

Galois action on knots I: Action of the absolute Galois group

Hidekazu Furusho

Abstract. Our aim of this and subsequent papers is to enlighten (a part of, presumably) arithmetic structures of knots. This paper introduces a notion of profinite knots which extends topological knots and shows its various basic properties. Particularly an action of the absolute Galois group of the rational number field on profinite knots is rigorously established, which is attained by our extending the action of Drinfel’d’s Grothendieck–Teichmüller group on profinite braid groups into on profinite knots.

Mathematics Subject Classification (2010). Primary: 14G32; Secondary: 11R, 57M25.

Keywords. Profinite knots, absolute Galois group, Grothendieck–Teichmüller group.

Contents

0	Introduction	296
1	Profinite braids	297
2	Profinite knots	309
	Appendices	348
A	Two-bridge profinite knots	348
B	Profinite framed knots	349
	References	357

0. Introduction

It is known that there are analogies between algebraic number theory and 3-dimensional topology. It is said that Mazur and Manin among others spotted them in 1960's and, after a long silence, in 1990's, Kapranov and Reznikov ([20], [35] and their lecture in MPI) who took up their ideas and explored them jointly and Morishita [31] whose works started independently in a more sophisticated aspect, settled the new area of mathematics, arithmetic topology. Lots of analogies between algebraic number theory and 3-dimensional topology are suggested in arithmetic topology, however, as far as we know, no direct relationship seems to be known. Our attempt of this and subsequent [10] papers is to give a direct one particularly between Galois groups and knots.

A *profinite tangle diagram*, a profinite analogue of an oriented tangle diagram, is introduced in Definition 2.3 as a consistent finite sequence of fundamental profinite tangle diagrams; symbols of three types A , \widehat{B} and C (Definition 2.2). A *profinite knot diagram*, a profinite analogue of an oriented knot diagram, is defined by a profinite tangle diagram without endpoints and with a single connected component in Definition 2.4. A profinite version of Turaev moves is given in our Definition 2.8 (T1)–(T6), which determines an equivalence, called isotopy, for profinite tangle diagrams. The set $\widehat{\mathcal{T}}$ of profinite tangles means the quotient of the set of profinite tangle diagrams by the equivalence and the set $\widehat{\mathcal{K}}$ of profinite knots is the subspace of $\widehat{\mathcal{T}}$ consisting of isotopy classes of profinite knot diagrams (Definition 2.9). Our first theorem is:

Theorem A (Theorem 2.11 and 2.16). (1) *The space $\widehat{\mathcal{K}}$ carries a structure of a topological commutative monoid whose product is given by the connected sum (2.1).*

(2) *Let \mathcal{K} denote the set of isotopy classes of (topological) oriented knots. Then there is a natural map*

$$h: \mathcal{K} \longrightarrow \widehat{\mathcal{K}}.$$

The map is with dense image and is a monoid homomorphism with respect to the connected sum.

The map h is naturally conjectured to be injective (Conjecture 2.12). It is because, if it is not injective, the Kontsevich invariant fail to be a perfect knot invariant (cf. Remark 2.13 and 2.34).

The topological group $\text{Frac } \widehat{\mathcal{K}}$ of profinite knots is introduced as the group of fraction of the topological monoid $\widehat{\mathcal{K}}$ in Definition 2.35. A continuous action of the profinite Grothendieck–Teichmüller group $\widehat{\text{GT}}$ [8] on $\text{Frac } \widehat{\mathcal{K}}$ is rigorously

established in Definition 2.39 - Theorem 2.41. By using the inclusion from the absolute Galois group $G_{\mathbb{Q}}$ of the rational number field \mathbb{Q} into \widehat{GT} , our second theorem is obtained.

Theorem B (Theorem 2.45). *Fix an embedding from the algebraic closure $\overline{\mathbb{Q}}$ of the rational number field \mathbb{Q} into the complex number field \mathbb{C} . Then the group $\text{Frac } \widehat{\mathcal{K}}$ of profinite knots admits a non-trivial topological $G_{\mathbb{Q}}$ -module structure. Namely, there is a non-trivial continuous Galois representation on profinite knots*

$$\rho_0: G_{\mathbb{Q}} \longrightarrow \text{Aut Frac } \widehat{\mathcal{K}}.$$

It is explained that particularly the complex conjugation sends each knot to its mirror image (Example 2.46). The validity of a knot analogue of the so-called Belyi's theorem, i.e. the injectivity of ρ_0 , is posed in Problem 2.49. Several projects and problems on the Galois action are posted in the end of this paper.

Our construction of the Galois action could be said as a lift of the action of Kassel–Turaev [22] given in a proalgebraic setting into a profinite setting. Our Galois action on knots might also be related to the ‘Galois relations’ suggested in Gannon–Walton [11]. Our discussion on profinite knots in this paper may be linked to Mazur’s discussion on profinite equivalence of knots in [28]

The contents of the paper is as follows. §1 is devoted to a review of Drinfel’d’s work on \widehat{GT} and $G_{\mathbb{Q}}$ -actions on profinite braids. Main results are presented in §2: The ABC-construction of profinite knots and their basic properties are introduced in §2.1. We also introduce and discuss the notion of pro- l knots in §2.2. It serves for our arguments in §2.3 where an action of the absolute Galois group on profinite knots is established. In Appendix A, a two-bridge profinite knot, a profinite knot with a specific type, is introduced and its Galois behavior is investigated. A profinite analogue of a framed knot is introduced and the above two theorems are extended into those for profinite framed knots in Appendix B.

1. Profinite braids

This section is a review mainly on Drinfel’d’s work [8] of his profinite Grothendieck–Teichmüller group \widehat{GT} and its action on profinite braids. Definitions of the profinite braid group \widehat{B}_n and the absolute Galois group $G_{\mathbb{Q}}$ are recalled in Example 1.4. The definition of \widehat{GT} is presented in Definition 1.6. In Theorem 1.9, it is explained that \widehat{GT} contains $G_{\mathbb{Q}}$. The action of \widehat{GT} on \widehat{B}_n is explained in Theorem 1.12. Specific properties of the action which will be required to next section are shown in Proposition 1.16 and Proposition 1.18.

Definition 1.1. The Artin braid group B_n with n -strings ($n \geq 2$) is the group generated by σ_i ($1 \leq i \leq n - 1$) with two relations

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if } |i - j| > 1. \end{aligned}$$

The unit of B_n is denoted by e_n . For $n = 1$, put $B_1 = \{e_1\}$: the trivial group. The pure braid group P_n with n -strings is the kernel of the natural projection from B_n to the symmetric group \mathfrak{S}_n of degree n .

When $n = 2$, there is an identification $B_2 \simeq \mathbb{Z}$ and the subgroup P_2 corresponds to $2\mathbb{Z}$ under the identification. When $n = 3$, P_3 contains a free group F_2 generated by σ_1^2 and σ_2^2 .

Notation 1.2. The generator σ_i in B_n is depicted as in Figure 1.1. And for b and $b' \in B_n$, we draw the product $b \cdot b' \in B_n$ as in Figure 1.2 (the order of product $b \cdot b'$ is chosen to combine the bottom endpoints of b with the top endpoints of b').



Figure 1.1. σ_i .

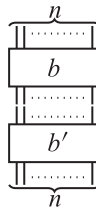


Figure 1.2. $b \cdot b'$.

For $1 \leq i < j \leq n$, special elements

$$x_{ij} = \sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \sigma_j^2 \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i$$

generate P_n . For $1 \leq a \leq a + \alpha < b \leq b + \beta \leq n$, we define

$$\begin{aligned} x_{a \dots a + \alpha, b \dots b + \beta} &:= (x_{a,b} x_{a,b+1} \cdots x_{a,b+\beta}) \cdot (x_{a+1,b} x_{a+1,b+1} \cdots x_{a+1,b+\beta}) \cdots \\ &\quad (x_{a+\alpha,b} x_{a+\alpha,b+1} \cdots x_{a+\alpha,b+\beta}) \in P_n. \end{aligned}$$

They are drawn in Figures 1.3 and 1.4.

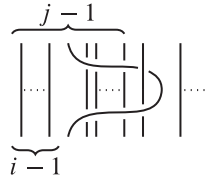


Figure 1.3. x_{ij} .

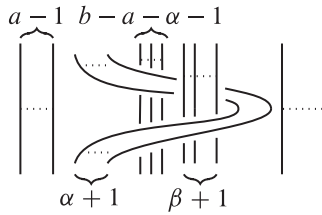


Figure 1.4. $x_{a\dots a+\alpha, b\dots b+\beta}$.

Next we briefly review the definition and a few examples of profinite groups.

Definition 1.3. A topological group G is called a *profinite group* if it is a projective limit $\varprojlim G_i$ of a projective system of finite groups $\{G_i\}_{i \in I}$. For a discrete group Γ , its *profinite completion* $\hat{\Gamma}$ is the profinite group defined by the projective limit

$$\hat{\Gamma} = \varprojlim \Gamma/N$$

where N runs over all normal subgroups of Γ with finite indices.

For profinite groups, consult [36] for example. We note there is a natural homomorphism $\Gamma \rightarrow \hat{\Gamma}$. In the paper, we employ the same symbol when we express the image of elements of Γ by the map if there is no confusion.

Example 1.4. (1) The set $\hat{\mathbb{Z}}$ of *profinite integers* is the profinite completion of \mathbb{Z} . There is an identification $\hat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p$. Here p runs over all primes and \mathbb{Z}_p stands for the ring of p -adic integers.

(2) The *absolute Galois group* $G_{\mathbb{Q}}$ of the rational number field \mathbb{Q} is the profinite group

$$G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) := \varprojlim \text{Gal}(F/\mathbb{Q}).$$

Here the limit runs over all finite Galois extension F of \mathbb{Q} and $\text{Gal}(F/\mathbb{Q})$ means its Galois group.

(3) The *profinite braid group* \widehat{B}_n means the profinite completion of B_n . It contains the *profinite pure braid group* \widehat{P}_n (the profinite completion of P_n), which is equal to the kernel of the natural projection $\widehat{B}_n \rightarrow \mathfrak{S}_n$. It is known that both B_n and P_n are *residually finite*, i.e., their natural maps are both injective;

$$B_n \hookrightarrow \widehat{B}_n \quad \text{and} \quad P_n \hookrightarrow \widehat{P}_n. \tag{1.1}$$

When $n = 2$, we have $\widehat{B}_2 \simeq \widehat{\mathbb{Z}}$. For profinite braid groups, we employ the same figures as in Notation 1.2 to express such elements. Further we also depict $\sigma_i^c \in \widehat{B}_n$ ($c \in \widehat{\mathbb{Z}}$) as Figure 1.5.

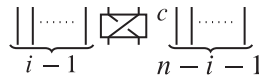


Figure 1.5. $\sigma_i^c \in \widehat{B}_n$ ($c \in \widehat{\mathbb{Z}}$).

To state the definition of $\widehat{\text{GT}}$, we need fix several notations.

Notation 1.5. Let F_2 be the free group of rank 2 with two variables x and y and \widehat{F}_2 be its profinite completion. For any $f \in \widehat{F}_2$ and any group homomorphism $\tau: \widehat{F}_2 \rightarrow G$ sending $x \mapsto \alpha$ and $y \mapsto \beta$, the symbol $f(\alpha, \beta)$ stands for the image $\tau(f)$. Particularly for the (actually injective) group homomorphism $\widehat{F}_2 \rightarrow \widehat{P}_n$ sending $x \mapsto x_{a\dots a+\alpha, b\dots b+\beta}$ and $y \mapsto x_{b\dots b+\beta, c\dots c+\gamma}$ ($1 \leq a \leq a + \alpha < b \leq b + \beta < c \leq c + \gamma \leq n$), the image of $f \in \widehat{F}_2$ is denoted by $f_{a\dots a+\alpha, b\dots b+\beta, c\dots c+\gamma}$.

The profinite Grothendieck–Teichmüller group $\widehat{\text{GT}}$ which is a main character of our paper is defined by Drinfel'd [8] to be a profinite subgroup of the topological automorphism group of \widehat{F}_2 .

Definition 1.6 ([8]). The *profinite Grothendieck–Teichmüller group*¹ $\widehat{\text{GT}}$ is the profinite subgroup of $\text{Aut } \widehat{F}_2$ defined by

$$\widehat{\text{GT}} := \left\{ \sigma \in \text{Aut } \widehat{F}_2 \mid \begin{array}{l} \sigma(x) = x^\lambda, \sigma(y) = f^{-1}y^\lambda f \text{ for some } (\lambda, f) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}_2 \\ \text{satisfying the three relations below.} \end{array} \right\},$$

$$f(x, y)f(y, x) = 1 \quad \text{in } \widehat{F}_2, \tag{1.2}$$

¹ It is named after Grothendieck’s project of *un jeu de Teichmüller-Lego* posted in *Esquisse d’un programme* [13]. A construction of such group was suggested there though Drinfel’d came to the group independently in his subsequent works [7, 8] on deformation of specific type of quantum groups.

$$f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1 \quad \text{in } \widehat{F}_2 \text{ with } z = (xy)^{-1} \text{ and } m = \frac{\lambda - 1}{2}, \tag{1.3}$$

$$f_{1,2,34}f_{12,3,4} = f_{2,3,4}f_{1,23,4}f_{1,2,3} \quad \text{in } \widehat{P}_4. \tag{1.4}$$

Remark 1.7. (1) In some literatures, (1.2), (1.3), and (1.4) are called *2-cycle*, *3-cycle*, and *5-cycle relation* respectively. The author often calls (1.2) and (1.3) by *two hexagon equations* and (1.4) by *one pentagon equation* because they reflect the three axioms, two hexagon and one pentagon axioms, of braided monoidal (tensor) categories [19]. We remind that (1.4) represents

$$\begin{aligned} &f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) \\ &= f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23}) \quad \text{in } \widehat{P}_4. \end{aligned}$$

In several literatures such as [16] and [37], equation (1.4) is replaced by a different (more symmetric) formulation.

(2) For the *proalgebraic* Grothendieck–Teichmüller group $\text{GT}(\mathbf{k})$ in [8] (\mathbf{k} : an algebraically closed field of characteristic 0), it is shown that its one pentagon equation implies its two hexagon equations in [9]. But as for our profinite group $\widehat{\text{GT}}$, it is not known if the pentagon equation (1.4) implies two hexagon equations (1.2) and (1.3).

(3) We remark that each $\sigma \in \widehat{\text{GT}}$ determines a pair (λ, f) uniquely because the pentagon equation (1.4) implies that f belongs to the topological commutator subgroup of \widehat{F}_2 . By abuse of notation, we occasionally express the pair (λ, f) to represent σ and denote as $\sigma = (\lambda, f) \in \widehat{\text{GT}}$. The above set-theoretically defined $\widehat{\text{GT}}$ forms indeed a profinite group whose product is induced from that of $\text{Aut } \widehat{F}_2$ and is given by²

$$(\lambda_2, f_2) \circ (\lambda_1, f_1) = (\lambda_2\lambda_1, f_2 \cdot f_1(x^{\lambda_2}, f_2^{-1}y^{\lambda_2}f_2)). \tag{1.5}$$

The next lemma will be used later.

Lemma 1.8. *Let $p_i: \widehat{P}_3 \rightarrow \widehat{P}_2 (= \widehat{\mathbb{Z}})$ ($i = 1, 2, 3$) denote the map of omission of i -th strand in \widehat{P}_3 . Let $(\lambda, f) \in \widehat{\text{GT}}$. Then $p_i(f_{1,2,3}) = 0$ ($i = 1, 2, 3$).*

Proof. It is easy because f belongs to the commutator of \widehat{F}_2 as mentioned above. □

² For our purpose, we change the order of the product in [8].

One of the important property of the profinite Grothendieck–Teichmüller group $\widehat{\text{GT}}$ is that it contains the absolute Galois group $G_{\mathbb{Q}}$.

Theorem 1.9 ([8] and [17]). *Fix an embedding from $\overline{\mathbb{Q}}$ into \mathbb{C} . Then there is an embedding*

$$G_{\mathbb{Q}} \hookrightarrow \widehat{\text{GT}}. \tag{1.6}$$

We briefly review its proof below.

Proof. As is explained in [17], an action of the absolute Galois group on \widehat{F}_2 , i.e.

$$G_{\mathbb{Q}} \longrightarrow \text{Aut } \widehat{F}_2 \tag{1.7}$$

is derived from the action on the profinite (scheme-theoretical, cf. [12]) fundamental group $\hat{\pi}_1(\mathbb{P}_{\overline{\mathbb{Q}}}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01})$ of the algebraic curve $\mathbb{P}_{\overline{\mathbb{Q}}}^1 \setminus \{0, 1, \infty\}$ with Deligne’s [6] tangential base point $\overrightarrow{01}$ (this is achieved in the method explained in Remark 1.14 below). The so-called *Belyi’s theorem* [3] claims that the map (1.7) is injective:

$$G_{\mathbb{Q}} \hookrightarrow \text{Aut } \widehat{F}_2. \tag{1.8}$$

Equations (1.2)–(1.4) are checked for $\sigma \in G_{\mathbb{Q}}$ in [8] and [18], which means that $G_{\mathbb{Q}}$ is contained in $\widehat{\text{GT}} \subset \text{Aut } \widehat{F}_2$. \square

Example 1.10. Fix an embedding from $\overline{\mathbb{Q}}$ into \mathbb{C} . Then the complex conjugation sending $z \in \mathbb{C}$ to $\bar{z} \in \mathbb{C}$ determines an element $\zeta_0 \in G_{\mathbb{Q}}$. It is mapped to the pair $(-1, 1) \in \widehat{\text{GT}}$ by (1.6).

Asking its surjectivity on the injection (1.6) is open for many years:

Problem 1.11. Is $G_{\mathbb{Q}}$ equal to $\widehat{\text{GT}}$?

The following Drinfel’d’s [8] $\widehat{\text{GT}}$ -action on \widehat{B}_n (a detailed description is also given in [18]) plays a fundamental role of our results, $\widehat{\text{GT}}$ -action on knots.

Theorem 1.12 ([8, 18]). *Let $n \geq 2$. There is a continuous $\widehat{\text{GT}}$ -action on \widehat{B}_n*

$$\rho_n: \widehat{\text{GT}} \longrightarrow \text{Aut } \widehat{B}_n$$

given by

$$\sigma = (\lambda, f): \begin{cases} \sigma_1 \mapsto \sigma_1^\lambda, \\ \sigma_i \mapsto f_{1\dots i-1, i, i+1}^{-1} \sigma_i^\lambda f_{1\dots i-1, i, i+1} \quad (2 \leq i \leq n-1). \end{cases}$$

We denote $\rho_n(\sigma)(b)$ simply by $\sigma(b)$ when there is no confusion.

Remark 1.13. (1) According to the method to calculate the action in [8] (explicitly presented in the appendix of [18]), particularly we have

$$\begin{cases} (\sigma_1 \cdots \sigma_i) \mapsto f_{[1],[i],[n-i-1]}^{-1} \cdot (\sigma_1 \cdots \sigma_i) \cdot x_{1\dots i,i+1}^m, \\ (\sigma_i \cdots \sigma_1) \mapsto x_{1\dots i,i+1}^m \cdot (\sigma_i \cdots \sigma_1) \cdot f_{[1],[i],[n-i-1]}. \end{cases} \tag{1.9}$$

Here $m = \frac{\lambda-1}{2}$ and for $f_{[1],[i],[n-i-1]}$, see (1.13).

