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DOI 10.4171/QT/82 **Quantum Topology**

Families of Legendrian submanifolds via generating families

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Abstract. We investigate families of Legendrian submanifolds in 1-jet spaces by developing and applying a theory of families of generating family homologies. This theory allows us to detect an infinite family of loops of Legendrian n-spheres embedded in the standard contact $\mathbb{R}^{2n+1}$ (for $n > 1$) that are contractible in the smooth, but not Legendrian, category.

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1. Introduction

A central motivating question in contact and symplectic topology is the search for the boundary between flexibility (when contact objects behave like smooth objects) and rigidity (when behavior is more restrictive). This search tends to take the form of distinguishing or classifying contact objects (such as contact structures or Legendrian submanifolds) up to isotopy. Phrased in terms of the space of all such contact objects on a given manifold, investigating isotopy classes can be thought of as trying to understand the set of path components. Flexibility results tend to give information about higher homotopy groups as well as path components: Eliashberg proved, for example, that there is a homotopy equivalence between the space of overtwisted contact structures and the set of smooth 2-plane distributions on a 3-manifold [8], and Gromov proved that there is a homotopy equivalence between the space of Lagrangian immersions $L \to (W, \omega)$ and a space of bundle maps $TL \to TW$; see [12].

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Rigidity results for higher homotopy groups are less common, though examples do exist. Bourgeois uses the cylindrical contact homology invariant to construct non-trivial examples of elements in π_m of the space of contact structures on unit cotangent bundles of negatively curved manifolds [1]; see also [10]. Kálmán uses the Chekanov–Eliashberg DGA invariant to construct a non-trivial example in π_1 of the space of Legendrian knots in standard contact $\mathbb{R}^3$; see [17]. Kálmán's example is especially interesting because his loop of Legendrian knots is contractible as a loop of smooth knots.

In this article, we study the space of Legendrian submanifolds in the 1-jet space J^1M with its canonical contact structure. The template for finding nontrivial elements in higher homotopy groups is the same as that used in the rigidity results above: first, to an object X in the space $\mathcal{X}$, associate some (graded) group $H(X)$ which is an invariant of the path component of $X \in \mathcal{X}$. Next, to an element $\gamma \in \pi_m(\mathcal{X}; X)$, associate an element $\Phi(\gamma) \in \mathrm{End}_{m-1}(H_*(X))$, and attempt to prove that this endomorphism is non-trivial. In contrast to the results above, which use flavors of the holomorphic-curve-based contact homology, we use the generating family homology as our invariant; see [9, 24]. Because generating family homology is based on finite-dimensional Morse theory, the advantage of this choice is two-fold: first, our proofs do not have to deal with the technical analysis of a holomorphic curve theory or the complicated combinatorics of the Chekanov–Eliashberg algebra; and second, families of Morse-theory-based homologies have been elegantly packaged in Hutchings' language of spectral sequences [15].

Suppose the Legendrian $\Lambda \subset J^1M$ has a generating family f with generating family homology $\mathrm{GH}_*(f)$. Let $\mathcal{L}$ denote the space of Legendrian embeddings in J^1M. Let $\pi_m(\mathcal{L}; \Lambda, f)$ denote the subgroup of $\pi_m(\mathcal{L}, \Lambda)$ consisting of (homotopy classes of) m-spheres of Legendrians based at Λ that, up to an equivalence to be defined in Section 2.2, lift to m-spheres of generating families based at f. The main technical application of the families framework developed in this article is the following:

Theorem 1.1. *Under the conditions above, there exists a morphism*

$$\Psi\colon \pi_m(\mathcal{L}; \Lambda, f) \longrightarrow \mathrm{End}_{m-1}(\mathrm{GH}_*(f)).$$

If $m = 1$, the codomain is, in fact, $\mathrm{Aut}(\mathrm{GH}_*(f))$.

The domain of Ψ can be expanded to $\pi_m(\mathcal{L}, \Lambda)$ if we expand its codomain. For example, if $m = 1$, lifts of loops of Legendrians may result in paths of generating families, inducing homomorphisms between the homologies of the generating families at the ends of the path. Thus, the codomain must expand to endomorphisms of the direct sum of the homologies of all possible generating families for Λ.

For the space of Legendrian submanifolds of $\mathbb{R}^{2n+1}$, with $n > 1$, we find that the morphism is nontrivial.

Theorem 1.2. *There exists an infinite family of Legendrian n-spheres in $\mathbb{R}^{2n+1}$ such that for each sphere Λ, there exists an element $\alpha \in \pi_1(\mathcal{L}; \Lambda)$ which is contractible as a smooth loop of spheres but is not contractible in the space of Legendrian submanifolds.*

We remark that recently a similar map has been announced by Bourgeois and Brönnle. Their map counts certain holomorphic curves, and it is unclear if the two maps are related.

In Section 2, we review generating families and generating family homology. In Section 3, we review Hutchings' families framework for families of Morse functions, and adapt it to our set-up of generating families. In Section 4, we prove the main results, finishing by rephrasing Theorem 1.1 in slightly more general terms. In Section 5, we apply the families framework in several ways; for example, to computing generating family homology of higher dimensional Legendrians via a bootstrap argument, as well as to showing how the morphism in Theorem 1.1 factors through front-spinning.

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2. Background notions

In this section, we briefly review the notion of a generating family for a Legendrian submanifold and the (Morse theoretic) generating family homology.

2.1. Spaces of Legendrian submanifolds. Let J^1M denote the $(2n + 1)$-dimensional 1-jet space of a n-dimensional smooth manifold M. We assume that M is closed, or else diffeomorphic to $\mathbb{R}^n$ outside of a compact set. The 1-jet space is equipped with the standard contact structure. Let $\Lambda \subset J^1M$ be an n-dimensional compact Legendrian submanifold. We are interested in the topology of the space of Legendrian submanifolds, which is formed by taking the quotient of the function space of Legendrian embeddings by orientation-preserving self-diffeomorphisms of the domain. The space of submanifolds inherits the quotient topology from the weak C^∞ topology on the function space, as in [14]. Let $\mathcal{L}$ denote this space of submanifolds.

2.2. Generating families for Legendrian submanifolds. Generating families generalize the fact that the 1-jet of a function $f: M \to \mathbb{R}$ is a Legendrian submanifold of J^1M. To see how, begin by considering the trivial fiber bundle $M \times \mathbb{R}^N$ with coordinates (x, η). A function $f: M \times \mathbb{R}^N \to \mathbb{R}$ is a *generating family* if 0 is a regular value of the function $\partial_\eta f: M \times \mathbb{R}^N \to \mathbb{R}^N$. We restrict our attention to generating families that are *linear at infinity*. The condition requires the generating family f to agree with a nonzero linear function $A(\eta)$ outside a compact set in $M \times \mathbb{R}^N$. If f is linear at infinity, then it may be represented as $f = f_0 + A$, where f_0 has compact support and A is linear; the *support* of f is the support of f_0. From here on, we assume that our functions are linear at infinity. Denote by $\mathcal{F}_N$ the set of all linear-at-infinity generating families with fiber dimension N.

A generating family yields a Legendrian submanifold as follows: consider the *fiber critical set*

$$\Sigma_f = \{(x, \eta) \in M \times \mathbb{R}^N : \partial_\eta f(x, \eta) = 0\}.$$

The Legendrian submanifold Λ_f defined by f is then the 1-jet of f along Σ_f:

$$\Lambda_f = \{(x, \partial_x f(x, \eta), f(x, \eta)) : (x, \eta) \in \Sigma_f\}.$$

Said another way, the Cerf diagram for the family of functions f_x parametrized by $x \in M$ is the front diagram for Λ_f.

A given Legendrian submanifold Λ may have many different generating families of fiber dimension N; call that set $\mathcal{F}_N^\Lambda$. We will use the notation $\mathcal{F}^\Lambda$ when we do not wish to specify the fiber dimension and the notation $\mathcal{F}$ when we do not wish to fix the Legendrian submanifold. There are two operations on generating families that preserve the Legendrian submanifold generated.

- Fiber-preserving diffeomorphism. Suppose that a smooth map

  $$\Phi: M \times \mathbb{R}^N \longrightarrow M \times \mathbb{R}^N$$

  has the form $\Phi(x, v) = (x, \phi_x(v))$ for a family of diffeomorphisms ϕ_x. Then $f \circ \Phi$ also generates Λ_f.

- Stabilization. Suppose $Q\colon \mathbb{R}^k \to \mathbb{R}$ is a non-degenerate quadratic function. Then the function

$$f \oplus Q\colon M \times \mathbb{R}^{N+k} \longrightarrow \mathbb{R},$$

defined by

$$(f \oplus Q)(x, \eta_N, \eta_k) = f(x, \eta_N) + Q(\eta_k),$$

also generates Λ_f. Note that if f is linear-at-infinity, then after a fiber-preserving diffeomorphism, so is $f \oplus Q$ [20, Lemma 3.8].

We say that two generating families for a given Legendrian are *equivalent* if they can be made equal by a succession of fiber-preserving diffeomorphisms and stabilizations.

