Lemma 5.6. Consider one such term in the sum above:

$$(E_{\mathbf{i}_{k,1}}, E_{\mathbf{j}_{1,2}})(E_{\mathbf{i}_{k-1,1}}\diamond\cdots E_{\mathbf{i}_{1,1}}, E_{\mathbf{j}_{1,1}}).\tag{5.8}$$

Assume this term is nonzero. Since $|\mathbf{i}_{k,1}| \le |\mathbf{i}_k| < |\mathbf{i}|$ and $|\mathbf{j}_{2,1}| \le |\mathbf{j}_1| < |\mathbf{j}|$ in the dominance ordering on Q^+, induction on Q^+-grading implies that

$$(\mathrm{E}_{\mathbf{i}_{k,1}}, \mathrm{E}_{\mathbf{j}_{1,2}}) = 0 \quad \text{unless } \mathbf{i}_{k,1} = \mathbf{j}_{1,2}.$$

Therefore, $\mathbf{j}_{1,2} \neq \emptyset$ and $\mathbf{j}_{1,2} = \mathbf{i}_{k,1} \le \mathbf{i}_k \le \mathbf{i}_1 \le \mathbf{j}_1 \le \mathbf{j}_{1,2}$. Hence $\mathbf{j}_{1,2} = \mathbf{j}_1$ and $\mathbf{j}_{1,1} = \emptyset$. Since (5.8) is nonzero,

$$(\mathrm{E}_{\mathbf{i}_{k-1,1}} \diamond \cdots \mathrm{E}_{\mathbf{i}_{1,1}}, \mathrm{E}_{\mathbf{j}_{1,1}}) = (\mathrm{E}_{\mathbf{i}_{k-1,1}} \diamond \cdots \mathrm{E}_{\mathbf{i}_{1,1}}, 1) \neq 0,$$

so $\mathbf{i}_{k-1,1} = \cdots = \mathbf{i}_{1,1} = \emptyset$. Claim $(\star\star)$ follows.

Now, assume that

$$(\mathrm{E}_{\mathbf{i}_{s,1}} \diamond \cdots \diamond \mathrm{E}_{\mathbf{i}_{1,1}}, \mathrm{E}_{\mathbf{j}_1}) \neq 0.$$

Then, there is a unique k such that $\mathbf{i}_{k,1} = \mathbf{j}_1$ and $\mathbf{i}_{t,1} = \emptyset$ for $t \neq k$. Since $\mathbf{j}_1 = \mathbf{i}_{k,1} \le \mathbf{i}_k \le \mathbf{i}_1 \le \mathbf{j}_1$, it follows that $\mathbf{i}_{k,1} = \mathbf{i}_k = \mathbf{i}_1 = \mathbf{j}_1$.

Let $n_1 \ge 1$ be maximal such that $\mathbf{i}_1 = \mathbf{i}_2 = \cdots = \mathbf{i}_{n_1}$. Then, it follows from the previous arguments and the algebra structure on $\mathsf{U} \otimes \mathsf{U}$ that (5.7) becomes

$$\begin{aligned}
&(\mathrm{E}_{\mathbf{i}_s} \diamond \cdots \diamond \mathrm{E}_{\mathbf{i}_1}, \mathrm{E}_{\mathbf{j}_r} \diamond \cdots \diamond \mathrm{E}_{\mathbf{j}_1}) \\
&\quad = (1 + \pi^{p(\mathbf{i}_1)} q^{-(|\mathbf{i}_1|,|\mathbf{i}_1|)} + \cdots + \pi^{(n_1-1)p(\mathbf{i}_1)} q^{-(n_1-1)(|\mathbf{i}_1|,|\mathbf{i}_1|)}) \\
&\qquad (\mathrm{E}_{\mathbf{i}_s} \diamond \cdots \diamond \mathrm{E}_{\mathbf{i}_2}, \mathrm{E}_{\mathbf{j}_t} \diamond \cdots \diamond \mathrm{E}_{\mathbf{j}_2})(\mathrm{E}_{\mathbf{i}_1}, \mathrm{E}_{\mathbf{i}_1}).
\end{aligned}$$

We may now complete by induction the computation of

$$(\mathrm{E}_{\mathbf{i}_s} \diamond \cdots \diamond \mathrm{E}_{\mathbf{i}_1}, \mathrm{E}_{\mathbf{j}_r} \diamond \cdots \diamond \mathrm{E}_{\mathbf{j}_1})$$

and then $(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{j}})$, which yields the formula as stated in the theorem. □

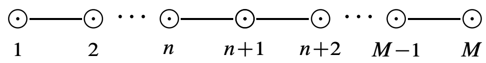
Now we define the *dual PBW basis* for U

$$\mathrm{E}^*_{\mathbf{i}} = \mathrm{E}_{\mathbf{i}} / (\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}}), \quad \text{for } \mathbf{i} \in \mathsf{W}^+. \tag{5.9}$$

6. Computations of dominant Lyndon words and root vectors

In this section we will compute the dominant Lyndon words, Lyndon and (dual) PBW root vectors explicitly for general Dynkin diagrams of type A-D. Throughout this section, we will set $M = m + n + 1$ and continue to order $\mathrm{I} = \{1, \ldots, M\}$ as specified in Table 1. We also remind the reader of the notation s_{ij} from (2.2).

6.1. Type $A(m,n)$. A general Dynkin diagram of type $A(m,n)$ is of the form



The next proposition computes the set of dominant Lyndon words inductively using Proposition 4.16.

Proposition 6.1. *The set of dominant Lyndon words is*

$$\mathsf{L}^+ = \{(i,\dots,j) \mid 1 \le i \le j \le M\}.$$

Having computed L^+, we compute the Lyndon basis using Proposition 4.12. For $\mathbf{i} = (i,\dots,j)$ with $1 \le i \le j \le M$, we set

$$\varpi_A(\mathbf{i}) = \prod_{k=i}^{j-1} s_{k,k+1}.$$

Proposition 6.2. *For* $\mathbf{i} = (i,\dots,j)$ *with* $1 \le i \le j \le M$, *the Lyndon root vector is*

$$\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}\pi^{j-i}\varpi_A(\mathbf{i})(q-q^{-1})^{j-i}(i,\dots,j).$$

Proof. We proceed by induction on $j-i$, the case $j-i=0$ being trivial. Note that if $\mathbf{i}=(i,\dots,j)$, and $\mathbf{i}=\mathbf{i}_1\mathbf{i}_2$ is the co-standard factorization of $\mathbf{i}$, then $\mathbf{i}_1=(i,\dots,j-1)$ and $\mathbf{i}_2=j$. By induction, we compute

$$\begin{aligned}
\mathrm{R}_{\mathbf{i}} &= \mathrm{R}_{\mathbf{i}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{i}_2}\\
&= \pi^{P(|\mathbf{i}_1|)}\pi^{j-i+1}\varpi_A(\mathbf{i}_1)(q-q^{-1})^{j-i-1}(i,\dots,j-1)\diamond_{q,q^{-1}} j\\
&= \pi^{P(|\mathbf{i}_1|)}\pi^{j-i+1}\varpi_A(\mathbf{i}_1)(q-q^{-1})^{j-i-1}((i,\dots,j-2)\diamond_{q,q^{-1}} j)(j-1)\\
&\qquad + \pi^{P(|\mathbf{i}_1|)}\pi^{j-i+1}\varpi_A(\mathbf{i}_1)(q-q^{-1})^{j-i-1}\\
&\qquad\qquad \pi^{p(i,\dots,j-1)p(j)}(q^{-(\alpha_{j-1},\alpha_j)}-q^{(\alpha_{j-1},\alpha_j)})(i,\dots,j)\\
&= \pi^{P(|\mathbf{i}_1|)}\pi^{j-i+1}\varpi_A(\mathbf{i}_1)(q-q^{-1})^{j-i-1}\\
&\qquad \pi^{p(i,\dots,j-1)p(j)}(q^{-(\alpha_{j-1},\alpha_j)}-q^{(\alpha_{j-1},\alpha_j)})(i,\dots,j).
\end{aligned}$$

The proof now follows by the observations

$$q^{-(\alpha_{j-1},\alpha_j)}-q^{(\alpha_{j-1},\alpha_j)} = -s_{j-1,j}(q-q^{-1})$$

and

$$P(|\mathbf{i}_1|)+p(i,\dots,j-1)p(j) = P(|\mathbf{i}|). \qquad \square$$

Corollary 6.3. *Let* $\mathbf{i} = (i, \dots, j)$ *with* $1 \le i \le j \le M$. *Then*

(1) *the PBW root vector is*

$$\mathrm{E}_{\mathbf{i}} = \varpi_A(\mathbf{i})(q - q^{-1})^{j-i} q^{-N(|\mathbf{i}|)}(i, \dots, j);$$

(2) $(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}}) = \varpi_A(\mathbf{i})(q - q^{-1})^{j-i} q^{-N(|\mathbf{i}|)}$;

(3) $\mathrm{E}_{\mathbf{i}}^* = (i, \dots, j)$.

Proof. The formula (1) for $\mathrm{E}_{\mathbf{i}}$ is clear from the definitions, and part (3) follows immediately from (1) and (2). So it remains to prove (2). To this end, let $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ be the co-standard factorization of $\mathbf{i}$, $\mathbf{i}_1 = (i, \dots, j-1)$, $\mathbf{i}_2 = j$. Note that

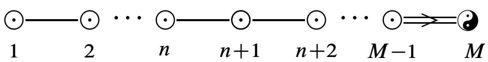
$$\mathrm{E}_{\mathbf{i}} = \mathrm{E}_j \diamond \mathrm{E}_{\mathbf{i}_1} - \pi^{p(j)p(\mathbf{i}_1)} q^{-(\alpha_{j-1}, \alpha_j)} \mathrm{E}_{\mathbf{i}_1} \diamond \mathrm{E}_j.$$

Therefore, using Proposition 3.13,

$$\begin{aligned}(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}}) =& \varpi_A(\mathbf{i})(q - q^{-1})^{j-i} q^{-N(|\mathbf{i}|)}(\mathbf{i}, \mathrm{E}_{\mathbf{i}}) \\ =& \varpi_A(\mathbf{i})(q - q^{-1})^{j-i} q^{-N(|\mathbf{i}|)}(j \otimes (i, \dots, j-1), \mathrm{E}_j \otimes \mathrm{E}_{\mathbf{i}_1}) \\ =& s_{j-1,j}(q - q^{-1}) q^{\frac{1}{2}(2(|\mathbf{i}_1|, \alpha_j) - (\alpha_j, \alpha_j))} (\mathrm{E}_j, \mathrm{E}_j)(\mathrm{E}_{\mathbf{i}_1}, \mathrm{E}_{\mathbf{i}_1}).\end{aligned}$$

Therefore, (2) follows by induction. □

6.2. Type $B(m, n+1)$. A general Dynkin diagram of type $B(m, n+1)$ is of the form



In order to facilitate computations below, we note the following properties of the signs s_{ij} $(i, j \in \mathrm{I})$ given in (2.2).

