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$\mathrm{SO}(N)_2$ braid group representations are Gaussian

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Abstract. We give a description of the centralizer algebras for tensor powers of spin objects in the pre-modular categories $\mathrm{SO}(N)_2$ (for N odd) and $\mathrm{O}(N)_2$ (for N even) in terms of quantum $(n-1)$-tori, via non-standard deformations of $\mathrm{U}\,\mathfrak{so}_N$. As a consequence we show that the corresponding braid group representations are Gaussian representations, the images of which are finite groups. This verifies special cases of a conjecture that braid group representations coming from weakly integral braided fusion categories have finite image.

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Contents

1. Introduction

Let V be the N-dimensional vector representation of the quantum group $\mathrm{U}_q\,\mathfrak{g}$, where $\mathfrak{g} \in \{\mathfrak{sl}_N, \mathfrak{so}_N, \mathfrak{sp}_N\}$, $N > 0$ is a positive integer and where q is an indeterminate. Let $\mathcal{B}_n$ be the braid group on n strands, for a natural number $n > 0$.

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The centralizer algebras $\mathrm{End}_{\mathrm{U}_q\,\mathfrak{g}}(V^{\otimes n})$ have complete descriptions in terms of semisimple quotients of braid group algebras $\mathbb{C}(q)\mathcal{B}_n$, namely Hecke and BMW-algebras ([17], [34]). For $q = e^{\pi \mathrm{i}/\ell}$ the representation categories $Rep(\mathrm{U}_q\,\mathfrak{g})$ are not semisimple, but have semisimple sub-quotients $\mathcal{C}(\mathfrak{g}, \ell)$ obtained via a process called "purification" in [31]. Continuing to denote by V the image of V in the sub-quotient $\mathcal{C}(\mathfrak{g}, \ell)$, the centralizer algebras $\mathrm{End}(V^{\otimes n})$ in the fusion category $\mathcal{C}(\mathfrak{g}, \ell)$ are still quotients of Hecke or BMW-algebras, so that the description in terms of the braid group algebras persists. The closures of the images of these braid group representations were analyzed in [6, 30], which provided evidence for the following conjecture (see [26, 28]).

Conjecture 1.1. *Let $\mathcal{C}$ be a braided fusion category and let X be a simple object in $\mathcal{C}$. The braid group representations $\mathcal{B}_n$ on $\mathrm{End}(X^{\otimes n})$ have finite image for all $n > 0$ if and only if $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$.*

Here $\mathrm{FPdim}(X) \in \mathbb{R}$ is the Frobenius–Perron dimension, which coincides with the categorical dimension for unitary fusion categories. Categories with $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$ for all simple objects X are called *weakly integral*. One large class of weakly integral braided fusion categories for which this conjecture has been verified are the so-called *group theoretical* categories related to (the doubles of) finite groups (see [5]). The main difficulty in verifying Conjecture 1.1 for arbitrary objects X in a braided fusion category is that a sufficiently explicit description of the braid group action on $\mathrm{End}(X^{\otimes n})$ is usually lacking.

For the class of categories $\mathcal{C}(\mathfrak{g}, \ell)$ associated with quantum groups at roots of unity the "only if" part of the conjecture has been confirmed (see [29]); more precisely, it was shown that for any simple object X in a fusion category $\mathcal{C}(\mathfrak{g}, \ell)$ for which $\mathrm{FPdim}(X)^2 \notin \mathbb{Z}$ the associated braid representations have infinite image. Conversely, for simple objects X in $\mathcal{C}(\mathfrak{g}, \ell)$ for which $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$ the only remaining open cases are in $\mathcal{C}(\mathfrak{so}_N, 2N)$ for N odd and $\mathcal{C}(\mathfrak{so}_N, N)$ for N even. We will adopt the uniform notation $\mathrm{SO}(N)_2$ for these two families (this notation conforms with the physics literature, where the subscript 2 is the level). For N odd, the (fundamental) spinor object $S \in \mathrm{SO}(N)_2$ has dimension $\sqrt{N}$, whereas for N even the two (fundamental) spinor objects $S_\pm \in \mathrm{SO}(N)_2$ have dimension $\sqrt{N/2}$. In particular, Conjecture 1.1 predicts that the $\mathcal{B}_n$ representations associated with S and $S_\pm$ have finite image. The main result of our paper verifies this (see Theorem 4.9):

Theorem. *Conjecture 1.1 is true for the categories $\mathrm{SO}(N)_2$ for all positive integers N.*

This had been proved for $N \leq 8$ using low rank coincidences and for the cases where $\dim(S), \dim(S_\pm) \in \mathbb{Z}$ by appealing to the results of [5], see [26]. For general N, it seems to be difficult to give an intrinsic description of the braid representations as the number of eigenvalues of the image of a standard generator increases with N. We overcome these difficulties by the following approach. We denote by S the quantum version of the spinor representation of $\mathbf{U} = \mathrm{U}_q\, \mathfrak{so}_N$ (for N odd) as well as its image in $\mathrm{SO}(N)_2$ and by $S_\pm$ the analogous fundamental spinor objects for N even and their images in $\mathrm{SO}(N)_2$. For N even we denote by $\mathbf{U}$ a certain semidirect (smash) product $\mathrm{U}_q\, \mathfrak{so}_N \rtimes \mathbb{Z}_2$ and we will also denote by S the irreducible $\mathbf{U}$-module whose restriction to $\mathrm{U}_q\, \mathfrak{so}_N$ is $S_- \oplus S_+$. For generic parameter q, the centralizer algebras $\mathrm{End}_{\mathbf{U}}(S^{\otimes n})$ are described ([36, Theorem 4.8]) in terms of a non-standard deformation $\mathrm{U}'_q\, \mathfrak{so}_n$ of $\mathrm{U}\, \mathfrak{so}_n$, for both N odd and even. Although $\mathrm{Rep}(\mathbf{U})$ carries a braiding, the image of $\mathcal{B}_n$ inside $\mathrm{End}_{\mathbf{U}}(S^{\otimes n})$ does not generate these algebras. On the other hand, for q a $2N$th root of unity, we show that the algebra $\mathrm{U}'_q\, \mathfrak{so}_n$ admits a homomorphism into the quantum $(n-1)$-torus $T_q(n)$, which contains an isomorphic copy of $\mathrm{End}(S^{\otimes n})$. The key observation now is that this homomorphism identifies the image of the $\mathcal{B}_n$-representations in $\mathrm{End}(S^{\otimes n})$ for the braided fusion category $\mathrm{SO}(N)_2$ with so-called Gaussian braid representations (so named because the coefficients are Gaussian functions of the form $Ke^{2\pi i a^2/\ell}$, defined in Proposition 4.7(a)) which live in the quantum torus. These explicitly realized braid representations can be shown to have finite images, which implies the conjecture for $\mathrm{SO}(N)_2$. So the identification of these different braid representations is achieved using the representation theory of the algebra $\mathrm{U}'_q\, \mathfrak{so}_n$.

Here is a more detailed outline of the contents of this article. In Section 2 we review results about the centralizer algebras $\mathrm{End}(S^{\otimes n})$ where S is a spinor representation of $\mathrm{U}_q\, \mathfrak{so}_N$ respectively $\mathrm{U}_q\, \mathfrak{so}_N \rtimes \mathbb{Z}_2$, or the corresponding object in one of the associated fusion categories. Most of these results have already more or less appeared before in [12, 36]. In Section 3, we reprove and extend several results by Klimyk and his coauthors ([9, 15]) concerning the representation theory of $\mathrm{U}'_q\, \mathfrak{so}_n$. In contrast with *loc. cit.*, we use a Verma module approach which also has the advantage of proving (crucial, for our paper) uniqueness results at roots of unity, for certain types of modules. In Section 4 we construct representations of $\mathrm{U}'_q\, \mathfrak{so}_n$ into algebras called quantum tori. The main result of this section is the identification of these representations with those in $\mathrm{End}(S^{\otimes n})$ for fusion categories $\mathrm{SO}(N)_2$ (N odd) and $\mathrm{O}(N)_2$ (N even). This allows us to describe the corresponding tower of centralizer algebras in terms of the quantum tori using Jones' basic construction. Finally, we identify the braid group representations corresponding to the object S

in $\mathrm{SO}(N)_2$ (respectively $\mathrm{O}(N)_2$) for N odd (respectively N even) with the Gaussian braid representations first encountered in the work of Jones and Goldschmidt [11, 19] for N odd. The easy generalization to N even is worked out in [7]. From this follows Conjecture 1.1 in our case.

2. Duality for spinor representations

2.1. Deformations of $\mathbf{U}\,\mathfrak{so}_n$. The algebra $\mathrm{U}'_q\,\mathfrak{so}_n$ is defined (see [9]) via generators $B_1, \ldots, B_{n-1}$ satisfying the relations $B_iB_j = B_jB_i$ for $|i-j| \neq 1$ and the q-Serre relations:

$$B_i^2B_{i\pm1} - (q+q^{-1})B_iB_{i\pm1}B_i + B_{i\pm1}B_i^2 = B_{i\pm1}. \tag{2.1}$$

It is well-known that in the classical limit $q = 1$ we obtain a presentation of the universal enveloping algebra $\mathrm{U}\,\mathfrak{so}_n$ of the orthogonal Lie algebra $\mathfrak{so}_n$, and for this reason $\mathrm{U}'_q\,\mathfrak{so}_n$ is sometimes called the *non-standard* deformation of $\mathrm{U}\,\mathfrak{so}_n$. It follows from the definitions that the elements $B_1, B_3, \ldots, B_{n-1}$ (for n even, respectively B_{n-2} for n odd), generate an abelian subalgebra A of $\mathrm{U}'_q\,\mathfrak{so}_n$. We define a weight vector of a $\mathrm{U}'_q\,\mathfrak{so}_n$-module V to be a common eigenvector of the generators of A. We call a weight regular if all the eigenvalues of generators B_{2i-1} of A are of the form $[r]$ with r an integer or a half integer, and $[r] = (q^r - q^{-r})/(q-q^{-1})$ the usual q-number.

In the following we denote by $\mathbf{U}$ a semidirect product of the (standard) Drinfeld–Jimbo quantum group $\mathrm{U}_q\,\mathfrak{so}_N$ with $\mathbb{Z}_2$. For N odd, $\mathbf{U}$ is just the direct sum of the corresponding $\mathbb{C}$-algebras, while in the N even case, the nontrivial element t of $\mathbb{Z}_2$ acts via the obvious type $D_{N/2}$ graph automorphism. This completely determines the defining relations for $\mathbf{U}$. It is also easy to check that the map $\Delta(t) = t \otimes t$ extends the bialgebra structure of $\mathrm{U}_q\,\mathfrak{so}_N$ to $\mathbf{U}$. Indeed, by [25, Theorem 2.1] $\mathbf{U}$ (called the *smash product algebra* in *loc. cit.*) is a ribbon Hopf algebra as the action of t preserves the braiding. For N odd, it is clear that $\mathrm{Rep}(\mathbf{U}) \cong \mathrm{Rep}(\mathrm{U}_q\,\mathfrak{so}_N) \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$ (Deligne tensor product) as ribbon categories. Note that by [8] $\mathrm{Rep}(\mathbf{U})$ is the $\mathbb{Z}_2$-equivariantization of $\mathrm{Rep}(\mathrm{U}_q\,\mathfrak{so}_N)$. We shall also be interested in the case where q is a root of unity. In this case we consider the subcategory of tilting modules in $\mathrm{Rep}(\mathbf{U})$ which is again a ribbon category (see e.g. [35] for details). As such, we may consider the quotient category by negligible morphisms (see [31, Section XI.4]) to obtain ribbon fusion categories $\mathrm{SO}(N)_r$ and $\mathrm{O}(N)_r$, which we describe below.