Let $p\colon \mathcal{F} \to \mathcal{L}$ denote the map that sends a generating family f to the Legendrian submanifold Λ_f that it generates. A key fact for this paper is the following

Theorem 2.1 ([16], cf. [23]). *The map $p\colon \mathcal{F} \to \mathcal{L}$ is a Serre fibration up to equivalence. That is, if the smooth map $h\colon \Delta_n \to \mathcal{L}$ has a smooth lift $H\colon \Delta_n \to \mathcal{F}$ and if h_t is a smooth homotopy with $h_0 = h$, then there is a smooth homotopy H_t, with $H_0 = H$ up to equivalence, that lifts h_t.*

More is true. The discussion in [16, Section 3.2] and [23, Section 5] on the persistence of the uniqueness of generating families for certain Legendrians may be viewed as a proof that the homotopy lifting in Theorem 2.1 is unique up to equivalence. To be precise, we have

Proposition 2.2. *If the smooth homotopy $h_t\colon \Delta_n \times [0,1] \to \mathcal{L}$ has smooth lifts $H_t, \bar{H}_t\colon \Delta_n \times [0,1] \to \mathcal{F}$ with $H_0 = \bar{H}_0$, then, possibly after stabilization, there exists a fiber-preserving isotopy Φ_t such that $H_t \circ \Phi_t = \bar{H}_t$ for all $t \in [0,1]$.*

Proof. We combine the work of [16, Section 3.2] and [23, Section 5] with the ideas in the standard proof that a fibration has unique path lifting if every fiber has no non-constant paths (see [22, Section 2.2], for example).

It suffices to show that we can uniquely lift paths, up to equivalence. Let H and $\bar{H}$ be paths in $\mathcal{F}$ (possibly after stabilization) that satisfy $p \circ H = p \circ \bar{H}$ and $H(0) = \bar{H}(0)$. For each $t \in [0,1]$, define a map $\eta\colon [0,1] \times [0,1] \to \mathcal{F}$ by

$$\eta(s,t) = \begin{cases} H((1-2s)t) & s \in [0,1/2], \\ \bar{H}((2s-1)t) & s \in [1/2,1]. \end{cases}$$

Notice that for fixed t, $\eta(s,t)$ is a path from $H(t)$ to $\bar{H}(t)$ whose projection to $\mathcal{L}$ is a loop that is homotopic to the constant loop based at $p \circ H(t) = p \circ \bar{H}(t)$. As t varies, these homotopies fit together into a homotopy $k\colon [0,1]^3 \to \mathcal{L}$ such that $k(s,t,0) = p \circ \eta(s,t)$ and $k(s,t,1) = p \circ H(t)$. Lifting k to a map $K\colon [0,1]^3 \to \mathcal{F}$ (possibly after another equivalence), we obtain a path of paths $K(s,t,1)$ from $H(t)$ to $\bar{H}(t)$, each in a single fiber. As remarked in [16, p. 909], there then exists a path of fiber-preserving diffeomorphisms Φ_t such that $H(t) \circ \Phi_t = \bar{H}(t)$, as required. □

2.3. Generating family homology. Generating families may be used to define a Morse–Floer-type theory for Legendrian submanifolds; see [9, 24] as well as [20]. The first step in the definition of generating family homology is to introduce the *difference function* on the fiber product of the domain of f with itself:

$$\begin{aligned} \delta\colon M \times \mathbb{R}^N \times \mathbb{R}^N &\longrightarrow \mathbb{R}, \\ (x,\eta,\tilde{\eta}) &\longmapsto f(x,\tilde{\eta}) - f(x,\eta). \end{aligned}$$

The critical points of δ with positive critical values correspond to the Reeb chords of Λ_f, and we capture this geometric information with the following definition of *generating family homology* with coefficients in $\mathbb{Z}/2$:

$$\mathrm{GH}_k(f) = H_{N+1+k}(\delta^\omega, \delta^\epsilon; \mathbb{Z}/2),$$

where ω is a number larger than any critical value of δ and where there are no critical values of δ in $(0,\epsilon)$.

Remark 2.3. It is not hard to prove (see [20, §3]) that the groups $\mathrm{GH}_k(f)$ are independent of the choices of ω and ϵ. Nor is it hard to see that the chain complexes underlying the generating family homologies associated to two equivalent generating families are identical.

It is worth noting that 0 is a critical value for δ whose critical points form a Morse–Bott submanifold diffeomorphic to the Legendrian itself. Further, if a generating family f is linear-at-infinity, then, after a fiberwise change of coordinates, so is its difference function δ [9].

The basic invariance property of generating family homology is the following

Theorem 2.4 (Traynor [24]). *If $F\colon [0,1] \times M \times \mathbb{R}^N$ is a 1-parameter family of generating families joining f_0 to f_1 that generate a Legendrian isotopy, then there exists an isomorphism*

$$\Phi_F\colon \mathrm{GH}_k(f_0) \simeq \mathrm{GH}_k(f_1).$$

Combining this theorem with Theorem 2.1, we see that the set of all generating family homologies for a Legendrian submanifold Λ is invariant under Legendrian isotopy.

3. Hutchings' spectral sequence

We review Hutchings' construction in [15, §6] of a spectral sequence for smooth families of Morse functions and submanifolds in the context of generating families. Up to some small modifications, his constructions and results apply to difference functions of generating families. We slightly extend the theory developed in [15, §6] to include parameter spaces that have non-empty boundary.

Our first task is to set notation for the family of difference functions we plan to analyze using Hutchings' scheme. Fix $0 < \epsilon \ll 1$. Let B be a finite-dimensional compact manifold, thought of as a parameter space. Unlike in [15, §6], we allow B to have nonempty boundary. Let $\pi\colon Z \to B$ be a fiber bundle whose fiber over $b \in B$ is $Z_b = M \times \mathbb{R}^N \times \mathbb{R}^N$. Let $\Delta\colon Z \to \mathbb{R}$ be a family of smooth functions whose restrictions δ_b to the fiber Z_b satisfy the following properties.

- Genericity. In the complement of a codimension one subvariety of B, all critical points of δ_b with critical value at least ϵ are non-degenerate.
- Linear-at-infinity. Outside a compact set in $M \times \mathbb{R}^N \times \mathbb{R}^N$, δ_b agrees with a fixed nonzero linear function on $\mathbb{R}^N \times \mathbb{R}^N$.

Let ∇ be a connection on $Z \to B$.

To work with Morse homology in this setting, we need to introduce metrics and gradient flows. We begin by introducing a Morse–Smale pair (K, g^B) on the base space B, requiring the additional property that all critical points of δ_b with positive critical value are non-degenerate for all $b \in \mathrm{Crit}(K)$. If $\partial B \neq \emptyset$, we assume that the component of the negative gradient flow of K with respect to g^B, orthogonal to ∂B, is non-zero and points inward. Let W be the horizontal lift to Z of this negative gradient flow lifted using ∇. Let g^Z denote a fiberwise metric on Z that induces a negative fiberwise gradient flow ξ_b of δ_b with respect to g^Z; let ξ be the vector field over Z whose restriction to Z_b is ξ_b. Finally, we define the vector field

$$V = \xi + W, \tag{3.1}$$

which we will use to define differentials in a spectral sequence. We label this geometric data by the tuple

$$\mathcal{Z} := (Z \longrightarrow B, \Delta, K, V).$$

The zeroes of V are pairs $p = (b, x)$, where $b \in B$ is a critical point of K and $x \in Z_b$ is a critical point of δ_b. We will consider two complementary gradings: the base grading $i(b; K)$ and the fiber grading $i(x; \delta_b)$. The total grading of a zero p of V is $i(p) = i(b; K) + i(x; \delta_b)$.

Hutchings proves in [15, Proposition 3.4 and p. 461] that, generically, the stable and unstable manifolds of the zeroes of V intersect transversally under a slightly different set-up: his fiber Z_b is compact, his base B cannot have boundary, and 0 is not a degenerate critical value. Even so, since Hutchings' proof works by examining one pair of non-degenerate critical points at a time, his proof still applies to pairs of critical points with positive critical value in our set-up, with the linear at infinity condition taking the place of compactness. We say that $\mathcal{Z}$ is *admissible* (over B) if the choices above are sufficiently generic so that the stable and unstable manifolds of zeroes of V are transverse.

To make the intersections of the stable and unstable manifolds easier to work with, we set some additional notation. Fix zeroes p and q of V. Define $\widetilde{\mathcal{M}}(p,q)$ to be the space of flowlines $u \in C^\infty(\mathbb{R}, Z)$ of V, i.e. smooth maps $u\colon \mathbb{R} \to Z$ that satisfy $\frac{d}{dt}u(t) = V(u(t))$, with the property that $\lim_{t\to-\infty} u(t) = p$ and $\lim_{t\to\infty} u(t) = q$. We use this set to define the *moduli space of flowlines*

$$\mathcal{M}(p,q) = \{u \in \widetilde{\mathcal{M}}(p,q)\}/\sim$$

where $u \sim u'$ if $u(t) = u'(t+\tau)$ for some $\tau \in \mathbb{R}$.

Proposition 3.1. *For a generic choice of V, $\mathcal{M}(p,q)$ is a pre-compact manifold of dimension $i(p) - i(q)$. The boundary of the compactification is given by*

$$\partial\overline{\mathcal{M}}(p,q) = \bigsqcup_{r \in \mathrm{Crit}(V)} \mathcal{M}(p,r) \times \mathcal{M}(r,q)$$

Proof. This is a rephrasing of the standard argument in Morse homology. Note that even though the space Z need not be compact, the linear-at-infinity condition on Δ means that V satisfies the Palais–Smale condition as set down in [21, §2.4.2].