Lemma 6.4. (1) *if* $a_{ii} = 0$, *then*

$$s_{i-1,i} = \pi s_{i,i+1};$$

(2) *if* $a_{ii} \neq 0$, *then*

$$s_{i-1,i} = s_{i,i+1} = \pi s_{ii};$$

(3) *for any* $k, l \in \mathrm{I}$ *with* $k \neq l$,

$$(\alpha_k, \alpha_l) \in \{2s_{kl}, 0\}.$$

Proof. This follows immediately using the standard $\varepsilon\delta$-notation for the root system and the simple systems of type B; cf. [13, 7]. The factor 2 in part (3) is due to the normalization of $(\cdot,\cdot)$ adopted in §2.1. □

Proposition 6.5. *The set of dominant Lyndon words is*

$$\mathsf{L}^+ = \{(i,\dots,j) \mid 1 \le i \le j \le M\} \\ \cup \{(i,\dots,M,M,\dots,j+1) \mid 1 \le i \le j < M\}.$$

We set

$$\varpi_B(\mathbf{i}) = \begin{cases} \varpi_A(\mathbf{i}) & \text{if } \mathbf{i} = (i,\dots,j), \\ & \text{for } 1 \le i \le j \le M, \\ \varpi_A(i,\dots j)\pi^{p(M)} & \text{if } \mathbf{i} = (i,\dots,M,M,\dots,j+1), \\ & \text{for } 1 \le i \le j < M. \end{cases}$$

Proposition 6.6. (1) *For* $\mathbf{i} = (i,\dots,j)$*, with* $1 \le i \le j \le M$*, the Lyndon root vector is*

$$\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}\pi^{j-i}\varpi_B(\mathbf{i})(q^2-q^{-2})^{j-i}(i,\dots,j).$$

(2) *For* $\mathbf{i} = (i,\dots,M,M,\dots,j+1)$*, with* $1 \le i \le j < M$*, the Lyndon root vector is*

$$\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}\pi^{i+j}\varpi_B(\mathbf{i})(q^2-q^{-2})^{2M-i-j}(i,\dots,M,M,\dots,j+1).$$

Proof. The proof of part (1) is same as for type A in Proposition 6.2.

We prove (2) by downward induction on j. For $j = M-1$, $\mathbf{i} = (i,\dots,M,M)$ and the co-standard factorization is $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ where $\mathbf{i}_1 = (i,\dots,M)$ and $\mathbf{i}_2 = M$. Therefore,

$$\begin{aligned}\mathrm{R}_{\mathbf{i}} &= \pi^{P(|\mathbf{i}_1|)}\pi^{M-i}\varpi_A(\mathbf{i}_1)(q^2-q^{-2})^{M-i}(i,\dots,M)\diamond_{q,q^{-1}} M \\ &= \pi^{P(|\mathbf{i}_1|)}\pi^{M-i}\varpi_A(\mathbf{i}_1)(q^2-q^{-2})^{M-i}((i,\dots,M-1)\diamond_{q,q^{-1}} M)M \\ &\quad + \pi^{P(|\mathbf{i}_1|)}\pi^{M-i}\varpi_A(\mathbf{i}_1)(q^2-q^{-2})^{M-i} \\ &\qquad \pi^{p(i,\dots,M)p(M)}((i,\dots M,M)-(i,\dots,M,M)) \\ &= \pi^{P(|\mathbf{i}_1|)}\pi^{M-i}\varpi_A(\mathbf{i}_1)(q^2-q^{-2})^{M-i}((i,\dots,M-1)\diamond_{q,q^{-1}} M)M \\ &= \pi^{P(|\mathbf{i}_1|)}\pi^{M-i}\varpi_A(\mathbf{i}_1)(q^2-q^{-2})^{M-i}\pi^{p(i,\dots,M-1)p(M)} \\ &\quad (q^{-(\alpha_{M-1},\alpha_M)}-q^{(\alpha_{M-1},\alpha_M)})(i,\dots,M,M).\end{aligned}$$

This case now follows since

$$q^{-(\alpha_{M-1},\alpha_M)} - q^{(\alpha_{M-1},\alpha_M)} = \pi s_{M-1,M}(q^2 - q^{-2})$$

by Lemma 6.4(3) and

$$\pi^{P(|\mathbf{i}_1|)+p(i,\ldots,M-1)p(M)} = \pi^{P(|\mathbf{i}|)+p(M)}.$$

We now proceed to the general case

$$\mathbf{i} = (i, \ldots, M, M, \ldots, j+1).$$

Let

$$\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$$

be the co-standard factorization, with

$$\mathbf{i}_1 = (i, \ldots, M, M, \ldots, j+2)$$

and

$$\mathbf{i}_2 = j+1.$$

Then,

$$\begin{aligned}
\mathrm{R}_{\mathbf{i}} = {} & \pi^{P(|\mathbf{i}_1|)}\pi^{i+j-1}\varpi_B(\mathbf{i}_1)(q^2-q^{-2})^{2M-i-j-1} \\
& \quad (i, \ldots, M, M, \ldots, j+2) \diamond_{q,q^{-1}} (j+1) \\
= {} & \pi^{P(|\mathbf{i}_1|)}\pi^{i+j-1}\varpi_B(\mathbf{i}_1)(q^2-q^{-2})^{2M-i-j-1} \\
& \quad (i, \ldots, M, M, \ldots, j+3) \diamond_{q,q^{-1}} (j+1))(j+2) \\
& + \pi^{P(|\mathbf{i}_1|)}\pi^{i+j-1}\varpi_B(\mathbf{i}_1)(q^2-q^{-2})^{2M-i-j-1} \\
& \quad \pi^{p(j+1)p(\mathbf{i}_1)}(q^{-(|\mathbf{i}_1|,\alpha_{j+1})} - q^{(|\mathbf{i}_1|,\alpha_{j+1})})(i, \ldots, M, M, \ldots, j+1).
\end{aligned} \tag{6.1}$$

Using Lemma 6.4, we have that

$$-(|\mathbf{i}_1|, \alpha_{j+1})) = -(\alpha_j + \alpha_{j+1} + 2\alpha_{j+2}, \alpha_{j+1}) = -2s_{j+1,j+2}.$$

Also, $P(|\mathbf{i}_1|) + p(j+1)p(\mathbf{i}_1) = P(|\mathbf{i}|)$. Therefore, last term in (6.1) above is

$$\pi^{P(|\mathbf{i}|)}\pi^{2M-i-j}\varpi_B(\mathbf{i})(q^2-q^{-2})^{2M-i-j-1}(q^2-q^{-2})(i, \ldots, M, M, \ldots, j+1).$$

Hence, the proposition will follow if we can show that

$$((i, \ldots, M, M, \ldots, j+3) \diamond_{q,q^{-1}} (j+1))(j+2) = 0. \tag{6.2}$$

Indeed, since $(\alpha_k, \alpha_{j+1}) = 0$ for $j+2 < k \le M$,

$$\begin{aligned}
&((i,\dots,M,M,\dots,j+3)\diamond_{q,q^{-1}}(j+1))(j+2)\\
&\quad= ((i,\dots,j+2)\diamond_{q,q^{-1}}(j+1))(j+3,\dots,M,M,\dots,j+2)\\
&\quad= ((i,\dots,j+1)\diamond_{q,q^{-1}}(j+1))(j+2,\dots,M,M,\dots,j+2)\\
&\qquad+\pi^{p(j+1)(p(i)+\dots+p(j+2))}(q^{-(\alpha_i+\dots+\alpha_{j+2},\alpha_{j+1})}-q^{(\alpha_i+\dots+\alpha_{j+1},\alpha_{j+1})})\\
&\qquad\quad(i,\dots,j+2,j+1,j+3\dots,M,M,\dots,j+2).
\end{aligned}$$

But, using Lemma 6.4 again, we have $(\alpha_i+\dots+\alpha_{j+2},\alpha_{j+1}) = 0$, so

$$\begin{aligned}
&((i,\dots,M,M,\dots,j+3)\diamond_{q,q^{-1}}(j+1))(j+2)\\
&\quad= ((i,\dots,j+1)\diamond_{q,q^{-1}}(j+1))(j+2,\dots,M,M,\dots,j+2)\\
&\quad= ((i,\dots,j)\diamond_{q,q^{-1}}(j+1))(j+1,\dots,M,M,\dots,j+2))\\
&\qquad+\pi^{p(i,\dots,j+1)p(j+1)}(q^{-(\alpha_j+\alpha_{j+1},\alpha_{j+1})}-q^{(\alpha_j+\alpha_{j+1},\alpha_{j+1})})\\
&\qquad\quad(i,\dots,j+1,j+1,\dots,M,M,\dots,j+2)\\
&\quad= ((i,\dots,j-1)\diamond_{q,q^{-1}}(j+1))(j,\dots,M,M,\dots,j+2))\\
&\qquad+\pi^{p(i,\dots,j)p(j+1)}(q^{-(\alpha_j,\alpha_{j+1})}-q^{(\alpha_j,\alpha_{j+1})})\\
&\qquad\quad(i,\dots,j+1,j+1,\dots,M,M,\dots,j+2)\\
&\qquad+\pi^{p(i,\dots,j+1)p(j+1)}(q^{-(\alpha_j+\alpha_{j+1},\alpha_{j+1})}-q^{(\alpha_j+\alpha_{j+1},\alpha_{j+1})})\\
&\qquad\quad(i,\dots,j+1,j+1,\dots,M,M,\dots,j+2).
\end{aligned}$$

Obviously,

$$(i,\dots,j-1)\diamond_{q,q^{-1}}(j+1) = 0$$

since $(\alpha_i+\ldots+\alpha_{j-r},\alpha_{j+1}) = 0$ for $r \ge 1$. To treat the last two summands above, note that either $p(j+1) = 0$, or $a_{j+1,j+1} = 0$. If $p(j+1) = 0$, then

$$\pi^{p(i,\dots,j+1)p(j+1)} = \pi^{p(i,\dots,j)p(j+1)},$$

$$(q^{-(\alpha_j,\alpha_{j+1})}-q^{(\alpha_j,\alpha_{j+1})}) = \pi(q^{-(\alpha_j+\alpha_{j+1},\alpha_{j+1})}-q^{(\alpha_j+\alpha_{j+1},\alpha_{j+1})}),$$

and hence (6.2) holds. If $a_{j+1,j+1} = 0$, then

$$\pi^{p(i,\dots,j+1)p(j+1)} = \pi^{p(i,\dots,j)p(j+1)+1},$$

$$(q^{-(\alpha_j,\alpha_{j+1})}-q^{(\alpha_j,\alpha_{j+1})}) = (q^{-(\alpha_j+\alpha_{j+1},\alpha_{j+1})}-q^{(\alpha_j+\alpha_{j+1},\alpha_{j+1})}),$$

and hence (6.2) still holds. The proposition is proved. □

Corollary 6.7. *The following formulas hold for* $1 \le i \le j \le M$:

(1) *for* $\mathbf{i} = (i, \dots, j)$, *the PBW root vector is*

$$\mathrm{E}_{\mathbf{i}} = \varpi_B(\mathbf{i})(q^2 - q^{-2})^{j-i} q^{-N(|\mathbf{i}|)}(i, \dots, j);$$

for $\mathbf{i} = (i, \dots, M, M, \dots, j+1)$, *the PBW root vector is*

$$\mathrm{E}_{\mathbf{i}} = \varpi_B(\mathbf{i})(q^2 - q^{-2})^{2M-i-j} q^{-N(|\mathbf{i}|)}[2]_M^{-1}(i, \dots, M, M, \dots, j+1);$$

(2) *we have*

$$(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}}) = \begin{cases} \varpi_B(\mathbf{i})(q^2 - q^{-2})^{j-i} q^{-N(|\mathbf{i}|)}, & \\ \qquad\qquad \textit{if } \mathbf{i} = (i, \dots, j), & \\ \varpi_B(\mathbf{i})(q^2 - q^{-2})^{2M-i-j} q^{-N(|\mathbf{i}|)}[2]_N^{-2}, & \\ \qquad\qquad \textit{if } \mathbf{i} = (i, \dots, N, N, \dots, j+1); & \end{cases}$$

(3) *we have*

$$\mathrm{E}_{\mathbf{i}}^* = \begin{cases} (i, \dots, j), & \textit{if } \mathbf{i} = (i, \dots, j), \\ [2]_M(i, \dots, M, M, \dots, j+1), & \textit{if } \mathbf{i} = (i, \dots, M, M, \dots, j+1). \end{cases}$$

Proof. Parts (1) and (3) are proved in the same way as in the type A case.

