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Categorifications of the extended affine Hecke algebra and the affine q-Schur algebra $\widehat{\mathrm{S}}(n,r)$ for $3 \leq r < n$

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Abstract. We categorify the extended affine Hecke algebra and the affine quantum Schur algebra $\widehat{\mathrm{S}}(n,r)$ for $3 \leq r < n$, using results on diagrammatic categorification in affine type A by Elias-Williamson, that extend the work of Elias–Khovanov for finite type A, and Khovanov-Lauda respectively. We also define 2-representations of these categorifications on an extension of the 2-category of affine (singular) Soergel bimodules. These results are the affine analogue of the results in [28].

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Contents

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Introduction

Khovanov and Lauda [16, 18, 17], and Rouquier [34] following a slightly different approach, defined a graded additive 2-category $\mathcal{U}(\mathfrak{g})$ with "nice properties" for any Cartan datum describing a Kac–Moody algebra $\mathfrak{g}$. The 2-morphisms are defined by string diagrams with regions labeled by $\mathfrak{g}$-weights. They are generated by a finite set of elementary diagrams which obey a finite set of relations. The split Grothendieck group of the Karoubi envelope of $\mathcal{U}(\mathfrak{g})$ is isomorphic to the idempotented version of the corresponding quantum group $\dot{\mathbf{U}}(\mathfrak{g})$. In Crane and Frenkel's [3] terminology, we say that $\mathcal{U}(\mathfrak{g})$ *categorifies* $\dot{\mathbf{U}}(\mathfrak{g})$.

Khovanov and Lauda only proved this *categorification theorem* for $\mathfrak{g} = \mathfrak{sl}_n$. A key ingredient of that proof was a 2-representation of $\mathcal{U}(\mathfrak{sl}_n)$ on a 2-category build out of the cohomology rings of partial flag varieties. The equivariant cohomology rings of these varieties, which also give rise to a 2-representation, are equivalent to the singular Soergel bimodules of type A, introduced and studied by Williamson in his Ph.D. thesis in 2008 and published in [37]. The general categorification theorem was proved by Webster [36].

In [28], Mackaay, Stošić and Vaz defined a quotient of $\mathcal{U}(\mathfrak{sl}_n)$, denoted $\mathcal{S}(n, r)$, and proved that it categorifies the quantum Schur algebra $\mathbf{S}(n, r)$, for any $r \in \mathbb{Z}_{>0}$. If $n \geq r$, then $\mathcal{S}(n, r)$ contains a full sub-2-category which categorifies the Hecke algebra $\mathcal{H}_{A_{r-1}}$. This sub-2-category is equivalent to the 2-category of (ordinary) Soergel bimodules of type A_{r-1}, as was proved in [28] using Elias and Khovanov's diagrammatic presentation of the Soergel 2-category [8].

In the same paper, Mackaay, Stošić and Vaz also showed that Khovanov and Lauda's 2-representation of $\mathcal{U}(\mathfrak{sl}_n)$ on the singular Soergel bimodules descends to $\mathbf{S}(n, r)$. Its restriction to the aforementioned sub-2-category of $\mathbf{S}(n, r)$ is exactly Elias and Khovanov's 2-equivalence of their diagrammatic 2-category and the 2-category of (ordinary) Soergel bimodules.

Naturally the question arises whether the results in [28] extend to affine type A. In this paper, we show that this is indeed the case for $3 \leq r < n$:

- As Libedinsky explained in [22], one can define Soergel bimodules using the geometric representation of the affine Weyl group $\mathcal{W}_{\hat{A}_{r-1}}$ or Soergel's extension of that representation [35]. The geometric representation is not reflection faithful in Soergel's sense, whereas Soergel's representation is. Both representations give rise to categories of Soergel bimodules which categorify the affine Hecke algebra $\mathcal{H}_{\hat{A}_{r-1}}$, as shown in [13, 23, 22, 35]. For more information on this topic, see also [9].

However, the extension of the geometric representation to the *extended* affine Weyl group $\widehat{W}_{\hat{A}_{r-1}}$ is “too degenerate” and cannot be used to categorify the *extended* affine Hecke algebra $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$. Soergel’s representation extends nicely to $\widehat{W}_{\hat{A}_{r-1}}$ and we show that there is a corresponding category of bimodules, denoted $\mathcal{EBim}_{\hat{A}_{r-1}}$, which categorifies $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$.

Soergel’s representation of $\widehat{W}_{\hat{A}_{r-1}}$ has dimension $r+1$ (see [35, Section 2], [9, Example 3.2]) and the corresponding bimodules are defined over $\mathbb{Q}[y, x_1, \ldots, x_r]$ where $\deg(y) = \deg(x_1) = \cdots = \deg(x_r) = 2$. We note that Soergel’s representation fixes y, so the left and the right action of y on any bimodule coincide.

- We also define a diagrammatic 2-category $\mathcal{DEBim}_{\hat{A}_{r-1}}$, similar to the ones in [8, 9], and show that it is 2-equivalent to $\mathcal{EBim}_{\hat{A}_{r-1}}$ (actually they are equivalent as monoidal categories, i.e. 2-categories with one object). Here we use the corresponding result for the non-extended category of affine bimodules and its diagrammatic analogue due to Elias and Williamson [9].

- We define a y-deformation of the level-zero 2-category $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$. In order to do that, the homogeneous 2-morphisms are defined over $\mathbb{Q}[y]$ instead of $\mathbb{Q}$, with y a formal variable of degree two. We denote this 2-category by $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and prove that its Karoubi envelope is Krull–Schmidt.

  We recover $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ when we quotient by the 2-ideal generated by y. This 2-ideal is virtually nilpotent, so the Grothendieck groups of the Karoubi envelopes of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ are isomorphic.

- We define a quotient of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$, which we denote $\widehat{\mathcal{S}}(n,r)_{[y]}$. We prove that $\widehat{\mathcal{S}}(n,r)_{[y]}$ categorifies the affine quantum Schur algebra $\widehat{\mathbf{S}}(n,r)$. Again, the 2-ideal generated by y is virtually nilpotent, so the quotient of $\widehat{\mathcal{S}}(n,r)_{[y]}$ by this 2-ideal also categorifies $\widehat{\mathbf{S}}(n,r)$. We denote this quotient by $\widehat{\mathcal{S}}(n,r)$, which can also be obtained as a quotient of Khovanov and Lauda’s original $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$.

- We define a 2-functor

$$\Sigma_{n,r} \colon \mathcal{DEBim}^*_{\hat{A}_{r-1}} \longrightarrow \widehat{\mathcal{S}}^*(n,r)_{[y]},$$

  prove it to be faithful and conjecture it to be full.[1]

[1] We will explain the * notation in Section 2. It basically allows us to consider 2-morphisms of arbitrary degree.

- We define the 2-category of extended affine singular Soergel bimodules $\mathcal{ESBim}_{\hat{A}_{r-1}}$ and give the affine analogue (and y-deformation) of Khovanov and Lauda's 2-representation, i.e. a 2-functor

$$\mathcal{F}'\colon \widehat{\mathcal{S}}(n,r)^*_{[y]} \longrightarrow \mathcal{ESBim}^*_{\hat{A}_{r-1}}.$$

Remark 0.1. We do not consider the case $r = 2$, because we heavily rely on the results on affine Schur–Weyl duality due to Doty and Green [6], which they proved for $r \geq 3$.

The case $n = r$ is different, because $\widehat{\mathbf{S}}(n,n)$ is not a quotient of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ but only of a strictly larger algebra. Therefore, one has to extend the Khovanov-Lauda affine calculus in order to define $\widehat{\mathcal{S}}(n,n)$. This case is dealt with in a follow-up paper [29].

The case $n < r$ cannot be dealt with at present, because even the decategorified story has not been worked out (see Problem 2.4.5 in [5]).

Remark 0.2. There is a technical detail, which will be fully explained in Section 3 but should be mentioned here already. Just as in [28], we actually define $\widehat{\mathcal{S}}(n,r)_{[y]}$ as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, which is a 2-category obtained from $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ by switching to $\widehat{\mathfrak{gl}}_n$-weights of level zero for the labels of the regions in the string diagrams. We conjecture that $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ categorifies the level-zero $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, but do not need that fact for the rest of this paper.

The results in this paper have several points of interest. The categories $\mathcal{EBim}_{\hat{A}_{r-1}}$ and $\mathcal{DEBim}_{\hat{A}_{r-1}}$ contain the new objects $B_{\rho^{\pm}}$ and $\pm$, respectively, and some corresponding morphisms. As our results show, these objects and morphisms also show up naturally in $\widehat{\mathcal{S}}(n,r)_{[y]}$ as 1 and 2-morphisms.

The y-deformation $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and its Schur quotient $\widehat{\mathcal{S}}(n,r)_{[y]}$ are new. As the results in this paper show, they both show up naturally when considering Soergel bimodules over $\mathbb{Q}[y, x_1, \ldots, x_r]$, which is the ring of polynomial functions on Soergel's reflection faithful representation of the affine Weyl group.

Furthermore, there are interesting (possible) links with other categorifications of the (extended) affine Hecke algebra and the quantum affine Schur algebra. Lusztig [25, 26] and Ginzburg and Vasserot [11] gave a categorification of $\widehat{\mathbf{S}}(n,r)$ using perverse sheaves, extending Grojnowski and Lusztig's approach to the categorification of $\mathbf{S}(n,r)$. It would be interesting to find the precise relation with the categorification presented here and in our follow-up paper [29].

In this paper we also define an extended version of the affine singular Soergel bimodules. Williamson introduced and studied the 2-category of singular Soergel bimodules for any Coxeter group in his PhD thesis in 2008, the results of which were published in [37], and proved that it categorifies a certain "new" algebra, which he called the *Schur algebroid*. In finite type A the Schur algebroid is isomorphic to the quantum Schur algebra. Williamson's affine type A Schur algebroid should also be closely related to the affine quantum Schur algebra. Whatever the precise relation turns out to be, the 2-representation of $\widehat{\mathcal{S}}(n,r)_{[y]}$ on the extended affine singular Soergel bimodules establishes an interesting relation between Khovanov and Lauda's work and Williamson's.

Another point of interest is related to the possibility of categorifying the so called *Kirillov-Reshetikhin modules* of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$. These level zero modules can be defined for any affine quantum group and have been intensively studied (see [2, 5, 7, 15] for more information and references).

In affine type A (and only in that type), they are special examples of *evaluation modules* $V_{\lambda,a}$, where λ is a dominant weight and $a \in \mathbb{C}^*$. If λ is an n-part partition of r, then $V_{\lambda,a}$ descends to a representation of $\widehat{\mathbf{S}}(n,r)$. More precisely, $V_{\lambda,a}$ is defined by pulling back (the technical term is *inflating*) the action of $\mathbf{S}(n,r)$ on the irrep V_λ via the so called *evaluation map*

$$\mathrm{ev}_a \colon \widehat{\mathbf{S}}(n,r) \longrightarrow \mathbf{S}(n,r).$$

If $\lambda = (m^i)$, i.e. m times the i-th fundamental $\mathfrak{gl}_n$-weight, then it is known that $V_{\lambda,q^{i-m+2}}$ is isomorphic to a Kirillov-Reshetikhin module and has a canonical basis. It seems likely that one can categorify the evaluation map

$$\mathrm{ev}_{q^{i-m+2}} \colon \widehat{\mathbf{S}}(n,mi) \longrightarrow \mathbf{S}(n,mi)$$

and therefore $V_{\lambda,q^{i-m+2}}$, but such a categorification is beyond the scope of this paper.

1. The affine setting

1.1. Affine roots of level zero. We use the well-known realization of $\widehat{\mathfrak{sl}}_r$ as the central extension of the loop algebra of $\mathfrak{sl}_r$, defined as $\mathcal{L}(\mathfrak{sl}_r) := \mathfrak{sl}_r \otimes_{\mathbb{Q}} \mathbb{Q}[t,t^{-1}]$, together with a derivation (see for example [33, 14, 10]), i.e. the underlying vector space is isomorphic to

$$\widehat{\mathfrak{sl}}_r = \mathcal{L}(\mathfrak{sl}_r) \oplus \mathbb{Q}c \oplus \mathbb{Q}d.$$

In order to express its root system, consider the Cartan subalgebra

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{Q}c \oplus \mathbb{Q}d = \mathbb{Q}\langle h_i, c, d \mid i = 1, \dots, r-1\rangle$$

with $\mathfrak{h}$ being the Cartan subalgebra of $\mathfrak{sl}_r$. The roots with respect to $\hat{\mathfrak{h}}$ are

$$\alpha = (\bar{\alpha}, 0, m), \quad \bar{\alpha} \in \Phi(\mathfrak{sl}_r) \text{ i.e. } \bar{\alpha} \text{ is a root of } \mathfrak{sl}_r,\ m \in \mathbb{Z},$$

and

$$\alpha = (0, 0, m), \quad m \in \mathbb{Z} \setminus \{0\}.$$

The roots of the first family are called the *real roots* and the ones of the second family are called the *imaginary roots*.

The simple roots are

$$\alpha_i = (\bar{\alpha_i}, 0, 0), \quad i = 1, \dots, r-1,$$

and

$$\alpha_r = (-\bar{\theta}, 0, 1) = \delta - \bar{\theta},$$

where $\bar{\alpha}_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \dots, r-1$ are the simple roots of $\mathfrak{sl}_r$, $\bar{\theta} = \bar{\alpha}_1 + \cdots + \bar{\alpha}_{r-1} = \varepsilon_1 - \varepsilon_r$ is the highest root and δ is the dual element of d. The elements ε_i for $i = 1, \dots, r$ are the canonical basis vectors in $\mathbb{Z}^r$.

Weights are triples of the form

$$\kappa = (\bar{\kappa}, k, m),$$

where $\bar{\kappa}$ is an $\mathfrak{sl}_r$-weight and k and m are integers. The integer k is called the *level* of κ. The inner product between two weights $\kappa = (\bar{\kappa}, k, m)$ and $\kappa' = (\bar{\kappa}', k', m')$ is given by

$$(\kappa, \kappa') = (\bar{\kappa}, \bar{\kappa}') + km' + k'm,$$

where $(\bar{\kappa}, \bar{\kappa}')$ is the usual inner product of $\mathfrak{sl}_r$-weights. In particular, we have

$$(\alpha_i, \alpha_i) = 2, \quad \text{for all } i = 1, \dots, r,$$

and

$$(\alpha_r, \alpha_1) = (\alpha_i, \alpha_{i+1}) = -1, \quad \text{for all } i = 1, \dots, r-1.$$

In the following sections, we will also use $\widehat{\mathfrak{gl}_r}$-weights

$$\kappa = (\bar{\kappa}, k, m),$$

where $\bar{\kappa}$ denotes a non-affine $\mathfrak{gl}_r$-weight.

Remark 1.1. In this paper, we will only consider $\widehat{\mathfrak{sl}_r}$ and $\widehat{\mathfrak{gl}_r}$-weights of level zero.

1.2. The Weyl group action. For any real root $\alpha \in \Phi(\widehat{\mathfrak{sl}}_r)$ and $\widehat{\mathfrak{sl}}_r$-weight κ, the Weyl reflection σ_α is defined by $\sigma_\alpha(\kappa) = \kappa - \langle\kappa,\alpha^\vee\rangle\alpha$. So if $\alpha = (\bar{\alpha}, 0, n)$ and $\kappa = (\bar{\kappa}, k, m)$, then $\alpha^\vee = \bar{\alpha}^\vee + \frac{2n}{(\alpha,\alpha)}c$ and we can express $\sigma_\alpha(\kappa)$ as follows:

$$\sigma_\alpha(\kappa) = \Big(\bar{\kappa} - \langle\bar{\kappa},\bar{\alpha}^\vee\rangle\bar{\alpha} - \frac{2kn}{(\alpha,\alpha)}\bar{\alpha}, k, m - \langle\bar{\kappa},\bar{\alpha}^\vee\rangle n - \frac{2kn^2}{(\alpha,\alpha)}\Big).$$

The *affine Weyl group* $\mathcal{W}_{\widehat{A}_{r-1}}$ is the group generated by all these reflections. For any simple root α_i, we write $\sigma_i := \sigma_{\alpha_i}$.

Similarly, one can consider the action of σ_i on a level zero $\widehat{\mathfrak{gl}}_r$-weight of the form $(\varepsilon_j, 0, m)$ with $j = 1, \ldots, r$. If $i \neq r$, one gets

$$\sigma_i(\varepsilon_j, 0, m) = \begin{cases} (\varepsilon_{i+1}, 0, m) & \text{if } j = i, \\ (\varepsilon_i, 0, m) & \text{if } j = i+1, \\ (\varepsilon_j, 0, m) & \text{otherwise.} \end{cases}$$

The action of σ_r is given by

$$\sigma_r(\varepsilon_j, 0, m) = \begin{cases} (\varepsilon_r, 0, m+1) & \text{if } j = 1, \\ (\varepsilon_1, 0, m-1) & \text{if } j = r, \\ (\varepsilon_j, 0, m) & \text{otherwise.} \end{cases}$$

For each simple $\mathfrak{sl}_r$-root $\bar{\alpha}_i$, $i = 1, \ldots, r-1$, there also exists a translation $t_{\bar{\alpha}_i}$, which acts on the level zero $\widehat{\mathfrak{gl}}_r$-weights as follows:

$$t_{\bar{\alpha}_i}(\bar{\kappa}, 0, m) = (\bar{\kappa}, 0, m - (\kappa_i - \kappa_{i+1})),$$

where $\bar{\kappa} = (\kappa_1, \ldots, \kappa_r)$ and the indices are taken to be modulo r, e.g. $\kappa_{r+1} = \kappa_1$ by definition.

One can prove that $\mathcal{W}_{\widehat{A}_{r-1}}$ is the semidirect product of the finite Weyl group $\mathcal{W}_{A_{r-1}}$, generated by the reflections σ_i for $i = 1, \ldots, r-1$, and of the abelian group $\langle t_{\bar{\alpha}_1}, \ldots, t_{\bar{\alpha}_{r-1}}\rangle$ of translations along the root lattice of $\mathfrak{sl}_r$.

1.3. The extended affine Weyl group. See [24], [6] or [5] for more details about the extended affine Weyl group. For example in [6], there is a definition of this group different from the following, it is described as a subgroup of permutations of $\mathbb{Z}$.

Let us now consider the translations t_{ε_i} along the basic $\mathfrak{gl}_r$-weights ε_i, $i = 1, \ldots, r$. Their action on a level zero $\widehat{\mathfrak{gl}}_r$-weight κ is given by

$$t_{\varepsilon_i}(\bar{\kappa}, 0, m) = (\bar{\kappa}, 0, m - \kappa_i).$$

The *extended affine Weyl group* $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is defined as the semidirect product of the finite Weyl group $\mathcal{W}_{A_{r-1}}$ and the abelian group $\langle t_{\varepsilon_1}, \dots, t_{\varepsilon_r} \rangle$ of translations along the weight lattice of $\mathfrak{gl}_r$. It contains the affine Weyl group $\mathcal{W}_{\widehat{A}_{r-1}}$ as a normal subgroup.

The group $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is generated by $\sigma_1, \dots, \sigma_{r-1}$ and $t_{\varepsilon_1}, \dots, t_{\varepsilon_r}$, which satisfy the following relations:

$$\sigma_i t_{\varepsilon_j} \sigma_i = t_{\sigma_i(\varepsilon_j)} \quad \text{for } i = 1, \dots, r-1 \text{ and } j = 1, \dots, r.$$

Hence the set of generators is not minimal, e.g. one can obtain any t_{ε_j} for $j = 2, \dots, r$ by conjugating t_{ε_1} by certain reflections.

There is another presentation of $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$, which is important for this paper. It involves the following specific element

$$\rho = t_{\varepsilon_1} \sigma_1 \dots \sigma_{r-1},$$

which acts on a level zero $\widehat{\mathfrak{gl}}_r$-weight $\kappa = (\bar{\kappa}, 0, m)$ by

$$\rho(\bar{\kappa}, 0, m) = ((\kappa_r, \kappa_1, \dots, \kappa_{r-1}), 0, m - \kappa_r).$$

The action of its inverse $\rho^{-1} = \sigma_{r-1}^{-1} \dots \sigma_1^{-1} t_{-\varepsilon_1}$ is given by

$$\rho^{-1}(\bar{\kappa}, 0, m) = ((\kappa_2, \dots, \kappa_r, \kappa_1), 0, m + \kappa_1).$$

One then sees that $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ is generated by

$$\sigma_1, \dots, \sigma_r, \rho,$$

subject to the relations

$$\sigma_i^2 = 1, \qquad \text{for } i = 1, \dots, r, \tag{1.1a}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad \text{for distant } i, j = 1, \dots, r, \tag{1.1b}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \text{for } i = 1, \dots, r, \tag{1.1c}$$

$$\rho \sigma_i \rho^{-1} = \sigma_{i+1}, \qquad \text{for } i = 1, \dots, r, \tag{1.1d}$$

where the indices have to be understood modulo r, as before. We say that i and j are *distant* if $j \not\equiv i \pm 1 \mod r$. Using this set of generators, any element $w \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ can be written in the following way:

$$w = \rho^k w' = \rho^k \sigma_{i_1} \cdots \sigma_{i_l} \tag{1.2}$$

where $k \in \mathbb{Z}$ is unique and $\sigma_{i_1} \cdots \sigma_{i_l}$ is a reduced expression of the element $w' \in \mathcal{W}_{\widehat{A}_{r-1}}$.

Note that the conventions here are opposite to the ones chosen by Doty and Green [6].

1.4. The extended affine braid group and Hecke algebra. One can form the *extended affine braid group* $\widehat{\mathcal{B}}_{\hat{A}_{r-1}}$ associated to $\widehat{W}_{\hat{A}_{r-1}}$. It admits the same presentation (1.1a)–(1.1d) as $\widehat{W}_{\hat{A}_{r-1}}$ except that one omits the involutivity relations (1.1a) of the generators σ_i for $i = 1, \dots, r$.

One can also define the *extended affine Hecke algebra* $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$, which is the quotient of the $\mathbb{Q}(q)$-group algebra of $\widehat{\mathcal{B}}_{\hat{A}_{r-1}}$ by the relations

$$T_{\sigma_i}^2 = (q^2-1)T_{\sigma_i} + q^2, \quad \text{for all } i = 1, \dots, r,$$

with q being a formal parameter. For more details about this algebra, see [30, 31, 4, 32, 5].

A $\mathbb{Q}(q)$-basis of $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$ is given by the set

$$\{T_w, w \in \widehat{W}_{\hat{A}_{r-1}}\},$$

where

$$T_w = T_\rho^k T_{w'} = T_\rho^k T_{\sigma_{i_1}} \cdots T_{\sigma_{i_l}},$$

with w, k, w' and σ_{i_j} as in (1.2). See [12] and [6].

The above shows that $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$ is generated by

$$\{T_\rho, T_{\rho^{-1}}, T_{\sigma_i}, i = 1, \dots, r\}$$

as an algebra. An alternative set of algebraic generators of $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$ is given by

$$\{T_\rho, T_\rho^{-1}, b_i, i = 1, \dots r\},$$

where the $b_i := C'_{\sigma_i} = q^{-1}(1 + T_{\sigma_i})$ are the Kazhdan–Lusztig generators. The relations satisfied by these generators are the following:

$$\begin{aligned} b_i^2 &= (q+q^{-1})b_i, && \text{for } i = 1, \dots, r, \\ b_i b_j &= b_j b_i, && \text{for distant } i, j = 1, \dots, r, \\ b_i b_{i+1} b_i + b_{i+1} &= b_{i+1} b_i b_{i+1} + b_i, && \text{for } i = 1, \dots, r, \\ T_\rho b_i T_\rho^{-1} &= b_{i+1}, && \text{for } i = 1, \dots, r. \end{aligned} \tag{1.3}$$

Let C'_w be the Kazhdan–Lusztig basis elements of the (non-extended) affine Hecke algebra associated to the affine Weyl group $W_{\hat{A}_{r-1}}$. Then by (1.3) it follows that

$$T_\rho C'_w T_\rho^{-1} = C'_{\rho w \rho^{-1}},$$

for any $w \in W_{\hat{A}_{r-1}}$. Therefore, $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$ has the following Kazhdan–Lusztig type basis

$$\{T_\rho^k C'_w, k \in \mathbb{Z} \text{ and } w \in W_{\hat{A}_{r-1}}\}, \tag{1.4}$$

with the usual positive integrality property.

2. A categorification of the extended affine Hecke algebra

Notation 2.1. Let $\mathcal{C}$ be a $\mathbb{Q}$-linear $\mathbb{Z}$-graded additive category (resp. 2-category) with translation (see Section 5.1 in [21] for the technical definitions). In all examples in this paper, the vector space of morphisms (resp. 2-morphisms) of any fixed degree is finite-dimensional.

If $\mathcal{C}$ is a category (resp. 2-category), its hom-spaces (resp. hom-categories) are denoted $\mathcal{C}(x, y)$, for any objects x, y in $\mathcal{C}$.

The Karoubi envelope of $\mathcal{C}$ is denoted by $\operatorname{Kar}\mathcal{C}$.

By $\mathcal{C}^*$ we denote the category (resp. 2-category) with the same objects (resp. same objects and 1-morphisms) as $\mathcal{C}$, but whose hom-spaces (resp. 2-hom-spaces) are defined by

$$\mathcal{C}^*(x, y) := \bigoplus_{t\in\mathbb{Z}} \mathcal{C}(x\{t\}, y).$$

These graded vector spaces of morphisms are referred to as HOM-spaces (resp. 2-HOM-spaces) following Khovanov and Lauda's notations, see [17, Remark 3.2].

A degree preserving functor $\mathcal{F}\colon\mathcal{C} \to \mathcal{D}$ between two such categories $\mathcal{C}$ and $\mathcal{D}$ lifts to a functor between the categories enriched in vector spaces $\mathcal{C}^* \to \mathcal{D}^*$ and to a functor between the Karoubi envelopes $\operatorname{Kar}\mathcal{C} \to \operatorname{Kar}\mathcal{D}$, which are both also denoted $\mathcal{F}$.

2.1. An extension of Soergel's categorification

2.1.1. Action on polynomial rings. Consider $R = \mathbb{Q}[y][x_1, \dots, x_r]$. The extended affine Weyl group $\widehat{W}_{\hat{A}_{r-1}}$ acts $\mathbb{Q}[y]$-linearly and faithfully on R as follows:

$$\rho(x_i) = \begin{cases} x_{i+1} & \text{for } i = 1, \dots, r-1, \\ x_1 - y & \text{for } i = r, \end{cases}$$

$$\rho^{-1}(x_i) = \begin{cases} x_{i-1} & \text{for } i = 2, \dots, r, \\ x_r + y & \text{for } i = 1, \end{cases}$$

$$t_{\varepsilon_j}(x_i) = \begin{cases} x_j - y & \text{for } i = j, \\ x_i & \text{otherwise}, \end{cases}$$

$$\sigma_j(x_i) = \begin{cases} x_{j+1} & \text{for } i = j, \\ x_j & \text{for } i = j+1, \quad \text{for } j = 1, \dots, r-1, \\ x_i & \text{otherwise}, \end{cases}$$

$$\sigma_r(x_i) = \begin{cases} x_r + y & \text{for } i = 1, \\ x_1 - y & \text{for } i = r, \\ x_i & \text{otherwise.} \end{cases}$$

Note that in particular ρ^r simply translates all x_i's by $-y$ and that multiplication by y commutes with the action of $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$ on R.

Remark 2.2. The action above naturally extends the action of the (non-extended) affine Weyl group $\mathcal{W}_{\widehat{A}_{r-1}}$. We are working here with Soergel's original reflection faithful realization of the affine type A Coxeter system $\mathcal{W}_{\widehat{A}_{r-1}}$ considered in [35] and [13]. Indeed, when generalized to the extended affine Weyl group $\widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$, this representation remains faithful which explains why we choose to use this precise realization to achieve a categorification of the extended affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ in the present paper.

Let us set $X_i = x_{i+1} - x_i$ for $i = 1, \dots, r-1$ and $X_r = x_1 - x_r - y$.

2.1.2. Extended Soergel bimodules. We introduce a grading on R, inducing a grading on the R-bimodules considered in the sequel, by setting

$$\deg(y) = \deg(x_k) = 2$$

for all $k = 1, \dots, r$. Curly brackets will indicate a shift of the grading: if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a $\mathbb{Z}$-graded bimodule and p an integer, then the $\mathbb{Z}$-graded bimodule $M\{p\}$ is defined by $M\{p\}_i = M_{i-p}$ for all $i \in \mathbb{Z}$.