If $\partial B \neq \emptyset$, we augment the standard argument as follows. Extend the family to be over a slightly larger open base manifold B' where the fiber Z_b for $b \in B' \setminus B$ is constant in the direction orthogonal to ∂B. Extend the function K to K' such that for a generic metric $g^{B'}$ which extends g^B, the negative gradient flow projected orthogonally to ∂B points towards $\partial B \subset B'$ in any component of $B' \setminus B$. Even though B' is not compact, there are no negative gradient flow lines starting at any critical point that flow into $B' \setminus B$; thus, the usual arguments that show that the moduli spaces are manifolds with corners from Morse theory, applied to B, hold. □

Following Hutchings, the data $\mathcal{Z}$ yield a bigraded chain complex with coefficients in $\mathbb{Z}/2$:

$$\Big(C_{l,m} = C_{l,m}(\mathcal{Z}, \epsilon),\ d = \sum_{n\geq 0} d_n(\mathcal{Z})\Big), \tag{3.2}$$

where the generators are the critical points (b, x) of V with $\delta_b(x) > \epsilon$. The generator (b, x) has bigrading $(i(b; K), i(x; \delta_b))$. The differential

$$d_n \colon C_{l,m} \longrightarrow C_{l-n,m+n-1}$$

counts modulo 2 the number of flow lines of V. Specifically, we define

$$d_n((b, x)) := \sum_{(c,y)\in C_{l-n,m+n-1}} \#\mathcal{M}((b, x), (c, y))(c, y). \tag{3.3}$$

That the map d is a genuine differential follows from Proposition 3.1. We filter the complex $C_\nu := \bigoplus_{l+m=\nu} C_{l,m}$ by the first grading, $F_l C_\nu := \bigoplus_{l'\leq l} C_{l',\nu-l'}$, and let $E^*_{*,*} = E^*_{*,*}(\mathcal{Z}, \epsilon)$ be its associated spectral sequence.

The proof of Theorem 2.4 applies to the current situation, and implies that the fiberwise generating family homologies $\mathrm{GH}_*(f_b)$ can be assembled into a local coefficient system, which we denote by $\mathcal{F}_*(\mathcal{Z})$.

Theorem 3.2. *Consider the admissible family of generating families*

$$\mathcal{Z} = (Z \longrightarrow B, \Delta, K, V).$$

(1) E^2 TERM. *The E^2 term of the spectral sequence is*

$$E^2_{l,m} = H_l(\mathcal{F}_m(Z)).$$

(2) HOMOTOPY INVARIANCE. *If $\mathcal{Z}$ is admissible over $B \times [0, 1]$ with the restrictions $\mathcal{Z}_0 := \mathcal{Z}|_{\{0\}\times B}$ and $\mathcal{Z}_1 := \mathcal{Z}|_{\{1\}\times B}$ also admissible, then there is an isomorphism of spectral sequences*

$$E^*_{*,*}(\mathcal{Z}_0) = E^*_{*,*}(\mathcal{Z}_1).$$

On the E^2 term, this is the isomorphism

$$H_l(\mathcal{F}_j(\mathcal{Z}_0)) \cong H_l(\mathcal{F}_j(\mathcal{Z}_1))$$

induced by the isomorphism of local coefficient systems

$$\mathcal{F}_j(\mathcal{Z}_0) \cong \mathcal{F}_j(\mathcal{Z}_1)$$

defined by Φ in Theorem 2.4.

(3) Naturality. *If $\phi\colon (B', \partial B') \to (B, \partial B)$ is sufficiently generic so that $\phi^*\mathcal{Z}$ is admissible, then the pushforward in homology*

$$\phi_*\colon H_*(B'; \mathcal{F}_*(\phi^*\mathcal{Z})) \longrightarrow H_*(B; \mathcal{F}_*(\mathcal{Z}))$$

extends to a morphism of spectral sequences

$$E^*_{*,*}(\phi^*\mathcal{Z}) = E^*_{*,*}(\mathcal{Z}).$$

(4) Triviality. *If (δ_b, ξ_b) is Morse–Smale for all $b \in B$, then the spectral sequence collapses at the E^2 page.*

Proof. When $\partial B = \emptyset$, the properties stated in the theorem follow with little or no modifications from Hutchings' arguments. In outline, Hutchings first establishes the theorem for spectral sequences defined using singular chains in the base (for *any* base); see Propositions 4.1, 4.3, 4.6 and Remark 1.5 in [15]. Hutchings then extends the isomorphism from singular homology to Morse homology in [15, Section 2.3] to an isomorphism of singular spectral sequences and Morse spectral sequences over closed manifold base spaces in [15, Proposition 6.1].

When $\partial B \neq \emptyset$, we need to supplement the arguments connecting singular and Morse homology. The key idea in the argument is that the descending manifold of a critical point is a manifold with corners [15, equations (2.6) and (2.7)]. That these equations extend to the case of a base manifold with boundary comes from repeating the argument given in the proof of Proposition 3.1. □

Remark 3.3. There are several other properties of Hutchings' spectral sequence that we have not included in the theorem above. The most interesting is a Poincaré duality statement, which holds in our set-up for some cases. In particular, compare [20, Lemma 7.1] with [15, Proposition 7.1]. A more general duality principle for generating family (co)homology is, however, unclear.

4. Algebra of homotopies

In this section, we use the ideas of Section 3 to investigate the homotopy groups of the space of Legendrian submanifolds. In Section 4.1, we discuss how to interpret a family of n-dimensional Legendrians parameterized by the m-manifold B as a single $(m+n)$-dimensional Legendrian. We also discuss relationships to the generating family homology. In Section 4.2, where $B = S^m$ is a (based) m-sphere, we interpret Theorem 3.2 as a morphism from the based homotopy groups of the space of Legendrian embeddings to the space of endomorphisms of generating family homology. In Section 4.3, we study this morphism further to find examples of loops of Legendrian embeddings which are non-contractible as Legendrians submanifolds, but contractible as smooth submanifolds. In Section 4.4, we construct a more general morphism from the free homotopy classes of $\mathcal{L}$.

4.1. Tracing families of Legendrian submanifolds. We begin by rephrasing the main concept of Section 3 in the language of Legendrian submanifolds and generating family homology.

Let $\{\Lambda_b \subset J^1 M : b \in B\}$ be a smooth family of n-dimensional Legendrian submanifolds parameterized by a compact manifold B, possibly with boundary. Assume there is a family $F: B \times M \times \mathbb{R}^N \to \mathbb{R}$ of generating families. Define

$$\Delta: B \times M \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$$

by

$$\Delta(b, m, \eta, \tilde{\eta}) = F(b, m, \eta) - F(b, m, \tilde{\eta}). \tag{4.1}$$

Here and later in the article, we denote by f_b and δ_b the restrictions of F and Δ to the fiber over $b \in B$. Let $\Lambda \subset J^1(B \times M)$ be the $(n + \dim(B))$-dimensional Legendrian **trace**; that is, the front of Λ over the point b is the front of Λ_b. As in Section 3, let $K: B \to \mathbb{R}$ be a generic function on the base, let V be the vector field from equation (3.1), and let $\mathcal{Z} = (Z = B \times M \times \mathbb{R}^N \times \mathbb{R}^N, \Delta, K, V)$.

Lemma 4.1. *The function F is a generating family for Λ. If K is a sufficiently C^2-small Morse function and $\mathcal{Z}$ is admissible, then*

$$\mathrm{GH}_k(F) = \bigoplus_{i+j=k+N+1} E^{\infty}_{i,j}(\mathcal{Z}).$$

Proof. This result is straightforward after making two observations. First, in local coordinates, the differential of the fiber derivative of F at (b, m, η) contains the differential of the derivative of f_b as a full-rank submatrix. Thus, F also satisfies the transversality condition for generating families. Second, the quasi-isomorphism type (which determines its homology) of $CM_*((\Delta+K)^{\omega}, (\Delta+K)^{\epsilon})$ is independent of the choice of generic K, assuming K is C^2-small, and hence perturbing by K does not change the topology of the level ϵ sublevel set. $\square$

We next consider two examples. The first will be used in Sections 4.2 and 4.3, while the second appears in Section 4.4.

Example 4.2 (based m-sphere). Let $\Lambda \subset J^1M$ be an n-dimensional Legendrian submanifold with generating family f. Let ρ be a smooth S^m-family of Legendrian submanifolds with the properties that $[\rho] \in \pi_m(\mathcal{L}; \Lambda, f)$ and that for a small contractible neighborhood U of $b \in S^m$, we have $\rho(U) = \Lambda$. Construct a Morse function $K\colon S^m \to \mathbb{R}$ that has two critical points, a maximum at $a \in U$ and a minimum at b. Assume that $\|K\|_{C^2} < \epsilon$ as in Lemma 4.1. Let Λ^ρ be the trace of this m-isotopy and define a generating family F and difference function Δ for Λ^ρ as in equation (4.1). Perturb V if necessary so that

$$\mathcal{Z} = (Z \longrightarrow S^m, \Delta, K, V)$$

is an admissible family.

Example 4.3 (based homotopy). Let $\Lambda \subset J^1(M)$ be an n-dimensional Legendrian submanifold with generating family f. Let $\tilde{\rho}\colon [0,1]^m \to \mathcal{L}$ be a smooth $[0,1]^m$-family of Legendrian submanifolds such that $\rho(0,\dots,0) = \Lambda$. Extend $\tilde{\rho}$ to $\rho\colon I^m := [-1,1]^m \to \mathcal{L}$ by smoothing the function

$$\rho(b_1,\dots,b_m) = \tilde{\rho}(\max(b_1,0),\dots,\max(b_m,0)).$$

Assume that $\tilde{\rho}|_{\partial[0,1]^{m-1}\times b_m}$ is independent of b_m. Define the Morse function on the base to be

$$K\colon I^m \to \mathbb{R}, \quad K(b_1,\dots,b_m) = \sigma \sum_{i=1}^{m} (b_i+1)^2(b_i-1)^2, \tag{4.2}$$

where $0 < \sigma \ll \epsilon \ll 1$. Note that for any metric, the negative gradient of K projects to the outward normal direction on ∂I^m.