(2) We note that ρ_n is injective when $n \geq 4$.

By Theorem 1.9 and Theorem 1.12, we obtain the absolute Galois representation

$$\rho_n: G_{\mathbb{Q}} \longrightarrow \text{Aut } \widehat{B}_n. \tag{1.10}$$

The below is an algebraic-geometrical interpretation of the Galois action in terms of Grothendieck’s theory [12] on profinite (scheme-theoretical) fundamental groups.

Remark 1.14. We have a well-known identification between the braid group B_n with the topological fundamental group $\pi_1(X_n(\mathbb{C}), *)$. Here $X_n(\mathbb{C}) = \text{Conf}_{\mathfrak{S}_n}^n(\mathbb{C})$ means the quotient of the *configuration space*

$$\text{Conf}^n(\mathbb{C}) = \{(z_1 \dots z_n) \in \mathbb{C}^n \mid z_i \neq z_j (i \neq j)\}$$

by the symmetric group \mathfrak{S}_n action and $*$ is a basepoint.

Let $\widehat{\pi}_1(X_n \times \overline{\mathbb{Q}}, *)$ denote the profinite (scheme-theoretical) fundamental group of $X_n \times \overline{\mathbb{Q}}$ in the sense of Grothendieck [12]. Here the scheme X_n means the \mathbb{Q} -structure of $X_n(\mathbb{C})$ and $*$ is a basepoint defined over $\overline{\mathbb{Q}}$ in the sense of loc. cit. Fix an embedding from $\overline{\mathbb{Q}}$ into \mathbb{C} , then, by the so-called Riemann’s existence theorem (loc.cit. VII.Théorème 5.1), the group $\widehat{\pi}_1(X_n \times \overline{\mathbb{Q}}, *)$ is identified with the profinite completion of $\pi_1(X_n(\mathbb{C}), *)$. Hence we have an identification

$$\widehat{B}_n \simeq \widehat{\pi}_1(X_n \times \overline{\mathbb{Q}}, *). \tag{1.11}$$

Next assume that $*$ is defined over \mathbb{Q} . Then by [12] IX.Théorème 6.1, we have the homotopy exact sequence of the profinite fundamental group

$$1 \longrightarrow \widehat{\pi}_1(X_n \times \overline{\mathbb{Q}}, *) \longrightarrow \widehat{\pi}_1(X_n, *) \longrightarrow \widehat{\pi}_1(\text{Spec } \mathbb{Q}, *) \longrightarrow 1.$$

The last $\widehat{\pi}_1(\text{Spec } \mathbb{Q}, *)$ is nothing but the absolute Galois group $G_{\mathbb{Q}}$. A point here is that each basepoint $*$ determines a section s_* of the exact sequence. By (1.11), the section s_* yields a continuous Galois representation on \widehat{B}_n

$$\rho_{n,*}: G_{\mathbb{Q}} \longrightarrow \text{Aut } \widehat{B}_n$$

by inner conjugation, i.e., $\rho_{n,*}(\sigma)(b) = s_*(\sigma) \cdot b \cdot s_*(\sigma)^{-1}$ ($\sigma \in G_{\mathbb{Q}}$ and $b \in \widehat{B}_n$). A specific (tangential in the sense of Deligne [6]) basepoint t_n is constructed in [18], where they showed that the resulting ρ_{n,t_n} is equal to our ρ_n in (1.10).

Special properties of the $\widehat{\text{GT}}$ -action in Theorem 1.12 are presented in the following Proposition 1.16 and Proposition 1.18 (though they are implicitly suggested in [8]). They will be employed several times in our paper.

Notation 1.15. Put $n > 0$ and $m_1, m_2 \geq 0$. On the continuous homomorphism

$$e_{m_1} \otimes \cdot \otimes e_{m_2}: \widehat{B}_n \longrightarrow \widehat{B}_{m_1+n+m_2}$$

which is defined by $\sigma_i \mapsto \sigma_{m_1+i}$ (obtained by placing the trivial braids e_{m_1} and e_{m_2} on the left and right respectively), we denote the image of $b \in \widehat{B}_n$ by $e_{m_1} \otimes b \otimes e_{m_2}$.

Proposition 1.16. Put $n > 0$ and $m_1, m_2 \geq 0$. Let $\sigma = (\lambda, f) \in \widehat{\text{GT}}$ and $b \in \widehat{B}_n$. Then

$$\sigma(e_{m_1} \otimes b \otimes e_{m_2}) = f_{[m_1],[n],[m_2]}^{-1} \cdot (e_{m_1} \otimes \sigma(b) \otimes e_{m_2}) \cdot f_{[m_1],[n],[m_2]}. \quad (1.12)$$

Here

$$\begin{aligned} & f_{[m_1],[n],[m_2]} \\ & := f_{1\dots m_1, m_1+1\dots m_1+n-1, m_1+n} \\ & \quad \cdot f_{1\dots m_1, m_1+1\dots m_1+n-2, m_1+n-1} \cdots f_{1\dots m_1, m_1+1, m_1+2} \in \widehat{B}_{m_1+n+m_2}. \end{aligned} \quad (1.13)$$

Proof. It is enough to check (1.12) for $b = \sigma_i$ ($1 \leq i \leq n-1$). By Theorem 1.12,

$$\begin{aligned} & f_{[m_1],[n],[m_2]}^{-1} \cdot (e_{m_1} \otimes \sigma(\sigma_i) \otimes e_{m_2}) \cdot f_{[m_1],[n],[m_2]} \\ & = f_{[m_1],[n],[m_2]}^{-1} \cdot f_{m_1+1\dots m_1+i-1, m_1+i, m_1+i+1}^{-1} \\ & \quad \cdot \sigma_{m_1+i}^\lambda \cdot f_{m_1+1\dots m_1+i-1, m_1+i, m_1+i+1} \cdot f_{[m_1],[n],[m_2]}. \end{aligned} \quad (1.14)$$

- When $M \geq i+2$, both $f_{m_1+1\dots m_1+i-1, m_1+i, m_1+i+1}$ and σ_{m_1+i} commute with $f_{1\dots m_1, m_1+1\dots m_1+M-1, m_1+M}$ because $x_{m_1+1\dots m_1+i-1, m_1+i}$, x_{m_1+i, m_1+i+1} , and σ_{m_1+i} commute with $x_{1\dots m_1, m_1+1\dots m_1+M-1}$ and $x_{m_1+1\dots m_1+M-1, m_1+M}$. Therefore

$$\begin{aligned} (1.14) & = f_{[m_1],[i+1],[m_2]}^{-1} \cdot f_{m_1+1\dots m_1+i-1, m_1+i, m_1+i+1}^{-1} \cdot \sigma_{m_1+i}^\lambda \\ & \quad \cdot f_{m_1+1\dots m_1+i-1, m_1+i, m_1+i+1} \cdot f_{[m_1],[i+1],[m_2]}. \end{aligned}$$

- When $M = i, i + 1$, our calculation goes as follows.

$$\begin{aligned}
 (1.14) &= f_{[m_1],[i-1],[m_2]}^{-1} \cdot f_{1\dots m_1, m_1+1\dots m_1+i-1, m_1+i}^{-1} \\
 &\quad \cdot f_{1\dots m_1, m_1+1\dots m_1+i, m_1+i+1}^{-1} \cdot f_{m_1+1\dots m_1+i-1, m_1+i, m_1+i+1}^{-1} \\
 &\quad \cdot \sigma_{m_1+i}^\lambda \cdot f_{m_1+1\dots m_1+i-1, m_1+i, m_1+i+1} \\
 &\quad \cdot f_{1\dots m_1, m_1+1\dots m_1+i, m_1+i+1} \cdot f_{1\dots m_1, m_1+1\dots m_1+i-1, m_1+i} \\
 &\quad \cdot f_{[m_1],[i-1],[m_2]},
 \end{aligned}$$

(by the pentagon equation (1.4))

$$\begin{aligned}
 &= f_{[m_1],[i-1],[l+m_2]}^{-1} \cdot f_{1\dots m_1+i-1, m_1+i, m_1+i+1}^{-1} \\
 &\quad \cdot f_{1\dots m_1, m_1+1\dots m_1+i-1, m_1+i, m_1+i+1}^{-1} \\
 &\quad \cdot \sigma_{m_1+i}^\lambda \cdot f_{1\dots m_1, m_1+1\dots m_1+i-1, m_1+i, m_1+i+1} \\
 &\quad \cdot f_{1\dots m_1+i-1, m_1+i, m_1+i+1} \cdot f_{[m_1],[i-1],[l+m_2+2]}
 \end{aligned}$$

(since σ_{m_1+i} commutes with $f_{1\dots m_1, m_1+1\dots m_1+i-1, m_1+i, m_1+i+1}$)

$$\begin{aligned}
 &= f_{[m_1],[i-1],[l+m_2]}^{-1} \cdot f_{1\dots m_1+i-1, m_1+i, m_1+i+1}^{-1} \\
 &\quad \cdot \sigma_{m_1+i}^\lambda \cdot f_{1\dots m_1+i-1, m_1+i, m_1+i+1} \cdot f_{[m_1],[i-1],[l+m_2+2]}.
 \end{aligned}$$

- When $M \leq i - 1$, both $f_{1\dots m_1+i-1, m_1+i, m_1+i+1}$ and σ_{m_1+i} commute with $f_{1\dots m_1, m_1+1\dots m_1+M-1, m_1+M}$. Therefore

$$\begin{aligned}
 (1.14) &= f_{1\dots m_1+i-1, m_1+i, m_1+i+1}^{-1} \cdot \sigma_{m_1+i}^\lambda \cdot f_{1\dots m_1+i-1, m_1+i, m_1+i+1} \\
 &= \sigma(\sigma_{m_1+i}) \\
 &= \sigma(e_{m_1} \otimes \sigma_i \otimes e_{m_2}).
 \end{aligned}$$

Hence we get equality (1.12). □

Notation 1.17. Let $l, n \geq 1$ and $1 \leq k \leq l$. We consider the continuous group homomorphism

$$P_l \longrightarrow P_{l+n-1} \tag{1.15}$$

sending, for $1 \leq i < j \leq l$,

$$x_{i,j} \longmapsto \begin{cases} x_{i+k-1, j+k-1} & (k < i), \\ x_{i\dots i+k-1, j+k-1} & (k = i), \\ x_{i, j+k-1} & (i < k < j), \\ x_{i, j\dots j+k-1} & (k = j), \\ x_{i, j} & (j < k). \end{cases}$$

We obtain the map by replacing the k -th string (from the left) by the trivial braid e_n with n strings, hence it naturally extends to two maps (not homomorphisms)

$$\text{ev}_{k,e_n}: B_l \longrightarrow B_{l+n-1} \quad \text{and} \quad \text{ev}^{k,e_n}: B_l \longrightarrow B_{l+n-1}$$

which replaces the k -th string (from the bottom and the above left respectively) by the trivial braid e_n with n strings. Both of their restrictions into P_l are equal to the above map (1.15). Since the map (1.15) also continuously extends into the homomorphism $\widehat{P}_l \rightarrow \widehat{P}_l$, our two maps naturally extend to the maps (not homomorphisms)

$$\text{ev}_{k,e_n}: \widehat{B}_l \longrightarrow \widehat{B}_{l+n-1} \quad \text{and} \quad \text{ev}^{k,e_n}: \widehat{B}_l \longrightarrow \widehat{B}_{l+n-1}.$$

Here we employ the same symbols because there would be no confusion.

Proposition 1.18. *Put $l \geq 1$. Let $\sigma = (\lambda, f) \in \widehat{\text{GT}}$ and $b \in \widehat{B}_l$. Set $k' = b(k)$. Here $b(k)$ stands for the image of k by the permutation corresponding to b by the projection $B_l \rightarrow \mathfrak{S}_l$. Then, for each k with $1 \leq k \leq l$, we have*

$$\begin{cases} \sigma(\text{ev}_{k,e_n}(b)) = f_{[k'-1],[n],[l-k']}^{-1} \cdot \text{ev}_{k,e_n}(\sigma(b)) \cdot f_{[k-1],[n],[l-k]}, \\ \sigma(\text{ev}^{k',e_n}(b)) = f_{[k'-1],[n],[l-k']}^{-1} \cdot \text{ev}^{k',e_n}(\sigma(b)) \cdot f_{[k-1],[n],[l-k]}. \end{cases} \quad (1.16)$$

Proof. It suffices to prove the first equality, for the validity of the second equality is immediate once we have the first equality. Firstly we prove (1.16) for $b = \sigma_i$ ($1 \leq i \leq l-1$).

- When $k < i$, we have

$$\text{ev}_{k,e_n}(\sigma_i) = \sigma_{i+n-1}$$

and $k' = k$. Therefore

$$\begin{aligned} \text{RHS} &= f_{[k-1],[n],[l-k]}^{-1} \cdot \text{ev}_{k,e_n}(f_{1\dots i-1,i,i+1}^{-1} \sigma_i^\lambda f_{1\dots i-1,i,i+1}) \cdot f_{[k-1],[n],[l-k]} \\ &= f_{[k-1],[n],[l-k]}^{-1} \cdot (f_{1\dots i+n-2,i+n-1,i+n}^{-1} \sigma_{i+n-1}^\lambda f_{1\dots i+n-2,i+n-1,i+n}) \\ &\quad \cdot f_{[k-1],[n],[l-k]}. \end{aligned}$$

By $k-1+n \leq i+n-2$, $f_{[k-1],[n],[l-k]}$ commutes with $f_{1\dots i+n-2,i+n-1,i+n}$ and σ_{i+n-1} . Thus

$$\begin{aligned} \text{RHS} &= f_{1\dots i+n-2,i+n-1,i+n}^{-1} \sigma_{i+n-1}^\lambda f_{1\dots i+n-2,i+n-1,i+n} \\ &= \sigma(\sigma_{i+n-1}) \\ &= \sigma(\text{ev}_{k,e_n}(\sigma_i)) \\ &= \text{LHS}. \end{aligned}$$

- When $k = i$, we have

$$\text{ev}_{k,e_n}(\sigma_i) = \text{ev}_{i,e_n}(\sigma_i) = \sigma_i \sigma_{i+1} \cdots \sigma_{i+n-1}$$

and $k' = k + 1 = i + 1$. Therefore

$$\begin{aligned} \text{RHS} &= f_{[i],[n],[l-i-1]}^{-1} \cdot \text{ev}_{i,e_n}(f_{1\dots i-1,i,i+1}^{-1} \sigma_i^\lambda f_{1\dots i-1,i,i+1}) \cdot f_{[i-1],[n],[l-i]} \\ &= f_{[i],[n],[l-i-1]}^{-1} \cdot f_{1\dots i-1,i,i+1\dots i+n}^{-1} \cdot \text{ev}_{i,e_n}(\sigma_i^\lambda) \\ &\quad \cdot f_{1\dots i-1,i\dots i+n-1,i+n} \cdot f_{[i-1],[n],[l-i]} \\ &= f_{[i],[n],[l-i-1]}^{-1} \cdot f_{1\dots i-1,i,i+1\dots i+n}^{-1} \cdot \text{ev}_{i,e_n}(\sigma_i \cdot x_{i,i+1}^m) \\ &\quad \cdot f_{[i-1],[n+1],[l-i-1]} \\ &= f_{[i],[n],[l-i-1]}^{-1} \cdot f_{1\dots i-1,i,i+1\dots i+n}^{-1} \cdot (\sigma_i \cdots \sigma_{i+n-1}) \cdot x_{i\dots i+n-1,i+n}^m \\ &\quad \cdot f_{[i-1],[n+1],[l-i-1]} \\ &= f_{[i],[n],[l-i-1]}^{-1} \cdot f_{1\dots i-1,i,i+1\dots i+n}^{-1} \\ &\quad \cdot (e_{i-1} \otimes (\sigma_1 \cdots \sigma_n) \cdot x_{1\dots n,n+1}^m \otimes e_{l-i-1}) \\ &\quad \cdot f_{[i-1],[n+1],[l-i-1]}. \end{aligned}$$

By (1.9),

$$\begin{aligned} \text{RHS} &= f_{[i],[n],[l-i-1]}^{-1} \cdot f_{1\dots i-1,i,i+1\dots i+n}^{-1} \\ &\quad \cdot (e_{i-1} \otimes f_{[1],[n],[0]} \cdot \sigma(\sigma_1 \cdots \sigma_n) \otimes e_{l-i-1}) \\ &\quad \cdot f_{[i-1],[n+1],[l-i-1]} \\ &= f_{[i],[n],[l-i-1]}^{-1} \cdot f_{1\dots i-1,i,i+1\dots i+n}^{-1} \cdot (e_{i-1} \otimes f_{[1],[n],[0]} \cdot \otimes e_{l-i-1}) \\ &\quad \cdot (e_{i-1} \otimes \sigma(\sigma_1 \cdots \sigma_n) \otimes e_{l-i-1}) \cdot f_{[i-1],[n+1],[l-i-1]}. \end{aligned}$$

By Lemma 1.19 below,

$$\begin{aligned} \text{RHS} &= f_{[i-1],[n+1],[l-i-1]}^{-1} \cdot (e_{i-1} \otimes \sigma(\sigma_1 \cdots \sigma_n) \otimes e_{l-i-1}) \\ &\quad \cdot f_{[i-1],[n+1],[l-i-1]}. \end{aligned}$$

By Proposition 1.16,

$$\begin{aligned} \text{RHS} &= \sigma(e_{i-1} \otimes (\sigma_1 \cdots \sigma_n) \otimes e_{l-i-1}) \\ &= \sigma(\sigma_i \cdots \sigma_{i+n-1}) \\ &= \sigma(\text{ev}_{i,e_n}(\sigma_i)) \\ &= \text{LHS}. \end{aligned}$$

- When $k = i + 1$, we have

$$\text{ev}_{k,e_n}(\sigma_i) = \text{ev}_{i+1,e_n}(\sigma_i) = \sigma_{i+n-1} \cdots \sigma_{i+1} \sigma_i$$

and $k' = k - 1 = i$. Therefore

$$\begin{aligned} \text{RHS} &= f_{[i-1],[n],[l-i]}^{-1} \cdot \text{ev}_{i+1,e_n}(f_{1\dots i-1,i,i+1}^{-1} \sigma_i^\lambda f_{1\dots i-1,i,i+1}) \\ &\quad \cdot f_{[i],[n],[l-i-1]} \\ &= f_{[i-1],[n],[l-i]}^{-1} \cdot f_{1\dots i-1,i\dots i+n-1,i+n}^{-1} \cdot \text{ev}_{i+1,e_n}(\sigma_i^\lambda) \\ &\quad \cdot f_{1\dots i-1,i,i+1\dots i+n} \cdot f_{[i],[n],[l-i-1]} \\ &= f_{[i-1],[n+1],[l-i-1]}^{-1} \cdot \text{ev}_{i+1,e_n}(x_{i,i+1}^m \cdot \sigma_i) \\ &\quad \cdot f_{1\dots i-1,i,i+1\dots i+n} \cdot f_{[i],[n],[l-i-1]} \\ &= f_{[i-1],[n+1],[l-i-1]}^{-1} \cdot x_{i\dots i+n-1,i+n}^m \cdot (\sigma_{i+n-1} \cdots \sigma_{i+1} \sigma_i) \\ &\quad \cdot f_{1\dots i-1,i,i+1\dots i+n} \cdot f_{[i],[n],[l-i-1]} \\ &= f_{[i-1],[n+1],[l-i-1]}^{-1} \cdot (e_{i-1} \otimes x_{1\dots n,n+1}^m \cdot (\sigma_n \cdots \sigma_1) \otimes e_{l-i-1}) \\ &\quad \cdot f_{1\dots i-1,i,i+1\dots i+n} \cdot f_{[i],[n],[l-i-1]}. \end{aligned}$$