For any $i = 1, \dots, r$, we define the R-bimodule

$$B_i = R \otimes_{R^{\sigma_i}} R\{-1\}$$

where R^{σ_i} is the subalgebra of elements of R fixed by the reflection $\sigma_i \in \widehat{\mathcal{W}}_{\widehat{A}_{r-1}}$:

$$R^{\sigma_i} = \mathbb{Q}[y][x_1, \dots, x_i + x_{i+1}, x_i x_{i+1}, \dots, x_r] \quad \text{for } i = 1, \dots, r-1,$$

$$R^{\sigma_r} = \mathbb{Q}[y][x_2, \dots, x_{r-1}, x_r + x_1, (x_r + y/2)(x_1 - y/2)].$$

We also define the *twisted R-bimodule* B_ρ (resp. $B_{\rho^{-1}}$), which coincides with R as a left R-module but is twisted by ρ (resp. by ρ^{-1}) as a right R-module, i.e. any $a \in R$ acts on B_ρ on the right by multiplication by $\rho(a)$ (resp. $\rho^{-1}(a)$).

Form now the category monoidally generated by the graded R-bimodules defined above. Then allow direct sums and grading shifts of these objects and consider only morphisms which are degree preserving morphisms of R-bimodules, and denote this category $\mathcal{E}\mathcal{B}im_{\widehat{A}_{r-1}}$. Its Karoubi envelope $\mathrm{Kar}\,\mathcal{E}\mathcal{B}im_{\widehat{A}_{r-1}}$ is a $\mathbb{Q}$-linear graded additive monoidal category with translation, which we call *the category of extended Soergel bimodules of type* $\widehat{A}_{r-1}$.

As mentioned in Remark 2.2, we are precisely using Soergel's original realization, so the bimodules B_i considered here are the ones constructed by Härterich [13] and Soergel [35]. Therefore Soergel's category $\mathrm{Kar}\,\mathcal{B}im_{\widehat{A}_{r-1}}$ of affine type A is equivalent to the full subcategory of $\mathrm{Kar}\,\mathcal{E}\mathcal{B}im_{\widehat{A}_{r-1}}$ generated by the B_i, for $i = 1, \ldots, r$. Let us recall a special case of Härterich's categorification result [13]:

Theorem 2.3 (Härterich). *We have*

$$\mathcal{H}_{\widehat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\,\mathcal{B}im_{\widehat{A}_{r-1}})$$

where

$$K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\,\mathcal{B}im_{\widehat{A}_{r-1}}) := K_0(\mathrm{Kar}\,\mathcal{B}im_{\widehat{A}_{r-1}}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q).$$

For each $w \in \mathcal{W}_{\widehat{A}_{r-1}}$, there exists a unique indecomposable bimodule B_w in $\mathrm{Kar}\,\mathcal{B}im_{\widehat{A}_{r-1}}$ *such that, under the isomorphism above, the Kazhdan–Lusztig basis element C'_w is mapped to $[B_w]$. Conversely, any indecomposable bimodule in this category is isomorphic to $B_w\{t\}$, for a certain $w \in \mathcal{W}_{\widehat{A}_{r-1}}$ and a certain grading shift $t \in \mathbb{Z}$.*

In the rest of the paper, we keep the convention that subscripts are considered to be modulo r, e.g. the bimodule B_{r+1} is by definition equal to B_1. We will also use the notation $B_\rho^{\otimes k}$ for any $k \in \mathbb{Z}$, where this bimodule is defined to be the tensor product of $|k|$ copies of B_ρ if $k \geq 0$ and of $B_{\rho^{-1}}$ if $k \leq 0$. In both cases $B_\rho^{\otimes k}$ is isomorphic to B_{ρ^k}.

2.1.3. Categorification of $\mathcal{H}_{\widehat{A}_{r-1}}$

Lemma 2.4. *For any $i = 1, \ldots, r$, there exists an R-bimodule isomorphism*

$$B_\rho \otimes_R B_i \cong B_{i+1} \otimes_R B_\rho. \tag{2.1}$$

Applying these isomorphisms r times gives an isomorphism

$$B_\rho^{\otimes r} \otimes_R B_i \cong B_i \otimes_R B_\rho^{\otimes r}, \tag{2.2}$$

for any $i = 1, \ldots, r$.

Proof. • *Isomorphism* (2.1). First note that there exist natural isomorphisms of R-bimodules:

$$B_\rho \otimes_R B_i \cong B_\rho \otimes_{R^{\sigma_i}} R\{-1\} \quad \text{and} \quad B_{i+1} \otimes_R B_\rho \cong R \otimes_{R^{\sigma_{i+1}}} B_\rho\{-1\}.$$

Define the isomorphism of R-bimodules $\psi\colon B_\rho \otimes_{R^{\sigma_i}} R \to R \otimes_{R^{\sigma_{i+1}}} B_\rho$ by

$$\psi(a \otimes b) = a \otimes \rho(b).$$

This isomorphism is well–defined, because ρ defines an isomorphism between R^{σ_i} and $R^{\sigma_{i+1}}$.

• *Isomorphism* (2.2). Note that $B_\rho^{\otimes r} \cong B_{\rho^r}$ and that ρ^r leaves the ring R^{σ_i} invariant. □

Theorem 2.5. *The category* $\mathrm{Kar}\,\mathcal{EBim}_{\hat{A}_{r-1}}$ *categorifies the extended affine Hecke algebra, i.e.*

$$\widehat{\mathcal{H}}_{\hat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\,\mathcal{EBim}_{\hat{A}_{r-1}}),$$

where

$$K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\,\mathcal{EBim}_{\hat{A}_{r-1}}) := K_0(\mathrm{Kar}\,\mathcal{EBim}_{\hat{A}_{r-1}}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q).$$

Under this isomorphism, the set of isomorphism classes of the indecomposables $\{B_\rho^k B_w, k \in \mathbb{Z}$ *and* $w \in \mathcal{W}_{\hat{A}_{r-1}}\}$ *in* $\mathrm{Kar}\,\mathcal{EBim}_{\hat{A}_{r-1}}$ *corresponds exactly to the Kazhdan–Lusztig basis* (1.4) *of* $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$*, with* B_w *as in Theorem* 2.3*. In particular,* $[B_i]$ *corresponds to* b_i *and* $[B_{\rho^{\pm 1}}]$ *corresponds to* $T_\rho^{\pm 1}$.

Proof. Recall that

$$\mathcal{H}_{\hat{A}_{r-1}} \cong K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\,\mathcal{Bim}_{\hat{A}_{r-1}}),$$

by Theorem 2.3. The indecomposables in $\mathrm{Kar}\,\mathcal{Bim}_{\hat{A}_{r-1}}$, which are denoted B_w for $w \in \mathcal{W}_{\hat{A}_{r-1}}$, correspond exactly to the Kazhdan–Lusztig basis elements $\{C'_w, w \in \mathcal{W}_{\hat{A}_{r-1}}\}$ of $\mathcal{H}_{\hat{A}_{r-1}}$. Moreover, B_w appears as a direct summand of the tensor product $B_{i_1} \otimes_R \cdots \otimes_R B_{i_k}$ where $\sigma_{i_1} \ldots \sigma_{i_k}$ is a reduced expression of w.

We define the homomorphism of algebras

$$\widehat{\mathcal{H}}_{\hat{A}_{r-1}} \longrightarrow K_0^{\mathbb{Q}(q)}(\mathrm{Kar}\,\mathcal{EBim}_{\hat{A}_{r-1}})$$

by

$$b_i \longmapsto [B_i\{-1\}] \quad \text{and} \quad T_\rho^{\pm 1} \longmapsto [B_{\rho^{\pm 1}}].$$

The homomorphism is well-defined, as follows from the following isomorphisms in $\mathcal{E}\mathcal{B}im_{\hat{A}_{r-1}}$:

$$
\begin{aligned}
B_i \otimes_R B_i &\cong B_i \oplus B_i\{2\}, \\
B_i \otimes_R B_j &\cong B_j \otimes_R B_i, \quad \text{for distant } i, j, \\
B_i \otimes_R B_{i+1} \otimes_R B_i \oplus B_{i+1}\{2\} &\cong B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \oplus B_i\{2\}, \\
B_\rho \otimes_R B_i &\cong B_{i+1} \otimes_R B_\rho,
\end{aligned}
\tag{2.3}
$$

for $i, j = 1, \dots, r$.

Note that any tensor product of $B_{\rho^{\pm 1}}$'s and B_i's can be rewritten in the following way

$$B_\rho^{\otimes k} \otimes_R B_{i_1} \otimes_R \cdots \otimes_R B_{i_l},$$

by sliding all the $B_{\rho^{\pm 1}}$'s to the left using the isomophism (2.3).

Let us now look at the indecomposables of the category Kar $\mathcal{E}\mathcal{B}im_{\hat{A}_{r-1}}$. First observe that if the bimodule M is indecomposable in Kar $\mathcal{E}\mathcal{B}im_{\hat{A}_{r-1}}$ then, for any $k \in \mathbb{Z}$, the tensor product $B_\rho^{\otimes k} \otimes_R M$ is indecomposable as well. Indeed assume that

$$B_\rho^{\otimes k} \otimes_R M \cong P \oplus Q,$$

then tensoring on the left by $B_\rho^{\otimes -k}$ gives

$$M \cong B_\rho^{\otimes -k} \otimes_R P \oplus B_\rho^{\otimes -k} \otimes_R Q,$$

which contradicts the fact that M is supposed to be indecomposable.

So let M be an indecomposable of Kar $\mathcal{E}\mathcal{B}im_{\hat{A}_{r-1}}$. It is a direct summand of some tensor product $B_\rho^{\otimes k} \otimes_R B_{i_1} \otimes_R \cdots \otimes_R B_{i_l}$. Then the indecomposable bimodule $B_\rho^{\otimes -k} \otimes_R M$ is a direct summand of $B_{i_1} \otimes_R \cdots \otimes_R B_{i_l}$. The latter tensor product belongs to the subcategory Kar $\mathcal{B}im_{\hat{A}_{r-1}}$ of Kar $\mathcal{E}\mathcal{B}im_{\hat{A}_{r-1}}$. Thus $B_\rho^{\otimes -k} \otimes_R M$ is of the form B_w for some $w \in \mathcal{W}_{\hat{A}_{r-1}}$. We can conclude that the indecomposables of the category Kar $\mathcal{E}\mathcal{B}im_{\hat{A}_{r-1}}$ are all of the form

$$B_\rho^{\otimes k} \otimes_R B_w \quad \text{for } k \in \mathbb{Z} \text{ and } w \in \mathcal{W}_{\hat{A}_{r-1}}.$$

Their Grothendieck classes correspond bijectively to the elements of

$$\{T_\rho^k C'_w, k \in \mathbb{Z} \text{ and } w \in \mathcal{W}_{\hat{A}_{r-1}}\},$$

which is precisely the Kazhdan–Lusztig basis of $\widehat{\mathcal{H}}_{\hat{A}_{r-1}}$ in (1.4). □

Proposition 2.6. *For any $k, l, i_1, \dots, i_m, j_1, \dots, j_n$, we have*

$$\begin{aligned}&\mathcal{E}\mathcal{B}im^*_{\widehat{A}_{r-1}}(B_\rho^{\otimes k} \otimes_R B_{i_1} \otimes_R \cdots \otimes_R B_{i_m}, B_\rho^{\otimes l} \otimes_R B_{j_1} \otimes_R \cdots \otimes_R B_{j_n})\\ &\quad\cong \delta_{k,l}\, \mathcal{B}im^*_{\widehat{A}_{r-1}}(B_{i_1} \otimes_R \cdots \otimes_R B_{i_m}, B_{j_1} \otimes_R \cdots \otimes_R B_{j_n}),\end{aligned}$$

where $\delta_{k,l}$ is the Kronecker delta.

Proof. In the category $\mathcal{E}\mathcal{B}im_{\widehat{A}_{r-1}}$, the B_i are self-adjoint and B_ρ and B_ρ^{-1} form a biadjoint pair. Therefore, there exist isomorphisms

$$\begin{aligned}&\mathcal{E}\mathcal{B}im^*_{\widehat{A}_{r-1}}(B_\rho^{\otimes k} \otimes_R B_{i_1} \otimes_R \cdots \otimes_R B_{i_m}, B_\rho^{\otimes l} \otimes_R B_{j_1} \otimes_R \cdots \otimes_R B_{j_n})\\ &\quad\cong \mathcal{E}\mathcal{B}im^*_{\widehat{A}_{r-1}}(R, B_\rho^{\otimes l-k} \otimes_R B_{j_1} \otimes_R \cdots \otimes_R B_{j_n} \otimes_R B_{i_m} \otimes_R \cdots \otimes_R B_{i_1}).\end{aligned}$$

The latter HOM-space is equal to zero except when $k = l$, in which case it is isomorphic to the corresponding HOM-space in $\mathcal{B}im^*_{\widehat{A}_{r-1}}$.

Indeed any morphism

$$f \in \mathcal{E}\mathcal{B}im^*_{\widehat{A}_{r-1}}(R, B_\rho^{\otimes l-k} \otimes_R B_{j_1} \otimes_R \cdots \otimes_R B_{j_n} \otimes_R B_{i_m} \otimes_R \cdots \otimes_R B_{i_1})$$

is completely determined by the image p of 1. Since f is a morphism of R-bimodules, we have $pa = ap$ for any $a \in R$. In particular, for $a = \sum_{i=1}^r x_i$ we have

$$p\Big(\sum_{i=1}^r x_i\Big) = \Big(\sum_{i=1}^r x_i\Big)p.$$

Since $\sum_{i=1}^r x_i$ is invariant under all the reflections σ_j for $j = 1, \dots, r$, we also have

$$p\Big(\sum_{i=1}^r x_i\Big) = \rho^{l-k}\Big(\sum_{i=1}^r x_i\Big)p = \Big(\sum_{i=1}^r x_i - (l-k)y\Big)p.$$

This implies that p, and therefore the morphism f, has to be zero unless $k = l$. □

2.2. The diagrammatic version. The category of Soergel bimodules of finite type A is described via planar diagrams by Elias and Khovanov in [8]. They associate planar diagrams to certain generating bimodule maps and give a complete set of relations on them. Elias and Williamson [9] worked out the generalization of the diagrammatic approach to Soergel bimodules for any Coxeter group which does not contain a standard parabolic subgroup isomorphic to H_3.

Our aim is to define and study an extension of Elias and Williamson's diagrammatic category for extended affine type A [9]. In affine type A Elias and Williamson's diagrammatic category is a straightforward generalization of Elias and Khovanov's original category, which was defined for finite type A. In addition to Elias and Williamson's diagrams, we also have to introduce a new type of strand. These new strands are oriented and their endpoints are labeled $+$ or $-$ depending on orientations. The Karoubi envelope of the diagrammatic category $\mathcal{DEB}im_{\hat{A}_{r-1}}$ obtained in this way is equivalent to the category of extended Soergel bimodules Kar $\mathcal{EB}im_{\hat{A}_{r-1}}$ of affine type A, as we will show.

2.2.1. Definition of $\mathcal{DEB}im_{\hat{A}_{r-1}}$. First start with the category whose objects are graded finite sequences of integers belonging to $\{1, \ldots, r\}$ and the symbols $+$ and $-$. Graphically we represent these sequences by sequences of colored points (read from left to right) of the x-axis of the real plane $\mathbb{R}^2$. The morphisms are then equivalence classes of $\mathbb{Q}$-linear combinations of graded planar diagrams in $\mathbb{R} \times [0, 1]$ (read from bottom to top) and composition is defined by vertically glueing the diagrams and rescaling the vertical coordinate. These morphisms are defined by generators and relations listed below. This category possesses a monoidal structure given by stacking sequences and diagrams next to each other.

Let $\mathcal{DEB}im_{\hat{A}_{r-1}}$ be the category containing all direct sums and grading shifts of these objects and let its morphisms be the degree preserving diagrams. The diagrammatic extended Soergel category is by definition its Karoubi envelope Kar $\mathcal{DEB}im_{\hat{A}_{r-1}}$.

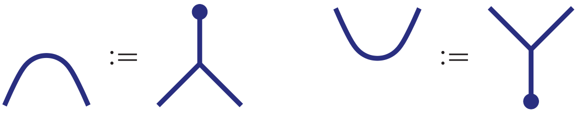
In the diagrams, the strands whose endpoints are $+$ or $-$-signs are oriented and the other strands are non-oriented. The non-oriented strands can be colored with integers belonging to $\{1, \ldots, r\}$. Two colors i and j are called *adjacent* (resp. *distant*) if $i \equiv j \pm 1 \mod r$ (resp. $i \not\equiv j \pm 1 \mod r$). By convention, no label means that the equation holds for any color $i \in \{1, \ldots, r\}$.

The morphisms of $\mathcal{DEB}im_{\hat{A}_{r-1}}$ are built out of the following generating diagrams. The non-oriented diagrams are the affine analogues of Elias and Khovanov's diagrams, the ones involving oriented strands are new.

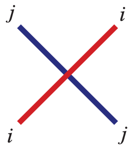
- Generators involving only one color:

Degree	1	1	-1	-1
	enddot_i	startdot_i	merge_i	split_i

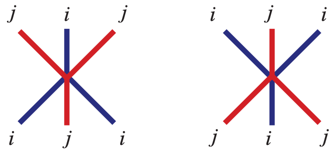
It is useful to define the cap and cup as follows:



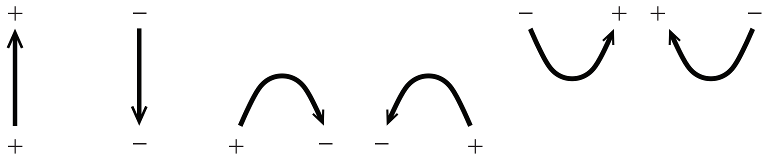
- Generators involving two colors:
  - the 4-valent vertex with distant colors, of degree 0, denoted $4\,\mathrm{vert}_{i,j}$



  - and the 6-valent vertices with adjacent colors i and j, of degree 0, denoted $6\,\mathrm{vert}_{i,j}$ and $6\,\mathrm{vert}_{j,i}$

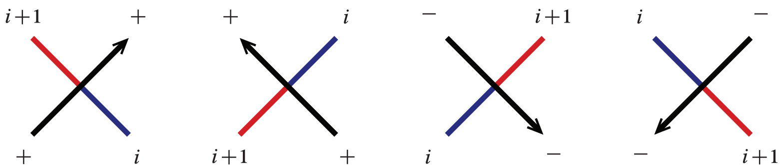


- Generators involving only oriented strands, of degree 0:



$+$strand $-$strand $+$cap $-$cap $-$cup $+$cup

- Generators involving oriented strands and adjacent colored strands. The mixed 4-valent vertex of degree 0:



$4\,\mathrm{vert}_{+,i}$ $4\,\mathrm{vert}_{i+1,+}$ $4\,\mathrm{vert}_{i,-}$ $4\,\mathrm{vert}_{-,i+1}$

- Generators involving boxes, of degree 2, denoted box_i

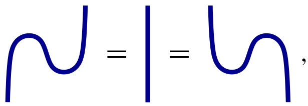
$$\boxed{i}$$

for all $i = 1, \ldots, r$, and denoted box_y

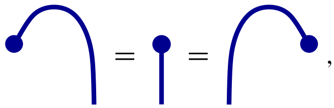
$$\boxed{y}$$

The generating diagrams are subject to the following relations. The relations involving only non-oriented diagrams are the obvious affine analogues of Elias and Khovanov's relations, the ones with diagrams involving oriented strands are new. The labels of the colored strands are omitted here, but the reader can figure what they are from which generating morphisms are involved in the relations.

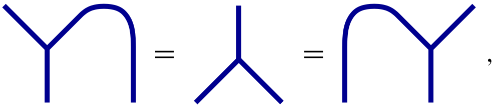
- Isotopy relations:



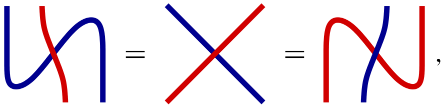
(2.4a)



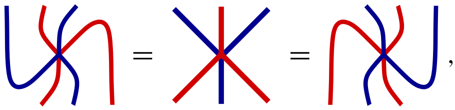
(2.4b)



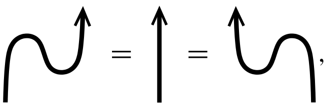
(2.4c)



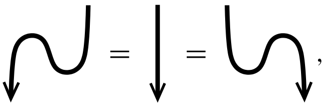
(2.4d)



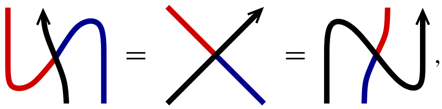
(2.4e)



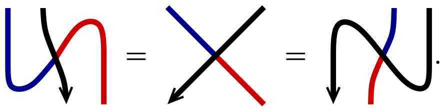
(2.4f)



(2.4g)

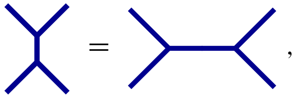


(2.4h)

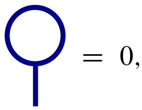


(2.4i)

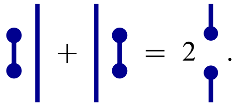
- Relations involving one color:



(2.5a)

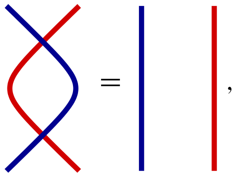


(2.5b)

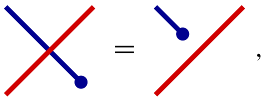


(2.5c)

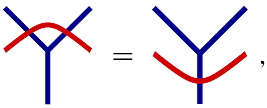
- Relations involving two distant colors:



(2.6a)

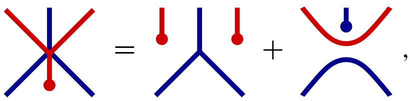


(2.6b)

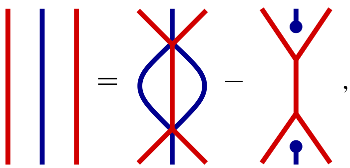


(2.6c)

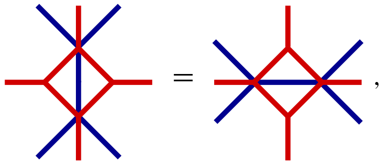
- Relations involving two adjacent colors:



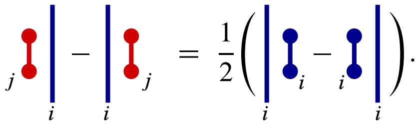
(2.7a)



(2.7b)

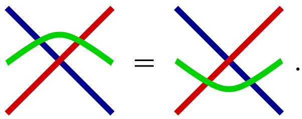


(2.7c)



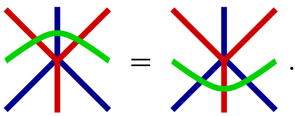
(2.7d)

- Relation involving three distant colors (i.e. three colors forming a parabolic subgroup of type $A_1 \times A_1 \times A_1$):



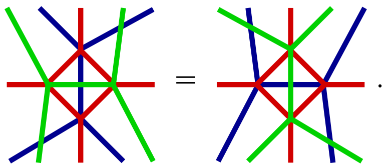
(2.8)

- Relation involving two adjacent colors and one distant from the other two (i.e. three colors forming a parabolic subgroup of type $A_2 \times A_1$):



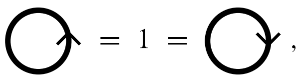
(2.9)

- Relation involving three adjacent colors forming a parabolic subgroup of type A_3 (i.e. , the case $\widehat{A}_2$ is excluded):

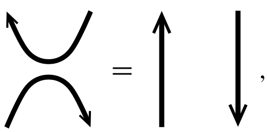


(2.10)

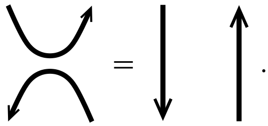
- Relations involving only oriented strands:



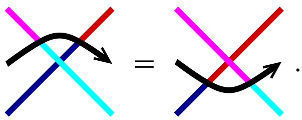
(2.11a)



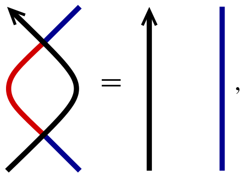
(2.11b)

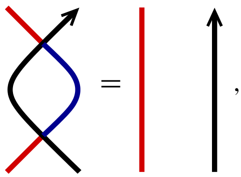
 (2.11c)

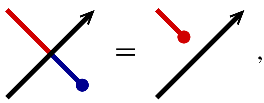
- Relations involving oriented strands and distant colored strands:

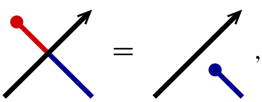
 (2.12)

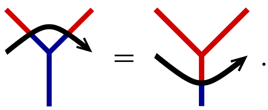
- Relations involving oriented strands and two adjacent colored strands:

 (2.13a)

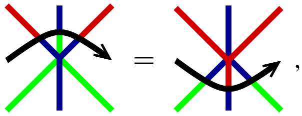
 (2.13b)

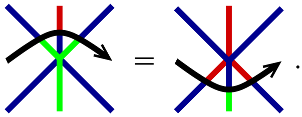
 (2.13c)

 (2.13d)

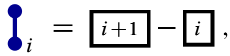
 (2.13e)

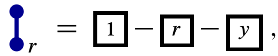
- Relations involving oriented strands and three adjacent colored strands:

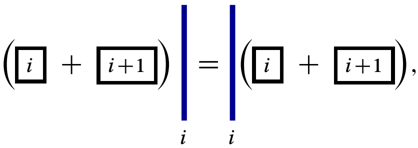

(2.14a)

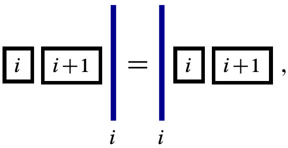

(2.14b)

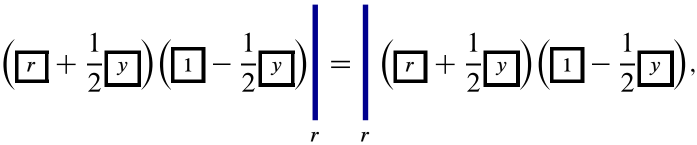
- Relations involving boxes:

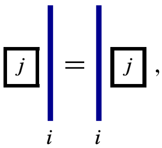

for $i \neq r$, (2.15a)

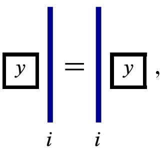

(2.15b)

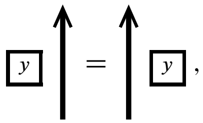

(2.15c)


for $i \neq r$, (2.15d)

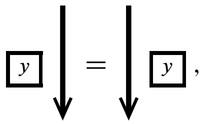

(2.15e)


for $j \neq i, i+1$, (2.15f)

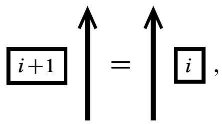

(2.15g)



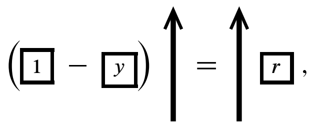
(2.15h)



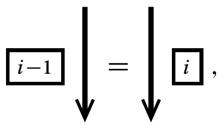
(2.15i)



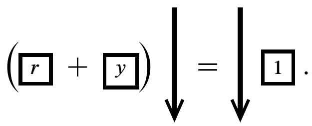
for $i \neq r$, (2.15j)



(2.15k)



for $i \neq 1$, (2.15l)



(2.15m)

Remark 2.7. Note that the relations (2.12)–(2.14b) also hold when the oriented strand has the opposite orientation. This follows from relations (2.12)–(2.14b) and relations of isotopy.