Let Λ be the trace of this m-isotopy and define a generating family F and its difference function Δ as in equation (4.1). Perturb V if necessary such that

$$\mathcal{Z} = (Z \longrightarrow I^m, \Delta, K, V)$$

is an admissible family.

4.2. From homotopy groups of the space of Legendrians to generating family homology. We revisit the map $\rho\colon S^m \to \mathcal{L}$ from Example 4.2, using it to relate the homotopy groups of $\mathcal{L}$ to morphisms of generating family homology.

If f is a generating family for the basepoint Λ, then we will construct a morphism

$$\Psi\colon \pi_m(\mathcal{L}; \Lambda, f) \longrightarrow \mathrm{End}_{m-1}(\mathrm{GH}_*(f)).$$

Recall from Remark 2.3 that "up to equivalence" suffices to completely specify the generating family chain complex over the basepoint Λ. For this reason, we will stop making remarks about our lifts to the space of generating families being well-defined only up to equivalence.

To define the map Ψ, construct the generating family F as in Example 4.2. Lemma 4.1 implies that the differential of the generating family chain complex $GC_*(F)$ in degree l can be written as $d = \sum_{k=0}^{l+1} d_k(\mathcal{Z})$, as in equations (3.2) and (3.3). For an element $c \in \mathrm{Crit}(K)$, and a generator $(e, p) \in GC_*(F)$, define $\langle (e, p), c\rangle$ to be $p \in GC_*(f_c)$ if $e = c$ and 0 otherwise. (Recall that $f_c = F|_c$.) Extend this pairing bilinearly.

Finally, define a map $\psi_\rho\colon GC_*(f) \to GC_{*+m-1}(f)$ by

$$\psi_\rho(x) = \begin{cases} \langle d_m(a, x), b\rangle + x, & m = 1, \\ \langle d_m(a, x), b\rangle, & m > 1. \end{cases} \tag{4.3}$$

We can now restate (and prove) Theorem 1.1 in more detail.

Proposition 4.4. *The map ψ_ρ defined above has the following properties.*

(1) *The map induces a homomorphism*

$$\Psi_\rho\colon \mathrm{GH}_*(f) \longrightarrow \mathrm{GH}_{*+m-1}(f).$$

(2) *If ρ and ρ' are homotopic through maps that send $U \subset S^m$ to Λ, then $\Psi_\rho = \Psi_{\rho'}$. In particular, given $[\rho] \in \pi_m(\mathcal{L}; \Lambda, f)$, we may refer to the map $\Psi_{[\rho]}$.*

(3) *The map $\rho \mapsto \Psi_\rho$ induces, for $m > 1$, a morphism from $\pi_m(\mathcal{L}; \Lambda, f)$ to $\mathrm{End}_{m-1}(\mathrm{GH}_*(f))$ or, for $m = 1$, from $\pi_1(\mathcal{L}; \Lambda, f)$ to $\mathrm{Aut}(\mathrm{GH}_*(f))$. In particular, we have*

$$\Psi_{[\rho][\sigma]} = \Psi_{[\rho]}\Psi_{[\sigma]} \qquad \text{if } m = 1,$$
$$\Psi_{[\rho]+[\sigma]} = \Psi_{[\rho]} + \Psi_{[\sigma]} \quad \text{if } m > 1.$$

For the $m = 1$ case, the equation above and the fact that $\Psi_{[Id]}$ is the identity imply that $\Psi_{[\rho]}$ is invertible.

Proof. The general principle of this proof is outlined in [15]. For the convenience of the reader, we present some of the details here when considering generating families.

To prove the first property, note that $d^2(c,x)=0$ if and only if $\langle d^2(c,x),e\rangle=0$ for all $e\in \mathrm{Crit}(K)$. Since the base function K has critical points of index 0 and m only, we see that $d_k=0$ unless $k=0,m$. In particular, for all $x\in\mathrm{Crit}(\delta_a)$, we have

$$0=\langle d^2(a,x),b\rangle=\langle (d_0d_m+d_md_0)(a,x),b\rangle.$$

Thus, ψ_ρ is a chain map and induces a map

$$\Psi_\rho\colon \mathrm{GH}_*(f_a)\longrightarrow \mathrm{GH}_{*+m-1}(f_b).$$

Next, we take two homotopic maps $\rho,\rho'\colon S^m\to\mathcal{L}$ which lift to S^m-families of generating families with admissible data $\mathcal{Z}$ and $\mathcal{Z}'$, respectively. Combining Examples 4.2 and 4.3 and Proposition 2.2, we construct an admissible $\mathcal{Z}[-1,1]$ over $I\times S^m=[-1,1]\times S^m$ such that $\mathcal{Z}|_{-1}=\mathcal{Z}=\mathcal{Z}|_0$ and $\mathcal{Z}|_1=\mathcal{Z}'$. We then apply Lemma 4.1 to define $d=d(\mathcal{Z}[-1,1])$. Denote by F^I and Δ^I the generating family and difference function of the trace of this homotopy. There are six critical points of the difference function over $I\times S^m$, which we denote by (n,c) where $n\in\{-1,0,1\}$ and $c\in\{a,b\}$. Since the base indices lie in the set $\{0,1,m,m+1\}$, the equation $d^2=0$ now implies

$$0=\langle (d_0d_{m+1}+d_{m+1}d_0+d_1d_m+d_md_1)((0,a),x),(1,b)\rangle. \tag{4.4}$$

Since we are working with a *based* homotopy between ρ and ρ', Proposition 2.2 implies the map d_1 corresponds to the identity map; in particular, we have

$$d_1((c,0),x)=((c,1),x)+((c,-1),x)$$

for $c\in\{a,b\}$ and $x\in\mathrm{Crit}(\delta^I_{(c,0)})=\mathrm{Crit}(\delta^I_{(c,\pm1)})$. Thus, equation (4.4) indicates that the map

$$H\colon GC_*(f^I_{(a,0)})\longrightarrow GC_{*+m-2}(f^I_{(b,1)})$$

defined by

$$H(x)=\langle d_{m+1}((a,0),x),(b,1)\rangle,$$

is a chain homotopy between ψ_ρ and $\psi_{\rho'}$.

The proof of the third statement for $m\geq 2$ essentially appears in [15, Example 1.9], as Hutchings' proof relies on a based homotopy similar to the one we just explicitly constructed.

For $m=1$, we are unaware how to apply Theorem 3.2 to prove that $\Psi_{[\rho][\rho']}=\Psi_{[\rho]}\Psi_{[\rho']}$. Instead, this follows from the traditional "broken-curves" argument of the well-studied continuation methods in Morse–Floer theory. □

4.3. A constructive proof of Theorem 1.2. In this section, we prove Theorem 1.2, namely that for every $n > 1$, there is an infinite family of Legendrian submanifolds, $\Lambda^{n,r} \subset \mathbb{R}^{2n+1}$ parametrized by sufficiently large $r \in \mathbb{N}$ so that $\pi_1(\mathcal{L}^n; \Lambda^{n,r})$ is non-trivial. Further, the non-trivial homotopy classes we produce in $\pi_1(\mathcal{L}^n; \Lambda^{n,r})$ are trivial in the smooth category.

We begin by constructing $\Lambda^{n,r}$. Consider the Legendrian link in $\mathbb{R}^3$ whose front projection appears in Figure 1. This link, which is isotopic to the Hopf link, has a generating family $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ with a difference in index of $r + 2$ between the critical points generating the top strand of the top component and the bottom strand of the bottom component. Spin the front about its central axis into $\mathbb{R}^{n+1}$ as in [11] to get two Legendrian spheres. Then perform a 0-surgery along the horizontal dotted 1-disk in Figure 1 to get a connected Legendrian sphere $\widetilde{\Lambda}^{n,r}$. That the spinning and surgery constructions yield Legendrian surfaces with generating families is a simple generalization of facts proven in [2].

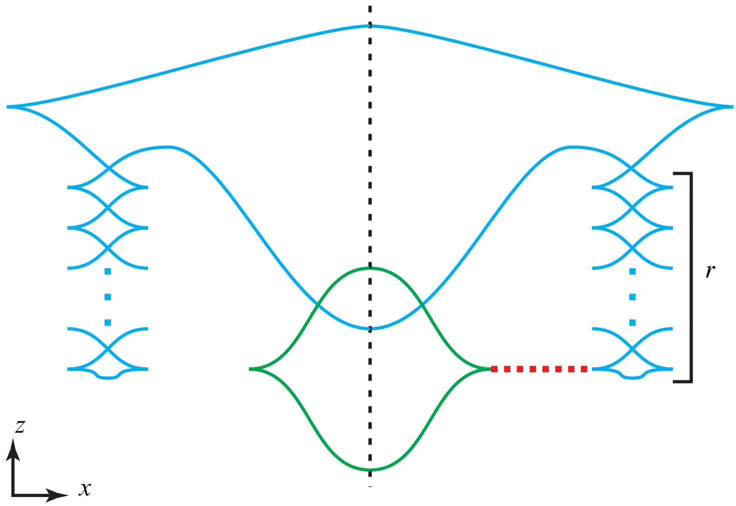

Figure 1. By spinning this front around the central z axis and then performing a 0-surgery along the dotted horizontal disk, we obtain the Legendrian surface $\widetilde{\Lambda}^{2,r}$.

To construct $\Lambda^{n,r}$ itself, we take two copies of $\widetilde{\Lambda}^{n,r}$, positioned sufficiently far apart along the x_1 axis so that the pair can be generated by a single generating family that is equal to a linear function in η in a neighborhood of the hyperplane $x_1 = 0$; see [20, §3.3]. Finally, perform another 0-surgery to connect the two copies along their topmost cusps; once again, the result has a generating family which we will call $f^{n,r}$. It is important that the three 0-surgeries performed thus far line up as in Figure 2.

