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Quantum Topology

Positive half of the Witt algebra acts on triply graded link homology

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Abstract. The positive half of the Witt algebra is the Lie algebra spanned by vector fields $x^{m+1}\frac{d}{dx}$ acting as differentiations on the polynomial algebra $\mathbb{Q}[x]$ upon which the Soergel bimodule construction of triply graded link homology is based. We show that this action of the Witt algebra can be extended to link homology.

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Contents

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1. Introduction

1.1. Motivation. Stable cohomological operations on cohomology of topological spaces exhibit the deepest structure, that of the Steenrod algebra, when the ring of coefficients is $\mathbb{Z}/p$ for a prime p, while nontrivial operations don't even exist when the coefficient ring is $\mathbb{Q}$. The work of Lipshitz and Sarkar [14, 15], see also Kriz, Kriz, and Po [9], shows that the Steenrod algebra acts, in the homological direction, on Khovanov $\mathfrak{sl}(2)$ link homology with coefficients in $\mathbb{Z}/p$, and this action is already highly nontrivial when $p = 2$ for the subalgebra generated by Sq^1, Sq^2, see [15, 20].

Gorsky [6] and Gorsky, Oblomkov, Rasmussen [7] conjectured that large algebras related to affine Lie algebras act on $\mathfrak{sl}(2)$ homology and triply graded homology of torus knots. In this conjecture the ground ring is $\mathbb{Z}$ rather than torsion. One wonders what part of this structure will act on cohomology groups of arbitrary links.

A variety of spectral sequences acting between various link and 3-manifold homologies, including Ozsvath-Szabo 3-manifold homology, Ozsvath–Rasmussen–Szabo link homology and $\mathfrak{sl}(k)$ homology indicates the existence of a rich structure of cohomological operations in these theories. Baston–Seed spectral sequence [2] from $\mathfrak{sl}(2)$ homology of a link to tensor product of homology groups of its components comes from one family of such operations.

Reduced Khovanov homology of alternating and quasi-alternating knots lies on one diagonal in the bigrading plane; for general knots this diagonal direction is the preferred one for the growth of homology group ranks. It would be strange if the groups living on the same diagonal were completely unrelated, and we conjecture the existence of cohomological operations acting along this direction. The knot Floer homology case is similar, and such homological operations, if found, might help to prove the Fox conjecture that coefficients of the Alexander polynomial of an alternating knot constitute a trapezoidal sequence.

Categorifications of quantum groups at roots of unity of prime order p over a field $\Bbbk$ of characteristic p, investigated in [12, 5], use derivation $\partial = x^2\partial/\partial x$ acting first on the polynomial algebra $\Bbbk[x]$ in one variable x, then on $\Bbbk[x_1, \dots, x_n]$, and later on suitable endomorphism algebras of these polynomial spaces, including the nil-Hecke algebras, KLR and Webster algebras. Characteristic p is needed to make ∂ nilpotent, $\partial^p = 0$, and work in the hopfological setting, that is, over the base monoidal triangulated category of stable graded modules over the Hopf algebra $\Bbbk[\partial]/(\partial^p)$. The latter category generalizes the homotopy category of complexes.

This fails in characteristic zero, but one can at least study the symmetries $L_m = x^{m+1}\partial/\partial x$, $m \geq -1$, acting on $\Bbbk[x]$. Symmetries L_m span the positive half $\mathfrak{W}^+$ of the Witt Lie algebra, and their action induces an action of $\mathfrak{W}^+$ on nil-Hecke, KLR, and other algebras categorifying quantum groups and their representations. One might hope that the entire Webster's categorification of Reshetikhin–Turaev link invariants for arbitrary simple Lie algebras and their representations [23, 24] can be done $\mathfrak{W}^+$-equivariantly, leading to an action of the latter on all these link homology theories.

In finite characteristic one can expect an action of a larger algebra. Recent papers of Beliakova, Cooper [3] and Kitchloo [13] exhibit a Steenrod algebra action on the nil-Hecke algebra. If this action extends naturally to Webster algebras, associated bimodules, and link homology in an invariant way, there will be a Steenrod algebra action on bigraded link homology groups. Curiously, the direction of that action (if it exists) in the bigrading plane will not match that of the Lipshitz-Sarkar Steenrod algebra action [14, 15]. Overall, we expect the existence of very large algebras of cohomological operations on link homology for various homology theories and coefficient rings.

In this paper we work in the simplest instance of the categorified Schur–Weyl dual set-up. Instead of looking at the action of $\mathfrak{W}^+$ on rings categorifying quantum groups, we consider an action of $\mathfrak{W}^+$ on Soergel bimodules serving as building blocks for the Soergel category, which categorifies the Hecke algebra of the symmetric group. We make this action compatible with the Rouquier braid group action [18, 19] via complexes of Soergel bimodules and with the Hochschild homology description of triply-graded categorification of the HOMFLYPT polynomial, resulting in an action of $\mathfrak{W}^+$ (and its universal enveloping algebra) on the triply-graded link homology of [11, 10].

1.2. Notations and definitions. Throughout the paper we use notation $\mathbf{a} = a_1, a_2, \ldots$ for a finite or infinite sequence of elements of a set $\mathfrak{A}$ and $|\mathbf{a}|$ for the length of $\mathbf{a}$. We also use a shortcut $\mathbf{a} \in \mathfrak{A}$ for $a_i \in \mathfrak{A}$, $i \geq 1$. In particular, $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of variables with $|\mathbf{x}| = n$.

We often use a special sequence of two-variable polynomials $\boldsymbol{\pi}(x, y)$ defined as divided differences

$$\pi_m(x, y) = \frac{y^{m+1} - x^{m+1}}{y - x}, \quad m \geq -1. \tag{1.1}$$

We also use a one-variable version of these polynomials:

$$\boldsymbol{\pi}(x) = \boldsymbol{\pi}(x, x), \quad \pi_m(x) = (m+1)x^m. \tag{1.2}$$

In addition we define two more polynomial sequences:

$$\boldsymbol{\pi}'(x,y) = \frac{\boldsymbol{\pi}(y) - \boldsymbol{\pi}(x)}{y-x}, \quad \boldsymbol{\pi}'(x) = \boldsymbol{\pi}'(x,x). \tag{1.3}$$

All modules in this paper are $\mathbb{Z}$-graded; we refer to this grading as q-grading and denote it $\deg_q$. The q-degree of all main variables such as $\mathbf{x}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ is 2. We work over the base field $\mathbb{Q}$ of rational numbers. The positive half $\mathfrak{W}^+$ of the Witt algebra is a q-graded Lie algebra with generators L_m, $m \geq -1$, $\deg_q L_m = 2m$, and relations

$$[L_m, L_n] = (n-m)L_{m+n}. \tag{1.4}$$

This algebra acts by differentiations on the polynomial algebra $\mathbb{Q}[x]$:

$$\widehat{L}_m = x^{m+1}\partial_x.$$

More generally, it acts on the algebra $\mathbb{Q}[\mathbf{x}]$, $|\mathbf{x}| = k$, as

$$\widehat{L}_m = \sum_{i=1}^{k} x_i^{m+1}\partial_{x_i}.$$

We refer to this action as the standard action.

Let $\mathcal{W}^+$ denote the universal enveloping algebra of $\mathfrak{W}^+$. The standard action of $\mathcal{W}^+$ on $\mathbb{Q}[\mathbf{x}]$ allows us to define a semi-direct product $\mathcal{W}^+_{\mathbf{x}} = \mathbb{Q}[\mathbf{x}] \rtimes \mathcal{W}^+$ with the multiplication

$$(p \otimes L_m)(q \otimes L_n) = pq \otimes L_m L_n + p(\widehat{L}_m q) \otimes L_n, \tag{1.5}$$

where $p, q \in \mathbb{Q}[\mathbf{x}]$. For two sequences of variables $\boldsymbol{\alpha}$ and $\mathbf{x}$ we also use notation

$${}_{\boldsymbol{\alpha}}\mathcal{W}^+_{\mathbf{x}} = \mathcal{W}^+_{\boldsymbol{\alpha},\mathbf{x}} = \mathbb{Q}[\mathbf{x}, \boldsymbol{\alpha}] \rtimes \mathcal{W}^+,$$

if we want to separate $\boldsymbol{\alpha}$ from $\mathbf{x}$ in order to emphasize that the variables $\boldsymbol{\alpha}$ are treated differently from $\mathbf{x}$. Note that $\mathcal{W}^+$ acts on both $\mathbf{x}$ and $\boldsymbol{\alpha}$.

We use a subscript $\mathbf{x}$, as in $M_{\mathbf{x}}$, to denote a $\mathcal{W}^+_{\mathbf{x}}$-module. Then for another sequence of variables $\mathbf{y}$, $|\mathbf{y}| = |\mathbf{x}|$, $M_{\mathbf{y}}$ denotes the corresponding module over $\mathcal{W}^+_{\mathbf{y}}$. Similarly, we use notation ${}_{\boldsymbol{\alpha}}M_{\mathbf{x}}$ for a ${}_{\boldsymbol{\alpha}}\mathcal{W}^+_{\mathbf{x}}$-module. We denote by $\mathbb{Q}_{\mathbf{x}}$ the algebra $\mathbb{Q}[\mathbf{x}]$ considered as a $\mathcal{W}^+_{\mathbf{x}}$-module with the standard action of $\mathcal{W}^+$.

Define the curvature of an infinite sequence of polynomials $\mathbf{a} = a_0, a_1, \ldots \in \mathbb{Q}[\mathbf{x}]$ as a double sequence

$$\mathrm{F}_{m,n}[\mathbf{a}] = \widehat{L}_m a_n - \widehat{L}_n a_m - (n-m)\, a_{m+n}. \tag{1.6}$$

A sequence $\mathbf{a}$ is called $\mathfrak{W}^+$-flat if its curvature is zero:

$$\mathrm{F}_{m,n}[\mathbf{a}] = 0, \quad m, n \geq -1. \tag{1.7}$$

For example, the sequence $\boldsymbol{\pi}(x)$ defined by eq. (1.2) is obviously $\mathfrak{W}^+$-flat.

A $\mathfrak{W}^+$-flat sequence determines an automorphism of $\mathcal{W}^+_{\mathbf{x}}$:

$$p \otimes L_m \longmapsto p \otimes (L_m + a_m). \tag{1.8}$$

This automorphism, in turn, determines an invertible endo-functor $\langle \mathbf{a} \rangle$ of the category $\mathcal{W}^+_{\mathbf{x}} - \mathbf{gm}$ of graded left $\mathcal{W}^+_{\mathbf{x}}$-modules and its derived category $\mathbf{D}(\mathcal{W}^+_{\mathbf{x}} - \mathbf{gm})$: the action of $\mathcal{W}^+_{\mathbf{x}}$ on a module $M_{\mathbf{x}}\langle \mathbf{a} \rangle$ is the action on $M_{\mathbf{x}}$ modified by the automorphism (1.8).

1.3. Main result. Let L be an oriented framed link with m ordered components. We associate to it an object ${}_{\boldsymbol{\lambda}}[\![L]\!]$ of the bounded derived category of bigraded $\mathcal{W}^+_{\boldsymbol{\lambda}}$-modules $\mathbf{D}(\mathcal{W}^+_{\boldsymbol{\lambda}}-\mathbf{gm}_2)$, where $|\boldsymbol{\lambda}| = m$ so that every variable λ_i is associated to the corresponding component L_i of L. The homology of the complex ${}_{\boldsymbol{\lambda}}[\![L]\!]$ is isomorphic to the triply graded homology of L as a triply graded vector space.

1.4. Nested derived categories. In this paper we consider only small categories. Let $\mathbf{Ad}$ be an additive category. Following the notations of Weibel's book [25], $\mathbf{Ch}(\mathbf{Ad})$ denotes the chain category, whose objects are complexes over $\mathbf{Ad}$, its morphisms being chain maps, while $\mathbf{K}(\mathbf{Ad})$ denotes the homotopy category, its morphisms being chain maps modulo null-homotopies.

Since old and new constructions of triply graded link homology involve 'nested' homotopy and derived categories, we will use the following shortcut notations for derived categories of graded modules over algebras:

$$\mathbf{Q}_{\mathbf{x}} = \mathbf{D}(\mathbb{Q}[\mathbf{x}]-\mathbf{gm}), \quad \mathbf{W}^+_{\mathbf{x}} = \mathbf{D}(\mathcal{W}^+_{\mathbf{x}}-\mathbf{gm}), \quad {}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x}} = \mathbf{D}({}_{\boldsymbol{\alpha}}\mathcal{W}^+_{\mathbf{x}}-\mathbf{gm}). \tag{1.9}$$

The categories (1.9) are additive, hence we may use the categories of complexes and homotopy categories over them, such as $\mathbf{Ch}(\mathbf{W}^+_{\mathbf{x}})$ and $\mathbf{K}(\mathbf{W}^+_{\mathbf{x}})$. We call these categories 'nested'.

In addition to outer chain and homotopy categories, we also use 'relative' nested derived categories $\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x}})$ and $\mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x}})$. First, consider a general setup. Suppose that two additive categories $\mathbf{A}$ and $\mathbf{B}$ are related by an additive 'forgetful' functor $\mathcal{F}\colon \mathbf{A} \to \mathbf{B}$ which is extended to the chain categories: $\mathcal{F}\colon \mathbf{Ch}(\mathbf{A}) \to \mathbf{Ch}(\mathbf{B})$.

Definition 1.1. *A morphism $f \in \mathrm{Hom}_{\mathbf{Ch}(\mathbf{A})}(A_\bullet, B_\bullet)$ between two complexes in $\mathbf{Ch}(\mathbf{A})$ is called a **$\mathcal{F}$-isomorphism** if $\mathcal{F}(f)\colon \mathcal{F}(A_\bullet) \to \mathcal{F}(B_\bullet)$ is an isomorphism in $\mathbf{K}(\mathbf{B})$.*

If $f, g\colon A_\bullet \to B_\bullet$ are homotopic and f is a $\mathcal{F}$-isomorphism, then g is also $\mathcal{F}$-isomorphism. Hence the notion of $\mathcal{F}$-isomorphism extends to the homotopy category $\mathbf{K}(\mathbf{A})$.

The collection of $\mathcal{F}$-isomorphisms in $\mathbf{K}(\mathbf{A})$ is a saturated localizing system compatible with triangulation. Denote by $\mathbf{D}_{\mathrm{r}}(\mathbf{A})$ the localization of $\mathbf{K}(\mathbf{A})$ relative to $\mathcal{F}$-isomorphisms (the index r means "relative"). It is a triangulated category and the localization functor $\mathcal{Q}\colon \mathbf{K}(\mathbf{A}) \to \mathbf{D}_{\mathrm{r}}(\mathbf{A})$ is exact (see Lemma 5.3 and Proposition 5.5 in [1, Section 13.5]). The image $\mathcal{Q}(A_\bullet)$ of an object $A_\bullet$ in $\mathbf{K}(\mathbf{A})$ is isomorphic to the zero object in $\mathbf{D}_{\mathrm{r}}(\mathbf{A})$ iff $0 \to A_\bullet$ is a $\mathcal{F}$-isomorphism. This is also equivalent to the existence of a distinguished triangle

$$A'_\bullet \xrightarrow{f} A''_\bullet \longrightarrow A_\bullet \longrightarrow A'_\bullet[1]$$

in $\mathbf{K}(\mathbf{A})$ such that $f\colon A'_\bullet \to A''_\bullet$ is a $\mathcal{F}$-isomorphism ([1, Lemma 6.7]).

The chain, homotopy, and relative derived categories are related by functors:

$$\begin{tikzcd} \mathbf{Ch}(\mathbf{A}) \arrow[r] \arrow[rr, bend left, "\mathcal{F}_\heartsuit"] & \mathbf{K}(\mathbf{A}) \arrow[r, "\mathcal{Q}"'] & \mathbf{D}_{\mathrm{r}}(\mathbf{A}), \end{tikzcd} \tag{1.10}$$

where we denote by $\mathcal{F}_\heartsuit$ the composition of the quotient functor and the localization functor $\mathcal{Q}$.

The definition of $\mathbf{D}_{\mathrm{r}}(\mathbf{A})$ as the localization relative to $\mathcal{F}$-isomorphisms specializes to the definition of the derived category in two familiar cases. The first case is when $\mathbf{A}$ is the category of modules over an algebra $\mathcal{A}$ (over $\mathbb{Q}$), $\mathbf{B}$ is the category of $\mathbb{Q}$-vector spaces and $\mathcal{F}$ forgets the $\mathcal{A}$-module structure. Then $\mathbf{D}_{\mathrm{r}}(\mathbf{A})$ is the derived category of $\mathcal{A}$-modules. The second case is when $\mathbf{A}$ is again the category of modules over $\mathcal{A}$, $\mathbf{B}$ is the category of modules over its subalgebra $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{F}\colon \mathcal{A}-\mathbf{mod} \longrightarrow \mathcal{B}-\mathbf{mod}$ is the restriction functor. Then $\mathbf{D}_{\mathrm{r}}(\mathbf{A})$ is the relative derived category $\mathbf{D}\big((\mathcal{A},\mathcal{B})-\mathbf{mod}\big)$.

In this paper we use the 'nested' version of the second example. We set

$$\mathbf{A} = \mathbf{D}(\mathcal{A}-\mathbf{gm}), \quad \mathbf{B} = \mathbf{D}(\mathcal{B}-\mathbf{gm})$$

(algebras $\mathcal{A}$, $\mathcal{B}$ are now graded), and $\mathcal{F}$ is the restriction functor extended to these derived categories. The result is the nested relative derived category $\mathbf{D}_{\mathrm{r}}\big(\mathbf{D}(\mathcal{A}-\mathbf{gm})\big)$. Note that its objects are triply graded: the first grading comes from $\mathcal{A}$, the second is the homological grading of $\mathbf{D}$ and the third is the homological grading of $\mathbf{D}_{\mathrm{r}}$.

More specifically, we take $\mathcal{A} = {}_{\boldsymbol{\alpha}}\mathcal{W}^+_{\mathbf{x}}$, $\mathcal{B} = \mathbf{Q}_{\mathbf{x}}$, and $\mathcal{F}$ restricts ${}_{\boldsymbol{\alpha}}\mathcal{W}^+_{\mathbf{x}}$-modules to $\mathbb{Q}[\mathbf{x}]$-modules. Then we get the nested relative derived category $\mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x}})$.

Overall, there is a chain of functors

$$\begin{tikzcd}[column sep=small] \underbrace{{}_{\boldsymbol{\alpha}}\mathcal{W}^{+}_{\mathbf{x}}-\mathbf{gm}}_{\text{abelian}} \arrow[r] & \underbrace{{}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x}} = \mathbf{D}({}_{\boldsymbol{\alpha}}\mathcal{W}^{+}_{\mathbf{x}}-\mathbf{gm})}_{\text{derived}} \\ \arrow[r] & \underbrace{\mathbf{Ch}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x}}) \arrow[r] \arrow[rr, bend left, "\mathcal{F}_{\heartsuit}"] & \mathbf{K}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x}}) \arrow[r, "\mathcal{Q}"] & \mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x}})}_{\text{nested}}. \end{tikzcd} \tag{1.11}$$

The category $\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x}})$ is a particular case of $\mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x}})$ when the sequence $\boldsymbol{\alpha}$ is empty. Often we use a combined sequence of variables $\mathbf{x}, \mathbf{y}$ instead of $\mathbf{x}$, thus getting the categories $\mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}})$ and $\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}})$.

Since objects in nested categories are 'complexes of complexes', we have to distinguish between homotopy equivalences in the inner and outer categories. Hence we use notations $\underset{\text{in}}{\sim}$ and $\underset{\text{in}}{\simeq}$ for homotopy equivalence and quasi-isomorphism in the inner category, such as $\mathbf{Ch}({}_{\boldsymbol{\alpha}}\mathcal{W}^{+}_{\mathbf{x}}\text{–}\mathbf{gm})$, and $\sim$, $\simeq$ for homotopy equivalence and $\mathcal{F}$-isomorphism in the outer category, such as $\mathbf{Ch}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x}})$. When we need to distinguish between complexes in the inner and outer categories, we use a square bracket notation

$$\left[\cdots \to M_i \to M_{i+1} \to \cdots\right]$$

for a complex in the inner category and a boxed notation

$$\boxed{\cdots \to M_i \to M_{i+1} \to \cdots}$$

for a complex in the outer category, with each M_i itself a complex.

1.5. Derived partial tensor products. In this paper we use two versions of the partial tensor product:

$$\begin{tikzcd} & (\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}-\mathbf{gm}) \times (\mathcal{W}^{+}_{\mathbf{y},\mathbf{z}}-\mathbf{gm}) \arrow[dl, "\otimes_{\bar{\mathbf{y}}}"'] \arrow[dr, "\otimes_{\mathbf{y}}"] & \\ \mathcal{W}^{+}_{\mathbf{x},\mathbf{y},\mathbf{z}}-\mathbf{gm} & & \mathcal{W}^{+}_{\mathbf{x},\mathbf{z}}-\mathbf{gm}. \end{tikzcd}$$

On the level of underlying vector spaces, both versions agree with the usual tensor product $\otimes_{\mathbb{Q}[\mathbf{y}]}$. The action of the $\mathfrak{W}^{+}$ generators L_m is defined by the Leibniz rule: L_m acts on a tensor product $M_{\mathbf{x},\mathbf{y}} \otimes_{\mathbb{Q}[\mathbf{y}]} N_{\mathbf{y},\mathbf{z}}$ of $\mathcal{W}^{+}$-modules as $L_m \otimes \mathbb{1} + \mathbb{1} \otimes L_m$. The difference between $\otimes_{\bar{\mathbf{y}}}$ and $\otimes_{\mathbf{y}}$ is that the former remembers the $\mathbb{Q}[\mathbf{y}]$-module structure, while the latter forgets it. In other words, the functor $\otimes_{\mathbf{y}}$

is the composition of $\otimes_{\bar{\mathbf{y}}}$ and the functor of forgetting the $\mathbb{Q}[\mathbf{y}]$-module structure. The following properties are shared by both products $\otimes_{\bar{\mathbf{y}}}$ and $\otimes_{\mathbf{y}}$, and we will formulate them mostly for $\otimes_{\mathbf{y}}$.

Tensor products $\otimes_{\mathbf{y}}$ and $\otimes_{\bar{\mathbf{y}}}$ extend in left-derived form to derived categories, *e.g.*

$$\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}} \times \mathbf{W}^{+}_{\mathbf{y},\mathbf{z}} \xrightarrow{\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}} \mathbf{W}^{+}_{\mathbf{x},\mathbf{z}} \tag{1.12}$$

and, further, to the chain and homotopy categories built over them:

$$\mathbf{K}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}) \times \mathbf{K}(\mathbf{W}^{+}_{\mathbf{y},\mathbf{z}}) \xrightarrow{\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}} \mathbf{K}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{z}})\ . \tag{1.13}$$

Theorem 2.6 says that the latter tensor product descends directly to the nested relative derived categories such as $\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}})$ without the need for any additional derivation:

$$\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}) \times \mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{y},\mathbf{z}}) \xrightarrow{\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}} \mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{z}}). \tag{1.14}$$

When extra variables $\boldsymbol{\alpha}$ are present, we use the 'left' version of the tensor product (1.12):

$$_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}} \times {}_{\boldsymbol{\beta}}\mathbf{W}^{+}_{\mathbf{y},\mathbf{z}} \xrightarrow{\ {}_{1}\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}\ } {}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x},\mathbf{z}}\ . \tag{1.15}$$

The name 'left' has nothing to do with derivation. It stems from the fact that the product ${}_{1}\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}$ keeps the $\mathbb{Q}[\boldsymbol{\alpha}]$-module structure, while forgetting the $\mathbb{Q}[\boldsymbol{\beta}]$-module structure. This tensor product also extends to nested relative derived categories

$$\mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}) \times \mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\beta}}\mathbf{W}^{+}_{\mathbf{y},\mathbf{z}}) \xrightarrow{\ {}_{1}\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}\ } \mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x},\mathbf{z}}).$$

For two finite sequences of variables $\mathbf{x}$, $\mathbf{y}$ of the same length we use notation

$$(\mathbf{y}-\mathbf{x}) = (y_1 - x_1, \dots, y_n - x_n)$$

for the ideal in $\mathbb{Q}[\mathbf{x},\mathbf{y}]$ generated by the pairwise differences of corresponding variables. We introduce a special 'diagonal' $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-module

$$\Delta_{\mathbf{x};\mathbf{y}} = \mathbb{Q}_{\mathbf{x},\mathbf{y}}/(\mathbf{y}-\mathbf{x}),$$

which has an obvious property: for any $\mathcal{W}^{+}_{\mathbf{x}}$-module $M_{\mathbf{x}}$ there is an isomorphism $M_{\mathbf{x}} \otimes_{\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}} \cong M_{\mathbf{y}}$. This property persists at the derived level:

$$M_{\mathbf{x}} \overset{\mathrm{L}}{\otimes}_{\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}} \simeq M_{\mathbf{y}}.$$

1.6. Gradings. For an algebra $\mathcal{A}$, let $\mathcal{A}-\mathbf{gm}_i$ denote the category of $\mathbb{Z}^i$-graded $\mathcal{A}$-modules. If $\mathcal{A}$ is graded, then one grading matches the grading of $\mathcal{A}$, while others have external origin. Sometimes we drop i and use notation $\mathcal{A}-\mathbf{gm}$.

The triply graded link homology construction involves three gradings called q-, a- and t-gradings. The corresponding degrees are denoted $\deg_{\mathrm{q}}$, $\deg_{\mathrm{a}}$ and $\deg_{\mathrm{t}}$ and the degree shifting functors are denoted q, a and t (*e.g.* q^2 is the functor that shifts the q-degree up by two).

We have already introduced the q-degree. The algebras $\mathbb{Q}[\mathbf{x}]$ and ${}_{\boldsymbol{\alpha}}\mathcal{W}^{+}_{\mathbf{x}}$ and their modules are q-graded:

$$\deg_{\mathrm{q}}\mathbf{x} = \deg_{\mathrm{q}}\boldsymbol{\alpha} = 2, \quad \deg_{\mathrm{q}} L_m = 2m. \tag{1.16}$$

Within the nested categories such as $\mathbf{Ch}(\mathbf{W}^{+}_{\mathbf{x}})$ and $\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x}})$ the t-degree is the 'total' homological degree appearing in both inner and outer complexes and determining all sign factors, while a-degree is the 'pseudo-homological' degree associated with the complexes of the inner categories and having no impact on sign factors.