Note also that the generators and relations listed above are redundant. This redundancy helps us to simplify some of the proofs later on. More specifically, let us list some of these unnecessary generators and relations:

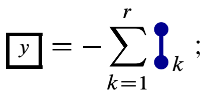
(i) relation (2.11c) follows from relation (2.11b) and isotopy invariance;

(ii) relation (2.15l) follows from relation (2.15j) and isotopy invariance;

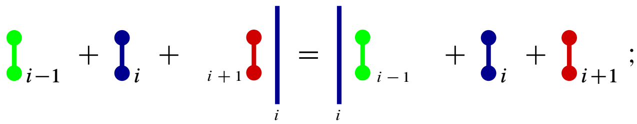
(iii) relation (2.15k) follows from relations (2.15j) for $i = r - 1$, (2.15a) for $i = r - 1$, (2.15g), (2.13c), and (2.13d);

(iv) relation (2.15m) follows from relations (2.15i), (2.15k), and isotopy invariance;

(v) if we sum relation (2.15a) for all $i = 1, \dots, r-1$ and relation (2.15b), we obtain that



(vi) if we sum relation (2.7d) for $j = i-1$ and for $j = i+1$, we obtain that

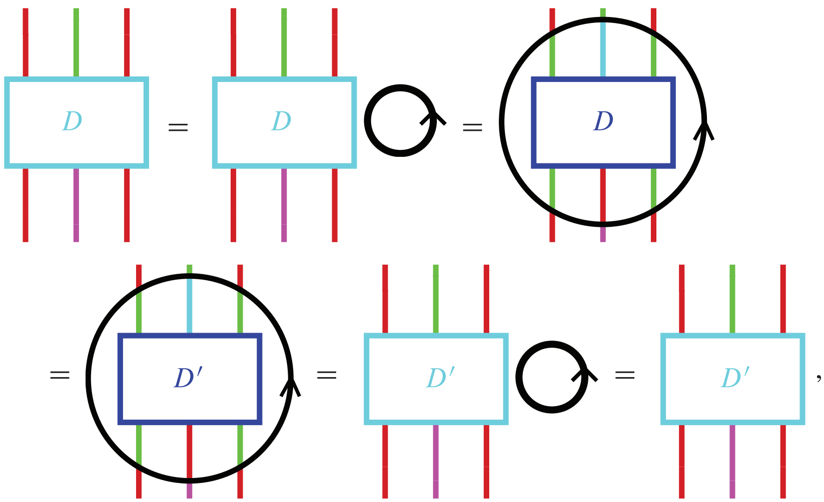


(vii) relation (2.15g) follows from relation (2.6b), the two relations exhibited just above in (v) and (vi) and isotopy invariance;

(viii) relation (2.15h) follows from relations (2.13c) and (2.13d) and the relation exhibited in (v);

(ix) relation (2.15i) follows from relation (2.15h) and isotopy invariance;

(x) relations (2.11a)–(2.14b) imply the following: suppose that two diagrams without boxes D and D' are equal, then the diagrams obtained by shifting all their colors by -1 (mod r) are also equal. Indeed,



where the first and last equalities follow from relation (2.11a), the second and one but last equalities follow from relations (2.12)–(2.14b), and the third equality is our assumption.

As a consequence, provided that relations (2.5a)–(2.10) are each satisfied for one set of colors, they are each satisfied for all the colors. In particular, since none of them involve all the colors $\{1,\dots,r\}$ at the same time, relations (2.5a)–(2.10) with strand(s) colored r follow by this argument from the same relations without strands colored r.

2.2.2. Functor from $\mathcal{DEBim}^*_{\widehat{A}_{r-1}}$ to $\mathcal{EBim}^*_{\widehat{A}_{r-1}}$. Let us construct a degree preserving functor

$$\mathcal{F}\colon \mathcal{DEBim}^*_{\widehat{A}_{r-1}} \longrightarrow \mathcal{EBim}^*_{\widehat{A}_{r-1}}$$

which extends the one by Elias and Khovanov. On objects, it is defined as follows: it maps each integer $i \in \{1,\dots,r\}$ to B_i, and the symbols $+$ and $-$ to B_ρ and $B_{\rho^{-1}}$ respectively. Sequences of these are mapped to tensor products. The empty sequence is sent to R.

For the morphisms one only needs to specify $\mathcal{F}$ on the generators. A sequence of vertical strands is mapped to the identity of the corresponding bimodule and box_i (resp. box_y) is mapped to multiplication by x_i (resp. y). We define

$$\mathcal{F}(\mathrm{enddot}_i) = \mathrm{br}_i\colon a \otimes b \longmapsto ab,$$

$$\mathcal{F}(\mathrm{startdot}_i) = \mathrm{rb}_i\colon a \longmapsto \frac{a}{2}(X_i \otimes 1 + 1 \otimes X_i),$$

$$\mathcal{F}(\mathrm{merge}_i) = \mathrm{pr}_i\colon \begin{cases} a \otimes 1 \otimes b \longmapsto 0, \\ a \otimes X_i \otimes b \longmapsto 2a \otimes b, \end{cases}$$

$$\mathcal{F}(\mathrm{split}_i) = \mathrm{inj}_i\colon a \otimes b \longmapsto a \otimes 1 \otimes b,$$

$$\mathcal{F}(4\,\mathrm{vert}_{i,j}) = \mathrm{f}_{i,j}\colon a \otimes 1 \otimes b \longmapsto a \otimes 1 \otimes b,$$

$$\mathcal{F}(6\,\mathrm{vert}_{i,i+1}) = \mathrm{f}_{i,i+1}\colon \begin{cases} a \otimes 1 \otimes 1 \otimes b \longmapsto a \otimes 1 \otimes 1 \otimes b, \\ a \otimes (X_{i+1} \otimes 1 + 1 \otimes X_{i+1}) \otimes b \longmapsto 0, \end{cases}$$

$$\mathcal{F}(6\,\mathrm{vert}_{i+1,i}) = \mathrm{f}_{i+1,i}\colon \begin{cases} a \otimes 1 \otimes 1 \otimes b \longmapsto a \otimes 1 \otimes 1 \otimes b, \\ a \otimes (X_i \otimes 1 + 1 \otimes X_i) \otimes b \mapsto 0, \end{cases}$$

$$\mathcal{F}(+\,\mathrm{cap}) =_{+,-} \mathrm{r}\colon a \otimes b \longmapsto a\rho(b),$$

$$\mathcal{F}(-\,\mathrm{cap}) =_{-,+} \mathrm{r}\colon a \otimes b \longmapsto a\rho^{-1}(b),$$

$$\mathcal{F}(-\,\mathrm{cup}) = \mathrm{r}_{-,+}\colon a \longmapsto a \otimes 1,$$

$$\mathcal{F}(+\,\mathrm{cup}) = \mathrm{r}_{+,-}\colon a \longmapsto a \otimes 1,$$

$$\mathcal{F}(4\,\mathrm{vert}_{+,i}) = \mathrm{fl}_{+,i}\colon a \otimes b \mapsto a \otimes \rho(b),$$

$$\mathcal{F}(4\,\mathrm{vert}_{i+1,+}) = \mathrm{fl}_{i+1,+}\colon a \otimes b \mapsto a \otimes \rho^{-1}(b),$$

$$\mathcal{F}(4\,\mathrm{vert}_{i,-}) = \mathrm{fl}_{i,-}\colon a \otimes b \mapsto a \otimes \rho(b),$$

$$\mathcal{F}(4\,\mathrm{vert}_{-,i+1}) = \mathrm{fl}_{-,i+1}\colon a \otimes b \mapsto a \otimes \rho^{-1}(b),$$

$$\mathcal{F}(\mathrm{box}_i) = \mathrm{m}_i\colon a \longmapsto a x_i,$$

$$\mathcal{F}(\mathrm{box}_y) = \mathrm{m}_y\colon a \longmapsto a y,$$

see the end of Section 2.1.1 for the definition of the X_i's.

Proposition 2.8. *The functor $\mathcal{F}$ is well-defined, degree preserving and essentially surjective.*

Proof. The fact that the functor $\mathcal{F}$ is well-defined and degree preserving amounts to a straightforward verification that it preserves the relations (2.4a)–(2.15m) and the degrees of the morphisms. For the relations involving only non-oriented strands, this is completely analogous to Elias and Khovanov's case. For the relations involving oriented strands, the calculations are new but easy.

Furthermore, in view of the definitions of the objects of $\mathcal{DEB}im^*_{\hat{A}_{r-1}}$ and $\mathcal{EB}im^*_{\hat{A}_{r-1}}$, the functor $\mathcal{F}$ is clearly essentially surjective. □

Proposition 2.9. *The functor $\mathcal{F}$ is full.*

Proof. Libedinsky has proved in [23] that all the morphisms of the category $\mathcal{B}im^*_{\hat{A}_{r-1}}$ are generated by the following ones:

- br_i, rb_i, pr_i and inj_i, for all $i = 1, \ldots, r$,
- $\mathrm{f}_{i,j}$, for all $i, j = 1, \ldots, r$ with $i \neq j$,
- multiplication by x_i, for all $i = 1, \ldots, r$.

In view of Proposition 2.6, this implies that all the morphisms of the category $\mathcal{EB}im^*_{\hat{A}_{r-1}}$ are generated by the ones listed by Libedinsky and copied above, together with multiplication by y plus $\mathrm{fl}_{+,i}$ and $\mathrm{fl}_{i+1,+}$, giving

$$B_\rho \otimes_R B_i \cong B_{i+1} \otimes_R B_\rho \quad \text{for } i = 1, \ldots, r,$$

and ${}_{+,-}\mathrm{r}$, ${}_{-,+}\mathrm{r}$, $\mathrm{r}_{-,+}$ and $\mathrm{r}_{+,-}$, giving

$$B_\rho \otimes_R B_{\rho^{-1}} \cong R \cong B_{\rho^{-1}} \otimes_R B_\rho.$$

Thus the functor $\mathcal{F}$ is full, since all the morphisms generating $\mathcal{EB}im^*_{\hat{A}_{r-1}}$ are in the image of $\mathcal{F}$. □

Theorem 2.10. *The categories* $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$ *and* $\mathcal{EB}im^*_{\widehat{A}_{r-1}}$ *are equivalent and so are their Karoubi envelopes* Kar $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ *and* Kar $\mathcal{EB}im_{\widehat{A}_{r-1}}$.

Proof. Only the faithfulness of $\mathcal{F}$ remains to be proved, which is an easy consequence of the far more general and important Theorem 6.28 in [9].

For a given object X in $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$, let $k_X \in \mathbb{Z}$ denote the difference between the number of pluses and minuses in X. Given two objects X, Y in $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$, the hom-space between X and Y is non-zero only if $k_X = k_Y$ (see Proposition 2.6). Any object X in $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$ is isomorphic to the object EX', where E is the string of signs $\text{sign}(k_X)^{|k_X|}$ and X' does not contain any signs, and is obtained from X by applying the commutation isomorphisms $(\pm, i) \cong (i \pm 1, \pm)$ and the isomorphisms $(\pm, \mp) \cong \emptyset$.

Besides, the mixed relations (2.12)–(2.14b) imply that an oriented strand can be pulled over any diagram involving only colored strands. In particular, any diagram with only signs at the boundary is equal to a diagram with only oriented strands times a closed colored diagram, i.e. times an element of R, cf [9]. Moreover, due to relations (2.11a)–(2.11c), all diagrams containing only oriented strands with a fixed boundary are equal. Therefore, the endomorphism ring of any string of signs is isomorphic to R.

Let D be a diagram representing a morphism from X to Y, such that $k_X = k_Y$. Since

$$\mathcal{DEB}im^*_{\widehat{A}_{r-1}}(\text{sign}(k_X)^{|k_X|}, \text{sign}(k_X)^{|k_X|}) \cong R$$

and HOM-spaces in $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$ have a structure of R-modules, it follows that

$$\mathcal{DEB}im^*_{\widehat{A}_{r-1}}(X, Y) \cong \mathcal{DEB}im^*_{\widehat{A}_{r-1}}(X', Y'),$$

Let us illustrate this by the example in Figure 1.

Since

$$\mathcal{DEB}im^*_{\widehat{A}_{r-1}}(X', Y') \cong \mathcal{DB}im^*_{\widehat{A}_{r-1}}(X', Y'),$$

the faithfulness of $\mathcal{F}$ follows from the faithfulness of Elias and Williamson's analogous functor for non-extended affine type A, which they proved in their aforementioned theorem. □

3. The affine Schur quotient

3.1. Notations. Let n, r be integers, with $n > r$ and $r \ge 3$, and let q be a formal parameter.

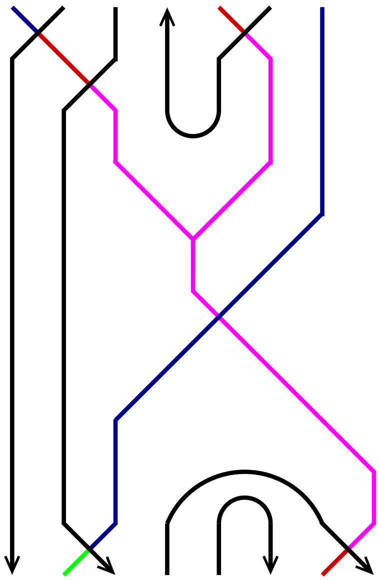


Figure 1. Example of a decomposition of a diagram D

A generic $\mathfrak{gl}_n$-weight will be denoted $\lambda = (\lambda_1, \dots, \lambda_n)$, and we set $\bar{\lambda}_i = \lambda_i - \lambda_{i+1}$. Define

$$\Lambda(n,r) = \left\{\lambda \in \mathbb{N}^n \colon \sum_{i=1}^{n} \lambda_i = r\right\}$$

and

$$\Lambda^+(n,r) = \{\lambda \in \Lambda(n,r) \colon r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$

We will denote by (1^r) the weight $\varepsilon_1 + \cdots + \varepsilon_r$.

Let us formally denote by $\bar{\alpha}_n$ the opposite of the highest root of $\mathfrak{sl}_n$, *i.e.* $\bar{\alpha}_n = -\bar{\theta} = -\bar{\alpha}_1 - \cdots - \bar{\alpha}_{n-1} = \varepsilon_n - \varepsilon_1$.

In the definitions below of the affine quantum enveloping algebras and the affine Schur algebras, we use the convention that the indices appearing in the relations are considered modulo n. Perhaps it is a bit confusing that until now indices were considered modulo r, but that was for the affine Hecke algebra.

3.2. The (extended) affine algebras. In the following definitions we do not need to consider the derivation, which we used above.

Definition 3.1. The *extended quantum general linear algebra* $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the associative unital $\mathbb{Q}(q)$-algebra generated by $R^{\pm 1}$, $K_i^{\pm 1}$ and $E_{\pm i}$, for $i = 1, \dots, n$, subject to the relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$E_i E_{-j} - E_{-j} E_i = \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}},$$

$$K_i E_{\pm j} = q^{\pm \langle \varepsilon_i, \bar{\alpha}_j \rangle} E_{\pm j} K_i,$$

$$E_{\pm i}^2 E_{\pm(i\pm 1)} - (q + q^{-1}) E_{\pm i} E_{\pm(i\pm 1)} E_{\pm i} + E_{\pm(i\pm 1)} E_{\pm i}^2 = 0,$$

$$E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i} = 0, \quad \text{for distant } i, j,$$

$$R R^{-1} = R^{-1} R = 1,$$

$$R X_i R^{-1} = X_{i+1} \quad \text{for } X_i \in \{E_{\pm i}, K_i^{-1}\}.$$

Definition 3.2. The *affine quantum general linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n) \subseteq \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is the unital $\mathbb{Q}(q)$-subalgebra generated by $E_{\pm i}$ and $K_i^{\pm 1}$, for $i = 1, \dots, n$.

The *affine quantum special linear algebra* $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n) \subseteq \mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ is the unital $\mathbb{Q}(q)$-subalgebra generated by $E_{\pm i}$ and $K_i K_{i+1}^{-1}$, for $i = 1, \dots, n$.

Remark 3.3. Note that $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ is the quantum group associated to the affine Lie algebra $\widehat{\mathfrak{gl}}_n$ without the well-known central extension. In other words, we will only consider level-zero representations in this paper. The same holds for $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$. The algebra $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is a Hopf algebra, so in that sense it can be considered to be a quantum group.

We will in fact only need the bialgebra structure on $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, so we do not explicit the antipode here.

Definition 3.4. $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ is a bialgebra with counit $\varepsilon\colon \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \to \mathbb{Q}(q)$ defined by

$$\varepsilon(E_{\pm i}) = 0, \quad \varepsilon(R^{\pm 1}) = \varepsilon(K_i^{\pm 1}) = 1$$

and coproduct $\Delta\colon \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \to \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \otimes \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, defined by

$$\Delta(1) = 1 \otimes 1,$$

$$\Delta(E_i) = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i,$$

$$\Delta(E_{-i}) = K_i^{-1} K_{i+1} \otimes E_{-i} + E_{-i} \otimes 1,$$

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1},$$

$$\Delta(R^{\pm 1}) = R^{\pm 1} \otimes R^{\pm 1}.$$

Note that Δ and ε can be restricted to $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ and $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$, which are bialgebras too.

At level 0 we can work with the $\mathbf{U}_q(\mathfrak{sl}_n)$-weight lattice, when considering $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$-weight representations. The corresponding root-lattice is degenerate, because $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0$, but that does not matter in this section. Similarly, we can work with the $\mathbf{U}_q(\mathfrak{gl}_n)$-weight lattice, when considering $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ and $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$-weight representations.

Suppose that V is a $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$-weight representation with weights $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, i.e.

$$V \cong \bigoplus_{\lambda} V_\lambda$$

and K_i acts as multiplication by q^{λ_i} on V_λ. Then V is also a $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$-weight representation with weights $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$ such that $\bar{\lambda}_j = \lambda_j - \lambda_{j+1}$ for $j = 1, \ldots, n-1$.

Conversely, there is not a unique choice of $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$-action on a given $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$-weight representation with weights $\mu = (\mu_1, \ldots, \mu_{n-1})$. We first have to fix the action of $K_1 \cdots K_n$. In terms of weights, this corresponds to the observation that, for any given $r \in \mathbb{Z}$ the equations

$$\lambda_i - \lambda_{i+1} = \mu_i, \tag{3.1a}$$

$$\sum_{i=1}^{n} \lambda_i = r \tag{3.1b}$$

determine $\lambda = (\lambda_1, \ldots, \lambda_n)$ uniquely, if there exists a solution to (3.1a) and (3.1b) at all. We therefore define the map $\varphi_{n,r} \colon \mathbb{Z}^{n-1} \to \mathbb{Z}^n \cup \{*\}$ by

$$\varphi_{n,r}(\mu) = \lambda$$

if (3.1a) and (3.1b) have a solution, and put $\varphi_{n,r}(\mu) = *$ otherwise.

As far as weight representations are concerned, we can restrict our attention to the Beilinson–Lusztig–MacPherson idempotented version of these quantum groups [1], denoted $\hat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$, $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ and $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ respectively. For each $\lambda \in \mathbb{Z}^n$ adjoin an idempotent 1_λ to $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ and add the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda,\mu} 1_\lambda, \\ E_{\pm i} 1_\lambda &= 1_{\lambda \pm \bar{\alpha}_i} E_{\pm i}, \\ K_i 1_\lambda &= q^{\lambda_i} 1_\lambda, \\ R 1_{(\lambda_1,\ldots,\lambda_n)} &= 1_{(\lambda_n,\lambda_1,\ldots,\lambda_{n-1})} R. \end{aligned}$$

Definition 3.5. The *idempotented extended affine quantum general linear algebra* is defined by

$$\hat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^n} 1_\lambda \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) 1_\mu .$$

Of course one defines $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \subset \hat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$ as the idempotented subalgebra generated by 1_λ and $E_{\pm i} 1_\lambda$, for $i = 1, \ldots, n$ and $\lambda \in \mathbb{Z}^n$.

Similarly for $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$, adjoin an idempotent 1_λ for each $\lambda \in \mathbb{Z}^{n-1}$ and add the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda,\mu} 1_\lambda, \\ E_{\pm i} 1_\lambda &= 1_{\lambda \pm \bar{\alpha}_i} E_{\pm i}, \\ K_i K_{i+1}^{-1} 1_\lambda &= q^{\lambda_i} 1_\lambda . \end{aligned}$$

Definition 3.6. The *idempotented quantum special linear algebra* is defined by

$$\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^{n-1}} 1_\lambda \mathbf{U}_q(\widehat{\mathfrak{sl}}_n) 1_\mu .$$

Any weight-representation of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$ or $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)$ is also a representation of $\hat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$, $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ or $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, respectively. This is not true for non-weight representations, of which there are many. There are also other differences of course, e.g. $\hat{\dot{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$, $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ and $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ are not unital, because they have infinitely many orthogonal idempotents. For that same reason, they are not bialgebras, although their action on tensor products of weight representations is well-defined.

3.3. The affine q-Schur algebra. Let us first recall Green's [12, 6] tensor space and the action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ on it. We will also recall some basic results about this action and add some of our own. The proofs of most results can be found in [6] and its references. Some other results cannot be found in the literature, but are probably well-known among experts, e.g. the inner product on tensor space.

Let V be the $\mathbb{Q}(q)$-vector space freely generated by $\{e_t \mid t \in \mathbb{Z}\}$.

Definition 3.7. The following defines an action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ on V :

$$\begin{aligned}
E_i e_{t+1} &= e_t && \text{if } i \equiv t \mod n, \\
E_i e_{t+1} &= 0 && \text{if } i \not\equiv t \mod n, \\
E_{-i} e_t &= e_{t+1} && \text{if } i \equiv t \mod n, \\
E_{-i} e_t &= 0 && \text{if } i \not\equiv t \mod n, \\
K_i^{\pm 1} e_t &= q^{\pm 1} e_t && \text{if } i \equiv t \mod n, \\
K_i^{\pm 1} e_t &= e_t && \text{if } i \not\equiv t \mod n, \\
R^{\pm 1} e_t &= e_{t\pm 1} && \text{for all } t \in \mathbb{Z}.
\end{aligned}$$

Note that V is clearly a weight representation of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, with e_t having weight ε_i, for $i \equiv t \mod n$. Therefore V is also a representation of $\dot{\widehat{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$.

From now on, let $r \in \mathbb{N}_{>0}$ be arbitrary but fixed. As usual, one extends the above action to $V^{\otimes r}$ using the coproduct in $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$. Again, this is a weight representation and therefore a representation of $\dot{\widehat{\mathbf{U}}}(\widehat{\mathfrak{gl}}_n)$, which we call Green's *tensor space*.

We also define a $\mathbb{Q}(q)$-bilinear form on V by $\langle e_s, e_t\rangle = \delta_{st}$, which extends to $V^{\otimes r}$ factorwise, i.e.

$$\langle v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_r\rangle := \langle v_1, w_1\rangle \cdots \langle v_r, w_r\rangle.$$

We clearly have the following non-degeneracy result.

Lemma 3.8. *For any* $0 \neq v \in V^{\otimes r}$, *we have*

$$\langle v, v\rangle \neq 0.$$

There is a right action of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ on $V^{\otimes r}$ which commutes with the left action of $\widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$. Its precise definition, which can be found in [12, 6], is not relevant here.

Definition 3.9. The *affine q-Schur algebra* $\widehat{\mathbf{S}}(n,r)$ is by definition the centralizing algebra

$$\mathrm{End}_{\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}}(V^{\otimes r}).$$

By affine Schur–Weyl duality (see [6] for example), the image of

$$\psi_{n,r} \colon \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \longrightarrow \mathrm{End}(V^{\otimes r})$$

is always isomorphic to $\widehat{\mathbf{S}}(n,r)$. If $n > r$, we can even restrict to $\mathbf{U}_q(\widehat{\mathfrak{sl}}_n) \subset \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$, i.e.

$$\psi_{n,r}(\mathbf{U}_q(\widehat{\mathfrak{sl}}_n)) \cong \widehat{\mathbf{S}}(n,r).$$

For $n = r$, this is no longer true.

Definition 3.10. Let $\eta \colon \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n) \to \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ be the $\mathbb{Q}(q)$-linear algebra anti-involution defined by

$$\eta(E_i) = qK_iK_{i+1}^{-1}E_{-i}, \quad \eta(E_{-i}) = qK_i^{-1}K_{i+1}E_i,$$
$$\eta(K_i) = K_i, \qquad \eta(R) = R^{-1},$$

for $1 \le i \le n$.

The proof of the following lemma is a straightforward check, which we leave to the reader.

Lemma 3.11. *We have*

$$\Delta\eta = (\eta \otimes \eta)\Delta.$$

Lemma 3.12. *For any $X \in \widehat{\mathbf{U}}_q(\widehat{\mathfrak{gl}}_n)$ and any $v, w \in V^{\otimes r}$, we have*

$$\langle Xv, w\rangle = \langle v, \eta(X)w\rangle.$$

Proof. By Lemma 3.11, it suffices to check the above for $r = 1$ and $v = e_i$ and $w = e_j$, for any $i, j \in \mathbb{Z}$. This is straightforward and left to the reader. □

Note that η can also be defined on $\dot{\mathrm{U}}(\widehat{\mathfrak{gl}}_n)$, such that $\eta(1_\lambda) = 1_\lambda$ for any $\lambda \in \mathbb{Z}^r$, and that it descends to $\widehat{\mathbf{S}}(n,r)$.

3.4. A presentation of $\widehat{\mathbf{S}}(n, r)$ for $n > r$. In this subsection, let $n > r$. Recall that in this case

$$\psi_{n,r} \colon \dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n) \longrightarrow \mathrm{End}_{\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}}(V^{\otimes r}) \cong \widehat{\mathbf{S}}(n,r)$$

is surjective. This gives rise to the following presentation of $\widehat{\mathbf{S}}(n, r)$. The proof can be found in [6].

Theorem 3.13. [6] *$\widehat{\mathbf{S}}(n, r)$ is isomorphic to the associative unital $\mathbb{Q}(q)$-algebra generated by 1_λ, for $\lambda \in \Lambda(n, r)$, and $E_{\pm i}$, for $i = 1, \ldots, n$, subject to the relations*

$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda,$$

$$\sum_{\lambda \in \Lambda(n,r)} 1_\lambda = 1,$$

$$E_{\pm i} 1_\lambda = 1_{\lambda \pm \bar{\alpha}_i} E_{\pm i},$$

$$E_i E_{-j} - E_{-j} E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,r)} [\lambda_i - \lambda_{i+1}] 1_\lambda,$$

$$E_{\pm i}^2 E_{\pm(i\pm1)} - (q + q^{-1}) E_{\pm i} E_{\pm(i\pm1)} E_{\pm i} + E_{\pm(i\pm1)} E_{\pm i}^2 = 0,$$

$$E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i} = 0 \quad \text{for distant } i, j.$$

We use the convention that $1_\mu X 1_\lambda = 0$ whenever μ or λ is not contained in $\Lambda(n, r)$. Recall that $[a]$ is the q-integer $(q^a - q^{-a})/(q - q^{-1})$.

We will use signed sequences $\underline{\boldsymbol{i}} = (\mu_1 i_1, \ldots, \mu_m i_m)$, with $m \in \mathbb{N}$, $\mu_j \in \{\pm 1\}$ and $i_j \in \{1, \ldots, n\}$. The set of signed sequences we denote SSeq. For a signed sequence $\underline{\boldsymbol{i}} = (\mu_1 i_1, \ldots, \mu_m i_m)$ we define

$$\underline{\boldsymbol{i}}_\Lambda := \mu_1 \bar{\alpha}_{i_1} + \cdots + \mu_m \bar{\alpha}_{i_m}.$$

We write $E_{\underline{i}}$ for the product $E_{\mu_1 i_1} \ldots E_{\mu_m i_m}$. For any $\lambda \in \mathbb{Z}^n$ and $\underline{\boldsymbol{i}} \in \mathrm{SSeq}$, we have

$$E_{\underline{i}} 1_\lambda = 1_{\lambda + \underline{i}_\Lambda} E_{\underline{i}}.$$

The surjection $\psi_{n,r} \colon \dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) \to \widehat{\mathbf{S}}(n, r)$ can also be given explicitly in terms of the generators in Theorem 3.13. For any $\lambda \in \mathbb{Z}^{n-1}$, we have

$$\psi_{n,r}(E_{\pm i} 1_\lambda) = E_{\pm i} 1_{\varphi_{n,r}(\lambda)}, \tag{3.2}$$

where $\varphi_{n,r} \colon \mathbb{Z}^{n-1} \to \Lambda(n, r) \cup \{*\}$ is the map defined in 3.2. By convention, we put $1_* = 0$.

Recall the definition of η in 3.10. Since η is an algebra anti-homomorphism, we get

$$\eta\colon 1_\lambda \widehat{\mathbf{S}}(n,r) 1_\mu \longrightarrow 1_\mu \widehat{\mathbf{S}}(n,r) 1_\lambda,$$

for any $\lambda, \mu \in \Lambda(n,r)$.