To compute the generating family homology of $\Lambda^{n,r}$, we first assume that $r \geq n+2$. We begin by computing the generating family homology of the Hopf link. The Hopf link has $\binom{4}{2} = 6$ Reeb chords, which correspond to generators of degrees

$$r, \quad n+1-r, \quad n, \quad n, \quad n+r, \quad n+r+1.$$

The computation of degrees comes from the fact that the index of a critical point of the difference function may be computed to be the sum of N, the index difference of the generating family between the top and bottom strands, and the index of a function that measures the distance in z heights between the top and bottom strands; see the proof of Proposition 3.2 in [20]. The fact that $r \geq n+2$ implies that the generators in degrees $r, n+1-r, n, n$ must survive in homology. On the other hand, the Duality Exact Sequence of [20] and the fact that we are working over a field tells us that $\dim \mathrm{GH}_{n+r}(f) = \dim \mathrm{GH}_{-r-1}(f) = 0$; a similar argument, or a simple consequence of the definition of a chain complex, shows that $\dim \mathrm{GH}_{n+r+1}(f) = 0$ as well. Thus, we obtain:

$$\mathrm{GH}_m(f) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & m = n, \\ \mathbb{Z}/2 & m = r,\ n+1-r, \\ 0 & \text{otherwise.} \end{cases}$$

The generating family homologies for $\tilde{f}^{n,r}$ and $f^{n,r}$ may then be computed using the Cobordism Exact Sequence [20, Theorem 1.1], which states that when there exists a generating family-compatible Lagrangian cobordism[1] L between (Λ_-, f_-) and (Λ_+, f_+), then there exists a long exact sequence

$$\cdots \longrightarrow H_{k+1}(L, \Lambda_+) \longrightarrow \mathrm{GH}_k(f_+) \longrightarrow \mathrm{GH}_k(f_-) \longrightarrow \cdots. \tag{4.5}$$

Since $H_{k+1}(L, \Lambda_+)$ is nonzero only for $k+1 = n$ when attaching a 1-handle, we see that

$$\mathrm{GH}_m(\tilde{f}^{n,r}) = \begin{cases} \mathbb{Z}/2 & m = n,\ r,\ n+1-r, \\ 0 & \text{otherwise.} \end{cases}$$

Taking two copies of $\widetilde{\Lambda}^{n,r}$ in the construction yields the direct sum of two copies of the groups above for the generating family homology. Applying the

[1] A precise definition of "generating family-compatible Lagrangian cobordism" is not necessary for this paper; we need only note that the surgery constructions in [2] produce such objects.

Cobordism Exact Sequence again to the last 0-surgery gives us

$$\mathrm{GH}_m(f^{n,r}) = \begin{cases} \mathbb{Z}/2 & m = n, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & m = r,\ n+1-r, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see from the computation that the group $\mathrm{GH}_r(f^{n,r})$ is generated by two chains β_L and β_R, each of which is arises from a sum of critical points that lie in exactly one of the copies of $\widetilde{\Lambda}^{n,r}$.

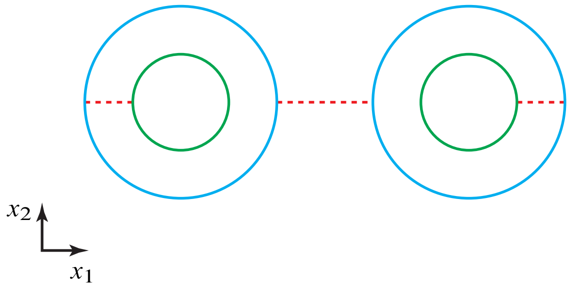

Figure 2. The three 0-surgeries in the construction of $\Lambda^{2,r}$ must line up as in the figure.

With the Legendrian spheres $\Lambda^{n,r}$ in hand, we proceed to construct a non-contractible loop in $\mathcal{L}$ based at $\Lambda^{n,r}$. The idea is to effect a rotation by π in the first two coordinates of the base manifold $\mathbb{R}^n$, which yields a loop in $\mathcal{L}$ because of the symmetry of $\Lambda^{n,r}$. To be more precise, fix $\tau \ll 1$ and choose a smooth function $\sigma\colon [0, 2\pi] \to [0, \pi]$ with the properties that σ is non-decreasing, $\sigma^{-1}\{0\} = [0, \tau]$, and $\sigma^{-1}\{\pi\} = [\pi - \tau, 2\pi]$. Define a path $\rho\colon [0, 2\pi] \to \mathrm{SO}(n)$ of rotations of the base $\mathbb{R}^n$ to be the identity except for the following elements of SO(2) in the upper left corner:

$$\begin{bmatrix} \cos\sigma(s) & \sin\sigma(s) \\ -\sin\sigma(s) & \cos\sigma(s) \end{bmatrix}.$$

Finally, let $f_s = f^{n,r} \circ \rho^{-1}(s)$, where we have implicitly extended ρ to be the identity on the fiber component. The symmetry of the function $f^{n,r}$ implies that this is actually a smooth family of generating families over the base S^1 even though ρ does not descend to a smooth function on S^1. In particular, we obtain a smooth loop $\hat{\rho}$ of Legendrian spheres with a loop of generating families, i.e., an element in $\pi_1(\mathcal{L}; \Lambda^{n,r}, f^{n,r})$.

To place the construction above in the families context, consider the following geometric data $\mathcal{Z}$: $Z = S^1 \times \mathbb{R}^{n+2N}$ is the trivial bundle over S^1; δ_s is the fiber-wise difference function; K is the base function as constructed in Example 4.2

with maximum at 0 and minimum at π; W is the lift of $-\nabla K$ to Z via the trivial connection; ξ_s is the negative fiber-wise gradient flow of δ_s; ξ is the vector field whose restriction to Z_s is ξ_s; and $V = W + \xi$.

Proposition 4.5. *The loop $\hat{\rho}$ based at $\Lambda^{n,r}$ is not contractible in $\mathcal{L}^n$.*

Proof. It suffices to show that $\Psi_{\hat{\rho}}$ is not the identity.

Let ξ_0 be the fiber-wise gradient of δ_0. In equation (4.6) below, $a \cdot b$ denotes matrix multiplication of a and b, ab denotes scalar multiplication where a is a scalar, and a^T denotes the transpose of a. We abuse notation in the equation, treating $W(s)$ first as a vector field in $T(S^1 \times \mathbb{R}^{n+2N})$ and then as a scalar. Define

$$\begin{aligned}\bar{V}(s, x, \eta, \tilde{\eta}) = W(s) + W(s)\rho'(s) \cdot (\rho^{-1}(s) \cdot (x, \eta, \tilde{\eta})^T) \\ + \rho(s) \cdot \xi_0(\rho^{-1}(s) \cdot (x, \eta, \tilde{\eta})^T).\end{aligned} \tag{4.6}$$

The definition of σ in the construction of ρ, above, implies that $\bar{V}$ and V agree in an open neighborhood of their (identical) sets of critical points. Note that if we replace V with $\bar{V}$ in the computation of the differential, we still get a bigraded complex $(C_{l,m}, d^{\bar{V}})$ where the $C_{l,m}$ is same one as yielded by the data $\mathcal{Z}$ above. (Review equation (3.2) and its accompanying discussion for a description of the bigrading.) In particular, we can define a $d_1^{\bar{V}}$-map as introduced in equation (3.3).

We now construct a filtered chain homotopy equivalence between the two bigraded complexes using standard continuation methods, following the outline in [15, §6]. Let $\beta\colon [0, 1] \to \mathbb{R}$ be a smooth function such that $\beta(t) \geq 0$, $\beta^{-1}(0) = \{0, 1\}$, $\beta'(0) > 0$ and $\beta'(1) < 0$. Let V_t be a smooth (in t) family of vector fields on $S^1 \times \mathbb{R}^{n+2N}$ such that $V_0 = V$, $V_1 = \bar{V}$, and V_t is independent of t in a neighborhood of $\{0, \pi\} \times \mathbb{R}^{n+2N}$. Define a vector field $\mathcal{V}$ on $S^1 \times [0, 1]_t \times \mathbb{R}^{n+2N}$ by $\mathcal{V} = -\beta(t)\partial_t + V_t$. It is not hard to see that $\mathcal{V}$ then determines the desired chain map; the chain map is filtered since its projection to the base S^1 is parallel to $-\nabla K$.

The existence of $\mathcal{V}$ implies that $d_1^{\bar{V}} = d_1$ where d_1 is determined by V and determines $\Psi_{\hat{\rho}}$. The vector field $\bar{V}$ constructed in equation (4.6) is designed so that a flow line $\gamma(t) = (\gamma_S(t), \gamma_{\mathbb{R}}(t)) \in S^1 \times \mathbb{R}^{n+2N}$ has the following properties:

(1) the component $\gamma_S(t)$ satisfies the decoupled one-dimensional equation $\gamma_S'(t) = W(\gamma_S(t))$;

(2) the component $\gamma_{\mathbb{R}}(t)$ is of the form $\gamma_{\mathbb{R}}(t) = \rho(\gamma_S(t))\zeta(t)$ for some flow line $\zeta(t)$ of the vector field ξ_0.