By (1.9),

$$\begin{aligned} \text{RHS} &= f_{[i-1],[n+1],[l-i-1]}^{-1} \cdot (e_{i-1} \otimes \sigma(\sigma_n \cdots \sigma_1) \cdot f_{[1],[n],[0]}^{-1} \otimes e_{l-i-1}) \\ &\quad \cdot f_{1\dots i-1,i,i+1\dots i+n} \cdot f_{[i],[n],[l-i-1]} \\ &= f_{[i-1],[n+1],[l-i-1]}^{-1} \cdot (e_{i-1} \otimes \sigma(\sigma_n \cdots \sigma_1) \otimes e_{l-i-1}) \\ &\quad \cdot (e_{i-1} \otimes f_{[1],[n],[0]}^{-1} \otimes e_{l-i-1}) \\ &\quad \cdot f_{1\dots i-1,i,i+1\dots i+n} \cdot f_{[i],[n],[l-i-1]}. \end{aligned}$$

By Lemma 1.19 below,

$$\begin{aligned} \text{RHS} &= f_{[i-1],[n+1],[l-i-1]}^{-1} \cdot (e_{i-1} \otimes \sigma(\sigma_n \cdots \sigma_1) \otimes e_{l-i-1}) \\ &\quad \cdot f_{[i-1],[n+1],[l-i-1]}. \end{aligned}$$

By Proposition 1.16,

$$\begin{aligned} \text{RHS} &= \sigma(e_{i-1} \otimes (\sigma_n \cdots \sigma_1) \otimes e_{l-i-1}) \\ &= \sigma(\sigma_{i+n-1} \cdots \sigma_i) \\ &= \sigma(\text{ev}_{i+1,e_n}(\sigma_i)) \\ &= \text{LHS}. \end{aligned}$$

- When $k > i + 1$, we have

$$\text{ev}_{k,e_n}(\sigma_i) = \sigma_i$$

and $k' = k$. Therefore

$$\begin{aligned} \text{RHS} &= f_{[k-1],[n],[l-k]}^{-1} \cdot \text{ev}_{k,e_n}(f_{1\dots i-1,i,i+1}^{-1} \sigma_i^\lambda f_{1\dots i-1,i,i+1}) \cdot f_{[k-1],[n],[l-k]} \\ &= f_{[k-1],[n],[l-k]}^{-1} \cdot (f_{1\dots i-1,i,i+1}^{-1} \sigma_i^\lambda f_{1\dots i-1,i,i+1}) \cdot f_{[k-1],[n],[l-k]}. \end{aligned}$$

By $i + 1 \leq k - 1$, $f_{[k-1],[n],[l-k]}$ commutes with $f_{1\dots i-1,i,i+1}$ and σ_i . Thus

$$\begin{aligned} \text{RHS} &= f_{1\dots i-1,i,i+1}^{-1} \sigma_i^\lambda f_{1\dots i-1,i,i+1} \\ &= \sigma(\sigma_i) \\ &= \sigma(\text{ev}_{k,e_n}(\sigma_i)) \\ &= \text{LHS}. \end{aligned}$$

When equation (1.16) for $b = \sigma_i$ is obtained.

The validity for $b = \sigma_i$ implies the validity for $b \in B_l$ because each element of B_l is a finite product of σ_i 's. Whence particularly we have the validity for P_l . Then by continuity we have for \hat{P}_l . Since we have the validity for B_l and \hat{P}_l , we have the validity for \hat{B}_l . \square

The auxiliary lemma below is required to prove the above proposition.

Lemma 1.19. For $\sigma = (\lambda, f) \in \widehat{\text{GT}}$ and $i, n, l > 0$ with $l > i$, the following equation holds in \hat{B}_{l+n-1} :

$$\begin{aligned} (e_{i-1} \otimes f_{[1],[n],[0]} \otimes e_{l-i-1}) \cdot f_{[i-1],[n+1],[l-i-1]} \\ = f_{1\dots i-1,i,i+1\dots i+n} \cdot f_{[i],[n],[l-i-1]}. \end{aligned}$$

Proof. The above equation can be read as

$$\begin{aligned} (f_{i,i+1\dots i+n-1,i+n} \cdots f_{i,i+1,i+2}) \cdot f_{1\dots i-1,i\dots i+n-1,i+n} \cdots f_{1\dots i-1,i,i+1} \\ = f_{1\dots i-1,i,i+1\dots i+n} \cdot (f_{1\dots i,i+1\dots i+n-1,i+n} \cdots f_{1\dots i,i+1,i+2}). \end{aligned}$$

It can be proved by successive applications of (1.4). \square

2. Profinite knots

This section is to present our main results. Our ABC-construction of profinite knots is introduced and the basic properties of profinite knots are shown in §2.1. An action of the absolute Galois group on profinite knots is rigorously established in §2.3, where the notion of pro- l knots introduced in §2.2 serves to show its property.

2.1. ABC-construction. Profinite tangle diagrams, profinite analogues of tangle diagrams, are introduced as consistent finite sequences of symbols of three types A , \widehat{B} and C in Definition 2.3. Profinite link diagrams mean profinite tangle diagrams without endpoints and profinite knot diagrams mean profinite link diagrams with a single connected component (Definition 2.4). The notion of isotopy for them are given by a profinite analogue of Turaev moves in Definition 2.8. Two fundamental properties for the set $\widehat{\mathcal{K}}$ of profinite knots (isotopy classes of profinite knot diagrams) are presented; Theorem 2.11 explains that there is a natural map from the set \mathcal{K} of isotopy classes of (topological) knots to our set $\widehat{\mathcal{K}}$ and our Theorem 2.16 says that our $\widehat{\mathcal{K}}$ carries a structure of a topological commutative monoid.

Here is a brief review of tangles and knots.

Notation 2.1. Let $k, l \geq 0$. Let $\epsilon = (\epsilon_1 \dots \epsilon_k)$ and $\epsilon' = (\epsilon'_1 \dots \epsilon'_l)$ be sequences (including the empty sequence \emptyset) of symbols \uparrow and \downarrow . An (oriented)³ *tangle* of type (ϵ, ϵ') means a smooth embedded compact oriented one-dimensional real manifolds in $[0, 1] \times \mathbb{C}$ (hence it is a finite disjoint union of embedded one-dimensional intervals and circles), whose boundaries are $\{(1, 1) \dots (1, k), (0, 1) \dots (0, l)\}$ such that ϵ_i (resp. ϵ'_j) is \uparrow or \downarrow if the tangle is oriented upwards or downwards at $(1, i)$ (resp. at $(0, j)$) respectively. A *link* is a tangle of type (\emptyset, \emptyset) , i.e. $k = l = 0$, and a *knot* means a link with a single connected component.

A tangle is like ‘a braid’ whose each connected component is allowed to be a circle and have endpoints on the same plane. Figure 2.1 might help the readers to have a good understanding of the definition.

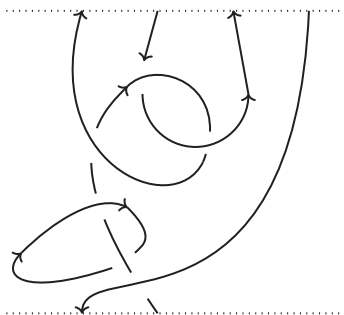


Figure 2.1. A tangle of type $(\uparrow\downarrow\downarrow, \downarrow\uparrow)$.

³ We occasionally omit to mention it. Throughout the paper all tangles are assumed to be oriented.

The following notion of profinite fundamental tangle diagrams plays a role of composite elements of the notion of profinite tangles.

Definition 2.2. The set of *fundamental profinite tangle diagrams* means the disjoint union of the following three sets A , \hat{B} and⁴ C of symbols:

$$A := \{a_{k,l}^\epsilon \mid k, l = 0, 1, 2, \dots, \epsilon = (\epsilon_i)_{i=1}^{k+l+1} \in \{\uparrow, \downarrow\}^k \times \{\curvearrowright, \curvearrowleft\} \times \{\uparrow, \downarrow\}^l\},$$

$$\hat{B} := \{b_n^\epsilon \mid b_n^\epsilon = (b_n, \epsilon = (\epsilon_i)_{i=1}^n) \in \hat{B}_n \times \{\uparrow, \downarrow\}^n, n = 1, 2, 3, 4, \dots\},$$

$$C := \{c_{k,l}^\epsilon \mid k, l = 0, 1, 2 \dots \epsilon = (\epsilon_i)_{i=1}^{k+l+1} \in \{\uparrow, \downarrow\}^k \times \{\cup, \cap\} \times \{\uparrow, \downarrow\}^l\}.$$

Here all arrows are merely regarded as symbols.

We occasionally depict these fundamental profinite tangle diagrams with ignorance of arrows (which represent orientation of each strings) as the pictures in Figure 2.2, which we call their topological pictures.

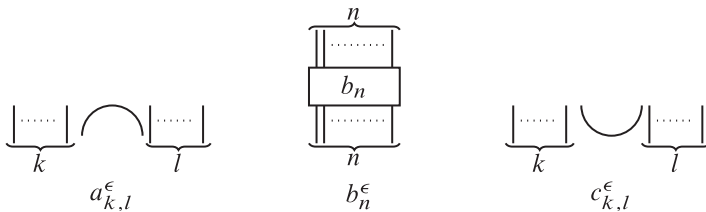


Figure 2.2. Topological picture of fundamental profinite tangle diagrams.

For a fundamental profinite tangle diagram γ , its *source* $s(\gamma)$ and its *target* $t(\gamma)$ are sequences of \uparrow and \downarrow defined below:

- (1) When $\gamma = a_{k,l}^\epsilon$, $s(\gamma)$ is the sequence of \uparrow and \downarrow replacing \curvearrowright (resp. \curvearrowleft) by $\uparrow\downarrow$ (resp. $\downarrow\uparrow$) in ϵ and $t(\gamma)$ is the sequence omitting \curvearrowright and \curvearrowleft in ϵ (cf. Figure 2.3).



Figure 2.3. $a_{2,1}^\epsilon$ with $s(a_{2,1}^\epsilon) = \uparrow\downarrow\uparrow\uparrow$ and $t(a_{2,1}^\epsilon) = \uparrow\downarrow\uparrow$.

- (2) When $\gamma = b_n^\epsilon$, $s(\gamma) = \epsilon$ and $t(\gamma)$ is the permutation of ϵ induced by the image of b_n^ϵ of the projection \hat{B}_n to the symmetric group \mathfrak{S}_n (cf. Figure 2.4).

⁴ A , B , and C stand for “annihilations,” “braids,” and “creations,” respectively.

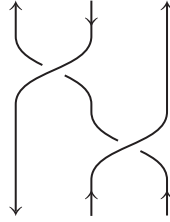


Figure 2.4. An example of b_3^ϵ with $s(b_3^\epsilon) = \epsilon = \downarrow\uparrow\uparrow$ and $t(b_3^\epsilon) = \uparrow\downarrow\uparrow$.

- (3) When $\gamma = c_{k,l}^\epsilon$, $s(\gamma)$ is the set omitting \cup and \smile in ϵ and $t(\gamma)$ is the set replacing \cup (resp. \smile) by $\downarrow\uparrow$ (resp. $\uparrow\downarrow$) in ϵ .

Definition 2.3. A *profinite (oriented) tangle diagram* means a finite *consistent*⁵ sequence $T = \{\gamma_i\}_{i=1}^n$ of fundamental profinite tangles (which we denote by $\gamma_n, \dots, \gamma_2 \cdot \gamma_1$). Its source and its target are defined by

$$s(T) := s(\gamma_1) \quad \text{and} \quad t(T) := t(\gamma_n).$$

A *profinite (oriented) link diagram* means a profinite tangle diagram T with $s(T) = t(T) = \emptyset$.

For a fundamental profinite tangle diagram γ , its *skeleton* $\mathbb{S}(\gamma)$ is the graph consisting of finitely many vertices and finitely many edges connecting them as follows:

- (1) When $\gamma = a_{k,l}^\epsilon$ or $c_{k,l}^\epsilon$, $\mathbb{S}(\gamma)$ is nothing but the graph of the topological picture of γ in Figure 2.2 whose set of vertices is the collection of its endpoints and whose set of edges is given by the arrows connecting them (cf. Figure 2.5).



Figure 2.5. Skeleton of Figure 2.3.

- (2) When $\gamma = b_n^\epsilon$, $\mathbb{S}(\gamma)$ is the graph describing the permutation $p(b_n^\epsilon)$ (cf. Figure 2.6), where p means the projection $\widehat{B}_n \rightarrow \mathfrak{S}_n$. Namely its set of vertices is the collection of its endpoints and its set of edges is the set of diagonal lines combining the corresponding vertices.

⁵Here ‘consistent’ means successively composable, that is, $s(\gamma_{i+1}) = t(\gamma_i)$ holds for all $i = 1, 2, \dots, n - 1$.

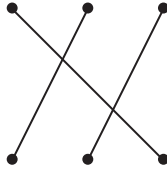


Figure 2.6. Skeleton of Figure 2.4.

For a profinite tangle diagram $T = \gamma_n \cdots \gamma_2 \cdot \gamma_1$, its skeleton

$$\mathbb{S}(T) = \mathbb{S}(\gamma_n) \cdots \mathbb{S}(\gamma_2) \cdot \mathbb{S}(\gamma_1)$$

means the graph obtained by the composition of $\mathbb{S}(\gamma_i)$ ($1 \leq i \leq n$) (cf. Figures 2.7 and 2.8).

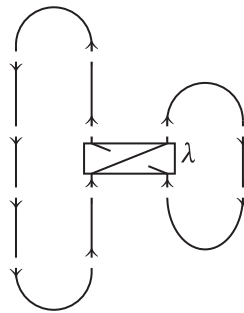


Figure 2.7. Is this a profinite knot?

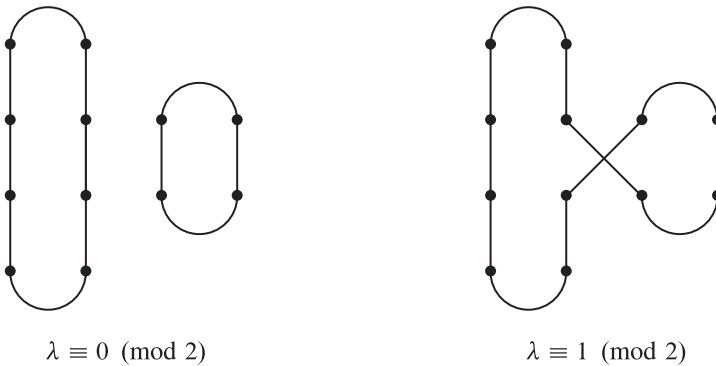


Figure 2.8. Skeleton of Figure 2.7.

Definition 2.4. A *connected component* of a profinite tangle diagram T means a connected component of the skeleton $\mathbb{S}(T)$ as a graph. A *profinite (oriented) knot diagram* means a profinite (oriented) link diagram with a single connected component.

It is easy to see that the number of connected components of any profinite tangle diagram is always finite. A profinite knot diagram is a profinite version of a (topological) oriented knot diagram.

Problem 2.5. Can we regard a profinite knot as a wild knot?⁶

It might be nice if we could give any topological meaning for all (or a part of) profinite knot diagrams.

Example 2.6. The profinite link diagram

$$a_{0,0}^{\curvearrowright} \cdot a_{2,0}^{\downarrow\uparrow\curvearrowright} \cdot (\sigma_2^{\downarrow\uparrow\uparrow\downarrow})^\lambda \cdot c_{2,0}^{\downarrow\uparrow\curvearrowleft} \cdot c_{0,0}^{\curvearrowup} \quad (\lambda \in \widehat{\mathbb{Z}})$$

depicted in Figure 2.7 (here $\sigma_2^{\downarrow\uparrow\uparrow\downarrow}$ is the generator σ_2 of \widehat{B}_4) is with 2 connected components if $\lambda \equiv 0 \pmod{2}$ and is with a single connected component (hence is a profinite knot) if $\lambda \equiv 1 \pmod{2}$ (cf. Figure 2.8).

Notation 2.7. (1) The symbol $e_n^\epsilon = (e_n, \epsilon)$ stands for the fundamental profinite tangle diagram in \widehat{B} with $s(e_n^\epsilon) = \epsilon$ which corresponds to the trivial braid e_n in \widehat{B}_n . For a fundamental profinite tangle diagram γ , we mean $e_{n_1}^{\epsilon_1} \otimes \gamma \otimes e_{n_2}^{\epsilon_2}$ by the fundamental profinite tangle diagram obtained by putting $e_{n_1}^{\epsilon_1}$ and $e_{n_2}^{\epsilon_2}$ on the left and on the right of γ respectively. So, $e_{n_1}^{\epsilon_1} \otimes a_{k,l}^\epsilon \otimes e_{n_2}^{\epsilon_2} = a_{n_1+k,l+n_2}^{\epsilon_1, \epsilon, \epsilon_2}$, for instance. For a profinite tangle $T = \{\gamma_i\}_{i=1}^n$ (γ_i : a fundamental profinite tangle diagram), $e_{n_1}^{\epsilon_1} \otimes T \otimes e_{n_2}^{\epsilon_2}$ means the profinite tangle diagram $\{e_{n_1}^{\epsilon_1} \otimes \gamma_i \otimes e_{n_2}^{\epsilon_2}\}_{i=1}^n$.

(2) Let $\epsilon_0 \in \{\uparrow, \downarrow\}^n$ for some n . For a fundamental profinite tangle diagram $b_l^\epsilon \in \widehat{B}$ and k with $1 \leq k \leq l$, the symbol $\text{ev}_{k, \epsilon_0}(b_l^\epsilon)$ (resp. $\text{ev}^{k, \epsilon_0}(b_l^\epsilon)$) means the element in \widehat{B} which represents the profinite braid replacing the k -th string (from its bottom (resp. above) left) of b_l^ϵ by the trivial braid $e_n^{\epsilon_0}$ with n strings whose source is ϵ_0 (cf. Notation 1.17). For instance, the profinite tangle diagram in Figure 2.4 is described as $\text{ev}_{1, \downarrow\uparrow}(\curvearrowright)$ or $\text{ev}^{2, \downarrow\uparrow}(\curvearrowright)$.

⁶ A *wild knot* means a *topological* embedding of the oriented circle into S^3 (or \mathbb{R}^3).

Definition 2.8. For profinite tangle diagrams, the moves (T1)–(T6) are defined as follow.

(T1) *Trivial braids invariance:* for a profinite tangle diagram T with⁷ $|s(T)| = m$ (resp. $|t(T)| = n$),

$$e_n^{t(T)} \cdot T = T = T \cdot e_m^{s(T)}.$$

For e_n , see Notation 2.7. Figure 2.9 depicts the move.

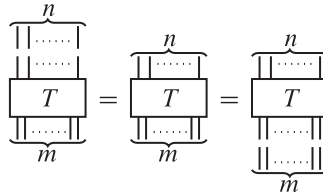


Figure 2.9. (T1): Trivial braids invariance.