Lemma 3.14. *For any non-zero element $X \in \widehat{\mathbf{S}}(n,r)$, we have*

$$X\eta(X) \neq 0 \quad \text{and} \quad \eta(X)X \neq 0.$$

Proof. By the definition of $\widehat{\mathbf{S}}(n,r)$, we know that there exist $e_{t_1}, \dots, e_{t_r} \in V$ such that

$$X(e_{t_1} \otimes \cdots \otimes e_{t_r}) \neq 0.$$

By Lemmas 3.12 and 3.8, we get

$$\begin{aligned} &\langle e_{t_1} \otimes \cdots \otimes e_{t_r}, \eta(X)X(e_{t_1} \otimes \cdots \otimes e_{t_r})\rangle \\ &\quad = \langle X(e_{t_1} \otimes \cdots \otimes e_{t_r}), X(e_{t_1} \otimes \cdots \otimes e_{t_r})\rangle \neq 0. \end{aligned}$$

Therefore, we see that

$$\eta(X)X \neq 0.$$

The other case follows automatically, because $X\eta(X) = \eta^2(X)\eta(X) = \eta(Y)Y$ for $Y = \eta(X)$. □

We can also give an explicit formula for the well-known embedding (see [6]) of $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ into $\widehat{\mathbf{S}}(n,r)$. Let $1_r = 1_{(1^r)}$. We define the following map

$$\sigma_{n,r}\colon \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \longrightarrow 1_r \widehat{\mathbf{S}}(n,r) 1_r$$

by

$$\sigma_{n,r}(b_i) = 1_r E_{-i} E_i 1_r = 1_r E_i E_{-i} 1_r,$$

for $i = 1, \dots, r-1$,

$$\sigma_{n,r}(b_r) = 1_r E_{-n} \dots E_{-r} E_r \dots E_n 1_r,$$

$$\begin{aligned} \sigma_{n,r}(T_\rho) &= 1_r E_{-n} \dots E_{-r-1} E_{-1} \dots E_{-r} 1_r \\ &\quad (= 1_r E_{-n} E_{-1} \dots E_{-r+1} E_{-n+1} \dots E_{-r} 1_r), \end{aligned}$$

and

$$\begin{aligned} \sigma_{n,r}(T_{\rho^{-1}}) &= 1_r E_r \dots E_1 E_{r+1} \dots E_n 1_r \\ &\quad (= 1_r E_r \dots E_{n-1} E_{r-1} \dots E_1 E_n 1_r). \end{aligned}$$

It is easy to check that $\sigma_{n,r}$ is well-defined. It turns out that $\sigma_{n,r}$ is actually an isomorphism, which induces the affine q-Schur functor

$$\widehat{\mathbf{S}}(n,r)\text{-mod} \longrightarrow \widehat{\mathcal{H}}_{\hat{A}_{r-1}}\text{-mod}$$

between the categories of finite-dimensional modules of the extended affine Hecke algebra and of the affine q-Schur algebra. This functor is an equivalence (see Theorem 4.1.3 in [5], for example).

The following result will be needed in Section 6.

Proposition 3.15. *Let $n > r$. Suppose that A is a $\mathbb{Q}(q)$-algebra and*

$$f\colon \widehat{\mathbf{S}}(n,r) \longrightarrow A$$

is a surjective $\mathbb{Q}(q)$-algebra homomorphism which is an embedding when restricted to $1_r\widehat{\mathbf{S}}(n,r)1_r \cong \widehat{\mathcal{H}}_{\hat{A}_{r-1}}$. Then f is a $\mathbb{Q}(q)$-algebra isomorphism

$$A \cong \widehat{\mathbf{S}}(n,r).$$

Proof. We first prove that f is an embedding when restricted to $1_r\widehat{\mathbf{S}}(n,r)$ and $\widehat{\mathbf{S}}(n,r)1_r$. Suppose that this is not true in the first case, then there exists a non-zero element $1_rX \in 1_r\widehat{\mathbf{S}}(n,r)$ in the kernel of f. By Lemma 3.14, we have $1_rX\eta(X)1_r \neq 0$. However, we have

$$f(1_rX\eta(X)1_r) = f(1_rX)f(\eta(X)1_r) = 0,$$

which leads to a contradiction, because by hypothesis f is an embedding when restricted to $1_r\widehat{\mathbf{S}}(n,r)1_r$. The second case can be proved similarly.

Now, let $X \in \widehat{\mathbf{S}}(n,r)$ be an arbitrary non-zero element. By affine Schur–Weyl duality, we have

$$\widehat{\mathbf{S}}(n,r) \cong \operatorname{Hom}_{\widehat{\mathcal{H}}_{\hat{A}_{r-1}}}(\widehat{\mathbf{S}}(n,r)1_r, \widehat{\mathbf{S}}(n,r)1_r),$$

where the isomorphism is induced by left composition. Therefore, there exists an element $Y1_r$ such that

$$XY1_r \neq 0.$$

By the above, we have

$$f(X)f(Y1_r) = f(XY1_r) \neq 0,$$

so

$$f(X) \neq 0.$$

This shows that f is an embedding and therefore an isomorphism. □

Let us end this section giving an embedding between affine q-Schur algebras which we will use in Section 5.

Proposition 3.16. *The $\mathbb{Q}(q)$-linear algebra homomorphism*

$$\iota_n \colon \widehat{\mathbf{S}}(n,r) \longrightarrow \widehat{\mathbf{S}}(n+1,r)$$

defined by

$$\begin{aligned} 1_\lambda &\longmapsto 1_{(\lambda,0)}, \\ E_{\pm i} 1_\lambda &\longmapsto E_{\pm i} 1_{(\lambda,0)}, \\ E_n 1_\lambda &\longmapsto E_n E_{n+1} 1_{(\lambda,0)}, \\ E_{-n} 1_\lambda &\longmapsto E_{-(n+1)} E_{-n} 1_{(\lambda,0)}, \end{aligned}$$

for any $1 \leq i \leq n-1$ and $\lambda \in \Lambda(n,r)$, is an embedding and gives an isomorphism of algebras

$$\widehat{\mathbf{S}}(n,r) \cong \bigoplus_{\lambda,\mu \in \Lambda(n,r)} 1_{(\lambda,0)} \widehat{\mathbf{S}}(n+1,r) 1_{(\mu,0)}.$$

3.5. The 2-categories $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ and $\widehat{\mathcal{S}}(n,r)_{[y]}$. In this section we define three 2-categories, $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$, $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ and $\widehat{\mathcal{S}}(n,r)_{[y]}$, using a graphical calculus analogous to Khovanov and Lauda's in [17].

3.5.1. The 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$. The 2-category $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ is defined just as $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ in [17], but the 2-HOM-spaces are tensored with $\mathbb{Q}[y]$, where y is a formal variable of degree two, and the defining KL-relation (3.17) is y-deformed, as shown below. In order to define $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, change the weights in the definition of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ into level-zero $\widehat{\mathfrak{gl}}_n$-weights (i.e. $\mathfrak{gl}_n$-weights). The 2-category $\widehat{\mathcal{S}}(n,r)_{[y]}$ is then defined as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$. This is precisely the affine analogue of what was done in [28].

Remark 3.17. We use the sign conventions from [28] in the relations on 2-morphisms, which differ from Khovanov and Lauda's sign conventions. For more details on this change of convention, see below.

Remark 3.18. We do not prove that the 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ provides a categorification of $\mathbf{U}_q(\widehat{\mathfrak{gl}}_n)$, although we conjecture that it does. We will prove that $\widehat{\mathcal{S}}(n,r)_{[y]}$ categorifies $\widehat{\mathbf{S}}(n,r)$.

In order to avoid giving several long definitions which are very similar, we only define $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ in detail. The 2-category $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ is defined exactly as $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, but using level-zero $\widehat{\mathfrak{sl}}_n$-weights. The 2-category $\widehat{\mathcal{S}}(n,r)_{[y]}$ is defined as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$. We will show that $\widehat{\mathcal{S}}(n,r)_{[y]}$ can also be obtained as a quotient of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$.

To be more precise, we first define the $\mathbb{Q}[y]$-linear graded 2-category with translation $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$, whose 2-morphisms are $\mathbb{Q}[y]$-linear combinations of homogeneous 2-morphisms of various degrees. The $\mathbb{Q}$-linear 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ is then obtained by restricting to the degree-zero 2-morphisms.

Definition 3.19. The additive $\mathbb{Q}[y]$-linear graded 2-category with translation $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$ consists of objects $\lambda \in \mathbb{Z}^n$ and 1 and 2-morphisms such that the hom-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}(\lambda,\lambda')$ between two objects λ, λ' is an additive $\mathbb{Q}[y]$-linear graded category with translation defined as follows.

- OBJECTS[2] OF $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}(\lambda,\lambda')$. A 1-morphism in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$ from λ to λ' is a formal finite direct sum of 1-morphisms

$$\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\} = \mathbf{1}_{\lambda'}\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\} := \mathcal{E}_{\mu_1 i_1}\cdots\mathcal{E}_{\mu_m i_m}\mathbf{1}_\lambda\{t\}$$

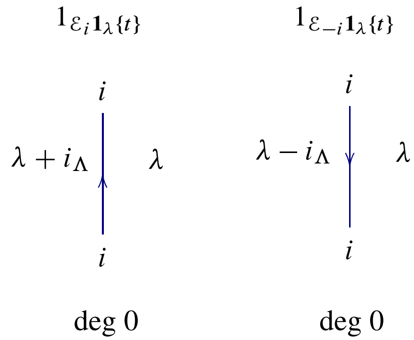
  for any $t \in \mathbb{Z}$ and signed sequence $\underline{\boldsymbol{i}} \in \mathrm{SSeq}$ such that $\lambda' = \lambda + \underline{\boldsymbol{i}}_\Lambda$ and λ, $\lambda' \in \mathbb{Z}^n$.

- MORPHISMS OF $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}(\lambda,\lambda')$. They are presented by generators and relations. Multiplication by y is indicated graphically by $\boxed{y}$ in the diagrams below. The $\mathbb{Q}[y]$-linearity of the 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$ implies that $\boxed{y}$ can freely slide through any line in a diagram.

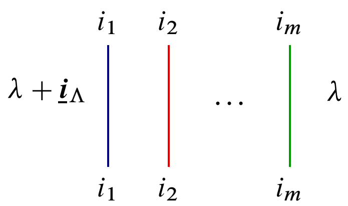
  - GENERATORS. For 1-morphisms $\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\}$ and $\mathcal{E}_{\underline{j}}\mathbf{1}_\lambda\{t'\}$ in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$, the hom-sets $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}(\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\},\mathcal{E}_{\underline{j}}\mathbf{1}_\lambda\{t'\})$ of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}(\lambda,\lambda')$ are graded $\mathbb{Q}[y]$-vector spaces given by linear combinations of diagrams of homogeneous degrees, modulo certain relations, built from composites of the following morphisms.

[2] We refer to objects of the category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}(\lambda,\lambda')$ as 1-morphisms of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$. Likewise, the morphisms of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}(\lambda,\lambda')$ are called 2-morphisms in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$.

i) Degree-zero identity 2-morphisms 1_x for each 1-morphism x in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$; in particular for any $i \in \{1,\dots,n\}$, $t \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}^n$, the identity 2-morphisms $1_{\mathcal{E}_i\mathbf{1}_\lambda}\{t\}$ and $1_{\mathcal{E}_{-i}\mathbf{1}_\lambda}\{t\}$ are represented graphically by



More generally, for a signed sequence $\underline{\boldsymbol{i}} = (\mu_1 i_1, \mu_2 i_2, \dots \mu_m i_m)$, the identity $1_{\mathcal{E}_{\underline{i}}\mathbf{1}_\lambda\{t\}}$ 2-morphism is represented as

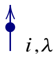
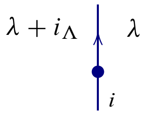
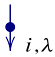
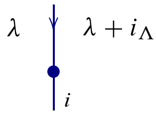
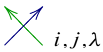
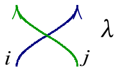
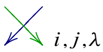
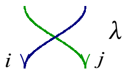
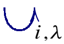
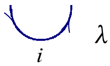
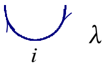
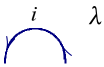
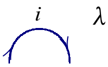


where the strand labeled i_k is oriented up if $\mu_k = +$ and oriented down if $\mu_k = -$. We will often place labels on the side of a strand and omit the labels at the top and bottom. The signed sequence can be recovered from the labels and the orientations on the strands. We might also forget the object on the left of the diagram which can be recovered from the object on the right and the signed sequence corresponding to the diagram.

ii) For any $\lambda \in \mathbb{Z}^n$ the 2-morphisms of Table 1, where the degrees are given by the symmetric $\mathbb{Z}$-valued bilinear form on $\mathbb{C}\{1,\dots,n\}$,

$$i \cdot j = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \equiv j \pm 1 \mod n, \\ 0 & \text{if } i \not\equiv j \pm 1 \mod n. \end{cases}$$

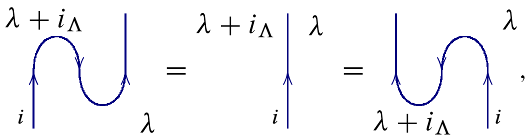
Table 1

Notation	2-morphism	Degree
		$i \cdot i$
		$i \cdot i$
		$-i \cdot j$
		$-i \cdot j$
		$1 + \bar{\lambda}_i$
		$1 - \bar{\lambda}_i$
		$1 + \bar{\lambda}_i$
		$1 - \bar{\lambda}_i$

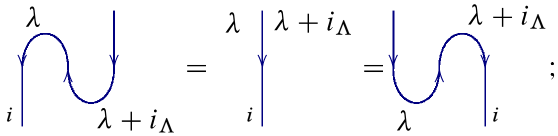
∘ Relations

⋆ Biadjointness and cyclicity

i) $\mathbf{1}_{\lambda+i_\Lambda}\mathcal{E}_i\mathbf{1}_\lambda$ and $\mathbf{1}_\lambda\mathcal{E}_{-i}\mathbf{1}_{\lambda+i_\Lambda}$ are biadjoint, up to grading shifts:

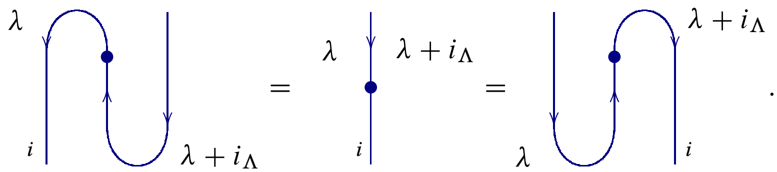


(3.3)



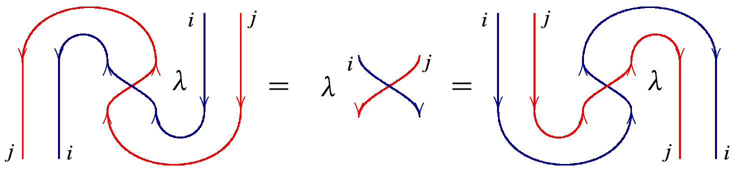
(3.4)

ii) as well for dotted lines:



(3.5)

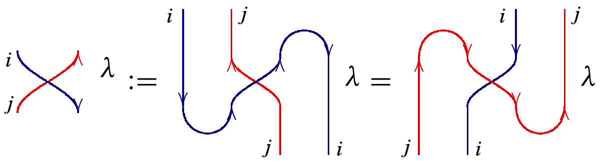
iii) All 2-morphisms are cyclic with respect to the above biadjoint structure. This is ensured by the relations (3.3)–(3.5), and the relations



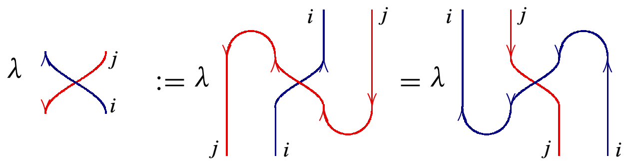
(3.6)

The cyclic condition on 2-morphisms expressed by (3.3)–(3.6) ensures that diagrams related by isotopy represent the same 2-morphism in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$.

It will be convenient to introduce degree zero 2-morphisms:



(3.7)

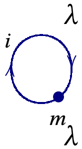


(3.8)

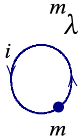
where the second equality in (3.7) and (3.8) follow from (3.6).

⋆ Bubble relations

i) All dotted bubbles of negative degree are zero. That is,



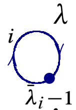
$$= 0 \quad \text{if } m < \bar{\lambda}_i - 1, \tag{3.9a}$$



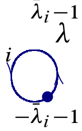
$$= 0 \quad \text{if } m < -\bar{\lambda}_i - 1, \tag{3.9b}$$

for all $m \in \mathbb{Z}_+$, where a dot carrying a label m denotes the m-fold iterated vertical composite of  $_{i,\lambda}$ or  $_{i,\lambda}$ depending on the orientation.

ii) A dotted bubble of degree zero equals ±1:

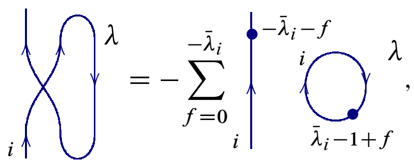


$$= (-1)^{\lambda_{i+1}} \qquad \text{for } \bar{\lambda}_i \geq 1, \tag{3.10a}$$

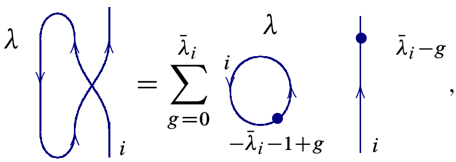


$$= (-1)^{\lambda_{i+1}-1} \qquad \text{for } \bar{\lambda}_i \leq -1. \tag{3.10b}$$

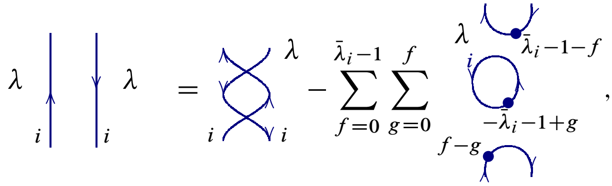
iii) For the following relations we employ the convention that all summations are increasing, so that a summation of the form $\sum_{f=0}^{m}$ is zero if $m < 0$:



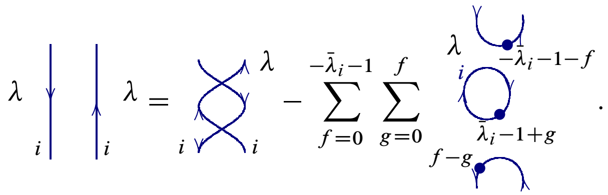
(3.11a)



(3.11b)

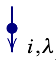


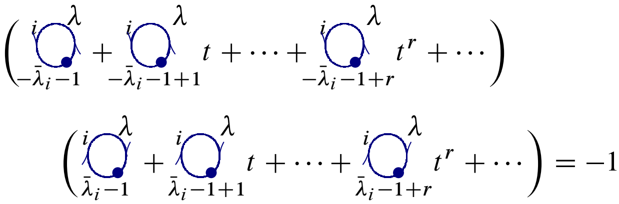
(3.12a)



(3.12b)

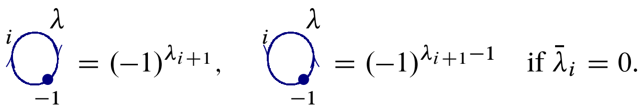
iiv) Fake bubbles

Notice that for some values of λ the dotted bubbles appearing above have negative labels. A composite of  or  with itself a negative number of times does not make sense. These dotted bubbles with negative labels, called fake bubbles, are formal symbols inductively defined by the equation



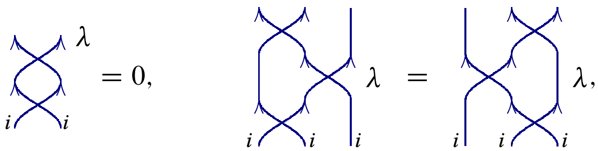
(3.13)

and the additional condition

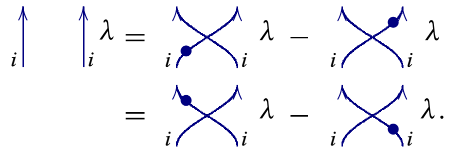


Although the labels are negative for fake bubbles, one can check that the overall degree of each fake bubble is still positive, so that these fake bubbles do not violate the positivity of dotted bubble axiom. The above equation, called the infinite Grassmannian relation, remains valid even in high degree when most of the bubbles involved are not fake bubbles.

⋆ Nil-Hecke relations

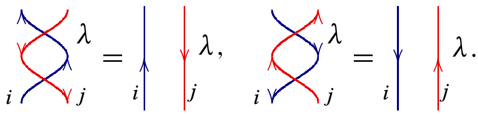


(3.14)



(3.15)

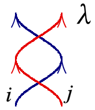
★ For $i \neq j$



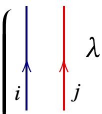
(3.16)

★ The analogue of the $R(\nu)$-relations

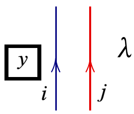
i) For $i \neq j$



$=$

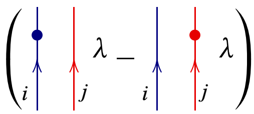
 if $i \cdot j = 0$,

$\Delta_{i,j}$ if $i \cdot j = -1$ and $\{i, j\} \neq \{1, n\}$,

$\Delta_{i,j} -$  if $\{i, j\} = \{1, n\}$,

(3.17)

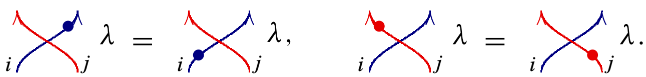
where

$\Delta_{i,j} := \varepsilon(i, j)$ .

For $i \cdot j = -1$, we define

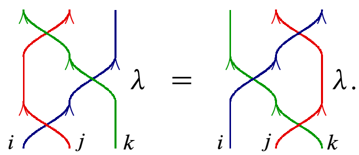
$$\varepsilon(i, j) := \begin{cases} 1 & \text{if } i \equiv j + 1 \mod n, \\ -1 & \text{if } i \equiv j - 1 \mod n. \end{cases}$$

Note that this sign takes into account the standard orientation of the Dynkin diagram:



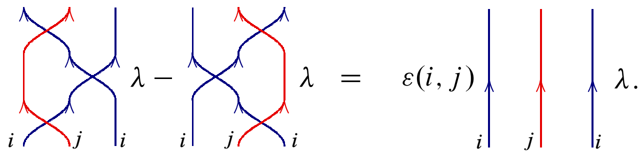
(3.18)

ii) Unless $i = k$ and $i \cdot j = -1$



(3.19)

For $i \cdot j = -1$



(3.20)

Composition of 1-morphisms is defined by multiplication and horizontal and vertical composition of 2-morphisms are defined by juxtaposition and glueing, as in [17].

Definition 3.20. The 2-category $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ is the full 2-subcategory of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^*_{[y]}$ with the same objects and 1-morphisms but in which the 2-morphisms are only the ones of degree zero.

Remark 3.21. Note that the 2-hom-spaces in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ are no longer $\mathbb{Q}[y]$-linear, because $\deg(y) = 2$. They are finite-dimensional as $\mathbb{Q}$-vector spaces, because the original KL 2-HOM-spaces are finite-dimensional in each degree and their grading is bounded below.

As already remarked, $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ is defined similarly. Only some signs in the relations involving right cups and caps have to be changed, so that all relations really depend on $\widehat{\mathfrak{sl}}_n$-weights and not on $\widehat{\mathfrak{gl}}_n$-weights. We use the convention which is the affine analogue of the signed-version in [17]. For more information on this change of signs, see (3.36).

Definition 3.22. Let $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ denote the $\mathbb{Q}$-linear 2-categories obtained by modding out $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by the ideal generated by y.

Note that $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ is isomorphic to the original KL 2-category and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ isomorphic to its $\widehat{\mathfrak{gl}}_n$-analogue.

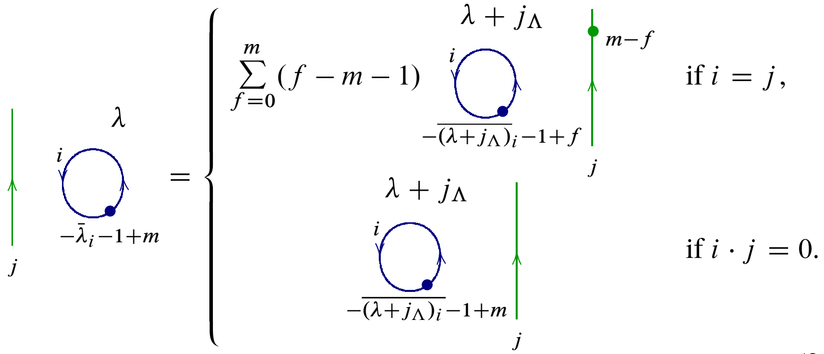
Recall that Khovanov and Lauda [17] defined a basis of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ over $\mathbb{Q}$. The following theorem will be proved below Proposition 5.10.

Theorem 3.23. *$\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ is freely generated over $\mathbb{Q}[y]$ by the KL basis, i.e. by the same diagrams as Khovanov and Lauda used for their basis, so the deformation is flat. Furthermore, the deformation is non-trivial, i.e. there exists no $\mathbb{Q}[y]$-linear degree preserving 2-equivalence between $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\mathcal{U}(\widehat{\mathfrak{sl}}_n) \otimes_{\mathbb{Q}} \mathbb{Q}[y]$.*

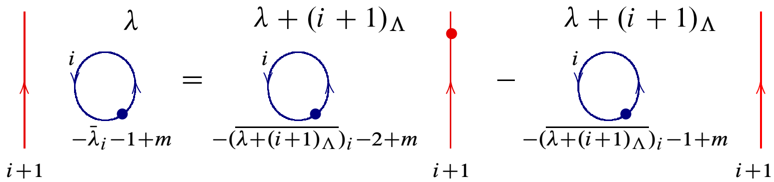
3.5.2. Further relations in $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$. The other $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$-relations expressed below follow from the relations in Definition 3.19 and are going to be used in the sequel.

⋆ Bubble slides

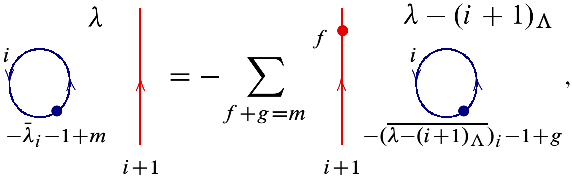
If $\{i, j\} \neq \{1, n\}$, we have



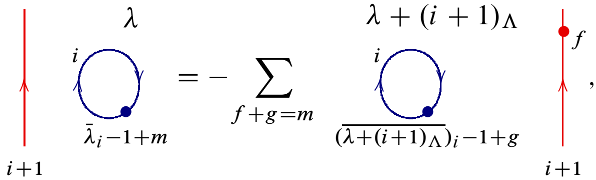
(3.21)



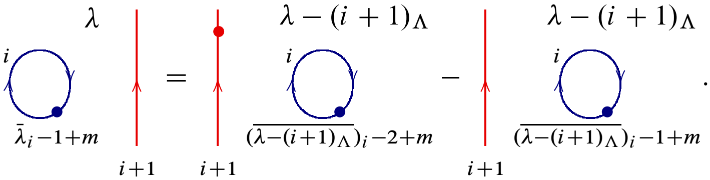
, (3.22)



, (3.23)



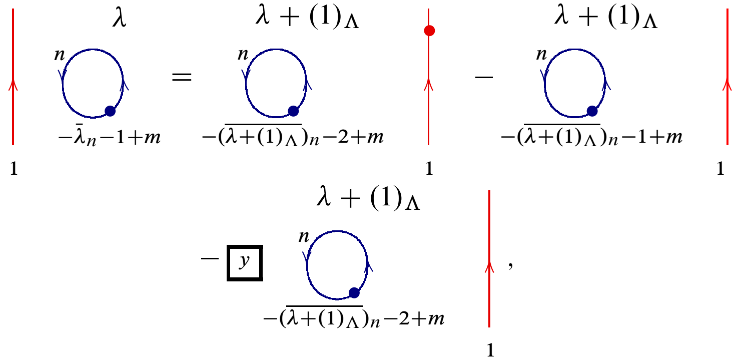
, (3.24)



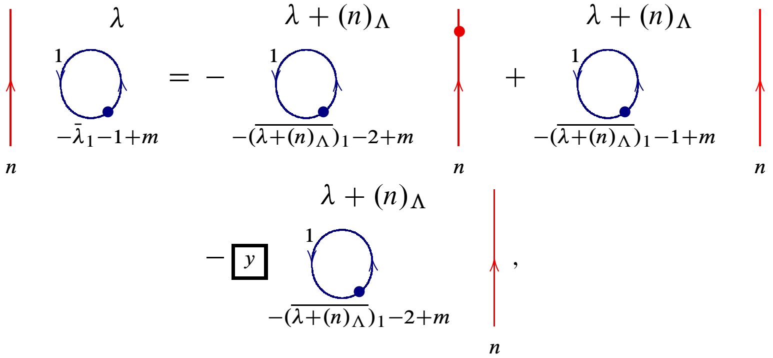
. (3.25)

If we switch labels i and $i+1$, then the right hand side of the above equations gets a minus sign. Bubble slides with the vertical strand oriented downwards can easily be obtained from the ones above by rotating the diagrams 180 degrees.