To see this how this second statement follows from equation (4.6), note that

$$\begin{aligned}\gamma_{\mathbb{R}}'(t) &= \gamma_S'(t)\rho'(\gamma_S(t)) \cdot \zeta(t) + \rho(\gamma_S(t)) \cdot \zeta'(t)\\ &= \gamma_S'(t)\rho'(\gamma_S(t)) \cdot (\rho^{-1}(\gamma_S(t)) \cdot \gamma_{\mathbb{R}}(t))\\ &\qquad + \rho(\gamma_S(t)) \cdot \xi_0 \left(\rho^{-1}(\gamma_S(t)) \cdot \gamma_{\mathbb{R}}(t)\right).\end{aligned}$$

So we see that $(\gamma_S'(t), \gamma_{\mathbb{R}}'(t)) = \bar{V}(\gamma_S(t), \gamma_{\mathbb{R}}(t))$.

Condition (1) and the gradings of the generators on $\mathrm{GH}_r(f^{n,r})$ imply that the rigid flow lines that compute the map $\Psi_{\hat{\rho}}$ on $\mathrm{GH}_*(f^{n,r})$ send a class of $\mathrm{GH}_*(f^{n,r})$ represented by critical points with $x_1 < 0$ to the symmetric class represented by critical points with $x_1 > 0$. Condition (2) implies that the component $\gamma_{\mathbb{R}}(t)$ is constant, hence there is a unique flow between this class at $x_1 < 0$ and its symmetric one at $x_1 > 0$. By construction, this map is not the identity in degree r, and hence the loop $\hat{\rho}$ is not contractible. □

While the loop $\hat{\rho}$ is non-trivial in $\pi_1(\mathcal{L}; \Lambda^{n,r})$, it is smoothly trivial. More precisely, we have the following

Proposition 4.6. *The loop $\hat{\rho}$ is null-homotopic in the space of smooth embedded n-spheres in $\mathbb{R}^{2n+1}$.*

Proof. The null-homotopy is constructed in two stages. First, note that the space of long n-knots in $\mathbb{R}^{2n+1}$ is connected for $n > 1$ [3]. Further, as noted in [3, Definition 1], the space of long n-knots in $\mathbb{R}^{2n+1}$ is homotopy equivalent to the space of embeddings of D^n into D^{2n+1} that agree with a fixed linear function on the boundary. Thus, there is a smooth isotopy of the left lobe of $\Lambda^{n,r}$ that satisfies the following:

(1) it fixes the attaching region of the 0-surgery joining the left to the right lobes;

(2) it is supported in the left half-space of $\mathbb{R}^{2n+1}$; and

(3) it takes the left lobe to a flying saucer.

Performing this isotopy on the left lobe and its rotation on the right, we obtain a smooth isotopy H that takes $\Lambda^{n,r}$ down to a flying saucer; note that this isotopy is symmetric about the z axis.

We are now ready for the first stage of the homotopy $\Theta\colon [0,2] \to \mathcal{L}^2$ that connects ρ to the identity. We work entirely with the front diagram. At time $t = 0$, we simply take Θ to be ρ. As t increases to 1, for each fixed t, we perform $H(x, 3s)$ to gradually transform $\Lambda^{2,r}$ into the flying saucer over $s \in [0, \frac{t}{3}]$, then rotate the result by π, and then perform the reverse homotopy $H(x, 3(1-s))$ for $s \in [1-\frac{t}{3}, 1]$. See Figure 3 for a schematic picture of this construction. At $t = 1$, the loop ρ has been transformed into a loop that starts by doing H over $[0, \frac{1}{3}]$, then fixes the flying saucer over $[\frac{1}{3}, \frac{2}{3}]$, and then undoes H over $[\frac{2}{3}, 1]$. This loop is clearly null-homotopic, and we append this null homotopy to the homotopy constructed above. □

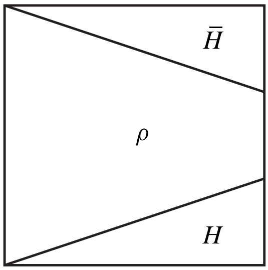

Figure 3. A schematic picture of the first part of the homotopy between ρ and the constant loop in $\mathcal{L}^n$.

Propositions 4.5 and 4.6 together imply Theorem 1.2.

The proof above shows that the element $[\hat{\rho}] \in \pi_1(\mathcal{L}; \Lambda^{n,r})$ has order at least two. We can modify the construction to produce elements $\hat{\rho}_m \in \pi_1(\mathcal{L}; \Lambda^{n,r})$ that have order at least m for any $m > 1$. Instead of connecting two copies of $\widetilde{\Lambda}^{2,r}$ with a 0-surgery, we begin with a central flying saucer centered on the z axis. We then take m copies of $\widetilde{\Lambda}^{n,r}$, arrayed as in Figure 4, and let $\rho^{m,r}$ be a rotation about the z axis by $\frac{2\pi}{m}$.

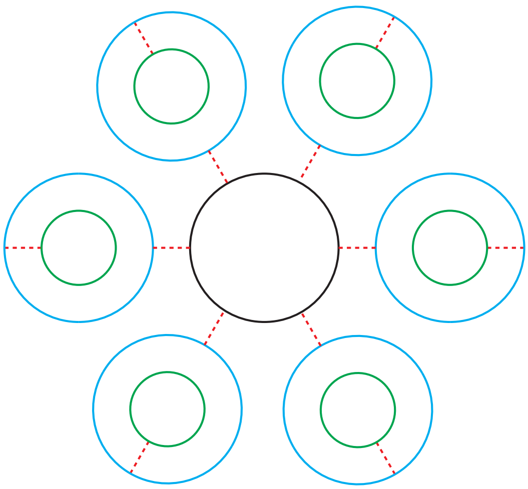

Figure 4. The fundamental group of $\mathcal{L}^2$ based at this surface has an element of order at least 6.

The computations of the generating family homology have the same form as those for $\Lambda^{n,r}$, and a slight generalization of the proof of Proposition 4.5 shows that all powers $\rho^{m,r}, (\rho^{m,r})^2, \dots, (\rho^{m,r})^{m-1}$ are nontrivial maps. In fact, this argument proves the following

Proposition 4.7. *For any subgroup $G < \mathrm{SO}(n)$ that acts transitively and without fixed points on a finite set $S \subset S^{n-1}$, there exists an n-dimensional Legendrian submanifold $\Lambda_G \subset \mathbb{R}^{2n+1}$ such that there is an injection $G \hookrightarrow \pi_1(\mathcal{L}; \Lambda_G)$.*

4.4. Free homotopies. One can also consider relative versions of the map Ψ: instead of m-spheres of Legendrians up to basepoint-preserving homotopy, consider m-cubes of Legendrians up to homotopy relative to their boundary. One way to algebraically package this, before passing to homology, is as a *fundamental ∞-groupoid*, which we sketch below. This groupoid is an example of a so-called *$(\infty, 0)$-category*. Essentially, an $(\infty, 0)$-category is a category with objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, etc. The "$(\cdot, 0)$"-label indicates that all k-morphisms for $k > 0$ have homotopy inverses. The "$(\infty, \cdot)$"-label indicates that operations and relations, such as the composition of two composable 1-morphisms and associativity of composition, only hold up to "homotopy." For a rigorous definition of an $(\infty, 0)$-category in terms of Kan complexes and simplicial sets, see [19, Remark 1.1.2.3 and Example 1.1.2.5]

Example 4.8. As mentioned, an example of an $(\infty, 0)$-category is $\pi_{\leq\infty}(X)$, the fundamental ∞-groupoid of a topological space X. The objects of $\pi_{\leq\infty}(X)$ are the points in X. The 1-morphisms $\mathrm{Mor}_1(x, y)$ are the (possibly empty set of) paths from x to y. Composition of composable 1-morphisms is concatenation of paths. Note that we are unconcerned with how to parameterize the composite path since all choices are homotopic. This leads to the 2-morphisms $\mathrm{Mor}_2(\alpha, \beta)$ between paths α, β which start and end at $x, y \in X$: they are the based homotopies connecting α, β. Note that all (≥ 1)-morphisms have homotopy inverses.

Example 4.9. We define another $(\infty, 0)$-category, $\mathcal{GH}(\mathcal{L})$, based on the generating family chain complexes of points in $\mathcal{L}$. The objects are $GC_*(\mathcal{Z}) := GC_*(f)$ with differentials $d = d(\mathcal{Z})$. Note if $GC_*(\mathcal{Z}) = GC_*(\mathcal{Z}')$, but the Legendrians that f and f' generate are not the same, the chain complexes are considered the same object in this category. Given a Legendrian isotopy Λ_b, $-1 \leq b \leq 1$ which is constant for $-1 \leq b \leq 0$, let $\mathcal{Z}$ be the admissible family associated to the trace Λ. (See Section 4.2.) Define a 1-morphisms

$$\alpha = \alpha(\mathcal{Z}) \in \mathrm{Mor}_1(GC_*(f_{-1}), GC_*(f_1)), \quad \alpha(x) := \langle d_1(\mathbf{0}, x), \mathbf{1}\rangle.$$

(using the notation of the proof of Proposition 4.4). Note that when defining $\mathrm{Mor}_1(GC_*(\mathcal{Z}), GC_*(\mathcal{Z}'))$, we are considering all families $\mathcal{Z}[-1,1]$ between *all* pairs $\mathcal{Z}$ and $\mathcal{Z}'$ (as in the proof of Proposition 4.4) such that $GC_*(\mathcal{Z}) = GC_*$ and $GC_*(\mathcal{Z}') = GC'_*$. We continue in this manner, defining the 2-morphisms with the d_2-map, et cetera.