In addition to shift functors t, a and q we introduce a framing shift functor

$$\mathsf{fr}_x M = \mathsf{atq}^{-2} M \,\langle -\boldsymbol{\pi}(\lambda)\rangle, \tag{1.17}$$

which combines a degree shift atq^{-2} and a connection shift $\langle -\boldsymbol{\pi}(\lambda)\rangle$. This functor is related to a change of framing of a link component (see (1.31) and (5.6)).

In order to avoid overloading the formulas, we will ignore the gradings everywhere, except in the most important formulas which define our construction: the categorification of elementary braids (3.15) and the application of Hochschild homology (1.23) and (1.28). We also keep all degree shift functors in Section 6 where we compute homology of two-strand torus knots and links.

1.7. Outline of the homology construction. For a set $\mathfrak{A}$ and a small category $\mathbf{C}$ notation $f : \mathfrak{A} \to \mathbf{C}$ means a map from the set $\mathfrak{A}$ to the set of isomorphism classes of objects of $\mathbf{C}$. If $\mathfrak{A}$ is a semi-group and $\mathbf{C}$ has a monoidal structure, then we assume that f is a homomorphism: it intertwines the product in $\mathfrak{A}$ with the product on the set of isomorphism classes induced by the monoidal structure.

For a topological object (a braid word, a braid or a link) τ we use notation $[\![\tau]\!]$ for the associated object or complex constructed though *categorification* of an algebraic invariant such as the HOMFLY-PT polynomial [8, 16]. The upper-left 'check' decoration ${}^{\vee}[\![\tau]\!]$ indicates that this is the original categorification of [10]. The upper-right dot $[\![\tau]\!]^{\cdot}$ indicates that a map $[\![-]\!]^{\cdot} : \mathfrak{T} \to \mathbf{C}$ from a set of topological objects $\mathfrak{T}$ to the (isomorphism classes of) objects of a category $\mathbf{C}$ has a built-in invariance: there is an equivalence relation within $\mathfrak{T}$ and an equivalence relation between the objects of $\mathbf{C}$, so that if $\tau_1 \sim \tau_2$, then $[\![\tau_1]\!]^{\cdot} \sim [\![\tau_2]\!]^{\cdot}$, and as a result there is an unmarked bracket map $[\![-]\!]$ between the two quotient sets $\mathfrak{T}/\sim$ and $\mathbf{C}/\sim$ (see two vertical arrows in the diagram(1.18) which map braid words into chain complexes and braids into chain complexes up to homotopy equivalence).

1.7.1. Original construction. The following commutative diagram summarizes the Soergel bimodule [21, 22] construction of the triply graded homology in [10]:

$$\begin{tikzcd}
\widetilde{\mathfrak{B}}_n \arrow[r, two heads, "\mathrm{br}"] \arrow[d, "\check{[\![-]\!]}^{\cdot}_{\mathbf{x},\mathbf{y}}"] & \mathfrak{B}_n \arrow[r, "\mathrm{cl}"] \arrow[d, "\check{[\![-]\!]}_{\mathbf{x},\mathbf{y}}"] & \mathfrak{L} \arrow[d, dashed] \arrow[dr, "\check{\mathcal{H}}"] & \\
\mathbf{Ch}(\mathbf{Q}_{\mathbf{x},\mathbf{y}}) \arrow[r] & \mathbf{K}(\mathbf{Q}_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathsf{a}^n\,\mathrm{HH}"] & \mathbf{K}(\mathbb{Q}-\mathbf{gm}_2) \arrow[r, "\mathrm{H}"] & \mathbb{Q}-\mathbf{gm}_3
\end{tikzcd} \tag{1.18}$$

where the category $\mathbf{Q}_{\mathbf{x},\mathbf{y}} = \mathbf{D}(\mathbb{Q}[\mathbf{x},\mathbf{y}]-\mathbf{gm})$, see also formula (1.9).

The top row of the diagram is purely topological. $\widetilde{\mathfrak{B}}_n$ is the semigroup of n-strand braid words, that is, a semigroup freely generated by the elements σ_i and $\sigma_i^{(-1)}$, $1 \le i \le n-1$. $\mathfrak{B}_n$ is the braid group and the map br turns σ_i and $\sigma_i^{(-1)}$ into a braid group generator σ_i and its inverse σ_i^{-1}. $\mathfrak{L}$ is the set of framed oriented links and the map cl performs the circular closure of a braid.

Sequences $\mathbf{x}$ and $\mathbf{y}$ appearing in $\mathbf{Q}_{\mathbf{x},\mathbf{y}}$ contain n variables each, one per braid strand position: $|\mathbf{x}| = |\mathbf{y}| = n$. The bracket $\check{[\![-]\!]}^{\cdot}_{\mathbf{x},\mathbf{y}}$ maps braid words into Rouquier complexes [18] of Soergel bimodules. The category $\mathbf{Q}_{\mathbf{x},\mathbf{y}}$ has a monoidal structure coming from the derived tensor product over intermediate variables:

$$\mathbf{Q}_{\mathbf{x},\mathbf{y}} \times \mathbf{Q}_{\mathbf{y},\mathbf{z}} \xrightarrow{\overset{\mathrm{L}}{\otimes}_{\mathbb{Q}[\mathbf{y}]}} \mathbf{Q}_{\mathbf{x},\mathbf{z}}, \tag{1.19}$$

and this structure extends to the chain category $\mathbf{Ch}(\mathbf{Q}_{\mathbf{x},\mathbf{y}})$. The bracket $\check{[\![-]\!]}^{\cdot}_{\mathbf{x},\mathbf{y}}$ turns the product in $\widetilde{\mathfrak{B}}_n$ into the tensor product (1.19), hence it is determined by its value on generators σ_i and $\sigma_i^{(-1)}$.

In order to prove the existence of the map $\check{[\![-]\!]}_{\mathbf{x},\mathbf{y}}$ taking braids to isomorphism classes of objects in $\mathbf{K}(\mathbf{Q}_{\mathbf{x},\mathbf{y}})$ one has to verify that if two braid words $\mathfrak{w}_1$ and $\mathfrak{w}_2$ represent the same braid, $\mathrm{br}(\mathfrak{w}_1) = \mathrm{br}(\mathfrak{w}_2)$, then their complexes are homotopy equivalent: $\check{[\![\mathfrak{w}_1]\!]}^{\cdot} \sim \check{[\![\mathfrak{w}_2]\!]}^{\cdot}$.

The bracket $\check{[\![-]\!]}_{\mathbf{x},\mathbf{y}}$ has a 'variable sliding' property. Let $\check{\mathfrak{b}} \in S_n$ be the symmetric group element corresponding to a braid $\mathfrak{b}$, so that x_i and $y_{\check{\mathfrak{b}}(i)}$ are variables at opposite ends of the same braid strand. Then endomorphisms $\hat{x}_i, \hat{y}_{\check{\mathfrak{b}}(i)} \in \mathrm{End}_{\mathbf{Ch}(\mathbf{Q}_{\mathbf{x},\mathbf{y}})}\big(\check{[\![\mathfrak{b}]\!]}_{\mathbf{x},\mathbf{y}}\big)$ of multiplication by x_i and $y_{\check{\mathfrak{b}}(i)}$ are homotopic to each other:

$$\hat{x}_i \sim \hat{y}_{\check{\mathfrak{b}}(i)}. \tag{1.20}$$

Hochschild homology is a functor HH: $\mathbf{Q}_{\mathbf{x},\mathbf{y}} \to \mathbb{Q}{-}\mathbf{gm}_2$, and the functor HH in the diagram is its extension to the homotopy categories over $\mathbf{Q}_{\mathbf{x},\mathbf{y}}$ and $\mathbb{Q}{-}\mathbf{gm}_2$. Finally, homology functor $\mathrm{H}(-)$ establishes an equivalence between categories $\mathbf{K}(\mathbb{Q}{-}\mathbf{gm}_2)$ and $\mathbb{Q}{-}\mathbf{gm}_3$, so there is no additional benefit in introducing a dashed arrow mapping links into complexes of $\mathbb{Q}$-vector spaces. In order to prove the existence of the homology map $\check{\mathcal{H}}(-)$, one has to verify that if the closures of two braids β_1 and β_2 represent isotopic framed oriented links, then their homologies are isomorphic: $\mathrm{H}\big(\mathrm{HH}(\check{[\![\beta_1]\!]}_{\mathbf{x},\mathbf{y}})\big) \cong \mathrm{H}\big(\mathrm{HH}(\check{[\![\beta_2]\!]}_{\mathbf{x},\mathbf{y}})\big)$.

Property (1.20) allows one to endow the link homology $\check{\mathcal{H}}(L)$ of a m-component link with the structure of a $\mathbb{Q}[\boldsymbol{\lambda}]$-module, where the variables $\boldsymbol{\lambda}$, $|\boldsymbol{\lambda}| = m$ are assigned bijectively to link components. Since $\mathrm{HH}(-)$ and $\mathrm{H}(-)$ are functors, there is a map

$$\mathrm{End}_{\mathbf{K}(\mathbf{Q}_{\mathbf{x},\mathbf{y}})}(\check{[\![\mathfrak{b}]\!]}_{\mathbf{x},\mathbf{y}}) \longrightarrow \mathrm{End}_{\mathbb{Q}-\mathbf{gm}}(\check{\mathcal{H}}(L)),$$

where $L = \mathrm{cl}(\mathfrak{b})$. This map turns $\hat{x}_i$ and $\hat{y}_i$ into linear operators acting on $\check{\mathcal{H}}(L)$. For each link component L_k, $1 \leq k \leq m$ we choose a strand in the braid $\mathfrak{b}$ which is a part of this component and define the action of λ_k either as $\hat{x}_i$ or as $\hat{y}_{\check{\mathfrak{b}}(i)}$, the variables x_i and $y_{\check{\mathfrak{b}}(i)}$ corresponding to the endpoints of that strand. Equation (1.20) guarantees that the action of $\boldsymbol{\lambda}$ does not depend on the choice of strands which represent link components.

1.7.2. New construction. The following commutative diagram outlines our modification of the previous triply graded homology construction:

$$\begin{tikzcd}
\widetilde{\mathfrak{B}}_n \arrow[r, two heads, "\mathrm{br}"] \arrow[d, "{[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}}"] \arrow[dd, bend right, "{\boldsymbol{\alpha}[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}}"'] & \mathfrak{B}_n \arrow[d, "{[\![-]\!]_{\mathbf{x},\mathbf{y}}}"'] \arrow[dd, bend left, "{\boldsymbol{\alpha}[\![-]\!]_{\mathbf{x},\mathbf{y}}}"] & \mathfrak{B}_n^{(s)} \arrow[l, hook'] \arrow[r, "\mathrm{cl}"] \arrow[dr, "{\boldsymbol{\lambda}[\![-]\!]^{\cdot}}"] & \mathfrak{L}_m \arrow[d, "{\boldsymbol{\lambda}[\![-]\!]}"] \arrow[dr, bend left, "\mathcal{H}"] & \\
\mathbf{Ch}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathcal{F}_{\heartsuit}"] \arrow[d, "\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}"] & \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathsf{a}^n\,\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}"] \arrow[d, "\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}"'] & \mathbf{W}^+_{\mathbf{x}} \arrow[r, "\mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}"] \arrow[d, "\mathcal{F}_{\mathbf{x}\to\boldsymbol{\alpha}}"'] & \mathbf{W}^+_{\boldsymbol{\lambda}} \arrow[r, "\mathrm{H}"] & \mathcal{W}^+_{\boldsymbol{\lambda}}{-}\mathbf{gm}_3 \\
\mathbf{Ch}({}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathcal{F}_{\heartsuit}"] & \mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathsf{a}^n\,\mathrm{HH}_{\mathbf{x},\mathbf{y}}"] & \mathbf{W}^+_{\boldsymbol{\alpha}}. \arrow[ur, "\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}"'] & &
\end{tikzcd} \tag{1.21}$$

The relative complexity of this diagram is due to the appearance of extra variables $\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}| = n$, associated with braid strands and variables $\boldsymbol{\lambda}$, $|\boldsymbol{\lambda}| = m$, associated with link components.

All categories in the diagram (1.21) have q-, a- and t-gradings. The grading of nested categories is described in subsection 1.6. t-grading is also the homological grading of derived categories and $\mathbf{W}^+_{\mathbf{x}}$, $\mathbf{W}^+_{\boldsymbol{\lambda}}$ and $\mathbf{W}^+_{\boldsymbol{\alpha}}$, while their modules have q-grading coming from (1.16) as well as a-grading.

The top line in the diagram (1.21) is almost the same as in the diagram (1.18), except that in the middle we had to restrict the braid group $\mathfrak{B}_n$ to the subset of braids $\mathfrak{B}_n^{(s)} \subset \mathfrak{B}_n$ which correspond to a permutation $s \in \mathrm{S}_n$. This restriction is convenient, because the permutation s determines the replacement of variables $\mathbf{x}$ and braid strand variables $\boldsymbol{\alpha}$ by link component variables $\boldsymbol{\lambda}$ performed by the functors $\mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}$ and $\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}$. $\mathfrak{L}_m$ is the set of m-component framed oriented links.

The bracket $[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}$ is essentially the same as $\check{[\![-]\!]}^{\cdot}_{\mathbf{x},\mathbf{y}}$ in the previous diagram (1.18): it turns a braid word into a complex of Soergel bimodules (with the differential acting in the second homological direction) representing an object of the category $\mathbf{Ch}\big(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}\big)$, while intertwining the products of braid words and braids with the tensor products (1.13) and (1.14)

$$[\![\mathfrak{w}_1\mathfrak{w}_2]\!]^{\cdot}_{\mathbf{x},\mathbf{z}} \cong [\![\mathfrak{w}_1]\!]^{\cdot}_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\mathfrak{w}_2]\!]^{\cdot}_{\mathbf{y},\mathbf{z}} \tag{1.22}$$

(according to Remark 3.1, the ordinary tensor product in the right hand side is $\mathcal{F}$-isomorphic to the derived one with respect to $\mathbf{W}^+_{\mathbf{x},\mathbf{y}}$). However, this time Soergel bimodules are endowed with the $\mathbf{W}^+_{\mathbf{x},\mathbf{y}}$-module structure coming from the standard action of $\mathfrak{W}^+$ on $\mathbf{x}$ and $\mathbf{y}$. As a result of this additional structure, the map $[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}$ preserves the braid relation only up to $\mathcal{F}$-isomorphism, hence the definition of the map $[\![-]\!]_{\mathbf{x},\mathbf{y}}$ requires taking a quotient over $\mathcal{F}$-isomorphisms performed by the functor $\mathcal{F}_{\heartsuit}$.

The functor $\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}$ in the diagram is a Hochschild homology, that is, $\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}$ is a functor

$$\mathbf{W}^+_{\mathbf{x},\mathbf{y}} \xrightarrow{\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}} \mathcal{W}^+_{\mathbf{x}}-\mathbf{gm}_2, \quad \mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(-) = \mathrm{H}(- \overset{\mathrm{L}}{\otimes}_{\bar{\mathbf{x}},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}}), \tag{1.23}$$

where the bar over $\mathbf{x}$ indicates that we remember the action of the whole $\mathcal{W}^+_{\mathbf{x}}$, including $\mathbb{Q}[\mathbf{x}]$, on the Hochschild homology. This functor extends to the functor from nested relative derived category to an ordinary relative derived category:

$$\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \xrightarrow{\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}} \mathbf{D}((\mathcal{W}^+_{\mathbf{x}}, \mathbb{Q}[\mathbf{x}])-\mathbf{gm}_2).$$

The functor $\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}$ is the composition of the latter functor with the standard forgetful functor from the relative to the absolute derived category:

$$\begin{tikzcd} \mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}"'] \arrow[rr, bend left, "\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}"] & \mathbf{D}((\mathcal{W}^{+}_{\mathbf{x}}, \mathbb{Q}[\mathbf{x}])-\mathbf{gm}_2) \arrow[r] & \mathbf{W}^{+}_{\mathbf{x}}. \end{tikzcd} \tag{1.24}$$

Keeping the $\mathbb{Q}[\mathbf{x}]$-module structure within the Hochschild homology might violate its trace-like property important for ensuring the invariance under the first Markov move. However, we will show that this problem disappears for Soergel bimodules after the application of the renaming functor $\mathcal{F}^{(s)}_{\mathbf{x};\boldsymbol{\lambda}}$. The permutation $s \in S_n$ corresponding to a braid determines the split of initial strand positions $\{1,\ldots,n\}$ into cycle subsets:

$$\{1,\ldots,n\} = \bigcup_{i=1}^{m} \mathfrak{c}_i,$$

each subset corresponding to a component of the link constructed by the cyclic closure of the braid. We choose an element $k_i \in \mathfrak{c}_i$ in each cycle subset thus forming a sequence $\mathbf{k}$, $|\mathbf{k}| = m$. The functor $\mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}$ renames variables x_{k_i} into λ_i, while forgetting the action of all other variables $\{\mathbf{x}\} \setminus \{x_{k_1},\ldots,x_{k_m}\}$. We will show that the application of this functor to the Hochschild homology (1.24) of the bracket of a braid

$$\mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}(\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}([\![-]\!]_{\mathbf{x},\mathbf{y}})) \tag{1.25}$$

does not depend on the choice of the representatives k_i within each cycle.

1.7.3. Braid strand variables. We introduce auxiliary variables $\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}| = n$, into the categorification construction. They help to prove that the composition of maps (1.25) does not depend on the choice of cycle representatives in the definition of $\mathcal{F}^{(s)}_{\mathbf{x};\boldsymbol{\lambda}}$. In addition, we will use them to capitalize on the property (1.20) of the original categorification complex $\check{}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}}$.

The braid strand variables $\boldsymbol{\alpha}$ are introduced through the functors

$$\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}-\mathbf{gm} \xrightarrow{\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}} {}_{\boldsymbol{\alpha}}\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}-\mathbf{gm}, \tag{1.26}$$

which turn a $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-module $M_{\mathbf{x},\mathbf{y}}$ into a ${}_{\boldsymbol{\alpha}}\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-module ${}_{\boldsymbol{\alpha}}M_{\mathbf{x},\mathbf{y}} = M_{\mathbf{x},\mathbf{y}}$ by making generators $\boldsymbol{\alpha}$ act as $\mathbf{x}$. This functor $\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}(-) = - \otimes_{\bar{\mathbf{x}}} \Delta_{\boldsymbol{\alpha};\mathbf{x}}$ essentially doubles variables $\mathbf{x}$ to variables $\mathbf{x},\boldsymbol{\alpha}$. We will also use a similarly defined functor $\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{y}}$ which doubles variables $\mathbf{y}$. Functors $\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}$ appearing in the diagram (1.21) are the extensions of the original functor (1.26) to the nested categories.

The new feature of $\boldsymbol{\alpha}$ is that these variables can 'slide' along braid strands. Let $\check{\mathfrak{b}}$ be a permutation associated with a braid $\mathfrak{b}$ and let $\check{\mathfrak{b}}(\boldsymbol{\alpha})$ denote the variables $\boldsymbol{\alpha}$ permuted by $\check{\mathfrak{b}}$. In subsection 5.1 we prove the stronger version of the relation(1.20):

Theorem 1.2. *The result of doubling the action of* $\mathbf{x}$ *on* $[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}}$ *with* $\boldsymbol{\alpha}$ *is* $\mathcal{F}$*-isomorphic* (*in the outer category* **Ch**) *to the doubling of* $\mathbf{y}$ *with* $\check{\mathfrak{b}}(\boldsymbol{\alpha})$:

$$\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}([\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}}) \simeq \mathcal{G}_{\check{\mathfrak{b}}(\boldsymbol{\alpha})=\mathbf{y}}([\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}}). \tag{1.27}$$

Monoidal structure within the categories $\mathbf{Ch}({}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x},\mathbf{y}})$ and $\mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x},\mathbf{y}})$ is defined by the left tensor product ${}_1\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}$ of the diagram (1.15). Since this tensor product does not impact the action of $\boldsymbol{\alpha}$, the doubling functor $\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}$ intertwines the tensor products $\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}$ and ${}_1\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}$: for any two modules $M_{\mathbf{x},\mathbf{y}}$ and $N_{\mathbf{x},\mathbf{y}}$,

$$\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}(M_{\mathbf{x},\mathbf{y}})\ {}_1\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}\ \mathcal{G}_{\boldsymbol{\beta}=\mathbf{y}}(N_{\mathbf{y},\mathbf{z}}) \cong \mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}(M_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}}).$$

As a consequence, the composite brackets

$${}_{\boldsymbol{\alpha}}[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}} = \mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}([\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}), \quad {}_{\boldsymbol{\alpha}}[\![-]\!]_{\mathbf{x},\mathbf{y}} = \mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}([\![-]\!]_{\mathbf{x},\mathbf{y}})$$

intertwine the products within $\widetilde{\mathfrak{B}}_n$ and $\mathfrak{B}_n$ with ${}_1\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}$.

The Hochschild homology in the presence of braid strand variables is a functor

$${}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x},\mathbf{y}} \xrightarrow{\mathrm{HH}_{\mathbf{x},\mathbf{y}}} \mathcal{W}^+_{\boldsymbol{\alpha}}-\mathbf{gm}, \quad \mathrm{HH}_{\mathbf{x},\mathbf{y}}(-) = (\mathsf{a}\mathsf{t}^{-1})^{\frac{1}{2}n}\,\mathrm{H}(-\overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}}), \tag{1.28}$$

extended to derived categories:

$$\begin{tikzcd} \mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \arrow[r, "-\overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}}\Delta_{\mathbf{x};\mathbf{y}}"] \arrow[rr, bend right, "\mathrm{HH}_{\mathbf{x},\mathbf{y}}"'] & \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\boldsymbol{\alpha}}) \arrow[r, "(\mathsf{a}\mathsf{t}^{-1})^{\frac{1}{2}n}\mathrm{H}_{\mathbf{W}^+_{\boldsymbol{\alpha}}}"] & \mathbf{W}^+_{\boldsymbol{\alpha}}, \end{tikzcd} \tag{1.29}$$

where $\mathrm{H}_{\mathbf{W}^+_{\boldsymbol{\alpha}}}$ is the homology taken with respect to the differential of the inner category $\mathbf{W}^+_{\boldsymbol{\alpha}}$ in the nested category $\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\boldsymbol{\alpha}})$.

Note that the diagonal module $\Delta_{\mathbf{x};\mathbf{y}}$ has no $\boldsymbol{\alpha}$ action. The functor $\mathcal{F}_{\mathbf{x}\to\boldsymbol{\alpha}}$ renames the variables $\mathbf{x}$ into $\boldsymbol{\alpha}$, and the commutativity of the subdiagram

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}"] \arrow[d, "\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}"'] & \mathbf{W}^{+}_{\mathbf{x}} \arrow[r, "\mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}"] \arrow[d, "\mathcal{F}_{\mathbf{x}\to\boldsymbol{\alpha}}"'] & \mathbf{W}^{+}_{\boldsymbol{\lambda}} \\
\mathbf{D}_{\mathrm{r}}({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathrm{HH}_{\mathbf{x},\mathbf{y}}"] & \mathbf{W}^{+}_{\boldsymbol{\alpha}} \arrow[ur, "\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}"'] &
\end{tikzcd}$$

of the diagram (1.21) is obvious.

The trace-like property of this Hochschild homology:

$$\mathrm{HH}_{\mathbf{x},\mathbf{z}}\big({}_{\boldsymbol{\alpha}}[\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}}\ {}_1\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}\ {}_{\boldsymbol{\alpha}}[\![\mathfrak{b}_2]\!]_{\mathbf{y},\mathbf{z}}\big) \simeq \mathrm{HH}_{\mathbf{x},\mathbf{z}}\big({}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})}[\![\mathfrak{b}_2]\!]_{\mathbf{x},\mathbf{y}}\ {}_1\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}\ {}_{\boldsymbol{\alpha}}[\![\mathfrak{b}_1]\!]_{\mathbf{y},\mathbf{z}}\big)$$

is established with the help of the sliding property (1.27). The sliding property also leads to the following theorem proved in subsection 5.2:

Theorem 1.3. *The action of the renaming functor $\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}$ on the Hochschild homology of the braid bracket does not depend on the choice of braid strands k_i representing link components*: *for two sequences of strands* $\mathbf{k}$ *and* $\mathbf{k}'$ *there is a $\mathcal{F}$-isomorphism*

$$\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}\big(\mathrm{HH}_{\mathbf{x},\mathbf{y}}({}_{\boldsymbol{\alpha}}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}})\big) \simeq \mathcal{F}^{(\mathbf{k}')}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}\big(\mathrm{HH}_{\mathbf{x},\mathbf{y}}({}_{\boldsymbol{\alpha}}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}})\big).$$

Corollary 1.4. *The object $\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}(\mathrm{HH}_{\mathbf{x},\mathbf{y}}({}_{\boldsymbol{\alpha}}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}}))$ of $\mathbf{W}^{+}_{\boldsymbol{\alpha}}$ is determined up to isomorphism only by the braid $\mathfrak{b}$, so there exists a map ${}_{\boldsymbol{\lambda}}[\![-]\!]^{\cdot}$ which makes the following diagram commutative*:

$$\begin{tikzcd}
\mathfrak{B}_n & & \mathfrak{B}^{(s)}_n \arrow[ll, hook'] \arrow[dd, "{}_{\boldsymbol{\lambda}}[\![-]\!]^{\cdot}"] \\
& & \\
\mathbf{D}_{\mathrm{r}}\big({}_{\boldsymbol{\alpha}}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}\big) \arrow[r, "\mathrm{HH}_{\mathbf{x},\mathbf{y}}"'] \arrow[from=1-1, "{}_{\boldsymbol{\alpha}}[\![-]\!]_{\mathbf{x},\mathbf{y}}"'] & \mathbf{W}^{+}_{\boldsymbol{\alpha}} \arrow[r, "\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}"'] & \mathbf{W}^{+}_{\boldsymbol{\lambda}}.
\end{tikzcd}$$

Finally, we show in subsection 5.3 that the bracket ${}_{\boldsymbol{\lambda}}[\![-]\!]^{\cdot}$ is invariant under Markov moves up to a connection shift $\langle\boldsymbol{\pi}(\lambda_i)\rangle$, where $\boldsymbol{\pi}(x)$ is defined by (1.2), caused by the change in framing of the i-th link component, so this bracket is an invariant of the framed link L constructed by closing the braid $\mathfrak{b}$.