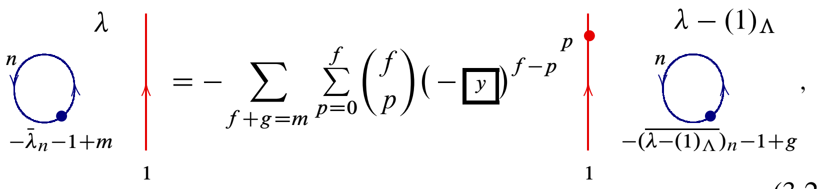
If $\{i, j\} = \{1, n\}$, we get



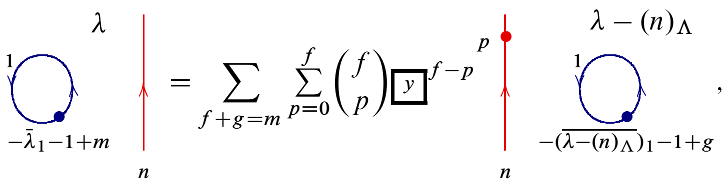
(3.26)



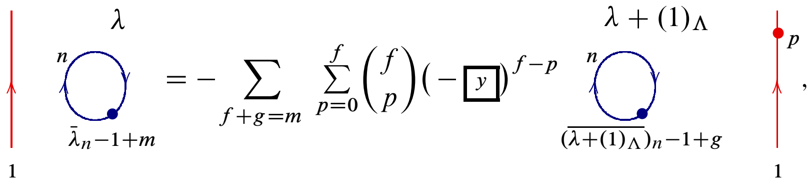
(3.27)



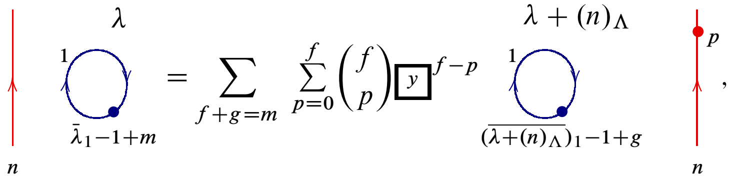
(3.28)



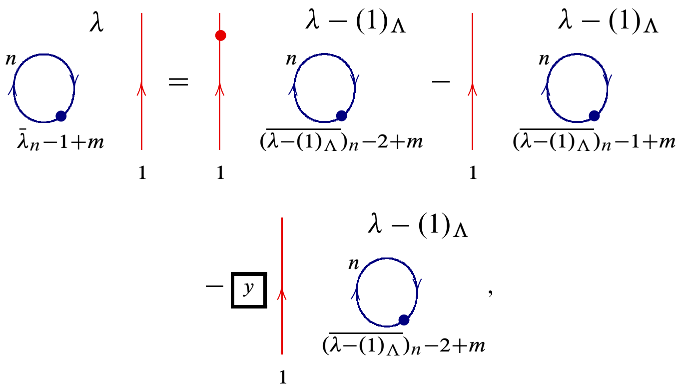
(3.29)



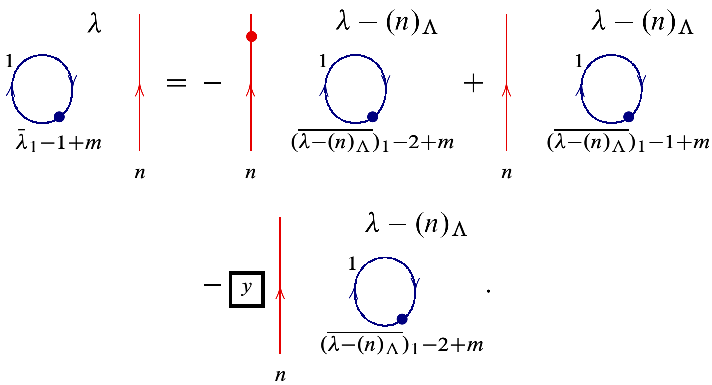
(3.30)



(3.31)



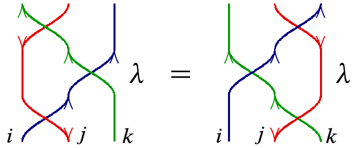
(3.32)



(3.33)

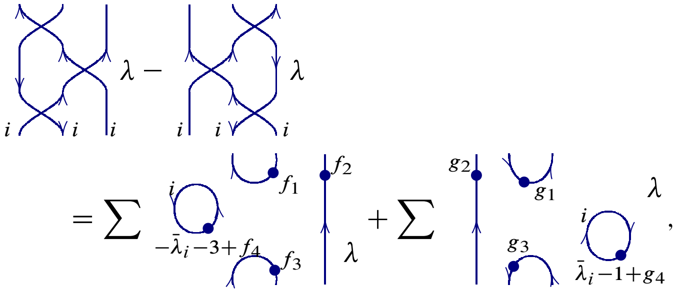
⋆ More Reidemeister 3 like relations

Unless $i = k = j$ we have



(3.34)

and when $i = j = k$ we have



(3.35)

where the first sum is over all $f_1, f_2, f_3, f_4 \ge 0$ with $f_1+f_2+f_3+f_4 = \bar{\lambda}_i$ and the second sum is over all $g_1, g_2, g_3, g_4 \ge 0$ with $g_1+g_2+g_3+g_4 = \bar{\lambda}_i - 2$. Note that the first summation is zero if $\bar{\lambda}_i < 0$ and the second is zero when $\bar{\lambda}_i < 2$.

Reidemeister 3 like relations for all other orientations are determined from relations (3.19), (3.20), and the above relations using biadjunction and cyclicity.

3.5.3. The 2-category $\widehat{\mathcal{S}}(n,r)_{[y]}$. As explained in Section 3.4, the q-Schur algebra $\widehat{\mathbf{S}}(n,r)$ can be seen as a quotient of $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ by the ideal generated by all idempotents corresponding to the weights that do not belong to $\Lambda(n,r)$.

It is then natural to define the 2-category $\widehat{\mathcal{S}}(n,r)_{[y]}$ as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ as follows.

Definition 3.24. The 2-category $\widehat{\mathcal{S}}(n,r)_{[y]}$ is the quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ by the 2-ideal generated by all 2-morphisms containing a region with a label not in $\Lambda(n,r)$.

We remark that we only put real bubbles, whose interior has a label outside $\Lambda(n,r)$, equal to zero. To see what happens to a fake bubble, one first has to write it in terms of real bubbles with the opposite orientation using the infinite Grassmannian relation (3.13).

As in [28], we define $\widehat{\mathcal{S}}(n,r)_{[y]}$ as a quotient of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, rather than $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$. Therefore, just as in [28] (see the introduction of Sections 3 and 4.3 in that paper), we have to show that there exists a full and essentially surjective 2-functor

$$\Psi_{n,r}\colon \mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]} \longrightarrow \widehat{\mathcal{S}}(n,r)_{[y]},$$

which categorifies the surjective homomorphism

$$\psi_{n,r}\colon \dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) \longrightarrow \widehat{\mathbf{S}}(n,r)$$

defined in (3.2).

On objects $\Psi_{n,r}$ maps μ to $\lambda := \varphi_{n,r}(\mu)$, which was defined in Section 3.2. On 1 and 2-morphisms $\Psi_{n,r}$ is defined to be the identity except for the left cups and caps, on which it is given by

$$\frown_{i,\mu} \longmapsto (-1)^{\lambda_{i+1}+1} \frown_{i,\lambda} \quad \text{and} \quad \smile_{i,\mu} \longmapsto (-1)^{\lambda_i+1} \smile_{i,\lambda}. \tag{3.36}$$

Note that here we are simply extending the 2-functor used in [28].

Just for completeness, we now state the following result without proof.

Proposition 3.25. *The* 2*-functor*

$$\Psi_{n,r}\colon \mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]} \longrightarrow \widehat{\mathcal{S}}(n,r)_{[y]}$$

is well-defined, full and essentially surjective.

Just as for $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$, we can put $y = 0$.

Definition 3.26. Let $\widehat{\mathcal{S}}(n,r)$ be the quotient of $\widehat{\mathcal{S}}(n,r)_{[y]}$ by the 2-ideal generated by y.

Of course there also exists a full and essentially surjective 2-functor

$$\Psi_{n,r}\colon \mathcal{U}(\widehat{\mathfrak{sl}}_n) \longrightarrow \widehat{\mathcal{S}}(n,r),$$

which is defined and denoted just as above.

4. A functor from $\mathcal{DEB}im^*_{\widehat{A}_{r-1}}$ to $\widehat{\mathcal{S}}(n,r)^*_{[y]}$

In this section, we define a functor

$$\Sigma_{n,r}\colon \mathcal{DEB}im^*_{\widehat{A}_{r-1}} \longrightarrow \widehat{\mathcal{S}}(n,r)^*_{[y]},$$

which categorifies the embedding

$$\sigma_{n,r}\colon \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \longrightarrow 1_r \widehat{\mathbf{S}}(n,r) 1_r.$$

Actually its target will be the one-object sub-2-category $\widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r))$ of $\widehat{\mathcal{S}}(n,r)^*_{[y]}$. Since $\widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r))$ has only one object, it can be seen as a monoidal category.

This functor is the affine analogue of $\Sigma_{n,d}$ in Section 6.5 in [28]. For diagrams with only unoriented i-colored strands for $i = 1,\dots,r-1$ the definitions in that paper and in this one coincide, for diagrams with unoriented r-colored strands or oriented strands the definition here is new.

In Section 6, we will prove that

$$\Sigma_{n,r} : \mathcal{DEB}im^*_{\widehat{A}_{r-1}} \longrightarrow \widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r))$$

is faithful. We conjecture that $\Sigma_{n,r}$ is also full and, therefore, that the two categories $\mathcal{DEB}im_{\widehat{A}_{r-1}}$ and $\widehat{\mathcal{S}}(n,r)_{[y]}((1^r),(1^r))$ are equivalent. The latter equivalence would the affine analogue of the one proved in Proposition 6.9 in [28] for finite type A.

4.1. The definition of the functor. The functor $\Sigma_{n,r}$ is $\mathbb{Q}$-linear and monoidal, so it is sufficient to specify the image of the generating objects and morphisms.

The functor $\Sigma_{n,r}$ is defined on objects by

$$\emptyset \longmapsto \mathbf{1}_r = \mathbf{1}_{(1^r)},$$

$$i \longmapsto \mathcal{E}_{-i}\mathcal{E}_i\mathbf{1}_r,$$

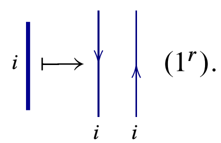
$$r \longmapsto \mathcal{E}_{-n}\dots\mathcal{E}_{-r}\mathcal{E}_r\dots\mathcal{E}_n\mathbf{1}_r,$$

$$+ \longmapsto \mathcal{E}_{-n}\dots\mathcal{E}_{-r-1}\mathcal{E}_{-1}\dots\mathcal{E}_{-r}\mathbf{1}_r,$$

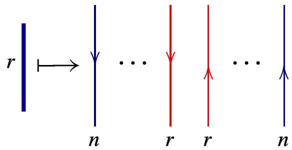
$$- \longmapsto \mathcal{E}_r\dots\mathcal{E}_1\mathcal{E}_{r+1}\dots\mathcal{E}_n\mathbf{1}_r.$$

The functor $\Sigma_{n,r}$ is defined on morphisms as follows, where we use rotation through 180°, denoted ω, and reflection in the x-axis plus orientation reversal, denoted τ, to shorten the definition.

- The empty diagram is sent to the empty diagram in the region labeled (1^r).
- The vertical line colored i is sent to the identity 2-morphism on the morphism $\mathcal{E}_{-i}\mathcal{E}_i\mathbf{1}_r$:

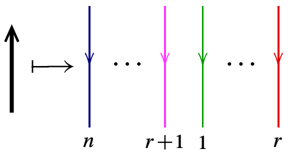


- The vertical line colored r is sent to the identity 2-morphism on the morphism $\mathcal{E}_{-n}\dots\mathcal{E}_{-r}\mathcal{E}_r\dots\mathcal{E}_n\mathbf{1}_r$:



(1^r).

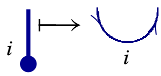
- The vertical line colored $+$ is sent to the identity 2-morphism on the morphism $\mathcal{E}_{-n}\dots\mathcal{E}_{-r-1}\mathcal{E}_{-1}\dots\mathcal{E}_{-r}\mathbf{1}_r$:



(1^r).

  The vertical line colored $-$ is sent to the identity 2-morphism on the morphism $\mathcal{E}_r\dots\mathcal{E}_1\mathcal{E}_{r+1}\dots\mathcal{E}_n\mathbf{1}_r$, which is obtained from the one above by applying ω.

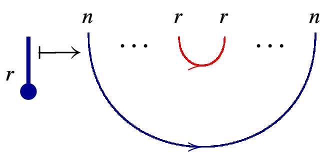
- The image of startdot$_i$ for $i\neq r$:



(1^r).

  The image of enddot$_i$ is obtained by applying ω or τ.

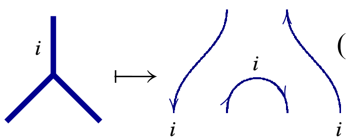
- The image of startdot$_r$:



(1^r).

  The image of enddot$_r$ is obtained by applying ω or τ.

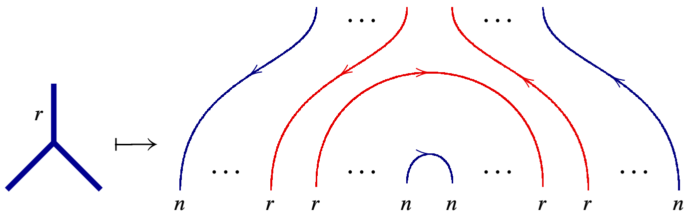
- The image of merge$_i$ for $i\neq r$:



(1^r).

  The image of split$_i$ is obtained by applying ω or τ.

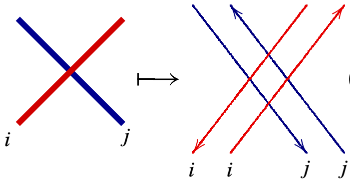
- The image of merge_r:



(1^r).

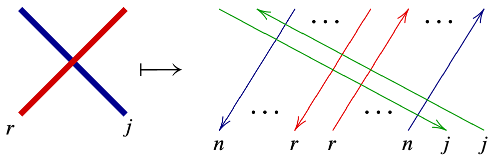
The image of split_r is obtained by applying ω or τ.

- The image of $4\mathrm{vert}_{i,j}$ with distant colors i and j different from r:



(1^r).

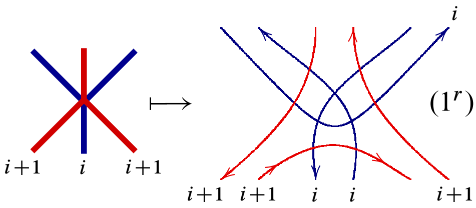
- The image of $4\mathrm{vert}_{r,j}$ with distant colors r and j:



(1^r).

The image of $4\mathrm{vert}_{j,r}$ is obtained by applying τ (not ω!).

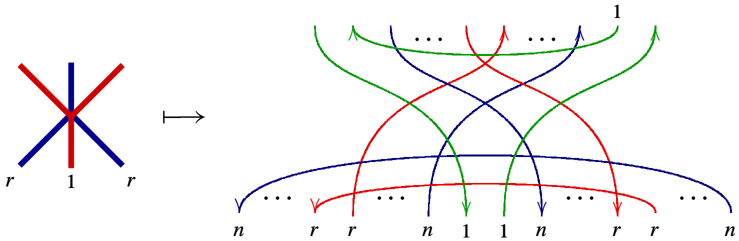
- The image of $6\mathrm{vert}_{i+1,i}$ with colors i and $i+1$ different from r:



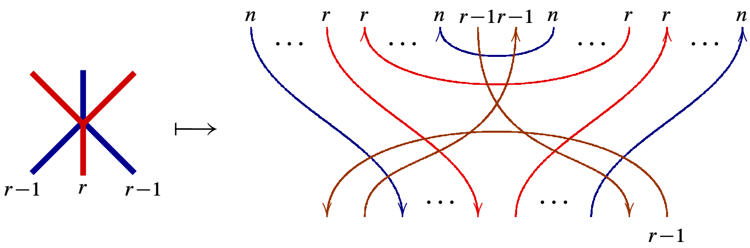
(1^r). (4.1)

The image of $6\mathrm{vert}_{i,i+1}$ is obtained by applying ω or τ.

- The images of $6\mathrm{vert}_{r,1}$ and $6\mathrm{vert}_{r-1,r}$ respectively:



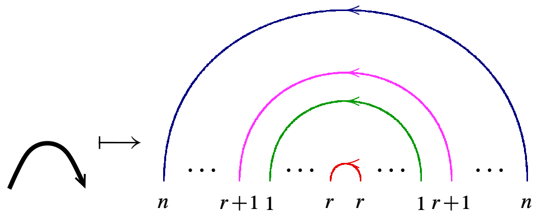
(1^r),



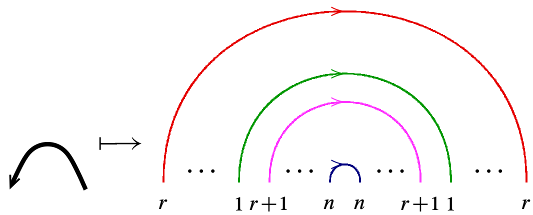
(1^r).

The images of $6\mathrm{vert}_{r,r-1}$ and $6\mathrm{vert}_{1,r}$, respectively, are obtained by ω or τ.

- The images of $+$cap and $-$cap respectively:



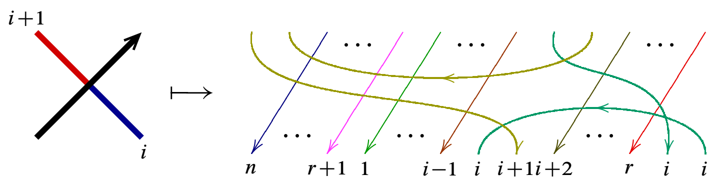
(1^r),



(1^r).

The images of $+$cup and $-$cup are obtained from those of $-$cap and $+$cap, respectively, by applying ω or τ.

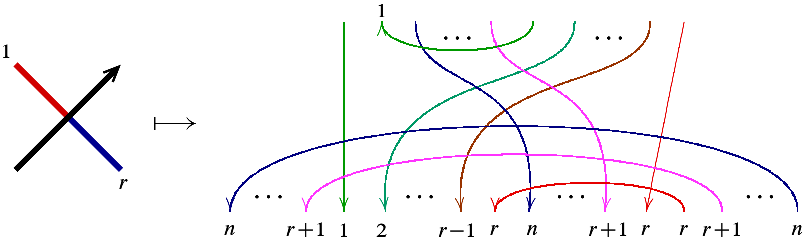
- The image $4\mathrm{vert}_{+,i}$ with colors i different from $r-1$ and r and $i+1$ different from r and 1:



(1^r).

The images of $4\mathrm{vert}_{i+1,+}$, $4\mathrm{vert}_{i,-}$ and $4\mathrm{vert}_{-,i+1}$ are obtained by applying τ, $\tau\omega$ and ω respectively.

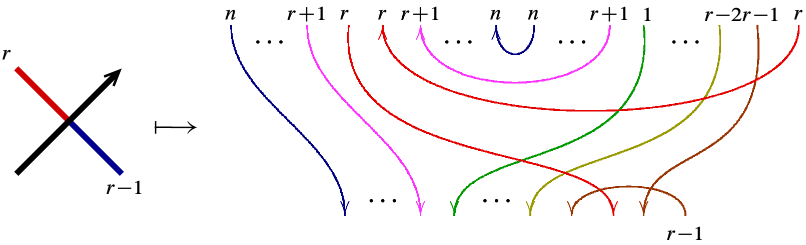
- The image of $4\text{vert}_{+,r}$:



(1^r).

The images of $4\text{vert}_{1,+}$, $4\text{vert}_{r,-}$ and $4\text{vert}_{-,1}$ are obtained by applying τ, $\tau\omega$ and ω respectively.

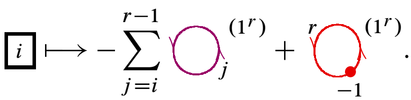
- The image of $4\text{vert}_{+,r-1}$:



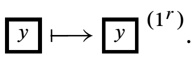
(1^r).

The images of $4\text{vert}_{r,+}$, $4\text{vert}_{r-1,-}$ and $4\text{vert}_{-,r}$ are obtained by applying τ, $\tau\omega$ and ω respectively.

- The image of box_i for $i = 1, \dots, r$:



- The image of the box morphism box_y:



It is easy to check that $\Sigma_{n,r}$ is degree preserving and monoidal.

Lemma 4.1. $\Sigma_{n,r}$ *is well-defined.*

Proof. We check that $\Sigma_{n,r}$ preserves all relations in $\mathcal{DEB}im^*_{\hat{A}_{r-1}}$.

First of all, note that all "finite type A" relations, i.e. the relations between diagrams without r-colored or oriented strands, are preserved by precisely the same arguments as in [28]. And this fact, provided that relations (2.11a)–(2.14b) are preserved, implies that relations (2.5a)–(2.10) with at least one r-colored strand do not need to be checked separately, as mentioned in (x) of Remark 2.7.

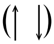
Let us also remark that, except in the checks of the box relations, we will use neither relation (3.17) nor the bubble slides relations (3.26)–(3.33) for $\{i, j\} = \{1, n\}$.

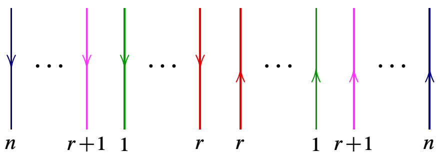
Let us now go through the list of the remaining relations and explain why they are preserved.

- The Isotopy relations (2.4a)–(2.4i) are straightforward.

- Relation (2.11a) follows directly from the fact that in $\hat{S}(n,r)^*_{[y]}((1^r),(1^r))$ all dotted bubbles of degree zero are equal to ± 1. Indeed we can apply successively relations (3.10) to the nested bubbles in the image of  and .

- For relations (2.11b) and (2.11c) use repeatedly relations (3.12). We only give the details for relation (2.11b), the other relation being completely analogous.

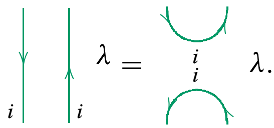
  The diagram $\Sigma_{n,r}($$)$ is as follows



(1^r).

  Apply successively relation (3.12b) to each pair of up and down i-strands, with the region on the right of the pair labelled by λ. For $i = r+1, \ldots, n$, we have $\lambda = (0, 1, \ldots, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the entries which are equal to one are the 2nd until the rth and the $i+1$st (mod n). For $i = 1, \ldots, r$, we have $\lambda = (1, \ldots, 1, 0, 1, \ldots, 1, 0, \ldots, 0)$ where the entries which are equal to zero are the $r+2$nd until the nth and the ith.

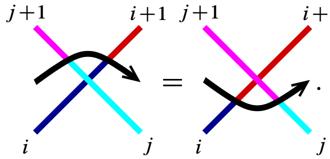
  For these λ, relation (3.12b) becomes

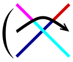
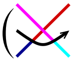


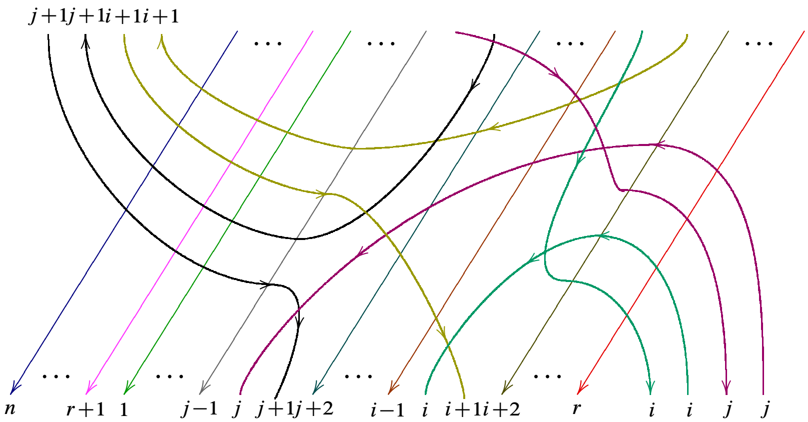
(4.2)

  for all i. In this way, we get exactly the nested cups and caps which together form $\Sigma_{n,r}($$)$.

- Relation (2.12):



The basic idea of the proof is the same for all cases, so we only show one case in detail. For $j < i$ and $i, j \neq r$, this relation follows from the fact that, using repeatedly relations (3.16), (3.17) for distant i, j, (3.19) and (3.34), one can reduce both $\Sigma_{n,r}\Big($  $\Big)$ and $\Sigma_{n,r}\Big($  $\Big)$ to the diagram



(1^r).

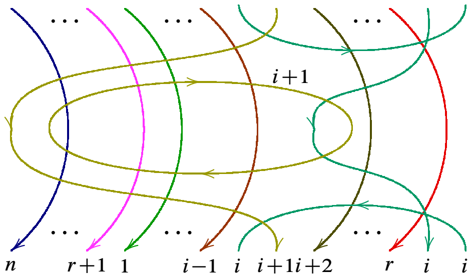
The case for $j > i$ and $i, j \neq r$ can be proved similarly using exactly the same relations.

For the remaining cases, in which one of the integers i or j is equal to $r-1$ or r, use repeatedly relations (3.16), (3.17) for $i \cdot j = 0$, (3.19) and (3.34).

- Relations (2.13a) and (2.13b).

First when the colors $(i, i+1)$ differ from $(r, 1)$ and $(r-1, r)$. We only give the details for relation (2.13a), because the proof of the other relation is very similar.

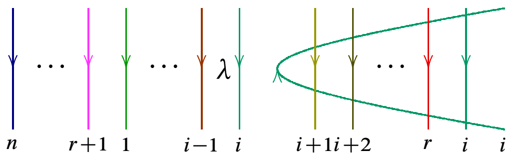
The image under $\Sigma_{n,r}$ of  is equal to



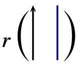
(1^r).

Thanks to relation (3.16), the left part of the central bubble can be slid to the right until the whole bubble is contained in the region immediately to the right of the vertical i-strand, which has label $\lambda = (1, \ldots, 1, 0, 1, \ldots, 1, 0, \ldots, 0)$ where the first 0 is in $i + 2$nd position and the last 1 in $r + 1$st position. That degree-zero bubble is equal to one, by relation (3.10).

After removing the bubble, we can slide the i-colored strand over the $r - i - 1$ rightmost strands and slide the $i + 1$-colored strand over the $n - r + i - 1$ leftmost strands, using relation (3.17). Apply relation (4.2) to the two horizontal i-strands in the region immediately left of the $i + 1$-strand, which has only one non-zero term as before because $\lambda = (1, \ldots, 1, 0, 1, \ldots, 1, 0, \ldots, 0)$, where the first 0 is in ith position and the last 1 in $r + 1$st position. We get



(1^r).

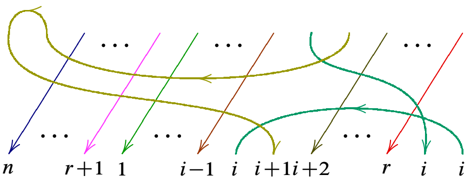
Then the strand with the rightmost endpoints can be slid all the way to the right, first using relation (3.16) and then relation (3.12b), which here is simply equal to an ordinary Reidemeister 2 relation (i.e. without extra bubble terms) because $\lambda = (1^r)$. Finally, we end up with a diagram which is indeed equal to $\Sigma_{n,r}$ .

We can prove relations (2.13a) and (2.13b) with colors $(r, 1)$ and $(r - 1, r)$ in almost the same way. We can slide bubbles and strands using relations (3.16) and (3.17), evaluate bubbles using relation (3.10) and use relations (3.12a) and (3.12b), which again are particularly simple for the relevant labels. The difference here is that we have to iterate some of the steps that we used above, e.g. because we get nested bubbles or various pairs of strands to which we can apply relations (3.12a) and (3.12b).

- Relations (2.13c) and (2.13d).

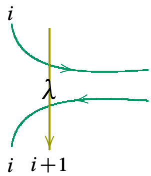
We only prove relation (2.13d) when $(i, i+1)$ differs from $(r, 1)$ and $(r-1, r)$. The other cases for this relation and all cases for the other relation are proved in the same way.

The image under $\Sigma_{n,r}$ of  is equal to



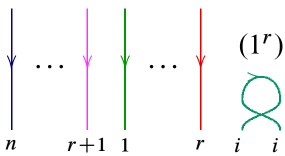
(1^r).