Proposition 4.10. *There is a functor from $\pi_{\leq\infty}(\mathcal{L}_n(J^1M))$ to $\mathcal{GH}(\mathcal{L}_n(J^1M))$.*

Proof. The proposition follows from almost identical arguments to the proof of Proposition 4.4. □

5. Further applications

In this section, we examine several explicit constructions of families of Legendrian submanifolds with generating families, teasing out the implications of the families machinery of Section 3 for each construction.

5.1. Product families. Suppose that $\Lambda \subset J^1M$ is a Legendrian submanifold with generating family f. Given a closed manifold B, we form the *product family* $\Lambda \times B \subset J^1(M \times B)$ simply by taking the generating family F with fiber $f_b = f$. This construction, together with a choice of a C^2-small Morse function K on B and a metric g on $M \times \mathbb{R}^N$, induces a family $(Z \to B, \Delta, K, V)$. We then use Theorem 3.2 to compute the generating family homology of the constant family F on the total space $\Lambda \times B$ using a Künneth-type formula. Note that our techniques are not necessary to make this computation, as one can apply the (topological) Künneth theorem directly to the sublevel sets of the difference function, but the following proposition is a good first application of Theorem 3.2.

Proposition 5.1. *The generating family homology of the total space of a product family is computed by*

$$\mathrm{GH}_k(F) = \bigoplus_{l=0}^{\dim B} \mathrm{GH}_l(f) \otimes H_{k-l}(B).$$

Proof. The E^2 property of Theorem 3.2 implies that

$$E^2_{i,j} = H_i(B; \mathrm{GH}_j(f)).$$

The triviality property of Theorem 3.2 implies that the spectral sequence $E^*_{*,*}$ collapses at the E^2 page, and we recover the generating family homology of the family F as in the statement of the theorem. □

Corollary 5.2. *Suppose that the Legendrian submanifolds $\Lambda_1, \Lambda_2 \subset J^1M$ have different sets of generating family homologies. If B is any closed manifold, then $\Lambda_1 \times B$ and $\Lambda_2 \times B$ are not Legendrian isotopic in $J^1(M \times B)$.*

While the result of this corollary has been obtained when $M = \mathbb{R}^n$ and B is the k-torus [5], this is a new result for all other cases.

To see an application of the corollary, one may take any pair of twist knots in $J^1\mathbb{R}$ that Chekanov distinguished using linearized Legendrian contact homology [4]. In this case, since the twist knots have only one possible linearized contact homology group, it is easy to use Fuchs and Rutherford's results in [9] to show that Chekanov's twist knots have different generating family homology.

Remark 5.3. The product families construction is a special case of Lambert-Cole's Legendrian product construction [18]. The 1-jet of F in J^1B is a Legendrian Λ_B isotopic to the zero section, and the product above is then Lambert-Cole's Legendrian product $\Lambda \times \Lambda_B$.

5.2. Front spinning. In the next few subsections, we bring the front spinning constructions of [6, 11], their adaptation to generating families [2], and their generalization to twist spinning [2] into the families context.

For the simplest version of this construction, suppose that a Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$ is contained in the half-space H defined by $x_n > 1$. This can always be achieved via a translation in the x_n direction, which is a Legendrian isotopy. Suppose further that Λ has a linear-at-infinity generating family f whose support (Section 2.3) also lies in the half-space H. As alluded to in Section 2.3, we may also assume that δ is linear-at-infinity, i.e. that $\delta = \delta_0 + A$ where A is linear and δ_0 has compact support in the half-space H. Moreover, we can assume that the support lies in the set defined by $x_n > 1$; see [20].

We define a new generating family for an $(n+m)$-dimensional Legendrian in $\mathbb{R}^{2(n+m)+1}$ as follows: let $(\rho, \boldsymbol{\theta})$ denote generalized spherical coordinates on $\mathbb{R}^{m+1}$; hence, we may represent a point in $\mathbb{R}^{n+m} = \mathbb{R}^{n-1} \times \mathbb{R}^{m+1}$ by $(x_1, \dots, x_n, \rho, \boldsymbol{\theta})$. Define the generating family for the spun Legendrian by

$$F^{\Sigma,m}(x_1, \dots, x_{n-1}, \rho, \boldsymbol{\theta}, \eta) = f(x_1, \dots, x_{n-1}, \rho, \eta). \tag{5.1}$$

It is straightforward to check, as noted in [2], that $F^{\Sigma,m}$ is still a generating family. We call the new Legendrian the *m-spinning* of Λ and denote it by $\Sigma^m\Lambda$; it clearly has the diffeomorphism type of $\Lambda \times S^m$.

A small generalization of the proof of Proposition 5.1 yields:

Proposition 5.4. *The generating family homology of the m-spun generating family $F^{\Sigma,m}$ may be computed as*

$$\mathrm{GH}_k(F^{\Sigma,m}) = \mathrm{GH}_k(f) \oplus \mathrm{GH}_{k-m}(f).$$

Proof. The proof is structured around a relative Mayer–Vietoris argument in the domain of $\Delta^{\Sigma,m}$, where we take the set A^h to consist of points $(x, \rho, \boldsymbol{\theta}, \eta) \in \mathbb{R}^{n+m} \times \mathbb{R}^{2N}$ with $\rho < 1$ and $\Delta^{\Sigma,m} < h$ and the set B^h to consist of points with $\rho > \frac{1}{2}$ and $\Delta^{\Sigma,m} < h$. Since $\Delta^{\Sigma,m}$ is a linear function for $\rho < 1$, we see that the pairs (A^ω, A^ϵ) and $(A^\omega \cap B^\omega, A^\epsilon \cap B^\epsilon)$ are both acyclic. Thus, a Mayer–Vietoris argument shows that $\mathrm{GH}_*(F^{\Sigma,m})$ is isomorphic to $H_{*+N+1}(B^\omega, B^\epsilon)$, which, by examination of equation 5.1, is precisely the generating family homology of the product family $\Lambda \times S^m$ constructed in the previous section. □

We conclude, as in the previous section, that if two Legendrians may be distinguished by their generating family homology, then their m-spins are so distinguished as well; see [5, Section 5] for a comparable computation for Legendrian Contact Homology when $m = 1$.

5.3. Twist spinning. To generalize the spinning construction of Section 5.2, consider a representative α of an element in $\pi_m(\mathcal{L}; \Lambda)$. Suppose that Λ has a generating family f, and let $f_{\boldsymbol{\theta}}$ denote the lift of α to the set of generating families for $\Lambda_{\boldsymbol{\theta}}$ starting at f. As before, we explicitly assume that the lifting procedure yields an m-sphere of generating families, that is, $\alpha \in \pi_m(\mathcal{L}; \Lambda, f)$. As a common generalization of [2] and [11], and in parallel to [7] for $m = 1$, we define a generating family for the *twist-spun* Legendrian $(n+m)$-submanifold Λ_α by

$$F^\alpha(x_1, \ldots, x_{n-1}, \rho, \boldsymbol{\theta}, \eta) = f_{\boldsymbol{\theta}}(x_1, \ldots, x_{n-1}, \rho, \eta). \tag{5.2}$$

Front spinning is obviously a special case of twist spinning: simply twist-spin the constant isotopy.

To compute $\mathrm{GH}_*(F^\alpha)$, we return to the setup in Example 4.2, where the base function $K\colon S^m \to \mathbb{R}$ has a maximum at $a \in S^m$, a minimum at $b \in S^m$, and no other critical points. Theorem 3.2 implies that the E^2 term of the families spectral sequence for the family $f_{\boldsymbol{\theta}}$ is $\mathrm{GH}_*(f) \oplus \mathrm{GH}_*(f)[m-1]$ with the differential defined as follows. If x is a generator of $\mathrm{GH}_*(f)$, then in the notation of Sections 3 and 4, the generators of the E^2 term are of the form (a, x) and (b, x). The definition of the map Ψ then implies that the differential is

$$d(a,x) = \begin{cases} (b, \Psi_{[\alpha]}(x) + x) & m = 1, \\ (b, \Psi_{[\alpha]}(x)) & m > 1, \end{cases}$$

$$d(b,x) = 0.$$

Proposition 5.5. *The generating family homology* $\mathrm{GH}_*(F^\alpha)$ *is independent of the choice of representative of* α *and may be computed from the chain complex* $(\mathrm{GH}_*(f) \oplus \mathrm{GH}_*(f)[1-m], d)$ *described above.*

Proof. The proof is parallel to that of Proposition 5.4, above, with the construction of Ψ in equation (4.3) and Proposition 4.4 taking the place of Proposition 5.1. □

The theorem above can give us information in two ways: first, it allows us to use distinct elements of $\pi_m(\mathcal{L}; \Lambda_0, f)$ to produce pairs of distinct $(n+m)$-dimensional Legendrian submanifolds. For example, twist-spinning the Legendrian Λ constructed in Section 4.3 by the non-trivial element in $\pi_1(\mathcal{L}; \Lambda_0, f)$ yields a Legendrian $(n+1)$-submanifold distinct from the ordinary spin of Λ.

The theorem above also provides a potential mechanism to distinguish elements of $\pi_m(\mathcal{L}; \Lambda, f)$: if the twist-spins of two loops of Legendrian with a common base point have different generating family homology, then the difference must have arisen from the Ψ maps. Thus, if one can compute the generating family homology by some other means — surgery [20] or a generating family version of the Mayer–Vietoris sequence of [13], for example — then one has a chance of finding new examples of non-trivial elements of $\pi_m(\mathcal{L}; \Lambda, f)$ without directly computing the Ψ maps directly. Unfortunately, as of this writing, we know of no implementations of this technique.

5.4. Factoring Ψ through spinning. In this section, we study the relationship between the morphism Ψ and the 1-spinning construction. Unlike in Section 5.2, we need the analyze the chain complex more closely, but along the way, we reprove Proposition 5.4 in the 1-spun case.