1.8. Results. Our main results are Theorems 4.1 and 5.6 which state the existence and uniqueness of the maps

$$\mathfrak{B}_n \xrightarrow{\alpha[\![-]\!]_{\mathbf{x},\mathbf{y}}} \mathbf{D}_{\mathrm{r}}({}_{\alpha}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}), \qquad \mathfrak{L}_m \xrightarrow{[\![-]\!]} \mathbf{W}^{+}_{\boldsymbol{\lambda}}$$

in the diagram (1.21) after we define properly the map

$$\widetilde{\mathfrak{B}}_n \xrightarrow{\alpha[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}} \mathbf{Ch}({}_{\alpha}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}}). \tag{1.30}$$

Moreover, we show that if a framed link L' is obtained from a link L by increasing the framing of its i-th component by one, then

$$[\![L']\!] \simeq \mathfrak{fr}_{\lambda_i}[\![L]\!], \tag{1.31}$$

where the framing shift functor $\mathfrak{fr}_{\lambda_i}$ is defined by eq. (1.17).

2. Preliminaries

2.1. Simplifying objects in a relative derived category. Consider a general definition of a relative derived category $\mathbf{D}_{\mathrm{r}}(\mathbf{A})$ described in subsection 1.4: $\mathbf{A}$ and $\mathbf{B}$ are additive categories related by an additive functor $\mathcal{F}\colon \mathbf{A} \to \mathbf{B}$.

Theorem 2.1. *Suppose that for three complexes $A_\bullet$, $B_\bullet$ and $C_\bullet$ of* $\mathbf{Ch}(\mathbf{A})$ *there exists a sequence of chain maps*

$$B_\bullet \xrightarrow{f} A_\bullet \xrightarrow{g} C_\bullet$$

such that after the application of $\mathcal{F}$ it splits:

$$\begin{tikzcd} \mathcal{F}(B_\bullet) \arrow[r, "\mathcal{F}(f)"] \arrow[dr, "(\mathbb{1},0)"'] & \mathcal{F}(A_\bullet) \arrow[r, "\mathcal{F}(g)"] \arrow[d, "\cong"] & \mathcal{F}(C_\bullet) \\ & \mathcal{F}(B_\bullet) \oplus \mathcal{F}(C_\bullet). \arrow[ur, "(0,\mathbb{1})"'] & \end{tikzcd}$$

Then

$$A_\bullet \simeq \begin{cases} C_\bullet, & \text{if } \mathcal{F}(B_\bullet) \text{ is contractible,} \\ B_\bullet, & \text{if } \mathcal{F}(C_\bullet) \text{ is contractible,} \end{cases} \tag{2.1}$$

where $\simeq$ denotes a $\mathcal{F}$-isomorphism in $\mathbf{Ch}(\mathbf{A})$.

Proof. If $\mathcal{F}(C_\bullet)$ is contractible, then $\mathcal{F}(f)$ is a homotopy equivalence in $\mathbf{Ch}(\mathbf{B})$, hence f is a $\mathcal{F}$-isomorphism by definition.

If $\mathcal{F}(B_\bullet)$ is contractible, then $\mathcal{F}(g)$ is a homotopy equivalence in $\mathbf{Ch}(\mathbf{B})$, hence g is a $\mathcal{F}$-isomorphism by definition. $\square$

2.2. Derived partial tensor product

2.2.1. Equivariant resolutions. The derived partial tensor product (1.12) can be computed with the help of a projective resolution. It is sufficient to resolve one factor: for example,

$$M_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}} \simeq \mathcal{P}^{\bullet}(M_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}},$$

the resolution complex $\mathcal{P}^{\bullet}(M_{\mathbf{x},\mathbf{y}})$ consisting of projective $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-modules.

Definition 2.2. *Consider a semi-direct product $\mathcal{A} \rtimes \mathrm{U}\mathfrak{g}$, where $\mathfrak{g} \subset \mathrm{Der}(\mathcal{A})$ is a Lie algebra acting on an algebra $\mathcal{A}$ by derivations. A $\mathfrak{g}$-equivariant $\mathcal{A}$-projective resolution of an $\mathcal{A} \rtimes \mathrm{U}\mathfrak{g}$-module M is a complex in $\mathbf{Ch}(\mathcal{A} \rtimes \mathrm{U}\mathfrak{g} - \mathbf{mod})$ which is a projective resolution of M in $\mathbf{Ch}(\mathcal{A} - \mathbf{mod})$.*

Theorem 2.3. *If $\mathcal{P}^{\bullet}_{\mathcal{W}^{+}}(M_{\mathbf{x},\mathbf{y}})$ is a $\mathfrak{W}^{+}$-equivariant $\mathbb{Q}[\mathbf{x},\mathbf{y}]$-projective resolution of $M_{\mathbf{x},\mathbf{y}}$, then*

$$M_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}} \simeq \mathcal{P}^{\bullet}_{\mathcal{W}^{+}}(M_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}}.$$

In other words, a $\mathcal{W}^{+}$-equivariant resolution is sufficient for the computation of the derived partial tensor product $\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}$.

Proof. If $C_{\bullet}$ is the Chevalley-Eilenberg resolution of the trivial $\mathcal{W}^{+}$-module, then the tensor product $C_{\bullet} \otimes \mathcal{P}^{\bullet}_{\mathcal{W}^{+}}(M_{\mathbf{x},\mathbf{y}})$ (with $\mathcal{W}^{+}$ acting on this tensor product by the Leibnitz rule) is a $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-projective resolution of $M_{\mathbf{x},\mathbf{y}}$, hence

$$\begin{aligned} M_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}} &\simeq (C_{\bullet} \otimes \mathcal{P}^{\bullet}_{\mathcal{W}^{+}}(M_{\mathbf{x},\mathbf{y}})) \otimes_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}} \\ &\cong C_{\bullet} \otimes (\mathcal{P}^{\bullet}_{\mathcal{W}^{+}}(M_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}}) \\ &\simeq \mathcal{P}^{\bullet}_{\mathcal{W}^{+}}(M_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}}. \end{aligned}$$ □

2.2.2. Partial tensor products of semi-free modules. A $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-module $M_{\mathbf{x},\mathbf{y}}$ is called semi-free if it is free as a $\mathbb{Q}[\mathbf{x}]$-module and as a $\mathbb{Q}[\mathbf{y}]$-module. All elementary Soergel bimodules introduced in subsection 3.1 are semi-free, and their partial derived tensor products can be replaced by the ordinary ones in view of the following corollary of Theorem 2.3:

Corollary 2.4. *If modules $M_{\mathbf{x},\mathbf{y}}$ and $N_{\mathbf{x},\mathbf{y}}$ are semi-free then*

$$M_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}} \simeq M_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}},$$

and this tensor product is also semi-free as a $\mathcal{W}^{+}_{\mathbf{x},\mathbf{z}}$-module.

By definition, a free finitely generated q-graded $\mathbb{Q}[\mathbf{x}]$-module M has a presentation

$$M = \bigoplus_{i\in\mathbb{Z}} m_i \mathsf{q}^i \mathbb{Q}[\mathbf{x}],$$

where q is the q-degree shift functor and $m_i \in \mathbb{Z}$ are the multiplicities. We define the q-rank of M as a Laurent polynomial

$$\operatorname{rank}_q M = \sum_{i\in\mathbb{Z}} m_i q^i. \tag{2.2}$$

Let $\operatorname{rank}_{\mathbf{x};q} M_{\mathbf{x},\mathbf{y}}$ denote the $\mathbb{Q}[\mathbf{x}]$ q-rank of a semi-free module $M_{\mathbf{x},\mathbf{y}}$. The following is obvious:

Proposition 2.5. *If modules $M_{\mathbf{x},\mathbf{y}}$ and $N_{\mathbf{y},\mathbf{z}}$ are semi-free then*

$$\operatorname{rank}_{\mathbf{x};q}(M_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}}) = (\operatorname{rank}_{\mathbf{x};q} M_{\mathbf{x},\mathbf{y}})(\operatorname{rank}_{\mathbf{y};q} N_{\mathbf{y},\mathbf{z}}).$$

2.2.3. Partial tensor product in nested derived categories

Theorem 2.6. *There exists a unique* (*up to isomorphism*) *functor* (1.14) *which makes the following diagram commutative*:

$$\begin{tikzcd}
\mathbf{Ch}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \times \mathbf{Ch}(\mathbf{W}^+_{\mathbf{y},\mathbf{z}}) \arrow[r, "\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}"] \arrow[d, "\mathcal{F}_{\heartsuit}"] & \mathbf{Ch}(\mathbf{W}^+_{\mathbf{x},\mathbf{z}}) \arrow[d, "\mathcal{F}_{\heartsuit}"] \\
\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \times \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{y},\mathbf{z}}) \arrow[r, "\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}"] & \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{z}})
\end{tikzcd}$$

This theorem is a particular case of the following lemma:

Lemma 2.7. *Suppose that we have two pairs of additive categories related by additive functors*: $\mathbf{A}_1 \xrightarrow{\mathcal{F}_1} \mathbf{B}_1$ *and* $\mathbf{A}_2 \xrightarrow{\mathcal{F}_2} \mathbf{B}_2$. *If for a functor* $\mathbf{A}_1 \xrightarrow{\mathcal{F}_{\mathbf{A}}} \mathbf{A}_2$ *there exists a functor* $\mathbf{B}_1 \xrightarrow{\mathcal{F}_{\mathbf{B}}} \mathbf{B}_2$ *such that the following square is commutative*:

$$\begin{tikzcd}
\mathbf{A}_1 \arrow[r, "\mathcal{F}_{\mathbf{A}}"] \arrow[d, "\mathcal{F}_1"'] & \mathbf{A}_2 \arrow[d, "\mathcal{F}_2"] \\
\mathbf{B}_1 \arrow[r, "\mathcal{F}_{\mathbf{B}}"] & \mathbf{B}_2
\end{tikzcd} \tag{2.3}$$

(*functors* $\mathcal{F}_2 \circ \mathcal{F}_{\mathbf{A}}$ *and* $\mathcal{F}_{\mathbf{B}} \circ \mathcal{F}_1$ *are isomorphic*), *then the functor* $\mathcal{F}_{\mathbf{A}}$ *extends to the derived categories*:

$$\mathbf{D}_{\mathcal{F}_1}(\mathbf{A}_1) \xrightarrow{\mathcal{F}_{\mathbf{A}}} \mathbf{D}_{\mathcal{F}_2}(\mathbf{A}_2).$$

Proof. It is sufficient to show that if there is a $\mathcal{F}_1$-isomorphism

$$A_\bullet \xrightarrow{f} A'_\bullet$$

between the objects of $\mathbf{Ch}(\mathbf{A}_1)$, then its image

$$\mathcal{F}_{\mathbf{A}}(A_\bullet) \xrightarrow{\mathcal{F}_{\mathbf{A}}(f)} \mathcal{F}_{\mathbf{A}}(A'_\bullet)$$

is a $\mathcal{F}_2$-isomorphism. Indeed, by Definition 1.1 $\mathcal{F}_1(f)$ is an isomorphism in $\mathbf{K}(\mathbf{B}_1)$, hence $\mathcal{F}_{\mathbf{B}} \circ \mathcal{F}_1(f)$ is an isomorphism in $\mathbf{K}(\mathbf{B}_2)$. Commutativity of the square (2.3) means that $\mathcal{F}_2 \circ \mathcal{F}_{\mathbf{A}}(f) = \mathcal{F}_{\mathbf{B}} \circ \mathcal{F}_1(f)$, hence $\mathcal{F}_2 \circ \mathcal{F}_{\mathbf{A}}(f)$ is an isomorphism in $\mathbf{K}(\mathbf{B}_2)$ and by Definition 1.1 $\mathcal{F}_{\mathbf{A}}(f)$ is a $\mathcal{F}_1$-isomorphism in $\mathbf{D}_{\mathcal{F}_2}(\mathbf{A}_2)$. □

Proof of Theorem 2.6. The theorem follows from the previous lemma when we observe that the original derived tensor product (1.19) plays the role of the functor $\mathcal{F}_{\mathbf{B}}$. □

2.3. $\mathcal{W}^+_{\mathbf{x}}$-modules and $\mathfrak{W}^+$-connections

2.3.1. Notations for connections If two $\mathcal{W}^+_{\mathbf{x}}$-modules M and N have finite numbers of generators as $\mathbb{Q}[\mathbf{x}]$-modules, then we describe the homomorphisms between them by matrices relative to these generators: $M \xrightarrow{A} N$, where A is a matrix with polynomial entries

If a $\mathcal{W}^+_{\mathbf{x}}$-module M has a finite number of generators $\mathbf{v} = v_1, \ldots, v_k \in M$ as a $\mathbb{Q}[\mathbf{x}]$-module, then the action of $\mathfrak{W}^+$ on M can be described by a sequence of $k \times k$ matrices with polynomial entries $\mathbf{A} = A_0, A_1, \ldots, A_m = \|a_{m;ij}\|$ giving the action of derivations ∇_m, representing the algebra generators L_m, on module generators $\mathbf{v}$:

$$\nabla_m v_i = \sum_{j=1}^{k} a_{m;ij} v_j.$$

We refer to the sequence $\mathbf{A}$ as *connection* and use notation $(M, \mathbf{A})$ for this $\mathcal{W}^+_{\mathbf{x}}$-module, reserving a simple notation M for the case when all matrices $\mathbf{A}$ are zero. If M has a single generator v, that is, $M \cong \mathbb{Q}[\mathbf{x}]/I$, where I is a $\mathfrak{W}^+$-invariant $\mathbb{Q}[\mathbf{x}]$-ideal, then the connection matrices $\mathbf{A}$ are reduced to numbers $\mathbf{a}$ such that $L_m v = a_m v$, and the notation for the module becomes $(M, \mathbf{a})$.

Proposition 2.8. *A $\mathbb{Q}[\mathbf{x}]$-homomorphism between two $\mathcal{W}^+_{\mathbf{x}}$-modules with single generators*

$$f:(M,\mathbf{a}) \xrightarrow{p} (M',\mathbf{a}')$$

is a $\mathcal{W}^+_{\mathbf{x}}$-homomorphism iff the polynomial $p \in \mathbb{Q}[\mathbf{x}]$ satisfies the condition

$$\hat{L}_m p = (a_m - a'_m)\, p \mod I', \tag{2.4}$$

where $I' \subset \mathbb{Q}[\mathbf{x}]$ is the ideal such that $M' \cong \mathbb{Q}[\mathbf{x}]/I'$.

Proof. It is sufficient to verify $\mathfrak{W}^+$-equivariance of the action of f on the generator v of M:

$$\begin{aligned}\nabla'_m f(v) - f(\nabla_m v) &= (\hat{L}_m p)\, v' + p a'_m\, v' - p a_m v' \\ &= (\hat{L}_m p - p(a_m - a'_m))v',\end{aligned}$$

where v' is the generator of M'. □

2.3.2. Flat sequences. As we explained in the introduction, a sequence of polynomials $\mathbf{a} \in \mathbb{Q}[\mathbf{x}]$ is called *flat* if it satisfies the property (1.7). A flat sequence $\mathbf{a}$ determines an automorphism (1.8) of $\mathcal{W}^+_{\mathbf{x}}$ and, consequently, an endo-functor $\langle\mathbf{a}\rangle$. Obviously, $(M,\mathbf{A})\,\langle\mathbf{a}\rangle = (M,\mathbf{A}+\mathbf{a}\mathbb{1})$, where $\mathbb{1}$ is the $k\times k$ identity matrix.

We have two examples of flat sequences. The first one is described in the introduction: it is $\boldsymbol{\pi}'(x) \in \mathbb{Q}[x]$, where $\pi'_m(x) = (m+1)x^m$, and its flatness is verified by a direct calculation.

The second example originates from a 'gauge transformation.'

Theorem 2.9. *If for a sequence of polynomials $\mathbf{a} \in \mathbb{Q}[\mathbf{x}]$ there exists a polynomial $p \in \mathbb{Q}[\mathbf{x}]$ such that*

$$\hat{L}_m p = a_m\, p \tag{2.5}$$

for all $m \geq -1$, then the sequence $\mathbf{a}$ is flat.

Proof. Consider the action of $\mathfrak{W}^+$ on the field of fractions of $\mathbb{Q}[\mathbf{x}]$. Then the modified generators $L'_m = L_m + a_m$ are the result of a 'conjugation' of L_m by p:

$$\hat{L}'_m = \widehat{p^{-1}} \circ \hat{L}_m \circ \hat{p},$$

where $\hat{p}$ denotes the operator of multiplication by p. Hence the commutation relations between the generators L'_m are the same as those between L_m. □

The polynomial $y - x \in \mathbb{Q}[x, y]$ satisfies the condition (2.5):

$$\widehat{L}_m(y-x) = \pi_m(x,y)\,(y-x),$$

where

$$\pi_m(x,y) = \frac{y^{m+1} - x^{m+1}}{y-x} = \sum_{i=0}^{m} x^i\, y^{m-i}, \tag{2.6}$$

hence $\boldsymbol{\pi}(x,y)$ is a flat sequence.

Note that flat sequences for $\mathbb{Q}[\mathbf{x}]$ form a $\mathbb{Q}$-vector space, since a linear combination of flat sequences is flat.

2.3.3. Connections in Koszul complexes. For a sequence of polynomials $\mathbf{p} \in \mathbb{Q}[\mathbf{x}]$, $|\mathbf{x}| = n$, $|\mathbf{p}| = k$, consider the quotient $\mathbb{Q}[\mathbf{x}]$-module $M = \mathbb{Q}[\mathbf{x}]/(\mathbf{p})$. This module has a single generator v, which is represented by $1 \in \mathbb{Q}[\mathbf{x}]$.

Suppose that the ideal $(\mathbf{p})$ is invariant under the standard action of $\mathfrak{W}^+$ on $\mathbb{Q}[\mathbf{x}]$, that is, there exist $k \times k$ matrices $\mathbf{A} = A_{-1}, A_0, A_1, \ldots$ with polynomial entries: $A_m = \|a_{m;ij}\|$, $a_{m;ij} \in \mathbb{Q}[\mathbf{x}]$, $m \geq 0$, $1 \leq i, j \leq k$, which satisfy the relation

$$\widehat{L}_m \vec{p} = A_m\, \vec{p}, \qquad \vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_k \end{pmatrix}, \qquad \vec{p} \in \mathbb{Q}[\mathbf{x}]^k. \tag{2.7}$$

Then $(\mathbf{p})$ is invariant under the full action of $\mathcal{W}^+_{\mathbf{x}}$ and M has a structure of $\mathcal{W}^+_{\mathbf{x}}$-module with trivial connection: $\nabla_m v = 0$.

As a $\mathbb{Q}[\mathbf{x}]$-module, $M = \mathbb{Q}[\mathbf{x}]/(\mathbf{p})$ has an associated Koszul complex $\mathcal{P}^\bullet(M) = \mathbb{Q}[\mathbf{x}, \boldsymbol{\theta}]$, where $\boldsymbol{\theta}$, $|\boldsymbol{\theta}| = k$, are odd variables ($\theta_i\theta_j + \theta_j\theta_i = 0$, $\theta_i^2 = 0$) of homological degree -1 and the differential is

$$d = \sum_{i=1}^{n} p_i\, \partial_{\theta_i}, \tag{2.8}$$

where $\partial_{\theta_i}\theta_j = \delta_{ij}$. Define an action of the generators L_m on $\mathcal{P}^\bullet(M)$ by the formula

$$\nabla_m = \widehat{L}_m + \hat{A}_m, \qquad \hat{A}_m = \sum_{i,j=1}^{k} a_{m;ij}\, \theta_j\, \partial_{\theta_i}. \tag{2.9}$$

Proposition 2.10. *The derivations* (2.9) *commute with the differential* d*:*

$$[d, \nabla_m] = 0.$$

Proof. The proposition is proved by a direct calculation:

$$-[d, \widehat{L}_m] = [d, \hat{A}_m] = \sum_{i,j=1}^{k} a_{m;ij}\, p_j \partial_{\theta_i}. \qquad \square$$

Define the curvature of a sequence $\mathbf{A}$ of $k \times k$ matrices with polynomial entries as a double sequence of matrices

$$\mathrm{F}_{m,n}[\mathbf{A}] = \widehat{L}_m A_n - \widehat{L}_n A_m + [A_n, A_m] - (n-m) A_{m+n}$$

(*cf.* eq. (1.6)).

Theorem 2.11. *The Koszul complex $\mathcal{P}^\bullet(M) = \mathbb{Q}[\mathbf{x}, \boldsymbol{\theta}]$ with the differential* (2.8) *and the action of $\mathfrak{W}^+$ given by the derivations* (2.9) *is $\mathfrak{W}^+$-equivariant if the matrices $\mathbf{A}$ satisfy the relation*

$$\mathrm{F}_{m,n}[\mathbf{A}] = 0, \quad m, n \geq -1. \tag{2.10}$$

Proof. Is is easy to check that the relation (2.10) is equivalent to the condition $[\nabla_m, \nabla_n] = (n-m)\nabla_{m+n}$, the latter guaranteeing that derivations ∇_m represent the algebra $\mathfrak{W}^+$. $\square$

Remark 2.12. *The condition* (2.7) *does not define the matrices A_m uniquely.*

Remark 2.13. *Combining the structure relations* (1.4) *with relations* (2.7) *we find that the matrices A_m satisfy the relations*

$$\mathrm{F}_{m,n}[\mathbf{A}]\, \vec{p} = 0. \tag{2.11}$$

This relation is weaker than the condition (2.10) *for the Koszul complex $\mathcal{P}^\bullet(M)$ to be $\mathfrak{W}^+$-equivariant.*

Remark 2.14. *The 'diagonal' bimodule $\Delta_{\mathbf{x};\mathbf{y}}$ which appears in eq.* (1.23) *defining the Hochschild homology of bimodules in the derived category $\mathbf{W}^+_{\mathbf{x},\mathbf{y}}$ has a $\mathfrak{W}^+$-equivariant Koszul resolution*

$$\mathcal{P}^\bullet(\Delta_{\mathbf{x};\mathbf{y}}) = \bigotimes_{i=1}^{|\mathbf{x}|} \Big[\, \mathsf{a}^{-2}\mathsf{t}^{-1}\mathsf{q}^2 \left(\mathbb{Q}_{x_i,y_i}, \boldsymbol{\pi}(x_i, y_i)\right) \xrightarrow{y_i - x_i} \mathbb{Q}_{x_i,y_i} \,\Big]. \tag{2.12}$$

Hence, in accordance with eq. (1.23)*, one can use the following formula for the Hochschild homology*:

$$\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(-) = \mathrm{H}\Big(- \otimes_{\bar{\mathbf{x}},\mathbf{y}} \bigotimes_{i=1}^{|\mathbf{x}|} \Big[\, \mathsf{a}^{-2}\mathsf{t}^{-1}\mathsf{q}^2 \left(\mathbb{Q}_{x_i,y_i}, \boldsymbol{\pi}(x_i, y_i)\right) \xrightarrow{y_i - x_i} \mathbb{Q}_{x_i,y_i} \,\Big]\Big). \tag{2.13}$$

3. Categorification of the braid word semi-group by $\mathfrak{W}^+$-equivariant Soergel bimodules

3.1. Elementary $\mathfrak{W}^+$-equivariant Soergel bimodules. For a positive integer m, the elementary $\mathfrak{W}^+$-equivariant Soergel bimodule $\mathcal{S}_m = \mathcal{S}_{m;\mathbf{x},\mathbf{y}}$ is a $\mathcal{W}^+_{\mathbf{x},\mathbf{y}}$-module

$$\mathcal{S}_m = \mathbb{Q}_{\mathbf{x},\mathbf{y}}/I_{S_m}, \quad I_{S_m} = (e_1(\mathbf{y}) - e_1(\mathbf{x}), \dots, e_m(\mathbf{y}) - e_m(\mathbf{x})),$$

where $|\mathbf{x}| = |\mathbf{y}| = m$ and the $\mathbb{Q}[\mathbf{x},\mathbf{y}]$-ideal I_{S_m} is generated by the differences of elementary symmetric polynomials $e_k(\mathbf{x}) = \sum_{1\le i_1<\dots<i_k\le m} x_{i_1}\cdots x_{i_k}$. $L_n\, e_i(\mathbf{x})$ is also a symmetric polynomial, hence $L_n(e_i(\mathbf{y}) - e_i(\mathbf{x})) \in I_{S_m}$, and $\mathcal{S}_m$ has a $\mathcal{W}^+_{\mathbf{x},\mathbf{y}}$-module structure.

For $|\mathbf{x}| = |\mathbf{y}| = n$ an extended elementary Soergel bimodule $\mathcal{S}^i_{m;\mathbf{x},\mathbf{y}}$ ($i + m \le n + 1$) is a $\mathcal{W}^+_{\mathbf{x},\mathbf{y}}$-module defined as the following tensor product:

$$\mathcal{S}^i_{m;\mathbf{x},\mathbf{y}} = \Delta_{\mathbf{x}',\mathbf{y}'} \otimes \mathcal{S}_{m;\mathbf{x}'',\mathbf{y}''} \otimes \Delta_{\mathbf{x}''',\mathbf{y}'''}, \tag{3.1}$$

where $\mathbf{x}' = x_1,\dots,x_{i-1}$, $\mathbf{x}'' = x_i,\dots,x_{i+m-1}$, $\mathbf{x}''' = x_{i+m},\dots,x_n$, $\mathbf{x}$ is the combined list: $\mathbf{x} = \mathbf{x}',\mathbf{x}'',\mathbf{x}'''$, while $\mathbf{y}',\mathbf{y}'',\mathbf{y}'''$ and $\mathbf{y}$ are defined analogously.