The algebra $\mathbf{U}$ should be viewed as a quantum version of $\mathrm{Pin}(N)$. Indeed, $\mathbf{U}$ is well-defined in the classical limit $q = 1$, and its finite dimensional simple representations are in 1-1 correspondence with the simple representations of $\mathrm{Pin}(N)$. It is easy to see that we obtain a well-defined quantum version of the spinor $\mathrm{Pin}(N)$-module S for $\mathbf{U}$ (where the matrices of the generators E_i, F_i and t do not depend on q). As any finite-dimensional simple $\mathrm{Pin}(N)$-module does appear in some tensor power of S, we can also make it into a $\mathbf{U}$-module. This deformation also works for roots of unity. It is well-known and easy to check that if the restriction of a simple $\mathrm{U}_q\,\mathfrak{so}_N \rtimes \mathbb{Z}_2$-module to $\mathrm{U}_q\,\mathfrak{so}_N$ does not remain simple, it decomposes into the direct sum of two irreducible $\mathrm{U}_q\,\mathfrak{so}_N$-modules with the same q-dimension. Hence we also obtain a well-defined fusion tensor category associated to $\mathrm{U}_q\,\mathfrak{so}_N \rtimes \mathbb{Z}_2$, with the usual restriction rules to $\mathrm{U}_q\,\mathfrak{so}_N$.

Recall the construction of spinors in the classical setting. Consider a simple module of the Clifford algebra on $V = \mathbb{C}^N$. It is well-known that for N even we get an irreducible representation S of $\mathrm{Pin}(N)$ which decomposes as a $\mathrm{Spin}(N)$-module into a direct sum $S \cong S_+ \oplus S_-$ of two non-isomorphic irreducible representations. If N is odd, we have two non-isomorphic simple modules of the Clifford algebra, say S_0 and S_1, both of which restrict to the same irreducible $\mathrm{Spin}(N)$-module. We will just denote them (both) as S, consistent with the notation above. For N odd, we will also need the (reducible) $\mathrm{Pin}(N)$-module $\widetilde{S} = S_0 \oplus S_1$ at some point.

The relationships between the spinor representations of $\mathbf{U}$ and $\mathrm{U}_q\,\mathfrak{so}_N$ are analogous to those of $\mathrm{Pin}(N)$ and $\mathrm{Spin}(N)$. That is, for N even, we have a $\mathbf{U}$-module S which is irreducible and decomposes as $S \cong S_+ \oplus S_-$ as a $\mathrm{U}_q\,\mathfrak{so}_N$-module. For N odd, there are two non-isomorphic $\mathbf{U}$-modules S_0 and S_1 which are isomorphic upon restriction to $\mathrm{U}_q\,\mathfrak{so}_N$ (S_0 and S_1 differ only on the $\mathbb{Z}_2$-action).

2.2. Classical case. We first check some well-known identities in the classical case, where $\mathbf{U}$ is replaced by $\mathrm{Pin}(N)$ and $\mathrm{U}'_q\,\mathfrak{so}_n$ is replaced by $\mathrm{SO}(n)$. Most of these results have already more or less explicitly appeared, as special cases of a more general approach, see [12].

We consider the case where $\mathrm{Pin}(N)$ representations are also $\mathrm{O}(N)$ representations. We remark that our symmetric bilinear form on the root lattice is normalized so that $\langle \beta, \beta \rangle = 2$ for *long* roots for all N, for uniformity's sake. Recall (see e.g. [38]) that simple $\mathrm{O}(N)$ representations are labeled by the Young diagrams λ for which $\lambda'_1 + \lambda'_2 \leq N$ (here λ'_i denotes the number of boxes in the i-th column). The representations of the Lie algebra $\mathfrak{so}_n$ for $n = 2j$ are labeled by the dominant integral weights $\mu = (\mu_i)_i$ such that $\mu_1 \geq \mu_2 \geq \ \dots \ \mu_{j-1} \geq |\mu_j|$, where either all μ_i are integers or all $\mu_i \equiv 1/2 \bmod \mathbb{Z}$. Then it is easy to check that the map

$$\lambda \longmapsto \bar{\lambda}, \quad \text{where } \bar{\lambda}_i = N/2 - \lambda'_{j+1-i} \tag{2.2}$$

defines a bijection between the set of simple representations V_λ of $\mathrm{O}(N)$ for which $\lambda_1 \leq n/2 = j$ and the set of simple $\mathfrak{so}_n$ representations $V_{\bar{\lambda}}$ for which $\bar{\lambda}_1 \leq N/2$ and $N/2 - \bar{\lambda}_i$ is an integer for $1 \leq i \leq n/2$. Now consider the obvious action of $\mathrm{O}(N) \times \mathrm{SO}(n)$ on $\mathbb{C}^N \otimes \mathbb{C}^n$. This induces commuting actions of $\mathrm{O}(N)$ and $\mathrm{SO}(n)$ via automorphisms on $\mathrm{Cliff}(\mathbb{C}^N \otimes \mathbb{C}^n)$, and hence to projective actions of these groups on a simple module S_{Nn} of $\mathrm{Cliff}(\mathbb{C}^N \otimes \mathbb{C}^n)$ i.e. proper actions of the corresponding covering groups, the spinor groups.

Lemma 2.1. (a) *Let n be even and let S_{Nn} be a simple module of the simple algebra* $\mathrm{Cliff}(\mathbb{C}^N \otimes \mathbb{C}^n)$*. Then S_{Nn} decomposes as an* $\mathrm{O}(N) \times \mathrm{Spin}(n)$ *module as*

$$S_{Nn} \cong \bigoplus_\lambda V_\lambda \otimes V_{\bar{\lambda}},$$

where V_λ and $V_{\bar{\lambda}}$ are simple $\mathrm{O}(N)$ *and* $\mathrm{Spin}(n)$*-modules and λ runs through the set of Young diagrams as in equation* (2.2)*.*

(b) *If both N and n are even, $S^{\otimes n}$ is isomorphic as a* $\mathrm{Pin}(N) \times \mathrm{Spin}(n)$ *module to the module S_{Nn} in* (a)*. If N is odd and n is even, $\widetilde{S}^{\otimes n}$ is isomorphic as a* $\mathrm{Pin}(N) \times \mathrm{Spin}(n)$ *module to the direct sum of $2^{n/2}$ copies of S_{Nn} as in* (a)*.*

(c) *Regardless of parity of N and n, the irreducible representations of* $\mathrm{Spin}(n)$ *in cases* (a) *and* (b) *are labeled by the dominant integrals weights μ satisfying $\mu_1 \leq N/2$ and such that $\mu_i - N/2$ is an integer for $1 \leq i \leq k$.*

Proof. It suffices to calculate the $\mathrm{Pin}(N) \times \mathrm{Spin}(n)$ characters of the various modules. Let $n = 2k$ and $\mathrm{i} = (i_1, \dots, i_k) \in \mathbb{Z}^k_{\geq 0}$. We denote by $\omega(\mathrm{i})$ the $\mathrm{Spin}(n)$ weight given by the vector $(i_j - N/2)_j$. Then we claim that the $\mathrm{Spin}(n)$ character of a simple $\mathrm{Cliff}(\mathbb{C}^N \otimes \mathbb{C}^n)$ module is given by

$$\chi^{S_{Nn}} = \sum_{i_1=0,\dots,\ i_k=0}^{N} \Big(\prod_{j=1}^{k} \chi_{i_j}\Big) e^{\omega(\mathrm{i})}, \tag{2.3}$$

where χ_i is the $\mathrm{O}(n)$ character for the i-th antisymmetrization $\bigwedge^i V$ of the vector representation of $\mathrm{O}(N)$, for $0 \leq i \leq N$. This can be seen as follows. As Nn is even by assumption, we can describe the character of the full spinor representation of $\mathrm{O}(Nn)$ (which is a simple $\mathrm{Cliff}(Nn)$-module) by

$$(z_1 z_2 \cdots z_{Nn/2})^{-1/2} \sum_{j=0}^{Nn/2} e_j(z),$$

where $e_j(z)$ is the j-th elementary symmetric function in $z_1, \ldots, z_{Nn/2}$. To view this as a character of $\mathrm{Spin}(n)$ we replace the z-variables by variables $x_i y_j$, $1 \le i \le n/2$, $1 \le j \le N$. We regard the result as a polynomial in the x_i variables over the ring of polynomials in the y_i variables. As every x_i variable comes with all possible y_j variables, and our formula is obviously symmetric in the z-variables, and hence also in the x and y variables, a monomial in the x-variables containing the variable x_i with the power m_i must also have the factor $e_{m_i}(y)$, the elementary symmetric function in the variables $y_1, \ldots, y_N$. Now it is well-known that the elementary symmetric functions are the characters of the antisymmetrizations of the vector representation which remain irreducible as $\mathrm{O}(N)$-modules. This proves equation (2.3).

We can now prove statement (a) by induction with respect to inverse alphabetical order of the weights ω(i). It is clear that the highest possible weight occurring in equation (2.3) is $\omega = N\varepsilon$. Then the coefficient of e^{ω} is equal to the trivial character, which proves (a) for $\lambda = 0$. The general claim follows by induction, using the formula

$$\prod \chi_{i_j} = \chi_\lambda + \text{lower characters},$$

where i_j is a nonincreasing sequence of integers, λ is the Young diagram whose j-th column has exactly i_j boxes, and lower characters refers to a sum of simple $\mathrm{O}(N)$ characters labeled by Young diagrams smaller than λ in alphabetical order.

To prove the corresponding formulas for the tensor product representations, we check it first for $n = 2$. Here for N even, the second tensor product of the spinor representation S is a direct sum of all possible antisymmetrizations of the vector representation $\mathbb{C}^N$. For N odd, we similarly get that $\widetilde{S}^{\otimes 2}$ decomposes into the direct sum of two copies of the exterior algebra of $\mathbb{C}^N$. It was shown in [36] that the i-th antisymmetrization in $S^{\otimes 2}$ (respectively in $\widetilde{S}^{\otimes 2}$, where it appears with multiplicity 2) is an eigenspace of the $\mathfrak{so}_2$ generator B_1 with eigenvalue $N/2 - i$. This proves that the $\mathfrak{so}_2$ character of $S^{\otimes 2}$ (respectively of $\widetilde{S}^{\otimes 2}$) is given by equation (2.3) for N even (respectively by twice the value of equation (2.3) for N odd). For $n = 2k > 2$, we write $S^{\otimes n} = (S^{\otimes 2})^{\otimes k}$ and observe that the i-th factor $S^{\otimes 2}$ gives us the eigenspaces of B_{2i-1}, to which we can apply the same arguments as before. Comparing with equation (2.3) (with the χ_{i_j} evaluated at the identity element) we see that the $\mathrm{SO}(n)$ character of $S^{\otimes n}$ is the same as the one for S_{Nn} for N even, and the $\mathrm{SO}(n)$ character of $\widetilde{S}^{\otimes n}$ is $2^{n/2}$ times the character of

S_{Nn} for N odd. From this follow statements (b) and (c) (for n even). For n odd, the corresponding statements follow from the results for $n+1$ from the restriction rules of representations of $\mathfrak{so}_{n+1}$. $\square$

2.3. Quantum and fusion cases. By the main result of [36], we have commuting actions of $\mathbf{U} = \mathrm{U}_q\,\mathfrak{so}_N \rtimes \mathbb{Z}/2$ and $\mathrm{U}'_q\,\mathfrak{so}_n$ on $S^{\otimes n}$ (for N even) and $\widetilde{S}^{\otimes n}$ for N odd. Not surprisingly, the decomposition in the Lemma 2.1 carries over to this setting if q is not a root of unity. If q is a primitive 2ℓ–th root of unity, we have a similar relationship in the corresponding ribbon fusion category $\mathrm{O}(N)_r$ where $r = \ell + 2 - N$ is the level. This is the quotient category of the (ribbon) category of tilting modules in $\mathbf{U} = \mathrm{U}_q\,\mathfrak{so}_N \rtimes \mathbb{Z}_2$ by negligible morphisms. Adopting the notation from the affine Lie algebra literature, we denote this category by $\mathrm{O}(N)_r$ where $r = \ell+2-N$. In the case N is odd, we have $\mathrm{O}(N)_r \cong \mathrm{SO}(N)_r \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$, whereas in the case N is even $\mathrm{O}(N)_r$ is the $\mathbb{Z}_2$-equivariantization of $\mathrm{SO}(N)_r$. The simple objects in $\mathrm{O}(N)_r$ corresponding to $\mathrm{O}(N)$-representations are labeled by Young diagrams λ satisfying $\lambda'_1 + \lambda'_2 \leq N$ and $\lambda_1 + \lambda_2 \leq \ell + 2 - N$ and the additional Young diagram $\lambda = [\ell - N + 2, 1^N]$. The objects with half-integer spin can be described by similar inequalities. A more explicit description is given below in the case $r = 2$. We will again denote the images of the corresponding tilting modules in $\mathbf{U}$ by S (respectively $\widetilde{S}$) in the fusion category $\mathrm{O}(N)_r$. We have the following results, most of which were already proved in [36]:

Theorem 2.2. (a) *Let n be even. Then we can define an action of $\mathbf{U} \times \mathrm{U}'_q\,\mathfrak{so}_n$ on $S^{\otimes n}$ for N even (respectively $\widetilde{S}^{\otimes n}$ for N odd) whose decomposition into irreducibles is the same as in the classical case, if q is not a root of unity.*

(b) *If q is a primitive 2ℓ-th root of unity, then the objects $S^{\otimes n}$ for N even (respectively $\widetilde{S}^{\otimes n}$ for N odd) decompose in $\mathrm{O}(N)_{\ell-N+2}$ as a direct sum $\bigoplus_\lambda V_\lambda \otimes V_{\bar{\lambda}}$. Here now V_λ ranges over the objects as in the classical case, subject to the additional condition $\lambda_1 + \lambda_2 \leq \ell + 2 - N$, and the additional diagram $[\ell - N + 2, 1^N]$, and $V_{\bar{\lambda}}$ is the (via (2.2)) corresponding $\mathrm{U}'_q\,\mathfrak{so}_n$ module with highest weight $\bar{\lambda}$.*

Proof. Part (a) follows from Lemma 2.1, using the explicit representations in [36] and the fact that for q not a root of unity the representation theory of Drinfeld–Jimbo quantum groups is essentially the same as for the corresponding Lie algebra. For part (b) we just use the fact that tensor powers of S and $\widetilde{S}$ can be written as a direct sum of indecomposable tilting modules; the objects in the fusion category are obtained by taking the quotient module by the tensor ideal generated by those tilting modules which have q-dimension equal to 0. The representations of $\mathrm{U}'_q\,\mathfrak{so}_n$

into these tensor powers are still well-defined at a root of unity, and they factor over the fusion quotient. As these $U'_q\,\mathfrak{so}_n$ modules usually have smaller dimensions at a root of unity than in the generic case, we still need to check that they have the same highest weight vector. But this follows from the restriction rule: restricting the action to $\mathfrak{so}_{n-1}$, the highest weight vector is again a highest weight vector in an $\mathfrak{so}_{n-1}$-module which also exists in the fusion category. The explicit combinatorics can be checked either directly by using Gelfand-Tseitlin bases for the orthogonal case (see e.g. [9]), or by using the tensor product rules for spinor representations (see e.g. [36]) via the correspondence (2.2). □

We use the notation $\varepsilon = (1/2, 1/2, \ldots, 1/2) \in \mathbb{R}^j$ and ϵ_i for the i-th standard basis vector of $\mathbb{R}^j$. We associate these vectors with weights of $\mathfrak{so}_n$ for $n = 2j$ or $n = 2j + 1$ in the usual way. Let ρ be half the sum of the positive roots of $\mathfrak{so}_n$, and let $q^{2\rho}$ be the operator on a finite dimensional $\mathbf{U}$-module defined by $q^{2\rho}v_\mu = q^{(2\rho,\mu)}v_\mu$ for a weight vector v_μ of weight μ. We define, as usual, the q-dimension of a $\mathbf{U}$-module V by $\dim_q V = \mathrm{Tr}(q^{2\rho})$. As we have commuting actions of $\mathbf{U}$ and $U'_q\,\mathfrak{so}_n$ on $S^{\otimes n}$ (respectively $\widetilde{S}^{\otimes n}$), we can define the virtual $U'_q\,\mathfrak{so}_n$ character χ_n^ρ by

$$\chi_n^\rho(u) = \mathrm{Tr}(uq^{2\rho}),$$

where u is in the Cartan algebra of $U'_q\,\mathfrak{so}_n$, and Tr is the usual trace of $S^{\otimes n}$ (respectively $\widetilde{S}^{\otimes n}$). The following lemma follows from the multiplicativity of the trace for tensor factors, using a similar argument as in the proof of Lemma 2.1.

Lemma 2.3. *If N is even then the character χ_n^ρ is uniquely determined by*

$$\chi_n^\rho\Big(\prod B_{2i-1}^{e_{2i-1}}\Big) = \prod \chi_2^\rho(B_{2i-1}^{e_{2i-1}}),$$

and $\chi_2^\rho(B_1^e) = \sum_{j=1}^N \dim_q V_{[1^j]}[N/2 - j]^e$. If N is odd, the same formulas hold, except that we have to add a factor 2 on the right hand side of the formula for each $\chi_2^\rho(B_{2i-1}^{e_i})$, $1 \leq i < N/2$.

2.4. Weakly Integral Cases. In the rest of this paper we will mostly focus on the case $q = e^{\pi i/N}$ corresponding to $O(N)_2$.

The special cases $O(N)_2$ correspond to the quotient by negligible morphisms of the categories of tilting $\mathbf{U}$- modules for q a $2N$th root of unity. These $O(N)_2$ are weakly integral unitary ribbon fusion categories, i.e. $(\dim_q V)^2 \in \mathbb{Z}$ for simple objects V.

The related categories $\mathrm{SO}(N)_2$ (see e.g. [26]) obtained from $\mathrm{U}_q\,\mathfrak{so}_N$ at $q = e^{\pi \mathrm{i}/N}$ are also weakly integral modular categories and have simple objects labeled by highest weights for $\mathfrak{so}_N$. We will describe these categories in some detail.

Setting $N = 2k + 1$ for N odd and $N = 2k$ for N even, we denote the fundamental weights for $\mathfrak{so}_N$ by $\Lambda_1, \dots, \Lambda_k$. No confusion should arise as we deal with N even and N odd separately. For later use we define for $0 \le j \le k$ the highest weight $\gamma_j = (1, \dots, 1, 0, \dots, 0)$ with the first j entries equal to 1.

For N odd $\Lambda_k = (1/2, \dots, 1/2)$ labels the simple object S associated with the fundamental spin representation for $\mathfrak{so}_N$ and $\Lambda_j = (1, \dots, 1, 0, \dots, 0)$ for $1 \le j \le k - 1$.

For N even the two fundamental spin objects $S_\pm$ are labeled by $\Lambda_k = (1/2, \dots, 1/2)$ and $\Lambda_{k-1} = (1/2, \dots, 1/2, -1/2)$, while $\Lambda_j = (1, \dots, 1, 0 \dots, 0)$ for $1 \le j \le k - 2$.

2.4.1. *N* odd. The fusion category $\mathrm{SO}(N)_2$ for N odd has two simple (self-dual) objects $S = V_{\Lambda_k}$ and $S' = V_{\Lambda_k + \Lambda_1}$ of dimension $\sqrt{N}$, 2 simple objects $\mathbf{1} = V_{\gamma_0}$ and $V_{2\Lambda_1}$ of dimension 1, and $\frac{N}{2}$ simple objects V_{γ_s} of dimension 2 where $1 \le s \le \frac{N-1}{2}$. Thus, for N odd, the rank of $\mathrm{SO}(N)_2$ is $\frac{N-1}{2} + 4$ and the categorical dimension is $4N$.

As we have noted above, for N odd $\mathrm{O}(N)_2 \cong \mathrm{SO}(N)_2 \boxtimes \mathrm{Rep}(\mathbb{Z}_2)$ as ribbon fusion categories, so that the structure of $\mathrm{O}(N)_2$ is easily determined from that of $\mathrm{SO}(N)_2$. Here $\mathrm{Rep}(\mathbb{Z}_2)$ is regarded as the ribbon category with trivial twists and symmetric braiding. We will denote the two objects in $\mathrm{Rep}(\mathbb{Z}_2)$ be $\mathbf{1}$ and $-\mathbf{1}$ where $-\mathbf{1}$ coresonds to the non-trivial representation of $\mathbb{Z}_2$. In particular we have a $\mathbb{Z}_2$ grading of $\mathrm{O}(N)_2$ with components corresponding to $(V, \pm\mathbf{1})$. In this notation we have $S_0 = (S, \mathbf{1})$ and $S_1 = (S, -\mathbf{1})$. For example we have $\tilde{S}^{\otimes 2} = [(S, \mathbf{1}) \oplus (S, -\mathbf{1})]^{\otimes 2} \cong 2[(S^{\otimes 2}, \mathbf{1}) \oplus (S^{\otimes 2}, -\mathbf{1})]$. Moreover, the (forgetful) functor $F\colon \mathrm{O}(N)_2 \to \mathrm{SO}(N)_2$ by $F(V, \pm\mathbf{1}) \to V$ is obviously faithful and is braided since the braiding on $\mathrm{Rep}(\mathbb{Z}_2)$ is symmetric.

2.4.2. *N* even. For N even the fusion category $\mathrm{SO}(N)_2$ has 4 simple objects $S_\pm$ (labeled by Λ_k and Λ_{k-1}) and $S'_\pm$ (labeled by $\Lambda_k + \Lambda_1$ and $\Lambda_{k-1} + \Lambda_1$) of dimension $\sqrt{N/2}$, 4 simple objects $\mathbf{1}$, $V_{2\Lambda_1}$, $V_{2\Lambda_k}$, and $V_{2\Lambda_{k-1}}$ of dimension 1 and $\frac{N}{2} - 1$ simple objects V_{γ_s} of dimension 2 where $1 \le s \le \frac{N}{2} - 1$. Thus, for N even, the rank of $\mathrm{SO}(N)_2$ is $\frac{N}{2} + 7$ and the categorical dimension is $4N$.

The simple objects in $\mathrm{O}(N)_2$ are the images (under purification) of the simple $\mathrm{U}_q\,\mathfrak{so}_N \rtimes \mathbb{Z}_2$-tilting modules with non-zero q-dimension. Using [36, Section 3.4] we find that the simple objects in $\mathrm{O}(N)_2$ are S and S' of dimension $2\sqrt{N/2}$, $\mathbf{1}$, $V_{[2]}$, $V_{[1^N]}$ and $V_{[1^{N-1},1]}$ of dimension 1, and $V_{[1^s]}$ of dimension 2 with $1 \leq s \leq N-1$. The restriction map $\mathrm{Rep}(\mathrm{U}_q\,\mathfrak{so}_N \rtimes \mathbb{Z}_2) \to \mathrm{Rep}(\mathrm{U}_q\,\mathfrak{so}_N)$ induces a braided tensor functor $F\colon \mathrm{O}(N)_2 \to \mathrm{SO}(N)_2$ with images

$$\begin{aligned}
F(S) &= S_+ \oplus S_-,\\
F(S') &= S'_+ \oplus S'_-,\\
F(V_{[2]}) &= F(V_{[1^{N-1},1]}) = V_{2\Lambda_1},\\
F(V_{[1^N]}) &= F(\mathbf{1}) = \mathbf{1},\\
F(V_{[1^s]}) &= F(V_{[1^{N-s}]}) = V_{\gamma_s}, \quad 1 \leq s \leq k-1.
\end{aligned}$$

Observe that the objects S and S' in $\mathrm{O}(N)_2$ are self-dual, although $S_\pm$ are not. We now can proof the following corollary to Theorem 2.2(b).

Corollary 2.4. *Let q be a primitive $2N$-th root of unity. Then the representations Φ of $\mathrm{U}'_q\,\mathfrak{so}_n$ into the $\mathrm{O}(N)_2$ centralizer algebras $\mathrm{End}(S^{\otimes n})$ for N even (respectively $\mathrm{End}(\widetilde{S}^{\otimes n})$ for N odd) are labelled by the weights $N\varepsilon$ and $N\varepsilon - \epsilon_j$ if $n = 2j+1$ is odd, and by the weights $N\varepsilon - r\epsilon_j$, $0 \leq r \leq N$, $N\varepsilon - \epsilon_{j-1} - \epsilon_j$ and $N\varepsilon - \epsilon_{j-1} - (N-1)\epsilon_j$ if $n = 2j$ is even. For N even, Φ is surjective.*

Proof. This follows from Theorem 2.2(b) and from the restriction rules for representations of $\mathrm{U}'_q\,\mathfrak{so}_n$ (see [36, Lemma 4.2 and Proposition 4.3]) and tensor product rules of $\mathrm{O}(N)_2$. The surjectivity for N even follows from a dimension count (simply compute the Bratteli diagram for the object S). □

Remark 2.5. (a) One can check that already for $n = 2$ the map Φ is not surjective for N odd. It follows from the explicit representation of $\mathrm{U}'_q\,\mathfrak{so}_n$ in [36] that the image of B_1 in $\widetilde{S}^{\otimes 2} = (S_0 \oplus S_1)^{\otimes 2}$ = permutes $S_1^{\otimes 2}$ and $S_0^{\otimes 2}$, and similarly for the mixed terms. In particular, $\Phi(B_1)$ does not commute with the projections $p \in \mathrm{Hom}(\widetilde{S}^{\otimes 2}, V)$ where V is a simple subobject of $\widetilde{S}^{\otimes 2}$. However, $\Phi(B_1^2)$ leaves $S_i^{\otimes 2}$, $S_0 \otimes S_1$ and $S_1 \otimes S_0$ invariant (see [36]), and more generally $\Phi(B_i^2)$ leaves invariant any tensor product of any number of copies of S_0 and S_i.