It remains to prove (2), the case $\mathbf{i} = (i, \dots, j)$ also being the same as in type $A(m, n)$. Assume that $\mathbf{i} = (i, \dots, M, M)$. Then $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ is the co-standard factorization where $\mathbf{i}_1 = (i, \dots, M)$ and $\mathbf{i}_2 = M$. We have

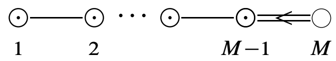
$$\begin{aligned}
(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}}) &= [2]_M^{-1}(\mathrm{E}_{\mathbf{i}}, M \diamond \mathrm{E}_{\mathbf{i}_1} - \pi^{p(M)}\mathrm{E}_{\mathbf{i}} \diamond M) \\
&= \varpi_B(\mathbf{i})(q^2 - q^{-2})^{M-i+1} q^{-N(|\mathbf{i}|)}[2]_M^{-2}(\mathbf{i}, M \diamond \mathrm{E}_{\mathbf{i}_1} - \pi^{p(M)}\mathrm{E}_{\mathbf{i}_1} \diamond M) \\
&= \varpi_B(\mathbf{i})(q^2 - q^{-2})^{M-i+1} q^{-N(|\mathbf{i}|)}[2]_M^{-2}(M \otimes \mathbf{i}_1, M \otimes \mathrm{E}_{\mathbf{i}_1}) \\
&= \varpi_B(\mathbf{i})(q^2 - q^{-2})^{M-i+1} q^{-N(|\mathbf{i}|)}[2]_M^{-2}(\mathbf{i}_1, \mathrm{E}_{\mathbf{i}_1}) \\
&= s_{M-1,M} q^{-N(|\mathbf{i}|)+N(|\mathbf{i}_1|)}[2]_M^{-2}(\mathrm{E}_M, \mathrm{E}_M)(\mathrm{E}_{\mathbf{i}_1}, \mathrm{E}_{\mathbf{i}_1}) \\
&= \varpi_B(\mathbf{i})(q^2 - q^{-2})^{M-i+1} q^{-N(|\mathbf{i}|)}[2]_M^{-2}.
\end{aligned}$$

Finally, assume that $\mathbf{i} = (i, \dots, M, M, \dots, j+1)$ with $i \le j < M - 1$. Then, $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ is the co-standard factorization, where $\mathbf{i}_1 = (i, \dots, M, M, \dots, j+2)$ and $\mathbf{i}_2 = j + 1$. Hence,

$$\begin{aligned}
(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}}) &= (\mathrm{E}_{\mathbf{i}}, (j+1) \diamond \mathrm{E}_{\mathbf{i}_1} - \pi^{p(\mathbf{i}_1)p(j+1)} q^{-(|\mathbf{i}_1|, \alpha_j)}\mathrm{E}_{\mathbf{i}_1} \diamond (j+1)) \\
&= \varpi_B(\mathbf{i})(q^2 - q^{-2})^{2M-i-j} q^{-N(|\mathbf{i}|)}[2]_M^{-1}(\mathrm{E}_{j+1}, \mathrm{E}_{j+1})(\mathbf{i}_1, \mathrm{E}_{\mathbf{i}_1}) \\
&= s_{j,j+1}(q^2 - q^{-2}) q^{-N(|\mathbf{i}|)+N(|\mathbf{i}_1|)}(\mathrm{E}_{\mathbf{i}_1}, \mathrm{E}_{\mathbf{i}_1}).
\end{aligned}$$

Therefore, (2) follows by induction. □

6.3. Types $C(M)$ and $D(m,n+1)$, I. We regard the type $C(M)$ as a limiting case of the type $D(m,n+1)$ with $m = 1$ (and $M = n+2$), and will treat them simultaneously. The Dynkin diagrams arise in two different shapes, with or without a branching node. We separate the discussion into 2 parts, according to the shape of the Dynkin diagrams. Here we consider a general Dynkin diagram without a branching node of the form



The root system is given in [7, Chapter 1] and [33, §3]. We have the following properties regarding the system of signs.

Lemma 6.8. (1) *We have*

$$s_{M-1,M} = 1;$$

(2) *if* $a_{ii} = 0$, *then*

$$s_{i-1,i} = \pi s_{i,i+1};$$

(3) *if* $a_{ii} \neq 0$, *then*

$$s_{i-1,i} = s_{i,i+1} = \pi s_{ii};$$

(4) *for any* $k, l \in \mathrm{I}$ *with* $k \neq l$,

$$(\alpha_k, \alpha_l) \in \{(1 + \delta_{kN} + \delta_{lN}) s_{kl}, 0\}.$$

Proof. The lemma can be checked readily case-by-case by using the standard $\varepsilon\delta$-notation for root systems and simple systems. □

The set of dominant Lyndon words are computed in the usual way.

Proposition 6.9. *The set of dominant Lyndon words is*

$$\begin{aligned} \mathsf{L}^+ = &\{(i,\ldots,j) \mid 1 \le i \le j \le M\} \\ &\cup \{(i,\ldots,M,\ldots,j+1) \mid 1 \le i \le j < M\} \\ &\cup \{(i,\ldots,M-1,i,\ldots,M) \mid 1 \le i < M \text{ and } p(i,\ldots,M-1) = 0\}. \end{aligned}$$

Note the parity condition $p(i,\ldots,M-1)=0$ above corresponds to the fact that there is no non-isotropic odd root for type C and D. Set

$$\varpi_C(\mathbf{i}) = \begin{cases} \varpi_A(\mathbf{i}) & \text{if } \mathbf{i}=(i,\ldots,j) \\ & \text{for } 1\le i\le j\le M, \\ \varpi_A(i,\ldots,j+1) & \text{if } \mathbf{i}=(i,\ldots,M,\ldots,j+1) \\ & \text{for } 1\le i<j<M-1. \end{cases}$$

Proposition 6.10. *The Lyndon root vectors are given as follows*:

(1) *for* $\mathbf{i}=(i,\ldots,j)$ *with* $j<M$,

$$\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}\pi^{j-i}\varpi_C(\mathbf{i})(q-q^{-1})^{j-i}(i,\ldots,j);$$

(2) *for* $\mathbf{i}=(i,\ldots,M)$,

$$\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}\pi^{M-i}\varpi_C(\mathbf{i})(q-q^{-1})^{M-i-1}(q^2-q^{-2})(i,\ldots,M);$$

(3) *for* $\mathbf{i}=(i,\ldots,M,\ldots,j+1)$,

$$\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}\pi^{i+j+1}\varpi_C(\mathbf{i})(q-q^{-1})^{2M-i-j-1}(q^2-q^{-2})(i,\ldots,M,\ldots,j+1);$$

(4) *for* $\mathbf{i}=(i,\ldots,M-1,i,\ldots,M)$,

$$\mathrm{R}_{\mathbf{i}} = q^{-1}(q-q^{-1})^{2M-2i-1}(q^2-q^{-2})((i,\ldots,M-1)\diamond(i,\ldots,M-1))M.$$

Proof. The proof of (1)–(3) are similar to the cases treated in types A and B, and we omit the details.

We prove (4). To this end, note that $(i,\ldots,M-1)\diamond(i,\ldots,M-1)\in\mathsf{U}$ since $(i,\ldots,M-1)\in\mathsf{U}$ by (1). Now, by Proposition 3.9, we deduce that

$$x = ((i,\ldots,M-1)\diamond(i,\ldots,M-1))M\in\mathsf{U}.$$

Evidently, $\max(x)=\mathbf{i}$ and, therefore, $\max(x)=\max(\mathrm{R}_{\mathbf{i}})$ by Lemma 4.19. Hence, we may express x as

$$x=\sum_{\mathbf{j}\le\mathbf{i}}\lambda_{\mathbf{j}}\mathrm{R}_{\mathbf{j}}.$$

But, by Corollary 4.17, $\mathbf{i}$ is the smallest dominant word of its degree, so $x=\lambda_{\mathbf{i}}\mathrm{R}_{\mathbf{i}}$.

We now compute the coefficient $\lambda_{\mathbf{i}}$. To this end, note that the co-standard factorization of $\mathbf{i}$ is $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$, where $\mathbf{i}_1 = (i, \dots, M-1)$ and $\mathbf{i}_2 = (i, \dots, M)$. Hence, since $p(M) = p(i, \dots, M-1) = 0$ and $s_{M-1,M} = 1$,

$$\begin{aligned}
\mathrm{R}_{\mathbf{i}} &= \mathrm{R}_{\mathbf{i}_1} \diamond_{q,q^{-1}} \mathrm{R}_{\mathbf{i}_2} \\
&= -\pi^{P(|\mathbf{i}_1|)+P(|\mathbf{i}_2|)} \varpi_C(\mathbf{i}_1)\varpi_C(\mathbf{i}_2)(q-q^{-1})^{2M-2i-2} \\
&\qquad (q^2-q^{-2})(i,\dots,M-1) \diamond_{q,q^{-1}} (i,\dots,M) \\
&= -(q-q^{-1})^{2M-2i-2}(q^2-q^{-2})((i,\dots,M-2) \diamond_{q,q^{-1}} (i,\dots,M))(M-1) \\
&\quad - (q-q^{-1})^{2M-2i-2}(q^2-q^{-2}) \\
&\qquad (q^{-(\alpha_i+\dots+\alpha_{M-1},\alpha_M)}((i,\dots,M-1) \diamond (i,\dots,M-1))M \\
&\quad - q^{-(\alpha_i+\dots+\alpha_{M-1},\alpha_M)}((i,\dots,M-1) \bar{\diamond} (i,\dots,M-1))M).
\end{aligned}$$

By the argument in the previous paragraph,

$$((i,\dots,M-2) \diamond_{q,q^{-1}} (i,\dots,M))(M-1) = 0.$$

Therefore, applying the identity

$$\begin{aligned}
&(i,\dots,M-1) \bar{\diamond} (i,\dots,M-1) \\
&\quad = q^{(\alpha_i+\dots+\alpha_{M-1},\alpha_i+\dots+\alpha_{M-1})}(i,\dots,M-1) \diamond (i,\dots,M-1),
\end{aligned}$$

and Lemma 6.8, we see that

$$\begin{aligned}
\mathrm{R}_{\mathbf{i}} &= -(q-q^{-1})^{2M-2i-2}(q^2-q^{-2}) \\
&\qquad (q^{-(\alpha_{M-1},\alpha_M)} - q^{(\alpha_{M-1},\alpha_M)+(\alpha_i+\dots+\alpha_{M-1},\alpha_i+\dots+\alpha_{M-1})}) \\
&\qquad ((i,\dots,M-1) \diamond (i,\dots,M-1))M \\
&= -(q-q^{-1})^{2M-2i-2}(q^2-q^{-2})(q^{-2}-1) \\
&\qquad ((i,\dots,M-1) \diamond (i,\dots,M-1))M \\
&= q^{-1}(q-q^{-1})^{2M-2i-1}(q^2-q^{-2})((i,\dots,M-1) \diamond (i,\dots,M-1))M.
\end{aligned}$$