Remark 3.1. *All extended elementary Soergel bimodules are semi-free with q-ranks*

$$\operatorname{rank}_{\mathbf{x};q} \mathcal{S}^i_{m;\mathbf{x},\mathbf{y}} = \operatorname{rank}_{\mathbf{y};q} \mathcal{S}^i_{m;\mathbf{x},\mathbf{y}} = [m]_q!, \tag{3.2}$$

where

$$[m]_q = \frac{q^{2m}-1}{q^2-1}, \quad [m]_q! = \prod_{i=1}^m [i]_q.$$

According to Corollary 2.4, all other Soergel bimodules defined as their partial tensor products, are also semi-free, their derived partial tensor products are quasi-isomorphic to the ordinary ones and their ranks are determined by Proposition 2.5

First three $\mathfrak{W}^+$-equivariant elementary Soergel bimodules are especially important and we denote them as

$$M_{|} = \mathcal{S}_1, \quad M_{\times} = \mathcal{S}_2, \quad M_{*} = \mathcal{S}_3. \tag{3.3}$$

The bimodule $M_{|} = \Delta_{x,y} = \mathbb{Q}_{x,y}/(y-x)$ represents the identity endofunctor acting on the category of $\mathcal{W}^+_x$-modules: for any such module M_x there is a canonical isomorphism

$$M_y \cong M_x \otimes_{\mathbb{Q}[x]} M_{|;x,y}.$$

Now we set $n = 2$, that is, $\mathbf{x} = x_1, x_2$ and $\mathbf{y} = y_1, y_2$, and consider two $\mathcal{W}^+_{\mathbf{x},\mathbf{y}}$-modules:

$$M_{\shortparallel} = \Delta_{\mathbf{x};\mathbf{y}} = \mathbb{Q}_{\mathbf{x},\mathbf{y}}/I_{\shortparallel}, \quad I_{\shortparallel} = (y_1 - x_1, y_2 - x_2), \tag{3.4a}$$

$$M_{\times} = \mathcal{S}_2 = \mathbb{Q}_{\mathbf{x},\mathbf{y}}/I_{\times}, \quad I_{\times} = (\rho, y_1 y_2 - x_1 x_2), \tag{3.4b}$$

where $\rho = y_1 + y_2 - x_1 - x_2$. Sometimes it is convenient to choose an 'asymmetric' set of generators for the ideals:

$$I_{\shortparallel} = (\rho, y_2 - x_2), \quad I_{\times} = (\rho, (y_2 - x_2)(y_2 - x_1)). \tag{3.5}$$

For brevity, we use notation $\boldsymbol{\pi} = \boldsymbol{\pi}(x_1, x_2)$, where the polynomials $\boldsymbol{\pi} = \pi_{-1}, \pi_0, \pi_1, \ldots$ are defined in (1.1). Note the relations between various polynomials π as they act on these bimodules:

$$\boldsymbol{\pi} = \boldsymbol{\pi}(x_1, x_2) = \boldsymbol{\pi}(y_1, y_2) \qquad \mod I_{\times}, \tag{3.6}$$

$$\boldsymbol{\pi} = \boldsymbol{\pi}(x_1, x_2) = \boldsymbol{\pi}(y_1, y_2) = \boldsymbol{\pi}(x_1, y_2) \quad \mod I_{\shortparallel}. \tag{3.7}$$

There are two important homomorphisms between the $\mathcal{W}^+_{\mathbf{x},\mathbf{y}}$-modules $M_{\shortparallel}$ and $M_{\times}$:

$$\chi_-\colon M_{\times} \xrightarrow{1} M_{\shortparallel}, \qquad \chi_+\colon M_{\shortparallel} \xrightarrow{\frac{1}{2}(y_2 - y_1 + x_2 - x_1)} (M_{\times}, -\boldsymbol{\pi})\,. \tag{3.8}$$

Sometimes it is convenient to describe the second homomorphism equivalently as

$$\chi_+\colon M_{\shortparallel} \xrightarrow{y_2 - x_1} (M_{\times}, -\boldsymbol{\pi})\,. \tag{3.9}$$

Obviously χ_- commutes with derivations of $\mathfrak{W}^+$. The same is true for χ_+: condition (2.4) is satisfied in view of the following chain of equalities (the first one is modulo $I_{\times}$):

$$\begin{aligned}
&\frac{1}{2}(y_2 - y_1 + x_2 - x_1)\,\pi_m(x_1, x_2) \\
&\quad = \frac{1}{2}(y_2 - y_1)\,\pi_m(y_1, y_2) + \frac{1}{2}(x_2 - x_1)\,\pi_m(x_1, x_2) \\
&\quad = \frac{1}{2}(y_2^{m+1} - y_1^{m+1} + x_2^{m+1} - x_1^{m+1}) \\
&\quad = \frac{1}{2}L_m\,(y_2 - y_1 + x_2 - x_1) \quad \mod I_{\times}.
\end{aligned}$$

3.2. Categorification bracket. In order to define the map

$$[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}\colon \widetilde{\mathfrak{B}}_n \longrightarrow \mathbf{Ch}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}})$$

which categorifies braid words, we introduce extended versions of bimodules $M_{\shortparallel}$ and $M_{\times}$ as $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-modules with $|\mathbf{x}| = |\mathbf{y}| = n$ for any $n \geq 2$.

First, we define the bimodule

$$M_{|\cdot|} = \Delta_{\mathbf{x};\mathbf{y}} = \bigotimes_{j=1}^{n} M_{|;x_j,y_j},$$

which determines the identity endo-functor in the category of $\mathfrak{W}^{+}$-equivariant $\mathbb{Q}[\mathbf{x}]$-modules: for any module $M_{\mathbf{x}}$ there is a canonical isomorphism

$$M_{\mathbf{y}} \cong M_{\mathbf{x}} \otimes_{\mathbb{Q}[\mathbf{x}]} M_{|\cdot|;\mathbf{x},\mathbf{y}}. \tag{3.10}$$

Second, we define the bimodule

$$M^{i}_{\times} = \Delta_{\mathbf{x}',\mathbf{y}'} \otimes M_{\times;\mathbf{x}'',\mathbf{y}''} \otimes \Delta_{\mathbf{x}''',\mathbf{y}'''} \tag{3.11}$$

(*cf.* the general definition (3.1)). Note that $M_{|\cdot|}$ has a similar presentation

$$M_{|\cdot|} = \Delta_{\mathbf{x}',\mathbf{y}'} \otimes M_{\shortparallel;\mathbf{x}'',\mathbf{y}''} \otimes \Delta_{\mathbf{x}''',\mathbf{y}'''}. \tag{3.12}$$

Finally, we define the $\mathfrak{W}^{+}$-invariant homomorphisms

$$\chi_{i;-}\colon M^{i}_{\times} \longrightarrow M_{|\cdot|}, \qquad \chi_{i;-} = \mathbb{1}' \otimes \chi_{-} \otimes \mathbb{1}''', \tag{3.13}$$

$$\chi_{i;+}\colon M_{|\cdot|} \longrightarrow \left(M^{i}_{\times}, -\boldsymbol{\pi}^{i}\right), \quad \chi_{i;+} = \mathbb{1}' \otimes \chi_{+} \otimes \mathbb{1}''', \tag{3.14}$$

where $\boldsymbol{\pi}^{i} = \boldsymbol{\pi}(x_i, x_{i+1})$, $\mathbb{1}'$ and $\mathbb{1}'''$ are the identity endomorphism of $\Delta_{\mathbf{x}',\mathbf{y}'}$ and $\Delta_{\mathbf{x}''',\mathbf{y}'''}$ and the formulas for $\chi_{i;-}$ and $\chi_{i;+}$ are written relative to the presentations (3.11) and (3.12).

Now to elementary braid words σ_i and $\sigma_i^{(-1)}$ we associate cones in the category **Ch** of the homomorphisms (3.13) and (3.14):

$$[\![\sigma_i]\!]^{\cdot} = \boxed{M_{|\cdot|} \xrightarrow{\chi_{i;+}} \mathsf{q}^{-2}\,\mathsf{t}(M^{i}_{\times}, -\boldsymbol{\pi}^{i})}\,, \quad [\![\sigma_i^{(-1)}]\!]^{\cdot} = \boxed{\mathsf{t}^{-1} M^{i}_{\times} \xrightarrow{\chi_{i;-}} M_{|\cdot|}}\,. \tag{3.15}$$

The action of the map $[\![-]\!]^{\cdot}$ on other elements of $\widetilde{\mathfrak{B}}_n$ is defined by the relation (1.22).

4. $\mathfrak{W}^+$-equivariant categorification of the braid group

Throughout the paper, by generators of a $\mathcal{W}^+_{\mathbf{x},\mathbf{y}}$-module $M_{\mathbf{x},\mathbf{y}}$ we mean the generators of $M_{\mathbf{x},\mathbf{y}}$ as a $\mathbb{Q}[\mathbf{x},\mathbf{y}]$-module.

4.1. Main theorem

Theorem 4.1. *There exists a unique homomorphism* $[\![-]\!]_{\mathbf{x},\mathbf{y}}\colon \mathfrak{B}_n \to \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}})$ *which makes the following diagram commutative*:

$$\begin{tikzcd} \widetilde{\mathfrak{B}}_n \arrow[r, two heads, "\mathrm{br}"] \arrow[d, "{[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}}"'] & \mathfrak{B}_n \arrow[d, "{[\![-]\!]_{\mathbf{x},\mathbf{y}}}"] \\ \mathbf{Ch}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \arrow[r, "\mathcal{F}_{\heartsuit}"] & \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}). \end{tikzcd}$$

Proof. The uniqueness of $[\![-]\!]_{\mathbf{x},\mathbf{y}}$ follows from the surjectivity of the map br. In order to establish its existence we have to prove that the complex $[\![\sigma_i]\!]^{\cdot}$ is the inverse of the complex $[\![\sigma_i^{(-1)}]\!]^{\cdot}$ with respect to the monoidal structure of $\mathbf{K}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}})$, and that the map $[\![-]\!]^{\cdot}$ respects the braid relations. For obvious reasons we will refer to these properties as second and third Reidemeister moves invariance. The factorization of bimodules (3.11) and (3.12) and the local nature of the homomorphisms (3.13), (3.14) implies that it is sufficient to prove the second Reidemeister move invariance for the 2-strand braid and the third Reidemeister move for the 3-strand braid. This will be done in the next two subsections. □

4.2. Second Reidemeister move invariance. Let $\mathbf{x} = x_1, x_2$, $\mathbf{y} = y_1, y_2$ and $\mathbf{z} = z_1, z_2$. Recall that by the definition of the map $[\![-]\!]^{\cdot}_{\mathbf{x},\mathbf{y}}$ the complex associated to the product of elementary braid words $\sigma_1\sigma_1^{(-1)}$ is the tensor product of elementary complexes over the intermediate algebra:

$$[\![\sigma_1\sigma_1^{(-1)}]\!]^{\cdot}_{\mathbf{x},\mathbf{z}} = [\![\sigma_1]\!]^{\cdot}_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma_1^{(-1)}]\!]^{\cdot}_{\mathbf{y},\mathbf{z}}. \tag{4.1}$$

Theorem 4.2. *There is a homotopy equivalence of complexes*

$$[\![\sigma_1\sigma_1^{(-1)}]\!]^{\cdot}_{\mathbf{x},\mathbf{z}} \sim M_{\shortparallel;\mathbf{x},\mathbf{z}} \tag{4.2}$$

within the category $\mathbf{Ch}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}})$.

Lemma 4.3. *The following tensor product splits as a $\mathcal{W}^+_{\mathbf{x},\mathbf{z}}$-module*:

$$M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\times;\mathbf{y},\mathbf{z}} \cong M_{\times;\mathbf{x},\mathbf{z}} \oplus (M_{\times;\mathbf{x},\mathbf{z}}, \boldsymbol{\pi}). \tag{4.3}$$

The generators of the modules $M_{\times;\mathbf{y},\mathbf{z}}$ in the right hand side may be chosen as $v_1 = v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}}$ and $v_2 = (y_2 - y_1)v_1$, where $v_{\times;\mathbf{x},\mathbf{y}}$ and $v_{\times;\mathbf{y},\mathbf{z}}$ are the generators of $M_{\times;\mathbf{x},\mathbf{y}}$ and $M_{\times;\mathbf{y},\mathbf{z}}$ respectively. The following diagram is commutative:

$$\begin{tikzcd}
M_{\parallel;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\times;\mathbf{y},\mathbf{z}} \arrow[r, "\chi_+\otimes\mathbb{1}"] \arrow[d, "\cong"] & (M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\times;\mathbf{y},\mathbf{z}}, -\boldsymbol{\pi}) \arrow[r, "\mathbb{1}\otimes\chi_-"] \arrow[d, "\cong"] & (M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\parallel;\mathbf{y},\mathbf{z}}, -\boldsymbol{\pi}) \arrow[d, "\cong"] \\
(\mathbf{x},\mathbf{z}) \arrow[r, "\frac{1}{2}(x_2-x_1{,}\mathbb{1})"] & (M_{\times;\mathbf{x},\mathbf{z}}, -\boldsymbol{\pi}) \oplus M_{\times;\mathbf{x},\mathbf{z}} \arrow[r, "(\mathbb{1}{,}z_2-z_1)"] & (M_{\times;\mathbf{x},\mathbf{z}}, -\boldsymbol{\pi}).
\end{tikzcd} \tag{4.4}$$

Proof. It is easy to establish the isomorphism (4.3) in the category of $\mathbb{Q}[\mathbf{x},\mathbf{z}]$-modules, the right hand side ones having generators v_1 and v_2. Indeed, the tensor product in the left hand side can be presented as the quotient $\mathbb{Q}[\mathbf{x},\mathbf{y},\mathbf{z}]$-module

$$M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbb{Q}[\mathbf{y}]} M_{\times;\mathbf{y},\mathbf{z}} \cong \mathbb{Q}[\mathbf{x},\mathbf{y},\mathbf{z}]/I_{\mathbf{x},\mathbf{y},\mathbf{z}},$$

where

$$I_{\mathbf{x},\mathbf{y},\mathbf{z}} = \begin{pmatrix} e_1(\mathbf{y}) - e_1(\mathbf{x}) \\ e_2(\mathbf{y}) - e_2(\mathbf{x}) \\ e_1(\mathbf{z}) - e_1(\mathbf{y}) \\ e_2(\mathbf{z}) - e_2(\mathbf{y}) \end{pmatrix} = \begin{pmatrix} e_1(\mathbf{z}) - e_1(\mathbf{x}) \\ e_2(\mathbf{z}) - e_2(\mathbf{x}) \\ e_1(\mathbf{y}) - e_1(\mathbf{x}) \\ e_2(\mathbf{y}) - e_2(\mathbf{x}) \end{pmatrix},$$

while e_1 and e_2 are elementary symmetric polynomials of degrees one and two. The second choice of generators of the ideal $I_{\mathbf{x},\mathbf{y},\mathbf{z}}$ implies the isomorphism of $\mathbb{Q}[\mathbf{x},\mathbf{z}]$-modules:

$$M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbb{Q}[\mathbf{y}]} M_{\times;\mathbf{y},\mathbf{z}} \cong M_{\times;\mathbf{x},\mathbf{z}} \otimes_{\mathbb{Q}[\mathbf{x}]} M_{\times;\mathbf{x},\mathbf{y}}. \tag{4.5}$$

At the same time, there is an isomorphism of $\mathbb{Q}[\mathbf{x}]$-modules

$$M_{\times;\mathbf{x},\mathbf{y}} \cong \mathbb{Q}[\mathbf{x}] \oplus \mathbb{Q}[\mathbf{x}] \tag{4.6}$$

and the generators of the latter modules $\mathbb{Q}[\mathbf{x}]$ can be chosen as $v_1' = v_{\mathbf{x},\mathbf{y}}$ and $v_2' = (y_2 - y_1)v_{\mathbf{x},\mathbf{y}}$, where $v_{\mathbf{x},\mathbf{y}}$ is the generator of $M_{\times;\mathbf{x},\mathbf{y}}$.

The action of a $\mathfrak{W}^+$ generator L_m on the generator v_2' of the second module in the right hand side of eq. (4.6) is

$$\nabla_m v_2' = (L_m\,(y_2 - y_1))v_{\mathbf{x},\mathbf{y}} = \pi_m(y_1, y_2)\,v_2' = \pi_m(x_1, x_2)\,v_2',$$

hence the splitting (4.6) is $\mathfrak{W}^+$-equivariant if we add connection $\boldsymbol{\pi}$ to the second summand in its right hand side:

$$M_{\times;\mathbf{x},\mathbf{y}} \cong \mathbb{Q}_{\mathbf{x}} \oplus (\mathbb{Q}_{\mathbf{x}}, \boldsymbol{\pi}).$$

Combining this splitting with the isomorphism (4.5) we get the splitting (4.3).

In order to verify the commutativity of the diagram (4.4) we check the action of the upper horizontal homomorphisms on the generators of modules:

$$\begin{aligned}
(\chi_+ \otimes \mathbb{1})\,(v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}}) &= \frac{1}{2}(x_2 - x_1 + y_2 - y_1)\,v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}}\\
&= \frac{1}{2}(x_2 - x_1)\,v_1 + \tfrac{1}{2}v_2,\\
(\mathbb{1} \otimes \chi_-)\,(v_1) &= (\mathbb{1} \otimes \chi_-)\,(v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}})\\
&= v_{\mathbf{x},\mathbf{z}},\\
(\mathbb{1} \otimes \chi_-)\,(v_2) &= (\mathbb{1} \otimes \chi_-)\,((y_2 - y_1)\,v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}})\\
&= (y_2 - y_1)\,v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}}\\
&= (z_2 - z_1)\,v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}}.
\end{aligned}$$

□

Proof of Theorem 4.2. We prove the theorem by a sequence of isomorphisms and homotopy equivalences:

$$\begin{aligned}
&[\![\sigma_1\sigma_1^{(-1)}]\!]^{\cdot}_{\mathbf{x},\mathbf{z}}\\
&\quad= [\![\sigma_1]\!]^{\cdot}_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma_1^{(-1)}]\!]^{\cdot}_{\mathbf{y},\mathbf{z}}
\end{aligned}$$

$$= \boxed{\begin{tikzcd}[column sep=tiny]
& (M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\times;\mathbf{y},\mathbf{z}}, -\boldsymbol{\pi}) \arrow[dr, "\mathbb{1}\otimes\chi_-"] & \\
M_{\parallel;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\times;\mathbf{y},\mathbf{z}} \arrow[ur, "\chi_+\otimes\mathbb{1}"] \arrow[dr, "\mathbb{1}\otimes\chi_-"'] & & (M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\parallel;\mathbf{y},\mathbf{z}}, -\boldsymbol{\pi}) \\
& M_{\parallel;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\parallel;\mathbf{y},\mathbf{z}} \arrow[ur, "-\chi_+\otimes\mathbb{1}"'] &
\end{tikzcd}}$$

$$\cong \boxed{\begin{tikzcd} & (M_{\times;\mathbf{x},\mathbf{z}}, -\boldsymbol{\pi}) \oplus M_{\times;\mathbf{x},\mathbf{z}} \arrow[dr, "-(\mathbb{1}, z_2 - z_1)"] & \\ M_{\times;\mathbf{x},\mathbf{z}} \arrow[ur, "\frac{1}{2}(x_2 - x_1, \mathbb{1})"] \arrow[dr, "1"'] & & (M_{\times;\mathbf{x},\mathbf{z}}, -\boldsymbol{\pi}) \\ & M_{\shortparallel;\mathbf{x},\mathbf{z}} \arrow[ur, "\frac{1}{2}(x_2 - x_1 + z_2 - z_1)"'] & \end{tikzcd}}$$

$$\sim \boxed{\begin{tikzcd} (M_{\times;\mathbf{x},\mathbf{z}}, -\boldsymbol{\pi}) \arrow[dr, "-\mathbb{1}"] & \\ & (M_{\times;\mathbf{x},\mathbf{z}}, -\boldsymbol{\pi}) \\ M_{\shortparallel;\mathbf{x},\mathbf{z}} \arrow[ur, "\frac{1}{2}(x_2 - x_1 + y_2 - y_1)"'] & \end{tikzcd}}$$

$$\sim \quad M_{\shortparallel;\mathbf{x},\mathbf{z}}.$$

Here the first isomorphism is the definition (4.1), the second isomorphism is the definition of the tensor product of complexes (3.15) and the third isomorphism comes from Lemma 4.3. The last two homotopy equivalences are contractions of a part of a cone within the homotopy category:

$$\boxed{A \longrightarrow B} \sim \begin{cases} A[1], & \text{if } B \text{ is contractible,} \\ B, & \text{if } A \text{ is contractible.} \end{cases} \qquad \square$$

4.3. Third Reidemeister move invariance. Let $\mathbf{x} = x_1, x_2, x_3$, $\mathbf{y} = y_1, y_2, y_3$, $\mathbf{z} = z_1, z_2, z_3$ and $\mathbf{w} = w_1, w_2, w_3$. Recall that by definition of the map $[\![-]\!]^{\cdot}$ the categorification of the triple products of braid words appearing at both sides of the third Reidemeister move takes the form

$$[\![\sigma_i \sigma_j \sigma_i]\!]^{\cdot}_{\mathbf{x},\mathbf{w}} = [\![\sigma_i]\!]^{\cdot}_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma_j]\!]^{\cdot}_{\mathbf{y},\mathbf{z}} \otimes_{\mathbf{z}} [\![\sigma_i]\!]^{\cdot}_{\mathbf{z},\mathbf{w}}.$$

Theorem 4.4. *The following complexes are $\mathcal{F}$-isomorphic in the category* $\mathbf{Ch}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{w}})$:

$$[\![\sigma_1 \sigma_2 \sigma_1]\!]^{\cdot}_{\mathbf{x},\mathbf{w}} \simeq [\![\sigma_2 \sigma_1 \sigma_2]\!]^{\cdot}_{\mathbf{x},\mathbf{w}}. \tag{4.7}$$

The algebra $\mathbb{Q}[\mathbf{x},\mathbf{y}]$ has an automorphism $f^{(13)}$ which swaps simultaneously x_1 with x_3 and y_1 with y_3: $f^{(13)}(x_1) = x_3$, $f^{(13)}(x_2) = x_2$, $f^{(13)}(x_3) = x_1$ and the same for y_1, y_2 and y_3. This automorphism generates an involutive endo-functor $\mathcal{F}^{(13)}_{\mathbf{x},\mathbf{y}}$ of the category $\mathbf{Ch}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}})$. The obvious isomorphism of bimodule complexes $\mathcal{F}^{(13)}_{\mathbf{x},\mathbf{y}}\big(\llbracket\sigma_1\rrbracket^{\cdot}_{\mathbf{x},\mathbf{y}}\big) \cong \llbracket\sigma_2\rrbracket^{\cdot}_{\mathbf{x},\mathbf{y}}$ and the commutativity of a diagram

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \times \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{y},\mathbf{z}}) \arrow[r, "\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}"] \arrow[d, "\mathcal{F}^{(13)}_{\mathbf{x},\mathbf{y}} \times \mathcal{F}^{(13)}_{\mathbf{y},\mathbf{z}}"'] & \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{z}}) \arrow[d, "\mathcal{F}^{(13)}_{\mathbf{x},\mathbf{z}}"] \\
\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{y}}) \times \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{y},\mathbf{z}}) \arrow[r, "\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}"] & \mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x},\mathbf{z}})
\end{tikzcd}$$

imply the isomorphism of complexes $\llbracket\sigma_2\sigma_1\sigma_2\rrbracket^{\cdot}_{\mathbf{x},\mathbf{w}} \cong \mathcal{F}^{(13)}_{\mathbf{x},\mathbf{y}}(\llbracket\sigma_1\sigma_2\sigma_1\rrbracket^{\cdot}_{\mathbf{x},\mathbf{w}})$, hence the homotopy equivalence (4.7) follows from the homotopy equivalence

$$\llbracket\sigma_1\sigma_2\sigma_1\rrbracket^{\cdot}_{\mathbf{x},\mathbf{w}} \simeq \mathcal{F}^{(13)}_{\mathbf{x},\mathbf{w}}(\llbracket\sigma_1\sigma_2\sigma_1\rrbracket^{\cdot}_{\mathbf{x},\mathbf{w}}). \tag{4.8}$$

In other words, it is sufficient to show that the left hand side of eq. (4.7) is symmetric under the transposition of indices 1 and 3 of the variables.

Before proceeding with the proof of eq. (4.8) we introduce new notations and establish decomposition of some bimodules appearing in the complex $\llbracket\sigma_1\sigma_2\sigma_1\rrbracket^{\cdot}_{\mathbf{x},\mathbf{w}}$. The simplest bimodules are

$$M_{|||;\mathbf{x},\mathbf{w}} = M_{|\cdot|;\mathbf{x},\mathbf{w}} = \Delta_{\mathbf{x},\mathbf{w}},$$

$$M_{\times|;\mathbf{x},\mathbf{w}} = M^1_{\times;\mathbf{x},\mathbf{w}},$$

$$M_{|\times;\mathbf{x},\mathbf{w}} = M^2_{\times;\mathbf{x},\mathbf{w}},$$

and the Soergel bimodule $M_{\ast;\mathbf{x},\mathbf{w}}$ of eq. (3.3). The action of the transposition endo-functor $\mathcal{F}^{(13)}$ on these bimodules is obvious:

$$\mathcal{F}^{(13)}_{\mathbf{x},\mathbf{w}}(M_{|||;\mathbf{x},\mathbf{w}}) \cong M_{|||;\mathbf{x},\mathbf{w}},$$

$$\mathcal{F}^{(13)}_{\mathbf{x},\mathbf{w}}(M_{\times|;\mathbf{x},\mathbf{w}}) \cong M_{|\times;\mathbf{x},\mathbf{w}},$$

$$\mathcal{F}^{(13)}_{\mathbf{x},\mathbf{w}}(M_{\ast;\mathbf{x},\mathbf{w}}) \cong M_{\ast;\mathbf{x},\mathbf{w}}.$$

Lemma 4.5. *The $\mathcal{W}^+_{\mathbf{x},\mathbf{w}}$-module*

$$M_{\#;\mathbf{x},\mathbf{w}} = M_{\times|;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{|||;\mathbf{y},\mathbf{z}} \otimes_{\mathbf{z}} M_{|\times;\mathbf{z},\mathbf{w}}$$

has a single generator $v_{\#;\mathbf{x},\mathbf{w}} = v_{\times|;\mathbf{x},\mathbf{y}} \otimes v_{|||;\mathbf{y},\mathbf{z}} \otimes v_{|\times;\mathbf{z},\mathbf{x}}$ and it can be presented as a quotient $M_{\#;\mathbf{x},\mathbf{w}} \cong \mathbb{Q}_{\mathbf{x},\mathbf{w}}/I_{\#;\mathbf{x},\mathbf{w}}$, where

$$I_{\#;\mathbf{x},\mathbf{w}} = (w_1+w_2+w_3-(x_1+x_2+x_3), (w_1-x_1)(w_1-x_2), (w_3-x_3)(w_3-x_2)).$$

The proof is obvious and we omit it. The bimodule $M_{\#';\mathbf{x},\mathbf{w}} \cong \mathcal{F}^{(13)}_{\mathbf{x},\mathbf{w}}(M_{\#;\mathbf{x},\mathbf{w}})$ has a similar property.