We can slide the $i+1$-colored strand over the $n-r+i-1$ leftmost strands using relation (3.17), so that we end up with the following picture in the central part of the diagram



with $\lambda = (1,\dots,1,0,1,\dots,1,0,\dots,0)$ where the first 0 is in ith position and the last 1 in $r+1$st position. We now use relation (3.12b), which again reduces to (4.2). Then we apply relation (3.16) to the two rightmost strands. After doing all this, the central part of the diagram becomes equal to three identity strands colored i, $i+1$, i, the two first being oriented down and the last up.

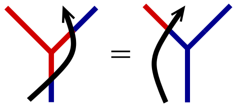
Then, in the full diagram, the rightmost i-colored strand above can be slid over the $r-i-1$ rightmost parallel strands using relation (3.16), so that we end up with



. (4.3)

By applying relation (3.11) to the curl and the fact that the bubble appearing then is equal to -1, the diagram in (4.3) becomes equal to the image under $\Sigma_{n,r}$ of .

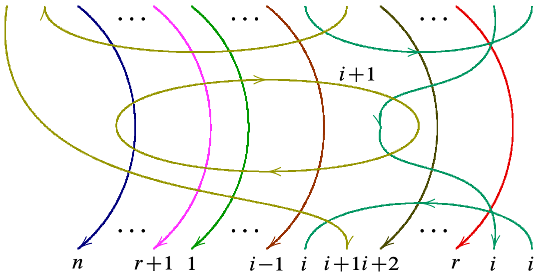
- Relation (2.13e). Actually, we will prove


(4.4)

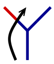
which is equivalent.

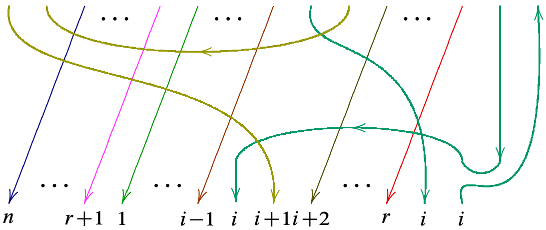
Let us start with colors $(i, i+1)$ different from $(r, 1)$ and $(r-1, r)$.

The image under $\Sigma_{n,r}$ of  is



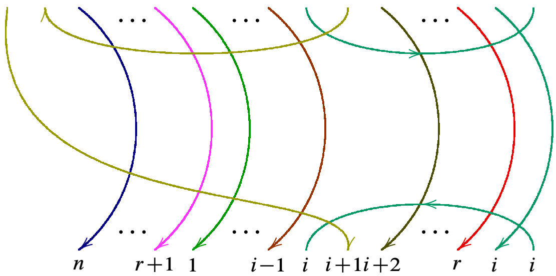
(1^r).

The image under $\Sigma_{n,r}$ of  is



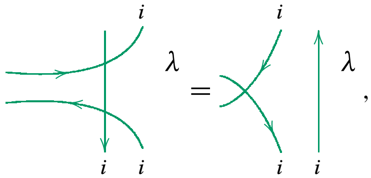
(1^r).

We can apply to the first of these two diagrams the same arguments as in the beginning of the proof of relation (2.13a), in order to simplify it to



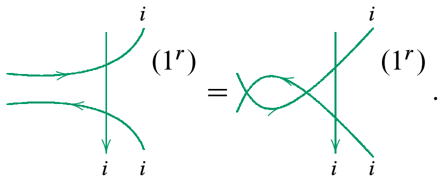
(1^r).

In order to prove that this diagram is equal to the second one, it only remains to check that

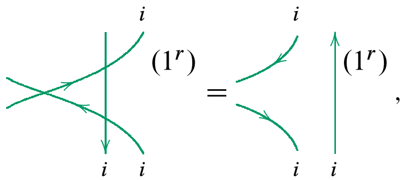


(4.5)

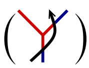
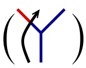
where $\lambda = (1^r)$. Because relation (3.12a) reduces to an ordinary Reidemeister 2 relation when $\lambda = (1^r)$, the left hand side of (4.5) is equal to



So proving (4.5) is equivalent to proving that



(4.6)

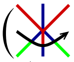
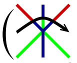
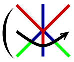
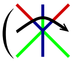
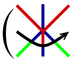
which follows directly from relation (3.35). Indeed in the case $\lambda = (1^r)$, the first term of the l.h.s. of (3.35) is killed since it has a region with a label possessing a negative entry. While the r.h.s. of (3.35) comes down to one term (the second sum does not appear since $\bar{\lambda}_i - 2 = -2$) whose bubble equals -1. Hence $\Sigma_{n,r}$  and $\Sigma_{n,r}$  are equal.

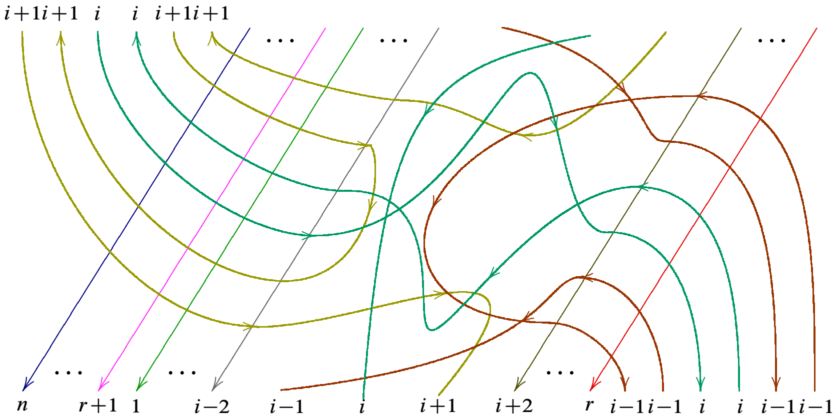
Proving relation (4.4) for colors $(r-1, r)$ is not much more complicated and is left to the reader.

- Relations (2.14a) and (2.14b) involving oriented strands and three adjacent colored strands.

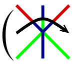
In order to prove these relations we bend the left end of the oriented strand downward and the right end upward in the diagrams.

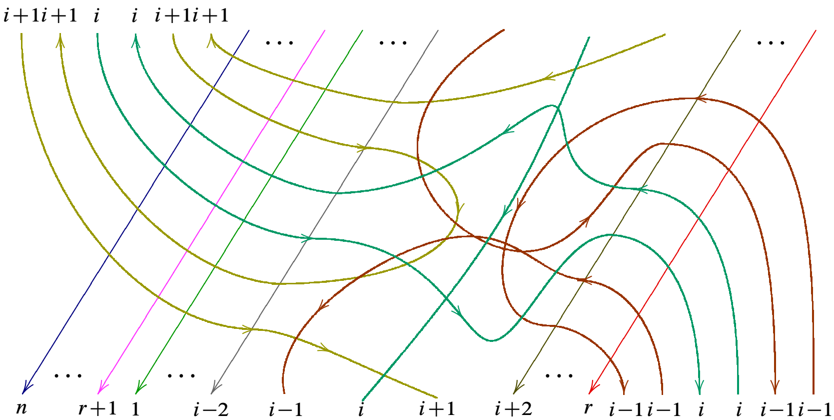
Let us start with the case when there are no r-colored strands. We prove relation (2.14a) for the case in which the bottom strands are colored $(i-1, i, i-1)$. Relation (2.14b) for the case in which the bottom strands are colored $(i, i-1, i)$ can be proved similarly.

By applying relations (3.16), (3.17), (3.19), and (3.34) to the diagrams $\Sigma_{n,r}$() and $\Sigma_{n,r}$(), we can slide the entangled parts of the strands colored $i-1$, i and $i+1$ to the middle of the diagrams. In this way, we can turn $\Sigma_{n,r}$() and $\Sigma_{n,r}$() into the following two diagrams: the diagram $\Sigma_{n,r}$() becomes



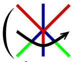
(1^r)

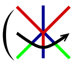
and $\Sigma_{n,r}$() becomes

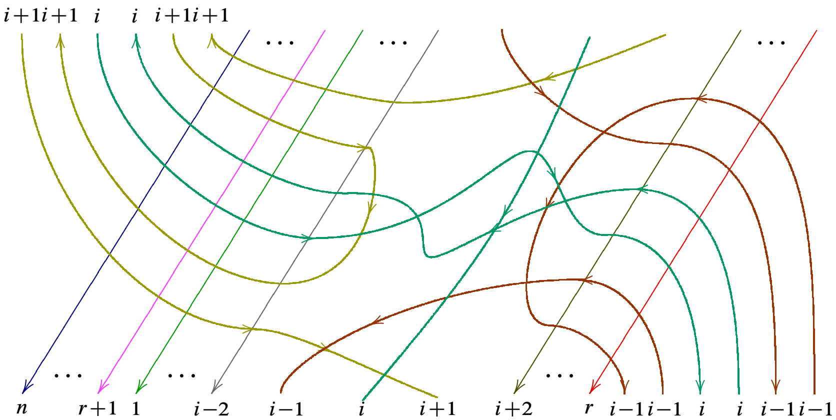


(1^r).

We have to prove that these two diagrams are equivalent.

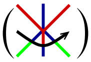
First consider the diagram above which is equivalent to $\Sigma_{n,r}$(). Apply relation (3.35) to the triangle in the center formed by three i-colored strands. Since $\lambda = (1,\ldots,1,0,1,2,0,1,\ldots,1,0,\ldots,0)$, where 2 is in the $i+1$st position and the last 1 in the $r+1$st position, that relation is equal to an ordinary Reidemeister 3 relation (i.e. without extra bubble terms).

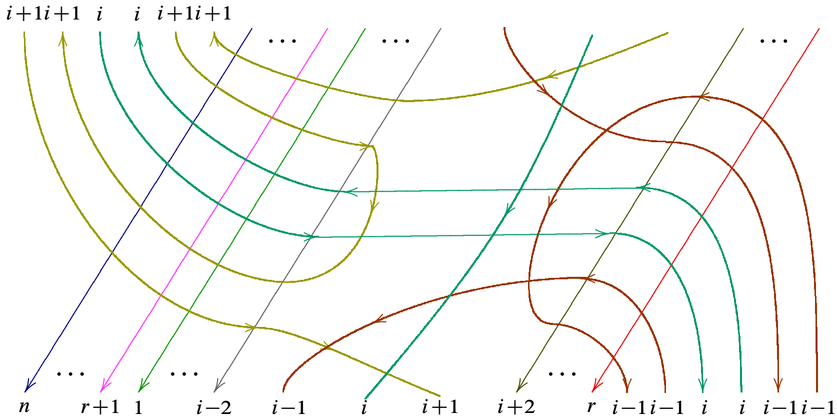
Then apply relation (3.19) to the two triangles formed by strands colored $(i-1,i,i+1)$. Both triangles are slightly to the right of the center and one is higher and the other is lower than the center. Sliding the i-colored strands to the left using this relation creates four bigons, two between strands colored i and $i+1$ and the other two between strands colored $i-1$ and $i+1$. The first two bigons can be solved using relation (3.16), the other two using relation (3.17). Finally, we apply relation (3.19) to the top and bottom central triangles of the diagram. This proves that $\Sigma_{n,r}$() is equivalent to



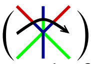
(1^r).

Now apply relation (3.20) to the triangle in the central right part of the diagram. This gives us two terms. The second term, with the identity strands (i.e. the extra term compared to the usual Reidemeister 3 relation), is killed because it contains a bigon between two i-colored strands with the same orientation, which is zero by relation (3.14). In the remaining term we can slide the vertical i-colored strand to the left using relation (3.35), which again simply reduces to an ordinary Reidemeister 3 relation. This leaves us with a bigon between two i-colored strands with opposite orientations. Use relation (3.12b) in order to remove this bigon. Note that this relation is equal to an ordinary Reidemeister 2 relation, since $\lambda = (1,\ldots,1,0,2,1,0,1,\ldots,1,0,\ldots,0)$ with 2 in the ith position and the last 1 in the $r+1$st position.

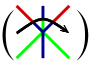
This tells us that $\Sigma_{n,r}\Big($  $\Big)$ is equivalent to

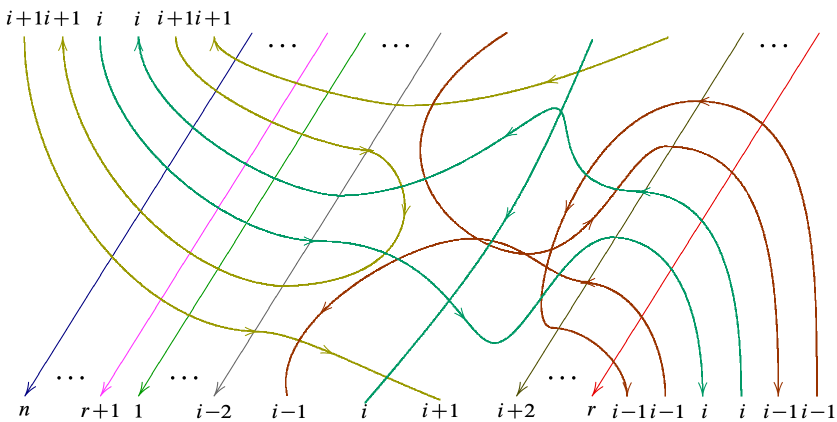


(1^r). (4.7)

Let us now prove that $\Sigma_{n,r}\Big($  $\Big)$ is equivalent to this same diagram. First apply relation (3.35) to the triangle formed by the three $i-1$-colored strands just right of the center. Since $\lambda = (1, \ldots, 1, 2, 0, 0, 1, \ldots, 1, 0, \ldots, 0)$ with 2 in the ith position and the last 1 in the $r+1$st position, this relation is equal to an ordinary Reidemeister 3 relation.

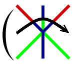
Then apply relation (3.19) to the two central triangles formed by strands colored $(i-1, i, i+1)$. Sliding the $i+1$-colored strands to the left using this relation creates two bigons between strands colored $i-1$ and $i+1$ respectively. These bigons can be removed using relation (3.17).

In this way, we have proved that $\Sigma_{n,r}\Big($  $\Big)$ is equivalent to



(1^r).

Next, apply relation (3.20) to the central left part of the diagram. Locally we end up with a sum of two terms, the usual term in the Reidemeister 3 relation and an extra term consisting in identity strands. This extra term is killed because it contains an $i-1$-colored curl which is zero by relation (3.14). Note that $\lambda = (1,\ldots,1,2,0,0,1,\ldots,1,0,\ldots,0)$ where 2 is in the ith position and the last 1 is in the $r+1$st position, so $\bar{\lambda}_{i-1} = -1$. The first term contains a bigon between two $i-1$-colored strands with opposite orientations. Remove this bigon using relation (3.12a), which in this case is just a Reidemeister 2 relation.

Now apply relation (3.19) to the top central and bottom central parts of the diagram which we have obtained so far. Apply relation (3.34) to the top right and bottom right parts of the diagram. In this way we get two more bigons between strands colored $i-1$ and i with opposite orientations, which we remove using relation (3.16). This finishes our proof that $\Sigma_{n,r}\Big($  $\Big)$ is equivalent to the same diagram in (4.7).

Relation (2.14a) when one of the strands has color r and relation (2.14b) can be dealt with in a similar way. Note that three cases have to be considered: when the bottom strands are colored $(r-2, r-1, r-2)$, $(r,1,r)$ or $(r-1, r, r-1)$. We omit the details.

- Box relations. Just as for relations (2.5a)–(2.10), some of the box relations with $i = r$ or $j = r$ follow from the same box relations for $i, j \neq r$ together with some other box relations. Taking into account the observations in Remark 2.7 too, we see that it suffices to prove relations (2.15b) and (2.15j) here.

Let us start with relation (2.15j). It is sufficient to prove this relation for $i = r-1$, because we can write

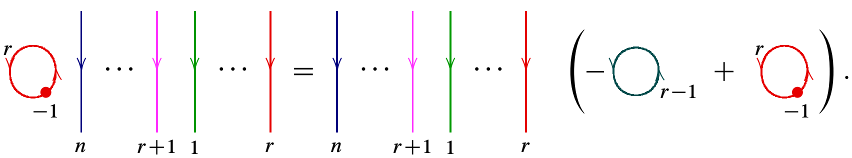
$$\boxed{i+1} = \boxed{i+1} - \boxed{i+2} + \boxed{i+2} - \boxed{i+3} + \cdots + \boxed{r-1} - \boxed{r} + \boxed{r}$$

and

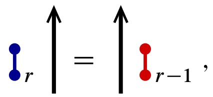
$$\boxed{i} = \boxed{i} - \boxed{i+1} + \boxed{i+1} - \boxed{i+2} + \cdots + \boxed{r-2} - \boxed{r-1} + \boxed{r-1}$$

and use relations (2.13c), (2.13d) and (2.15a).

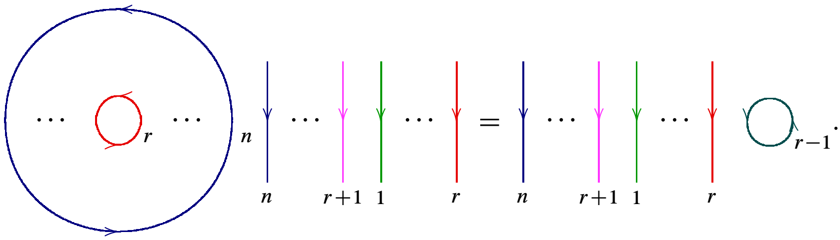
Let us prove relation (2.15j) for $i = r-1$, i.e.



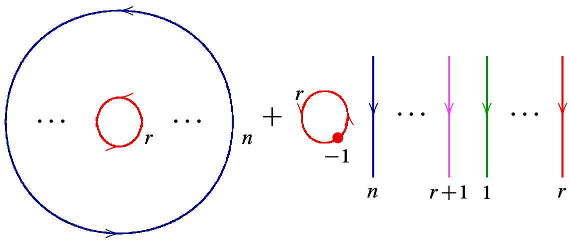
Relations (2.13c) and (2.13d) together imply that

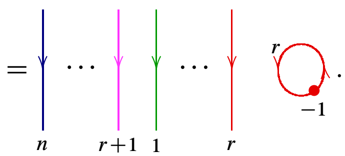


hence we have

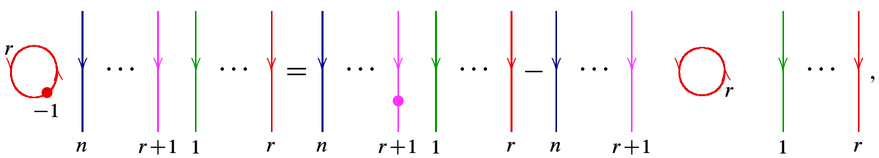


Therefore, it suffices to prove

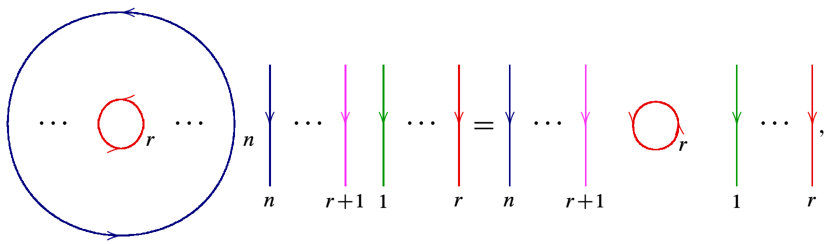




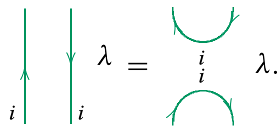
On the one hand, we observe that



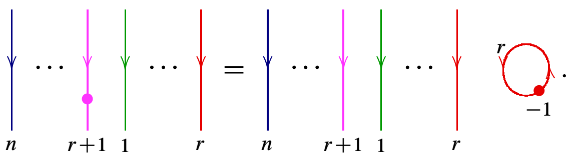
since the bubble can be slid through the first $n - r - 1$ left strands and then the bubble slide relation (3.22) can be applied to the bubble and the strand colored $r+1$. On the other hand, we have



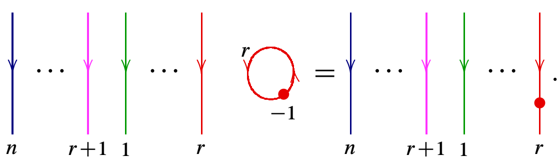
which is obtained by using repeatedly relation (3.12a), which for the relevant labels λ reduces to



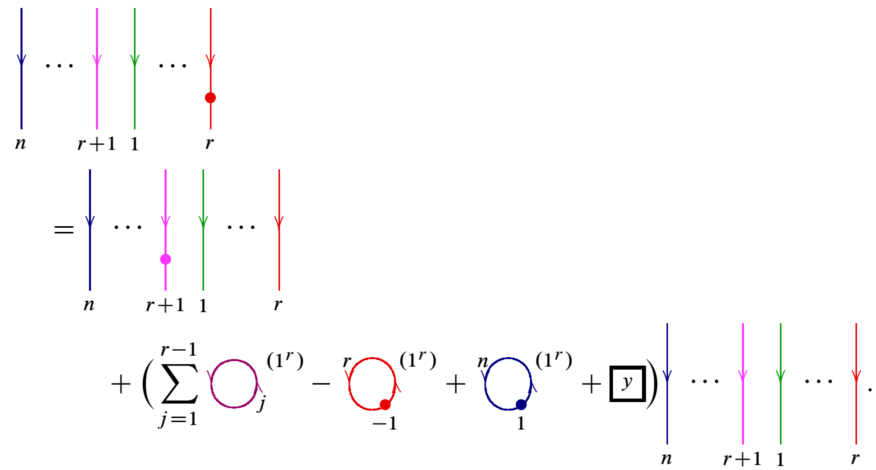
So it only remains to prove that



Notice that we can invert the orientation of the bubble, by the infinite Grassmannian relation. We can apply relation (3.12a) to the bubble and the strand colored r. The first term that appears is killed, because the weight inside the bigon has a negative entry. The bubble appearing in the second term is equal to -1 since $\lambda = (1, \ldots, 1, 0, 1, 0, \ldots, 0)$ with the last one in $r+1$st position. Thus we get

 (4.8)

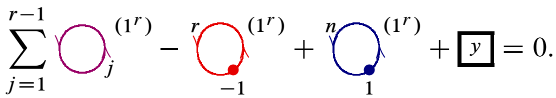
Finally, we have to show that we can move the dot from the r-strand to the $(r+1)$-strand. Just slide the left dotted strand over all the strands colored $1, \ldots, r-1$, using the first case of relation (3.17), and then apply the second case of relation (3.17). Observe that the term with the bigon is killed, because the weight inside the bigon has a negative entry. Note that this argument is not valid if $r = n-1$. Indeed in this case, $r+1$ and 1 are adjacent colors, hence the left dotted strand cannot be simply slid over the strand colored 1. In order to prove this remaining case together with relation (2.15b), let us remark that the right hand side of equation (4.8) can also be expressed as follows:



(4.9)

This expression is obtained using repeatedly kink resolutions and bubble slides. Indeed the kink on the left hand side of relation (3.11) for $i = r$ is equal to zero here since the label inside the kink possesses a negative entry. Hence one can express the dotted r strand as a non-dotted strand times a bubble colored r on the left. This bubble can then be slid through the $r-1$ strand using relation (3.25), creating two terms: one is a dotted $r-1$ strand while the other is a non-dotted $r-1$ strand times a bubble colored r on the left. This bubble can be slid all the way to the left using (3.25) making appear an extra term which is the dotted $r+1$ strand on the right hand side of (4.9). One applies this same trick successively to the dotted strands for colors $r-1$ to 1. The only exception is that, in the end, to slide the bubble colored 1 through the strand colored n, we have to use the deformed relation (3.33), which brings out the y term in (4.9). Finally the dotted n strand that thus appears can also be expressed as a non-dotted strand times a n-colored bubble on the left using relation (3.11).

Therefore, by (4.9), it suffices to prove

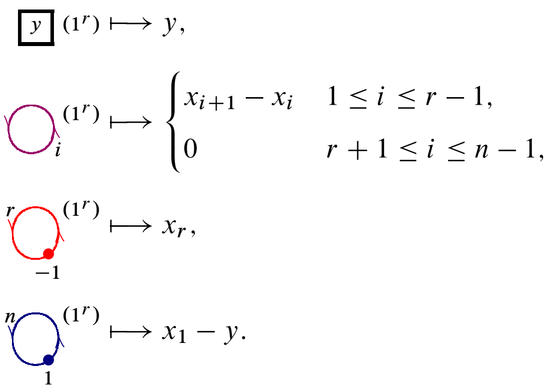


(4.10)

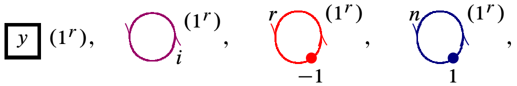
In Section 5 we construct a 2-representation of $\widehat{\mathcal{S}}(n,r)^*_{[y]}$. Its definition and the proof that is well-defined do not depend on the results we are proving here. So, let us just assume the well-definedness of this 2-representation for now. Its restriction to $\widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r))$ gives an algebra homomorphism

$$\mathcal{F}' \colon \widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r)) \longrightarrow \mathbb{Q}[y,x_1,\dots,x_r]$$

which on the elements of degree two is determined by



Note that $\mathcal{F}'$ maps the l.h.s. of (4.10) to zero. We are going to show that this implies (4.10) by showing that $\mathcal{F}'$ is an isomorphism. From the definition it is clear that $\mathcal{F}'$ is surjective. Injectivity follows if we can prove that $\widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r))$ is generated by

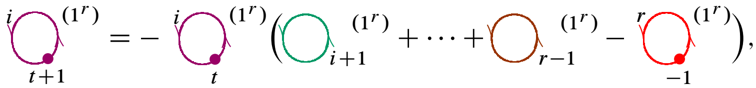


(4.11)

for $i = 1,\dots,r-1$, because that implies that $\widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r))$ is isomorphic to a quotient of $\mathbb{Q}[y,x_1,\dots,x_r]$ by the surjectivity of $\mathcal{F}'$, which means that the two algebras have to be isomorphic.

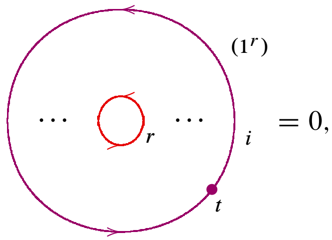
In order to prove that $\widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r))$ is indeed generated by the 2-morphisms in (4.11), first note that $\widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r))$ is generated by all counter-clockwise bubbles of arbitrary degree and . It therefore suffices to prove that any counter-clockwise bubble is in the span of the 2-morphisms in (4.11).

We first prove this fact for counter-clockwise i-bubbles with $1 \leq i \leq r-1$. The result follows from the recursive formula



(4.12)

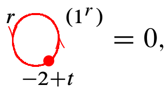
for $t \geq 0$. The equation in (4.12) can be obtained by unnesting the left hand side of



(4.13)

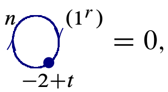
using bubble slides from the inside to the outside. Equation (4.13) holds, because the region in the center has label λ with $\lambda_{r+1} = -1$.

For $i = r$ the argument is simpler, because

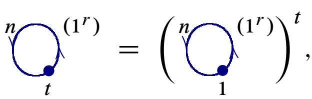


for any $t \geq 2$. This holds because the inner region has label λ with $\lambda_{r+1} = -1$.

Similarly, for $i = n$ we have



for any $t \geq 2$. By the infinite Grassmannian relation, this implies that



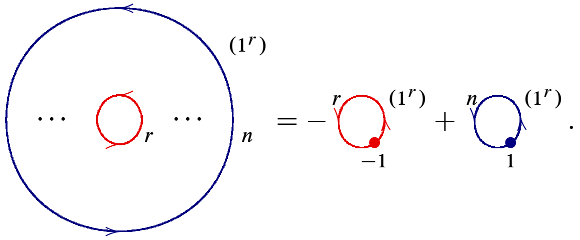
for any $t \geq 2$.

For $r+1 \leq i \leq n-1$ there is nothing to prove. In that case, the counter-clockwise i-bubbles of positive degree are all zero, because their interior is labeled by λ with $\lambda_{i+1} = -1$. This finishes the proof that

$$\widehat{S}(n,r)^*_{[y]}((1^r),(1^r)) \cong \mathbb{Q}[y, x_1, \ldots, x_r],$$

which implies (4.10).

Now let us consider relation (2.15b). The image under $\Sigma_{n,r}$ of $\,_r$ is



(4.14)

This expression can be obtained using repeatedly bubble slide relation (3.23). Observe that, at each step but the first one, only one term survives, the second being systematically zero since it includes a real bubble whose inside is a label with a negative entry.