(b) Using the notation $(V, \pm\mathbf{1})$ for objects in the two components of the $\mathbb{Z}_2$-grading on $\mathrm{O}(N)_2$, we have $\widetilde{S}^{\otimes n} \cong 2^{n-1}(S^{\otimes n}, \mathbf{1}) \oplus 2^{n-1}(S^{\otimes n}, -\mathbf{1})$. The (faithful braided tensor) functor $F\colon \mathrm{O}(N)_2 \to \mathrm{SO}(N)_2$ induces an algebra homomorphism $\Xi\colon \mathrm{End}(\widetilde{S}^{\otimes n}) \to \mathrm{End}((2S)^{\otimes n}) = \mathrm{End}(2^{n-1}S^{\otimes n} \oplus 2^{n-1}S^{\otimes n})$. The image of Ξ lies in the diagonal: $\mathrm{End}(2^{n-1}S^{\otimes n}) \times \mathrm{End}(2^{n-1}S^{\otimes n})$, since

$\mathrm{Hom}((S^{\otimes n},\mathbf{1}),(S^{\otimes n},-\mathbf{1})) = 0$. Moreover, as $\Phi(B_i^2) \in \mathrm{End}(\widetilde{S}^{\otimes n})$ leaves invariant each of the 2^{n-1} copies of $(S^{\otimes n},\pm\mathbf{1})$, we see that $\Xi(\Phi(B_i^2))$ lies in the diagonal $\prod_{i=1}^{2^n}\mathrm{End}(S^{\otimes n})$. Since the latter algebra is isomorphic to $\mathrm{End}(S^{\otimes n})$ we see that $\Xi\circ\Phi(\langle 1, B_1^2,\dots,B_{n-1}\rangle)$ is isomorphic to a subalgebra of $\mathrm{End}(S^{\otimes n})$.

2.5. $\mathcal{B}_n$ representations on End($S^{\otimes n}$). Denote by $\gamma_S\colon\mathcal{B}_n \to \mathrm{Aut}(S^{\otimes n})$ the representations of the braid group associated with the object S in $\mathrm{SO}(N)_2$ for N odd or $\mathrm{O}(N)_2$ for N even. Explicitly, γ_S is defined on generators by

$$\sigma_i \longrightarrow \mathrm{Id}_S^{\otimes(i-1)} \otimes c_{S,S} \otimes \mathrm{Id}_S^{\otimes(n-i-1)}.$$

For later use we compute the eigenvalues for the braiding operator $c_{S,S}$ for $\mathrm{SO}(N)_2$ when N is odd and $\mathrm{O}(N)_2$ for N even.

Remark 2.6. In the subsection only, we let $x = e^{\pi i/(2N)}$, and let $\langle\ ,\ \rangle$ be the symmetric bilinear form on the weight lattice normalized so that $\langle\alpha,\alpha\rangle = 2$ for short roots. This is to conform with the standard results, and is only different for N odd.

For $N = 2k+1$ odd, we have

$$S^{\otimes 2} \cong \bigoplus_{j=0}^{k} V_{\gamma_j}.$$

The eigenvalues of $c_{S,S}$ are easily computed, and we record them in the following

Lemma 2.7. *Let $N = 2k+1$ be odd. Up to an overall factor depending only on N, the eigenvalue of $c_{S,S}$ on the projection onto the simple object V_{γ_s} is*

$$\Psi(N,s) := i^{(k-s)^2} e^{-\pi i s^2/(2N)}. \tag{2.4}$$

Proof. It follows from Reshetikhin's formulas (see e.g. [23, Corollary 2.22]) that, up to an overall factor, $c_{S,S}$ acts on the projection onto V_λ by the scalar $\varsigma(\lambda)x^{\frac{c_\lambda}{2}}$ where $c_\lambda = \langle\lambda+2\rho,\lambda\rangle$ for any weight λ and the sign $\varsigma(\lambda) = 1$ if the corresponding $\mathfrak{so}_N$ representation appears in the symmetric tensor square of the fundamental spin representation and -1 otherwise. Observe that here $\langle\ ,\ \rangle$ is twice the usual Euclidean inner product and $2\rho = (2k-1,\dots,1)$. We compute $c_{\gamma_s} = 2(Ns-s^2)$ and note that

$$\varsigma(\gamma_s) = \begin{cases} -1 & (k-s)\equiv 1,2 \pmod 4,\\ 1 & (k-s)\equiv 0,3 \pmod 4,\end{cases}$$

from which the result follows. $\square$

In the case $N = 2k$ is even we have

$$S^{\otimes 2} = \bigoplus_{s=0}^{N} V_{[1^s]}$$

and the eigenvalues of $c_{S,S}$ are given in the following

Lemma 2.8. *Let $N = 2k$ be even. Up to an overall factor depending only on N, the eigenvalue of the* $\mathrm{O}(N)_2$ *braiding operator $c_{S,S}$ on the projection onto the simple object labeled by $[1^s]$ is $\eta(s)f(s)$, where $\eta(s) = e^{(N-2s)(N-2s+2)\pi\mathrm{i}/8}$ and $f(s) = i^s e^{-\pi\mathrm{i}s^2/(2N)}$.*

Proof. Since the functor $F\colon \mathrm{O}(N)_2 \to \mathrm{SO}(N)_2$ is a braided tensor functor we can compute the eigenvalues of $c_{S,S}$ from $F(c_{S,S})$. Up to signs these are just the eigenvalues of $c_{S_\pm,S_\pm}$ and the square roots of the eigenvalues of $c_{S_+,S_-}c_{S_-,S_+}$. These can be computed up to an overall factor using Drinfeld's quantum Casimir [4] (since $\langle\Lambda_k + 2\rho, \Lambda_k\rangle = \langle\Lambda_{k-1} + 2\rho, \Lambda_{k-1}\rangle$) as $q^{\frac{c_\lambda}{2}}$ with $q = e^{\pi\mathrm{i}/N}$ for any $V_\lambda \in F(S^{\otimes 2})$. Up to signs, the eigenvalues corresponding to $V_{[1^s]}$ and $V_{[1^{N-s}]}$ are (both) $q^{\frac{c_{\gamma_s}}{2}}$ for $0 \le s \le N/2$. We compute $c_{\gamma_s} = \langle\gamma_s + 2\rho, \gamma_s\rangle = Ns - s^2$ and set $f(s) = q^{\frac{Ns-s^2}{2}} = i^s e^{-\pi\mathrm{i}s^2/(2N)}$. Observe that $f(N-s) = f(s)$ so that $c_{S,S}$ has eigenvalue $\eta(s)f(s)$ on the projection onto $V_{[1^s]}$ for all $0 \le s \le N$, where $\eta(s)$ is a sign.

By continuity, it is enough to determine the signs for the classical case $q = 1$ for which the braiding is symmetric. One way to do this goes by induction on the dimension N, for N even (a similar argument also works for the slightly easier case N odd). One first observes that for $N = 4$ the signs are given by $\eta(0) = \eta(1) = \eta(4) = -1$ and $\eta(2) = \eta(3)$, using the fact that $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$.

The crucial observation now is that the sign for the representations $V_{[1^{N/2-s}]} \subset S_N^{\otimes 2}$ are the same as the ones for the representations $V_{[1^{N/2-s-1}]} \subset S_{N-2}^{\otimes 2}$, for $0 \le |s| < N/2$; here S_{2k} is the spinor representation in connection with $\mathrm{O}(2k)$. This follows from the fact that S_N decomposes as a $\mathrm{Pin}(N-2)$ module into the direct sum of two modules isomorphic to S_{N-2}, see e.g. the discussion in [36], Lemma 2.1. Using the eigenspace decomposition of the permutation $R_S \in \mathrm{End}(S^{\otimes 2})$, we obtain for the normalized trace tr on $\mathrm{End}(S^{\otimes 2})$

$$\frac{1}{2^{N/2}} = \mathrm{tr}(R_S) = \frac{1}{2^N}\sum_{s=0}^{N} \eta(N/2 - s)\dim\ V_{[1^{N/2-s}]}. \tag{2.5}$$

We remark that a similar formula also holds for the odd-dimensional case $\mathrm{Spin}(N+1)$, where now the summation only goes until $s = N/2$ and we have the antisymmetrizations of the $(N+1)$-dimensional vector representations on the right hand side. By induction assumption, $\eta(N/2-s)$ is known for $s < N/2$, and dim $V_{[1^{N/2-s}]}$ is equal to $\binom{N}{N/2-s}$. In the odd-dimensional case, we can now easily calculate the missing sign $\eta(0)$ from equation (2.5), as adjusted for the odd-dimensional case. To calculate the two remaining signs in the even-dimensional case, we consider Pin(N) as a subgroup of Spin$(N+1)$, which acts irreducibly via its spinor representation on the same vector space S; in particular, we can also identify the trivial subrepresentation in $S^{\otimes 2}$ for both groups, which hence has the same sign $\eta(0)$ for the permutation R_S at $q=1$. One now calculates $\eta(N)$ from equation (2.5). It is now easy to check that the signs can be given by the formula $\eta(s) = e^{\frac{(N-2s)(N-2s+2)\pi \mathrm{i}}{8}}$. □

3. Representation theory of $\mathrm{U}'_q\,\mathfrak{so}_n$

We review and (re)prove certain results of the representation theory of $\mathrm{U}'_q\,\mathfrak{so}_n$. Many of these results have already appeared in one form or another in work of Klimyk and his coauthors, see e.g. [9], [15]. However, in our case, we need these results for roots of unity where the situation is more complicated. Hence we have decided to give our own, quite different proofs by mimicking a Verma module construction. We will do this here only for what is called the classical series in [15], i.e. for representations which are deformations of representations of $\mathrm{U}\,\mathfrak{so}_n$, and those only for $n \le 5$. It is planned to give a more complete study of these representations in a separate paper [37].

3.1. Definitions. We identify roots and weights of $\mathrm{U}'_q\,\mathfrak{so}_n$ with vectors in $\mathbb{R}^k$, where $k = n/2$ or $(n-1)/2$ depending on the parity of n, as usual. So if ϵ_i is the i-th standard unit vector for $\mathbb{R}^k$, the roots are given by $\pm\epsilon_i \pm \epsilon_j$, $1 \le i < j \le k$, and, if $n = 2k+1$ is odd, also by $\pm\epsilon_i$, $1 \le i \le k$. Here the analog of the Cartan subalgebra is the algebra $\mathfrak{h}$ generated by $B_1, B_3, \ldots, B_{2k-1}$ for $n = 2k$ or $n = 2k+1$. A vector v in a $\mathrm{U}'_q\,\mathfrak{so}_n$-module is said to have weight λ if $B_{2i-1}v = [\lambda_i]v$ for all $B_{2i-1} \in \mathfrak{h}$; we shall often identify λ with the vector (λ_i). As usual, $[n] = (q^n - q^{-n})/(q - q^{-1})$. Let us first recall the following theorem, which has been proved in [15]; it also follows from the results in [36], as quoted in Theorem 2.2.