Consider three $\mathcal{W}^+_{\mathbf{x},\mathbf{w}}$-modules defined as tensor products

$$M^{\mathrm{d}}_{\times|;\mathbf{x},\mathbf{w}} = M_{\times|;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{|||;\mathbf{y},\mathbf{z}} \otimes_{\mathbf{z}} M_{|||;\mathbf{z},\mathbf{w}}, \tag{4.9}$$

$$M^{\mathrm{u}}_{\times|;\mathbf{x},\mathbf{w}} = M_{|||;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{|||;\mathbf{y},\mathbf{z}} \otimes_{\mathbf{z}} M_{\times|;\mathbf{z},\mathbf{w}}, \tag{4.10}$$

$$M_{\between|;\mathbf{x},\mathbf{w}} = M_{\times|;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{|||;\mathbf{y},\mathbf{z}} \otimes_{\mathbf{z}} M_{\times|;\mathbf{y},\mathbf{z}}, \tag{4.11}$$

and the homomorphisms

$$M^{\mathrm{d}}_{\times|;\mathbf{x},\mathbf{w}} \xleftarrow{\chi_-\otimes\mathbb{1}\otimes\mathbb{1}} M_{\between|;\mathbf{x},\mathbf{w}} \xrightarrow{\mathbb{1}\otimes\mathbb{1}\otimes\chi_-} M^{\mathrm{u}}_{\times|;\mathbf{x},\mathbf{w}}\ . \tag{4.12}$$

Canonical isomorphisms

$$M^{\mathrm{d}}_{\times|;\mathbf{x},\mathbf{w}} \cong M^{\mathrm{u}}_{\times|;\mathbf{x},\mathbf{w}} \cong M_{\times|;\mathbf{x},\mathbf{w}}$$

imply that $M^{\mathrm{d}}_{\times|;\mathbf{x},\mathbf{w}}$ and $M^{\mathrm{u}}_{\times|;\mathbf{x},\mathbf{w}}$ have single generators $v^{\mathrm{u}}_{\times|;\mathbf{x},\mathbf{w}}$ and $v^{\mathrm{d}}_{\times|;\mathbf{x},\mathbf{w}}$, which are tensor products of the generators of factor modules in the tensor products (4.9).

Lemma 4.6. *The $\mathcal{W}^+_{\mathbf{x},\mathbf{w}}$-module $M_{\between|;\mathbf{x},\mathbf{w}}$ splits*:

$$M_{\between|;\mathbf{x},\mathbf{w}} \cong M^1_{\times|;\mathbf{x},\mathbf{w}} \oplus (M^2_{\times|;\mathbf{x},\mathbf{w}}, \boldsymbol{\pi}_{12}), \tag{4.13}$$

where $M^1_{\times|;\mathbf{x},\mathbf{w}} \cong M^2_{\times|;\mathbf{x},\mathbf{w}} \cong M_{\times|;\mathbf{x},\mathbf{w}}$ as $\mathbb{Q}[\mathbf{x},\mathbf{w}]$-modules, while the connection $\boldsymbol{\pi}_{12}$ of $M^2_{\times|;\mathbf{x},\mathbf{w}}$ is

$$\boldsymbol{\pi}_{12} = \boldsymbol{\pi}(x_1,x_2) = \boldsymbol{\pi}(w_1,w_2) \mod I_{\times|}. \tag{4.14}$$

The generators of the modules $M^1_{\times|;\mathbf{x},\mathbf{w}}$ and $M^2_{\times|;\mathbf{x},\mathbf{w}}$ may be chosen as

$$v_1 = v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}} \otimes v_{\mathbf{z},\mathbf{w}}, \quad v_2 = (y_2-y_1)v_1 = (z_2-z_1)v_1, \tag{4.15}$$

where $v_{\times;\mathbf{x},\mathbf{y}}$ and $v_{\times;\mathbf{y},\mathbf{z}}$ are the generators of $M_{\times|;\mathbf{x},\mathbf{y}}$, $M_{|||;\mathbf{y},\mathbf{z}}$ and $M_{\times|;\mathbf{z},\mathbf{w}}$ respectively. The homomorphisms (4.12) *are presented by matrices*

$$\chi_-\otimes\mathbb{1}\otimes\mathbb{1} = (1 \quad w_2-w_1), \qquad \mathbb{1}\otimes\mathbb{1}\otimes\chi_- = (1 \quad x_2-x_1)$$

relative to the generators v_1, v_2 of the middle module and $v^{\mathrm{d}}_{\times|;\mathbf{x},\mathbf{w}}$, $v^{\mathrm{u}}_{\times|;\mathbf{x},\mathbf{w}}$ of the target modules.

Proof. This lemma follows easily from Lemma 4.3 if we tensor multiply its formulas by the bimodules $M_{|}$ over the polynomial algebras of variables with index 3. ☐

Consider the $\mathcal{W}^{+}_{\mathbf{x},\mathbf{w}}$-module

$$M_{\bigstar;\mathbf{x},\mathbf{w}} = M_{\times|;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{|\times;\mathbf{y},\mathbf{z}} \otimes_{\mathbf{z}} M_{\times|;\mathbf{z},\mathbf{w}}$$

and the homomorphism

$$M_{\bigstar;\mathbf{x},\mathbf{w}} \xrightarrow{\mathbb{1}\otimes\chi_{-}\otimes\mathbb{1}} M_{\asymp|;\mathbf{x},\mathbf{w}}\ . \tag{4.16}$$

Lemma 4.7. *There is an exact sequence of $\mathcal{W}^{+}_{\mathbf{x},\mathbf{w}}$-modules*

$$0 \longrightarrow M_{\ast;\mathbf{x},\mathbf{w}} \longrightarrow M_{\bigstar;\mathbf{x},\mathbf{w}} \longrightarrow (M_{\times|;\mathbf{x},\mathbf{w}}, \pi_{12}) \longrightarrow 0, \tag{4.17}$$

which splits in the category of $\mathbb{Q}[\mathbf{x},\mathbf{w}]$-modules:

$$M_{\bigstar;\mathbf{x},\mathbf{w}} \cong M_{\ast;\mathbf{x},\mathbf{w}} \oplus M_{\times|;\mathbf{x},\mathbf{w}}. \tag{4.18}$$

The generators of the modules in the right hand side may be chosen as

$$v_{\ast;\mathbf{x},\mathbf{w}} = v_{\mathbf{x},\mathbf{y}} \otimes v_{\mathbf{y},\mathbf{z}} \otimes v_{\mathbf{z},\mathbf{w}}, \quad v_{\times|;\mathbf{x},\mathbf{w}} = y\, v_{\ast;\mathbf{x},\mathbf{w}}, \tag{4.19}$$

where

$$y = 2(z_3 - y_2) = y_1 - y_2 - x_1 - x_2 + 2w_3, \tag{4.20}$$

while $v_{\mathbf{x},\mathbf{y}}$, $v_{\mathbf{y},\mathbf{z}}$ and $v_{\mathbf{z},\mathbf{w}}$ are generators of the bimodules $M_{\times|;\mathbf{x},\mathbf{y}}$, $M_{|\times;\mathbf{y},\mathbf{z}}$ and $M_{\times|;\mathbf{z},\mathbf{w}}$ respectively. The homomorphism (4.16) *is described by the matrix*

$$\mathbb{1} \otimes \chi_{-} \otimes \mathbb{1} = \begin{pmatrix} 1 & -x_1 - x_2 + 2w_3 \\ 0 & 1 \end{pmatrix} \tag{4.21}$$

relative to the generators (4.19) *of $M_{\bigstar;\mathbf{x},\mathbf{w}}$ and generators* (4.15) *of $M_{\asymp|;\mathbf{x},\mathbf{w}}$.*

Proof. The splitting (4.18) and the generators (4.19) are well-known in the theory of Soergel bimodules, but we will derive these results here for completeness. The rest of the lemma is an easy corollary: the matrix presentation (4.21) and the connections in the submodule and quotient module in the exact sequence (4.17) follow from the action of the homomorphism $\mathbb{1} \otimes \chi_{-} \otimes \mathbb{1}$ and the generators $L_m \in \mathfrak{W}^{+}$ on the module generators (4.19).

If we use the asymmetric formulas (3.5), then it is easy to see that the tensor product in the left hand side of eq. (4.18) has the following presentation as a $\mathbb{Q}[\mathbf{x},\mathbf{w}]$-module:

$$M_{\bigstar;\mathbf{x},\mathbf{w}} \cong \mathbb{Q}[\mathbf{x},\mathbf{w},y_1,y_2,z_2]/I_1, \tag{4.22}$$

where

$$I_1 = \begin{pmatrix} y_2 - (x_1 + x_2 - y_1) \\ (y_1 - x_1)(y_1 - x_2) \\ z_2 - (w_1 + w_2 - y_1) \\ (y_1 - w_1)(y_1 - w_2) \\ z_2 + w_3 - (y_2 + x_3) \\ z_2 w_3 - y_2 x_3 \end{pmatrix}, \tag{4.23}$$

and elements (4.19) become

$$v_{\ast;\mathbf{x},\mathbf{w}} = 1, \quad v_{\times|;\mathbf{x},\mathbf{w}} = y. \tag{4.24}$$

After eliminating the variables y_2 and z_2 with the help of the first and third lines and replacing y_1 with y in accordance with (4.20) we find the following presentation for the tensor product:

$$M_{\star;\mathbf{x},\mathbf{w}} \cong \mathbb{Q}[\mathbf{x}, \mathbf{w}, y]/I_2,$$

where

$$I_2 = \begin{pmatrix} e_1(\mathbf{w}) - e_1(\mathbf{x}) \\ e_2(\mathbf{w}) - e_2(\mathbf{x}) \\ p_1 \\ p_2 \end{pmatrix}, \quad p_1 = \tilde{x}_3\, y, \quad p_2 = y^2 + 2(\tilde{x}_1 + \tilde{x}_2)\, y + 4\tilde{x}_1\tilde{x}_2,$$

and we used shortcut notations $\tilde{x}_i = x_i - w_3$, $i = 1, 2, 3$. The other two modules of the sequence (4.17) have similar presentations:

$$M_{\ast;\mathbf{x},\mathbf{w}} \cong \mathbb{Q}[\mathbf{x}, \mathbf{w}]/I_{\ast}, \quad M_{\times|;\mathbf{x},\mathbf{w}} \cong \mathbb{Q}[\mathbf{x}, \mathbf{w}]/I_{\times|}, \tag{4.25}$$

where

$$I_{\ast} = \begin{pmatrix} e_1(\mathbf{w}) - e_1(\mathbf{x}) \\ e_2(\mathbf{w}) - e_2(\mathbf{x}) \\ \tilde{x}_1\tilde{x}_2\tilde{x}_3 \end{pmatrix}, \quad I_{\times|} = \begin{pmatrix} e_1(\mathbf{w}) - e_1(\mathbf{x}) \\ e_2(\mathbf{w}) - e_2(\mathbf{x}) \\ \tilde{x}_3 \end{pmatrix}$$

and we used an asymmetric choice for the generators of $I_{\ast}$ (*cf.* eq. (3.5)).

The modules (4.25) are both generated by $1 \in \mathbb{Q}[\mathbf{x}, \mathbf{w}]$. The relation $p_2 = 0$ appearing at the bottom of I_2 expresses y^2 in terms of lower powers of y, hence $M_{\star;\mathbf{x},\mathbf{w}}$ is generated as a $\mathbb{Q}[\mathbf{x}, \mathbf{w}]$-module by its elements (4.24) and the action of the homomorphism (4.16) is described by the matrix (4.21) relative to these generators and the target module generators (4.15).

Consider two special homomorphisms defined by their action on module generators:

$$M_{\times|;\mathbf{x},\mathbf{w}} \xrightarrow{f_-=\binom{0}{1}} M_{\bigstar;\mathbf{x},\mathbf{w}} \xrightarrow{f_+=(0\ 1)} M_{\times|;\mathbf{x},\mathbf{w}}.$$

The homomorphism f_- is well-defined because $yI_{\times|} \subset I_2$. The homomorphism f_+ is well-defined, because, in view of the matrix presentation (4.21), it can be expressed as a composition $f_+ = P_2 \circ (\mathbb{1} \otimes \chi_- \otimes \mathbb{1})$,

$$\begin{tikzcd} M_{\bigstar;\mathbf{x},\mathbf{w}} \arrow[r, "\mathbb{1}\otimes\chi_-\otimes\mathbb{1}"] \arrow[rr, bend right, "f_+"'] & M_{\between|;\mathbf{x},\mathbf{w}} \arrow[r, "P_2"] & M_{\times|;\mathbf{x},\mathbf{w}}, \end{tikzcd}$$

where P_2 is the projection on the second module in the decomposition (4.13). Since $f_+ \circ f_- = \mathbb{1}$, the $\mathbb{Q}[\mathbf{x},\mathbf{w}]$-module $M_{\bigstar;\mathbf{x},\mathbf{w}}$ decomposes:

$$M_{\bigstar;\mathbf{x},\mathbf{w}} = \operatorname{im} f_- \oplus \ker f_+. \tag{4.26}$$

Also it follows that f_- is injective, hence

$$M_{\times|;\mathbf{x},\mathbf{w}} \cong \operatorname{im} f_- = (y) \subset M_{\bigstar;\mathbf{x},\mathbf{w}},$$

where $(y) \subset M_{\bigstar;\mathbf{x},\mathbf{w}}$ is the $\mathbb{Q}[\mathbf{x},\mathbf{w}]$-submodule generated by y.

The matrix form of f_+ indicates that $f_+(1) = 0$, hence $(1) \subset \ker f_+$. At the same time, 1 and y generate $M_{\bigstar;\mathbf{x},\mathbf{w}}$, hence $(1) + (y) = M_{\bigstar;\mathbf{x},\mathbf{w}}$. Then it follows from the decomposition (4.26) that $\ker f_+ = (1)$, so (4.26) becomes

$$M_{\bigstar;\mathbf{x},\mathbf{w}} = (1) \oplus (y). \tag{4.27}$$

We have already established the isomorphism $(y) \cong M_{\times|;\mathbf{x},\mathbf{w}}$, so it remains to show that

$$(1) \cong M_{\ast;\mathbf{x},\mathbf{w}}. \tag{4.28}$$

We will use dimension counting arguments. For a q-graded module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ whose grading components M_i have finite dimension as $\mathbb{Q}$-vector spaces, define q-dimension as

$$\dim_q M = \sum_{i\in\mathbb{Z}} q^i \dim M_i.$$

If M is a free q-graded $\mathbb{Q}[\mathbf{x}]$-module then $\dim_q M = d(q) \operatorname{rank}_q M$, where $\operatorname{rank}_q M$ is q-rank defined by (2.2), while $d(q) = (1-q^2)^{-3}$, because $|\mathbf{x}| = 3$ and $\deg_{\mathrm{q}} \mathbf{x} = 2$.

According to Remark 3.1, the following $\mathbb{Q}[\mathbf{x}, \mathbf{w}]$-modules are free as $\mathbb{Q}[\mathbf{x}]$-modules with ranks

$$\operatorname{rank} M_{\ast;\mathbf{x},\mathbf{w}} = [3]_q!, \quad \operatorname{rank} M_{\star;\mathbf{x},\mathbf{w}} = [2]_q^3, \quad \operatorname{rank} M_{\times|;\mathbf{x},\mathbf{w}} = [2]_q.$$

In view of the decomposition (4.27), this means that

$$\dim_q(1) = d(q)([2]_q^3 - q^2[2]_q) = d(q)[3]_q!,$$

the factor q^2 in $q^2[2]_q$ being due to the fact that $\deg_{\mathrm{q}} y = 2$ in accordance with (1.16). Hence

$$\dim_q(1) = \dim_q M_{\ast;\mathbf{x},\mathbf{w}}.$$

A homomorphism

$$M_{\ast;\mathbf{x},\mathbf{w}} \xrightarrow{g=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} M_{\star;\mathbf{x},\mathbf{w}} \tag{4.29}$$

is well-defined, because $g(I_\ast) \subset I_2$ in view of relation

$$\tilde{x}_1\tilde{x}_2\tilde{x}_3 = \tilde{x}_3 p_2 - (y + \tilde{x}_1 + \tilde{x}_2)p_1.$$

A composition of g with the projection of $M_{\ast;\mathbf{x},\mathbf{w}}$ on (1) is a surjective homomorphism

$$M_{\ast;\mathbf{x},\mathbf{w}} \twoheadrightarrow (1).$$

Since $M_{\ast;\mathbf{x},\mathbf{w}}$ and (1) have equal q-dimensions, it follows that this is an isomorphism, and this proves (4.28).

Thus we established the splitting (4.18) with generators (4.19). Now we check the action of a $\mathfrak{W}^+$ generator L_m on these generators within the module $M_{\star;\mathbf{x},\mathbf{w}}$ in presentation (4.22). The module (4.22) has zero connection, hence $\nabla_m v_{\ast;\mathbf{x},\mathbf{w}} = \widehat{L}_m 1 = 0$ and the module $M_{\ast;\mathbf{x},\mathbf{w}}$ with zero connection is a submodule of $M_{\star;\mathbf{x},\mathbf{w}}$ as a $\mathcal{W}^+_{\mathbf{x},\mathbf{w}}$-module, the quotient module being $M_{\times|;\mathbf{x},\mathbf{w}}$.

In order to find the connection of $M_{\times|;\mathbf{x},\mathbf{w}}$, we compute the action of L_m on its generator y defined by eq. (4.20) as $y = y_1 - y_2 + q$, where $q = -x_1 - x_2 + 2w_3 \in \mathbb{Q}[\mathbf{x}, \mathbf{w}]$:

$$\begin{aligned}
\nabla_m v_{\times|;\mathbf{x},\mathbf{w}} &= \widehat{L}_m y \\
&= \widehat{L}_m\big((y_1 - y_2) + q\big) \\
&= \pi_m(y_1, y_2)(y_1 - y_2) + \widehat{L}_m q \\
&= \pi_m(x_1, x_2)(y_1 - y_2) + \widehat{L}_m q \\
&= \pi_m(x_1, x_2)\, y + \tilde{q} \\
&= \pi_m(x_1, x_2) v_{\times|;\mathbf{x},\mathbf{w}} + \tilde{q},
\end{aligned} \tag{4.30}$$

where

$$\tilde{q} = -\pi_m(x_1, x_2)\, q + \widehat{L}_m q \in M_{\ast;\mathbf{x},\mathbf{w}} \subset M_{\bigstar;\mathbf{x},\mathbf{w}}.$$

In deriving the formula (4.30) we used the relation

$$\pi_m(y_1, y_2) = \pi_m(x_1, x_2) \quad \mod I_1,$$

which is due to the first two rows in the presentation (4.23) of I_1 establishing the equality between symmetric polynomials in x_1, x_2 and in y_1, y_2. Since $\tilde{q} \in M_{\ast;\mathbf{x},\mathbf{w}} \subset M_{\bigstar;\mathbf{x},\mathbf{w}}$, the formula (4.30) shows that $\nabla_m v_{\times|;\mathbf{x},\mathbf{w}} = \pi_m(x_1, x_2) v_{\times|;\mathbf{x},\mathbf{w}}$ within the quotient module $M_{\times|;\mathbf{x},\mathbf{w}} \cong M_{\bigstar;\mathbf{x},\mathbf{w}}/M_{\ast;\mathbf{x},\mathbf{w}}$, hence this quotient has a connection $\boldsymbol{\pi}_{12}$ defined by eq. (4.14). □

Proof of Theorem 4.4. As we have observed, the $\mathcal{F}$-isomorphism (4.7) is equivalent to (4.8). In fact, it is easier to prove the same relation for the inverse generators:

$$[\![\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}]\!]^{\cdot}_{\mathbf{x},\mathbf{w}} \simeq \mathcal{F}^{(13)}_{\mathbf{x},\mathbf{w}}([\![\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}]\!]^{\cdot}_{\mathbf{x},\mathbf{w}}). \tag{4.31}$$

The left hand side of this relation is presented by a complex of $\mathfrak{W}^{+}$-equivariant $\mathbb{Q}[\mathbf{x},\mathbf{w}]$-modules:

$$[\![\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}]\!]^{\cdot}_{\mathbf{x},\mathbf{w}} = \boxed{\begin{tikzcd} & M_{\nparallel;\mathbf{x},\mathbf{w}} \arrow[r] \arrow[dr, "-1"] & M^{\mathrm{d}}_{\times|;\mathbf{x},\mathbf{w}} \arrow[dr] & \\ M_{\bigstar;\mathbf{x},\mathbf{w}} \arrow[ur] \arrow[r] \arrow[dr] & M_{\between|;\mathbf{x},\mathbf{w}} \arrow[ur, "-1"] \arrow[dr] & M^{\mathrm{m}}_{|\times;\mathbf{x},\mathbf{w}} \arrow[r] & M_{|||;\mathbf{x},\mathbf{w}} \\ & M_{\not\parallel;\mathbf{x},\mathbf{w}} \arrow[ur] \arrow[r, "-1"] & M^{\mathrm{u}}_{\times|;\mathbf{x},\mathbf{w}} \arrow[ur] & \end{tikzcd}}\ , \tag{4.32}$$

where $M^{\mathrm{m}}_{|\times;\mathbf{x},\mathbf{w}} = M_{|||;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{|\times;\mathbf{y},\mathbf{z}} \otimes_{\mathbf{z}} M_{|||;\mathbf{z},\mathbf{w}}$. All modules in this complex are tensor products of elementary Soergel bimodules over the intermediate algebras $\mathbb{Q}[\mathbf{y}]$ and $\mathbb{Q}[\mathbf{z}]$ and, in accordance with definitions (3.15), (3.13) and (3.8), all arrows represent either the homomorphisms $1 \otimes 1 \otimes 1$ (unmarked arrows) or $-1 \otimes 1 \otimes 1$ (arrows marked by -1).

According to lemmas 4.7 and 4.6, there are chain maps

$$B_\bullet \longrightarrow [\![\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}]\!]^{\cdot}_{\mathbf{x},\mathbf{w}} \longrightarrow C_\bullet, \tag{4.33}$$

where

$$
B_\bullet \quad = \quad \boxed{
\begin{tikzcd}
& M_{\nparallel;\mathbf{x},\mathbf{w}} \arrow[r] \arrow[dr, "-1"'] & M_{\times|;\mathbf{x},\mathbf{w}} \arrow[dr] & \\
M_{\ast;\mathbf{x},\mathbf{w}} \arrow[ur] \arrow[r] \arrow[dr] & M_{\times|;\mathbf{x},\mathbf{w}} \arrow[ur, "-1"] \arrow[dr] & M_{|\times;\mathbf{x},\mathbf{w}} \arrow[r] & M_{|||;\mathbf{x},\mathbf{w}} \\
& M_{\reflectbox{$\nparallel$};\mathbf{x},\mathbf{w}} \arrow[ur] \arrow[r, "-1"] & M_{\times|;\mathbf{x},\mathbf{w}} \arrow[ur] &
\end{tikzcd}
}
\tag{4.34}
$$

all unmarked arrows representing homomorphisms 1, and

$$
C_\bullet = \boxed{(M_{\times|;\mathbf{x},\mathbf{w}}, \boldsymbol{\pi}_{12}) \xrightarrow{\mathbb{1}} (M_{\times|;\mathbf{x},\mathbf{w}}, \boldsymbol{\pi}_{12})}\,,
$$

such that after the application of the forgetful functor $\mathcal{F}_{\heartsuit}$ the sequence (4.33) becomes a part of an exact triangle. Since the complex $C_\bullet$ is obviously contractible, by Theorem 2.1 there is a $\mathcal{F}$-isomorphism $[\![\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}]\!]^{\cdot}_{\mathbf{x},\mathbf{w}} \sim B_\bullet$.