This completes the proof. □

Corollary 6.11. (1) *The PBW root vectors are given as follows*:

(a) *for* $\mathbf{i} = (i, \ldots, j)$ *with* $j < M$,

$$\mathrm{E}_{\mathbf{i}} = \varpi_C(\mathbf{i})(q - q^{-1})^{j-i} q^{-N(|\mathbf{i}|)}(i, \ldots, j);$$

(b) *for* $\mathbf{i} = (i, \ldots, M)$,

$$\mathrm{E}_{\mathbf{i}} = \varpi_C(\mathbf{i})(q - q^{-1})^{M-i-1}(q^2 - q^{-2})q^{-N(|\mathbf{i}|)}(i, \ldots, M);$$

(c) *for* $\mathbf{i} = (i, \ldots, M, \ldots, j+1)$,

$$\mathrm{E}_{\mathbf{i}} = \varpi_C(\mathbf{i})(q - q^{-1})^{2M-i-j-1}(q^2 - q^{-2})q^{-N(|\mathbf{i}|)}(i, \ldots, M, \ldots, j+1);$$

(d) *for* $\mathbf{i} = (i, \ldots, M-1, i, \ldots, M)$,

$$\mathrm{E}_{\mathbf{i}} = \pi^{P(|\mathbf{i}_1|)} q(q - q^{-1})^{2M-2i} q^{-N(|\mathbf{i}|)}((i, \ldots, M-1) \diamond (i, \ldots, M-1))M,$$

where $\mathbf{i}_1 = (i, \ldots, M-1)$;

(2) *the values of* $(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}})$ *are given by*

$$\begin{cases} \varpi_C(\mathbf{i})(q - q^{-1})^{j-i-\delta_{jM}} q^{-N(|\mathbf{i}|)}, & \text{if } \mathbf{i} = (i, \ldots, j) \\ & \text{for } 1 \le i \le j \le M, \\ \varpi_C(\mathbf{i})(q - q^{-1})^{2M-i-j-1}(q^2 - q^{-2})q^{-N(|\mathbf{i}|)}, & \text{if } \mathbf{i} = (i, \ldots, M, \ldots, j+1), \\ \pi^{P(|\mathbf{i}_1|)}(q - q^{-1})^{2M-2i} q^{-N(|\mathbf{i}|)}, & \text{if } \mathbf{i} = (i, \ldots, M-1, i, \ldots, M), \end{cases}$$

where $\mathbf{i}_1 = (i, \ldots, M-1)$;

(3) *the dual PBW root vectors are given by*

$$\mathrm{E}^*_{\mathbf{i}} = \begin{cases} (i, \ldots, j), & \text{if } \mathbf{i} = (i, \ldots, j) \\ & \text{for } 1 \le i \le j \le M, \\ (i, \ldots, M, \ldots, j+1), & \text{if } \mathbf{i} = (i, \ldots, M, \ldots, j+1), \\ q\,((i, \ldots, M-1) \diamond (i, \ldots, M-1))M, & \text{if } \mathbf{i} = (i, \ldots, M-1, i, \ldots, M). \end{cases}$$

Proof. The formulas in (1) follow directly from the definitions.

We prove (2). Note that for

$$\mathbf{i} \in \{(i, \dots, j) \mid i \le j\} \cup \{(i, \dots, M, \dots, j+1) \mid i \le j < M\}$$

the computations are similar to those performed in types A and B, and we omit the details. Therefore, assume that $\mathbf{i} = (i, \dots, M-1, i, \dots, M)$. We have

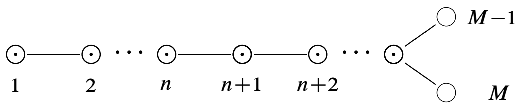
$$\begin{aligned} &\Delta(((i, \dots, M-1) \diamond (i, \dots, M-1))M) \\ &\quad = (\Delta(i, \dots, M-1) \diamond \Delta(i, \dots, M-1))(M \otimes 1) \\ &\qquad + 1 \otimes ((i, \dots, M-1) \diamond (i, \dots, M-1))M \end{aligned}$$

and, therefore, $(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}})$ is equal to

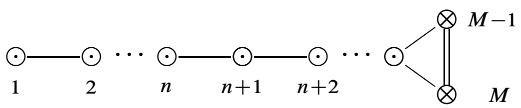
$$\begin{aligned} &\pi^{P(|\mathbf{i}_1|)} q(q-q^{-1})^{2M-2i-1}(q^2-q^{-2}) \\ &\quad q^{-N(|\mathbf{i}|)}[2]^{-1}(\mathrm{E}_{\mathbf{i}}, ((i, \dots, M-1) \diamond (i, \dots, M-1))M) \\ &\qquad = \pi^{P(|\mathbf{i}_1|)} q(q-q^{-1})^{2M-2i-1}(q^2-q^{-2}) q^{-N(|\mathbf{i}|)}[2]^{-2} \\ &\qquad\quad (E_{\mathbf{i}_2} \otimes \mathrm{E}_{\mathbf{i}_1}, (\Delta(i, \dots, M-1) \diamond \Delta(i, \dots, M-1))(M \otimes 1)) \\ &\qquad = \pi^{P(|\mathbf{i}_1|)} q(q-q^{-1})^{2M-2i-1}(q^2-q^{-2}) q^{-N(|\mathbf{i}|)}[2]^{-2} \\ &\qquad\quad (E_{\mathbf{i}_2} \otimes \mathrm{E}_{\mathbf{i}_1}, (q^{-2}+1)(i, \dots, M) \otimes (i, \dots, M-1)) \\ &\qquad = \pi^{P(|\mathbf{i}_1|)} q(q-q^{-1}) q^{-N(|\mathbf{i}|)+N(|\mathbf{i}_1|)+N(|\mathbf{i}_2|)} \\ &\qquad\quad [2]^{-2}(q^{-2}+1)(\mathrm{E}_{\mathbf{i}_1}, \mathrm{E}_{\mathbf{i}_1})(\mathrm{E}_{\mathbf{i}_2}, \mathrm{E}_{\mathbf{i}_2}) \\ &\quad = \pi^{P(\mathbf{i}_1)}(q-q^{-1})^{2M-2i-1}(q^2-q^{-2}) q^{-N(|\mathbf{i}|)}[2]^{-1} \\ &\quad = \pi^{P(\mathbf{i}_1)}(q-q^{-1})^{2M-2i} q^{-N(|\mathbf{i}|)}. \end{aligned}$$

This proves (2). Finally, (3) immediately follows from (2). □

6.4. Type $C(M)$ and $D(m, n+1)$, II. In this subsection, we consider the remaining simple systems of type $C(M)$ and $D(m, n+1)$, which correspond to Dynkin diagrams with a branching node as follows:



and



Proposition 6.12. *The set of dominant Lyndon words is*

$$\begin{aligned}\mathsf{L}^+ = \{&(i,\dots,j) \mid i \le j \le M-1\}\\ &\cup \{(i,\dots,M-2,M) \mid i \le M-2\}\\ &\cup \{(i,\dots,M-2,M,M-1,\dots,j+1) \mid i \le j \le M-2\}\\ &\cup \{(i,\dots,M-1,i,\dots,M-2,M) \mid i < M-1, p(i,\dots,M-1)=1\}.\end{aligned}$$

Set

$$\varpi_D(\mathbf{i}) = \begin{cases} \varpi_A(\mathbf{i}) & \text{if } \mathbf{i} = (i,\dots,j), \\ & i \le j \le M-1, \\ \varpi_A(i,\dots,M-1) & \text{if } \mathbf{i} = (i,\dots,M-2,M), \\ \varpi_A(i,\dots j+1) & \text{if } \mathbf{i} = (i,\dots,M-2,M,\dots,j+1), \\ & i < j < M-1. \end{cases}$$

Proposition 6.13. *The Lyndon root vectors are given as follows*:

(1) *for* $\mathbf{i} = (i,\dots,j)$, $j \le M-1$,

$$\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}\pi^{j-i}\varpi_D(\mathbf{i})(q-q^{-1})^{j-i}(i,\dots,j);$$

(2) *for* $\mathbf{i} = (i,\dots,M-2,M)$,

$$\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}\pi^{M-i-1}\varpi_D(\mathbf{i})(q-q^{-1})^{M-i-1}(i,\dots,M-2,M);$$

(3) *for* $\mathbf{i} = (i,\dots,M-2,M,M-1,\dots,j+1)$,

$$\begin{aligned}\mathrm{R}_{\mathbf{i}} = \pi^{P(|\mathbf{i}|)}&\pi^{i+j}\varpi_D(\mathbf{i})(q-q^{-1})^{2M-i-j-2}\\ &((i,\dots,M-1,M,M-2\dots,j+1)\\ &\qquad + (i,\dots,M-2,M,M-1,\dots,j+1));\end{aligned}$$

(4) *for* $\mathbf{i} = (i,\dots,M-1,i,\dots,M-2,M)$,

$$\mathrm{R}_{\mathbf{i}} = \pi(q-q^{-1})^{2M-2i-2}(q^2-q^{-2})((i,\dots,M-2)\diamond(i,\dots,M-1))M.$$

Proof. Formulas (1)–(3) can be obtained in the same way as in previous types and we omit the details.