Theorem 3.1. *Let λ be a dominant integral weight, and let q be generic. Then there exists a finite dimensional simple $\mathrm{U}'_q\,\mathfrak{so}_n$ module V_λ with highest weight λ and the same weight multiplicities as for the corresponding $\mathrm{U}\,\mathfrak{so}_n$ module.*

Lemma 3.2. *Let v be a vector in a* $\mathrm{U}_q' \,\mathfrak{so}_n$ *module with weight μ. Then*

(a) $(B_{2i-1} - [\mu_i + 1])(B_{2i-1} - [\mu_i - 1])B_{2i}v = 0$;

(b) $(B_{2i+1} - [\mu_{i+1} \pm 1])(B_{2i-1} - [\mu_i + 1])B_{2i}v$ *has weight* $\mu - (\epsilon_i \pm \epsilon_{i+1})$, *if it is nonzero.*

In particular, we can write $B_{2i}v$ as a sum of two eigenvectors of B_{2i-1} (if $[\mu_i+1] \neq [\mu_i - 1]$), and we can write $(B_{2i-1} - [\mu_i + 1])B_{2i}v$ as a sum of two weight vectors (if $[\mu_{i+1} + 1] \neq [\mu_{i+1} - 1]$).

Proof. These are straightforward calculations. e.g. for (a) we have

$$\begin{aligned} B_{2i-1}^2 B_{2i} v &= ([2]B_{2i-1}B_{2i}B_{2i-1} - B_{2i}B_{2i-1}^2 + B_{2i})v \\ &= [2][\mu_i]B_{2i-1}B_{2i}v - ([\mu_i]^2 - 1)B_{2i}v. \end{aligned}$$

We now get the claimed factorization in (a) using the identities

$$[2][\mu_i] = [\mu_i + 1] + [\mu_i - 1] \quad \text{and} \quad [\mu_i]^2 - 1 = [\mu_i + 1][\mu_i - 1].$$

For part (b) observe that a similar calculation also holds with i replaced by $i + 1$. The claim follows from this. □

For a given weight λ we define the left ideal

$$I_\lambda = \mathrm{U}_q' \,\mathfrak{so}_n \langle (B_{2i-1} - [\lambda_i]1), (B_{2i-1}B_{2i} - [\lambda_i - 1]B_{2i}) \rangle \tag{3.1}$$

for all values of i for which the indices $2i-1$ and $2i$ are between (including) 1 and $(n-1)$. Observe that one can show as in Lemma 3.2 that now B_{2i} is an eigenvector of B_{2i-1} with eigenvalue $[\lambda_i - 1]$ mod I_λ. Moreover, if $[\lambda_{i+1} + 1] \neq [\lambda_{i+1} - 1]$, we can write B_{2i} as a linear combination of the two vectors $(B_{2i+1} - [\lambda_{i+1} \pm 1])B_{2i}$ of weights $\lambda - (\epsilon_i \pm \epsilon_{i+1})$ mod I_λ; observe that these are weights of the form $\lambda - \alpha$ with α a positive root of $\mathrm{U}_q' \, so_n$.

3.2. Spanning property. It has already been observed in [9] that a PBW type theorem holds for the algebra $\mathrm{U}_q' \,\mathfrak{so}_n$, using its embedding into the quantum group $\mathrm{U}_q \,\mathfrak{sl}_n$. One can also prove the existence of an analogue of a Verma module. This, and more results, are planned to appear in a separate paper [37] by the second named author. For this paper, we will only give (or outline) *ad hoc* proofs for the special cases needed for our purpose.

Lemma 3.3. *Let λ be a weight of $\mathrm{U}'_q\,\mathfrak{so}_n$ for $n \leq 5$. Then the Verma module $M_\lambda = \mathrm{U}'_q\,\mathfrak{so}_n/I_\lambda$ is spanned by the ordered products of the form*

$$B_2^{e_1}(B_3B_2)^{e_2}(B_4B_3B_2)^{e_3}B_4^{e_4}v_0,$$

where the e_i are nonnegative integers which, for $n < 5$, are equal to 0 for those factors which are not in $\mathrm{U}'_q\,\mathfrak{so}_n$, and where $v_0 \equiv 1 \bmod I_\lambda$ is the highest weight vector.

Proof. The proof can be done via elementary, albeit somewhat tedious, calculations. A more general result will be proved in [37]. We give a fairly detailed outline for a proof of this lemma for the skeptical reader as follows.

For $n = 5$, the idea is to move the generators B_4 as far to the right as possible. To make this mathematically precise, we define an order on words in the generators B_i first by the length of the word, and then by reversed alphabetical order e.g. $B_4^2 < B_3B_4 < B_4B_3$ etc. We first prove that the claim holds if we only apply generators B_i, $2 \leq i \leq 4$, to the highest weight vector. As a first step one shows that any vector generated this way is a linear combination of vectors of the form $w(B_4B_3B_2)^{e_3}B_4^{e_4}v_0$, with the word $w \in \langle B_2, B_3\rangle$. This follows by moving generators B_4 as far to the right as possible, using the relation

$$B_4(B_4B_3B_2) = [2](B_4B_3B_2)B_4 - B_3B_2B_4^2 + B_3B_2.$$

It is not hard to show that one can express $B_3B_4^jv_0$ as a linear combination of vectors $B_4^iv_\lambda$, see Lemma 3.8 for details. Moreover, we also have the relation

$$B_3(B_4B_3B_2) = (B_4B_3B_2)B_3 + [B_3, B_2B_3B_4].$$

Using it, not only can we prove our claim, but we can also show that w may be assumed to end with a B_2, by induction on e_4 and e_3. It is now an easy induction on the number of B_3s in w to prove that it can be expressed as a linear combination of words of the form $B_2^{e_1}(B_3B_2)^{e_2}$ by moving the B_3s as far to the right as possible (taking into account that a B_3 on the right end of w will be absorbed, as just mentioned). To finish the proof for $n = 5$, it suffices to show that multiplying any of the words as in the statement by B_1 again results in a linear combination of words without a B_1; this follows by a similar induction on the order of the words. The claims for $n = 4$ and $n = 3$ are proved similarly, with the proofs being much easier. □

Corollary 3.4. *A weight appears in the highest weight module N_λ for $\mathrm{U}_q'\,\mathfrak{so}_n$ with at most the multiplicity as in the Verma module M_λ for the classical case $\mathrm{U}\,\mathfrak{so}_n$ at $q=1$, for $n \leq 5$.*

Proof. We give an outline of the proof for the most difficult case $n = 5$. As N_λ is a quotient of M_λ, it suffices to prove the statement for the latter module. It is standard to check that the elements B_2, B_3B_2, B_4 and $B_4B_3B_2$ form a basis of $(\mathfrak{so}_5 + I_\lambda)/I_\lambda$ for $q = 1$. Hence their ordered polynomials form a basis for $M_\lambda = \mathrm{U}\,\mathfrak{so}_n/I_\lambda$.

Let us consider the subspaces $M_\lambda(f_1, f_2)$ spanned by all the monomials in the generators with at most f_1 and f_2 factors equal to B_2 and B_4 respectively. It follows from Lemma 3.3 and its proof that any such element can be written as a linear combination of words which also contain $\leq f_1$ factors equal to B_1 and $\leq f_2$ factors equal to B_3. Hence this space is a module of the Cartan algebra generated by B_1 and B_3. By Lemma 3.2, the zeroes of the characteristic polynomial of B_{2i-1} acting on $M_\lambda(f_1, f_2)$ can only be of the form $[\lambda_i - j]$ for some integer j. Specializing at $q = 1$ gives us the estimates on the multiplicities of the zeroes. (In fact, with a little more effort, one could show that our basis for $q = 1$ extends to a basis for general q, which proves equality for the multiplicities.) The general claim now follows by letting f_1 and f_2 go to infinity. $\square$

Remark 3.5. Having an analog of Verma modules, one can show that there exists a unique simple $\mathrm{U}_q'\,\mathfrak{so}_n$ highest weight module with given highest weight, by the usual standard arguments, in the generic case. Unfortunately, we will need this at roots of unity. Results for the usual quantum groups at roots of unity would suggest that there could be many nonisomorphic simple modules with the same highest weight, see [3] and the papers quoted therein. This leads to the consideration of certain invariant forms.

3.3. Invariant forms.

We call a sesquilinear form $(\ ,\ )$ on a $\mathrm{U}_q'\,\mathfrak{so}_n$ module M *invariant* if $(B_i v, w) = (v, B_i w)$ for all $v, w \in M$ and $1 \leq i < n$. A $\mathrm{U}_q'\,\mathfrak{so}_n$ module M is called *unitarizable* if it admits a positive definite invariant form.

In the following, we will denote a highest weight module with highest weight λ by N_λ. If q is a root of unity, the action of the operators B_i on N_λ may no longer be diagonalizable. However, we only have finitely many (generalized) weight spaces. For a weight μ we let $N_\lambda[\mu]$ be the generalized weight space of N_λ, i.e. the set of all vectors v such that $(B_{2i-1} - [\mu_i]1)^k v = 0$ for sufficiently large k. Finally, if q is a primitive 2ℓ-th root of unity, with $\ell \geq n$, we say that λ is a *restricted dominant weight* for $\mathrm{U}_q'\,\mathfrak{so}_n$ if $\lambda_1 \leq \ell/2$.

Lemma 3.6. *Let λ be a dominant integral weight with corresponding highest weight module N_λ and highest weight vector v_λ.*

(a) *For q not a root of unity, there is at most one invariant bilinear form $(\ ,\ )$ on N_λ, up to scalar multiples.*

(b) *Let now q be arbitrary, and suppose N_λ admits an invariant bilinear form $(\ ,\ )$. For $a = B_{i_1}B_{i_2}\cdots B_{i_k}$, set $a^t = B_{i_k}\cdots B_{i_2}B_{i_1}$. Then the value of (av_λ, bv_λ) is uniquely determined by (v_λ, v_λ) whenever $a^t bv_\lambda$ can be written as a linear combination of generalized weight vectors such that the $N_\lambda[\lambda]$ component is a multiple of v_λ.*

Proof. Part (a) follows from a standard argument, which we omit. It follows from invariance that

$$(a_1v_\lambda, a_2v_\lambda) = (v_\lambda, a_1^ta_2v_\lambda).$$

If q is not a root of unity, all the weight spaces are mutually orthogonal with respect to an invariant bilinear form. Hence the value of $(a_1v_\lambda, a_2v_\lambda)$ is given by the scalar of v_λ in the expansion of $a_1^ta_2v_\lambda$ as a linear combination of weight vectors, times (v_λ, v_λ). Part (b) is proved the same way. □

Remark 3.7. The strategy now will be to show that for certain dominant weights λ there exists at most one unitarizable simple module with highest weight λ. The idea is to show that, loosely speaking, any additional vectors in the weight space of λ in M_λ already have to be in the annihilator ideal of a positive semidefinite form on M_λ.

3.4. $\mathfrak{so}_3$. We now give a detailed classification of certain $\mathrm{U}_q'\mathfrak{so}_3$ modules as these results will be used later for q a root of unity.

Lemma 3.8. *Suppose q is not a root of unity, and define $v_0 \in M_\lambda$ by $v_0 = 1$ mod I_λ for the $\mathrm{U}'\mathfrak{so}_3$ weight $\lambda \in \mathbb{R}$. Then the set $\{B_2^jv_0,\ j \geq 0\}$ forms a basis of M_λ. Moreover, M_λ also has a basis of weight vectors v_j with weight $[\lambda - j]$, $j = 0, 1, \ldots$ defined inductively by $v_1 = B_2v_0$ and*

$$v_{i+1} = B_2v_i - \alpha_{i-1,i}v_{i-1}, \quad \text{for } i > 1,$$

where

$$\alpha_{i-1,i} = \frac{[i][2\lambda - i + 1]}{(q^{\lambda-i} + q^{i-\lambda})(q^{\lambda-i+1} + q^{i-\lambda-1})}.$$

In particular, if λ is a half-integer, there exists a unique simple module with highest weight λ whose dimension is $2\lambda + 1$, and on which both B_1 and B_2 act with the same set of eigenvalues $\{[\lambda - j], 0 \leq j \leq 2\lambda\}$. Finally, there is at most one invariant form $(\ ,\ )$ on M_λ, up to scalar multiples. It is completely determined by $(v_j, v_i) = 0$ for $i \neq j$ and

$$(v_{j+1}, v_{j+1}) = \alpha_{j+1,j}(v_j, v_j).$$

Proof. Let us first consider a vector space V with a basis denoted by $(\tilde{v}_j)$. We define an action of B_1 and B_2 on V by substituting v_j by $\tilde{v}_j$ in the claim, i.e. by $B_1\tilde{v}_j = [\lambda - j]\tilde{v}_j$ and by

$$B_2\tilde{v}_j = \tilde{v}_{j+1} + \alpha_{j-1,j}\tilde{v}_{j-1}.$$

It is straightforward to check that this action indeed defines a representation of $\mathrm{U}'_q\,\mathfrak{so}_3$; just apply both sides of the given relation to a basis vector $\tilde{v}_j$. It also follows directly that the map $b \mapsto bv_0$ factors over the ideal I_λ of $\in \mathrm{U}'_q\,\mathfrak{so}_3$. Hence we obtain a map from M_λ onto V which maps v_j to $\tilde{v}_j$. This shows that the v_j are linearly independent. As $B_2^j v_0 = v_j + \sum_{i=0}^{j-2}\beta_i v_i$, it follows that also the vectors $B_2^j v_0$ are linearly independent. If λ is a half-integer, one checks easily that $v_{2\lambda+1}$ generates an ideal spanned by the vectors v_j with $j \geq 2\lambda + 1$. As M_λ has a basis of weight vectors, the maximality of this ideal follows from a well-known standard argument.