(T2) *Braids composition:* for $b_n^{\epsilon_1}, b_n^{\epsilon_2} \in \widehat{B}$ with $t(b_n^{\epsilon_1}) = s(b_n^{\epsilon_2})$,

$$b_n^{\epsilon_2} \cdot b_n^{\epsilon_1} = b_n^{\epsilon_3}.$$

Here $b_n^{\epsilon_3}$ means the element in \widehat{B} with $s(b_n^{\epsilon_3}) = s(b_n^{\epsilon_1})$ and $t(b_n^{\epsilon_3}) = t(b_n^{\epsilon_2})$ which represents the product $b_2 \cdot b_1$ of two braids in \widehat{B}_n . Figure 2.10 depicts the move.

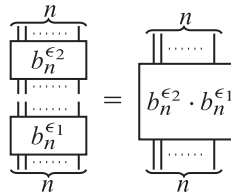


Figure 2.10. (T2): Braids composition.

(T3) *Independent tangles relation:* for profinite tangle diagrams T_1 and T_2 with $|s(T_1)| = m_1$, $|t(T_1)| = n_1$, $|s(T_2)| = m_2$ and $|t(T_2)| = n_2$,

$$(e_{n_1}^{t(T_1)} \otimes T_2) \cdot (T_1 \otimes e_{m_2}^{s(T_2)}) = (T_1 \otimes e_{n_2}^{t(T_2)}) \cdot (e_{m_1}^{s(T_1)} \otimes T_2).$$

We occasionally denote both hands side by $T_1 \otimes T_2$. For the symbol \otimes , see Notation 2.7. Figure 2.11 depicts the move.

⁷ For a set S , $|S|$ stands for its cardinality.

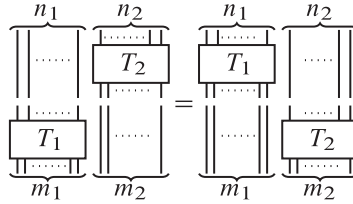


Figure 2.11. (T3): Independent tangles relation.

(T4) *Braid-tangle relations*: for $b_l^\epsilon \in \widehat{B}$, k with $1 \leq k \leq l$ and a profinite tangle diagram T with $|s(T)| = m$ and $|t(T)| = n$,

$$\text{ev}_{k,t(T)}(b_l^\epsilon) \cdot (e_{k-1}^{s_1} \otimes T \otimes e_{l-k}^{s_2}) = (e_{k'-1}^{t_1} \otimes T \otimes e_{l-k'}^{t_2}) \cdot \text{ev}^{k',s(T)}(b_l^\epsilon).$$

For ev , see Notation 2.7. For $s(b_l^\epsilon) = \epsilon = (\epsilon_i)_{i=1}^l$ we put $s_1 := (\epsilon_i)_{i=1}^{k-1}$ and $s_2 := (\epsilon_i)_{i=k+1}^l$. Put $k' = b_l^\epsilon(k)$. Here $b_l^\epsilon(k)$ stands for the image of k by the permutation which corresponds to b_l^ϵ by the projection $B_l \rightarrow \mathfrak{S}_l$. For $t(b_l^\epsilon) = (\epsilon'_i)_{i=1}^l$ we put $t_1 := (\epsilon'_i)_{i=1}^{k'-1}$ and $t_2 := (\epsilon'_i)_{i=k'+1}^l$. Figure 2.12 depicts the move.

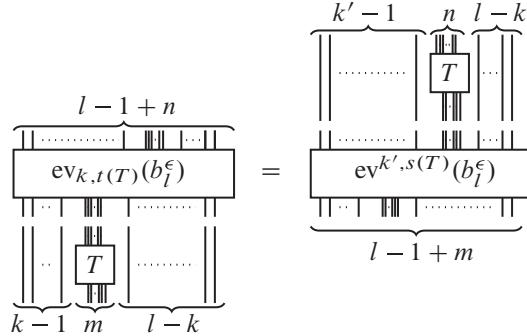


Figure 2.12. (T4): Braid-tangle relation.

(T5) *Creation–annihilation relation*: for $c_{k,l}^\epsilon \in C$ and $a_{k+1,l-1}^{\epsilon'}$ in A with $t(c_{k,l}^\epsilon) = s(a_{k+1,l-1}^{\epsilon'})$

$$a_{k+1,l-1}^{\epsilon'} \cdot c_{k,l}^\epsilon = e_{k+l}^{s(c_{k,l}^\epsilon)}.$$

And for $c_{k,l}^\epsilon \in C$ and $a_{k-1,l+1}^{\epsilon'}$ in A with $t(c_{k,l}^\epsilon) = s(a_{k-1,l+1}^{\epsilon'})$

$$a_{k-1,l+1}^{\epsilon'} \cdot c_{k,l}^\epsilon = e_{k+l}^{s(c_{k,l}^\epsilon)}.$$

Figure 2.13 depicts the move.



Figure 2.13. (T5): Creation–annihilation relations.

(T6) *First Reidemeister move*: for⁸ $c \in \widehat{\mathbb{Z}}$, $c_{k,l}^\epsilon \in C$ and $\sigma_{k+1}^{\epsilon'} \in \widehat{B}$ which represents $\sigma_{k+1} \in \widehat{B}_{k+l+2}$ and $t(c_{k,l}^\epsilon) = s((\sigma_{k+1}^{\epsilon'})^c)$

$$(\sigma_{k+1}^{\epsilon'})^c \cdot c_{k,l}^\epsilon = c_{k,l}^{\bar{\epsilon}}$$

where $\bar{\epsilon}$ is chosen to be $t(\bar{\epsilon}) = t((\sigma_{k+1}^{\epsilon'})^c)$.

For $c \in \widehat{\mathbb{Z}}$, $a_{k,l}^\epsilon \in A$ and $\sigma_{k+1}^{\epsilon'} \in \widehat{B}$ which represents $\sigma_{k+1} \in \widehat{B}_{k+l+2}$ and $s(a_{k,l}^\epsilon) = t((\sigma_{k+1}^{\epsilon'})^c)$

$$a_{k,l}^\epsilon \cdot (\sigma_{k+1}^{\epsilon'})^c = a_{k,l}^{\bar{\epsilon}}$$

where $\bar{\epsilon}$ is chosen to be $s(\bar{\epsilon}) = s((\sigma_{k+1}^{\epsilon'})^c)$. Figure 2.14 depicts the moves.

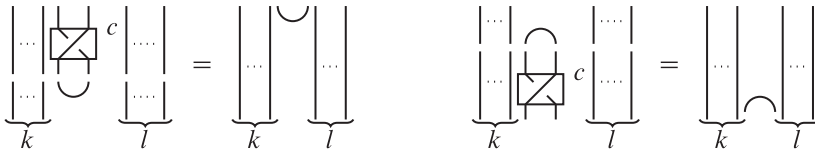


Figure 2.14. (T6): First Reidemeister move.

We note that in the first (resp. second) equation $c_{k,l}^\epsilon = c_{k,l}^{\bar{\epsilon}}$ (resp. $a_{k,l}^\epsilon = a_{k,l}^{\bar{\epsilon}}$) if and only if $c \equiv 0 \pmod{2}$.

These moves (T1)–(T6) are profinite analogues of the *Turaev moves* [39] for oriented tangles (consult also [5], [21], [32], etc.). Our above formulation is stimulated by the moves presented (R1)–(R11) in [1].

Definition 2.9. Two profinite (oriented) tangle diagrams T_1 and T_2 are *isotopic*, denoted $T_1 = T_2$ by abuse of notation, if they are related by a *finite* number of the moves (T1)–(T6). A *profinite tangle*, *profinite link* and *profinite knot* means an isotopy class of profinite tangle diagrams, link diagrams, knot diagrams respectively. The set of profinite tangles is denoted by $\widehat{\mathcal{T}}$. The set $\widehat{\mathcal{L}}$ of profinite links (resp. the set $\widehat{\mathcal{K}}$ of profinite knots) is the subset of $\widehat{\mathcal{T}}$ which consists of isotopy classes of profinite links (resp. of profinite knots).

⁸ It should be worthy to emphasize that c is assumed to be not in \mathbb{Z} but in $\widehat{\mathbb{Z}}$.

Note 2.10. Profinite topology on \widehat{B}_n ($n = 1, 2, \dots$) and the discrete topology on A and on C yield a topology on the space of profinite tangles. Hence $\widehat{\mathcal{T}}$ carries a structure of topological space.

Theorem 2.11. (1) Let \mathcal{T} be the set of isotopy classes of (topological) oriented tangles. There is a natural map

$$h: \mathcal{T} \longrightarrow \widehat{\mathcal{T}},$$

which we call the profinite realization map.

(2) The above profinite realization map induces the map

$$h: \mathcal{K} \longrightarrow \widehat{\mathcal{K}}.$$

Here \mathcal{K} stands for the set of isotopy classes of topological oriented knots.

Proof. (1) The result in [1] indicates that the set \mathcal{T} is described by the set of consistent finite sequences of fundamental tangles, elements of A ,

$$B := \{b_n^\epsilon \mid b_n^\epsilon = (b_n, \epsilon = (\epsilon_i)_{i=1}^n) \in B_n \times \{\uparrow, \downarrow\}^n, n = 1, 2, 3, 4, \dots\}$$

and C , modulo the (discrete) Turaev moves. Since the (discrete) Turaev moves in this case mean the moves replacing profinite tangles and braids by (discrete) tangles and braids in (T1)–(T6) and $c \in \widehat{\mathbb{Z}}$ by $c \in \mathbb{Z}$ in (T6). Because we have a natural map $B_n \rightarrow \widehat{B}_n$ and the Turaev moves are special case of our 6 moves, we have a natural map $h: \mathcal{T} \rightarrow \widehat{\mathcal{T}}$.

(2) It is easy because the set of profinite tangles isotopic to a given profinite knot consists of only profinite knots. □

We notice that the number of connected components is an isotopic invariant of profinite tangles. As a knot analogue of residually-finiteness (1.1) of the braid group B_n , we raise the conjecture below.

Conjecture 2.12. The map $h: \mathcal{K} \rightarrow \widehat{\mathcal{K}}$ is injective.

Remark 2.13. The Kontsevich invariant is an invariant of oriented knots which is conjectured to be a perfect invariant, i.e. an invariant detecting all oriented knots (cf. [33] etc). We note that if the invariant is perfect, then the map h is injective by the arguments given in Remark 2.34 below.

Since last century, lots of knot invariants have been investigated; such as the unknotting number, the Alexander polynomial, the Jones polynomial, quandles, the Khovanov homology, etc. It might be interesting to pose the following.

Problem 2.14. Extend various known knot invariants to those for profinite knots.

Profinite analogues of finite type knot invariants will be discussed in Remark 2.33.

Below we remind one of the most elementary results for oriented knots.

Proposition 2.15. *The space \mathcal{K} of topological oriented knots carries a structure of a commutative associative monoid by the connected sum (knot sum).*

Here the connected sum (knot sum) is a natural way to fuse two oriented knots, with an appropriate position of orientation, into one (an example is illustrated in Figure 2.15).

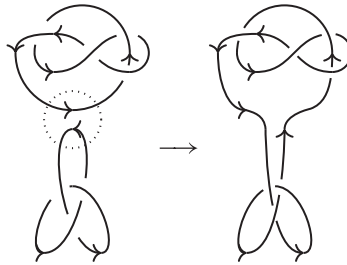


Figure 2.15. Connected sum (knot sum).

It can be done any points. It is actually well-defined. It yields a commutative monoid structure on \mathcal{K} . For more, consult the standard text book of knot theory.

The notion of connected sum can be extended into profinite knots.

Theorem 2.16. *For any two profinite knots $K_1 = \alpha_m \cdots \alpha_1$ and $K_2 = \beta_n \cdots \beta_1$ with $(\alpha_m, \alpha_1) = (\curvearrowright, \curvearrowleft)$ and $(\beta_n, \beta_1) = (\curvearrowright, \curvearrowleft)$, their connected sum means the profinite tangle defined by*

$$K_1 \# K_2 := \alpha_m \cdots \alpha_2 \cdot \beta_{n-1} \cdots \beta_1, \tag{2.1}$$

as illustrated in Figure 2.16. Then

- (1) the above connected sum induces a well-defined product

$$\# : \widehat{\mathcal{K}} \times \widehat{\mathcal{K}} \longrightarrow \widehat{\mathcal{K}};$$

- (2) by the product $\#$, the set $\widehat{\mathcal{K}}$ forms a topological (that is, the map $\#$ is continuous with respect to the topology given above) commutative associative monoid, whose unit is given by the oriented circle $\circlearrowleft := \curvearrowright \cdot \curvearrowleft$;
- (3) the profinite realization map $h : \mathcal{K} \rightarrow \widehat{\mathcal{K}}$ forms a monoid homomorphism whose image is dense in $\widehat{\mathcal{K}}$.

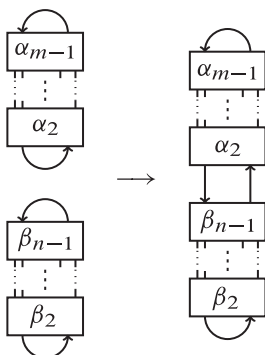


Figure 2.16. Connected sum for profinite knots.

Proof. (1) Since each isotopy class of profinite knot contains by (T6) a profinite knot K of the above type; a profinite knot starting with \cup and ending at \cap , we can show that the connected sum extends to $\widehat{\mathcal{K}}$ once we have the well-definedness.

Firstly we prove that $K_1 \# K_2$ is isotopic to $K'_1 \# K_2$ if K'_1 is isotopic to K_1 , i.e., K'_1 is obtained by a finite number of our moves (T1)–(T6) from K_1 . We may assume that K'_1 is obtained from K_1 by a single operation of one of the 6 moves. In the case where this move effects only on α_i 's for $i > 1$, it is easy to see our claim. If the moves effects on α_1 , it must be (T3) or (T6). Consider the latter case (T6). It suffices to show that $K_1 \# K_2$ is isotopic to $K_3 = \alpha_m \cdots \alpha_2 \cdot \sigma^{2c} \cdot \beta_{n-1} \cdots \beta_1$ for $c \in \widehat{\mathbb{Z}}$. The proof is depicted in Figure 2.17. Here $S_1 = \alpha_{m-1} \cdots \alpha_2$ and $S_2 = \beta_n \cdots \beta_2$. We note that the first and the fourth equalities follow from (T3) and (T5). We use (T4) in the second equality for σ^{2c} and the dashed box. We derived the third equality from (T6).

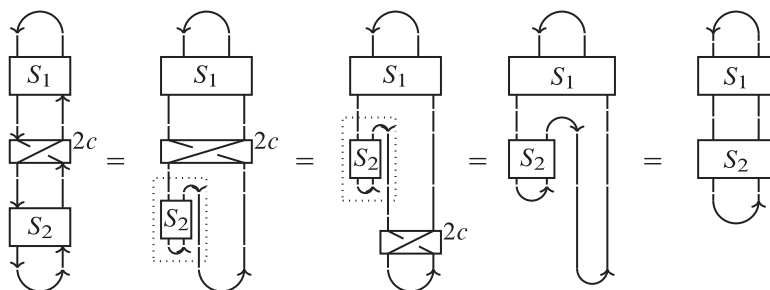


Figure 2.17. $K_3 = K_1 \# K_2$.

Next consider the former case (T3). Since K_1 is a profinite knot, m_1 and m_2 are both 0. By the above argument in case (T6), we may assume that both T_1 and T_2 in Figure 2.11 should start from \cup (i.e. $\alpha_1 = \beta_1 = \cup$). Define T as in Figure 2.18. A successive application of commutativity of profinite braids with T shown in (T4) and that of creations and annihilations with T shown in Lemma 2.18 lead the isotopy equivalence shown in Figure 2.19. Here T'_1 and T'_2 stand for emissions of the lowest \cup from T_1 and T_2 . We note that emission of T in the figure represents K_1 and K'_1 .

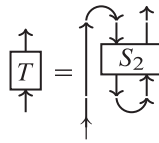


Figure 2.18. definition of T .

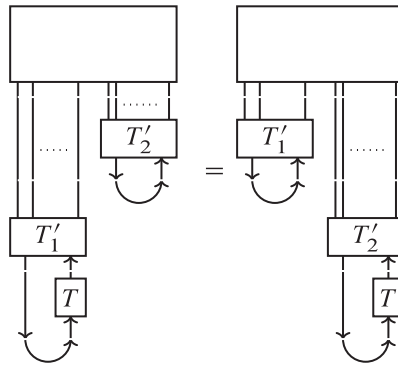


Figure 2.19. $K_1 \# K_2 = K'_1 \# K_2$.

Secondly we must prove that $K_1 \# K_2$ is isotopic to $K_1 \# K'_2$ if K'_2 is isotopic to K_2 . It can be proved in a completely same way to the above arguments. Thus our proof is finally completed.

(2) Associativity, i.e. $(K_1 \# K_2) \# K_3 = K_1 \# (K_2 \# K_3)$, is easy to see. A proof of commutativity is illustrated in Figure 2.20. We note that we use (T3) and (T5) in the first, the third and the fifth equalities and we apply (T4) and Lemma 2.18 for the dashed boxes in the second and the fourth equalities.

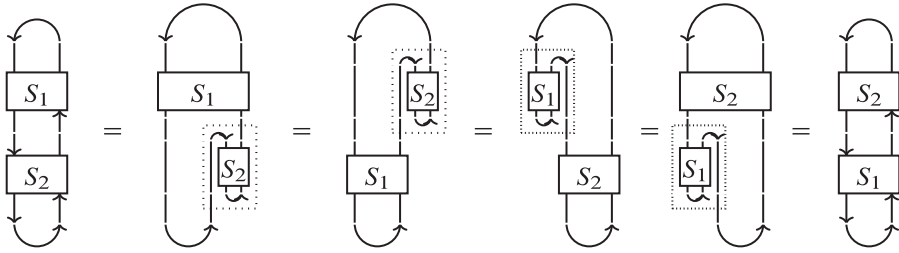


Figure 2.20. $K_1 \# K_2 = K_2 \# K_1$.

To show that $\#$ is continuous, we define by $\widehat{\mathcal{T}}^{\text{seq}}$ the set of finite consistent sequences of profinite fundamental tangles and by $\widehat{\mathcal{K}}^{\text{seq}}$ the set of finite consistent sequences of profinite fundamental tangles $\gamma_n \cdots \gamma_2 \cdot \gamma_1$ with a single connected component and with $(\gamma_n, \gamma_1) = (\cap, \cup)$. We note that the quotient set of $\widehat{\mathcal{T}}^{\text{seq}}$ by the equivalence of finite sequences of the moves (T1)–(T6) is equal to the set $\widehat{\mathcal{T}}$ of profinite tangles. We also note that $\widehat{\mathcal{K}}^{\text{seq}}$ is projected onto its subset $\widehat{\mathcal{K}}$ of profinite knots. The set $\widehat{\mathcal{T}}^{\text{seq}}$ carries a structure of a topological space by the profinite topologies on \widehat{B}_n (Note 2.10). It induces a subspace topology on $\widehat{\mathcal{K}}^{\text{seq}}$. The map $\widehat{\mathcal{T}}^{\text{seq}} \twoheadrightarrow \widehat{\mathcal{T}}$ is continuous, hence so the projection $\widehat{\mathcal{K}}^{\text{seq}} \twoheadrightarrow \widehat{\mathcal{K}}$ is. By the topology it is easy to see that the map

$$\#: \widehat{\mathcal{K}}^{\text{seq}} \times \widehat{\mathcal{K}}^{\text{seq}} \longrightarrow \widehat{\mathcal{K}}^{\text{seq}}$$

caused by (2.1) is continuous. Because the following diagram is commutative

$$\begin{array}{ccc} \widehat{\mathcal{K}}^{\text{seq}} \times \widehat{\mathcal{K}}^{\text{seq}} & \xrightarrow{\#} & \widehat{\mathcal{K}}^{\text{seq}} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{K}} \times \widehat{\mathcal{K}} & \xrightarrow{\#} & \widehat{\mathcal{K}} \end{array}$$

and the projection $\widehat{\mathcal{K}}^{\text{seq}} \twoheadrightarrow \widehat{\mathcal{K}}$ is continuous, the lower map is also continuous.