An obvious change of generators in the sum of modules $M_{\times|;\mathbf{x},\mathbf{w}} \oplus M_{\times|;\mathbf{x},\mathbf{w}}$ appearing in the third column of the complex (4.34) allows us to present it in the following form:

$$
B_\bullet \quad \cong \quad \boxed{
\begin{tikzcd}
& M_{\nparallel;\mathbf{x},\mathbf{w}} \arrow[r] \arrow[dr, "-1"'] & M_{\times|;\mathbf{x},\mathbf{w}} \arrow[dr] & \\
M_{\ast;\mathbf{x},\mathbf{w}} \arrow[ur] \arrow[r] \arrow[dr] & M_{\times|;\mathbf{x},\mathbf{w}} \arrow[dr] & M_{|\times;\mathbf{x},\mathbf{w}} \arrow[r] & M_{|||;\mathbf{x},\mathbf{w}} \\
& M_{\reflectbox{$\nparallel$};\mathbf{x},\mathbf{w}} \arrow[uur, "-1"] \arrow[ur] \arrow[r, "-1"] & M_{\times|;\mathbf{x},\mathbf{w}} &
\end{tikzcd}
}\,.
$$

This complex has a cone presentation: $B_\bullet \cong \boxed{D_\bullet \to E_\bullet}$, where

$$D_\bullet \quad = \quad \boxed{\begin{tikzcd}[column sep=large] & M_{\nparallel;\mathbf{x},\mathbf{w}} \arrow[r] \arrow[ddr, "-1"' near start] & M_{\times|;\mathbf{x},\mathbf{w}} \arrow[dr] & \\ M_{\ast;\mathbf{x},\mathbf{w}} \arrow[ur] \arrow[dr] & & & M_{|||;\mathbf{x},\mathbf{w}} \\ & M_{\nparallel;\mathbf{x},\mathbf{w}} \arrow[r] \arrow[uur, "-1" near start] & M_{|\times;\mathbf{x},\mathbf{w}} \arrow[ur] & \end{tikzcd}} \tag{4.35}$$

while

$$E_\bullet = \boxed{M_{\times|;\mathbf{x},\mathbf{w}} \xrightarrow{\mathbb{1}} M_{\times|;\mathbf{x},\mathbf{w}}}\,.$$

Since $E_\bullet$ is contractible, there is a homotopy equivalence $B_\bullet \sim D_\bullet$, hence

$$[\![\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}]\!]^{\cdot}_{\mathbf{x},\mathbf{w}} \simeq C_\bullet. \tag{4.36}$$

It is easy to verify that the complex (4.35) is symmetric under the transposition of indices $1 \leftrightarrow 3$ in the variables $\mathbf{x}$ and $\mathbf{w}$, that is, $\mathcal{F}^{(13)}_{\mathbf{x},\mathbf{w}}(C_\bullet) \cong C_\bullet$, hence the $\mathcal{F}$-isomorphism (4.31) follows from eq. (4.36). $\square$

5. Link invariant as an object of the derived category of $\mathcal{W}^+_\lambda$-modules

5.1. Categorification of the braid group with strand variables. Our immediate goal is to prove the $\mathcal{F}$-isomorphism (1.27) and its analogs. We begin by studying the sliding properties of a single braid strand variable α. Consider the functor

$$\mathcal{W}^+_{\mathbf{x}}-\mathbf{gm} \xrightarrow{\mathcal{G}_{\alpha=x_i}} {}_\alpha\mathcal{W}^+_{\mathbf{x}}-\mathbf{gm},$$

which defines the action of α on a $\mathcal{W}^+_{\mathbf{x}}$-module $M_{\mathbf{x}}$ as being equal to that of x_i. In other words,

$$\mathcal{G}_{\alpha=x_i} = - \otimes_{\bar{x}_i} \Delta_{\alpha-x_i}. \tag{5.1}$$

The functor $\mathcal{G}_{\alpha=x_i}$ extends along the chain of categories (1.11) to functors

$$\mathbf{W}^+_{\mathbf{x}} \xrightarrow{\mathcal{G}_{\alpha=x_i}} {}_\alpha\mathbf{W}^+_{\mathbf{x}},$$

$$\mathbf{Ch}(\mathbf{W}^+_{\mathbf{x}}) \xrightarrow{\mathcal{G}_{\alpha=x_i}} \mathbf{Ch}({}_\alpha\mathbf{W}^+_{\mathbf{x}}),$$

$$\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\mathbf{x}}) \xrightarrow{\mathcal{G}_{\alpha=x_i}} \mathbf{D}_{\mathrm{r}}({}_\alpha\mathbf{W}^+_{\mathbf{x}}).$$

Theorem 5.1. *For any braid* $\mathfrak{b}$ *there is an isomorphism in the category* $\mathbf{D}_{\mathrm{r}}({}_{\alpha}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}})$

$$\mathcal{G}_{\alpha=x_i}(\llbracket\mathfrak{b}\rrbracket_{\mathbf{x},\mathbf{y}}) \simeq \mathcal{G}_{\alpha=y_{\check{\mathfrak{b}}(i)}}(\llbracket\mathfrak{b}\rrbracket_{\mathbf{x},\mathbf{y}}), \tag{5.2}$$

where $\check{\mathfrak{b}} \in \mathrm{S}_n$ *is the permutation associated with* $\mathfrak{b}$.

We will prove this theorem by establishing $\mathcal{F}$-isomorphism (5.2) for elementary braids σ_i^{-1} in the category $\mathbf{Ch}({}_{\alpha}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}})$ and then use multiplicativity property (1.22) in order to extend it to all braids.

Lemma 5.2. *The* $\mathcal{F}$*-isomorphism* (5.2) *holds for the elementary braid* σ_i^{-1}.

Proof. We will prove the $\mathcal{F}$-isomorphism for a 2-strand elementary braid $\mathfrak{b} = \sigma^{-1}$, so that $|\mathbf{x}| = |\mathbf{y}| = 2$. The locality of the formulas (3.15) extends this result to the general case of n-strand elementary braid σ_i^{-1}. Thus we want to establish a $\mathcal{F}$-isomorphism

$$\mathcal{G}_{\alpha=x_1}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}}) \simeq \mathcal{G}_{\alpha=y_2}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}}), \tag{5.3}$$

where

$$\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}} = \boxed{M_{\times;\mathbf{x},\mathbf{y}} \xrightarrow{\chi_-} M_{\parallel;\mathbf{x},\mathbf{y}}}\,.$$

The case of $\alpha = x_2$ versus $\alpha = y_1$ is treated similarly.

The $\mathcal{F}$-isomorphism (5.3) is established by the diagram of Figure 1, in which we denote ${}_{\alpha}M_{\mathbf{x},\mathbf{y}} = M_{\mathbf{x},\mathbf{y}} \otimes \mathbb{Q}[\alpha]$. Each row in this diagram represents a complex in $\mathbf{Ch}({}_{\alpha}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}})$ (the rows have to be 'boxed' in our notations) and we used the following abbreviations for $\mathfrak{W}^{+}$ connections: $\mathbf{a} = \boldsymbol{\pi}(\alpha, x_1)$,

$$\mathbf{A} = \begin{pmatrix} 0 & (x_1 - y_2)\big(\boldsymbol{\pi}(\alpha, x_1) - \boldsymbol{\pi}(x_1, y_2)\big) \\ 0 & \boldsymbol{\pi}(\alpha, x_1) \end{pmatrix}.$$

That $f_{AB;\cong}$ is an isomorphism in $\mathbf{Ch}({}_{\alpha}\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}})$ is established by inspection. That $f_{A;\simeq}$ and $f_{B;\simeq}$ are $\mathcal{F}$-isomorphisms can be established similarly, but it is easier to observe this by explaining the origin of complexes $A_{\bullet}$ and $B_{\bullet}$.

There is an obvious $\mathcal{F}$-isomorphism $\mathrm{K}(\alpha - x) \simeq \Delta_{\alpha,x}$ within the category $\mathbf{Ch}({}_{\alpha}\mathbf{W}^{+}_{x})$ between a Koszul complex

$$\mathrm{K}(\alpha - x) = \boxed{\big(\mathbb{Q}_{\alpha,x}, \boldsymbol{\pi}(\alpha, x)\big) \xrightarrow{\alpha - x} \mathbb{Q}_{\alpha,x}}$$

and the diagonal bimodule $\Delta_{\alpha,x} = \mathbb{Q}_{\alpha,x}/(\alpha - x)$, where, as usual, $\mathbb{Q}_{\alpha,x}$ is $\mathbb{Q}[\alpha, x]$ viewed as a module over ${}_{\alpha}\mathcal{W}^{+}_{x}$. The complex $A_{\bullet}$ is the Koszul resolution of the relation $\alpha = x_1$ in the complex $\mathcal{G}_{\alpha=x_1}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}})$:

$$A_{\bullet} = \llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}} \otimes_{x_1} \mathrm{K}(\alpha - x_1), \qquad A_{\bullet} \simeq \mathcal{G}_{\alpha=x_1}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}}).$$

$$
\begin{tikzcd}
\mathcal{G}_{\alpha=x_1}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}}) & & \mathcal{G}_{\alpha=x_1}(M_{\times;\mathbf{x},\mathbf{y}}) \arrow[r, "1"] & \mathcal{G}_{\alpha=x_1}(M_{\shortparallel;\mathbf{x},\mathbf{y}}) \\
A_\bullet \arrow[u, "f_{A;\simeq}"] \arrow[d, "f_{AB;\cong}"'] & ({}_\alpha M_{\times;\mathbf{x},\mathbf{y}}, \mathbf{a}) \arrow[r, "\binom{\alpha-x_1}{1}"] \arrow[d, "1"'] & {}_\alpha M_{\times;\mathbf{x},\mathbf{y}} \oplus ({}_\alpha M_{\shortparallel;\mathbf{x},\mathbf{y}}, \mathbf{a}) \arrow[u, "(1\ \ 0)"] \arrow[r, "(1\ \ x_1-\alpha)"] \arrow[d, "\begin{pmatrix}1 & x_1-y_2\\ 0 & 1\end{pmatrix}"'] & {}_\alpha M_{\shortparallel;\mathbf{x},\mathbf{y}} \arrow[u, "1"'] \arrow[d, "1"'] \\
B_\bullet \arrow[d, "f_{B;\simeq}"'] & ({}_\alpha M_{\times;\mathbf{x},\mathbf{y}}, \mathbf{a}) \arrow[r, "\binom{\alpha-y_2}{1}"] & ({}_\alpha M_{\times;\mathbf{x},\mathbf{y}} \oplus {}_\alpha M_{\shortparallel;\mathbf{x},\mathbf{y}}, \mathbf{A}) \arrow[r, "(1\ \ y_2-\alpha)"] \arrow[d, "(1\ \ 0)"'] & {}_\alpha M_{\shortparallel;\mathbf{x},\mathbf{y}} \arrow[d, "1"] \\
\mathcal{G}_{\alpha=y_2}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}}) & & \mathcal{G}_{\alpha=y_2}(M_{\times;\mathbf{x},\mathbf{y}}) \arrow[r, "1"] & \mathcal{G}_{\alpha=y_2}(M_{\shortparallel;\mathbf{x},\mathbf{y}})
\end{tikzcd}
$$

Figure 1. Sliding of a strand variable.

A similar $\mathcal{F}$-isomorphism holds between $\mathcal{G}_{\alpha=y_2}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}})$ and $\widetilde{B}_\bullet = \llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}} \otimes_{y_2} \mathrm{K}(\alpha - y_2)$. However, $\widetilde{B}_\bullet$ is not isomorphic to $A_\bullet$ as a complex of ${}_\alpha\mathcal{W}^+_{\mathbf{x},\mathbf{y}}$-modules, unless we 'tweak' the $\mathfrak{W}^+$-connection in $\widetilde{B}_\bullet$. This tweaking transforms $\widetilde{B}_\bullet$ into $B_\bullet$, while keeping it $\mathcal{F}$-isomorphic to $\mathcal{G}_{\alpha=y_2}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{x},\mathbf{y}})$. □

Lemma 5.3. *$\mathcal{F}$-isomorphism* (5.2) *holds for the elementary braid σ_i.*

Proof. Again, it is sufficient to prove the lemma for the 2-strand braid generator σ. Consider the sequence of $\mathcal{F}$-isomorphisms:

$$
\begin{aligned}
\mathcal{G}_{\alpha=x_1}(\llbracket\sigma\rrbracket_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{y}} \llbracket\sigma^{-1}\rrbracket_{\mathbf{y},\mathbf{z}} &\simeq \mathcal{G}_{\alpha=x_1}\Delta_{\mathbf{x},\mathbf{z}} \cong \mathcal{G}_{\alpha=z_1}\Delta_{\mathbf{y};\mathbf{z}} \\
&\simeq \llbracket\sigma\rrbracket_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} \mathcal{G}_{\alpha=z_1}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{y},\mathbf{z}}) \\
&\simeq \llbracket\sigma\rrbracket_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} \mathcal{G}_{\alpha=y_2}(\llbracket\sigma^{-1}\rrbracket_{\mathbf{y},\mathbf{z}}) \\
&\cong (\mathcal{G}_{\alpha=y_2}\llbracket\sigma\rrbracket_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{y}} \llbracket\sigma^{-1}\rrbracket_{\mathbf{y},\mathbf{z}}.
\end{aligned}
$$

Finally, we apply the tensor multiplication $-\otimes_{\mathbf{z}} \llbracket\sigma\rrbracket_{\mathbf{z},\mathbf{w}}$ to the first and last complex in this chain and use the $\mathcal{F}$-isomorphism $\llbracket\sigma^{-1}\rrbracket_{\mathbf{y},\mathbf{z}} \otimes_{\mathbf{z}} \llbracket\sigma\rrbracket_{\mathbf{z},\mathbf{w}} \simeq \Delta_{\mathbf{y};\mathbf{w}}$ as well as $-\otimes_{\mathbf{y}} \Delta_{\mathbf{y};\mathbf{w}}$ being the identity functor. □

Proof of Theorem 5.1. We prove the theorem by induction over the length of the minimal braid word presentation of a braid. Lemmas 5.2 and 5.3 prove $\mathcal{F}$-isomorphism (5.2) for elementary braids. Suppose that the theorem holds for braids of minimal length k. If a braid $\mathfrak{b}$ has minimal length $k+1$ then it can be presented as a composition of a length k braid $\mathfrak{b}_1$ and an elementary braid $\mathfrak{b}_2$. Thus we have a sequence of isomorphisms and $\mathcal{F}$-isomorphisms:

$$\begin{aligned}
\mathcal{G}_{\alpha=x_i}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{z}} &\simeq \mathcal{G}_{\alpha=x_i}[\![\mathfrak{b}_1\mathfrak{b}_2]\!]_{\mathbf{x},\mathbf{y}}\\
&\cong \mathcal{G}_{\alpha=x_i}([\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}}\otimes_{\mathbf{y}}[\![\mathfrak{b}_2]\!]_{\mathbf{y},\mathbf{z}})\\
&\cong (\mathcal{G}_{\alpha=x_i}[\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}})\otimes_{\mathbf{y}}[\![\mathfrak{b}_2]\!]_{\mathbf{y},\mathbf{z}}\\
&\simeq (\mathcal{G}_{\alpha=y_{\check{\mathfrak{b}}_1(i)}}[\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}})\otimes_{\mathbf{y}}[\![\mathfrak{b}_2]\!]_{\mathbf{y},\mathbf{z}}\\
&\cong [\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}}\otimes_{\mathbf{y}}(\mathcal{G}_{\alpha=y_{\check{\mathfrak{b}}_1(i)}}[\![\mathfrak{b}_2]\!]_{\mathbf{y},\mathbf{z}})\\
&\simeq [\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}}\otimes_{\mathbf{y}}(\mathcal{G}_{\alpha=z_{\check{\mathfrak{b}}_2\check{\mathfrak{b}}_1(i)}}[\![\mathfrak{b}_2]\!]_{\mathbf{y},\mathbf{z}})\\
&\simeq \mathcal{G}_{\alpha=z_{\check{\mathfrak{b}}(i)}}([\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{z}}).
\end{aligned}$$ □

Theorem 1.2 is an obvious corollary of Theorem 5.1.

5.2. Proof of Theorem 1.3. Theorem 1.3 is an obvious corollary of the following theorem:

Theorem 5.4. *Suppose that i-th and j-th strands of a braid $\mathfrak{b}$ belong to the same component of the link constructed by circular closure. Then there is a $\mathcal{F}$-isomorphism in the category* $\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{\boldsymbol{\alpha}})$:

$$(\mathcal{G}_{\alpha=x_i}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}})\overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}}\Delta_{\mathbf{x};\mathbf{y}}\simeq(\mathcal{G}_{\alpha=x_j}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}})\overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}}\Delta_{\mathbf{x};\mathbf{y}}. \tag{5.4}$$

Indeed, this theorem means that a strand variable can slide around the closed braid, thus moving between all strands belonging to the same link component.

We prove Theorem 5.4 with the help of the following lemma:

Lemma 5.5. *For any object $M_{\mathbf{x},\mathbf{y}}$ of $\mathbf{W}^+_{\mathbf{x},\mathbf{y}}$ and any object $N_{\mathbf{y},\mathbf{z}}$ of $\mathbf{W}^+_{\mathbf{y},\mathbf{z}}$ there is a $\mathcal{F}$-isomorphism*

$$(\mathcal{G}_{\alpha=y_i}M_{\mathbf{x},\mathbf{y}})\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}N_{\mathbf{y},\mathbf{z}}\simeq M_{\mathbf{x},\mathbf{y}}\overset{\mathrm{L}}{\otimes}_{\mathbf{y}}(\mathcal{G}_{\alpha=y_i}N_{\mathbf{y},\mathbf{z}}).$$

Proof. Let $\mathcal{P}^\bullet(M_{\mathbf{x},\mathbf{y}})$ be a resolution of $M_{\mathbf{x},\mathbf{y}}$ which is a chain complex of $\mathcal{W}^+_{\mathbf{x},\mathbf{y}}$-modules, each of which is projective as a $\mathbb{Q}[\mathbf{y}]$-module. Then $\mathcal{G}_{\alpha=y_i}\mathcal{P}^\bullet(M_{\mathbf{x},\mathbf{y}})$ is a similar resolution of $\mathcal{G}_{\alpha=y_i}M_{\mathbf{x},\mathbf{y}}$, and there is a sequence of $\mathcal{F}$-isomorphisms, which proves the lemma:

$$\begin{aligned}(\mathcal{G}_{\alpha=y_i}M_{\mathbf{x},\mathbf{y}}) \overset{\mathrm{L}}{\otimes}_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}} &\simeq \mathcal{G}_{\alpha=y_i}\mathcal{P}^\bullet(M_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{y}} N_{\mathbf{y},\mathbf{z}} \\ &\cong \mathcal{P}^\bullet(M_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{y}} \mathcal{G}_{\alpha=y_i}N_{\mathbf{y},\mathbf{z}} \\ &\simeq M_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{y}} \mathcal{G}_{\alpha=y_i}N_{\mathbf{y},\mathbf{z}}. \qquad \square\end{aligned}$$

Proof of Theorem 5.4. Since $\check{\mathfrak{b}}$ performs a cyclic permutation of strand indices corresponding to the same link component, it is sufficient to prove the $\mathcal{F}$-isomorphism (5.4) in the case when $j = \check{\mathfrak{b}}(i)$. The latter is proved by the following sequence of $\mathcal{F}$-isomorphisms:

$$\begin{aligned}(\mathcal{G}_{\alpha=x_i}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}}) \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}} &\simeq (\mathcal{G}_{\alpha=y_{\check{\mathfrak{b}}(i)}}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}}) \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}} \\ &\underset{\mathrm{in}}{\simeq} [\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} (\mathcal{G}_{\alpha=y_{\check{\mathfrak{b}}(i)}}\Delta_{\mathbf{x};\mathbf{y}}) \\ &\cong [\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} (\mathcal{G}_{\alpha=x_{\check{\mathfrak{b}}(i)}}\Delta_{\mathbf{x};\mathbf{y}}) \\ &\underset{\mathrm{in}}{\simeq} (\mathcal{G}_{\alpha=x_{\check{\mathfrak{b}}(i)}}[\![\mathfrak{b}]\!]_{\mathbf{x},\mathbf{y}}) \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}}. \qquad \square\end{aligned}$$

5.3. Markov move invariance

Theorem 5.6. *There exists a unique map* $_{\boldsymbol{\lambda}}[\![-]\!]$ *which makes the upper right triangle in the following diagram commutative*:

$$\begin{tikzcd} \mathfrak{B}_n \arrow[d, "{}_{\boldsymbol{\alpha}}[\![-]\!]_{\mathbf{x},\mathbf{y}}"'] & \mathfrak{B}_n^{(s)} \arrow[l, hook'] \arrow[r, "\mathrm{cl}"] \arrow[dr, "{}_{\boldsymbol{\lambda}}[\![-]\!]^{\cdot}"'] & \mathfrak{L}_m \arrow[d, "{}_{\boldsymbol{\lambda}}[\![-]\!]"] \\ \mathrm{D}_{\mathrm{r}}\big({}_{\boldsymbol{\alpha}}\mathbf{W}^+_{\mathbf{x},\mathbf{y}}\big) \arrow[r, "\mathrm{HH}_{\mathbf{x},\mathbf{y}}(-)"'] & \mathbf{W}^+_{\boldsymbol{\alpha}} \arrow[r, "\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}"'] & \mathbf{W}^+_{\boldsymbol{\lambda}}. \end{tikzcd} \tag{5.5}$$

Moreover, if a framed link L' is isotopic to a framed link L, except that the i-th component of L' has an extra unit of framing, then their brackets are related by the connection shift endofunctor:

$$[\![L']\!] \simeq \mathsf{fr}_{\lambda_i}[\![L]\!] \tag{5.6}$$

where the framing shift functor fr_{λ_i} *is defined by eq.* (1.17).

Proof. The claim of this theorem follows from the invariance of ${}_{\boldsymbol{\lambda}}[\![-]\!]^{\cdot}$ under Markov moves up to the connection shift functor. We prove this invariance in the next two subsections. □

5.3.1. First Markov move

Theorem 5.7. *For two n-strand braids $\mathfrak{b}_1$ and $\mathfrak{b}_2$ there is an isomorphism in* $\mathbf{W}^+_{\boldsymbol{\lambda}}$:

$$ {}_{\boldsymbol{\lambda}}[\![\mathfrak{b}_1\mathfrak{b}_2]\!]^{\cdot} \simeq {}_{\boldsymbol{\lambda}}[\![\mathfrak{b}_2\mathfrak{b}_1]\!]^{\cdot}. \tag{5.7} $$

This theorem follows easily from the next lemma. Consider the functor $-\overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}}$ in the diagram (1.29).

Lemma 5.8. *There is a $\mathcal{F}$-isomorphism in the category* $\mathbf{Ch}(\mathbf{W}^+_{\boldsymbol{\alpha}})$:

$$ \mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}[\![\mathfrak{b}_1\mathfrak{b}_2]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}} \simeq \mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{x}}[\![\mathfrak{b}_2\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}}. \tag{5.8} $$

Proof. Consider a sequence of $\mathcal{F}$-isomorphisms:

$$ \begin{aligned} \mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_2\mathfrak{b}_1]\!]_{\mathbf{z},\mathbf{y}} &\simeq (\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_2]\!]_{\mathbf{z},\mathbf{w}}) \otimes_{\mathbf{w}} [\![\mathfrak{b}_1]\!]_{\mathbf{w},\mathbf{y}} \\ &\simeq (\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_2]\!]_{\mathbf{z},\mathbf{w}} \otimes [\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}}) \otimes_{\mathbf{x},\mathbf{w}} \Delta_{\mathbf{x};\mathbf{w}}. \end{aligned} $$

We substitute this $\mathcal{F}$-isomorphism into the right hand side of eq. (5.8) and rename some variables:

$$ \begin{aligned} &\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_2\mathfrak{b}_1]\!]_{\mathbf{z},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\mathbf{y},\mathbf{z}} \Delta_{\mathbf{y};\mathbf{z}} \\ &\quad\simeq (\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_2]\!]_{\mathbf{z},\mathbf{w}} \otimes [\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}}) \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}} (\Delta_{\mathbf{x};\mathbf{w}} \otimes \Delta_{\mathbf{y};\mathbf{z}}). \end{aligned} \tag{5.9} $$

Next consider the following sequence of $\mathcal{F}$-isomorphisms:

$$ \begin{aligned} \mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}[\![\mathfrak{b}_1\mathfrak{b}_2]\!]_{\mathbf{x},\mathbf{w}} &\simeq (\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}[\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{z}}) \otimes_{\mathbf{z}} [\![\mathfrak{b}_2]\!]_{\mathbf{z},\mathbf{w}} \\ &\simeq (\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{z}}) \otimes_{\mathbf{z}} [\![\mathfrak{b}_2]\!]_{\mathbf{z},\mathbf{w}} \\ &\simeq [\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{z}} \otimes_{\mathbf{z}} (\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_2]\!]_{\mathbf{z},\mathbf{w}}) \\ &\simeq ([\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{z}} (\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_2]\!]_{\mathbf{z},\mathbf{w}})) \otimes_{\mathbf{y},\mathbf{z}} \Delta_{\mathbf{y};\mathbf{z}}. \end{aligned} $$

We substitute it into the left hand side of eq. (5.8) and rename some variables:

$$\begin{aligned} &\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}[\![\mathfrak{b}_1\mathfrak{b}_2]\!]_{\mathbf{x},\mathbf{w}} \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{w}} \Delta_{\mathbf{x};\mathbf{y}} \\ &\quad \simeq ([\![\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{z}} (\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{z}}[\![\mathfrak{b}_2]\!]_{\mathbf{z},\mathbf{w}})) \overset{\mathrm{L}}{\otimes}_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}} (\Delta_{\mathbf{x};\mathbf{w}} \otimes \Delta_{\mathbf{y};\mathbf{z}}). \end{aligned} \tag{5.10}$$

Comparing the right hand side of equations (5.9) and (5.10), we come to eq. (5.8). □

Proof of Theorem 5.7. Applying the inner homology $\mathrm{H}_{\mathbf{W}^+_{\boldsymbol{\alpha}}}$ of diagram (1.29) to both sides of (5.8) we get the quasi-isomorphism of Hochschild homologies

$$\mathrm{HH}_{\mathbf{x},\mathbf{y}}(\mathcal{G}_{\boldsymbol{\alpha}=\mathbf{x}}[\![\mathfrak{b}_1\mathfrak{b}_2]\!]_{\mathbf{x},\mathbf{y}}) \sim \mathrm{HH}_{\mathbf{x},\mathbf{y}}(\mathcal{G}_{\check{\mathfrak{b}}_1(\boldsymbol{\alpha})=\mathbf{x}}[\![\mathfrak{b}_2\mathfrak{b}_1]\!]_{\mathbf{x},\mathbf{y}})$$

in $\mathbf{W}^+_{\boldsymbol{\alpha}}$. Applying the relabelling functor $\mathcal{F}^{(\mathbf{k})}_{\boldsymbol{\alpha};\boldsymbol{\lambda}}$ to both sides we get the isomorphism (5.7). □

5.3.2. Second Markov move. For a n_1-strand braid $\mathfrak{b}_1$ and a n_2-strand braid $\mathfrak{b}_2$ let $\mathfrak{b}_1 \sqcup \mathfrak{b}_2$ denote the $(n_1 + n_2)$-strand braid constructed by placing $\mathfrak{b}_1$ and $\mathfrak{b}_2$ side by side. In other words, $\mathfrak{b}_1 \sqcup \mathfrak{b}_2$ is the result of applying the injection $\mathfrak{B}_{n_1} \times \mathfrak{B}_{n_2} \hookrightarrow \mathfrak{B}_{n_1+n_2}$ to the pair $(\mathfrak{b}_1, \mathfrak{b}_2)$.

Let $\shortparallel_n$ denote the n-strand identity braid. The second Markov move relates an n-strand braid $\mathfrak{b}$ with two $(n+1)$-strand braids

$$\mathfrak{b}_{\pm 1} = (\sigma^{\pm 1} \sqcup \shortparallel_{n-1})(\shortparallel_1 \sqcup \mathfrak{b}), \tag{5.11}$$

where σ is the elementary 2-strand braid and σ^{-1} is its inverse. The cyclic closures of all three braids are isotopic links, and the first strand of $\mathfrak{b}$ as well as the first two strands of $\mathfrak{b}_{\pm 1}$ belong to the same link component. We refer to it as the first component, so that its component variable is λ_1. The framing of the first link component in the closure of $\mathfrak{b}_{\pm 1}$ differs by ± 1 from the one within the closure of $\mathfrak{b}$.