The proof of (4) is very similar to the long roots in type C and we only outline the proof, leaving the details to the interested reader. Indeed, let

$$\mathbf{i} = (i, \dots, M-1, i, \dots, M-2, M)$$

and let

$$\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2$$

be the co-standard factorization of $\mathbf{i}$. As in the type C case, we deduce from Proposition 3.9 that

$$x = ((i, \dots, M-2) \diamond (i, \dots, M-1))M \in \mathsf{U}.$$

Moreover, since $\max(x) = \mathbf{i}$, it follows that $\mathrm{R}_{\mathbf{i}}$ is proportional to x. To compute the coefficient, note that $P(|\mathbf{i}_1|) = P(|\mathbf{i}_2|)$ and $\varpi_D(\mathbf{i}_1) = \varpi_D(\mathbf{i}_2)$, so

$$\begin{aligned}
\mathrm{R}_{\mathbf{i}} &= (q-q^{-1})^{2M-2i-2}(i, \dots, M-1) \diamond_{q,q^{-1}} (i, \dots, M-2, M) \\
&= (q-q^{-1})^{2M-2i-2}\pi^{p(M)}(q(i, \dots, M-1) \diamond (i, \dots, M-2) \\
&\qquad - q^{-1}(i, \dots, M-1) \mathbin{\bar{\diamond}} (i, \dots, M-2))M \\
&= \pi(q-q^{-1})^{2M-2i-2}(q^2(i, \dots, M-2) \mathbin{\bar{\diamond}} (i, \dots, M-1) \\
&\qquad - q^{-2}(i, \dots, M-2) \diamond (i, \dots, M-1))M,
\end{aligned}$$

where we have used the fact that

$$p(M) + p(i, \dots, M-2) = 1 = p(i, \dots, M-1)$$

to obtain the factor π after the last equality. Finally, the computation follows upon observing that

$$(i, \dots, M-2) \diamond (i, \dots, M-1) = (i, \dots, M-2) \mathbin{\bar{\diamond}} (i, \dots, M-1).$$

This last statement can be proved as follows: first, we have

$$i \diamond (i, \dots, k) = i \mathbin{\bar{\diamond}} (i, \dots, k) \quad \text{for any } k > i,$$

by induction on k, and

$$(i, \dots, j) \diamond (i, \dots, k) = (i, \dots, j) \mathbin{\bar{\diamond}} (i, \dots, k) \quad \text{for } i \le j < k,$$

by induction on j. ◻

Corollary 6.14. (1) *The PBW root vectors are*:

(a) *for* $\mathbf{i} = (i, \dots, j)$, $j \le M - 1$,

$$\mathrm{E}_{\mathbf{i}} = \varpi_D(\mathbf{i})(q - q^{-1})^{j-i} q^{-N(|\mathbf{i}|)}(i, \dots, j);$$

(b) *for* $\mathbf{i} = (i, \dots, M-2, M)$,

$$\mathrm{E}_{\mathbf{i}} = \varpi_D(\mathbf{i})(q - q^{-1})^{M-i-1} q^{-N(|\mathbf{i}|)}(i, \dots, M-2, M);$$

(c) *for* $\mathbf{i} = (i, \dots, M-2, M, M-1, \dots, j+1)$,

$$\begin{aligned}\mathrm{E}_{\mathbf{i}} = \varpi_D(\mathbf{i})&(q - q^{-1})^{2M-i-j-2} \\ &q^{-N(|\mathbf{i}|)}((i, \dots, M-1, M, M-2 \dots, j+1) \\ &+ (i, \dots, M-2, M, M-1, \dots, j+1));\end{aligned}$$

(d) *for* $\mathbf{i} = (i, \dots, M-1, i, \dots, M-2, M)$,

$$\begin{aligned}\mathrm{E}_{\mathbf{i}} = &(q - q^{-1})^{2M-2i-2}(q^2 - q^{-2})[2]^{-1} \\ &q^{-N(|\mathbf{i}|)}((i, \dots, M-2) \diamond (i, \dots, M-1))M;\end{aligned}$$

(2) *The values of* $(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{i}})$ *are given by*

$$\begin{cases} \varpi_D(\mathbf{i})(q - q^{-1})^{j-i} q^{-N(|\mathbf{i}|)}, \\ \qquad \text{if } \mathbf{i} = (i, \dots, j)\ (j \le M-1) \text{ or } \mathbf{i} = (i, \dots, M-2, M), \\ \varpi_D(\mathbf{i})(q - q^{-1})^{2M-i-j-2} q^{-N(|\mathbf{i}|)}, \\ \qquad \text{if } \mathbf{i} = (i, \dots, M-2, M, M-1, \dots, j+1), \\ \dfrac{(q - q^{-1})^{2M-2i-1} q^{-N(|\mathbf{i}|)}}{q + q^{-1}}, \\ \qquad \text{if } \mathbf{i} = (i, \dots, M-1, i, \dots, M-2, M); \end{cases}$$

(3) *the dual PBW root vectors are*

(a) *for* $\mathbf{i} = (i, \dots, j)$, $j \le M - 1$,

$$\mathrm{E}^*_{\mathbf{i}} = (i, \dots, j);$$

(b) *for* $\mathbf{i} = (i, \dots, M-2, M)$,

$$\mathrm{E}^*_{\mathbf{i}} = (i, \dots, M-2, M);$$

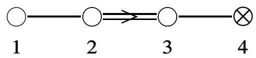
(*c*) *for* $\mathbf{i} = (i, \dots, M-2, M, M-1, \dots, j+1)$,

$$\mathrm{E}^*_{\mathbf{i}} = (i, \dots, M-1, M, M-2 \dots, j+1) \\ + (i, \dots, M-2, M, M-1, \dots, j+1));$$

(d) *for* $\mathbf{i} = (i, \dots, M-1, i, \dots, M-2, M)$,

$$\mathrm{E}^*_{\mathbf{i}} = (q + q^{-1})((i, \dots, M-2) \diamond (i, \dots, M-1))M.$$

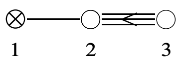
6.5. Type $F(3 \mid 1)$. Associated to the distinguished diagram



we have the following table of dominant Lyndon words.

Height	Dominant Lyndon Words
1	1, 2, 3, 4
2	(12), (23), (34)
3	(123), (233), (234)
4	(1233), (1234), (2343)
5	(12332), (12343)
6	(123432)
7	(1234323)
8	(12343234)

6.6. Type $G(3)$. Associated to the distinguished diagram



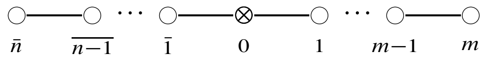
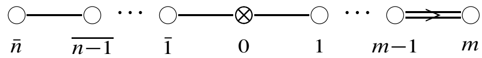
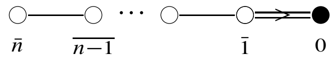
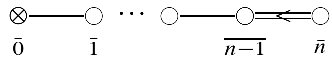
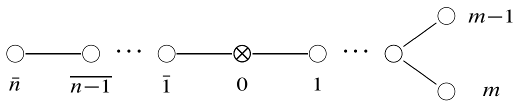
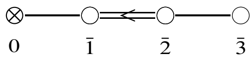
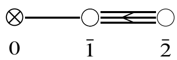
we have the following table of dominant Lyndon words.

Height	Dominant Lyndon Words
1	1, 2, 3
2	(12), (23)
3	(123), (223)
4	(1232), (2223)
5	(12322), (22323)
6	(123223)
7	(1232233)

7. Canonical bases

In this section, we shall formulate and construct the canonical basis of type $A(m,0)$, $B(0,n+1)$, and $C(n+1)$ for the standard simple system. Table 2 below compiles a list of standard simple systems for Lie superalgebras of basic type, with $D(2 \mid 1;\alpha)$ omitted.

Table 2. Dynkin diagrams for standard simple systems.

$A(m,n)$	
$B(m,n+1)$	
$B(0,n+1)$	
$C(n+1)$	
$D(m,n+1)$	
$F(3 \mid 1)$	
$G(3)$	

7.1. Integral forms. We start with some general discussions of root systems of basic type in order to define suitable integral forms of U_q.

We will restrict our attention to the standard simple systems in Table 2, and fix the ordered set

$$(\mathrm{I}, \leq) = \{\bar{n} < \cdots < \bar{1} < 0 < 1 < \cdots < m\}.$$

Following Lusztig, we call $i \in \mathrm{I}$ and $\alpha_i \in \Pi$ *special* if $c_i \leq 1$ in the expansions of every root β in $\widetilde{\Phi}^+$ in terms of Π, $\beta = \sum_{j\in \mathrm{I}} c_j \alpha_j$. We will call a Dynkin diagram (or the corresponding U) appearing in Table 2 *special* if any $i \in \mathrm{I}_{\mathsf{iso}}$ (which is unique if it exists) is special. Note that we take into account the entire (positive) root system $\widetilde{\Phi}^+$ as opposed to the reduced one. By inspection we have the following.

Proposition 7.1. *The Dynkin diagrams in Table 2 are special if and only if they are of type $A(m,n)$, $B(0,n+1)$, and $C(n+1)$.*

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and define $U_{\mathcal{A}}$ to be the $\mathcal{A}$-subalgebra of U_q generated by e_i ($i \in \mathrm{I}_{\mathsf{iso}}$) and the divided powers

$$e_i^{(k)} = e_i^k/[k]_i! \quad (i \in \mathrm{I}_{\bar{0}} \sqcup \mathrm{I}_{\mathsf{n\text{-}iso}}, k \geq 1).$$

Set

$$U_{\mathcal{A}}^* = \{u \in U_q \mid (u, v) \in \mathcal{A} \text{ for all } v \in U_{\mathcal{A}}\}.$$

Denote by W' the subset of words in $\mathbf{i} \in \mathsf{W}$ of the form $\mathbf{i} = i_1^{n_1} \cdots i_d^{n_d}$, where $i_k \neq i_{k+1}$ for all $1 \leq k < d$ and $n_l \in \{0, 1\}$ whenever $i_l \in \mathrm{I}_{\mathsf{iso}}$. For such $\mathbf{i} \in \mathsf{W}'$, we set

$$\varsigma_{\mathbf{i}} = [n_1]_{i_1}! \cdots [n_d]_{i_d}!$$

and write

$$e_{\mathbf{i}} = e_{i_1}^{n_1} \cdots e_{i_d}^{n_d}.$$

Then, $\varsigma_{\mathbf{i}}^{-1} e_{\mathbf{i}}$ is a product of divided powers. Consider the free $\mathcal{A}$-module

$$\mathsf{F}_{\mathcal{A}} = \bigoplus_{\mathbf{i} \in \mathsf{W}'} \mathcal{A} \varsigma_{\mathbf{i}} \mathbf{i}$$

and define

$$\mathsf{U}_{\mathcal{A}}^* = \mathsf{F}_{\mathcal{A}} \cap \mathsf{U}. \tag{7.1}$$

We have the following analogue of [23, Lemma 8] with an entirely similar proof.

Lemma 7.2. *We have*

$$\mathsf{U}^*_{\mathcal{A}} = \Psi(U^*_{\mathcal{A}}).$$

Proof. Any $u \in U_q$ belongs to $U^*_{\mathcal{A}}$ if and only if $(u, \varsigma_{\mathbf{i}}^{-1} e_{\mathbf{i}}) \in \mathcal{A}$ for all $\mathbf{i} \in \mathsf{W}'$. This holds if and only if $\Psi(u)$ is a linear combination of elements $\varsigma_{\mathbf{i}}\mathbf{i}$ for $\mathbf{i} \in \mathsf{W}'$, which is true if and only if $\Psi(u) \in \mathsf{F}_{\mathcal{A}}$. □

Corollary 7.3. *The free $\mathcal{A}$-module $U^*_{\mathcal{A}}$ is an $\mathcal{A}$-subalgebra of U_q.*

Proof. It is clear from the definitions that $(\mathsf{F}_{\mathcal{A}}, \diamond)$ is an $\mathcal{A}$-subalgebra of $(\mathsf{F}, \diamond)$ and, therefore, so is $\mathsf{U}^*_{\mathcal{A}}$. By Lemma 7.2, $U^*_{\mathcal{A}}$ is an $\mathcal{A}$-subalgebra of U_q. □

Let $\mathsf{U}_{\mathrm{PBW}}$ be the $\mathcal{A}$-lattice spanned by the PBW basis $\{\mathrm{E}_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}$, and $\mathsf{U}^*_{\mathrm{PBW}}$ the $\mathcal{A}$-lattice spanned by the dual PBW basis $\{\mathrm{E}^*_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}$ in (5.9).