To prove the statement about eigenvalues, we use the representations of $\mathrm{U}'_q\,\mathfrak{so}_3$ in [36]. They are given by mapping B_1 to $B \otimes 1$ and B_2 to $1 \otimes B$, where $B \in \mathrm{End}(S^{\otimes 2})$ and 1 stands for the identity of S, with S the spinor representation as described in previous sections. It is well-known that B_1 and B_2 are conjugated via certain braiding morphisms, and these braiding morphisms are in the algebra generated by B_1 and B_2 (see Section 2.5).

Let $(\ ,\ )$ be an invariant form on M_λ. If q is not a root of unity, then $[\lambda - j] \neq [\lambda - i]$ for $i \neq j$. Hence, by invariance, the v_j are pairwise orthogonal. But then we also have

$$\begin{aligned}(v_{j+1}, v_{j+1}) &= (B_2 v_j - \alpha_{j-1,j}v_{j-1}, v_{j+1})\\ &= (v_j, B_2 v_{j+1})\\ &= (v_j, v_{j+2} + \alpha_{j,j+1}v_j).\end{aligned}$$

The claim now follows from the fact that $(v_{j-1}, v_{j+1}) = (v_j, v_{j+2}) = 0$. $\square$

Lemma 3.9. *Let q be a primitive 2ℓ-th root of unity, and let $0 \le \lambda \le \ell/2$, with λ being a half-integer. Then there exists a unique simple unitary $\mathrm{U}'_q\,\mathfrak{so}_3$ module with highest weight λ.*

Proof. The proof goes along the lines of Lemma 3.8 by showing that any module as in the statement induces a unique form on M_λ. The main problem now is that B_1 has large eigenspaces on M_λ. First assume $\lambda < \ell/2$. Then we can construct vectors v_j, $0 \le j \le 2\lambda + 1$ with the same inner products as before. In particular, we have $(v_{2\lambda+1}, v_{2\lambda+1}) = 0$. As the pullback of the form $(\ ,\ )$ on M_λ is positive semidefinite, it follows that $v_{2\lambda+1}$ is in its annihilator ideal. Hence also the vectors $\tilde{v}_{2\lambda+1+j} = B_2^j v_{2\lambda+1}$ are in the annihilator ideal. As the vectors v_j respectively $\tilde{v}_j$ are of the form $B_2^j v_0 +$ *lower terms*, the form is uniquely determined on M_λ.

The same strategy also works for $\lambda = \ell/2$ until the construction of v_ℓ. We know from the generic case that, in M_λ, we have

$$v_{2\lambda+1} = \prod_{j=0}^{2\lambda} (B_2 - [\lambda - j])v_0,$$

see Lemma 3.8. As B_2 acts via a diagonalizable matrix in a unitary representation W, $v_{2\lambda+1}$ must be in the annihilator ideal of the pull-back of the positive definite form on W. So, in particular, also $(v_{\ell+1}, v_{\ell-1}) = 0$ if $\lambda = \ell/2$. Using this, we can prove the claim as before for $\lambda < \ell/2$. □

3.5. $\mathfrak{so}_4$ and $\mathfrak{so}_5$. First recall the weight structures for Verma modules for $\mathfrak{so}_4$. We have seen in the last subsection that there exist polynomials P_j of degree j such that $v_j = P_j(B_2)v_0$ is a weight vector of weight $\lambda - j$, where v_0 is the highest weight vector of the Verma module of $\mathrm{U}'_q\,\mathfrak{so}_3$ with highest weight λ. Then also $B_3^k P_j(B_2)v_\lambda$ is an eigenvector of B_1 with eigenvalue $[\lambda_1 - j]$, where v_λ is the highest weight vector of a $\mathrm{U}'_q\,\mathfrak{so}_4$ highest weight module. In view of Lemma 3.2, it follows by induction on j that the eigenvalues of B_3 are of the form $[\lambda_2 - j + 2i]$, $0 \le i \le j$. This can be written as

$$\prod_{i=0}^{j} (B_3 - [\lambda_2 - j + 2i])P_j(B_2)v_\lambda = 0.$$

Now leaving out the factor for a fixed $i = i_0$ gives us a weight vector of weight $(\lambda_1 - j, \lambda_2 - j + 2i_0)$, or, possibly the zero vector. As $(\lambda_2 - 1, \lambda_1 + 1)$ and $(-\lambda_2 - 1, -\lambda_1 - 1)$ are not weights of the simple $\mathrm{U}'_q\,\mathfrak{so}_4$ module with highest weight $\lambda = (\lambda_1, \lambda_2)$, the just mentioned expressions for these vectors have to be in an ideal of the Verma module. This means they are in the annihilator ideal of any invariant form in the generic case. Indeed, it follows from Harish-Chandra's theorem (see e.g. [32], Theorem 4.7.3) that these vectors generate the maximal ideal in the classical case. In view of our explicit basis, this can also be checked directly for $\mathrm{U}'_q\,\mathfrak{so}_4$ in the generic case.

If q is a primitive 2ℓ-th root of unity, and $0 \le \lambda_2 \le \lambda_1 \le \ell/2$, it is straightforward to check that the weight vectors mentioned in the last paragraph are also in the annihilator ideal of any invariant form, using Lemma 3.6, except possibly if $\lambda_1 = \ell/2$ and $|\lambda_2|$ is equal to $\ell/2$ or $\ell/2 - 1$. In the first case, we basically have a $\mathrm{U}'_q\,\mathfrak{so}_3$ module, as, e.g. for $\lambda_2 = \ell/2$ we have $B_3 B_2 v_\lambda = [\lambda_2 - 1] B_2 v_\lambda$ and the claim follows from the previous section. Similarly, if $\lambda_2 = \ell/2 - 1$, one considers the quotient of M_λ modulo the vector of weight $(\ell/2-2, \ell/2+1)$. It is not hard to check that it is the sum of two $\mathrm{U}'_q\,\mathfrak{so}_3$ modules with highest weights $\ell/2$ and $\ell/2-1$, and the claim again follows from Lemma 3.9. We have shown most of the following

Lemma 3.10. *Let $q = e^{\pm\pi i/\ell}$. There is at most one simple unitary $\mathrm{U}'_q\,\mathfrak{so}_4$ module with highest weight λ for any restricted dominant weight λ. The same uniqueness statement holds for a unitary $\mathrm{U}'_q\,\mathfrak{so}_5$ module with highest weight $\lambda = (\ell/2, \ell/2)$ or $\lambda = (\ell/2, \ell/2-1)$, provided its restriction to $\mathrm{U}'_q\,\mathfrak{so}_4$ is isomorphic to the corresponding restriction for the $\mathrm{U}'_q\,\mathfrak{so}_5$ module in Corollary 2.4 with the same highest weight λ.*

Proof. After the previous discussion, it only remains to check the claim for the two $\mathrm{U}'_q\,\mathfrak{so}_5$ modules. This can be done by a straightforward inspection as follows. One first checks that all the inner products for $\mathrm{U}'_q\,\mathfrak{so}_4$ highest weight vectors are uniquely determined by the value of (v_λ, v_λ), by Lemmas 3.6 and 3.9. To do this, one deduces from the character formulas in Lemma 2.1 and Theorem 2.2 that for $\lambda = (\ell/2, \ell/2)$, the corresponding $\mathrm{U}'_q\,\mathfrak{so}_5$ module decomposes as a direct sum of simple $\mathrm{U}'_q\,\mathfrak{so}_4$-modules with highest weights $(\ell/2, j)$ and highest weight vectors $P_j(B_4)v_\lambda$, for which the inner products are known by Lemma 3.9. The same method works for $\lambda = (\ell/2, \ell/2-1)$, except for the submodules with highest weights $(\ell/2-1, \pm(\ell/2-1))$. In the latter exceptional cases, the uniqueness of the norm can be deduced using Lemma 3.6. The claim now follows from this and and the already proven claim for unitary $\mathrm{U}'_q\,\mathfrak{so}_4$ modules. □

4. Quantum torus and braid representations

4.1. Quantum torus. Let $n > 1$ and let A be an $(n-1)\times(n-1)$ integer matrix defined by $a_{ij} = (j-i)$ if $|i-j| = 1$ and by $a_{ij} = 0$ otherwise. The quantum

$(n-1)$-torus associated with A is

$$T_q(n) := \mathbb{C}\langle u_1^{\pm 1}, \dots u_{n-1}^{\pm 1} : u_i u_j = q^{a_{ij}} u_j u_i \rangle.$$

For $q \in \mathbb{C}^*$ we may specialize $T_q(n)$ at q. In this situation we can give $T_q(n)$ the structure of a $*$-algebra by setting $u_i^* = u_i^{-1}$.

We have the following elementary lemma:

Lemma 4.1. *The algebra $T_q(n)$ has a basis consisting of the monomials*

$$u_1^{m_1}, \quad u_2^{m_2}, \dots, \quad u_{n-1}^{m_{n-1}}$$

with $m_j \in \mathbb{Z}$ for $1 \leq j < n$.

Proof. The spanning property is easy to check, using the fact that the generators u_i commute up to multiplication by a power of q. To prove linear independence, we define an action of u_i on the space of Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ by

$$u_i x^{\vec{m}} = q^{m_{i-1}} x_i x^{\vec{m}},$$

where $\vec{m} \in \mathbb{Z}^{n-1}$ and $x^{\vec{m}} = x_1^{m_1} x_2^{m_2} \dots x_{n-1}^{m_{n-1}}$. We leave it to the reader to check that this is indeed a representation of $T_q(n)$. The linear independence follows from $u^{\vec{m}} 1 = x^{\vec{m}}$ and the linear independence of the vectors $x^{\vec{m}}$. $\square$

4.2. Finite dimensional representations. If (ρ, V) is a d-dimensional representation of $T_q(n)$ for $n \geq 3$ then $u_i u_{i+1} u_i^{-1} = q u_{i+1}$ implies that $\mathrm{Spec}(\rho(u_i))$ is invariant under multiplication by q. This, in turn, implies that $q^k = 1$ for some k dividing d. Moreover, it is easy to check that $q^k = 1$ if and only if u_i^k is in the center of $T_q(n)$. We define for any $\vec{z} \in S^{n-1}$, where $S = \{z \in \mathbb{C}, |z| = 1\}$, the quotient $T_q^k(n, \vec{z})$ of $T_q(n)$ (specialized at a primitive kth root of unity) via the additional relations $u_i^k = z_i^k$, $1 \leq i \leq n-1$.

Proposition 4.2. (a) *The algebra $T_q(n)$ has nontrivial finite dimensional representations if and only if q is a root of unity of finite order.*

(b) *The algebra $T_q^k(n, \vec{z})$ has dimension k^{n-1}. It has one simple module of dimension $k^{(n-1)/2}$ for n odd, and k non-isomorphic simple modules of dimension $k^{(n-2)/2}$.*

Proof. Part (a) has been proved already.