Theorem 5.9. *For any n-strand braid $\mathfrak{b}$ there is a quasi-isomorphism in the category $\mathbf{W}^+_{\boldsymbol{\alpha}}$:*

$$_{\boldsymbol{\lambda}}[\![\mathfrak{b}_{\pm 1}]\!]^{\cdot} \simeq \mathfrak{fr}^{\pm 1}_{\lambda_1}\, {}_{\boldsymbol{\lambda}}[\![\mathfrak{b}]\!]^{\cdot},$$

where $\mathfrak{fr}_{\lambda_1}$ is the combined shift functor (1.17).

The proof of this theorem is based on the local version of the second Markov move:

Lemma 5.10. *Let* $|\mathbf{x}| = |\mathbf{y}| = 2$*. There is a* $\mathcal{F}$*-isomorphism in the category* $\mathbf{Ch}(\mathbf{W}^+_{x_2,y_2})$

$$[\![\sigma^{\pm 1}]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{x_1,y_1} \Delta_{x_1;y_1} \simeq \Delta_{x_2;y_2}\langle \mp\boldsymbol{\pi}(x_2)\rangle. \tag{5.12}$$

Proof of Theorem 5.9. In this case it is convenient to employ the presentation of the bracket ${}_{\boldsymbol{\lambda}}[\![-]\!]^{\cdot}$ which does not use strand variables:

$${}_{\boldsymbol{\lambda}}[\![-]\!]^{\cdot} = \mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}(\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}([\![-]\!]_{\mathbf{x},\mathbf{y}})) = \mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}(\mathrm{H}_{\mathbf{W}^+_{\mathbf{x}}}([\![-]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\bar{\mathbf{x}},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}})). \tag{5.13}$$

The first two strands of the closure of $\mathfrak{b}_{\pm 1}$ are parts of the same link component, so we can assume that the choice $\mathbf{k}$ of braid strands representing link components does not include the first strand. Hence applying the formula (5.13) to $\mathfrak{b}_{\pm 1}$ we may forget x_1 immediately after taking the derived tensor product:

$${}_{\boldsymbol{\lambda}}[\![\mathfrak{b}_{\pm 1}]\!]^{\cdot} = \mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}(\mathrm{H}_{\mathbf{W}^+_{\mathbf{x}'}}([\![\mathfrak{b}_{\pm 1}]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{x_1,\bar{\mathbf{x}}',\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}})), \tag{5.14}$$

where $\mathbf{x}' = x_2, \dots, x_{n+1}$.

The formula (5.11) for $\mathfrak{b}_{\pm 1}$ implies the tensor product presentation of its bracket:

$$[\![\mathfrak{b}_{\pm 1}]\!]_{\mathbf{x},\mathbf{y}} \simeq [\![\sigma^{\pm 1}]\!]_{x_1,x_2,y_1,z} \otimes_z [\![\mathfrak{b}]\!]_{z,\mathbf{x}'',\mathbf{y}'},$$

where $\mathbf{x}'' = x_3, \dots, x_{n+1}$ and $\mathbf{y}' = y_2, \dots, y_{n+1}$. Therefore,

$$\begin{aligned}
&[\![\mathfrak{b}_{\pm 1}]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{x_1,\bar{\mathbf{x}}',\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}} \\
&\quad \simeq ([\![\sigma^{\pm 1}]\!]_{x_1,x_2,y_1,z} \otimes_z [\![\mathfrak{b}]\!]_{z,\mathbf{x}'',\mathbf{y}'}) \overset{\mathrm{L}}{\otimes}_{x_1,\bar{\mathbf{x}}',\mathbf{y}} (\Delta_{x_1;y_1} \otimes \Delta_{\mathbf{x}';\mathbf{y}'}) \\
&\quad \simeq (([\![\sigma^{\pm 1}]\!]_{x_1,x_2,y_1,z} \overset{\mathrm{L}}{\otimes}_{x_1,y_1} \Delta_{x_1;y_1}) \otimes_z [\![\mathfrak{b}]\!]_{z,\mathbf{x}'',\mathbf{y}'}) \overset{\mathrm{L}}{\otimes}_{\bar{\mathbf{x}}',\mathbf{y}'} \Delta_{\mathbf{x}';\mathbf{y}'}.
\end{aligned}$$

According to Lemma 5.10

$$[\![\sigma^{\pm 1}]\!]_{x_1,x_2,y_1,z} \overset{\mathrm{L}}{\otimes}_{x_1,y_1} \Delta_{x_1;y_1} \simeq \Delta_{x_2;z}\langle \mp\boldsymbol{\pi}(x_2)\rangle,$$

while $\Delta_{x_2;z} \otimes_z [\![\mathfrak{b}]\!]_{z,\mathbf{x}'',\mathbf{y}'} \cong [\![\mathfrak{b}]\!]_{\mathbf{x}',\mathbf{y}'}$, hence

$$[\![\mathfrak{b}_{\pm 1}]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{x_1,\bar{\mathbf{x}}',\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}} \simeq ([\![\mathfrak{b}]\!]_{\mathbf{x}',\mathbf{y}'} \overset{\mathrm{L}}{\otimes}_{\bar{\mathbf{x}}',\mathbf{y}'} \Delta_{\mathbf{x}';\mathbf{y}'})\langle \mp\boldsymbol{\pi}(x_2)\rangle.$$

Substituting this relation to eq. (5.14) we find

$$\begin{aligned}
\boldsymbol{\lambda}[\![\mathfrak{b}_{\pm 1}]\!]^{\cdot} &\simeq \mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}\big(\mathrm{H}_{\mathbf{W}^{+}_{\mathbf{x}'}}([\![\mathfrak{b}]\!]_{\mathbf{x}',\mathbf{y}'} \overset{\mathrm{L}}{\otimes}_{\bar{\mathbf{x}}',\mathbf{y}'} \Delta_{\mathbf{x}';\mathbf{y}'})\langle \mp\boldsymbol{\pi}(x_2)\rangle\big) \\
&\cong \mathcal{F}^{(\mathbf{k})}_{\mathbf{x};\boldsymbol{\lambda}}\big(\mathrm{H}_{\mathbf{W}^{+}_{\mathbf{x}'}}([\![\mathfrak{b}]\!]_{\mathbf{x}',\mathbf{y}'} \overset{\mathrm{L}}{\otimes}_{\bar{\mathbf{x}}',\mathbf{y}'} \Delta_{\mathbf{x}';\mathbf{y}'})\big)\langle \mp\boldsymbol{\pi}(\lambda_1)\rangle \\
&\cong \boldsymbol{\lambda}[\![\mathfrak{b}]\!]^{\cdot}\langle \mp\boldsymbol{\pi}(\lambda_1)\rangle. \qquad \square
\end{aligned}$$

5.3.3. Proof of Lemma 5.10. According to Theorem 2.3 and Remark 2.14, in order to calculate the derived tensor product in the left hand side of eq. (5.12) we can use a $\mathfrak{W}^{+}$-equivariant $\mathbb{Q}[x_1, y_1]$-projective Koszul resolution of the $\mathcal{W}^{+}_{x_1,y_1}$-module $\Delta_{x_1;y_1}$:

$$\mathcal{P}^{\bullet}(\Delta_{x_1;y_1}) = \big[\,(\mathbb{Q}_{x_1,y_1}, \boldsymbol{\pi}(x_1, y_1)) \xrightarrow{\;y_1 - x_1\;} \mathbb{Q}_{x_1,y_1}\,\big]. \tag{5.15}$$

For any $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-module M the tensor product $M \otimes_{x_1,y_1} (\mathbb{Q}[x_1, y_1], \mathbf{a})$, $\mathbf{a} \in \mathbb{Q}[\mathbf{x}, \mathbf{y}]$ is isomorphic to $M\langle \mathbf{a}\rangle$ considered as a $\mathcal{W}^{+}_{x_2,y_2}$-module, that is, the tensor product functor

$$\mathbf{W}^{+}_{\mathbf{x},\mathbf{y}} \xrightarrow{\;-\otimes_{x_1,y_1}(\mathbb{Q}_{x_1,y_1},\mathbf{a})\;} \mathbf{W}^{+}_{x_2,y_2}$$

is isomorphic to the functor of shifting the $\mathcal{W}^{+}$ connection by $\mathbf{a}$ and forgetting the $\mathbb{Q}[x_1, y_1]$-module structure. It is convenient to replace the forgotten variables x_1 and y_1 with new variables

$$z = y_1 - x_2, \quad w = y_1 - x_1.$$

We present $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$-modules $M_{\parallel}$ and $M_{\times}$ as quotients

$$M_{\parallel} \cong \mathbb{Q}_{x_2,y_2,z,w}/(\rho, w), \quad M_{\times} \cong \mathbb{Q}_{x_2,y_2,z,w}/(\rho, zw),$$

where $\rho = w + y_2 - x_2$. Taking the quotient over ρ in these expressions implies that when $\mathbb{Q}[w]$-module structure is forgotten, the variable w can be eliminated from module generators with the help of the relation $w = x_2 - y_2$ in the quotient. Hence

$$M_{\parallel} \otimes_{x_1,y_1} \mathbb{Q}_{x_1,y_1} \cong \mathbb{Q}_{x_2,y_2,z}/(y_2 - x_2) \cong \Delta_{x_2;y_2}[z], \tag{5.16a}$$

$$M_{\times} \otimes_{x_1,y_1} \mathbb{Q}_{x_1,y_1} \cong \widetilde{\mathbb{Q}}_{x_2,y_2,z}, \tag{5.16b}$$

where we used shortcut notations

$$\widetilde{\mathbb{Q}}_{x_2,y_2,z} = \mathbb{Q}_{x_2,y_2,z}/((y_2 - x_2)z), \quad \Delta_{x_2;y_2}[z] = \Delta_{x_2;y_2} \otimes \mathbb{Q}[z],$$

and generators L_m of $\mathcal{W}^+$ have the standard action on x_2 and y_2, while

$$L_m\, z = \mathbf{a}_z\, z, \qquad \mathbf{a}_z = \boldsymbol{\pi}(x_2, y_1) = \boldsymbol{\pi}(x_2, x_2 + z). \tag{5.17}$$

$\mathbb{Q}[z]$ splits as a $\mathbb{Q}$-vector space: $\mathbb{Q}[z] = \mathbb{Q} \oplus z\mathbb{Q}[z]$, and there is a canonical isomorphism of $\mathbb{Q}[z]$-modules $z\mathbb{Q}[z] \cong \mathbb{Q}[z]$. The $\mathcal{W}^+_{x_2,y_2}$-modules (5.16) split accordingly:

$$M_{\shortparallel} \otimes_{x_1,y_1} \mathbb{Q}_{x_1,y_1} \cong \Delta_{x_2;y_2}[z] \cong \Delta_{x_2;y_2} \oplus (\Delta_{x_2;y_2}[z], \mathbf{a}_z),$$

$$M_{\times} \otimes_{x_1,y_1} \mathbb{Q}_{x_1,y_1} \cong \widetilde{\mathbb{Q}}_{x_2,y_2,z} \cong \mathbb{Q}_{x_2,y_2} \oplus (\Delta_{x_2;y_2}[z], \mathbf{a}_z),$$

the presence of connection $\mathbf{a}_z$ being due to (5.17). If the $\mathcal{W}^+$ action on the left hand side modules is modified by a connection $\mathbf{a}$, whose elements a_m depend on z, then the splitting remains only at the level of $\mathbb{Q}[x_2, y_2]$-modules, while $\mathcal{W}^+_{x_2,y_2}$-modules form non-split exact sequences

$$0 \longrightarrow (\Delta_{x_2;y_2}[z], \mathbf{a} + \mathbf{a}_z) \xrightarrow{\;z\;} (\Delta_{x_2;y_2}[z], \mathbf{a}) \xrightarrow{\;1\;} (\Delta_{x_2;y_2}, \mathbf{a}|_{z=0}) \longrightarrow 0, \tag{5.18}$$

$$0 \longrightarrow (\Delta_{x_2;y_2}[z], \mathbf{a} + \mathbf{a}_z) \xrightarrow{\;z\;} (\widetilde{\mathbb{Q}}_{x_2,y_2,z}, \mathbf{a}) \xrightarrow{\;1\;} (\mathbb{Q}_{x_2,y_2}, \mathbf{a}|_{z=0}) \longrightarrow 0. \tag{5.19}$$

Consider tensor products of the modules $M_{\shortparallel}$ and $M_{\times}$ with the Koszul resolution (5.15). We use shortcut notations

$$\boldsymbol{\pi} = \boldsymbol{\pi}(x_1, x_2) = \boldsymbol{\pi}(x_2, y_2 + z),$$

$$\mathbf{a}^{\shortparallel} = \boldsymbol{\pi}(y_1, y_1) = \boldsymbol{\pi}(x_2 + z),$$

$$\mathbf{a}^{\times} = \boldsymbol{\pi}(x_1, y_1) = \boldsymbol{\pi}(x_2 + z, y_2 + z),$$

$$\mathbf{a}^{\shortmid} = \boldsymbol{\pi}(x_2, y_2).$$

Since

$$y_1 - x_1 = 0, \quad \mathbf{a}^{\times} = \mathbf{a}^{\shortparallel} \bmod (\rho, w),$$

$$y_1 - x_1 = x_2 - y_2 \mod \rho,$$

then in view of presentations (5.16) we find

$$M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^{\bullet}(\Delta_{x_1;y_1}) \cong \big[(\Delta_{x_2;y_2}[z], \mathbf{a}^{\shortparallel}) \xrightarrow{\;0\;} \Delta_{x_2;y_2}[z]\big], \tag{5.20}$$

$$M_{\times} \otimes_{x_1,y_1} \mathcal{P}^{\bullet}(\Delta_{x_1;y_1}) \cong \big[(\widetilde{\mathbb{Q}}_{x_2,y_2,z}, \mathbf{a}^{\times}) \xrightarrow{\;y_2 - x_2\;} \widetilde{\mathbb{Q}}_{x_2,y_2,z}\big]. \tag{5.21}$$

Now we are ready to establish the $\mathcal{F}$-isomorphisms (5.12).

Proof of $\mathcal{F}$-isomorphism (5.12) *for* σ. Exact sequence (5.19) of $\mathcal{W}^+_{x_2,y_2}$-modules implies short exact sequence in the category of chain complexes $\mathbf{Ch}(\mathcal{W}^+_{x_2,y_2}-\mathbf{gm})$ represented by the first column of the diagram (we omitted zeroes at the top and bottom):

$$\begin{tikzcd}
(M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^\bullet(\Delta_{x_1;y_1}))\langle \mathbf{a}_z\rangle \arrow[d, "\chi_+\otimes\mathbb{1}=z"] & \cong & {[}(\Delta_{x_2;y_2}[z], \mathbf{a}^{\shortparallel}+\mathbf{a}_z) \arrow[r, "0"] \arrow[d, "z"] & (\Delta_{x_2;y_2}[z], \mathbf{a}_z){]} \arrow[d, "z"] \\
M_{\times} \otimes_{x_1,y_1} \mathcal{P}^\bullet(\Delta_{x_1;y_1}) \arrow[d, "f"] & \cong & {[}(\widetilde{\mathbb{Q}}_{x_2,y_2,z}, \mathbf{a}^{\times}) \arrow[r, "y_2-x_2"] \arrow[d, "1"] & \widetilde{\mathbb{Q}}_{x_2,y_2,z}{]} \arrow[d, "1"] \\
A_\bullet & = & {[}(\mathbb{Q}[x_2,y_2], \mathbf{a}^{\shortmid}) \arrow[r, "y_2-x_2"] & \mathbb{Q}[x_2,y_2]{]},
\end{tikzcd} \tag{5.22}$$

where $A_\bullet$ denotes the complex at the bottom right, f is a quotient homomorphism and we used the relation $\mathbf{a}^{\times}|_{z=0} = \mathbf{a}^{\shortmid}$ in the middle of the bottom row.

According to definition (3.15) and due to relation

$$\boldsymbol{\pi} = \mathbf{a}_z \mod y_1 - x_1,$$

the complex in the left hand side of the $\mathcal{F}$-isomorphism (5.12) corresponding to σ has a presentation as the cone of the morphism $\chi_+\otimes\mathbb{1}$ in the category $\mathbf{D}_{\mathrm{r}}(\mathbf{W}^+_{x_2,y_2})$:

$$\begin{aligned}
&[\![\sigma]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{x_1,y_1} \Delta_{x_1;y_1} \\
&\quad \cong \boxed{(M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^\bullet(\Delta_{x_1;y_1}))\langle \mathbf{a}_z\rangle \xrightarrow{\chi_+\otimes\mathbb{1}} M_{\times} \otimes_{x_1,y_1} \mathcal{P}^\bullet(\Delta_{x_1;y_1})}\,\langle -\boldsymbol{\pi}\rangle.
\end{aligned}$$

Since $f\circ(\chi_+\otimes\mathbb{1}) = 0$ in the diagram (5.22), and due to relation $\boldsymbol{\pi}|_{z=0} = \mathbf{a}^{\shortmid}$ there is a morphism in $\mathbf{Ch}(\mathbf{W}^+_{x_2,y_2})$:

$$\begin{tikzcd}
\boxed{(M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^\bullet(\Delta_{x_1;y_1}))\langle \mathbf{a}_z\rangle \xrightarrow{\chi_+\otimes\mathbb{1}} M_{\times} \otimes_{x_1,y_1} \mathcal{P}^\bullet(\Delta_{x_1;y_1})} \arrow[d, "f"] & \langle -\boldsymbol{\pi}\rangle. \\
A_\bullet\,\langle -\mathbf{a}^{\shortmid}\rangle &
\end{tikzcd} \tag{5.23}$$

The exact sequence (5.22) splits if we forget the $\mathcal{W}^+$-module structure, that is, it splits in the category $\mathbf{Ch}(\mathbb{Q}[x_2, y_2]-\mathbf{gm})$. Hence the morphism (5.23) is a homotopy equivalence in $\mathbf{Ch}(\mathbb{Q}[x_2, y_2]-\mathbf{gm})$ and, as a consequence, it is a $\mathcal{F}$-isomorphism in $\mathbf{Ch}(\mathbb{Q}[x_2, y_2]-\mathbf{gm})$, so there is a $\mathcal{F}$-isomorphism

$$[\![\sigma]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{x_1,y_1} \Delta_{x_1;y_1} \simeq A_{\bullet}\langle -\mathbf{a}^{\shortmid}\rangle.$$

Now the $\mathcal{F}$-isomorphism (5.12) follows from the quasi-isomorphism in $\mathbf{W}^+_{x_2,y_2}$

$$A_{\bullet}\langle -\mathbf{a}^{\shortmid}\rangle \simeq \Delta_{x_2;y_2}\langle -\boldsymbol{\pi}(x_2)\rangle$$

which is similar to (5.15). □

Proof of $\mathcal{F}$-isomorphism (5.12) *for* σ^{-1}. The exact sequence (5.18) of $\mathcal{W}^+_{x_2,y_2}$-modules leads to a short exact sequence in the category of chain complexes $\mathbf{Ch}(\mathcal{W}^+_{x_2,y_2}-\mathbf{gm})$ which is similar to (5.22):

$$\begin{array}{ccc}
(M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^{\bullet}(\Delta_{x_1;y_1}))\langle \mathbf{a}_z\rangle & \cong & [(\Delta_{x_2;y_2}[z], \mathbf{a}^{\shortparallel} + \mathbf{a}_z) \xrightarrow{0} (\Delta_{x_2;y_2}[z], \mathbf{a}_z)] \\
\Big\downarrow z & & \Big\downarrow z \qquad\qquad\qquad\qquad \Big\downarrow z \\
M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^{\bullet}(\Delta_{x_1;y_1}) & \cong & [(\Delta_{x_2;y_2}[z], \mathbf{a}^{\shortparallel}) \xrightarrow{\quad 0 \quad} \Delta_{x_2;y_2}[z]] \\
\Big\downarrow g & & \Big\downarrow 1 \qquad\qquad\qquad\qquad \Big\downarrow 1 \\
B_{\bullet} & = & [(\Delta_{x_2;y_2}, \boldsymbol{\pi}(x_2)) \xrightarrow{\quad 0 \quad} \Delta_{x_2;y_2}],
\end{array} \tag{5.24}$$

where $B_{\bullet}$ denotes the complex at the bottom right, g is a quotient homomorphism and we used a relation $\mathbf{a}^{\shortparallel}|_{z=0} = \boldsymbol{\pi}(x_2)$. Combining short exact sequences (5.22) and (5.24) we construct a chain of morphisms in the category $\mathbf{Ch}(\mathbf{W}^+_{x_2,y_2})$:

$$\begin{tikzcd}
(M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^{\bullet}(\Delta_{x_1;y_1}))\langle \mathbf{a}_z\rangle \arrow[r, "z"] \arrow[d, "\mathbb{1}"] & M_{\times} \otimes_{x_1,y_1} \mathcal{P}^{\bullet}(\Delta_{x_1;y_1}) \arrow[r, "f"] \arrow[d, "\chi_+ \otimes \mathbb{1} = 1"] & A_{\bullet} \arrow[d, "h"] \\
(M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^{\bullet}(\Delta_{x_1;y_1}))\langle \mathbf{a}_z\rangle \arrow[r, "z"] & M_{\shortparallel} \otimes_{x_1,y_1} \mathcal{P}^{\bullet}(\Delta_{x_1;y_1}) \arrow[r, "g"] & B_{\bullet}
\end{tikzcd} \tag{5.25}$$

where the chain map h is defined by the diagram

$$\begin{array}{ccc}
A_{\bullet} & = & [(\mathbb{Q}[x_2, y_2], \mathbf{a}^{\shortmid}) \xrightarrow{y_2 - x_2} \mathbb{Q}[x_2, y_2]] \\
\Big\downarrow h & & \Big\downarrow 1 \qquad\qquad\qquad \Big\downarrow 1 \\
B_{\bullet} & = & [(\Delta_{x_2;y_2}, \boldsymbol{\pi}(x_2)) \xrightarrow{\quad 0 \quad} \Delta_{x_2;y_2}].
\end{array} \tag{5.26}$$

Exact sequences of modules in the rows of the diagram (5.25) split if we forget the $\mathcal{W}^+$-module structure, and the complex in the first column is contractible. Hence, according to Theorem 2.1, the second and third columns are $\mathcal{F}$-isomorphic in $\mathbf{Ch}(\mathbf{W}^+_{x_2,y_2})$. Since the second column is isomorphic to $[\![\sigma^{\pm 1}]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{x_1,y_1} \Delta_{x_1;y_1}$, we have a $\mathcal{F}$-isomorphism

$$[\![\sigma^{\pm 1}]\!]_{\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{x_1,y_1} \Delta_{x_1;y_1} \simeq \boxed{A_\bullet \xrightarrow{h} B_\bullet}\,. \tag{5.27}$$

Diagram (5.26) shows that the complex in the right hand side of $\mathcal{F}$-isomorphism (5.27) can be presented as a cone in $\mathbf{Ch}(\mathbf{W}^+_{x_2,y_2})$:

$$\boxed{A_\bullet \xrightarrow{h} B_\bullet} \simeq \boxed{D_\bullet \longrightarrow (\Delta_{x_2;y_2}, \boldsymbol{\pi}(x_2))}\,, \tag{5.28}$$

where

$$D_\bullet = \boxed{\begin{array}{ccc} [(\mathbb{Q}[x_2,y_2], \mathbf{a}^{\mathsf{I}}) & \xrightarrow{y_2 - x_2} & \mathbb{Q}[x_2,y_2]] \\ & & \downarrow{\scriptstyle 1} \\ & & \Delta_{x_2;y_2} \end{array}}\,.$$

The complex $D_\bullet$ is contractible, because its upper row is quasi-isomorphic to $\Delta_{x_2;y_2}$ in $\mathbf{W}^+_{x_2,y_2}$, so

$$D_\bullet \simeq \boxed{\Delta_{x_2;y_2} \xrightarrow{\mathbb{1}} \Delta_{x_2;y_2}}\,.$$

Hence the complex in the right hand side of (5.28) is $\mathcal{F}$-isomorphic to $(\Delta_{x_2;y_2}, \boldsymbol{\pi}(x_2))$ and together with (5.27) this proves the $\mathcal{F}$-isomorphism (5.12) for σ^{-1}. □

6. Homology of two-strand torus knots and links

6.1. Results. Recall that to a oriented framed link L with m components we associate a complex ${}_{\boldsymbol{\lambda}}[\![L]\!]$ considered as an object in the derived category $\mathbf{W}^+_{\boldsymbol{\lambda}}$ of doubly-graded $\mathcal{W}^+_{\boldsymbol{\lambda}}$-modules, where the variables $\boldsymbol{\lambda} = \lambda_1, \ldots \lambda_m$ are associated with the components of L. Note that in all our examples the complex ${}_{\boldsymbol{\lambda}}[\![L]\!]$ turns out to be 'formal', that is, quasi-isomorphic to its homology $\mathcal{H}(L)$, although we do not know if this property holds in general. We refer to ${}_{\boldsymbol{\lambda}}[\![L]\!]$ simply as homology.

Theorem 6.1. *The homology* ${}_{\lambda}[\![\bigcirc]\!]$ *of the unknot* $\bigcirc$ *is*

$$ {}_{\lambda}[\![\bigcirc]\!] = \mathsf{a}^{-1}\mathsf{t}^{-1}\mathsf{q}^{2}\,(\mathbb{Q}_{\lambda}, \boldsymbol{\pi}(\lambda)) \oplus \mathsf{a}\,\mathbb{Q}_{\lambda}. \tag{6.1} $$

Proof. According to the defining relation (1.23) and formula (2.13),

$$ {}_{\lambda}[\![\bigcirc]\!] = \mathsf{a}\,\mathcal{F}_{x\to\lambda}\,\mathrm{H}(\Delta_{x,y} \otimes_{\bar{x},y} [\,\mathsf{a}^{-2}\mathsf{t}^{-1}\mathsf{q}^{2}(\mathbb{Q}_{x,y}, \boldsymbol{\pi}(x,y,)) \xrightarrow{y-x} \mathbb{Q}_{x,y}\,]), \tag{6.2} $$

where $\mathcal{F}_{x\to\lambda}$ is the functor renaming the variable x into λ. Tensoring with $\Delta_{x,y}$ amounts to taking a quotient over $y-x$, hence the right hand side of (6.2) is isomorphic to the right hand side of (6.1). □

The triply graded homology of a $(2,n)$ torus knots and links was computed in [10] (see also [17]). Here we establish the action of $\mathcal{W}^{+}_{\lambda}$ on it.