If one replaces, in this expression, the bubble colored n using equation (4.10), one recognizes precisely the image under $\Sigma_{n,r}$ of $\boxed{1} - \boxed{r} - \boxed{y}$.

We have checked that $\Sigma_{n,r}$ preserves all the relations of $\mathcal{DEB}im^*_{\hat{A}_{r-1}}$, so this ends the proof. □

5. A 2-representation of $\hat{\mathcal{S}}(n,r)^*_{[y]}$

In this section we define a 2-category $\mathcal{ESB}im_{\hat{A}_{r-1}}$ and a 2-functor

$$\mathcal{F}' : \hat{\mathcal{S}}(n,r)^*_{[y]} \longrightarrow \mathcal{ESB}im^*_{\hat{A}_{r-1}}.$$

The 2-category $\mathcal{ESB}im_{\hat{A}_{r-1}}$ is an extension of the category of singular Soergel bimodules in affine type A considered by Williamson in [37] (see also [27, 28]). The 2-functor $\mathcal{F}'$ is a generalization of Khovanov and Lauda's 2-representation Γ_r^G defined in [17, 19].

5.1. Extended singular bimodules. Let $R = \mathbb{Q}[y][x_1, \ldots, x_r]$. As in Section 2.1.2, there is a grading on R defined by $\deg y = \deg x_i = 2$, for $i = 1, \ldots, r$, and a degree preserving action of $\widehat{W}_{\hat{A}_{r-1}}$ on R. For any partition $(n_1, \ldots, n_k)$ of r, let

$$S_{n_1} \times \cdots \times S_{n_k} \subseteq W_{A_{r-1}} \subset \widehat{W}_{\hat{A}_{r-1}}$$

be the standard parabolic subgroup which is contained in the finite Weyl group. Let $R^{n_1 \cdots n_k} \subseteq R$ denote the subring of $S_{n_1} \times \cdots \times S_{n_k}$-invariant polynomials.

Using these rings of partially symmetric polynomials, we can construct bimodules by induction and restriction. Induction is defined as follows: suppose n_j splits into n_j^0 plus n_j^1, i.e. $n_j = n_j^0 + n_j^1$, then we define the *induction functor* by taking the tensor product on the left with $R^{n_1 \ldots n_j^0 n_j^1 \cdots n_k}$ over $R^{n_1 \cdots n_k}$. The *restriction functor* is defined by taking the tensor product on the left with $R^{n_1 \ldots n_j + n_{j+1} \cdots n_k} \otimes_{R^{n_1 \ldots n_j + n_{j+1} \cdots n_k}} R^{n_1 \cdots n_k}$ over $R^{n_1 \cdots n_k}$. In particular,

$$\mathrm{Ind}_{n_j}^{n_j^0, n_j^1}(R^{n_1 \cdots n_k}) = R^{n_1 \ldots n_j^0 n_j^1 \cdots n_k} \otimes_{n_1 \cdots n_k} R^{n_1 \cdots n_k},$$

and

$$\mathrm{Res}_{n_j, n_{j+1}}^{n_j + n_{j+1}}(R^{n_1 \cdots n_k}) = R^{n_1 \ldots n_j + n_{j+1} \cdots n_k} \otimes_{n_1 \ldots n_j + n_{j+1} \cdots n_k} R^{n_1 \cdots n_k}$$

where a subscript $n_1 \cdots n_k$ of a tensor product means that it is taken over $R^{n_1 \cdots n_k}$. Also note that $\mathrm{Ind}_{n_j}^{n_j^0, n_j^1}(M)$ is zero (resp. $\mathrm{Res}_{n_j, n_{j+1}}^{n_j + n_{j+1}}(M)$ is zero) if M is a $R^{m_1 \cdots m_k}$-left module such that $m_j \neq n_j$ (resp. such that $(m_j, m_{j+1}) \neq (n_j, n_{j+1})$).

Now some twisted bimodules are defined like the bimodules $B_{\rho^{\pm 1}}$ of Section 2.1.2, their well-definedness is proved in Lemma 5.2: $R_\rho^{n_k n_1 \cdots n_{k-1}}$ (resp. $R_{\rho^{-1}}^{n_2 \cdots n_k n_1}$) is equal to $R^{n_k n_1 \cdots n_{k-1}}$ (resp. $R^{n_2 \cdots n_k n_1}$) as a left $R^{n_k n_1 \cdots n_{k-1}}$-module (resp. as a left $R^{n_2 \cdots n_k n_1}$-module) whereas the action on the right is twisted. The right action of any $a \in R^{n_1 \cdots n_k}$ on $R_\rho^{n_k n_1 \cdots n_{k-1}}$ and $R_{\rho^{-1}}^{n_2 \cdots n_k n_1}$ is given by multiplication by $\rho^{n_k}(a)$ and $\rho^{-n_1}(a)$ respectively. Recall that the action of $\rho^{\pm 1}$ was defined in section 2.1.1. The *twisted functors* defined by taking the tensor product on the left with these twisted bimodules over $R^{n_1 \cdots n_k}$ are denoted by $R_{\rho^{n_k}}$ and $R_{\rho^{-n_1}}$. In particular,

$$R_{\rho^{n_k}}(R^{n_1 \cdots n_k}) = R_\rho^{n_k n_1 \cdots n_{k-1}} \otimes_{n_1 \cdots n_k} R^{n_1 \cdots n_k}$$

and

$$R_{\rho^{-n_1}}(R^{n_1 \cdots n_k}) = R_{\rho^{-1}}^{n_2 \cdots n_k n_1} \otimes_{n_1 \cdots n_k} R^{n_1 \cdots n_k}$$

respectively.

Definition 5.1. Let $\mathcal{ESBim}_{\hat{A}_{r-1}}$ be the 2-category with

- objects: the rings $R^{n_1 \cdots n_k}$, for all partitions $(n_1, \ldots, n_k)$ of r;
- 1-morphisms: given two partitions $(n_1, \ldots, n_k)$ and $(m_1, \ldots, m_l)$ of r, the 1-morphisms between $R^{n_1 \cdots n_k}$ and $R^{m_1 \cdots m_l}$ are the direct sums of shifts of tensor products of $R^{m_1 \cdots m_l}$-$R^{n_1 \cdots n_k}$-bimodules obtained by applying induction, restriction and twisted functors to these rings of partially symmetric polynomials;
- 2-morphisms: the degree preserving bimodule maps.

Lemma 5.2. *The map ρ^{n_k} gives an isomorphism between $R^{n_1\cdots n_k}$ and $R^{n_k n_1\cdots n_{k-1}}$, while ρ^{-n_1} gives an isomorphism between $R^{n_1\cdots n_k}$ and $R^{n_2\cdots n_k n_1}$.*

Proof. We only prove the lemma for ρ^{n_k}. The proof for ρ^{-n_1} is similar and is left to the reader.

It is clear that ρ^{n_k} is a bijection. What remains to be shown is that the image of $R^{n_1\cdots n_k}$ is indeed $R^{n_k n_1\cdots n_{k-1}}$.

For any generating reflection σ_l of $S_{n_k} \times S_{n_1} \times \cdots \times S_{n_{k-1}}$, relation (1.1d) implies that $\rho^{-n_k}\sigma_l\rho^{n_k} = \sigma_{l-n_k}$ (with the indices read modulo r) and hence this latter belongs to $S_{n_1} \times \cdots \times S_{n_k}$. In particular, for all $p \in R^{n_1\cdots n_k}$, we have

$$\rho^{-n_k}\sigma_l\rho^{n_k}(p) = \sigma_{l-n_k}(p) = p$$

i.e. $\rho^{n_k}(p) \in S_{n_k} \times S_{n_1} \times \cdots \times S_{n_k-1}$.

This shows that ρ^{n_k} sends the ring $R^{n_1\cdots n_k}$ isomorphically to $R^{n_k n_1\cdots n_{k-1}}$. □

This implies that the twisted bimodules $R_\rho^{n_k n_1\cdots n_{k-1}}$ and $R_{\rho^{-1}}^{n_2\cdots n_k n_1}$ are well-defined.

The proof of the following lemma is straightforward and is left to the reader.

Lemma 5.3. *We have the following isomorphisms of bimodules relating twisting, induction and restriction*:

$$R_{\rho^{-n_k}} R_{\rho^{n_k}}(R^{n_1\cdots n_k}) \cong R_{\rho^{n_1}} R_{\rho^{-n_1}}(R^{n_1\cdots n_k}) \cong R^{n_1\cdots n_k},$$

$$\mathrm{Ind}_{n_j}^{n_j^0,n_j^1} R_{\rho^{n_k}}(R^{n_1\cdots n_k}) \cong R_{\rho^{n_k}} \mathrm{Ind}_{n_j}^{n_j^0,n_j^1}(R^{n_1\cdots n_k}) \quad \textit{for } j \neq k,$$

$$\mathrm{Ind}_{n_k}^{n_k^0,n_k^1} R_{\rho^{n_k}}(R^{n_1\cdots n_k}) \cong R_{\rho^{n_k^0}} R_{\rho^{n_k^1}} \mathrm{Ind}_{n_k}^{n_k^0,n_k^1}(R^{n_1\cdots n_k}),$$

$$\mathrm{Res}_{n_j,n_{j+1}}^{n_j+n_{j+1}} R_{\rho^{n_k}}(R^{n_1\cdots n_k}) \cong R_{\rho^{n_k}} \mathrm{Res}_{n_j,n_{j+1}}^{n_j+n_{j+1}}(R^{n_1\cdots n_k}) \quad \textit{for } j \neq k-1,$$

$$\mathrm{Res}_{n_{k-1},n_k}^{n_{k-1}+n_k} R_{\rho^{n_{k-1}}} R_{\rho^{n_k}}(R^{n_1\cdots n_k}) \cong R_{\rho^{n_{k-1}+n_k}} \mathrm{Res}_{n_{k-1},n_k}^{n_{k-1}+n_k}(R^{n_1\cdots n_k}).$$

There exist analogous isomorphisms for the negative twists.

Lemma 5.4. *The category $\mathcal{EB}im_{\widehat{A}_{r-1}}$ is a full subcategory of $\mathcal{ESB}im_{\widehat{A}_{r-1}}$.*

Proof. For $i = 1, \ldots, r-1$, the full embedding of $\mathcal{EB}im_{\widehat{A}_{r-1}}$ into $\mathcal{ESB}im_{\widehat{A}_{r-1}}$ sends B_i to B_i and $B_{\rho^{\pm 1}}$ to $B_{\rho^{\pm 1}} = R^{(1^r)}_{\rho^{\pm 1}}$. For $i = r$, the bimodule B_r is sent to

$$B_\rho \otimes_R B_{r-1} \otimes_R B_{\rho^{-1}} \in \mathcal{ESB}im_{\widehat{A}_{r-1}}.$$

The fact that the isomorphism in $\mathcal{EB}im_{\widehat{A}_{r-1}}$ between B_r and $B_\rho \otimes_R B_{r-1} \otimes_R B_{\rho^{-1}}$ is unique up to scalar ensures that the category $\mathcal{EB}im_{\widehat{A}_{r-1}}$ is a full subcategory of $\mathcal{ESB}im_{\widehat{A}_{r-1}}$. □

5.2. The 2-representation. We will mostly refer the reader to [17, 19, 28] for the definition of

$$\mathcal{F}' : \widehat{\mathcal{S}}(n,r)^*_{[y]} \longrightarrow \mathcal{ESB}im^*_{\widehat{A}_{r-1}},$$

since $\mathcal{F}'$ is a straigthforward generalization of the equivariant Khovanov-Lauda 2-representations discussed in those papers.

Remark 5.5. Khovanov and Lauda used the equivariant cohomology rings of the varieties of partial flags in $\mathbb{C}^r$ for the definition of their equivariant 2-representations. These cohomology rings are isomorphic to the finite type A singular Soergel bimodules which were used in [28].

We do not know if the 2-representation in this paper, which we define using the extended affine singular Soergel bimodules, can be defined in terms of equivariant cohomology rings of the varieties of cyclic partial flags (or periodic lattices) in $\mathbb{C}[\varepsilon, \varepsilon^{-1}]^r$ defined in [26] and [11].

5.2.1. Definition of $\mathcal{F}'$. Note that for $y = 0$ the restriction of $\mathcal{F}' \circ \Psi_{n,r}$ to $\mathcal{U}(\mathfrak{sl}_n)$ is simply equal to Γ^G_r, where

$$\Psi_{n,r} \colon \mathcal{U}(\widehat{\mathfrak{sl}_n})^*_{[y]} \longrightarrow \widehat{\mathcal{S}}(n,r)^*_{[y]}$$

was defined just before Proposition 3.25.

We will define the 2-functor $\mathcal{F}'$ on all objects and 1-morphisms of $\widehat{\mathcal{S}}(n,r)^*_{[y]}$, give explicitly the images of the 2-morphisms for which the color n appears and explain afterwards how they are related to Khovanov and Lauda's 2-representation Γ^G_r, see Proposition 5.8. Here we are using their notation $k_i = \lambda_1 + \cdots + \lambda_i$, for $i = 1, \ldots, n$, with the convention that $k_0 = 0$. The origin of the shifts appearing in the images of the 1-morphisms is explained in Remark 5.7 and the well-definedness of $\mathcal{F}'$ is proved in Proposition 5.10.

Definition 5.6. The 2-functor

$$\mathcal{F}' \colon \widehat{\mathcal{S}}(n,r)^*_{[y]} \longrightarrow \mathcal{E}\mathcal{S}\mathcal{B}im^*_{\widehat{A}_{r-1}}$$

is defined as follows.

- On objects $\lambda \in \Lambda(n,r)$, the 2-functor $\mathcal{F}'$ is given by

$$\lambda = (\lambda_1, \cdots, \lambda_n) \longmapsto R^{\lambda_1 \cdots \lambda_n}.$$

- on 1-morphisms we define $\mathcal{F}'$ by

$$\mathbf{1}_\lambda\{t\} \longmapsto R^{\lambda_1 \cdots \lambda_n}\{t\}.$$

  For $i = 1, \ldots, n-1$ and $t \in \mathbb{Z}$ we define

$$\mathcal{E}_i \mathbf{1}_\lambda\{t\} \longmapsto \mathrm{Res}^{\lambda_i+1}_{\lambda_i,1} \, \mathrm{Ind}^{1,\lambda_{i+1}-1}_{\lambda_{i+1}} (R^{\lambda_1 \cdots \lambda_n}\{t + 1 + k_{i-1} + k_i - k_{i+1}\})$$

  and

$$\mathcal{E}_{-i} \mathbf{1}_\lambda\{t\} \longmapsto \mathrm{Res}^{\lambda_{i+1}+1}_{1,\lambda_{i+1}} \, \mathrm{Ind}^{\lambda_i-1,1}_{\lambda_i} (R^{\lambda_1 \cdots \lambda_n}\{t + 1 - k_i\}).$$

  For $i = n$ and $t \in \mathbb{Z}$ we define

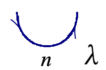
$$\mathcal{E}_n \mathbf{1}_\lambda\{t\} \longmapsto \mathrm{Res}^{\lambda_n+1}_{\lambda_n,1} \, R_{\rho^{-1}} \, \mathrm{Ind}^{1,\lambda_1-1}_{\lambda_1} (R^{\lambda_1 \cdots \lambda_n}\{t + n - (r + k_1) - (k_1 + \cdots + k_{n-2})\})$$

  and

$$\mathcal{E}_{-n} \mathbf{1}_\lambda\{t\} \longmapsto \mathrm{Res}^{\lambda_1+1}_{1,\lambda_1} \, R_\rho \, \mathrm{Ind}^{\lambda_n-1,1}_{\lambda_n} (R^{\lambda_1 \cdots \lambda_n}\{t + k_1 + \cdots + k_{n-1}\}).$$

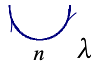
- On 2-morphisms, we define $\mathcal{F}'$ by giving the bimodule maps which correspond to the generating 2-morphisms of $\widehat{\mathcal{S}}(n,r)^*_{[y]}$. The ones not involving color n have the same image as under Khovanov and Lauda's 2-representation [17, Section 6.3.3] [19] [28, Section 4.2].

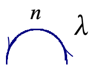
  When the color n occurs in a generating 2-morphism, the 2-functor $\mathcal{F}'$ is as follows. Here e_α and h_α denote the elementary and the complete symmetric polynomials respectively:

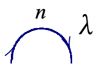
$\mathcal{F}'\Big($  $\Big)$

$$: 1 \longmapsto \sum_{f=0}^{\lambda_n} (-1)^{\lambda_n - f} \, x_1^f \otimes 1 \otimes 1 \otimes e_{\lambda_n - f}(x_{r-\lambda_n+1} + y, \ldots, x_r + y)$$

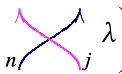
$$= \sum_{f=0}^{\lambda_n} (-1)^{\lambda_n - f} \, (x_1 - y)^f \otimes 1 \otimes 1 \otimes e_{\lambda_n - f}(x_{r-\lambda_n+1}, \ldots, x_r),$$

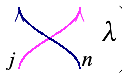
$$\mathcal{F}'\Big($$  $$\Big) : 1 \longmapsto (-1)^{\lambda_1} \sum_{f=0}^{\lambda_1} (-1)^{\lambda_1 - f}\ (x_r + y)^f \otimes 1 \otimes 1 \otimes e_{\lambda_1 - f}(x_1, \dots, x_{\lambda_1})$$
$$= (-1)^{\lambda_1} \sum_{f=0}^{\lambda_1} (-1)^{\lambda_1 - f}\ x_r^f \otimes 1 \otimes 1 \otimes e_{\lambda_1 - f}(x_1 - y, \dots, x_{\lambda_1} - y),$$

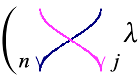
$$\mathcal{F}'\Big($$  $$\Big) : x_1^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}$$
$$\longmapsto (-1)^{\lambda_1 + 1} h_{\alpha_1 + \alpha_2 + 1 - \lambda_1}(x_1, \dots, x_{\lambda_1})$$
$$= (-1)^{\lambda_1 + 1} \sum_{p=0}^{\alpha_1} \sum_{q=0}^{\alpha_2} \binom{\alpha_1}{p} \binom{\alpha_2}{q} y^{\alpha_1 + \alpha_2 - p - q} h_{p+q+1-\lambda_1}(x_1 - y, \dots, x_{\lambda_1} - y),$$

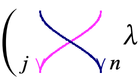
$$\mathcal{F}'\Big($$  $$\Big) : x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_r^{\alpha_2}$$
$$\longmapsto \sum_{p=0}^{\alpha_1} \sum_{q=0}^{\alpha_2} \binom{\alpha_1}{p} \binom{\alpha_2}{q} (-y)^{\alpha_1 + \alpha_2 - p - q} h_{\alpha_1 + \alpha_2 + 1 - \lambda_n}(x_{r - \lambda_n + 1} + y, \dots, x_r + y)$$
$$= h_{\alpha_1 + \alpha_2 + 1 - \lambda_n}(x_{r - \lambda_n + 1}, \dots, x_r).$$

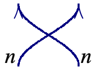
If $|n - j| > 1$,

$$\mathcal{F}'\Big($$  $$\Big) : x_r^{\alpha_1} \otimes 1 \otimes x_{k_j+1}^{\alpha_2} \longmapsto x_{k_j}^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1},$$

$$\mathcal{F}'\Big($$  $$\Big) : x_{k_j}^{\alpha_1} \otimes 1 \otimes x_1^{\alpha_2} \longmapsto (x_r + y)^{\alpha_2} \otimes 1 \otimes x_{k_j+1}^{\alpha_1},$$

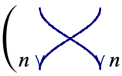
$$\mathcal{F}'\Big($$  $$\Big) : x_1^{\alpha_1} \otimes 1 \otimes x_{k_j}^{\alpha_2} \longmapsto x_{k_j+1}^{\alpha_2} \otimes 1 \otimes (x_r + y)^{\alpha_1},$$

$$\mathcal{F}'\Big($$  $$\Big) : x_{k_j 1}^{\alpha_1} \otimes 1 \otimes x_r^{\alpha_2} \longmapsto (x_1 - y)^{\alpha_2} \otimes 1 \otimes x_{k_j}^{\alpha_1},$$

$\mathcal{F}'\Bigg($ 

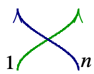
$\lambda\Bigg) : x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}$

$$\begin{aligned}
&\longmapsto (x_r + y)^{\alpha_2} \sum_{p=0}^{\alpha_1} \sum_{f=0}^{p-1} \binom{\alpha_1}{p} (-y)^{\alpha_1 - p} (x_r + y)^{p-1-f} \otimes 1 \otimes 1 \otimes x_1^f \\
&\qquad - x_r^{\alpha_1} \sum_{g=0}^{\alpha_2 - 1} (x_r + y)^{\alpha_2 - 1 - g} \otimes 1 \otimes 1 \otimes x_1^g \\
&\quad = (x_r + y)^{\alpha_2} \sum_{f=0}^{\alpha_1 - 1} x_r^{\alpha_1 - 1 - f} \otimes 1 \otimes 1 \otimes (x_1 - y)^f \\
&\qquad - x_r^{\alpha_1} \sum_{q=0}^{\alpha_2} \sum_{g=0}^{q-1} \binom{\alpha_2}{q} y^{\alpha_2 - q} x_r^{q-1-g} \otimes 1 \otimes 1 \otimes (x_1 - y)^g,
\end{aligned}$$

$\mathcal{F}'\Bigg($ 

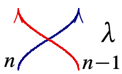
$\lambda\Bigg) : x_1^{\alpha_1} \otimes 1 \otimes 1 \otimes x_r^{\alpha_2}$

$$\begin{aligned}
&\longmapsto x_1^{\alpha_1} \sum_{p=0}^{\alpha_2} \sum_{f=0}^{p-1} \binom{\alpha_2}{p} (-y)^{\alpha_2 - p} x_1^{p-1-f} \otimes 1 \otimes 1 \otimes (x_r + y)^f \\
&\qquad - (x_1 - y)^{\alpha_2} \sum_{g=0}^{\alpha_1 - 1} x_1^{\alpha_1 - 1 - g} \otimes 1 \otimes 1 \otimes (x_r + y)^g \\
&\quad = x_1^{\alpha_1} \sum_{f=0}^{\alpha_2 - 1} (x_1 - y)^{\alpha_2 - 1 - f} \otimes 1 \otimes 1 \otimes x_r^f \\
&\qquad - (x_1 - y)^{\alpha_2} \sum_{q=0}^{\alpha_1} \sum_{g=0}^{q-1} \binom{\alpha_1}{q} y^{\alpha_1 - q} (x_1 - y)^{q-1-g} \otimes 1 \otimes 1 \otimes x_r^g,
\end{aligned}$$

$\mathcal{F}'\Bigg($ 

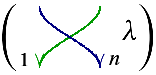
$\lambda\Bigg) : x_{\lambda_1}^{\alpha_1} \otimes 1 \otimes x_1^{\alpha_2}$

$$\begin{aligned}
&\longmapsto ((x_r + y)^{\alpha_2} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_1+1} - (x_r + y)^{\alpha_2+1} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_1})\{-1\} \\
&\quad = ((x_r + y)^{\alpha_2} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_1} (x_{\lambda_1+1} - y) \\
&\qquad - x_r (x_r + y)^{\alpha_2} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_1})\{-1\},
\end{aligned}$$

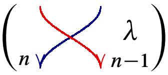
$\mathcal{F}'\Bigg($ 

$\lambda\Bigg) : x_r^{\alpha_1} \otimes 1 \otimes x_{r-\lambda_n+1}^{\alpha_2}$

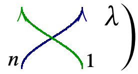
$$\begin{aligned}
&\longmapsto (x_{r-\lambda_n}^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1} x_1 \\
&\qquad - (x_{r-\lambda_n} + y) x_{r-\lambda_n}^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1})\{-1\} \\
&\quad = (x_{r-\lambda_n}^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1+1} - x_{r-\lambda_n}^{\alpha_2+1} \otimes 1 \otimes (x_1 - y)^{\alpha_1})\{-1\},
\end{aligned}$$

$\mathcal{F}'$ 

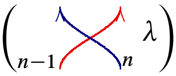
$$: x_{\lambda_1+1}^{\alpha_1} \otimes 1 \otimes x_r^{\alpha_2} \longmapsto ((x_1 - y)^{\alpha_2} \otimes 1 \otimes x_{\lambda_1}^{\alpha_1})\{-1\},$$

$\mathcal{F}'$ 

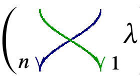
$$: x_1^{\alpha_1} \otimes 1 \otimes x_{r-\lambda_n}^{\alpha_2} \longmapsto (x_{r-\lambda_n+1}^{\alpha_2} \otimes 1 \otimes (x_r + y)^{\alpha_1})\{-1\},$$

$\mathcal{F}'$ 

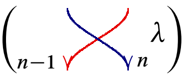
$$: x_r^{\alpha_1} \otimes 1 \otimes x_{\lambda_1+1}^{\alpha_2} \longmapsto (x_{\lambda_1}^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1})\{1\},$$

$\mathcal{F}'$ 

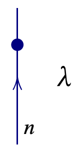
$$: x_{r-\lambda_n}^{\alpha_1} \otimes 1 \otimes x_1^{\alpha_2} \longmapsto ((x_r + y)^{\alpha_2} \otimes 1 \otimes x_{r-\lambda_n+1}^{\alpha_1})\{1\},$$

$\mathcal{F}'$ 

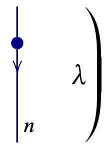
$$\begin{aligned} &: x_1^{\alpha_1} \otimes 1 \otimes x_{\lambda_1}^{\alpha_2} \\ &\longmapsto (x_{\lambda_1+1}^{\alpha_2+1} \otimes 1 \otimes (x_r + y)^{\alpha_1} - x_{\lambda_1+1}^{\alpha_2} \otimes 1 \otimes (x_r + y)^{\alpha_1+1})\{1\} \\ &\quad = (x_{\lambda_1+1}^{\alpha_2}(x_{\lambda_1+1} - y) \otimes 1 \otimes (x_r + y)^{\alpha_1} \\ &\qquad - x_{\lambda_1+1}^{\alpha_2} \otimes 1 \otimes x_r(x_r + y)^{\alpha_1})\{1\}, \end{aligned}$$

$\mathcal{F}'$ 

$$\begin{aligned} &: x_{r-\lambda_n+1}^{\alpha_1} \otimes 1 \otimes x_r^{\alpha_2} \\ &\longmapsto ((x_1 - y)^{\alpha_2} x_1 \otimes 1 \otimes x_{r-\lambda_n}^{\alpha_1} - (x_1 - y)^{\alpha_2} \otimes 1 \otimes (x_{r-\lambda_n} + y) x_{r-\lambda_n}^{\alpha_1})\{1\} \\ &\quad = ((x_1 - y)^{\alpha_2+1} \otimes 1 \otimes x_{r-\lambda_n}^{\alpha_1} - (x_1 - y)^{\alpha_2} \otimes 1 \otimes x_{r-\lambda_n}^{\alpha_1+1})\{1\}, \end{aligned}$$

$\mathcal{F}'$ 

$$: 1 \otimes 1 \longmapsto x_r \otimes 1 = 1 \otimes (x_1 - y),$$

$\mathcal{F}'$ 

$$: 1 \otimes 1 \mapsto (x_1 - y) \otimes 1 = 1 \otimes x_r.$$

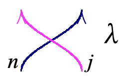
Remark 5.7. Let us explain where the shifts in the image of the new 1-morphisms come from. We will denote by $A_n(\lambda)$ the shift appearing in $\mathcal{F}'(\mathcal{E}_n \mathbf{1}_\lambda)$ and by $B_n(\lambda)$ the one appearing in $\mathcal{F}'(\mathcal{E}_{-n}\mathbf{1}_\lambda)$. To understand their origin, let us go back to the decategorified level (see Section 3.4), where the embedding of $\widehat{\mathbf{S}}(n,r)$ into $\widehat{\mathbf{S}}(n+1,r)$ sends $E_n 1_\lambda$ to $E_n E_{n+1} 1_{(\lambda,0)}$ and $E_{-n}1_\lambda$ to $E_{-(n+1)}E_{-n}1_{(\lambda,0)}$. We want this embedding to have a categorical analogue. Although we will not work out the details of the corresponding functor in this paper, a necessary condition for the existence of such a functor is that the shifts satisfy the following recurrence relations:

$$A_n(\lambda) = k_{n-1} - 1 + A_{n+1}(\lambda),$$

$$B_n(\lambda) = B_{n+1}(\lambda') + 1 - k_n,$$

where $\lambda' = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - 1, 1)$. This determines the value of $A_n(\lambda)$ and $B_n(\lambda)$ up to a constant which does not depend on n. To fix these constants, we use the fact that we want the triangle of lemma 6.1 to be commutative and the 2-functor $\mathcal{F}'$ to be degree preserving. Note that $\mathcal{F}(-)$ has a shift equal to zero, so $\mathcal{F}'(\mathcal{E}_r \ldots \mathcal{E}_1 \mathcal{E}_{r+1} \ldots \mathcal{E}_n \mathbf{1}_r) = \mathcal{F}' \Sigma_{n,r}(-)$ should have a shift equal to zero too. In this way we obtain a constraint on $A_n((1^r))$ and we deduce that the aforementioned overall constant is equal to $-(r+k_1)$. Similarly $\mathcal{F}(+)$ has a shift equal to zero, so $\mathcal{F}'(\mathcal{E}_{-n} \ldots \mathcal{E}_{-r-1}\mathcal{E}_{-1} \ldots \mathcal{E}_{-r}\mathbf{1}_r) = \mathcal{F}' \Sigma_{n,r}(+)$ has to have a zero-shift too. This gives us a constraint on $B_n((1^r) + \bar{\alpha}_n)$ and allows us to deduce that the aforementioned overall constant is equal to zero. The reader can verify that this choice of overall constants also fits with the -1-shift of $\mathcal{F}'(\mathcal{E}_{-n} \ldots \mathcal{E}_{-r}\mathcal{E}_r \ldots \mathcal{E}_n \mathbf{1}_r) = \mathcal{F}' \Sigma_{n,r}(r)$.