(3) The first statement is obvious. Let $\mathcal{K}'^{\text{seq}}$ be the subset of $\widehat{\mathcal{K}}^{\text{seq}}$ which consists of consistent finite sequences of ‘topological fundamental tangles’, that is, sequences of elements in A, C and B with $b_n \in B_n \subset \widehat{B}_n$. There are a natural inclusion $\mathcal{K}'^{\text{seq}} \rightarrow \widehat{\mathcal{K}}^{\text{seq}}$ and a surjection $\mathcal{K}'^{\text{seq}} \rightarrow \mathcal{K}$. Since the map $B_n \rightarrow \widehat{B}_n$ is with dense image, so the inclusion $\mathcal{K}'^{\text{seq}} \rightarrow \widehat{\mathcal{K}}^{\text{seq}}$ is. Therefore the map $\mathcal{K} \rightarrow \widehat{\mathcal{K}}$ is also with dense image. It is because the following diagram is commutative

$$\begin{array}{ccc} \mathcal{K}'^{\text{seq}} & \longrightarrow & \widehat{\mathcal{K}}^{\text{seq}} \\ \downarrow & & \downarrow \\ \mathcal{K} & \longrightarrow & \widehat{\mathcal{K}} \end{array}$$

and the projection $\widehat{\mathcal{K}}^{\text{seq}} \twoheadrightarrow \widehat{\mathcal{K}}$ is continuous. □

The following two lemmas are required to prove Theorem 2.16 (1).

Lemma 2.17. *For a profinite tangle T with $s(T) = t(T) = \uparrow$, define its transpose \mathcal{L} (occasionally also denoted by ${}^t T$) by*

$$\mathcal{L} := a_{0,1}^{\downarrow\uparrow} \cdot (e_1^\downarrow \otimes T \otimes e_1^\downarrow) \cdot c_{1,0}^{\downarrow\uparrow}.$$

Then we have

$$\mathcal{L} = a_{1,0}^{\downarrow\uparrow} \cdot (e_1^\downarrow \otimes T \otimes e_1^\downarrow) \cdot c_{0,1}^{\downarrow\uparrow}.$$

Similar claim holds for a profinite tangle T with $s(T) = t(T) = \downarrow$ by reversing all arrows.

Figure 2.21 describes our lemma.

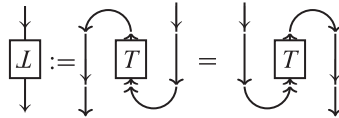


Figure 2.21. Transpose of T .

Proof. A proof is depicted in Figure 2.22. We note that we apply (T6) in the first equality and (T2) and (T4) in the second equality. □

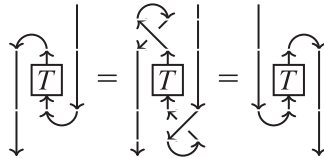


Figure 2.22. Proof of Lemma 2.17.

By (T5) and the above lemma, we see that ${}^{tt} T$ is isotopic to T .

Lemma 2.18. *For a profinite tangle T with $s(T) = t(T) = \uparrow$, the equalities in Figure 2.23 hold. The same claim also holds for a profinite tangle T with $s(T) = t(T) = \downarrow$ by reversing all arrows.*

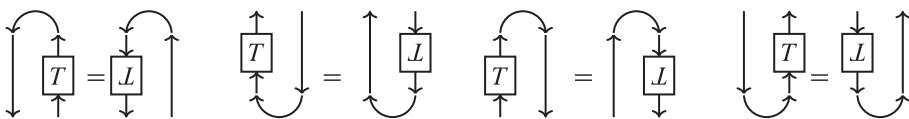


Figure 2.23. Creation and annihilation commute with T .

Proof. It can be checked by direct computation using Lemma 2.17. \square

In knot theory, the so-called Alexander–Markov’s theorem is fundamental on constructions of knot invariants. The theorem is to translate knots and links into purely algebraic objects:

Theorem 2.19 (Alexander–Markov’s theorem). *There is a one-to-one correspondence*

$$\mathcal{L} \longleftrightarrow \bigsqcup_n B_n / \sim_M$$

between the set \mathcal{L} of isotopy classes of oriented links and the (disjoint) union $\bigsqcup_n B_n$ of braids groups modulo the equivalence \sim_M given by the following Markov moves

$$(M1) \quad b_1 \cdot b_2 \sim_M b_2 \cdot b_1 \quad (b_1, b_2 \in B_n),$$

$$(M2) \quad b \in B_n \sim_M b \sigma_n^{\pm 1} \in B_{n+1} \quad (b \in B_n).$$

For more on the theorem, consult [5] and [32] for example. The question below is to ask a validity of profinite analogue of Alexander–Markov’s theorem.

Problem 2.20. Is there a ‘profinite analogue’ of the Alexander–Markov’s theorem which holds for the set $\widehat{\mathcal{L}}$ of isotopy classes of profinite links ?

There are several proofs of Alexander–Markov’s theorem for topological links ([4, 38, 40, 41] etc). But they look heavily based on a certain finiteness property, which we (at least the author) may not expect the validity for profinite links.

2.2. Pro- l knots. Pro- l tangle diagrams, pro- l knot diagrams and isotopy among them which are pro- l analogues of our corresponding notions given in the previous subsection, are introduced in Definition 2.25. A natural map from profinite tangles (resp. profinite knots) to pro- l tangles (resp. pro- l knots) is constructed in Proposition 2.26. Proalgebraic tangles and proalgebraic knots are recalled in Definition 2.30. A natural map from pro- l tangles (resp. pro- l knots) to proalgebraic tangles (resp. proalgebraic knots) is given in Proposition 2.31. It is explained that the Kontsevich invariant factors through these natural maps in Remark 2.34. Our discussion in this subsection will serve for a proof of non-triviality of the Galois action constructed in §2.3.

Let l be a prime. We may include $l = 2$.

Notation 2.21. A topological group G is called a *pro- l group* if it is a projective limit $\varprojlim G_i$ of a projective system of finite l -groups $\{G_i\}_{i \in I}$. For a discrete group Γ , its *pro- l completion* $\widehat{\Gamma}^l$ is the pro- l group defined by the projective limit

$$\widehat{\Gamma}^l = \varprojlim \Gamma/N$$

where N runs over all normal subgroups of Γ with finite indices of power of l .

For more on pro- l groups, consult [36] for example. We note there is a natural homomorphism $\Gamma \rightarrow \widehat{\Gamma}^l$.

Notation 2.22. Let $n \geq 2$. Let $\widehat{P}_n \rtimes B_n$ be the semi-direct product of \widehat{P}_n and B_n with respect to the B_n -action on \widehat{P}_n given by $p \mapsto bpb^{-1}$ ($p \in \widehat{P}_n$ and $b \in B_n$). consider the inclusion $P_n \hookrightarrow \widehat{P}_n \rtimes B_n$ sending $p \mapsto (p, p^{-1})$. Then it is easy to see the homomorphism sending $(p, b) \mapsto pb$ yields an isomorphism:

$$(\widehat{P}_n \rtimes B_n)/P_n \simeq \widehat{B}_n.$$

Definition 2.23. (1) The *pro- l pure braid group* \widehat{P}_n^l is the pro- l completion of P_n .

(2) The *pro- (l) braid group* $\widehat{B}_n^{(l)}$ is defined to be the induced quotient

$$\widehat{B}_n^{(l)} := (\widehat{P}_n^l \rtimes B_n)/P_n.$$

We encode a topological group structure on $\widehat{B}_n^{(l)}$ by the pro- l topology on \widehat{P}_n^l and the discrete topology on B_n . We note that this $\widehat{B}_n^{(l)}$ appears also in [27].

Remark 2.24. (1) Our $\widehat{B}_n^{(l)}$ is different from the pro- l completion \widehat{B}_n^l of B_n .

(2) There is an exact sequence:

$$1 \longrightarrow \widehat{P}_n^l \longrightarrow \widehat{B}_n^{(l)} \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$

(3) There are natural group homomorphisms:

$$B_n \longrightarrow \widehat{B}_n \longrightarrow \widehat{B}_n^{(l)}. \tag{2.2}$$

The map (2.2) is induced from $P_n \rightarrow \widehat{P}_n \rightarrow \widehat{P}_n^l$.

Definition 2.25. (1) A *pro-l tangle diagram* means a consistent finite sequence of *fundamental pro-l tangle diagrams*, which are elements in A, C (in Definition 2.2) or

$$\widehat{B}^l := \{b_n^\epsilon \mid b_n^\epsilon = (b_n, \epsilon = (\epsilon_i)_{i=1}^n) \in \widehat{B}_n^{(l)} \times \{\uparrow, \downarrow\}^n, n = 1, 2, 3, 4, \dots\}.$$

A *pro-l knot diagram* means a pro-l tangle diagram without endpoints (their sources and targets are both empty) and with a single connected component.

(2) Two pro-l tangle diagrams T_1 and T_2 are said to be *isotopic* if they are related by a finite number of the moves replacing profinite tangles and profinite braids by pro-l tangles and elements in \widehat{B}^l in (T1)–(T6) and $c \in \widehat{\mathbb{Z}}$ by $c \in \mathbb{Z}_l$ in (T6).⁹

(3) A *pro-l tangle* (resp. *pro-l*) means an isotopic class of pro-l tangle (resp. knot) diagram. We denote the set of pro-l tangles by $\widehat{\mathcal{T}}^l$ and the set of pro-l knots by $\widehat{\mathcal{K}}^l$.

Both $\widehat{\mathcal{T}}^l$ and $\widehat{\mathcal{K}}^l$ carry a structure of topological space by the method in Note 2.10.

Proposition 2.26. (1) *The set $\widehat{\mathcal{K}}^l$ forms a topological monoid with respect to the connected sum.*

(2) *There are continuous maps:*

$$\widehat{\mathcal{T}} \longrightarrow \widehat{\mathcal{T}}^l, \tag{2.3}$$

$$\widehat{\mathcal{K}} \longrightarrow \widehat{\mathcal{K}}^l. \tag{2.4}$$

(3) *The map (2.4) is monoid homomorphisms. The image of its composition with h in Theorem 2.11.(2)*

$$h_l : \mathcal{K} \xrightarrow{h} \widehat{\mathcal{K}} \longrightarrow \widehat{\mathcal{K}}^l. \tag{2.5}$$

is with dense image in $\widehat{\mathcal{K}}^l$.

Proof. (1) It is obtained by the same arguments to the proof of Theorem 2.16.

(2) The map (2.3) is induced from the second map in (2.2), whose continuity implies ours. It preserves each connected component, which yields the map (2.4).

(3) To see that they form homomorphisms are immediate. The density can be proved by the same arguments to the proof of Theorem 2.16. \square

⁹We note that, for $\sigma_i \in \widehat{B}_N^{(l)}$ and $c \in \mathbb{Z}_l$, the power σ_i^{2c+1} makes sense in $\widehat{B}_N^{(l)}$ because, by $\sigma_i^2 \in \widehat{P}_N^l$, we have $\sigma_i^{2c} \in \widehat{P}_N^l$.

Definition 2.27. Let R be a commutative ring.

- (1) Let I be the two-sided ideal of the group algebra $R[B_n]$ of B_n generated by $\sigma_i - \sigma_i^{-1}$ ($1 \leq i \leq n - 1$). The topological R -algebra $\widehat{R[B_n]}$ of *proalgebraic braids* means its completion with respect to the I -adic filtration, i.e.

$$\widehat{R[B_n]} := \varprojlim_N R[B_n]/I^N.$$

- (2) Put $I_0 := I \cap R[P_n]$. Then I_0 is the augmentation ideal of $R[P_n]$ (cf. [22]). The topological R -algebra $\widehat{R[P_n]}$ of *proalgebraic pure braids* means its completion

$$\widehat{R[P_n]} := \varprojlim_N R[P_n]/I_0^N.$$

It is a subalgebra of $\widehat{R[B_n]}$.

It is direct to see that both algebras naturally equip structures of co-commutative Hopf algebras.

Remark 2.28. (i) There is a short exact sequence

$$0 \longrightarrow \widehat{R[P_n]} \longrightarrow \widehat{R[B_n]} \longrightarrow R[\mathfrak{S}_n] \longrightarrow 0.$$

(ii) We remark that (the group-like part of) $\widehat{R[P_n]}$ is the unipotent (Malcev) completion of P_n and (the group-like part of) $\widehat{R[B_n]}$ is Hain's [15] relative completion of B_n with respect to the natural projection $B_n \rightarrow \mathfrak{S}_n$.

(iii) The natural morphisms $P_n \rightarrow R[P_n]$ and $B_n \rightarrow R[B_n]$ yield injections

$$P_n \hookrightarrow \widehat{R[P_n]} \quad \text{and} \quad B_n \hookrightarrow \widehat{R[B_n]} \tag{2.6}$$

(cf. [5], §12).

Since \widehat{P}_n^l is a pro- l group and I_0 is the augmentation ideal, we have a natural continuous homomorphism $\widehat{P}_n^l \rightarrow \widehat{Q_l[P_n]}$ (cf. [5] for example).

Proposition 2.29. *There is a natural continuous group homomorphism*

$$\widehat{B}_n^{(l)} \longrightarrow \widehat{Q_l[B_n]}. \tag{2.7}$$

Proof. It can be directly checked that the map induced from the above

$$\widehat{P}_n^l \longrightarrow \widehat{\mathbb{Q}_l[P_n]} (\subset \widehat{\mathbb{Q}_l[B_n]})$$

and the natural map

$$B_n \hookrightarrow \mathbb{Q}_l[B_n] \longrightarrow \widehat{\mathbb{Q}_l[B_n]}$$

holds the property. □

Next we discuss the corresponding notions in tangles and knots settings. The following notions go back to the idea of Vassiliev.

Definition 2.30 ([22]). Let R be a commutative ring. Let $\mathcal{T}_{\epsilon, \epsilon'}$ be the full set of isotopy classes of oriented tangles with type (ϵ, ϵ') .

- (1) Let $R[\mathcal{T}_{\epsilon, \epsilon'}]$ be the free R -module of finite formal sums of elements of $\mathcal{T}_{\epsilon, \epsilon'}$. A singular oriented tangle¹⁰ determines an element of $R[\mathcal{T}_{\epsilon, \epsilon'}]$ by the desingularization of each double point by the following relation

$$\begin{array}{c} \nearrow \searrow \\ \times \\ \nwarrow \nearrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \\ \nwarrow \nearrow \end{array} - \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array}.$$

Let \mathcal{T}_n ($n \geq 0$) be the R -submodule of $R[\mathcal{T}_{\epsilon, \epsilon'}]$ generated by all singular oriented tangles with type (ϵ, ϵ') and with n double points. The descending filtration $\{\mathcal{T}_n\}_{n \geq 0}$ is called the *singular filtration*. The topological R -module $\widehat{R[\mathcal{T}_{\epsilon, \epsilon'}]}$ of *proalgebraic tangles* means its completion with respect to the singular filtration:

$$\widehat{R[\mathcal{T}_{\epsilon, \epsilon'}]} := \varprojlim_N R[\mathcal{T}_{\epsilon, \epsilon'}] / \mathcal{T}_N.$$

- (2) Let $R[\mathcal{K}]$ be the R -submodule of $R[\mathcal{T}_{\emptyset, \emptyset}]$ generated by elements of \mathcal{K} . By Proposition 2.15, it forms a commutative R -algebra. Put $\mathcal{K}_n := \mathcal{T}_n \cap R[\mathcal{K}]$ ($n \geq 0$). Then \mathcal{K}_n forms an ideal of $R[\mathcal{K}]$ and the descending filtration $\{\mathcal{K}_n\}_{n \geq 0}$ is called the *singular knot filtration* (cf. loc. cit.). The topological commutative R -algebra $\widehat{R[\mathcal{K}]}$ of *proalgebraic knots* means its completion with respect to the singular knot filtration:

$$\widehat{R[\mathcal{K}]} := \varprojlim_N R[\mathcal{K}] / \mathcal{K}_N.$$

Actually it equips a structure of co-commutative and commutative Hopf algebra.

¹⁰ It is an ‘oriented tangle’ which is allowed to have a finite number of transversal double points (see [22] for precise).

The maps below are tangle and knot analogues of the map (2.7).

Proposition 2.31. (1) *There are continuous maps:*

$$\widehat{\mathcal{T}}^l \longrightarrow \widehat{\mathbb{Q}_l[\mathcal{T}]}, \tag{2.8}$$

$$\widehat{\mathcal{K}}^l \longrightarrow \widehat{\mathbb{Q}_l[\mathcal{K}]}. \tag{2.9}$$

(2) *The map (2.9) is a continuous monoid homomorphism and its image lies on the set $\widehat{\mathbb{Q}_l[\mathcal{K}]^\times}$ of invertible elements.*

Proof. (1) Since an element $b_n^\epsilon \in B_n \times \{\uparrow, \downarrow\}^n$ ($n \geq 1$), a braid $b_n \in B_n$ with an orientation $\epsilon \in \{\uparrow, \downarrow\}^n$ (namely its source), is naturally regarded as a special type of an oriented tangle, each orientation ϵ yields a natural inclusion

$$\mathbb{Q}_l[B_n] \hookrightarrow \mathbb{Q}_l[\mathcal{T}].$$

On the embedding, we have $\mathcal{T}_m \cap \mathbb{Q}_l[B_n] = I^m$ for $m \geq 0$. Therefore the above map and the map (2.7) induce

$$\widehat{B}_n^{(l)} \longrightarrow \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m.$$

Hence it determines the map of sets

$$\widehat{B}^l \longrightarrow \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m. \tag{2.10}$$

We also have the natural maps of sets

$$A \longrightarrow \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m \quad \text{and} \quad C \longrightarrow \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m. \tag{2.11}$$

As is described in the proof of Theorem 2.11, the set \mathcal{T} of (topological) oriented tangles is described by the set of consistent finite sequences of elements of A , B and C modulo the (discrete) Turaev moves. By \mathbb{Q}_l -linearly extending the description, we obtain the same description of $\mathbb{Q}_l[\mathcal{T}]$. Since our three maps (2.10) and (2.11) are consistent with the moves, we obtain

$$\widehat{\mathcal{T}}^l \longrightarrow \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m.$$

(Again we note that, for $\sigma_i \in B_N$ and $c \in \mathbb{Z}_l$, the power σ_i^c makes sense in $\mathbb{Q}_l[B_N]/I^m$ by the formula

$$\sigma_i^c := \exp \left\{ \frac{c}{2} \log \sigma_i^2 \right\},$$

when $l \neq 2$ or when $l = 2$ and $c \in 2\mathbb{Z}_2$, and

$$\sigma_i^c := \sigma_i \cdot \exp\left\{\frac{c-1}{2} \log \sigma_i^2\right\},$$

when $l = 2$ and $c \notin 2\mathbb{Z}_2$. Here \exp and \log are defined by the usual Taylor expansions. The RHS is well-defined by $\sigma_i^2 - 1 \in I$.)