First, we adapt a technique useful for gradient flow trees and holomorphic disks in Legendrian Contact Homology [6, 13] to generating family homology. We state the lemma more generally than is needed in this article for possible future applications. Let g be a metric on $M \times \mathbb{R}^N \times \mathbb{R}^N$, $S \subset M$ be a submanifold, and $N_\epsilon(S) \subset M$ be the ϵ-neighborhood of S. Let δ be the difference function of a generating family $f\colon M \times \mathbb{R}^N \to \mathbb{R}$. Let V be a (negative) gradient-like vector field for δ used to define the differential in $GC(f)$. Assume the support of V agrees with the support of δ.

Lemma 5.6. *For all sufficiently small $\epsilon > 0$, and for all $(x, \eta, \tilde{\eta})$ such that $x \in \partial N_\epsilon(S)$ and $\delta(x, \eta, \tilde{\eta}) > 0$, assume one of the following holds: either the component of V normal to $\partial N_\epsilon(S)$ is non-vanishing and points inwards; or, $(x, \eta, \tilde{\eta})$ is not in the support of δ. Fix points $p, q \in M \times \mathbb{R}^N \times \mathbb{R}^N$ with $\delta(p) > \delta(q) > 0$ and negative gradient-like flow line γ of δ connecting them.*

(1) *When S is a hypersurface, γ does not cross $S \times \mathbb{R}^N \times \mathbb{R}^N$.*

(2) *If both p and q lie in $S \times \mathbb{R}^N \times \mathbb{R}^N$, then γ sits entirely in $S \times \mathbb{R}^N \times \mathbb{R}^N$.*

(3) *If f_S is the restriction of f to $S \times \mathbb{R}^N$, then $GC(f_S)$ is naturally a subcomplex of $GC(f)$.*

If we replace "inwards" with "outwards" in the first assumption, then the first and second statements above still hold.

Proof. Note that if γ exits the support of V, it then stays within a single fiber $\{x\} \times \mathbb{R}^N \times \mathbb{R}^N$. Thus, for the first statement, it suffices to observe that the hypotheses imply that V is everywhere tangent to $S \times \mathbb{R}^N \times \mathbb{R}^N$.

For the second statement, since the normal component of V always points into $T(S \times \mathbb{R}^N \times \mathbb{R}^N)$ at p, or vanishes, even if p is a critical point of δ, the flow line cannot leave any ϵ neighborhood of $S \times \mathbb{R}^N \times \mathbb{R}^N$. Thus, the first observation implies that γ lies entirely in $S \times \mathbb{R}^N \times \mathbb{R}^N$. A similar proof, based at q, holds if we replace the "inwards" assumption by "outwards".

For the third statement, note that the vanishing normal component of V along $S \times \mathbb{R}^N \times \mathbb{R}^N$ implies that there is a one-to-one correspondence between the critical points of δ and those of δ_S. The equality of differentials then follows from the argument for the second statement which prevents a flow line from leaving $S \times \mathbb{R}^N \times \mathbb{R}^N$. $\square$

We now study the interaction of spinning and Proposition 4.4. Fix a Legendrian submanifold $\Lambda \subset \{\rho := x_n > 1\} \subset J^1\mathbb{R}^n$ with generating family f whose support lies in $\{\rho > 1/2\} \subset \mathbb{R}^n \times \mathbb{R}^N$. A 1-spin produces a Legendrian $\Sigma^1\Lambda \subset J^1\mathbb{R}^{n+1}$ with generating family $F^{\Sigma,1}$ as in equation (5.1). Choose a smooth monotonic function $\lambda(\rho)$ such that $\lambda|_{[0,1/2]} = 0$ and $\lambda|_{[1,\infty)} = 1$. Fix a small $\epsilon > 0$, and let V be the gradient vector field of the difference function with a C^2-small perturbation:

$$F^{\Sigma,1}(x_1, \ldots, x_n, \rho, \theta, \eta) - F^{\Sigma,1}(x_1, \ldots, x_n, \rho, \theta, \tilde{\eta}) + \epsilon\lambda(\rho)\sin(\theta).$$

All critical points of the gradient-like vector field V have coordinates $\rho > 1$ and $\theta = -\pi/2$ or $\pi/2$, which we distinguish by labeling as $c[-]$ and $c[+]$, respectively, where c is a critical point of the difference function of f. This induces a decomposition of the differential $d^{\Sigma,1}$ of $GC(F^{\Sigma,1}) = GC[-] \oplus GC[+]$:

$$d^{\Sigma,1} = \begin{bmatrix} d_{--} & d_{-+} \\ d_{+-} & d_{++} \end{bmatrix}.$$

We first prove a lemma which implies Proposition 5.4 for the 1-spin case.

Lemma 5.7. *For all critical points b, c of the difference function of f, we have*

$$\begin{aligned} d_{-+}c[-] &= 0, \\ d_{+-}c[+] &= 0, \\ \langle d_{--}c[-], b[-]\rangle = \langle dc, b\rangle &= \langle d_{++}c[+], b[+]\rangle, \end{aligned}$$

where d is the differential of $GC(f)$.

Proof. By the symmetry of V under the reflection through the $x_1 \cdots x_{n-1} z$ plane, any elements in any rigid moduli space $\mathcal{M}_0(c[+], b[-])$ appear in pairs; thus, $d_{+-} = 0$.

Let $S \subset \mathbb{R}^{n-1} \times \mathbb{R}^2$ be the open hypersurface satisfying $\theta = -\pi/2$ and $\rho > 1/2$. We see that the hypotheses (with "inward" specification) of Lemma 5.6 hold; therefore, the third statement of the lemma implies

$$d_{-+} = 0 \quad \text{and} \quad \langle d_{--}c[-], b[-]\rangle = \langle dc, b\rangle.$$

Finally, let $S' \subset \mathbb{R}^{n-1} \times \mathbb{R}^2$ be the hypersurface defined by $\theta = \pi/2$ and $\rho > 1/2$. The identity $\langle d_{++}c[+], b[+]\rangle = \langle dc, b\rangle$ now follows from the second statement of Lemma 5.6 (with the "outward" hypothesis). $\square$

Proposition 5.8. *Let Ψ be the map from Proposition 4.4. Let $\mathrm{Pr}_\pm$ be the projection map defined on generators as*

$$\mathrm{Pr}_\pm \colon \mathrm{GH}(F^{\Sigma,1}) \longrightarrow \mathrm{GH}(f), \quad c[\pm] \longrightarrow c, \; c[\mp] \longrightarrow 0.$$

Define the map $i \colon \pi_m(\mathcal{L}(J^1\mathbb{R}^n); \Lambda, f) \to \pi_m(\mathcal{L}(J^1\mathbb{R}^{n+1}); \Sigma^1\Lambda, F^{\Sigma,1})$ induced by 1-spinning S^m families of Legendrians. Then i is well-defined, and Ψ factors through 1-spinning, i.e. the following diagram commutes:

$$\begin{CD} \pi_m(\mathcal{L}(J^1\mathbb{R}^n); \Lambda, f) @>{\Psi}>> \mathrm{End}_{m-1}(\mathrm{GH}_*(f)) \\ @V{i}VV @AA{\mathrm{Pr}_\pm}A \\ \pi_m(\mathcal{L}(J^1\mathbb{R}^{n+1}); \Sigma^1\Lambda, F^{\Sigma,1}) @>{\Psi}>> \mathrm{End}_{m-1}(\mathrm{GH}_*(F^{\Sigma,1})). \end{CD}$$

Proof. Since the 1-spin of a homotopy of two Legendrian S^m-families is a homotopy of two 1-spun Legendrian S^m-families, 1-spinning induces a morphism $\pi_m(\mathcal{L}(J^1\mathbb{R}^n);\Lambda) \to \pi_m(\mathcal{L}(J^1\mathbb{R}^{n+1});\Sigma^1\Lambda)$. Since equation 5.1 (for $m=1$) can be extended to S^m families (for $m \geq 1$), the above induced morphism restricts to show that the map

$$i\colon \pi_m(\mathcal{L}(J^1\mathbb{R}^n);\Lambda, f) \longrightarrow \pi_m(\mathcal{L}(J^1\mathbb{R}^{n+1});\Sigma^1\Lambda, F^{\Sigma,1})$$

is well-defined.

Let d_m be the chain map which induces the upper arrow Ψ in the proposition, and $d_m^{\Sigma,1}$ be the chain map which induces the lower Ψ, both as in equation (4.3). Using the notation of Lemma 5.7, it suffices to show that

$$\langle d_m^{\Sigma,1} c[-], b[-]\rangle = \langle d_m c, b\rangle = \langle d_m^{\Sigma,1} c[+], b[+]\rangle. \tag{5.3}$$

We prove the first equality, as the second one follows from identical reasoning.

Let $\Lambda(t)$, $t \in S^m$, represent an arbitrary element in $\pi_m(\mathcal{L}^n;\Lambda, f)$ and $\Sigma^1\Lambda(t)$ be its front-spun counterpart. Recall the S^m-family described in Example 4.2. For $t \in S^m$, choose (smoothly in t) the half-hyperplane $S(t)$ from the proof of Lemma 5.7 (rotated according to t) which "cuts out" a copy of $\Lambda(t)$ from $\Sigma^1\Lambda(t)$. This defines a hypersurface S in $S^m \times \mathbb{R}^{n+1}$. Like in the proof of Lemma 5.7, we see that the hypotheses of Lemma 5.6 are satisfied. Equation (5.3) follows from the second statement of Lemma 5.6. ☐

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