Proposition 7.4. *Assume that* U *is special. Then*

$$\mathsf{U}^*_{\mathrm{PBW}} = \mathsf{U}^*_{\mathcal{A}} \quad \textit{and} \quad \mathsf{U}_{\mathrm{PBW}} = \mathsf{U}_{\mathcal{A}}.$$

Proof. The two identities are equivalent, and we shall prove that $\mathsf{U}^*_{\mathrm{PBW}} = \mathsf{U}^*_{\mathcal{A}}$. To this end, note that by the computations in Section 6, $\mathrm{E}^*_{\mathbf{i}} \in \mathsf{U}^*_{\mathcal{A}}$ for all $\mathbf{i} \in \mathsf{L}^+$. By Corollary 7.3, it follows that

$$\mathsf{U}^*_{\mathrm{PBW}} \subset \mathsf{U}^*_{\mathcal{A}}.$$

We will now prove that equality holds when U is special. To this end, suppose that

$$\sum_{\mathbf{i} \in \mathsf{W}^+} \lambda_{\mathbf{i}} \mathrm{E}^*_{\mathbf{i}} \in \mathsf{U}^*_{\mathcal{A}}.$$

We will prove that all $\lambda_{\mathbf{i}} \in \mathcal{A}$ by induction on $|\{\mathbf{i} \in \mathsf{W}^+ \mid \lambda_{\mathbf{i}} \neq 0\}|$.

First, suppose that $\lambda_{\mathbf{i}} \mathrm{E}^*_{\mathbf{i}} \in \mathsf{U}^*_{\mathcal{A}}$. Suppose that $\mathbf{i} = (i_1^{a_1}, \ldots, i_d^{a_d}) \in \mathsf{L}^+_{\bar{0}} \sqcup \mathsf{L}^+_{\text{n-iso}}$, $i_r \neq i_{r+1}$. Note that the coefficient of $\mathbf{i}$ in $\mathrm{E}_{\mathbf{i}}$ is $\varsigma_{\mathbf{i}}$ (except for the long roots in type C, where we instead consider the word $\mathbf{i}' = (i, i, i+1, i+1, \ldots, M-1, M-1, M)$ whose coefficient is $\varsigma_{\mathbf{i}'}$). For $n \geq 1$, let

$$\mathbf{i}^{(n)} = (i_1^{na_1}, \ldots, i_d^{na_d}).$$

Since the diagram for U is special, the coefficient of $\mathbf{i}^{(n)}$ in $(\mathrm{E}_{\mathbf{i}}^*)^{\diamond n}$ is nonzero, and (up to a power of q) equals

$$\varsigma_{\mathbf{i}}^n \begin{bmatrix} na_1 \\ a_1, \dots, a_1 \end{bmatrix}_{i_1} \cdots \begin{bmatrix} na_d \\ a_d, \dots, a_d \end{bmatrix}_{i_d} = \varsigma_{\mathbf{i}^{(n)}}$$

where, for $r \geq 1$,

$$\begin{bmatrix} na_r \\ a_r, \dots, a_r \end{bmatrix}_{i_r} = \frac{[na_r]_{i_r}!}{([a_r]_{i_r}!)^n}$$

is the quantum multinomial coefficient.

Now, if $\mathbf{i} \in \mathsf{W}^+$ and $\mathbf{i} = \mathbf{i}_1^{n_1} \cdots \mathbf{i}_r^{n_r}$, $\mathbf{i}_1 > \cdots > \mathbf{i}_r$,

$$\mathbf{i}_s = (i_{s1}^{a_{s1}}, \dots, i_{sd_s}^{a_{sd_s}}) \in \mathsf{L}^+,$$

then the coefficient of

$$\tilde{\mathbf{i}} := \mathbf{i}_1^{(n_1)} \cdots \mathbf{i}_r^{(n_r)}$$

in $\mathrm{E}_{\mathbf{i}}^*$ is nonzero (again, because the diagram is special) and (up to a power of q) equals

$$\prod_{s=1}^{r} \varsigma_{\mathbf{i}_s}^{n_s} \prod_{t=1}^{d_s} \begin{bmatrix} n_s a_{st} \\ a_{st}, \dots, a_{st} \end{bmatrix}_{i_{st}} = \prod_{s=1}^{r} \varsigma_{\mathbf{i}_s^{(n_s)}} = \varsigma_{\tilde{\mathbf{i}}}.$$

(Above, we make the appropriate adjustments in type C as in the last paragraph). Hence, if $\lambda_{\mathbf{i}} \mathrm{E}_{\mathbf{i}}^* \in \mathsf{U}_{\mathcal{A}}^*$, then $\lambda_{\mathbf{i}} \varsigma_{\tilde{\mathbf{i}}} \in \mathcal{A} \varsigma_{\tilde{\mathbf{i}}}$ which forces $\lambda_{\mathbf{i}} \in \mathcal{A}$ as required.

We now proceed to the inductive step. Let

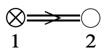
$$\mathbf{j} = \max\{\mathbf{i} \mid \lambda_{\mathbf{i}} \neq 0\}.$$

Then, the coefficient of $\tilde{\mathbf{j}}$ in $\mathrm{E}_{\mathbf{j}}^*$ (making the appropriate adjustments in type C) is $\varsigma_{\tilde{\mathbf{j}}}$. Moreover, $\tilde{\mathbf{j}}$ does not occur in $\mathrm{E}_{\mathbf{i}}^*$ for $\mathbf{i} < \mathbf{j}$. It follows that $\lambda_{\mathbf{j}} \in \mathcal{A}$, and induction applies to

$$\sum_{\mathbf{i} \neq \mathbf{j}} \lambda_{\mathbf{i}} \mathrm{E}_{\mathbf{i}}^* = \Big(\sum_{\mathbf{i}} \lambda_{\mathbf{i}} \mathrm{E}_{\mathbf{i}}^* \Big) - \lambda_{\mathbf{j}} \mathrm{E}_{\mathbf{j}}^* \in \mathsf{U}_{\mathcal{A}}^*.$$

This completes the proof. □

Example 7.5. It is not true that $\mathsf{U}_{\mathrm{PBW}} = \mathsf{U}_{\mathcal{A}}$ for non-special standard Dynkin diagrams in general. Indeed, consider type $B(1,1)$:



The root $\beta = \alpha_1 + \alpha_2$ non-isotropic. We have $\mathbf{i}(\beta) = (12)$, $\mathrm{E}_{(12)}^* = (12)$, and

$$\mathrm{E}_{(1212)}^* = q^{-1} \mathrm{E}_{(12)}^* \diamond \mathrm{E}_{(12)}^* = (\pi q + q^{-1})(1212).$$

In particular, $\frac{1}{\{2\}} \mathrm{E}_{(1212)} \in \mathsf{U}_{\mathcal{A}}^*$ showing that $\mathsf{U}_{\mathrm{PBW}}^* \neq \mathsf{U}_{\mathcal{A}}^*$.

7.2. Pseudo-canonical and canonical bases

Lemma 7.6. *For* $\mathbf{i} \in \mathsf{W}^+$, *write*

$$\overline{\mathrm{E}_{\mathbf{i}}} = \sum_{\mathbf{j} \in \mathsf{W}^+} a_{\mathbf{ij}} \mathrm{E}_{\mathbf{j}}, \quad \textit{for } a_{\mathbf{ij}} \in \mathbb{Q}(q). \tag{7.2}$$

Then,

$$a_{\mathbf{ii}} = 1 \quad \textit{for all } \mathbf{i} \in \mathsf{W}^+$$

and

$$a_{\mathbf{ij}} = 0 \quad \textit{if } \mathbf{i} > \mathbf{j}.$$

Proof. This proof is identical to that of [23, Lemma 37]. By Propositions 5.3 and 5.4 we have

$$\varepsilon_{\tau(\mathbf{i})} = \sum_{\mathbf{j} \geq \mathbf{i}} \beta_{\mathbf{ij}} \mathrm{E}_{\mathbf{j}}$$

with

$$\overline{\beta_{\mathbf{ii}}} = \beta_{\mathbf{ii}} = \kappa_{\mathbf{i}}.$$

As $\overline{\varepsilon_{\tau(\mathbf{i})}} = \varepsilon_{\tau(\mathbf{i})}$, substituting (7.2) into the equation above yields

$$a_{\mathbf{ij}} = \sum_{\mathbf{i} \leq \mathbf{k} \leq \mathbf{j}} \overline{\alpha_{\mathbf{ik}}}\, \beta_{\mathbf{kj}}.$$

Therefore, $a_{\mathbf{ij}} = 0$ if $\mathbf{i} > \mathbf{j}$ and $a_{\mathbf{ii}} = \overline{\alpha_{\mathbf{ii}}} \beta_{\mathbf{ii}} = \kappa_{\mathbf{i}}^{-1} \kappa_{\mathbf{i}} = 1$ by Proposition 5.3. ☐

Lemma 7.7. *Suppose that* U *is special. Then, the coefficients* $a_{\mathbf{ij}}$ *in* (7.2) *belong to* $\mathcal{A}$.

Proof. This is immediate since $\mathsf{U}_{\mathrm{PBW}} = \mathsf{U}_{\mathcal{A}}$ by Proposition 7.4 and $\mathsf{U}_{\mathcal{A}}$ is clearly bar invariant. ☐

It is well known that Lemmas 7.6 and 7.7 imply the existence of a unique basis of the form

$$\mathrm{b}_{\mathbf{i}} = \mathrm{E}_{\mathbf{i}} + \sum_{\mathbf{j} > \mathbf{i}} \theta_{\mathbf{ij}} \mathrm{E}_{\mathbf{j}} \tag{7.3}$$

such that

$$\theta_{\mathbf{ij}} \in q\mathbb{Z}[q] \quad \text{and} \quad \overline{\mathrm{b}_{\mathbf{i}}} = \mathrm{b}_{\mathbf{i}}.$$

We call the basis $\{\mathrm{b}_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}$ a *pseudo-canonical basis* for $\mathsf{U}_{\mathcal{A}}$ or for U.

A pseudo-canonical basis will be called a *canonical basis* if it is *almost orthogonal* in the sense that there exists $\epsilon \in \{1, -1\}$ such that, for all $\mathbf{i}, \mathbf{j} \in \mathsf{W}^+$,

(1) $(\mathrm{b}_{\mathbf{i}}, \mathrm{b}_{\mathbf{j}}) \in \mathbb{Z}[q^{\epsilon}]$, and

(2) $(b_{\mathbf{i}}, \mathrm{b}_{\mathbf{j}}) = \pi^{\theta}\delta_{\mathbf{ij}} \pmod{q^{\epsilon}}$ for some $\theta \in \{0, 1\}$.

Theorem 7.8. *When* U *is special it admits a pseudo-canonical basis. In types* $A(m, 0)$*,* $A(0, n)$*,* $B(0, n+1)$ *and* $C(n+1)$ *the pseudo-canonical basis is canonical.*

Proof. It has already been explained that U has a pseudo-canonical basis when it is special. By the computations in Section 6 to verify that in types $A(m, 0)$, $A(0, n)$, $B(0, n + 1)$ and $C(n + 1)$ one checks that the PBW basis is almost orthogonal. Hence, the pseudo-canonical basis is canonical. □

Remark 7.9. The constructions in this paper (see Lemma 4.19, and Theorems 5.1, 5.7, and 7.8) work equally well for U_q associated to semisimple Lie algebras, providing a new self-contained approach to the canonical basis in the non-super setting.