It yields the map (2.8) which is continuous. Since this map preserves each connected component, the map (2.9) is also obtained.

(2) It is immediate to see that it forms a continuous homomorphism.

Each oriented knot, an element of \mathcal{K} , is congruent to the unit, the trivial knot $\bigcirc \in \mathbb{Q}_l[\mathcal{K}]$, modulo \mathcal{K}_1 , because any knot can be untied by a finite times of changing crossings (consult for unknotting number, say, in [5]). Therefore the image of $h_l(\mathcal{K})$ ($\subset \widehat{\mathcal{K}}^l$) is contained in the subspace $\bigcirc + \mathcal{K}_1 \cdot \widehat{\mathbb{Q}_l[\mathcal{K}]}$. Hence the image of $\widehat{\mathcal{K}}^l$ should lie on the subspace. It is because the subspace is open in $\widehat{\mathbb{Q}_l[\mathcal{K}]}$, our map (2.9) is continuous as shown above and $h_l(\mathcal{K})$ is dense in $\widehat{\mathcal{K}}_l$ by Proposition 2.26.(3). All elements of the subspace are invertible because it is known that the quotient $\widehat{\mathbb{Q}_l[\mathcal{K}]} / \mathcal{K}_1$ is 1-dimensional and generated by \bigcirc . Thus the claim is obtained. \square

The author is not aware if our above two maps are injective or not.

Proposition 2.32. (1) For each prime l , there are continuous maps:

$$\widehat{\mathcal{T}} \longrightarrow \widehat{\mathbb{Q}_l[\mathcal{T}]}, \tag{2.12}$$

$$\widehat{\mathcal{K}} \longrightarrow \widehat{\mathbb{Q}_l[\mathcal{K}]}. \tag{2.13}$$

(2) The map (2.13) is a continuous monoid homomorphism and its image lies on the set $\widehat{\mathbb{Q}_l[\mathcal{K}]}^\times$ of invertible elements.

(3) The image of the composition of (2.13) with h in Theorem 2.11.(2)

$$\mathcal{K} \xrightarrow{h} \widehat{\mathcal{K}} \longrightarrow \widehat{\mathbb{Q}_l[\mathcal{K}]}. \tag{2.14}$$

lies on the rational invertible part $\widehat{\mathbb{Q}_l[\mathcal{K}]}^\times$ ($\subset \widehat{\mathbb{Q}_l[\mathcal{K}]}$). Furthermore the resulting morphism

$$i: \mathcal{K} \longrightarrow \widehat{\mathbb{Q}_l[\mathcal{K}]} \tag{2.15}$$

is independent of a prime l .

Proof. (1) and (2) follow from Proposition 2.26 and 2.31. Our claim (3) follows from that the map (2.14) is induced from the natural inclusion $\mathcal{K} \hookrightarrow R[\mathcal{K}]$ with $R = \mathbb{Q}$. \square

Relating to Problem 2.14, the following is obtained as an application of Proposition 2.32.

Remark 2.33. (1) Finite type knot invariants (resp. and their projective limits) valued on \mathbb{Q} are knot invariants which factor through $\mathcal{K} \rightarrow \widehat{\mathbb{Q}[\mathcal{K}]} \rightarrow \mathbb{Q}[\mathcal{K}]/\mathcal{K}_N$ for some N (resp. $\mathcal{K} \rightarrow \widehat{\mathbb{Z}[\mathcal{K}]}$). By using the map (2.14), we can extend all finite type knot invariants and their projective limits, such as each coefficient of the Jones polynomial substituting e^h (cf. [5]), into profinite knot invariants valued on \mathbb{Q}_l for each prime l . It is easy to see that the same holds for tangle invariants.

(2) The linking number is an invariant of two components links which values on \mathbb{Z} and which is known to be of finite type ([5], etc). Thus it can be extended to an invariant of two components profinite links which values on \mathbb{Q}_l , actually on \mathbb{Z}_l , for each prime l . The Milnor $\bar{\mu}$ -invariant ([29] and [30]) is an invariant of links which is known as a higher generalization of the linking number. It is defined on \mathbb{Z} but it has a certain indeterminacy. The author is not sure if it can be also extended to an invariant of profinite links, but he expects that the works, [14], [2] and [26], concerning its associated invariant of string links would help to detect that.

As a complementation of Remark 2.13, we have:

Remark 2.34. The Kontsevich invariant $Z: \mathcal{K} \rightarrow \widehat{CD}$ is a knot invariant which is a composition of i with the isomorphism $\widehat{\mathbb{Q}[\mathcal{K}]} \simeq \widehat{CD}$ constructed in [21]. Here \widehat{CD} stands for the \mathbb{Q} -vector space (completed by degree) of chord diagrams modulo 4T- and FI-relations (consult also [5] and [32]). The invariant is conjectured to be perfect, i.e., the map Z is conjectured to be injective (cf. [33], Conjecture 3.2). The conjecture is equivalent to saying that the above map i in (2.15) is injective. Hence it naturally leads us to conjecture that $h: \mathcal{K} \rightarrow \widehat{\mathcal{K}}$ is injective (Conjecture 2.12) because the non-injectivity of h imply the non-injectivity of i , that is, the non-perfectness of the Kontsevich invariant, by the above proposition. We may also say that Conjecture 2.12 is an assertion on a knot analogue of the injectivity of the maps (2.6).

2.3. Action of the absolute Galois group. The group $\text{Frac } \widehat{\mathcal{K}}$ of profinite knots is introduced as the group of fraction of the topological monoid $\widehat{\mathcal{K}}$ in Definition 2.35 and its basic property is shown in Theorem 2.36. A continuous action of the profinite Grothendieck–Teichmüller group \widehat{GT} (cf. Definition 1.6) on $\text{Frac } \widehat{\mathcal{K}}$ is established in Definition 2.39–Theorem 2.41. As a result of our construction, an action of the absolute Galois group $G_{\mathbb{Q}}$ of the rational number field on $\text{Frac } \widehat{\mathcal{K}}$ is obtained (Theorem 2.45). We post several projects and problems on this Galois representation in the end.

Definition 2.35. The *group of profinite knots* $\text{Frac } \widehat{\mathcal{K}}$ is defined to be the group of fraction of the monoid $\widehat{\mathcal{K}}$, i.e., the quotient space of $\widehat{\mathcal{K}}^2$ by the equivalent relations $(r, s) \approx (r', s')$ if $r \# s' \# t \sim r' \# s \# t$ for some profinite knot t , i.e. $r \# s' \# t = r' \# s \# t$ holds in $\widehat{\mathcal{K}}$. Occasionally we denote the equivalent class $[(r, s)]$ by $\frac{r}{s}$.

For $[(r_1, s_1)]$ and $[(r_2, s_2)] \in \text{Frac } \widehat{\mathcal{K}}$, define its product by

$$[(r_1, s_1)] \# [(r_2, s_2)] := [(r_1 \# r_2, s_1 \# s_2)] \in \text{Frac } \widehat{\mathcal{K}},$$

i.e.

$$\frac{r_1 \# r_2}{s_1 \# s_2} = \frac{r_1 \# r_2}{s_1 \# s_2} \in \text{Frac } \widehat{\mathcal{K}}.$$

We encode $\text{Frac } \widehat{\mathcal{K}}$ with the quotient topology of $\widehat{\mathcal{K}}^2$.

Theorem 2.36. (1) *The product $\#$ is well-defined on $\text{Frac } \widehat{\mathcal{K}}$. The set $\text{Frac } \widehat{\mathcal{K}}$ forms a topological commutative group.*

(2) *It is a non-trivial group. Actually it is an infinite group.*

Proof. (1) It is easy to see that $\#$ is well-defined and $\text{Frac } \widehat{\mathcal{K}}$ forms a commutative group with unit

$$e = (\bigcirc, \bigcirc)$$

by Theorem 2.16.

Consider the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{K}}^2 \times \widehat{\mathcal{K}}^2 & \xrightarrow{\#} & \widehat{\mathcal{K}}^2 \\ \downarrow & & \downarrow \\ \text{Frac } \widehat{\mathcal{K}} \times \text{Frac } \widehat{\mathcal{K}} & \xrightarrow{\#} & \text{Frac } \widehat{\mathcal{K}}. \end{array}$$

Since the upper map is continuous by Theorem 2.16 and the surjection

$$\widehat{\mathcal{K}}^2 \longrightarrow \text{Frac } \widehat{\mathcal{K}}$$

is continuous by definition, it follows that the map $\#$ is continuous.

Let $\tau: \widehat{\mathcal{K}}^2 \rightarrow \widehat{\mathcal{K}}^2$ be the switch map sending $(r, s) \mapsto (s, r)$. It is easy to see that it is continuous and it induces the inverse map on $\text{Frac } \widehat{\mathcal{K}}$. Then by the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{K}}^2 & \xrightarrow{\tau} & \widehat{\mathcal{K}}^2 \\ \downarrow & & \downarrow \\ \text{Frac } \widehat{\mathcal{K}} & \longrightarrow & \text{Frac } \widehat{\mathcal{K}}, \end{array}$$

the inverse map is also continuous.

(2) By Proposition 2.26 and 2.31, there is a continuous monoid homomorphism

$$\widehat{\mathcal{K}} \longrightarrow \widehat{\mathcal{K}}^l \longrightarrow \widehat{\mathbb{Q}_l[\mathcal{K}]} \tag{2.16}$$

for a prime l . By Proposition 2.31, the image lies on $\widehat{\mathbb{Q}_l[\mathcal{K}]^\times}$. Whence it induces a continuous group homomorphism

$$\text{Frac } \widehat{\mathcal{K}} \longrightarrow \widehat{\mathbb{Q}_l[\mathcal{K}]^\times}. \tag{2.17}$$

Thus it is enough to show that the image of the composition of the maps (2.16) and $h: \mathcal{K} \rightarrow \widehat{\mathcal{K}}$ is infinite set. The claim is obvious because this map is equal to i in (2.15) and the Kontsevich invariant takes infinite number of values (cf. Remark 2.34) □

A reason why we introduce $\text{Frac } \widehat{\mathcal{K}}$ is that we need to treat the inverse of Λ_f in Figure 2.26 when we let $\widehat{\text{GT}}$ act on profinite knots (cf. Definition 2.39).

We note that the natural morphism

$$h': \mathcal{K} \longrightarrow \text{Frac } \widehat{\mathcal{K}} \tag{2.18}$$

sending $K \mapsto [(K, \circlearrowleft)]$ is a homomorphism as monoid. By abuse of notations, we occasionally denote the image $h'(K)$ by the same symbol K . Related to Conjecture 2.12,

Problem 2.37. Is the map h' injective?

On a structure of $\text{Frac } \widehat{\mathcal{K}}$, we pose

Problem 2.38. Is $\text{Frac } \widehat{\mathcal{K}}$ a profinite group?

By [36], to show that $\text{Frac } \widehat{\mathcal{K}}$ is a profinite group, we must show that it is compact, Hausdorff and totally-disconnected. The author is not aware of any one of their validities. It is worthy to note that the set $\widehat{\mathcal{T}}$ of isotopy classes of profinite tangles is not compact, hence not a profinite space. It is because the map $|\pi_0|: \widehat{\mathcal{T}} \rightarrow \mathbb{N}$ taking the number of connected components of each profinite tangles is continuous and surjective to the non-compact space \mathbb{N} .

Definition 2.39. Let (r, s) be a pair of profinite knot diagrams with

$$r = \gamma_{1,m} \cdots \gamma_{1,2} \cdot \gamma_{1,1}$$

and

$$s = \gamma_{2,n} \cdots \gamma_{2,2} \cdot \gamma_{2,1}$$

$(\gamma_{i,j}$: profinite fundamental tangle diagram). For $\sigma = (\lambda, f) \in \widehat{\text{GT}}$ (hence $\lambda \in \widehat{\mathbb{Z}}^\times$, $f \in \widehat{F}_2$), define its action by

$$\sigma\left(\frac{r}{s}\right) := \frac{\sigma(r)}{\sigma(s)} := \frac{\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\} \# (\Lambda_f)^{\#\alpha(s)}}{\{\sigma(\gamma_{2,n}) \cdots \sigma(\gamma_{2,2}) \cdot \sigma(\gamma_{2,1})\} \# (\Lambda_f)^{\#\alpha(r)}} \in \text{Frac } \widehat{\mathcal{K}}. \quad (2.19)$$

It is well-defined by Proposition 2.40 and Theorem 2.41. Here

$$\sigma(r) := \frac{\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\}}{(\Lambda_f)^{\#\alpha(r)}} \in \text{Frac } \widehat{\mathcal{K}}$$

and

$$\sigma(s) := \frac{\{\sigma(\gamma_{2,n}) \cdots \sigma(\gamma_{2,2}) \cdot \sigma(\gamma_{2,1})\}}{(\Lambda_f)^{\#\alpha(s)}} \in \text{Frac } \widehat{\mathcal{K}}$$

are defined as follows:

- (1) When $\gamma_{i,j} = a_{k,l}^\epsilon$, we define

$$\sigma(\gamma_{i,j}) := \gamma_{i,j} \cdot f_{1\dots k, k+1, k+2}^{s(\gamma_{i,j})}$$

Here

$$f_{1\dots k, k+1, k+2}^{s(\gamma_{i,j})} = \text{ev}_{1, \epsilon_1}(f^{\uparrow \epsilon_2 \epsilon_3}) \otimes e_l^{\epsilon_4}$$

with

$$s(\gamma_{i,j}) = \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \in \{\uparrow, \downarrow\}^{k+l+2}$$

($\epsilon_1 \in \{\uparrow, \downarrow\}^k$, $\epsilon_2, \epsilon_3 \in \{\uparrow, \downarrow\}$, $\epsilon_4 \in \{\uparrow, \downarrow\}^l$). It is also described by

$$f_{1\dots k, k+1, k+2}^{s(\gamma_{i,j})} = (f_{1\dots k, k+1, k+2} \otimes e_l, s(\gamma_{i,j})) \in \widehat{B}$$

with

$$f_{1\dots k, k+1, k+2} \otimes e_l = f(x_{1\dots k, k+1}, x_{k+1, k+2}) \in \widehat{B}_{k+l+2},$$

where $x_{1\dots k, k+1}$ and $x_{k+1, k+2}$ are regarded as elements of \widehat{B}_{k+l+2} . We mean $f_{1\dots k, k+1, k+2} \otimes e_l$ by the trivial braid $e_{l+2} \in \widehat{B}_{l+2}$ when $k = 0$. Figure 2.24 depicts the action. Here the thickened black band stands for the trivial braid e_k with k -strings.

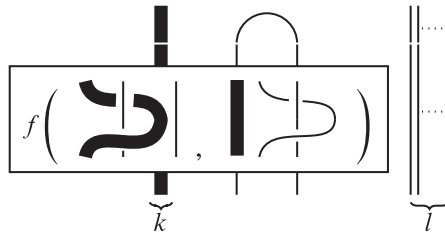


Figure 2.24. $\sigma(a_{k,l}^\epsilon)$.

(2) When $\gamma_{i,j} = b_n^\epsilon = (b_n, \epsilon) \in \widehat{B}$, we define

$$\sigma(\gamma_{i,j}) := (\sigma(b_n), \epsilon)$$

which is nothing but the image of $b_n \in \widehat{B}_n$ by the \widehat{GT} -action on \widehat{B}_n explained in §1.

(3) When $\gamma_{i,j} = c_{k,l}^\epsilon$, we define

$$\sigma(\gamma_{i,j}) := f_{1\dots k, k+1, k+2}^{-1, t(\gamma_{i,j})} \cdot \gamma_{i,j}$$

with

$$f_{1\dots k, k+1, k+2}^{-1, t(\gamma_{i,j})} = (f_{1\dots k, k+1, k+2}^{-1} \otimes e_l, t(\gamma_{i,j})) \in \widehat{B}.$$

Figure 2.25 depicts the action.

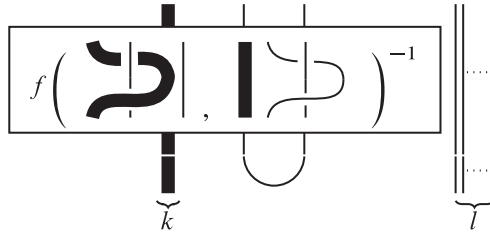


Figure 2.25. $\sigma(c_{k,l}^\epsilon)$.

The symbol Λ_f represents the profinite tangle

$$a_{0,0}^{\curvearrowright} \cdot a_{0,2}^{\curvearrowright\downarrow\uparrow} \cdot (e_1^{\downarrow} \otimes f) \cdot c_{1,1}^{\downarrow\curvearrowright\uparrow} \cdot c_{0,0}^{\curvearrowright}$$

(cf. Figure 2.26).

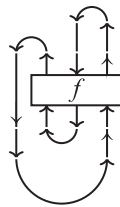


Figure 2.26. Λ_f .

The symbol $\alpha(r)$ (resp. $\alpha(s)$) means the number of annihilations; the cardinality of the set $\{j \mid \gamma_{i,j} \in A\}$ for $i = 1$ (resp. $i = 2$) and $(\Lambda_f)^{\sharp\alpha(r)}$ (resp. $(\Lambda_f)^{\sharp\alpha(s)}$) means the $\alpha(r)$ -th (resp. the $\alpha(s)$ -th) power of Λ_f with respect to \sharp . Particularly we have

$$\sigma(\mathcal{O}) \sharp \Lambda_f = \mathcal{O} \in \text{Frac } \widehat{\mathcal{K}} \tag{2.20}$$

Proposition 2.40. *Let $\sigma = (\lambda, f) \in \widehat{\text{GT}}$.*

- (1) *If $r = \gamma_{1,m} \cdots \gamma_{1,2} \cdot \gamma_{1,1}$ ($\gamma_{1,j}$: a profinite fundamental tangle) is a profinite knot, then $\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\}$ is again a profinite knot.*
- (2) *The profinite tangle Λ_f (Figure 2.26) is a profinite knot.*

Proof. (1) When $\gamma = b_n^\epsilon$, since the projection

$$p: \widehat{B}_n \longrightarrow \mathfrak{S}_n$$

is $\widehat{\text{GT}}$ -equivalent (the action on \mathfrak{S}_n means the trivial action), the skeleton never change, i.e. $\mathbb{S}(\sigma(\gamma)) = \mathbb{S}(\gamma)$.

When $\gamma = a_{k,l}^\epsilon$ (resp. $c_{k,l}^\epsilon$), the skeleton $\mathbb{S}(\sigma(\gamma))$ is obtained by connecting $k + l + 2$ straight bars on the top (resp. bottom) of $\mathbb{S}(\gamma)$ because $f \in \widehat{F}_2 \subset \widehat{P}_3$.

Therefore $\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\}$ is again a profinite knot.