Theorem 6.2. *The homology of the* $(2,-2n-1)$ *torus knot* $T_{2,-2n-1}$ *with zero framing is*

$$ {}_{\lambda}[\![T_{2,-2n-1}]\!] = \mathsf{fr}_{\lambda}^{2n+1}\,{}_{\lambda}[\![T_{2,-2n-1}]\!]_{\mathrm{bf}}, \tag{6.3} $$

where ${}_{\lambda}[\![T_{2,-2n-1}]\!]_{\mathrm{bf}}$ *is the homology in blackboard framing*:

$$ {}_{\lambda}[\![T_{2,-2n-1}]\!]_{\mathrm{bf}} = \left(\bigoplus_{i=1}^{n} \mathsf{t}^{-2i-1}\mathsf{q}^{4i}\; M_{\mathrm{md}}\,\langle 2i\,\boldsymbol{\pi}\rangle\right) \oplus\; \mathsf{fr}_{\lambda}^{-1}\,{}_{\lambda}[\![\bigcirc]\!], \tag{6.4} $$

while the 'middle module' M_{md} *is a sum of three modules*:

$$ M_{\mathrm{md}} = \mathsf{a}^{-2}\mathsf{t}^{-1}\mathsf{q}^{4}(\mathbb{Q}_{\lambda}, 2\boldsymbol{\pi}(\lambda)) \oplus (\mathbb{Q}_{\lambda} \oplus \mathsf{q}^{2}\mathbb{Q}_{\lambda}, \mathbf{A}_{\times}(\lambda)) \;\oplus\; \mathsf{a}^{2}\mathsf{t}\mathsf{q}^{-2}(\mathbb{Q}_{\lambda}, -\boldsymbol{\pi}(\lambda)), $$

the connection matrix $\mathbf{A}_{\times}(\lambda)$ *is*

$$ \mathbf{A}_{\times}(\lambda) = \begin{pmatrix} 0 & 0 \\ \boldsymbol{\pi}'(\lambda) & \boldsymbol{\pi}(\lambda) \end{pmatrix}, $$

and the sequence of polynomials $\boldsymbol{\pi}'(\lambda)$ *is defined by eq.* (1.3):

$$ \pi'_{m}(\lambda) = m(m+1)\lambda^{m-1}. $$

Let $\boldsymbol{\lambda} = \lambda_1, \lambda_2$. For a $\mathcal{W}^{+}_{\lambda}$-module M we define an associated $\mathcal{W}^{+}_{\boldsymbol{\lambda}}$-module $M^{(12)}$ which is the same M with the action of λ_1 and λ_2 being that of λ:

$$ M^{(12)} = M \otimes_{\lambda} \mathbb{Q}_{\lambda,\boldsymbol{\lambda}}/(\lambda_1 - \lambda, \lambda_2 - \lambda). $$

Theorem 6.3. *The homology* ${}_{\boldsymbol{\lambda}}[\![T_{2,-2n}]\!]$ *of the* $(2,-2n)$ *torus link* $T_{2,-2n}$ *with zero framing is*

$$ {}_{\boldsymbol{\lambda}}[\![T_{2,-2n}]\!] = \mathsf{t}^{-2n}\mathsf{q}^{4n-2}\, M_{\mathrm{tail}}\, \langle (2n-1)\rangle \oplus ({}_{\boldsymbol{\lambda}}[\![T_{2,-2n+1}]\!]_{\mathrm{bf}})^{(12)}, \tag{6.5}$$

where

$$\begin{aligned} M_{\mathrm{tail}} = {}& \mathsf{a}^{-4}\mathsf{t}^{-2}\mathsf{q}^{6}(\mathbb{Q}_{\boldsymbol{\lambda}}, \boldsymbol{\pi}(\lambda_1) + \boldsymbol{\pi}(\lambda_2) + \boldsymbol{\pi}(\lambda_1,\lambda_2)) \\ & \oplus\ \mathsf{a}^{-2}\mathsf{t}^{-1}\mathsf{q}^{2}(\mathbb{Q}_{\boldsymbol{\lambda}} \oplus \mathsf{q}^2\mathbb{Q}_{\boldsymbol{\lambda}}, \mathbf{A}'_{\times}(\lambda_1,\lambda_2))\ \oplus\ \mathbb{Q}_{\boldsymbol{\lambda}}, \end{aligned}$$

the connection matrices $\mathbf{A}'_{\times}(\lambda_1,\lambda_2)$ *being given by the formula*

$$\mathbf{A}'_{\times}(\lambda_1,\lambda_2) = \begin{pmatrix} \boldsymbol{\pi}(\lambda_1) & 0 \\ \boldsymbol{\pi}'(\lambda_1,\lambda_2) & \boldsymbol{\pi}(\lambda_2) + \boldsymbol{\pi}(\lambda_1,\lambda_2) \end{pmatrix},$$

while ${}_{\boldsymbol{\lambda}}[\![T_{2,-2n+1}]\!]_{\mathrm{bf}}$ *is the homology of the torus knot* $T_{2,-2n+1}$ *with blackboard framing defined by eq.* (6.4).

We prove this theorem in three stages. First, we construct a complex $[\![\sigma^{-n}]\!]$ associated to a two-strand braid with n negative crossings (Lemma 6.4). Then we calculate the Hochschild homology of its modules (Lemmas 6.8 and 6.10). Finally, we prove Theorem 6.3 by showing that the complex of Hochschild homologies is quasi-isomorphic to its homology and by computing this homology.

6.2. A chain complex of a 2-strand torus braid

Lemma 6.4. *A 2-strand braid with n negative crossings can be represented by a complex*

$$[\![\sigma^{-n}]\!] \sim \boxed{M_n \xrightarrow{p_n} \cdots \xrightarrow{p_{i+1}} M_i \xrightarrow{p_i} \cdots \xrightarrow{p_2} M_1 \xrightarrow{\chi_-} M_{\parallel}}, \tag{6.6}$$

where M_i *are shifted modules* $M_{\times}$:

$$M_i = \mathsf{t}^{-i}\mathsf{q}^{2(i-1)}\,(M_{\times}, (i-1)\boldsymbol{\pi}),$$

while

$$p_{2i} = \frac{1}{2}(y_2 - y_1) - \frac{1}{2}(x_2 - x_1), \quad p_{2i+1} = \frac{1}{2}(y_2 - y_1) + \frac{1}{2}(x_2 - x_1),$$

$\chi_- = 1$ *is defined by eq.* (3.8) *and* $\boldsymbol{\pi}$ *is defined by* (3.6).

The proof is based on three propositions.

Proposition 6.5. *There is a homotopy equivalence*

$$M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma^{-1}]\!]_{\mathbf{y},\mathbf{z}} \sim \mathsf{t}^{-1}\mathsf{q}^{2}\,(M_{\times;\mathbf{x},\mathbf{z}},\boldsymbol{\pi}). \tag{6.7}$$

Proof. According to the definition (3.15),

$$M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma^{-1}]\!]_{\mathbf{y},\mathbf{z}} = \boxed{\mathsf{t}^{-1} M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\times;\mathbf{y},\mathbf{z}} \xrightarrow{1\otimes 1} M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\shortparallel;\mathbf{y},\mathbf{z}}}\,. \tag{6.8}$$

Decomposition (4.3) implies that the complex in the right hand side of eq. (6.8) has a form

$$\boxed{M_{\times;\mathbf{x},\mathbf{z}} \oplus \mathsf{q}^{2}\left(M_{\times;\mathbf{x},\mathbf{z}},\boldsymbol{\pi}\right) \xrightarrow{(1\ \ z_2-z_1)} M_{\times;\mathbf{x},\mathbf{z}}}\,. \tag{6.9}$$

Contracting the subcomplex $M_{\times;\mathbf{x},\mathbf{z}} \xrightarrow{1} M_{\times;\mathbf{x},\mathbf{z}}$ we come to the homotopy equivalence (6.7). □

Since the modules in the left hand side of the complex (6.9) are generated by elements $1,(y_2-y_1) \in M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\times;\mathbf{y},\mathbf{z}}$, the chain map

$$\begin{array}{c}(M_{\times;\mathbf{x},\mathbf{z}},\boldsymbol{\pi})\\ \Big\downarrow{\scriptstyle f=\frac{1}{2}(y_2-y_1)+\frac{1}{2}(z_1-z_2)}\\ \boxed{M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\times;\mathbf{y},\mathbf{z}} \xrightarrow{1\otimes 1} M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} M_{\shortparallel;\mathbf{y},\mathbf{z}}}\end{array} \tag{6.10}$$

is the homotopy equivalence (6.7).

Homotopy equivalence (6.7) establishes an isomorphism between the spaces of endomorphisms

$$\mathrm{End}_{\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{z}})}(M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma^{-1}]\!]_{\mathbf{y},\mathbf{z}}) \xrightarrow{\hat{f}'} \mathrm{End}_{\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{z}})}(M_{\times;\mathbf{x},\mathbf{z}}\langle\boldsymbol{\pi}\rangle).$$

For a variable x let $\hat{x}$ denote the module endomorphism corresponding to multiplication by x.

Proposition 6.6. *The isomorphism $\hat{f}'$ converts the endomorphisms $\hat{y}_i$ into $\hat{z}_j$ with switched indices*:

$$\hat{f}'(\hat{y}_1 \otimes_{\mathbf{y}} \mathbb{1}) = \hat{z}_2, \qquad \hat{f}'(\hat{y}_2 \otimes_{\mathbf{y}} \mathbb{1}) = \hat{z}_1. \tag{6.11}$$

Proof. This relation follows immediately from the sliding property (1.20): for example,

$$\hat{y}_1 \otimes_{\mathbf{y}} \mathbb{1} = \mathbb{1} \otimes_{\mathbf{y}} \hat{y}_1 = \mathbb{1} \otimes_{\mathbf{y}} \hat{z}_2,$$

and $\hat{f}'(\mathbb{1} \otimes_{\mathbf{y}} \hat{z}_2) = \hat{z}_2$. □

Homotopy equivalence (6.7) combined with the isomorphism

$$M_{\parallel;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma^{-1}]\!]_{\mathbf{y},\mathbf{z}} \xrightarrow{\cong} [\![\sigma^{-1}]\!]_{\mathbf{x},\mathbf{z}} \tag{6.12}$$

establishes another isomorphism:

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{z}})}(M_{\times;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma^{-1}]\!]_{\mathbf{y},\mathbf{z}}, M_{\parallel;\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma^{-1}]\!]_{\mathbf{y},\mathbf{z}}) \\ &\qquad\qquad \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{D}_{\mathrm{r}}(\mathbf{W}^{+}_{\mathbf{x},\mathbf{z}})}((M_{\times;\mathbf{x},\mathbf{z}}, \boldsymbol{\pi}), [\![\sigma^{-1}]\!]_{\mathbf{x},\mathbf{z}}). \end{aligned} \tag{6.13}$$

Proposition 6.7. *This isomorphism converts $\chi_- \otimes_{\mathbf{y}} \mathbb{1}$ into the chain map*

$$\begin{array}{l} M_{\times;\mathbf{x},\mathbf{z}}\langle \boldsymbol{\pi} \rangle \\ \Big\downarrow f=\frac{1}{2}(x_2-x_1)+\frac{1}{2}(z_1-z_2) \\ \boxed{\mathsf{t}^{-1} M_{\times;\mathbf{x},\mathbf{z}} \xrightarrow{\chi_-} M_{\parallel;\mathbf{x},\mathbf{z}}} \end{array}.$$

Proof. The isomorphism (6.13) converts $\chi_- \otimes_{\mathbf{y}} \mathbb{1}$ into a composition of the homotopy equivalence map (6.10), the map $\chi_- \otimes_{\mathbf{y}} \mathbb{1}$ and the isomorphism (6.12), the latter replacing variables y_1 and y_2 with x_1 and x_2. □

Proof of Lemma 6.4. We prove this lemma by induction over n. The case of $n = 1$ follows obviously from the definition (3.15). Suppose that the presentation (6.6) hold for some value of n and consider the tensor product $[\![\sigma^{-(n+1)}]\!]_{\mathbf{x},\mathbf{z}} = [\![\sigma^{-n}]\!]_{\mathbf{x},\mathbf{y}} \otimes_{\mathbf{y}} [\![\sigma^{-1}]\!]_{\mathbf{y},\mathbf{z}}$. The tensor products of individual chain modules are contracted in accordance with homotopy equivalences (6.7) and (6.12), while the differentials p_i are replaced in accordance with eq. (6.11) and the last differential χ_- is replaced in accordance with Proposition 6.7. It is easy to see that t-grading precludes the appearance of 'secondary' differentials as a result of contractions. □

6.3. Hochschild homology of chain modules. The next step in computing the homology of two-strand torus knots and links is to find the Hochschild homology of chain bimodules of the complex (6.6). Recall that the Hochschild homology $\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}$ of a $\mathcal{W}^{+}_{\mathbf{x},\mathbf{y}}$ module is defined by eq. (1.23) as the homology of a derived tensor product with the diagonal bimodule $\Delta_{\mathbf{x};\mathbf{y}}$ considered as a module over $\mathcal{W}_{\mathbf{x}}^{+}$, while the action of $\mathbf{y}$ is forgotten. The Hochschild homology can be computed with the help of the resolution (2.12) by the formula (2.13). In the two-strand case the resolution (2.12) takes the form

$$\tilde{\mathsf{a}}^2(\mathbb{Q}_{\mathbf{x},\mathbf{y}}, \boldsymbol{\pi}_{(1)} + \boldsymbol{\pi}_{(2)}) \xrightarrow{\binom{y_2-x_2}{x_1-y_1}} \tilde{\mathsf{a}}((\mathbb{Q}_{\mathbf{x},\mathbf{y}}, \boldsymbol{\pi}_{(1)}) \oplus (\mathbb{Q}_{\mathbf{x},\mathbf{y}}, \boldsymbol{\pi}_{(2)})) \xrightarrow{(y_1-x_1 \;\; y_2-x_2)} \mathbb{Q}_{\mathbf{x},\mathbf{y}},$$

where we used shortcut notations

$$\boldsymbol{\pi}_{(i)} = \boldsymbol{\pi}(x_i, y_i), \quad i = 1, 2, \qquad \tilde{\mathsf{a}} = \mathsf{a}^{-2}\mathsf{t}^{-1}\mathsf{q}^2.$$

However in order to compute the Hochschild homology of both $M_{\times;\mathbf{x},\mathbf{y}}$ and $M_{\Vert;\mathbf{x},\mathbf{y}}$ it is more convenient to apply a simple change of basis in the middle module which results in an isomorphic resolution

$$\tilde{\mathsf{a}}^2(\mathbb{Q}_{\mathbf{x},\mathbf{y}}, \boldsymbol{\pi}_{(1)} + \boldsymbol{\pi}_{(2)}) \xrightarrow{\binom{x_2-y_2}{-\rho}} \tilde{\mathsf{a}}((\mathbb{Q}_{\mathbf{x},\mathbf{y}} \oplus \mathbb{Q}_{\mathbf{x},\mathbf{y}}, \mathbf{A}_{(12)})) \xrightarrow{(\rho \;\; y_2-x_2)} \mathbb{Q}_{\mathbf{x},\mathbf{y}}, \tag{6.14}$$

where $\rho = y_1 + y_2 - x_1 - x_2$, while

$$\mathbf{A}_{(12)} = \begin{pmatrix} \boldsymbol{\pi}_{(1)} & 0 \\ \boldsymbol{\pi}_{(2)} - \boldsymbol{\pi}_{(1)} & \boldsymbol{\pi}_{(2)} \end{pmatrix}$$

is a connection given by 2×2 matrices.

Lemma 6.8. *The Hochschild homology of the bimodule $M_{\times;\mathbf{x},\mathbf{y}}$ has the form*

$$\begin{aligned} \mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\times;\mathbf{x},\mathbf{y}}) \cong\ & \tilde{\mathsf{a}}^2\mathsf{q}^2(\mathbb{Q}_{\mathbf{x}}, \boldsymbol{\pi}(x_1) + \boldsymbol{\pi}(x_2) + \boldsymbol{\pi}) \\ & \oplus \tilde{\mathsf{a}}(\mathbb{Q}_{\mathbf{x}} \oplus \mathsf{q}^2\mathbb{Q}_{\mathbf{x}}, \mathbf{A}'_{\times}(x_1, x_2)) \oplus \mathbb{Q}_{\mathbf{x}}, \end{aligned} \tag{6.15}$$

where the connection matrices $\mathbf{A}'_{\times}(x_1, x_2)$ are given by the formula

$$\mathbf{A}'_{\times}(x_1, x_2) = \begin{pmatrix} \boldsymbol{\pi}(x_1) & 0 \\ \boldsymbol{\pi}'(x_1, x_2) & \boldsymbol{\pi}(x_2) + \boldsymbol{\pi} \end{pmatrix}$$

and $\boldsymbol{\pi}'(x_1, x_2)$ is defined by eq. (1.3).

Proof. Derived tensor product $M_{\times;\mathbf{x},\mathbf{y}} \overset{\mathrm{L}}{\otimes}_{\bar{\mathbf{x}},\mathbf{y}} \Delta_{\mathbf{x};\mathbf{y}}$ computed with the help of the free resolution (6.14) has a form

$$\tilde{\mathsf{a}}^2(M_{\times;\mathbf{x},\mathbf{y}}, \boldsymbol{\pi}_{(1)} + \boldsymbol{\pi}_{(2)}) \xrightarrow{\binom{x_2-y_2}{0}} \tilde{\mathsf{a}}((M_{\times;\mathbf{x},\mathbf{y}} \oplus M_{\times;\mathbf{x},\mathbf{y}}, \mathbf{A}_{(12)})) \xrightarrow{(0\ \ y_2-x_2)} M_{\times;\mathbf{x},\mathbf{y}}. \tag{6.16}$$

If we forget about the action of $\mathcal{W}^+$ and degree shifts, it splits into a sum of two isomorphic complexes

$$M_{\times;\mathbf{x},\mathbf{y}} \xrightarrow{y_2-x_2} M_{\times;\mathbf{x},\mathbf{y}}.$$

Is is easy to see that both kernel $\ker(\hat{y}_2-\hat{x}_2)$ and co-kernel $M_{\times;\mathbf{x},\mathbf{y}}/\operatorname{im}(\hat{y}_2-\hat{x}_2)$ are isomorphic to $\Delta_{\mathbf{x};\mathbf{y}}$, the former being generated by $\frac{1}{2}(y_2-y_1)+\frac{1}{2}(x_2-x_1)$. When the action of $\mathbf{y}$ is forgotten, $\Delta_{\mathbf{x};\mathbf{y}}$ becomes isomorphic to $\mathbb{Q}[\mathbf{x}]$ and the homology of the complex (6.16) turns into the expression (6.15). □

Remark 6.9. *Since all the modules $\mathbb{Q}_{\mathbf{x}}$ appearing in the Hochschild homology (6.15) are the result of taking kernel or co-kernel of endomorphisms $\hat{y}_i - \hat{x}_i$, $i = 1,2$, these endomorphisms act trivially on* $\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\times;\mathbf{x},\mathbf{y}})$.

Lemma 6.10. *The Hochschild homology of the bimodule $M_{\shortparallel;\mathbf{x},\mathbf{y}}$ has the form*

$$\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\shortparallel;\mathbf{x},\mathbf{y}}) \;\cong\; \tilde{\mathsf{a}}^2(\mathbb{Q}_{\mathbf{x}}, \boldsymbol{\pi}(x_1)+\boldsymbol{\pi}(x_2)) \oplus \tilde{\mathsf{a}}(\mathbb{Q}_{\mathbf{x}}\oplus\mathbb{Q}_{\mathbf{x}}, \mathbf{A}'_{\shortparallel}(x_1,x_2)) \oplus \mathbb{Q}_{\mathbf{x}}, \tag{6.17}$$

where

$$\mathbf{A}'_{\shortparallel}(x_1,x_2) = \begin{pmatrix} \boldsymbol{\pi}(x_1) & 0 \\ \boldsymbol{\pi}(x_2)-\boldsymbol{\pi}(x_1) & \boldsymbol{\pi}(x_2) \end{pmatrix}.$$

Proof. The proof is obvious. The reason for the appearance of the non-diagonal matrix $\mathbf{A}'_{\shortparallel}(x_1,x_2)$ is that we used the same resolution (6.14) that we have used for eq. (6.15). □

6.4. Proof of Theorems 6.2 and 6.3. In view of Remark 6.9, the even differentials p_{2i} of the complex (6.6) act trivially, whereas the odd differentials p_{2i+1} act as multiplication by $x_2 - x_1$. Therefore the Hochschild homology $\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(\llbracket\sigma^{-n}\rrbracket)$ splits into a sum of degree-shifted copies of short complexes

$$C_{\bullet,\times} = \boxed{\mathsf{t}^{-1}\,\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\times;\mathbf{x},\mathbf{y}}) \xrightarrow{x_2-x_1} \mathsf{q}^{-2}\,\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\times;\mathbf{x},\mathbf{y}})\langle-\boldsymbol{\pi}\rangle},$$

$$C_{\bullet,\shortparallel} = \boxed{\mathsf{t}^{-1}\,\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\times;\mathbf{x},\mathbf{y}}) \xrightarrow{\chi_-} \mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\shortparallel;\mathbf{x},\mathbf{y}})},$$

where we used the same notation χ_- for the homomorphism between Hochschild homologies of $M_{\times;\mathbf{x},\mathbf{y}}$ and $M_{\shortparallel;\mathbf{x},\mathbf{y}}$. If n is even, then there is an extra summand $\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\times;\mathbf{x},\mathbf{y}})$ coming from the first chain module. In other words,

$$\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}([\![\sigma^{-2n-1}]\!]) \cong \Big(\bigoplus_{i=1}^{n} \mathsf{t}^{-2i}\mathsf{q}^{4i}\, C_{\bullet,\times}\langle 2i\boldsymbol{\pi}\rangle\Big) \oplus C_{\bullet,\shortparallel}, \tag{6.18}$$

$$\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}([\![\sigma^{-2n}]\!]) \cong \mathsf{t}^{-2n}\mathsf{q}^{4n-2}\,\mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}(M_{\times;\mathbf{x},\mathbf{y}})\langle(2n-1)\boldsymbol{\pi}\rangle \oplus \mathrm{HH}_{\bar{\mathbf{x}},\mathbf{y}}([\![\sigma^{-2n+1}]\!]), \tag{6.19}$$

and the last summand of eq. (6.19) is defined by eq. (6.18).

If we forget about the $\mathfrak{W}^+$-module structure then, up to degree shifts, the complex $C_{\bullet,\times}$ splits into the sum of four degree-shifted complexes of the form

$$\mathbb{Q}[\mathbf{x}] \xrightarrow{x_2-x_1} \mathbb{Q}[\mathbf{x}]. \tag{6.20}$$

Their homology is concentrated in the last term, hence as an object of the derived category $\mathbf{W}^+_{\mathbf{x}}$, the complex $C_{\bullet,\times}$ is quasi-isomorphic to its homology:

$$\begin{aligned} C_{\bullet,\times} \simeq \mathrm{H}(C_{\bullet,\times}) \cong \tilde{\mathsf{a}}^2(\mathbb{Q}_x^{(12)}, 2\boldsymbol{\pi}(x)) \oplus \tilde{\mathsf{a}}\mathsf{q}^{-2}(\mathbb{Q}_x^{(12)} \oplus \mathsf{q}^2\mathbb{Q}_x^{(12)}, \mathbf{A}_\times(x)) \\ \oplus\, \mathsf{q}^{-2}\big(\mathbb{Q}_x^{(12)}, -\boldsymbol{\pi}(x)\big), \end{aligned} \tag{6.21}$$

where

$$\mathbf{A}_\times(x) = \begin{pmatrix} 0 & 0 \\ \boldsymbol{\pi}'(x) & \boldsymbol{\pi}(x) \end{pmatrix}$$

and we used a notation $\mathbb{Q}_x^{(12)} = \mathbb{Q}_{\mathbf{x}}/(x_2 - x_1)$. Since this module is isomorphic to $\mathbb{Q}_x$ with multiplication by x_1 and x_2 being identified with the multiplication by x, we use this single variable x as an argument of connections.

If we forget again the $\mathfrak{W}^+$-module structure, then, up to degree shifts, the complex $C_{\bullet,\shortparallel}$ splits into two contractible complexes $\mathbb{Q}[\mathbf{x}] \xrightarrow{1} \mathbb{Q}[\mathbf{x}]$ and two complexes (6.20), the differentials $x_2 - x_1$ appearing because $y_2 - x_1$ are the generators of the two modules $\mathbb{Q}[\mathbf{x}]$ in (6.15) which are kernels of $y_2 - x_2$ and hence are considered as submodules of $M_{\times;\mathbf{x},\mathbf{y}}$. This means again that the complex $C_{\bullet,\shortparallel}$ is quasi-isomorphic to its homology:

$$C_{\bullet,\shortparallel} \simeq \mathrm{H}(C_{\bullet,\shortparallel}) \cong \tilde{\mathsf{a}}^2(\mathbb{Q}_x^{(12)}, 2\boldsymbol{\pi}(x)) \oplus \tilde{\mathsf{a}}(\mathbb{Q}_x^{(12)}, \boldsymbol{\pi}(x)). \tag{6.22}$$

After we substitute the expressions (6.21) and (6.22) into (6.18) and (6.19), shift the a-degree by a^2 in view of closing two braid strands and rename the variables $\mathbf{x}$ into $\boldsymbol{\lambda}$, we obtain the torus link homology (6.5). In order to get the homology (6.3) we forget the action of x_2, rename x_1 into λ and apply the framing shift functor $\mathfrak{fr}_\lambda^{2n+1}$ which converts the blackboard framing of the closure of σ^{-2n-1} into the canonical framing of the torus knot $T_{2,-2n-1}$. □

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