It is natural to wonder how the previous images of the 2-morphisms in the definition of $\mathcal{F}'$ relate to Khovanov and Lauda's 2-representation Γ_r^G. Indeed, take a generating 2-morphism with n-strands. By Lemma 5.3, the images of its source and target 1-morphisms are isomorphic, up to a same shift, to conjugates of the images of 1-morphisms which do not contain factors $\mathcal{E}_{\pm n}$. Here conjugation means conjugation by certain invertible twisted bimodules. One can thus ask if the image of the 2-morphism can be obtained by conjugating a 2-morphism which does not contain strands of color n. Before answering this question in the "conjugation trick" Proposition 5.8, let us do an example to make things more concrete (we omit the shifts here). Consider the 2-morphism



with $|n-j|>1$. The image of its source and target 1-morphisms are

$$\mathrm{Res}^{\lambda_n+1}_{\lambda_n,1} R_{\rho^{-1}} \mathrm{Ind}^{1,\lambda_1-1}_{\lambda_1} \mathrm{Res}^{\lambda_j+1}_{\lambda_j,1} \mathrm{Ind}^{1,\lambda_{j+1}-1}_{\lambda_{j+1}} (R^{\lambda_1,\dots,\lambda_n})$$

and

$$\mathrm{Res}^{\lambda_j+1}_{\lambda_j,1} \mathrm{Ind}^{1,\lambda_{j+1}-1}_{\lambda_{j+1}} \mathrm{Res}^{\lambda_n+1}_{\lambda_n,1} R_{\rho^{-1}} \mathrm{Ind}^{1,\lambda_1-1}_{\lambda_1} (R^{\lambda_1,\dots,\lambda_n}).$$

These are isomorphic to

$$R_{\rho^{-(\lambda_n+1)}} \mathrm{Res}^{\lambda_n+1}_{\lambda_n,1} \mathrm{Ind}^{1,\lambda_1-1}_{\lambda_1} \mathrm{Res}^{\lambda_j+1}_{\lambda_j,1} \mathrm{Ind}^{1,\lambda_{j+1}-1}_{\lambda_{j+1}} R_{\rho^{\lambda_n}} (R^{\lambda_1,\dots,\lambda_n}) \tag{5.1}$$

and

$$R_{\rho^{-(\lambda_n+1)}} \mathrm{Res}^{\lambda_j+1}_{\lambda_j,1} \mathrm{Ind}^{1,\lambda_{j+1}-1}_{\lambda_{j+1}} \mathrm{Res}^{\lambda_n+1}_{\lambda_n,1} \mathrm{Ind}^{1,\lambda_1-1}_{\lambda_1} R_{\rho^{\lambda_n}} (R^{\lambda_1,\dots,\lambda_n}), \tag{5.2}$$

respectively, where in both cases the isomorphism is given by

$$a \otimes 1 \otimes b \longmapsto a \otimes 1 \otimes 1 \otimes \rho^{\lambda_n}(b) = 1 \otimes \rho^{\lambda_n+1}(a) \otimes \rho^{\lambda_n}(b) \otimes 1. \tag{5.3}$$

The inverse is given by

$$1 \otimes a \otimes b \otimes 1 = \rho^{-(\lambda_n+1)}(a) \otimes 1 \otimes \rho^{-\lambda_n}(b) = 1 \otimes \rho^{-\lambda_n}(a) \otimes \rho^{-\lambda_n}(b). \tag{5.4}$$

Note that the tensor factors

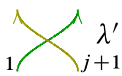
$$\mathrm{Res}^{\lambda_n+1}_{\lambda_n,1} \mathrm{Ind}^{1,\lambda_1-1}_{\lambda_1} \mathrm{Res}^{\lambda_j+1}_{\lambda_j,1} \mathrm{Ind}^{1,\lambda_{j+1}-1}_{\lambda_{j+1}} (R^{\lambda_n,\lambda_1\dots,\lambda_{n-1}})$$

and

$$\mathrm{Res}^{\lambda_j+1}_{\lambda_j,1} \mathrm{Ind}^{1,\lambda_{j+1}-1}_{\lambda_{j+1}} \mathrm{Res}^{\lambda_n+1}_{\lambda_n,1} \mathrm{Ind}^{1,\lambda_1-1}_{\lambda_1} (R^{\lambda_n,\lambda_1\dots,\lambda_{n-1}})$$

in the middle of (5.1) and (5.2) are, up to a same shift, the images of the morphisms $\mathcal{E}_1\mathcal{E}_{j+1}\mathbf{1}_{\lambda_n,\lambda_1,\dots,\lambda_{n-1}}$ and $\mathcal{E}_{j+1}\mathcal{E}_1\mathbf{1}_{\lambda_n,\lambda_1,\dots,\lambda_{n-1}}$ under $\mathcal{F}'$.

One can see that if one applies the isomorphism (5.3) to $x_r^{\alpha_1} \otimes 1 \otimes x_{k_j+1}^{\alpha_2}$ followed by the tensor product of the identity on the two twisted bimodules in (5.1) and of the bimodule map on the central tensor factor given by the image of



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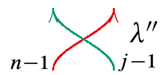
where $\lambda' = (\lambda_n, \lambda_1, \dots, \lambda_{n-1})$, and finally followed by the inverse isomorphism (5.4), one gets

$$\begin{aligned}
& x_r^{\alpha_1} \otimes 1 \otimes x_{k_j+1}^{\alpha_2} \\
& \longmapsto 1 \otimes (x_{\lambda_n+1} - y)^{\alpha_1} \otimes x_{k_j+\lambda_n+1}^{\alpha_2} \otimes 1 \\
& \qquad = \sum_{p=0}^{\alpha_1} \binom{\alpha_1}{p} (-y)^{\alpha_1-p} \otimes x_{\lambda_n+1}^{p} \otimes x_{k_j+\lambda_n+1}^{\alpha_2} \otimes 1 \\
& \longmapsto \sum_{p=0}^{\alpha_1} \binom{\alpha_1}{p} (-y)^{\alpha_1-p} \otimes x_{k_j+\lambda_n+1}^{\alpha_2} \otimes x_{\lambda_n+1}^{p} \otimes 1 \\
& \qquad = 1 \otimes x_{k_j+\lambda_n+1}^{\alpha_2} \otimes (x_{\lambda_n+1} - y)^{\alpha_1} \otimes 1 \\
& \longmapsto x_{k_j}^{\alpha_2} \otimes 1 \otimes (x_1 - y)^{\alpha_1},
\end{aligned}$$

which is precisely the image under $\mathcal{F}'$ of our original 2-morphism with the n-strand. So in this example one obtains indeed the same result using this conjugation trick turning n into 1.

Of course one could have used a similar conjugation trick turning n into $n-1$, i.e. writing everything as conjugates $R_{\rho^{\lambda_1-1}} \otimes - \otimes R_{\rho^{-\lambda_1}}$ and using the bimodule map corresponding to



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where $\lambda'' = (\lambda_2, \dots, \lambda_n, \lambda_1)$. In this case one obtains again the image under $\mathcal{F}'$ of our original 2-morphism.

Proposition 5.8. *The image under $\mathcal{F}'$ of any generating 2-morphism containing n-strands can be obtained by either one of the conjugation tricks, i.e. turning n into 1 or $n-1$, except for the dotted n-identities. The images of these two 2-morphisms (up and downward) can be obtained by the conjugation trick which turns n into $n-1$, but not by the one which turns n into 1.*

Proof. Let us start with the proof for the dotted n-identities. It is an easy check to see that applying the conjugation trick which turns n into $n-1$ gives indeed the expressions announced in our definition of $\mathcal{F}'$. While if one writes $\mathcal{F}'(\mathcal{E}_{\pm n}\mathbf{1}_\lambda)$ as $R_{\rho^{-(\lambda_n \pm 1)}} \otimes \mathcal{F}'(\mathcal{E}_{\pm 1}\mathbf{1}_{\lambda'}) \otimes R_{\rho^{\lambda_n}}$ and then apply the image under $\mathcal{F}'$ of the dotted 1-identity, one obtains

$$1 \otimes 1 \longmapsto (x_r + y) \otimes 1 = 1 \otimes x_1 \quad (\text{resp. } x_1 \otimes 1 = 1 \otimes (x_r + y)),$$

which differs from the image under $\mathcal{F}'$ of the oriented upward (resp. downward) dotted n-identity.

As for the non-dotted generating 2-morphisms of color n, the proof that both conjugation tricks give the same bimodule maps is trivial in certain cases and need some work in some others:

- whenever only one expression was given in our definition of $\mathcal{F}'$, it is because the expressions obtained from both conjugation tricks were obviously equal;
- whenever two expressions (obtained via the two different conjugation tricks) were given, let us prove that they are indeed equal.

We start with the image under $\mathcal{F}'$ of the right n-cup. Note that in the bimodule $\mathrm{Res}^{\lambda_1}_{1,\lambda_1-1} R_\rho \mathrm{Ind}^{\lambda_n,1}_{\lambda_n+1} \mathrm{Res}^{\lambda_n+1}_{\lambda_n,1} R_{\rho^{-1}} \mathrm{Ind}^{1,\lambda_1-1}_{\lambda_1}(R^{\lambda_1\cdots\lambda_n})$ we have $x_1 \otimes 1 \otimes 1 \otimes p = 1 \otimes (x_r + y) \otimes 1 \otimes p$, for any polynomial p. Therefore, we have to show that

$$\sum_{f=0}^{\lambda_n} (-1)^{\lambda_n-f} \otimes (x_r+y)^f \otimes 1 \otimes e_{\lambda_n-f}(x_{r-\lambda_n+1}+y,\ldots,x_r+y)$$
$$= \sum_{f=0}^{\lambda_n} (-1)^{\lambda_n-f} \otimes x_r^f \otimes 1 \otimes e_{\lambda_n-f}(x_{r-\lambda_n+1},\ldots,x_r).$$

For a fixed power of x_r, say k, this amounts to showing that

$$1 \otimes x_r^k \otimes 1 \otimes \sum_{i=0}^{\lambda_n-k} (-1)^i \binom{k+i}{i} y^i e_{\lambda_n-k-i}(x_{r-\lambda_n+1}+y,\ldots,x_r+y)$$
$$= 1 \otimes x_r^k \otimes 1 \otimes e_{\lambda_n-k}(x_{r-\lambda_n+1},\ldots,x_r).$$

This follows from Lemma 5.9.

Lemma 5.9. *For any* $0 \le k \le n$, *we have*

$$e_{n-k}(a_1,\ldots,a_n) = \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{i} y^i e_{n-k-i}(a_1+y,\ldots,a_n+y).$$

Proof. By induction with respect to n. For $n=0$ there is nothing to prove.

Suppose $n>0$. Write

$$e_{n-k}(a_1,\ldots,a_n) = e_{n-k}(a_1,\ldots,a_{n-1}) + a_n e_{n-1-k}(a_1,\ldots,a_{n-1}).$$

Note that $n-k = (n-1)-(k-1)$, so by induction the sum above is equal to

$$\sum_{i=0}^{n-k} (-1)^i \binom{k-1+i}{i} y^i e_{n-k-i}(a_1+y,\ldots,a_{n-1}+y)$$
$$+ a_n \sum_{i=0}^{n-1-k} (-1)^i \binom{k+i}{i} y^i e_{n-1-k-i}(a_1+y,\ldots,a_{n-1}+y).$$

Write $a_n = -y + a_n + y$. Then we get

$$\sum_{i=0}^{n-k}(-1)^i\binom{k-1+i}{i}y^i e_{n-k-i}(a_1+y,\ldots,a_{n-1}+y) \tag{5.5}$$

$$-\sum_{i=0}^{n-1-k}(-1)^i\binom{k+i}{i}y^{i+1}e_{n-1-k-i}(a_1+y,\ldots,a_{n-1}+y) \tag{5.6}$$

$$+(a_n+y)\sum_{i=0}^{n-1-k}(-1)^i\binom{k+i}{i}y^i e_{n-1-k-i}(a_1+y,\ldots,a_{n-1}+y). \tag{5.7}$$

After reindexing the sum in (5.6), the difference of the sums in (5.5) and (5.6) becomes

$$\begin{aligned}&e_{n-k}(a_1+y,\ldots,a_{n-1}+y)\\&\quad+\sum_{i=1}^{n-k}(-1)^i\left(\binom{k-1+i}{i}+\binom{k-1+i}{i-1}\right)y^i e_{n-k-i}(a_1+y,\ldots,a_{n-1}+y)\\&\quad=\sum_{i=0}^{n-k}(-1)^i\binom{k+i}{i}y^i e_{n-k-i}(a_1+y,\ldots,a_{n-1}+y).\end{aligned}$$

Together with the sum in (5.7), we now get

$$\begin{aligned}&\sum_{i=0}^{n-k}(-1)^i\binom{k+i}{i}y^i(e_{n-k-i}(a_1+y,\ldots,a_{n-1}+y)\\&\quad+(a_n+y)e_{n-1-k-i}(a_1+y,\ldots,a_{n-1}+y))\\&\qquad=\sum_{i=0}^{n-k}(-1)^i\binom{k+i}{i}y^i e_{n-k-i}(a_1+y,\ldots,a_{n-1}+y,a_n+y).\end{aligned}$$ △

The proof for the image of the left n-cup is similar, using $x_r \otimes 1 \otimes 1 \otimes 1 = 1 \otimes (x_1 - y) \otimes 1 \otimes 1$ and the lemma above with $-y$ instead of y.

The result for the cups implies the result for the caps, because of the biadjointness relations (3.3) and (3.4). If two maps corresponding to a cap both satisfy the biadjointness relations w.r.t. one fixed map associated to the corresponding cup, then the two maps have to be equal.

The result for the upward oriented n-crossing follows from the fact that both bimodule maps satisfy (easy calculations):

(1) $f(1 \otimes 1 \otimes 1 \otimes 1) = 0$;

(2) for any $\alpha_1, \alpha_2 \in \mathbb{N}$,

$$(x_r \otimes 1 \otimes 1 \otimes 1) f(x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}) - f(x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}(x_1 - y)) \\ = x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2};$$

(3) for any $\alpha_1, \alpha_2 \in \mathbb{N}$,

$$f(x_r^{\alpha_1+1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}) - (1 \otimes 1 \otimes 1 \otimes (x_1 - y)) f(x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}) \\ = x_r^{\alpha_1} \otimes 1 \otimes 1 \otimes x_1^{\alpha_2}.$$

These three properties determine the maps completely by recursion, so they have to be equal.

A similar argument proves the result for downward oriented n-crossings and the remaining cases of the crossings colored $(1, n)$ and $(n, n-1)$ are easy computations. □

Proposition 5.10. *$\mathcal{F}' \colon \widehat{\mathcal{S}}(n,r)^*_{[y]} \to \mathcal{E}\mathcal{S}\mathcal{B}im^*_{\hat{A}_{r-1}}$ is a well-defined degree preserving 2-functor.*

Proof. All relations between 2-morphisms which do not have n-colored strands are satisfied by the results in Section 6 in [17]. Since no relation in $\widehat{\mathcal{S}}(n,r)_{[y]}$ involves all colors at the same time, the proof that $\mathcal{F}'$ preserves a given relation can always be reduced to the fact that $\mathcal{F}'$ preserves the same relation with colors belonging to $\{1, \ldots, n-1\}$ by using the conjugation trick which turns n into $n-1$, except in the case of relations (3.17) and (3.18) for $\{i, j\} = \{1, n\}$. The proof that $\mathcal{F}'$ preserves these relations is straightforward and is left to the reader. □

Remark 5.11. Note that the twisted bimodules $B_\rho = R_\rho^{(1^r)}$ and $B_{\rho^{-1}} = R_{\rho^{-1}}^{(1^r)}$ are isomorphic to the images under $\mathcal{F}'$ of respectively

$$\mathcal{E}_{-n} \ldots \mathcal{E}_{-r-1} \mathcal{E}_{-1} \ldots \mathcal{E}_{-r} \mathbf{1}_r \quad \text{and} \quad \mathcal{E}_r \ldots \mathcal{E}_1 \mathcal{E}_{r+1} \ldots \mathcal{E}_n \mathbf{1}_r.$$

As a matter of fact, any twisted singular bimodule is isomorphic to the image under $\mathcal{F}'$ of a certain product of categorified divided powers (see [20] and [28] for more details on divided powers and extended graphical calculus). Indeed, let $n > r$ and let $\lambda \in \Lambda(n,r)$ be arbitrary. At least one entry of λ is equal to zero, let us assume it is λ_i. Then

$$R_{\rho}^{\lambda_n \lambda_1 \dots \lambda_{n-1}} \cong \mathcal{F}'(\mathcal{E}_{-i-1}^{(\lambda_i+1)} \dots \mathcal{E}_{-n}^{(\lambda_n)} \mathcal{E}_{-1}^{(\lambda_1)} \dots \mathcal{E}_{-i+1}^{(\lambda_{i-1})} \mathbf{1}_{\lambda})$$

and

$$R_{\rho^{-1}}^{\lambda_2 \dots \lambda_n \lambda_1} \cong \mathcal{F}'(\mathcal{E}_{i-2}^{(\lambda_i-1)} \dots \mathcal{E}_{1}^{(\lambda_2)} \mathcal{E}_{n}^{(\lambda_1)} \dots \mathcal{E}_{i}^{(\lambda_i+1)} \mathbf{1}_{\lambda}).$$

We now give the proof of Theorem 3.23.

Proof. The fact that the KL basis of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ over $\mathbb{Q}$ from [17] also generates $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ freely over $\mathbb{Q}[y]$, follows from exactly the same arguments as Khovanov and Lauda's. To avoid repeating them word by word, we are going to be very sketchy here and only briefly recall the main ideas and how they carry over to our case.

The first step required a polynomial representation of the positive half of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$, i.e. the algebra formed by the diagrams whose strands are all oriented upwards (in particular no cups or caps) modulo the relations between such diagrams. This was defined in Section 2.3 in [16]. This representation easily extends to the deformed setting in the following way: tensor the polynomial ring $\mathcal{P}o\ell_\nu$ with $\mathbb{Z}[y]$ (or $\mathbb{Q}[y]$). Then use the same representation, but in the last line add y if the colors are n and 1, i.e.

$$f \longmapsto (x_k(s_k \boldsymbol{i}) + x_{k+1}(s_k \boldsymbol{i}) + y)(s_k f)$$

if $i_k = n$ and $i_{k+1} = 1$. One just has to check that the deformed relations in $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ are preserved, which is straightforward. Note that Khovanov and Lauda used a different sign convention for the relations in [16], which we have copied here. In Theorem 2.5 in [16], Khovanov and Lauda proved that their polynomial representation is faithful and that the positive half of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ is freely generated by the positive half of their basis. Their proof also holds in the deformed case.

In Section 3.2 in [17], Khovanov and Lauda defined a spanning set of diagrams for the whole $\mathcal{U}(\mathfrak{g})$ for arbitrary $\mathfrak{g}$, but only proved linear independence for $\mathfrak{g} = \mathfrak{sl}_n$. In that case, their second step was to construct, for any integer $r > 0$, a 2-representation Γ_r of $\mathcal{U}(\mathfrak{sl}_n)$ on bimodules, which they did in Section 6 in [17]. The relation with step 1 is explained in the proof of Lemma 6.16 in [17]: when the positive half of $\mathcal{U}(\mathfrak{sl}_n)$ is considered as a 2-category, its polynomial representation, now seen as a 2-representation, can be embedded into the bimodule 2-representation Γ_r of the whole $\mathcal{U}(\mathfrak{sl}_n)$ for large enough $r > 0$. By the faithfulness of the polynomial representation, this implies that the obvious 2-functor from the positive half of $\mathcal{U}(\mathfrak{sl}_n)$ into the whole $\mathcal{U}(\mathfrak{sl}_n)$ is fully faithful.

In Proposition 5.10 we gave the analogous 2-representation $\mathcal{F}'$ of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ for any $r < n$ (to be very precise, we gave the analogue of the equivariant version of the KL 2-representation, denoted Γ_r^G and defined in Section 6.3 in [17]). Our definition of $\mathcal{F}'$ easily extends to arbitrary integers $r > 0$, the restriction $r < n$ in this paper is only needed for the link with the affine Schur algebra but not for the 2-representation. Just as Khovanov and Lauda did in Section 6.4 in [17], we can now conclude that the KL spanning set of diagrams of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$ is a basis of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ over $\mathbb{Q}[y]$. Khovanov and Lauda's arguments are literally the same, so we won't repeat them here.

And now the non-triviality of our deformation. Let us suppose, on the contrary, that there exists a $\mathbb{Q}[y]$-linear degree preserving 2-equivalence

$$f_y\colon \mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]} \longrightarrow \mathcal{U}(\widehat{\mathfrak{sl}}_n) \otimes_{\mathbb{Q}} \mathbb{Q}[y].$$

Then f_y induces a $\mathbb{Q}$-linear degree preserving 2-equivalence $f\colon \mathcal{U}(\widehat{\mathfrak{sl}}_n) \to \mathcal{U}(\widehat{\mathfrak{sl}}_n)$. So

$$(f^{-1} \otimes 1) f_y\colon \mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]} \longrightarrow \mathcal{U}(\widehat{\mathfrak{sl}}_n) \otimes_{\mathbb{Q}} \mathbb{Q}[y]$$

is a $\mathbb{Q}[y]$-linear degree preserving 2-equivalence which is the identity on objects and 1-morphisms, i.e. sends λ and $\mathcal{E}_{\pm i}$ to themselves for all $\lambda \in \mathbb{Z}^{n-1}$ and $i = 1, \ldots, n$.

So without loss of generality we can assume that the 2-equivalence

$$f\colon \mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]} \longrightarrow \mathcal{U}(\widehat{\mathfrak{sl}}_n) \otimes_{\mathbb{Q}} \mathbb{Q}[y]$$

is the identity on objects and 1-morphisms. This implies that cups and caps are sent to $\mathbb{Q}$-multiples of themselves, because they are generators of minimal degree of their respective 2-HOM-spaces. By rescaling cups and caps, which is also a $\mathbb{Q}[y]$-linear degree preserving 2-equivalence, we can therefore assume that f is the identity on non-dotted bubbles. Since f preserves weights, it induces a 2-equivalence between $\widehat{\mathcal{S}}(n,r)_{[y]}$ and $\widehat{\mathcal{S}}(n,r) \otimes_{\mathbb{Q}} \mathbb{Q}[y]$ satisfying the same assumptions. But this contradicts the $\mathbb{Q}[y]$-linearity of f, because in $\widehat{\mathcal{S}}(n,r) \otimes_{\mathbb{Q}} \mathbb{Q}[y]$ equation (4.10) only holds if we put $\boxed{y} = 0$. Indeed note that (4.10) can be written, using (4.14), in terms of non-dotted bubbles only. $\square$

6. The Grothendieck group of $\widehat{\mathcal{S}}(n,r)_{[y]}$

The following Lemma is the affine analogue of Lemma 6.6 in [28].

Lemma 6.1. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{DEB}im^*_{\widehat{A}_{r-1}} & \xrightarrow{\quad\mathcal{F}\quad} & \mathcal{ESB}im^*_{\widehat{A}_{r-1}} \\ & \searrow^{\Sigma_{n,r}} \qquad \nearrow^{\mathcal{F}'} & \\ & \widehat{\mathcal{S}}(n,r)^*_{[y]}((1^r),(1^r)) & \end{array}$$

Proof. The proof is straightforward and follows from checking the definitions carefully. □

Note that the 2-hom-spaces of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$ are finite-dimensional $\mathbb{Q}$-vector spaces, because the original 2-HOM-spaces in $\mathcal{U}(\widehat{\mathfrak{sl}}_n)^*$ are finite-dimensional in each degree, their grading is bounded below and $\deg(y) = 2 > 0$. Therefore, the Karoubi envelope (or idempotent completion) of $\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$, denoted $\operatorname{Kar}\mathcal{U}(\widehat{\mathfrak{sl}}_n)_{[y]}$, is Krull–Schmidt. The same holds for $\mathcal{U}(\widehat{\mathfrak{gl}}_n)_{[y]}$ and $\widehat{\mathcal{S}}(n,r)_{[y]}$, of course.

By the same arguments, we see that the 2-ideal generated by y is virtually nilpotent (for virtually nilpotent ideals and basic facts about them, see Section 3.8.1 and 3.8.2 in [17]). This proves that

$$K_0^{\mathbb{Q}(q)}(\operatorname{Kar}\mathcal{C}_{[y]}) \cong K_0^{\mathbb{Q}(q)}(\operatorname{Kar}\mathcal{C}), \tag{6.1}$$

where $\mathcal{C}$ is $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$, $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ or $\widehat{\mathcal{S}}(n,r)$.

Corollary 6.2. *The algebra homomorphism*

$$K_0^{\mathbb{Q}(q)}(\Sigma_{n,r})\colon \widehat{\mathcal{H}}_{\widehat{A}_{r-1}} \longrightarrow K_0^{\mathbb{Q}(q)}(\operatorname{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

is an embedding.

Proof. We already know that $K_0^{\mathbb{Q}(q)}(\mathcal{F})$ is injective, by Theorem 2.10 and the fact that $\mathcal{EB}im_{\widehat{A}_{r-1}}$ is a full sub-2-category of $\mathcal{ESB}im_{\widehat{A}_{r-1}}$. The result now follows from the commutativity of the diagram in Lemma 6.1. □

Theorem 2.10 and Lemma 6.1 also imply that $\Sigma_{n,r}$ is faithful. While in the finite type A case (Proposition 6.9 in [28]) we know that this functor is also full, we do not know if it is the case here, but we conjecture that to be true.

Conjecture 6.3. *The functor*

$$\Sigma_{n,r} : \mathcal{DEB}im_{\widehat{A}_{r-1}} \longrightarrow \widehat{\mathcal{S}}(n,r)_{[y]}((1^r),(1^r))$$

is an equivalence of 2-categories.

Theorem 6.4. *The algebra homomorphism*

$$\gamma\colon \widehat{\mathbf{S}}(n,r) \longrightarrow K_0^{\mathbb{Q}(q)}(\operatorname{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

defined by

$$\gamma(E_{\pm i}1_\lambda) := [\mathcal{E}_{\pm i}\mathbf{1}_\lambda]$$

is an isomorphism.

Proof. Khovanov and Lauda proved surjectivity of the homomorphism

$$\dot{\mathrm{U}}(\widehat{\mathfrak{sl}}_n) \longrightarrow K_0^{\mathbb{Q}(q)}(\mathcal{U}(\widehat{\mathfrak{sl}}_n))$$

in Theorem 1.1 in [17]. The same arguments which proved Lemma 7.7 in [28] can thus be used to prove that

$$\gamma\colon \widehat{\mathbf{S}}(n,r) \longrightarrow K_0^{\mathbb{Q}(q)}(\operatorname{Kar}\widehat{\mathcal{S}}(n,r))$$

is surjective. By (6.1) this implies that the analogous homomorphism

$$\gamma\colon \widehat{\mathbf{S}}(n,r) \longrightarrow K_0^{\mathbb{Q}(q)}(\operatorname{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$$

is surjective.

The rest of the proof follows from Lemma 3.15 and Corollary 6.2 with $A = K_0^{\mathbb{Q}(q)}(\operatorname{Kar}\widehat{\mathcal{S}}(n,r)_{[y]})$. □

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