- (2) By Figure 2.26, it is easy because $f \in \widehat{F}_2 \subset \widehat{P}_3$. □

Theorem 2.41. *Equation (2.19) determines a well-defined $\widehat{\text{GT}}$ -action on $\text{Frac } \widehat{\mathcal{K}}$. Namely,*

- (1) $\sigma\left(\frac{r_1}{s_1}\right) = \sigma\left(\frac{r_2}{s_2}\right) \in \text{Frac } \widehat{\mathcal{K}}$ if $r_1 \sim r_2$ and $s_1 \sim s_2$, i.e. if $r_1 = r_2$ and $s_1 = s_2$ in $\widehat{\mathcal{K}}$;
- (2) $\sigma\left(\frac{r_1}{s_1}\right) = \sigma\left(\frac{r_2}{s_2}\right) \in \text{Frac } \widehat{\mathcal{K}}$ if $(r_1, s_1) \approx (r_2, s_2)$, i.e. if $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ in $\text{Frac } \widehat{\mathcal{K}}$;
- (3) $\sigma_1(\sigma_2(x)) = (\sigma_1 \circ \sigma_2)(x)$ for any $\sigma_1, \sigma_2 \in \widehat{\text{GT}}$ and $x \in \text{Frac } \widehat{\mathcal{K}}$; furthermore $\text{Frac } \widehat{\mathcal{K}}$ forms a topological $\widehat{\text{GT}}$ -module, namely,
- (4) *the action is compatible with the group structure, i.e.*

$$\sigma(e) = e, \quad \sigma(x \sharp y) = \sigma(x) \sharp \sigma(y), \quad \sigma(x^{-1}) = \sigma(x)^{-1}$$

for any $\sigma \in \widehat{\text{GT}}$ and $x, y \in \text{Frac } \widehat{\mathcal{K}}$;

- (5) *the action is continuous.*

Proof. (1) Firstly we prove that $\sigma((r_1, s)) = \sigma((r_2, s)) \in \widehat{\mathcal{K}}^2$ when r_1 is isotopic to r_2 for $\sigma = (\lambda, f) \in \widehat{\text{GT}}$. We may further assume that r_1 is obtained from r_2 by a single operation of one of the moves (T1)–(T6).

- If it is (T1), it is clear.
- If it is (T2), it is immediate because $\sigma(b_2) \cdot \sigma(b_1) = \sigma(b_2 b_1)$ holds for $b_1, b_2 \in \widehat{B}_n$.

- If it is (T3), we may further assume that its T_1 and T_2 in (T3) are both fundamental profinite tangles. Then by Proposition 1.16 and Proposition 2.42

$$\begin{aligned} & \sigma(e_{n_1}^{t(T_1)} \otimes T_2) \cdot \sigma(T_1 \otimes e_{m_2}^{s(T_2)}) \\ &= f_{[n_1],[n_2],[0]}^{-1,(t(T_1),t(T_2))} \cdot (e_{n_1}^{t(T_1)} \otimes \sigma(T_2)) \cdot f_{[n_1],[m_2],[0]}^{(t(T_1),s(T_2))} \cdot (\sigma(T_1) \otimes e_{m_2}^{s(T_2)}), \end{aligned}$$

(by (T4))

$$\begin{aligned} &= f_{[n_1],[n_2],[0]}^{-1,(t(T_1),t(T_2))} \cdot (e_{n_1}^{t(T_1)} \otimes \sigma(T_2)) \cdot (\sigma(T_1) \otimes e_{m_2}^{s(T_2)}) \cdot f_{[m_1],[m_2],[0]}^{(s(T_1),s(T_2))} \\ &= f_{[n_1],[n_2],[0]}^{-1,(t(T_1),t(T_2))} \cdot (\sigma(T_1) \otimes e_{n_2}^{t(T_2)}) \cdot (e_{m_1}^{s(T_1)} \otimes \sigma(T_2)) \cdot f_{[m_1],[m_2],[0]}^{(s(T_1),s(T_2))} \\ &= (\sigma(T_1) \otimes e_{n_2}^{t(T_2)}) \cdot f_{[m_1],[n_2],[0]}^{-1,(s(T_1),t(T_2))} \cdot (e_{m_1}^{s(T_1)} \otimes \sigma(T_2)) \cdot f_{[m_1],[m_2],[0]}^{(s(T_1),s(T_2))} \\ &= \sigma(T_1 \otimes e_{n_2}^{t(T_2)}) \cdot \sigma(e_{m_1}^{s(T_1)} \otimes T_2). \end{aligned}$$

Whence (T3) is preserved by the $\widehat{\text{GT}}$ -action.

- If it is (T4), we may assume that T in (T4) is a fundamental profinite tangle. Then by Proposition 1.18 and Proposition 2.42

$$\begin{aligned} & \sigma(\text{ev}_{k,t(T)}(b_l^\epsilon)) \cdot \sigma(e_{k-1}^{s_1} \otimes T \otimes e_{l-k}^{s_2}) \\ &= f_{[k'-1],[n],[l-k']}^{-1,(t_1,t(T),t_2)} \cdot \text{ev}_{k,t(T)}(\sigma(b_l^\epsilon)) \cdot f_{[k-1],[n],[l-k]}^{(s_1,t(T),s_2)} \\ & \quad \cdot f_{[k-1],[n],[l-k]}^{-1,(s_1,t(T),s_2)} \cdot (e_{k-1}^{s_1} \otimes \sigma(T) \otimes e_{l-k}^{s_2}) \cdot f_{[k-1],[m],[l-k]}^{(s_1,s(T),s_2)} \\ &= f_{[k'-1],[n],[l-k']}^{-1,(t_1,t(T),t_2)} \cdot \text{ev}_{k,t(T)}(\sigma(b_l^\epsilon)) \cdot (e_{k-1}^{s_1} \otimes \sigma(T) \otimes e_{l-k}^{s_2}) \\ & \quad \cdot f_{[k-1],[m],[l-k]}^{(s_1,s(T),s_2)}, \end{aligned}$$

(by (T4))

$$\begin{aligned} &= f_{[k'-1],[n],[l-k']}^{-1,(t_1,t(T),t_2)} \cdot (e_{k'-1}^{t_1} \otimes \sigma(T) \otimes e_{l-k'}^{t_2}) \cdot \text{ev}^{k',s(T)}(\sigma(b_l^\epsilon)) \\ & \quad \cdot f_{[k-1],[m],[l-k]}^{(s_1,s(T),s_2)} \\ &= f_{[k'-1],[n],[l-k']}^{-1,(t_1,t(T),t_2)} \cdot (e_{k'-1}^{t_1} \otimes \sigma(T) \otimes e_{l-k'}^{t_2}) \cdot f_{[k'-1],[m],[l-k']}^{(t_1,s(T),t_2)} \\ & \quad \cdot f_{[k'-1],[m],[l-k']}^{-1,(t_1,s(T),t_2)} \cdot \text{ev}^{k',s(T)}(\sigma(b_l^\epsilon)) \cdot f_{[k-1],[m],[l-k]}^{(s_1,s(T),s_2)} \\ &= \sigma(e_{k'-1}^{t_1} \otimes T \otimes e_{l-k'}^{t_2}) \cdot \sigma(\text{ev}^{k',s(T)}(b_l^\epsilon)). \end{aligned}$$

Whence (T4) is preserved by the action.

- If it is (T5), we have

$$\sigma(a_{k+1,l-1}^{\epsilon'}) \cdot \sigma(c_{k,l}^{\epsilon}) = a_{k+1,l-1}^{\epsilon'} \cdot f_{1\dots k+1,k+2,k+3}^{s(c_{k,l}^{\epsilon})} \cdot f_{1\dots k+1,k+2}^{-1,s(c_{k,l}^{\epsilon})} \cdot c_{k,l}^{\epsilon}$$

(by the pentagon equation (1.4))

$$\begin{aligned} &= a_{k+1,l-1}^{\epsilon'} \cdot f_{1\dots k,k+1,k+2}^{-1,s(c_{k,l}^{\epsilon})} \cdot f_{k+1,k+2,k+3}^{s(c_{k,l}^{\epsilon})} \cdot f_{1\dots k,k+1,k+2,k+3}^{s(c_{k,l}^{\epsilon})} \cdot c_{k,l}^{\epsilon} \end{aligned}$$

(by (T4) and Lemma 1.8)

$$= a_{k+1,l-1}^{\epsilon'} \cdot f_{k+1,k+2,k+3}^{s(c_{k,l}^{\epsilon})} \cdot c_{k,l}^{\epsilon}.$$

It looks that (T5) is not preserved by the $\widehat{\text{GT}}$ -action. But actually it means that $\sigma(r_1)$ is obtained by an insertion of $f_{k+1,k+2,k+3}^{s(c_{k,l}^{\epsilon})}$ between $a_{k+1,l-1}^{\epsilon'}$ and $c_{k,l}^{\epsilon}$ in $\sigma(r_2)$. Thus $\sigma(r_1) = \sigma(r_2) \# \Lambda_f$. Because $\alpha(r_1) = \alpha(r_2) + 1$, we may say that (T5) is compatible with the action by (2.19). The second equality can be proved in the same way.

- If it is (T6), again by Proposition 1.16 and Proposition 2.42,

$$\begin{aligned} &\sigma(a_{k,l}^{\epsilon}) \cdot \sigma((\sigma_{k+1}^{\epsilon'})^c) \\ &= a_{k,l}^{\epsilon} \cdot f_{1\dots k,k+1,k+2}^{s(a_{k,l}^{\epsilon})} \cdot f_{1\dots k,k+1,k+2}^{-1,s(a_{k,l}^{\epsilon})} \cdot (\sigma_{k+1}^{\epsilon'})^{\lambda c} \cdot f_{1\dots k,k+1,k+2}^{s(a_{k,l}^{\bar{\epsilon}})} \\ &= a_{k,l}^{\epsilon} \cdot (\sigma_{k+1}^{\epsilon'})^{\lambda c} \cdot f_{1\dots k,k+1,k+2}^{s(a_{k,l}^{\bar{\epsilon}})} \end{aligned}$$

(by $\lambda \equiv 1 \pmod{2}$ and (T6))

$$= a_{k,l}^{\bar{\epsilon}} \cdot f_{1\dots k,k+1,k+2}^{s(a_{k,l}^{\bar{\epsilon}})} = \sigma(a_{k,l}^{\bar{\epsilon}}).$$

The case for $c_{k,l}^{\epsilon}$ can be checked in the same way.

Secondly we prove that $\sigma((r, s_1)) = \sigma((r, s_2)) \in \widehat{\mathcal{K}}^2$ when s_1 is isotopic to s_2 . But it can be proved in a similar way to the above. Hence our claim of (1) is obtained.

(2) By definition,

$$r_1 \# s_2 \# t = r_2 \# s_1 \# t$$

in \mathcal{K} for some profinite knot t . By the definition of $\#$,

$$\sigma(k_1 \# k_2) \# \sigma(\bigcirc) = \sigma(k_1) \# \sigma(k_2) \tag{2.21}$$

equivalently

$$\sigma(k_1 \# k_2) = \sigma(k_1) \# \sigma(k_2) \# \Lambda_f, \tag{2.22}$$

holds in $\text{Frac } \widehat{\mathcal{K}}$ for any profinite knot k_1 and k_2 . Therefore our claim is immediate because

$$\sigma(r_1 \# s_2 \# t) = \sigma(r_1) \# \sigma(s_2) \# \sigma(t) \# \Lambda_f \# \Lambda_f$$

and

$$\sigma(r_2 \# s_1 \# t) = \sigma(r_2) \# \sigma(s_1) \# \sigma(t) \# \Lambda_f \# \Lambda_f.$$

(3) For $\sigma_1 = (\lambda_1, f_1)$ and $\sigma_2 = (\lambda_2, f_2) \in \widehat{\text{GT}}$, put $\sigma_3 = \sigma_2 \circ \sigma_1 \in \widehat{\text{GT}}$. Hence, by (1.5), $\sigma_3 = (\lambda_3, f_3)$ with $\lambda_3 = \lambda_2 \lambda_1$ and

$$f_3 = f_2 \cdot \sigma_2(f_1) = f_2 \cdot f_1(x^{\lambda_2}, f_2^{-1} y^{\lambda_2} f_2) (= f_2 \circ f_1). \tag{2.23}$$

Firstly we note that

$$\sigma_3(\gamma_{i,j}) = \sigma_2(\sigma_1(\gamma_{i,j})).$$

When $\gamma_{i,j} = a_{k,l}^\epsilon$ or $c_{k,l}^\epsilon$, the equality is derived from (2.23). When $\gamma_{i,j} = b_n^\epsilon$, it is easy because of the $\widehat{\text{GT}}$ -action on \widehat{B}_n .

Secondly by definition we have

$$\sigma_2(\Lambda_{f_1}) = \{\sigma_2(a_{0,0}) \cdot \sigma_2(a_{0,2}) \cdot \sigma_2(e_1 \otimes f_1) \cdot \sigma_2(c_{1,1}) \cdot \sigma_2(c_{0,0})\} / \Lambda_{f_2}^{\#2}$$

(by Proposition 1.16 and Proposition 2.42)

$$\begin{aligned} &= \{a_{0,0} \cdot a_{0,2} \cdot (f_2^{-1})_{1,2,3} \cdot (f_2^{-1})_{1,23,4} \cdot (e_1 \otimes \sigma_2(f_1)) \\ &\quad \cdot (f_2)_{1,23,4} \cdot (f_2)_{1,2,3} \cdot (f_2^{-1})_{1,2,3} \cdot c_{1,1} \cdot c_{0,0}\} / \Lambda_{f_2}^{\#2} \\ &= \{a_{0,0} \cdot a_{0,2} \cdot (f_2^{-1})_{1,2,3} \cdot (f_2^{-1})_{1,23,4} \cdot (e_1 \otimes \sigma_2(f_1)) \\ &\quad \cdot (f_2)_{1,23,4} \cdot c_{1,1} \cdot c_{0,0}\} / \Lambda_{f_2}^{\#2} \end{aligned}$$

(by the pentagon equation (1.4))

$$\begin{aligned} &= \{a_{0,0} \cdot a_{0,2} \cdot (f_2^{-1})_{12,3,4} \cdot (f_2^{-1})_{1,2,34} \cdot (f_2)_{2,3,4} \cdot (e_1 \otimes \sigma_2(f_1)) \\ &\quad \cdot (f_2)_{1,23,4} \cdot c_{1,1} \cdot c_{0,0}\} / \Lambda_{f_2}^{\#2}, \end{aligned}$$

by a successive application of (T6) and Lemma 1.8

$$\begin{aligned} &= \{a_{0,0} \cdot a_{0,2} \cdot (f_2)_{2,3,4} \cdot (e_1 \otimes \sigma_2(f_1)) \cdot c_{1,1} \cdot c_{0,0}\} / \Lambda_{f_2}^{\#2} \\ &= \{a_{0,0} \cdot a_{0,2} \cdot (e_1 \otimes f_2 \cdot \sigma_2(f_1)) \cdot c_{1,1} \cdot c_{0,0}\} / \Lambda_{f_2}^{\#2} \\ &= \Lambda_{f_2 \circ f_1} \# \sigma_2(\bigcirc)^{\#2}. \end{aligned}$$

We note that in the above computation we omit the symbol ϵ of orientation.

Finally,

$$\begin{aligned}
& \sigma_2\left(\sigma_1\left(\frac{r}{s}\right)\right) \\
&= \sigma_2\left(\frac{\{\sigma_1(\gamma_{1,m}) \cdots \sigma_1(\gamma_{1,2}) \cdot \sigma_1(\gamma_{1,1})\} \# (\Lambda_{f_1})^{\# \alpha(s)}}{\{\sigma_1(\gamma_{2,n}) \cdots \sigma_1(\gamma_{2,2}) \cdot \sigma_1(\gamma_{2,1})\} \# (\Lambda_{f_1})^{\# \alpha(r)}}\right) \\
&= \frac{\sigma_2(\{\sigma_1(\gamma_{1,m}) \cdots \sigma_1(\gamma_{1,2}) \cdot \sigma_1(\gamma_{1,1})\}) \# \sigma_2(\Lambda_{f_1})^{\# \alpha(s)} \# \Lambda_{f_2}^{\# \alpha(s)}}{\sigma_2(\{\sigma_1(\gamma_{2,n}) \cdots \sigma_1(\gamma_{2,2}) \cdot \sigma_2(\gamma_{2,1})\}) \# \sigma_2(\Lambda_{f_1})^{\# \alpha(r)} \# \Lambda_{f_2}^{\# \alpha(r)}} \\
&= \frac{\{\sigma_2(\sigma_1(\gamma_{1,m})) \cdots \sigma_2(\sigma_1(\gamma_{1,1}))\} \# \Lambda_{f_2}^{\# \alpha(s)} \# (\Lambda_{f_3})^{\# \alpha(s)} \# \sigma_2(\bigcirc)^{\# 2\alpha(s)} \# \Lambda_{f_2}^{\# \alpha(s)}}{\{\sigma_2(\sigma_1(\gamma_{2,n})) \cdots \sigma_2(\sigma_1(\gamma_{2,1}))\} \# \Lambda_{f_2}^{\# \alpha(r)} \# (\Lambda_{f_3})^{\# \alpha(r)} \# \sigma_2(\bigcirc)^{\# 2\alpha(r)} \# \Lambda_{f_2}^{\# \alpha(r)}} \\
&= \frac{\{\sigma_3(\gamma_{1,m}) \cdots \sigma_3(\gamma_{1,2}) \cdot \sigma_3(\gamma_{1,1})\} \# (\Lambda_{f_3})^{\# \alpha(s)}}{\{\sigma_3(\gamma_{2,n}) \cdots \sigma_3(\gamma_{2,2}) \cdot \sigma_3(\gamma_{2,1})\} \# (\Lambda_{f_3})^{\# \alpha(r)}} \\
&= \sigma_3\left(\frac{r}{s}\right)
\end{aligned}$$

by (2.20) and (2.22).

(4) Let $x = r_1/s_1$ and $y = r_2/s_2$ with profinite knots r_1, r_2, s_1, s_2 . Then by (2.22) it is easy to see

$$\begin{aligned}
\sigma(x \# y) &= \sigma\left(\frac{r_1 \# r_2}{s_1 \# s_2}\right) \\
&= \frac{\sigma(r_1 \# r_2)}{\sigma(s_1 \# s_2)} \\
&= \frac{\sigma(r_1) \# \sigma(r_2) \# \Lambda_f}{\sigma(s_1) \# \sigma(s_2) \# \Lambda_f} \\
&= \frac{\sigma(r_1) \# \sigma(r_2)}{\sigma(s_1) \# \sigma(s_2)} \\
&= \frac{\sigma(r_1)}{\sigma(s_1)} \# \frac{\sigma(r_2)}{\sigma(s_2)} \\
&= \sigma\left(\frac{r_1}{s_1}\right) \# \sigma\left(\frac{r_2}{s_2}\right) \\
&= \sigma(x) \# \sigma(y).
\end{aligned}$$

The inverse is also easy to check.