Remark 7.10. For type $B(0, n)$, a canonical (sign) π-basis for U_q was constructed in [6] via a crystal basis approach. The canonical basis $\mathbf{B}$ for U_q of type $B(0, n)$ constructed in this paper is an honest basis. We expect that the associated π-basis $\mathbf{B} \cup \pi\mathbf{B}$ will be independent of the orderings and coincides with the one constructed in [6].

Given a (pseudo-)canonical basis $\mathrm{B} = \{\mathrm{b}_{\mathbf{i}}\}_{\mathbf{i} \in \mathsf{W}^+}$, let $\mathrm{B}^* = \{\mathrm{b}^*_{\mathbf{i}}\}_{\mathbf{i} \in \mathsf{W}^+}$ be the dual (pseudo) canonical basis satisfying $(\mathrm{b}^*_{\mathbf{i}}, \mathrm{b}_{\mathbf{j}}) = \delta_{\mathbf{ij}}$. Then, as in [23, Proposition 39, Theorem 40] we have the following.

Theorem 7.11. *The vector* $\mathrm{b}^*_{\mathbf{i}}$ *is characterized by the following two properties*:

(1) $\mathrm{b}^*_{\mathbf{i}} - \mathrm{E}^*_{\mathbf{i}}$ *is a linear combination of vectors* $\mathrm{E}^*_{\mathbf{j}}$, $\mathbf{j} < \mathbf{i}$*, with coefficients in* $q\mathbb{Z}[q]$*;*

(2) *The coefficients of* $\mathrm{b}^*_{\mathbf{i}}$ *in the word basis* W *of* F *are symmetric in* q *and* q^{-1}*.*

In particular,

$$\max(\mathrm{b}^*_{\mathbf{i}}) = \mathbf{i}, \quad \textit{for all } \mathbf{i} \in \mathsf{W}^+,$$

and

$$\mathrm{b}^*_{\mathbf{i}} = \mathrm{E}^*_{\mathbf{i}} \quad \textit{if } \mathbf{i} \in \mathsf{L}^+.$$

8. Canonical bases in the $\mathfrak{gl}(2 \mid 1)$ case

8.1. Canonical basis for $U_q^+(\mathfrak{gl}(2 \mid 1))$. We now compute canonical bases arising from quantum $\mathfrak{gl}(2 \mid 1)$ and its modules. The root datum in this case is given by

$$\underset{1}{\bigcirc}\!\!-\!\!\!-\!\!\!-\!\!\underset{2}{\otimes}, \qquad\qquad \Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}\,.$$

The algebra $U_q = U_q^+(\mathfrak{gl}(2 \mid 1))$ is generated by $\Psi^{-1}(\mathrm{E}_1), \Psi^{-1}(\mathrm{E}_2)$, with $\Psi^{-1}(\mathrm{E}_2)$ odd. Abusing notation slightly, we will identify these elements with E_1 and E_2, respectively. We note that by (5.1),

$$\mathrm{E}_{(12)} := \mathrm{E}_2\mathrm{E}_1 - q\mathrm{E}_1\mathrm{E}_2.$$

Then, since $\mathrm{E}_2^2 = 0$,

$$\begin{aligned}
\mathrm{E}_{(12)}^2 &= 0,\\
\mathrm{E}_2\mathrm{E}_{(12)} &= -q\mathrm{E}_{(12)}\mathrm{E}_2,\\
\mathrm{E}_1\mathrm{E}_{(12)} &= q\mathrm{E}_{(12)}\mathrm{E}_1,\\
\mathrm{E}_{(12)}\mathrm{E}_2 &= \mathrm{E}_2\mathrm{E}_1\mathrm{E}_2.
\end{aligned}$$

Moreover, we can verify that, for $r, s \geq 1$,

$$\mathrm{E}_1^{(r)}\mathrm{E}_2\mathrm{E}_1\mathrm{E}_2 = \mathrm{E}_2\mathrm{E}_1\mathrm{E}_2\mathrm{E}_1^{(r)}, \tag{8.1}$$

$$\mathrm{E}_2\mathrm{E}_1^{(r)} = q^r\mathrm{E}_1^{(r)}\mathrm{E}_2 + \mathrm{E}_1^{(r-1)}\mathrm{E}_{(12)},$$

$$\mathrm{E}_1^{(r)}\mathrm{E}_2\mathrm{E}_1^{(s)} = \begin{bmatrix} r+s-1 \\ s \end{bmatrix}\mathrm{E}_1^{(r+s)}\mathrm{E}_2 + \begin{bmatrix} r+s-1 \\ r \end{bmatrix}\mathrm{E}_2\mathrm{E}_1^{(r+s)}. \tag{8.2}$$

Formula (8.2) is the same as for quantum $\mathfrak{sl}(3)$, see [26]. One checks that

$$\mathrm{E}_2\mathrm{E}_1^{(r+1)}\mathrm{E}_2 = \mathrm{E}_1^{(r)}\mathrm{E}_{(12)}\mathrm{E}_2 = \mathrm{E}_1^{(r)}\mathrm{E}_2\mathrm{E}_1\mathrm{E}_2 = \mathrm{E}_2\mathrm{E}_1\mathrm{E}_2\mathrm{E}_1^{(r)}. \tag{8.3}$$

Now note that the Lyndon words are $2 > 12 > 1$, and so relative to this ordering we see that the PBW basis for U_q is

$$\{\mathrm{E}_1^{(r)}\mathrm{E}_{(12)}^b\mathrm{E}_2^a \mid 0 \leq a, b \leq 1, r \geq 0\}.$$

They span a $\mathbb{Z}[q]$-lattice $\mathcal{L}$ of U_q^+.

The following has appeared in [16], who works with quantum $\mathfrak{gl}(1 \mid 2)$ instead.

Proposition 8.1. $U_q^+(\mathfrak{gl}(2 \mid 1))$ *admits the following canonical basis*:

$$\mathrm{E}_1^{(r)}, \quad \mathrm{E}_1^{(r)}\mathrm{E}_2, \quad \mathrm{E}_2\mathrm{E}_1^{(r+1)}, \quad \mathrm{E}_2\mathrm{E}_1^{(r+1)}\mathrm{E}_2$$

for all $r \geq 0$

Proof. Now the first two elements $\mathrm{E}_1^{(r)}, \mathrm{E}_1^{(r)}\mathrm{E}_2$ are already bar-invariant PBW basis elements, whence pseudo-canonical basis elements. Similarly, the element $\mathrm{E}_2\mathrm{E}_1^{(r+1)}\mathrm{E}_2$ is bar-invariant and also a PBW element by (8.3), whence a pseudo-canonical basis element. One writes the remaining PBW elements as

$$\mathrm{E}_2\mathrm{E}_1^{(r+1)} = q^{r+1}\mathrm{E}_1^{(r+1)}\mathrm{E}_2 + \mathrm{E}_1^{(r)}\mathrm{E}_{(12)}, \quad \text{for } r \geq 0.$$

Hence $\mathrm{E}_2\mathrm{E}_1^{(r+1)}$ is a bar-invariant element, which equals a PBW element modulo $q\mathcal{L}$, whence a pseudo-canonical basis element.

Clearly the elements as in the proposition form a basis of the lattice $\mathcal{L}$, by comparing to the PBW basis, hence this is the promised pseudo-canonical basis. On the other hand, computing the norms of these elements proves that they are actually a canonical basis. □

Remark 8.2. In contrast to Proposition 8.1, $\mathrm{E}_2\mathrm{E}_1\mathrm{E}_2$ is not a canonical basis element for the positive half of quantum $\mathfrak{sl}(3)$.

Remark 8.3. When multiplying any canonical basis element for $U_q^+(\mathfrak{gl}(2 \mid 1))$ with $\mathrm{E}_1^{(s)}$ or E_2 (either on the left or on the right) and then expanding as a linear combination of the canonical basis, the coefficients are always in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$.

Denote by

$$\mathbf{B} = \{\mathrm{F}_1^{(r)},\ \mathrm{F}_2\mathrm{F}_1^{(r)},\ \mathrm{F}_1^{(r+1)}\mathrm{F}_2,\ \mathrm{F}_2\mathrm{F}_1^{(r+1)}\mathrm{F}_2 \mid r \geq 0\}$$

the canonical basis of U_q^-, which consists of the images of the elements in Proposition 8.1 under the anti-isomorphism $U_q^+ \to U_q^-$ defined by $\mathrm{E}_i \mapsto \mathrm{F}_i$. Below we often use the identifications

$$\mathrm{F}_2\mathrm{F}_1^{(r+1)}\mathrm{F}_2 = \mathrm{F}_2\mathrm{F}_{(12)}\mathrm{F}_1^{(r)}$$

and

$$\mathrm{F}_{(12)} = \mathrm{F}_1\mathrm{F}_2 - q\mathrm{F}_2\mathrm{F}_1.$$

8.2. Canonical basis for Kac modules. The subalgebra U_q^0 of U_q is generated by

$$K_1 = q^{e_{11}}, \quad K_2 = q^{e_{22}}, \quad K_3 = q^{e_{33}}.$$

Let $U_q^{2,1}$ be the subalgebra of U_q generated by U_q^0, E_1, and F_1, and let P_q be the subalgebra generated by $U_q^{2,1}$ and E_2. Denote by $\{\delta_1, \delta_2, \varepsilon_1\}$ the dual basis for $\{e_{11}, e_{22}, e_{33}\}$. Let

$$\mu = a\delta_1 + b\delta_2 + c\varepsilon_1,$$

with $a - b \in \mathbb{Z}_{\geq 0}$. Set $L^0(\mu)$ to be the simple $U_q^{2,1}$-module of highest weight μ. Then $L^0(\mu)$ is a P_q-module with trivial E_2-action. The *Kac module*

$$K(\mu) := U_q \otimes_{P_q} L^0(\mu)$$

over U_q is finite dimensional and has a simple quotient $L(\mu)$. Moreover,

$$\dim K(\mu) = 4 \dim L^0(\mu).$$

Denote by v_μ the highest weight vector of $K(\mu)$ and by v_μ^+ the image of v_μ in $L(\mu)$. Note that

$$K(\mu) \cong L^0(\mu) \oplus \mathrm{F}_2 L^0(\mu) \oplus \mathrm{F}_{(12)} L^0(\mu) \oplus \mathrm{F}_2\mathrm{F}_{(12)} L^0(\mu). \tag{8.4}$$

Hence, when applying elements in $\mathbf{B}$ to v_μ, the resulting elements are nonzero exactly when $0 \leq r \leq a - b$, thanks to $\mathrm{F}_1^{(a-b+1)} v_\mu = 0$.

Proposition 8.4. *Let*

$$\mu = a\delta_1 + b\delta_2 + c\varepsilon_1, \quad \textit{with } a - b \in \mathbb{Z}_{\geq 0}.$$

Then

$$\begin{aligned}&\{uv_\mu \mid uv_\mu \neq 0, u \in \mathbf{B}\}\\ &\quad = \{\mathrm{F}_1^{(r)} v_\mu, \mathrm{F}_2\mathrm{F}_1^{(r)} v_\mu, \mathrm{F}_1^{(r+1)}\mathrm{F}_2 v_\mu, \mathrm{F}_2\mathrm{F}_{(12)}\mathrm{F}_1^{(r)} v_\mu \mid 0 \leq r \leq a - b\},\end{aligned}$$

and this set forms a basis of the Kac module $K(\mu)$. It is canonical in the sense that it descends from the canonical basis.