It also follows easily that the dimension of $T_q(n,\vec{z})$ is at most as stated in (b). To prove the remainder of (b), suppose first that n is odd so that $T_q^k(n,\vec{z})$ has an even number of generators: $u_1,\ldots,u_{n-1}$. Let V be a $k^{(n-1)/2}$-dimensional vector space with basis $v(\vec{\imath})$, where $\vec{\imath} \in \{0,1,\ldots,k-1\}^{(n-1)/2}$. The action of u_{2s-1} on V is defined by $u_{2s-1}v(\vec{\imath}) = z_{2s-1}q^{i_s}v(\vec{\imath})$. The action of u_{2s} is given by the rule (indices modulo k)

$$u_{2s}(v(i_1,\ldots,i_s,i_{s+1},\ldots,i_{\frac{n-1}{2}})) = z_{2s}v(i_1,\ldots,i_s+1,i_{s+1}-1,i_{s+2},\ldots,i_{\frac{n-1}{2}});$$

in other words, the even indexed generators u_{2s} permute the vectors $v(i_1,\ldots,i_{\frac{n-1}{2}})$ by shifting the sth index up by 1 and the $(s+1)$th index down by 1, except for $s=(n-1)/2$ where there is no index left for shifting down.

It is straightforward to check that V is a $T_q^k(n,\vec{z})$-module. Standard arguments show that if W is a submodule of V, it must contain at least one common eigenvector of the elements u_{2s-1}, $1 \le s < n/2$, i.e. one of our basis vectors. It then follows for n odd that W contains all basis vectors, i.e. $W = V$ is simple. It follows that the image of $T_q(n,\vec{z})$ is the full matrix ring on V. This proves all the statements in (b) for n odd.

For n even, we look at the restriction of the just constructed representation of $T_q^k(n+1,\vec{z})$ to $T_q^k(n,\vec{z})$. It obviously must be faithful. On the other hand, it decomposes into the direct sum of V_r, $0 \le r < k$ of $T_q^k(n,\vec{z})$-modules, where each V_r is the span of vectors $v(\vec{\imath})$ for which the sum of the indices $i_1+i_2+\cdots+i_{(n-1)/2}$ is congruent to r mod k. From this follow the remaining statements of (b) for n even. □

In what follows we will only need to deal with the special case $\vec{z} = (1,\ldots,1)$ for which we set $T_q^k(n) = T_q^k(n,(1,\ldots,1))$.

4.3. $\mathrm{U}_q'\,\mathfrak{so}_n$ representations into the quantum torus.

Let B_i, $1 \le i < n$ be the generators of $\mathrm{U}_q'\,\mathfrak{so}_n$, as before.

Lemma 4.3. (a) *The assignments*

$$B_i \longrightarrow \pm\frac{u_i - u_i^{-1}}{q-q^{-1}} \quad \textit{and} \quad B_i \longrightarrow \pm\mathrm{i}\frac{u_i + u_i^{-1}}{q-q^{-1}}$$

extend to algebra homomorphisms

$$U_q'\mathfrak{so}_n \longrightarrow T_q(n)$$

(*for arbitrary* q).

(b) *For* $q = e^{2\pi \mathrm{i}/(2N)}$ *the assignments in* (*a*) *extend to algebra homomorphisms*

$$U_q'\mathfrak{so}_n \longrightarrow T_q^{2N}(n).$$

(c) Even case: $N = 2k$. *Denote by*

$$\Psi\colon U_q'\mathfrak{so}_n \longrightarrow T_q^{2N}(n)$$

the algebra homomorphism determined by

$$B_i \longrightarrow b_i := \mathrm{i}\frac{u_i + u_i^{-1}}{q - q^{-1}}$$

as in (*b*)*. Then*

$$\prod_{j=-\frac{N}{2}}^{\frac{N}{2}} (B_i - [j]) \in \ker(\psi_N)$$

where $[j] := \frac{q^j - q^{-j}}{q - q^{-1}}$*. Moreover, the set* $[j]$ *are distinct for* $-\frac{N}{2} \leq j \leq \frac{N}{2}$.

(d) Odd case: $N = 2k + 1$. *Denote by*

$$\Psi\colon U_q'\mathfrak{so}_n \longrightarrow T_q^{2N}(n)$$

the map determined by

$$B_i \longrightarrow b_i := \mathrm{i}\frac{u_i + u_i^{-1}}{q - q^{-1}}$$

as in (*b*)*. Then*

$$\prod_{j=-k-1}^{k} \left(B_i - \left[j + \frac{1}{2}\right]\right) \in \ker(\Psi).$$

In particular the image of the subalgebra of $U_q'\mathfrak{so}_n$ *generated by* B_i^2 *factors through the algebra* $Uo_q(n, k)$ (*see* [36, Definition 4.7(c)])*, so that* Ψ *induces*

$$\widehat{\psi}_N\colon Uo_q(n, k) \longrightarrow T_q^{2N}(n).$$

Moreover, the b_i *eigenvalues* $[j + \frac{1}{2}]$ *for* $-k - 1 \leq j \leq k$ *and the* b_i^2 *eigenvalues* $[j + \frac{1}{2}]^2$ *for* $0 \leq j \leq k$ *are distinct.*

Proof. Part (a) is a straight-forward calculation: the case $n = 3$ is sufficient since far-commutation is obvious, and writing out the q-Serre relations with $B_i = x(u_i \pm u_i^{-1})$ gives the specified values of x.

Part (b) is obvious since $q^2 \neq 1$.

For (c) first observe that for $-\frac{N}{2} \leq j \leq \frac{N}{2}$ the $N+1$ numbers $[j]$ are distinct since $\sin(x)$ is increasing on $[-\pi, \pi]$. We have $u_i^{2N} = 1$ so

$$\operatorname{Spec}(u_i) = \{q^j : 0 \leq j \leq 2N-1\},$$

where $q = e^{\pi \mathrm{i}/N}$ and $\mathrm{i} = q^{\frac{N}{2}}$. Thus

$$\operatorname{Spec}(b_i) = \left\{ q^{N/2} \frac{(q^j + q^{-j})}{q - q^{-1}} : 0 \leq j \leq 2N-1 \right\}.$$

Since $q^{N/2} q^{-j} = -q^{-j-N/2}$ and

$$\{j + N/2 \pmod{2N} : 0 \leq j \leq 2N-1\} = \{j \pmod{2N} : 0 \leq j \leq 2N-1\}$$

we have

$$\operatorname{Spec}(b_i) = \{[j] : -N/2 \leq j \leq N/2\}.$$

Since $b_i^* = -b_i$ the minimal polynomial of b_i is a product of distinct (linear) factors.

For (d) we note as above that $\{[j+1/2] : -k-1 \leq j \leq k\}$ is a set of $2k+2$ distinct numbers and $\{[j+1/2]^2 : 0 \leq j \leq k\}$ is a set of $k+1$ distinct numbers (since $[j+1/2] = -[-j-1/2]$). We have $u_i^{2N} = 1$ so

$$\operatorname{Spec}(u_i) = \{q^j : 0 \leq j \leq 2N-1\}$$

(where $q = e^{2\pi \mathrm{i}/(2N)}$) and $\mathrm{i} = q^{N/2}$. Thus $\mathrm{i}(q^j + q^{-j}) = q^{j+N/2} - q^{-j-N/2}$, and $\{j + N/2 \pmod{2N} : 0 \leq j \leq 2N-1\} = \{j + 1/2 \pmod{2N} : 0 \leq j \leq 2N-1\}$ so we have $\operatorname{Spec}(b_i) = \{[j+1/2] : -k-1 \leq j \leq k\}$. As in (c), the minimal polynomials of b_i and b_i^2 are products of distinct linear factors. $\square$

4.4. Basics from subfactor theory. In order to compare the representations defined in this section with the ones defined before in connection with fusion categories we shall need a few basic results from Jones' theory of subfactors (see [18, Section 3.1]). Let $\mathcal{A} \subset \mathcal{B}$ be finite or infinite dimensional unital von Neumann algebras with the same identity. Assume that $\mathcal{B}$ has a finite trace tr satisfying $tr(1) = 1$ and $(b, b) = tr(b^* b) > 0$ for $b \neq 0$. Let $L^2(\mathcal{B}, tr)$ be the Hilbert space completion of $\mathcal{B}$ under the inner product $(\ , \ )$, and let e_A be the orthogonal projection onto $L^2(\mathcal{A}, tr) \subset L^2(\mathcal{B}, tr)$. It can be shown that it maps any element $b \in \mathcal{B}$ to an element $\epsilon_{\mathcal{A}}(b) \in \mathcal{A}$. The algebra $\langle \mathcal{B}, e_A \rangle$ is called Jones' basic construction for $\mathcal{A} \subset \mathcal{B}$. If $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ are finite dimensional algebras and $e \in \mathcal{C}$ is such that $ebe = \epsilon_{\mathcal{A}}(b)e$ for all $b \in \mathcal{B}$ and the map $a \mapsto ae$ defines an algebra isomorphism between $\mathcal{A}$ and $\mathcal{A}e$, one can show that $\langle \mathcal{B}, e \rangle \cong \mathcal{B}e\mathcal{B} \oplus \mathcal{B}'$, where $\mathcal{B}e\mathcal{B}$ is isomorphic to a Jones basic construction for $\mathcal{A} \subset \mathcal{B}$, and $\mathcal{B}'$ is isomorphic to a subalgebra of $\mathcal{B}$.

4.5. Algebra isomorphisms. We consider the following set-up. Let A_i, $i \in \mathbb{N}$ be a sequence of self-adjoint operators acting on a Hilbert space, satisfying the following conditions.

(1) We have $[A_i, A_j] = 0$ for $|i - j| > 1$, and $\mathcal{A}_{i,j} = \langle A_i, A_{i+1}, \dots, A_{j-1}\rangle$ is a finite-dimensional algebra for all $i < j$.

(2) The map $A_i \mapsto A_{i+1}$, $1 \le i \le j-2$ induces an isomorphism between $\mathcal{A}_{1,j-1}$ and $\mathcal{A}_{2,j}$.

(3) There exists a unital trace on the algebra $\mathcal{A}$ generated by the elements A_i, $i \in \mathbb{N}$, and an $m > 0$ such that $\mathcal{A}_{i,j+1} = \langle \mathcal{A}_{i,j}, e_j\rangle$ is isomorphic to a Jones basic construction for $\mathcal{A}_{i,j-1} \subset \mathcal{A}_{i,j}$ whenever $j - i \ge m$, where e_i is an eigenprojection of A_i.

Remark 4.4. The conditions above are satisfied for any self-dual object X in a braided unitary fusion category for which $\mathrm{End}(X^{\otimes 2})$ is generated by an element A for which e is the projection onto the trivial object $\mathbf{1} \subset X^{\otimes 2}$, and where $\mathrm{End}(X^{\otimes n})$ is generated by the elements $A_i = 1_{i-1} \otimes A \otimes 1_{n-1-i}$ (see e.g. [36], Prop 2.2 and the references given there).

Lemma 4.5. *Let* (A_i) *and* $(\tilde{A}_i)$ *be operators satisfying conditions* (1)$-$(3) *above, with* $m \in \mathbb{N}$ *as in condition (3). Assume that* $\Phi\colon A_i \mapsto \tilde{A}_i$, $1 \le i \le m$ *defines an algebra isomorphism between* $\mathcal{A}_{1,m+1}$ *and* $\tilde{\mathcal{A}}_{1,m+1}$. *Then we can extend* Φ *to an algebra isomorphism between* $\mathcal{A}_{1,\infty}$ *and* $\tilde{\mathcal{A}}_{1,\infty}$ *such that* A_i *is mapped to* $\tilde{A}_i$ *for* $i \ge m$. *We call this an inclusion-respecting isomorphism between these algebras.*

Proof. It follows from our conditions that we can extend Φ to an algebra isomorphism between $\mathcal{A}_{1,\infty}$ and $\tilde{\mathcal{A}}_{1,\infty}$ by mapping e_i to $\tilde{e}_i$ for $i > m$, by uniqueness of the basic construction. It remains to show that it maps A_i to $\tilde{A}_i$ for $i > m$. We show this for the algebras $\mathcal{A}_{1,j}$ by induction on j, with $j \le m+1$ established by assumption. For the induction step $j \to j+1$, we extend Φ to $\mathcal{A}_{1,j+1}$ by mapping e_j to $\tilde{e}_j$. This also defines an injective homomorphism from $\mathcal{A}_{2,j+1}$ into the algebra generated by $\tilde{\mathcal{A}}_{2,j}$ and $\tilde{e}_j$, which is a subalgebra of $\tilde{\mathcal{A}}_{2,j+1}$. By injectivity and dimension count, the image actually is $\tilde{\mathcal{A}}_{2,j+1}$.