Proof. The equality of the two sets in the proposition follows by the two identities

$$\mathrm{F}_1^{(a-b+1)} v_\mu = 0 \quad \text{and} \quad \mathrm{F}_1^{(a-b+2)}\mathrm{F}_2 v_\mu = 0.$$

Note that $F_1^{(r)} v_\mu^0$ with $0 \leq r \leq a - b$ forms a basis of $L^0(\mu)$. Then by (8.4), the elements

$$\{\mathrm{F}_1^{(r)} v_\mu,\ \mathrm{F}_2\mathrm{F}_1^{(r)} v_\mu,\ \mathrm{F}_{(12)}\mathrm{F}_1^{(r)} v_\mu,\ \mathrm{F}_2\mathrm{F}_{(12)}\mathrm{F}_1^{(r)} v_\mu \mid 0 \leq r \leq a - b\}$$

form a basis of $K(\mu)$. Since the transition matrix from this basis to the set given in the proposition is upper-unitriangular, this set must form a basis of $K(\mu)$. □

8.3. Canonical basis for simple modules. Recall that the Weyl vector for $\mathfrak{gl}(2 \mid 1)$ is

$$\rho = -\delta_2 + \varepsilon_1.$$

A weight λ is called typical if $\langle \alpha, \lambda + \rho \rangle \neq 0$ for all $\alpha \in \Phi_1^+$; otherwise, we say the weight is atypical.

Let $\mu = a\delta_1 + b\delta_2 + c\varepsilon_1$, with $a - b \in \mathbb{Z}_{\geq 0}$. Then μ is typical only if $a \neq -c-1$ and $b \neq -c$. If μ is typical, then $K(\mu)$ is irreducible.

Corollary 8.5. *If μ is typical, then $L(\mu)$ has a canonical basis given by Proposition 8.4.*

Therefore, it remains to consider $L(\mu)$ when μ is atypical. The first step is to determine when canonical basis vectors are zero in $L(\mu)$.

Lemma 8.6. *Assume that*

$$\mu = a\delta_1 + b\delta_2 + c\varepsilon_1, \quad \textit{where } a - b \in \mathbb{Z}_{\geq 0},$$

is atypical; that is,

$$a = -c - 1 \quad \textit{or} \quad b = -c.$$

Then the following statements hold in $L(\mu)$:

(1) $\mathrm{F}_1^{(r)} v_\mu^+ \neq 0 \iff 0 \leq r \leq a - b$;

(2) *if $a = -1 - c$, then*

$$\mathrm{F}_2\mathrm{F}_1^{(r)} v_\mu^+ \neq 0 \iff 0 \leq r \leq a - b;$$

if $b = -c$, then

$$\mathrm{F}_2\mathrm{F}_1^{(r)} v_\mu^+ \neq 0 \iff 1 \leq r \leq a - b;$$

(3) $([r + b + c]\mathrm{F}_1^{(r)}\mathrm{F}_2 - [b + c]\mathrm{F}_2\mathrm{F}_1^{(r)})v_\mu^+ = 0$ *for all $r \geq 0$;*

(4) $\mathrm{F}_2\mathrm{F}_1^{(r+1)}\mathrm{F}_2 v_\mu^+ = \mathrm{F}_2\mathrm{F}_{(12)}\mathrm{F}_1^{(r)} v_\mu^+ = \mathrm{F}_1^{(r)}\mathrm{F}_2\mathrm{F}_1\mathrm{F}_2 v_\mu^+ = 0$ *for all $r \geq 0$;*

(5) $\mathrm{F}_1^{(r+1)}\mathrm{F}_2 v_\mu^+ \neq 0 \iff b \neq -c$ *and* $0 \leq r \leq a - b$.

Proof. We will use repeatedly the fact that a ν-weight vector in $L(\mu)$ with $\nu \neq \mu$ which is annihilated by E_1 and E_2 must be zero.

(1) It follows from the representation theory of $U_q(\mathfrak{sl}_2)$ generated by E_1 and F_1.

(2) By a direct computation we have that

$$E_2F_2F_1^{(r)}v_\mu^+ = [r+b+c]F_1^{(r)}v_\mu^+.$$

If $a = -1-c$, then $r+b+c = 0$ implies that $r = a-b+1$, and so $F_2F_1^{(r)}v_\mu^+ \neq 0$ if $0 \leq r \leq a-b$. Note that $F_2F_1^{(a-b+1)}v_\mu^+ = 0$ since this vector is annihilated by E_1 and E_2 simultaneously.

If $b = -c$, then $r+b+c = 0$ implies that $r = 0$, and so $F_2F_1^{(r)}v_\mu^+ \neq 0$ if $1 \leq r \leq a-b$. Note that $F_2v_\mu^+ = 0$ since $F_2v_\mu^+$ is annihilated by E_1 and E_2 simultaneously.

Hence (2) is proved when we take (1) into account.

(3) This is trivial for $b = -c$, since $F_2v_\mu^+ = 0$. So, we may assume $a = -1-c$. We shall proceed by induction, with the case $r = 0$ being trivial. Set

$$d = b + c.$$

Then (3) follows by the following computations (and by inductive assumption):

$$\begin{aligned} &E_1([d+r]F_1^{(r)}F_2 - [d]F_2F_1^{(r)})v_\mu^+ \\ &= -[d+r]([d+(r-1)]F_1^{(r-1)}F_2 - [d]F_2F_1^{(r-1)})F_2v_\mu^+ \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} &E_2([d+r]F_1^{(r)}F_2 - [d]F_2F_1^{(r)})v_\mu^+ \\ &= [d]([d+r] - [d+r])F_1v_\mu^+ \\ &= 0. \end{aligned}$$

(4) By an F-version of (8.1), we have

$$F_2F_1^{(r+1)}F_2v_\mu^+ = F_2F_{(12)}F_1^{(r)}v_\mu^+ = F_1^{(r)}F_2F_1F_2v_\mu^+.$$

It remains to show that $F_2F_1F_2v_\mu^+ = 0$. This follows from the computations below which use (4) in the second line:

$$\begin{aligned} E_1F_2F_1F_2v_\mu^+ &= F_2E_1F_1F_2v_\mu^+ = 0, \\ E_2F_2F_1F_2v_\mu^+ &= ([b+1+c]F_1F_2 - [b+c]F_2F_1)v_\mu^+ = 0. \end{aligned}$$

(5) Note that $b \neq -c$ if and only if $F_2 v_\mu \neq 0$. As in (1) the claim follows from the representation theory of $U_q(\mathfrak{sl}_2)$ generated by E_1 and F_1 (when applied to the highest weight vector $F_2 v_\mu^+$). □

Theorem 8.7. *Assume that $\mu = a\delta_1 + b\delta_2 + c\varepsilon_1$, where $a - b \in \mathbb{Z}_{\geq 0}$, is atypical; that is, $a = -c - 1$ or $b = -c$.*

(1) *If $b = -c$ or $b = a = -c - 1$, then $\{uv_\mu^+ \mid uv_\mu^+ \neq 0, u \in \mathbf{B}\}$ forms a (canonical) basis of $L(\mu)$. In particular,* $\dim L(\mu) = 2(a - b) + 1$.

(2) *It $b \neq a = -c - 1$, then $\{uv_\mu^+ \mid uv_\mu^+ \neq 0, u \in \mathbf{B}\}$ is linearly dependent in $L(\mu)$, but the subset $\{\mathrm{F}_1^{(r)} v_\mu^+ \ (0 \leq r \leq a-b),\ \mathrm{F}_1^{(r)}\mathrm{F}_2 v_\mu^+ \ (0 \leq r \leq a-b+1)\}$ is a basis for $L(\mu)$. In particular,* $\dim L(\mu) = 2(a - b) + 3$.

Proof. For (1), there are two cases. If $b = -c$, then Lemma 8.6 shows that

$$\{uv_\mu^+ \mid u \in \mathbf{B},\ uv_\mu^+ \neq 0\} = \{\mathrm{F}_1^{(r)} v_\mu^+ (0 \leq r \leq a-b),\ \mathrm{F}_2\mathrm{F}_1^{(r)} v_\mu^+ (1 \leq r \leq a-b)\}.$$

If $b = a = -1 - c$, then Lemma 8.6 implies

$$\{uv_\mu^+ \mid u \in \mathbf{B}, uv_\mu^+ \neq 0\} = \{v_\mu^+,\ \mathrm{F}_2 v_\mu^+,\ \mathrm{F}_1\mathrm{F}_2 v_\mu^+\}.$$

In either case, the set $\{u \in \mathbf{B} \mid uv_\mu^+ \neq 0\}$ spans $L(\mu)$; it is indeed a basis since each vector lies in a different weight space.

For (2), Lemma 8.6 implies that

$$\{uv_\mu^+ \mid u \in \mathbf{B}, uv_\mu^+ \neq 0\} = \{\mathrm{F}_1^{(r)} v_\mu^+, \mathrm{F}_1^{(r+1)}\mathrm{F}_2 v_\mu^+, \mathrm{F}_2\mathrm{F}_1^{(r)} v_\mu^+ \mid 0 \leq r \leq a-b\}.$$

All of these elements lie in different weight spaces except for $\mathrm{F}_1^{(r)}\mathrm{F}_2$ and $\mathrm{F}_2\mathrm{F}_1^{(r)}$ for $0 \leq r \leq a - b$. Now $(\mu - r\alpha_1 - \alpha_2)$-weight space is spanned by $\mathrm{F}_1^{(r)}\mathrm{F}_2 v_\mu^+$ and $\mathrm{F}_2\mathrm{F}_1^{(r)} v_\mu^+$. However, Lemma 8.6(4) shows that these vectors are linearly dependent. Then we may choose one of the vectors as a basis element, and (2) follows. □

We call $L(\mu)$ a *polynomial representation* of U_q if

$$\mu = a\delta_1 + b\delta_2 + c\varepsilon_1$$

with

$$(a, b, \underbrace{1, \ldots, 1}_{c})$$

being a partition (This is analogous to the polynomial representations of the Lie superalgebra $\mathfrak{gl}(m \mid n)$; see [7]). Note that a polynomial representation $L(\mu)$ is atypical if and only if $b = c = 0$. We have the following corollary from Theorem 8.7(1) and Corollary 8.5.

Corollary 8.8. *The set* $\{uv_\mu^+ \mid uv_\mu^+ \neq 0, u \in \mathbf{B}\}$ *forms a canonical basis for every polynomial representation* $L(\mu)$.

In a setting similar to Proposition 8.4, Theorem 8.7(1), Corollarys 8.5 and 8.8, we will simply say that the canonical basis of U_q^- descends to the canonical bases of the corresponding U_q-modules.

We end with formulating some general conjectures regarding canonical basis for representations of quantum supergroup of $\mathfrak{gl}(m+1 \mid 1)$. let U_q^- be the negative half of quantum $\mathfrak{gl}(m+1 \mid 1)$ of type $A(m,0)$, for $m \geq 1$. We transport the canonical basis of the positive half quantum supergroup U_q (see Theorem 7.8) to that for U_q^- via an (anti-)isomorphism sending E_i to F_i for all i.

Conjecture 8.9. *For type* $A(m,0)$, *the canonical basis of* U_q^- *descends to the canonical bases of the Kac modules as well as those of polynomial representations of* U_q.

For type $C(n)$, we also conjecture that the canonical basis of the negative half quantum supergroup descends to the canonical bases of the Kac modules.

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Received November 6, 2013

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