On the other hand, using the induction assumption and the isomorphisms of condition (2), there exists an isomorphism between $\mathcal{A}_{2,j+1}$ and $\tilde{\mathcal{A}}_{2,j+1}$ which maps A_i to $\tilde{A}_i$ for $2 \le i \le j$. As it also maps e_j to $\tilde{e}_j$, it must coincide with the restriction of Φ to $\mathcal{A}_{2,j}$. This shows the claim. $\square$

4.6. Identifying the representations. We use the notation

$$\varepsilon = (1/2, 1/2, \ldots, 1/2) \in \mathbb{R}^j$$

and ϵ_i for the i-th standard basis vector of $\mathbb{R}^j$. We associate these vectors with weights of $\mathfrak{so}_n$ for $n = 2k$ or $n = 2k + 1$ in the usual way.

Theorem 4.6. *We have the following inclusion-respecting isomorphisms* (*in the sense of Lemma 4.5*) *where* $\Psi\colon \mathrm{U}'_q\,\mathfrak{so}_n \to T_q^{2N}(n)$ *is as in Lemma 4.3 and* $\Phi\colon \mathrm{U}'_q\,\mathfrak{so}_n \to \mathrm{End}(S^{\otimes n})$ *for* N *even, respectively* $\Phi\colon \mathrm{U}'_q\,\mathfrak{so}_n \to \mathrm{End}(\widetilde{S}^{\otimes n})$:

(*a*) *for* N *even,*

$$\mathrm{End}(S^{\otimes n}) = \Phi(U'_q\mathfrak{so}_n) \cong \Psi(\mathrm{U}'_q\,\mathfrak{so}_n) = \langle 1, u_1 + u_1^{-1}, \ldots, u_{n-1} + u_{n-1}^{-1}\rangle;$$

(*b*) *for* N *odd,*

$$\mathrm{End}(\widetilde{S}^{\otimes n}) \supset \Phi(U'_q\mathfrak{so}_n) \cong \Psi(\mathrm{U}'_q\,\mathfrak{so}_n) = \langle 1, u_1 + u_1^{-1}, \ldots, u_{n-1} + u_{n-1}^{-1}\rangle;$$

(*c*) *for* N *odd,*

$$\mathrm{End}(S^{\otimes n}) \cong \Psi(\langle 1, B_1^2, \ldots, B_{n-1}^2\rangle) = \langle 1, u_1^2 + u_1^{-2}, \ldots, u_{n-1}^2 + u_{n-1}^{-2}\rangle,$$

where $S \in \mathrm{SO}(N)_2$ *is the fundamental spinor object.*

Proof. Parts (a) and (b) are proved by checking that conditions (1)-(3) of Subsection 4.2 and Lemma 4.5 are satisfied for $A_i = \Phi(B_i)$ and for $\widetilde{A}_i = \Psi(B_i)$. Conditions (1) and (2) are easy to check, using Remark 4.4 and the fact that $u_i \mapsto u_{i+1}$ also induces a homomorphism in the quantum torus with q a root of unity. Indeed, $\widetilde{S}$ is a self-dual object in $\mathrm{O}(N)_2$ and the element $A_1 \in \mathrm{End}(S^{\otimes 2})$ generates the image of $\Phi(U'_q\mathfrak{so}_2)$.

Observe that the representation Ψ of $\mathrm{U}'_q\,\mathfrak{so}_n$ into $T_q^{2N}(n)$ for $q = e^{2\pi \mathrm{i}/(2N)}$ in the previous section has the same simple components (though not with the same multiplicities) as its representation Φ into $\mathrm{End}(S^{\otimes n})$ respectively $\mathrm{End}(\widetilde{S}^{\otimes n})$ in Corollary 2.4 for $n \leq 5$. Indeed, for $n = 2$ it suffices to calculate the eigenvalues of B_1, which was done in Lemma 4.3. They coincide with the ones in the fusion representation, see [36, Lemma 4.2 and Proposition 4.3]. It is now easy to check that the usual trace for the standard representation of the quantum torus satisfies the same conditions as the functions χ_n^ρ of Lemma 2.3. Hence the same irreducible characters of $\mathrm{U}'_q\,\mathfrak{so}_n$ for n even appear in its representation into the quantum torus as in its representation into $\mathrm{End}(S^{\otimes n})$ respectively $\mathrm{End}(\widetilde{S}^{\otimes n})$. But as unitary representations are uniquely determined by their highest weights for $n \leq 5$, with the additional condition on the restriction for $n = 5$ by Lemma 3.10 (observe that all entries μ_i of our weights have absolute value $\leq \ell/2$), thus the irreducible representations of $\mathrm{U}'_q\,\mathfrak{so}_n$ in the quantum torus coincide with the ones in the fusion category, for $n \leq 5$.

Finally, condition (3) of Subsection 4.2 holds for the algebras $\mathcal{A}_{i,j}$ with $m = 4$ by Remark 4.4 and it was verified by Jones for the algebras $\widetilde{\mathcal{A}}_{i,j}$, see [19]. But now the conditions of Lemma 4.5 are satisfied for A_i and $\widetilde{A}_i$ with $m = 4$, and parts (a) and (b) follow.

Now suppose N is odd. Denote by $\mathcal{D} \subset U'_q\mathfrak{so}_n$ the algebra generated by the $(B_i)^2$. Clearly the inclusion-respecting isomorphism of (b) restricts to $\Phi(\mathcal{D}) \cong \Psi(\mathcal{D})$. Now it is an easy exercise in computing Bratteli diagrams (cf. [20, Section 5]) to see that $\dim \Psi(\mathcal{D}) = \dim \mathrm{End}(S^{\otimes n})$. It follows from this and Remark 2.5(b) that $\Phi(\mathcal{D}) \cong \mathrm{End}(S^{\otimes n})$. $\square$

4.7. Braid representations into quantum torus. The isomorphism in the last theorem transports the braid representations from the fusion categories to braid representations into the quantum torus. We determine precisely the images of the braid generators in these representations, up to an overall scalar factor.

Proposition 4.7. *Let $q = e^{\pi \mathrm{i}/N}$ and $\psi\colon \mathcal{B}_n \to T_q^{2N}(n)$ the braid group representations obtained as compositions of $\gamma_S\colon \mathcal{B}_n \to \mathrm{Aut}(S^{\otimes n})$ from Subsection 2.5 and the isomorphisms of Theorem 4.6(a,c). Then,*

(*a*) *for N odd,*

$$\psi(\sigma_i) = \frac{\gamma}{\sqrt{N}} \sum_{j=0}^{N-1} Q^{j^2} u_i^{2j},$$

where $Q = q^2 = e^{2\pi \mathrm{i}/N}$ and γ is a scalar of norm 1;

(*b*) *for N even,*

$$\psi(\sigma_i) = \frac{\gamma}{\sqrt{2N}} \sum_{j=0}^{2N-1} x^{\alpha j^2} u_i^{j},$$

where $x = e^{\pi \mathrm{i}/(2N)}$, $\alpha = 1 - N(-1)^{N/2}$ and γ is a scalar of norm 1.

Proof. Clearly $\psi(\sigma_i)$ must be a polynomial in $b_i = \frac{u_i + u_i^{-1}}{q - q^{-1}}$ for N even and b_i^2 for N odd. Since the isomorphisms of Theorem 4.6 respect inclusions it is enough to prove that

$$\psi(\sigma_1) = R_o := \frac{\gamma}{\sqrt{N}} \sum_{j=0}^{N-1} Q^{j^2} u_1^{2j} \in T_q^{2N}(n)$$

for N odd and

$$\psi(\sigma_1) = R_e := \frac{\gamma}{\sqrt{2N}} \sum_{j=0}^{2N-1} \tau(x)^{j^2} u_1^j \in T_q^{2N}(n)$$

for N even (for some scalars γ and some $\tau \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(x))$). Comparing the coefficients of u_1^j and u_1^{-j} one sees that R_o and R_e are indeed polynomials in b_1^2 respectively b_1. Since the number of distinct eigenvalues of b_1 and b_1^2 is equal to the dimension of $\mathrm{End}(S^{\otimes 2})$ (for N even, respectively odd) it is enough to verify that the eigenvalues of R_o and R_e coincide with those of $c_{S,S}$ in Lemmas 2.8 and 2.4 on each B_1-eigenspace. The eigenvalues of B_1 are computed in [36, Lemma 4.2]: the eigenvalue of B_1 on the projection onto $V_{[1^{N/2-j}]}$ is $[j]$ (note that in [36] the Young diagram in the subscript has a typo: $2k$ should be replaced by $k = N/2$ as we have here). For N odd, we must verify that $R_o v = \mathrm{i}^{(N/2-s)^2+s} e^{\frac{-s^2\pi \mathrm{i}}{2N}} v$ for any eigenvector v of b_1^2 with eigenvalue $[N/2-s]^2$, for $0 \le s \le (N-1)/2$ (up to a scalar independent of s) and for N even $R_e v = \eta(N/2-s) f(N/2-s) v$ for any eigenvector v of b_1 with eigenvalue $[s]$, for $-N/2 \le s \le N/2$ where η and f are functions defined in Lemma 2.8 (up to a scalar, independent of s, and some choice of τ).

We will give the details in the N even case and leave the N odd case to the reader.

For N even and $-N/2 \le s \le N/2$, u_1 acts on the $[s]$-eigenspace of b_1 by $x^{\pm(2s-N)}$. The corresponding eigenvalue of $\frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} (x)^{j^2} u_1^j$ is

$$\frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} x^{j^2 \pm (2s-N)j}.$$

Completing the square we have

$$\frac{x^{-(s-N/2)^2}}{\sqrt{2N}} \sum_{j=0}^{2N-1} x^{(j \pm (s-N/2))^2}.$$

Since x is a $4N$th root of unity and the set of residues modulo $4N$ of $(j \pm (s-N/2))^2$ is the same for $0 \le j \le 2N-1$ and $2N \le j \le 4N-1$ we double the sum to obtain:

$$\frac{x^{-(s-N/2)^2}}{2\sqrt{2N}} \sum_{j=0}^{4N-1} x^{j^2} = \frac{x^{-(s-N/2)^2}(1+\mathrm{i})}{\sqrt{2}}.$$

using Dirichlet's improvement on Gauss' result (see e.g. [2]).

Rescaling (independent of s) we obtain the eigenvalue $f(N/2-s)(-\mathrm{i})^{(N/2-s)}$ for R_e on these spaces. The result now follows by verifying, for $\alpha = 1-N(-1)^{N/2}$, that

$$\frac{[f(N/2-s)(-\mathrm{i})^{(N/2-s)}]^\alpha}{(\eta(N/2-s)f(N/2-s))}$$

is independent of s. □

Remark 4.8. The Gaussian representations of $\mathcal{B}_n$ in $T_n^{2N}(n)$ described in Proposition 4.7(a) go back at least to [11] in the case N is odd and were certainly known to Jones in the case $N = 3$ in the early 1980s. In the case N is even these representations seemed to explicitly appear only recently [7], in which results of [19] are employed, and their properties are studied in some detail.

As a consequence we can prove (a generalized version of) [28, Conjecture 5.4]:

Theorem 4.9. *The images of the braid group representations on* $\mathrm{End}_{\mathrm{SO}(N)_2}(S^{\otimes n})$ *for N odd and* $\mathrm{End}_{\mathrm{SO}(N)_2}(S_\pm^{\otimes n})$ *for N even are isomorphic to images of braid groups in Gaussian representations*; *in particular, they are finite groups.*

Proof. In [7] the Gaussian representations are shown to have finite image. Hence for N odd, the claim is immediate from Proposition 4.7. For N even the same analysis implies that the braid group representation on $\mathrm{End}_{\mathrm{O}(N)_2}(S^{\otimes n})$ for N even is a finite group. Since the forgetful functor $F\colon \mathrm{O}(N)_2 = (\mathrm{SO}(N)_2)^{\mathbb{Z}_2} \to \mathrm{SO}(N)_2$ is a braided tensor functor and the braiding is functorial we conclude that the image of the braid group acting on $\mathrm{End}_{\mathrm{SO}(N)_2}(S_\pm^{\otimes n})$ is a (finite) subquotient of the image of the braid group acting on $\mathrm{End}_{\mathrm{O}(N)_2}(S^{\otimes n})$. □

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