(5) We recall that $\widehat{\mathcal{K}}^{\text{seq}}$ (cf. the proof of Theorem 2.16. (2)) is the set of finite consistent sequences of profinite fundamental tangles $\gamma_n \cdots \gamma_2 \cdot \gamma_1$ with a single connected component and with $(\gamma_n, \gamma_1) = (\curvearrowright, \curvearrowleft)$. We define the map

$$A: \widehat{\text{GT}} \times \widehat{\mathcal{K}}^{\text{seq}} \times \widehat{\mathcal{K}}^{\text{seq}} \longrightarrow \widehat{\mathcal{K}}^{\text{seq}} \times \widehat{\mathcal{K}}^{\text{seq}}$$

by

$$A(\sigma, r, s) = (\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\} \# (\Lambda_f)^{\# \alpha(s)}, \\ \{\sigma(\gamma_{2,n}) \cdots \sigma(\gamma_{2,2}) \cdot \sigma(\gamma_{2,1})\} \# (\Lambda_f)^{\# \alpha(r)})$$

for $\sigma = (\lambda, f)$, $r = \gamma_{1,m} \cdots \gamma_{1,2} \cdot \gamma_{1,1}$ and $s = \gamma_{2,n} \cdots \gamma_{2,2} \cdot \gamma_{2,1}$ ($\gamma_{i,j}$: profinite fundamental tangle). We know that the $\widehat{\text{GT}}$ -action on \widehat{B}_n and the map $\widehat{\text{GT}} \rightarrow \widehat{B}_3$: $\sigma = (\lambda, f) \mapsto f$ are continuous, so the map A is continuous. Since the diagram below is commutative

$$\begin{array}{ccc} \widehat{\text{GT}} \times \widehat{\mathcal{K}}^{\text{seq}} \times \widehat{\mathcal{K}}^{\text{seq}} & \xrightarrow{A} & \widehat{\mathcal{K}}^{\text{seq}} \times \widehat{\mathcal{K}}^{\text{seq}} \\ \downarrow & & \downarrow \\ \widehat{\text{GT}} \times \text{Frac } \widehat{\mathcal{K}} & \longrightarrow & \text{Frac } \widehat{\mathcal{K}} \end{array}$$

and the projection $\widehat{\mathcal{K}}^{\text{seq}} \times \widehat{\mathcal{K}}^{\text{seq}} \rightarrow \text{Frac } \widehat{\mathcal{K}}$ is continuous, the lower map is also continuous. □

The following is required to prove Theorem 2.41.

Proposition 2.42. *Let $k, l, m_1, m_2 \geq 0$ and $\epsilon_i \in \{\uparrow, \downarrow\}^{m_i}$ ($i = 1, 2$). For any $\sigma \in \widehat{\text{GT}}$,*

$$\sigma(a_{m_1+k, l+m_2}^{\epsilon_1 \epsilon_2}) = f_{[m_1], [k+l], [m_2]}^{-1, \epsilon_t} \cdot (e_{m_1}^{\epsilon_1} \otimes \sigma(a_{k,l}^\epsilon) \otimes e_{m_2}^{\epsilon_2}) \cdot f_{[m_1], [k+l+2], [m_2]}^{\epsilon_s}$$

with

$$\epsilon_t = t(a_{m_1+k, l+m_2}^{\epsilon_1 \epsilon_2}) \quad \text{and} \quad \epsilon_s = s(a_{m_1+k, l+m_2}^{\epsilon_1 \epsilon_2}),$$

and

$$\sigma(c_{m_1+k, l+m_2}^{\epsilon_1 \epsilon_2}) = f_{[m_1], [k+l+2], [m_2]}^{-1, \epsilon_t} \cdot (e_{m_1}^{\epsilon_1} \otimes \sigma(c_{k,l}^\epsilon) \otimes e_{m_2}^{\epsilon_2}) \cdot f_{[m_1], [k+l], [m_2]}^{\epsilon_s}$$

with

$$\epsilon_t = t(a_{m_1+k, l+m_2}^{\epsilon_1 \epsilon_2}) \quad \text{and} \quad \epsilon_s = s(a_{m_1+k, l+m_2}^{\epsilon_1 \epsilon_2}).$$

Here $f_{[m_1],[M],[m_2]}^\epsilon \in \widehat{B}$ means

$$(f_{[m_1],[M],[m_2]}, \epsilon) \in \widehat{B}_{m_1+M+m_2} \times \{\uparrow, \downarrow\}^{m_1+M+m_2}$$

with (see also (1.13))

$$\begin{aligned} & f_{[m_1],[M],[m_2]} \\ & := f_{1\dots m_1, m_1+1\dots m_1+M-1, m_1+M} \\ & \quad \cdot f_{1\dots m_1, m_1+1\dots m_1+M-2, m_1+M-1} \cdots f_{1\dots m_1, m_1+1, m_1+2} \in \widehat{B}_{m_1+M+m_2}. \end{aligned}$$

Proof. We prove the first equality. To avoid the complexity, we again omit the symbol of orientations. By Definition 2.39.(1),

$$\begin{aligned} & f_{[m_1],[k+l],[m_2]}^{-1} \cdot (e_{m_1} \otimes \sigma(a_{k,l}) \otimes e_{m_2}) \cdot f_{[m_1],[k+l+2],[m_2]} \\ & = f_{[m_1],[k+l],[m_2]}^{-1} \cdot a_{m_1+k, l+m_2} \\ & \quad \cdot f_{m_1+1\dots m_1+k, m_1+k+1, m_1+k+2} \cdot f_{[m_1],[k+l+2],[m_2]}. \end{aligned} \tag{2.24}$$

- When $M \geq k + 3$, by (T4),

$$f_{m_1+1\dots m_1+k, m_1+k+1, m_1+k+2}$$

commutes with

$$f_{1\dots m_1, m_1+1\dots m_1+M-1, m_1+M}$$

and

$$\begin{aligned} & a_{m_1+k, l+m_2} \cdot f_{1\dots m_1, m_1+1\dots m_1+M-1, m_1+M} \\ & = f_{1\dots m_1, m_1+1\dots m_1+M-3, m_1+M-2} \cdot a_{m_1+k, l+m_2}. \end{aligned}$$

Therefore

$$\begin{aligned} (2.24) & = f_{[m_1],[k],[l+m_2]}^{-1} \cdot a_{m_1+k, l+m_2} \\ & \quad \cdot f_{m_1+1\dots m_1+k, m_1+k+1, m_1+k+2} \cdot f_{[m_1],[k+2],[l+m_2]}. \end{aligned}$$

- When $M = k + 1, k + 2$, our calculation goes as follows:

$$\begin{aligned} (2.24) & = f_{[m_1],[k],[l+m_2]}^{-1} \cdot a_{m_1+k, l+m_2} \cdot f_{m_1+1\dots m_1+k, m_1+k+1, m_1+k+2} \\ & \quad \cdot f_{1\dots m_1, m_1+1\dots m_1+k+1, m_1+k+2} \\ & \quad \cdot f_{1\dots m_1, m_1+1\dots m_1+k, m_1+k+1} \cdot f_{[m_1],[k],[l+m_2+2]} \end{aligned}$$

(by the pentagon equation (1.4))

$$\begin{aligned}
 &= f_{[m_1],[k],[l+m_2]}^{-1} \cdot a_{m_1+k,l+m_2} \\
 &\quad \cdot f_{1\dots m_1,m_1+1\dots m_1+k,m_1+k+1} \cdot f_{m_1+k+2} \\
 &\quad \cdot f_{1\dots m_1+k,m_1+k+1,m_1+k+2} \cdot f_{[m_1],[k],[l+m_2+2]}
 \end{aligned}$$

(by (T4) and Lemma 1.8)

$$\begin{aligned}
 &= f_{[m_1],[k],[l+m_2]}^{-1} \cdot a_{m_1+k,l+m_2} \cdot f_{1\dots m_1+k,m_1+k+1,m_1+k+2} \\
 &\quad \cdot f_{[m_1],[k],[l+m_2+2]}.
 \end{aligned}$$

- When $M \leq k$, by (T4) again,

$$f_{1\dots m_1+k,m_1+k+1,m_1+k+2}$$

commutes with

$$f_{1\dots m_1,m_1+1\dots m_1+M-1,m_1+M}$$

and

$$\begin{aligned}
 &a_{m_1+k,l+m_2} \cdot f_{1\dots m_1,m_1+1\dots m_1+M-1,m_1+M} \\
 &= f_{1\dots m_1,m_1+1\dots m_1+M-1,m_1+M} \cdot a_{m_1+k,l+m_2}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (2.24) &= f_{[m_1],[k],[l+m_2]}^{-1} \cdot a_{m_1+k,l+m_2} \cdot f_{[m_1],[k],[l+m_2+2]} \\
 &\quad \cdot f_{1\dots m_1+k,m_1+k+1,m_1+k+2}, \\
 &= a_{m_1+k,l+m_2} \cdot f_{1\dots m_1+k,m_1+k+1,m_1+k+2} = \sigma(a_{m_1+k,l+m_2}).
 \end{aligned}$$

Hence we get the equality.

The second equality can be proved in the same way. □

Thus by Theorem 2.41, the $\widehat{\text{GT}}$ -action

$$\widehat{\text{GT}} \longrightarrow \text{Aut Frac } \widehat{\mathcal{K}} \tag{2.25}$$

is established.

We note that due to the creation–annihilation relations (T5) we have to pass to the fractional group of $\widehat{\mathcal{K}}$ to construct $\widehat{\text{GT}}$ -action on profinite knots.

Remark 2.43. In [22], it is explained that the category $\widehat{\mathcal{T}}(R)$ (R : a commutative ring containing \mathbb{Q}) of pro- R -algebraic framed tangles forms an R -linear ribbon category. From which they deduced an action of the proalgebraic Grothendieck–Teichmüller group $\widehat{\text{GT}}(R)$ [8] on the space $R[\widehat{\mathcal{K}}]$ of pro- R -algebraic knots by a categorical arguments. In our forthcoming paper [10], it will be shown that our action (2.25) is a lift of their action. An analogous deduction of our Theorem 2.41 by such categorical arguments might be expectable. However a completely same argument does not seem to work. We may have a ‘ribbon’ category $\widehat{\mathcal{T}}$ of profinite (framed) tangles but a difficulty here is that the inverse $(\Lambda_f)^{-1}$ does not look to exist generally in $\widehat{\mathcal{T}}$, unlike the case of $\widehat{\mathcal{T}}(R)$. (That is why we introduced the group $\widehat{\text{Frac}} \widehat{\mathcal{K}}$ of the fraction of the monoid $\widehat{\mathcal{K}}$. A technical care to remedy this might be required.)

Remark 2.44. The Kontsevich knot invariant [23] is obtained by integrating a formal analogue of the Knizhnik–Zamolodchikov (KZ) equation. Bar-Natan [1], Kassel and Turaev [22], Le and Murakami [24], and Pieunikhin [34] gave a combinatorial reconstruction of the invariant by using an associator [8]. An *associator* means a pair (μ, φ) with $\mu \in R^\times$ and an R -coefficient non-commutative formal power series φ with two variables satisfying specific relations which are analogues of our pentagon and hexagon equations (1.3) and (1.4) ([8], see also [9]). One of striking results¹¹ in Le-Murakami [24] is the rationality of the Kontsevich invariant which follows from that the resulting invariant is, in fact, independent of φ (but depends on μ). Stimulated to their result, Kassel and Turaev [22] showed that their $\widehat{\text{GT}}(R)$ -action on $R[\widehat{\mathcal{K}}]$ factors through the cyclotomic action (cf. Appendix of [22]). The algebra $R[\widehat{\mathcal{K}}]$ looks ‘too linear’. In contrast, in our profinite setting, it is totally unclear if our above $\widehat{\text{GT}}$ -action (2.25) would depend only on $\lambda \in \widehat{\mathbb{Z}}^\times$ of $(\lambda, f) \in \widehat{\text{GT}}$, namely, the action would factor through $\widehat{\mathbb{Z}}^\times$. We remind that their proof of the above independency is based on certain linear algebraic arguments, actually a vanishing of a (Harrison) cohomology of a chain complex associated with chords. But here in our situation, we are working not on their proalgebraic setting but on the profinite setting where such vanishing result has not been established. And we do not know whether such a factorization would occur in our setting or not.

Finally we obtain a Galois representation on knots as an important consequence of Theorem 2.41.

¹¹ It might be amazing to know that Drinfel’d indicated it in [7].

Theorem 2.45. *Fix an embedding from $\widehat{\mathbb{Q}}$ in to \mathbb{C} . The group $\text{Frac } \widehat{\mathcal{K}}$ of profinite knots admits a non-trivial topological $G_{\mathbb{Q}}$ -module structure. Namely there is a non-trivial continuous Galois representation*

$$\rho_0: G_{\mathbb{Q}} \longrightarrow \text{Aut } \text{Frac } \widehat{\mathcal{K}}. \tag{2.26}$$

Proof. By Theorem 2.41, it is straightforward because in Theorem 1.9 we see that the absolute Galois group $G_{\mathbb{Q}}$ is mapped to $\widehat{\text{GT}}$. It is proved that $\text{Frac } \widehat{\mathcal{K}}$ is nontrivial in Theorem 2.36. The non-triviality of ρ_0 is a consequence of the example below because generally we have $K \neq \bar{K}$ in $\text{Frac } \widehat{\mathcal{K}}$. For instance, the left trefoil (the knot in the left below of Figure 2.15) and the right trefoil (its mirror image) are mapped to different elements by the map (2.17) because they are known to be separated by the Kontsevich invariant. (cf. Remark 2.34.) \square

Example 2.46. Especially when $\sigma \in G_{\mathbb{Q}}$ is equal to the complex conjugation morphism ς_0 , it corresponds to $(\lambda, f) = (-1, 1) \in \widehat{\text{GT}}$ whose action on \widehat{B}_n is given by $\sigma_i \mapsto \sigma_i^{-1}$ ($1 \leq i \leq n - 1$) (cf. Example 1.10). Whence the action of ς_0 on $\text{Frac } \widehat{\mathcal{K}}$ is particularly described by

$$\rho_0(\varsigma_0)(K) = \bar{K}$$

for $K \in \mathcal{K}$ because $\Lambda_1 = \circlearrowleft$. Here we denote the image of the map h' (2.18) on an oriented knot K by the same symbol K and we mean the mirror image of the knot K by \bar{K} . The easiest example is that the right trefoil knot is mapped to the left trefoil knot by the complex conjugation.

There is another type of involution. For each profinite oriented knot K , we define $\text{rev}(K)$ to be a profinite oriented knot which is obtained by reversing the orientation of K . It is easy to see that it induces a well-defined involution

$$\text{rev}: \text{Frac } \widehat{\mathcal{K}} \longrightarrow \text{Frac } \widehat{\mathcal{K}}$$

which is an automorphism as a topological group.

Problem 2.47. Is rev defined over \mathbb{Q} ? Namely does

$$\text{rev} \circ \sigma = \sigma \circ \text{rev} \tag{2.27}$$

hold for all $\sigma \in G_{\mathbb{Q}}$?

This problem would be proven affirmatively if we could show that $\Lambda_f = \text{rev}(\Lambda_f)$ in $\text{Frac } \widehat{\mathcal{K}}$ for all $\sigma = (\lambda, f) \in G_{\mathbb{Q}}$. In the proalgebraic setting (cf. Remark 2.44 and [10]), rev can be defined similarly for $\widehat{R[\mathcal{K}]}$. We can show the validity of an analogue of (2.27) for the action of $\sigma \in \text{GT}(R)$ on $\widehat{R[\mathcal{K}]}$ by transmitting the $\text{GT}(R)$ -action on $\widehat{R[\mathcal{K}]}$ into a $\text{GRT}(R)$ -action on \widehat{CD} .

Extending other standard operations on knots, such as mutation and cabling, into those on profinite knots and examining their Galois behaviors is also worthy to pursue.

Project 2.48. In §1, the actions of $\widehat{\text{GT}}$ and $G_{\mathbb{Q}}$ on the profinite braid group \widehat{B}_n are discussed. In Remark 1.14 it is explained in the language of algebraic geometry that the $G_{\mathbb{Q}}$ -action on \widehat{B}_n is caused by the homotopy exact sequence of the scheme-theoretic fundamental group of the quotient variety $\text{Conf}_{\mathfrak{S}_n}^n$ of the configuration space Conf^n . Whilst as for our Galois action on knots in Theorem 2.45, the author is not sure whether there is such kind of its ‘purely’ algebraic-geometrical interpretation (without a usage of $\widehat{\text{GT}}$ -factorization) or not. In other word, it is not clear if there exists any (co)homology theory H_{\star} (or any fundamental group theory π_1^{\star}) and any (pro-)variety X defined over \mathbb{Q} such that

$$\text{Frac } \widehat{\mathcal{K}} = H_{\star}(X_{\overline{\mathbb{Q}}})$$

(or $\text{Frac } \widehat{\mathcal{K}} = \pi_1^{\star}(X_{\overline{\mathbb{Q}}})$) and the right hand side naturally carries a $G_{\mathbb{Q}}$ -action which yields our $G_{\mathbb{Q}}$ -action on $\text{Frac } \widehat{\mathcal{K}}$. It would be our future research.

Asking the validity of an analogue of Belyi’s theorem [3] in (1.8) is particularly significant.

Problem 2.49. Is the action of the absolute Galois group (2.26) on profinite knots faithful? If not, then what is the corresponding kernel field? And what is the arithmetic meaning of this?

This is also related to the problem discussed in Remark 2.44 above. In [10], it will be explained that the corresponding kernel field is bigger than the maximal abelian extension $\mathbb{Q}(\mu_{\infty})$ of \mathbb{Q} .

Problem 2.50. What is the $G_{\mathbb{Q}}$ -invariant subspace of $\text{Frac } \widehat{\mathcal{K}}$?

In [10], we will settle a similar problem formulated in the proalgebraic setting. Asking the same type of questions for each given knot is also worthy to discuss.

Problem 2.51. (1) What is the Galois stabilizer of each given knot? And what is the corresponding Galois extension field of \mathbb{Q} ?

(2) Suppose that two topological knots K_1 and K_2 and an open normal subgroup \mathcal{N} of $\text{Frac } \widehat{\mathcal{K}}$ with finite index are given. When do two cosets $K_1 \cdot \mathcal{N}$ and $K_2 \cdot \mathcal{N}$ lie on a same Galois orbit? If so, then which $\sigma \in G_{\mathbb{Q}}$ connect them?

Example 2.46 tells that a knot and its mirror image lie on a same Galois orbit and that the stabilizer of an amphicheiral knot, say, the figure eight knot (the knot in the left above of Figure 2.15), contains the subgroup $\{id, \zeta_0\}$ of order 2.

Project 2.52. We can also consider Galois action not only on profinite knots but also on pro-solvable knots by replacing profinite pure braid groups \hat{P}_n by their pro-solvable completions on the definition of profinite knots. The direction might be also worthy to pursue. Ihara's profinite beta function B_σ [16] which arises from a description of the action of the absolute Galois group on the double commutator quotient of \hat{F}_2 might help to see the action on a certain quotient.

As is explained in Remark 2.34, we have an isomorphism $\widehat{\mathbb{Q}[\mathcal{K}]} \simeq \widehat{CD}$ between the linear space of pro-algebraic knots and that of chord diagrams.

Problem 2.53. What is a profinite analogue of the linear space of chord diagrams? Do we have a profinite analogue of the above isomorphism for $\widehat{\mathcal{K}}$ and this?

Grothendieck's dessins d'enfants [13] are combinatorial diagrams which describe the action of the absolute Galois group on \hat{F}_2 . The author wonders if a certain projective system of dessins attains the above profinite analogue of the space of chord diagrams.

Project 2.54. There are various notions of equivalences for (framed) knots (and links) such as the Kirby moves (the Fenn–Rourke moves), the knot cobordism, the knot concordance, etc. Extending these notions into those for our profinite links and examining their behaviors under our Galois action is worthy to pursue. Particularly the Kirby moves are known (consult the standard textbook such as [32]) to yield a one to one correspondence between the set of framed links modulo the equivalence generated by the moves and the set of isomorphism classes of closed connected oriented 3-manifolds (three dimensional manifolds) by Dehn surgery. Giving a nice formulation of profinite analogues of Kirby moves and a description of their Galois behaviors looks significant for a *realization* of the analogy between number rings and 3-manifolds, which is one of the most fundamental ones posted in arithmetic topology by Kapranov [20], Morishita [31] and Reznikov [35].

Appendices

A. Two-bridge profinite knots

We introduce profinite analogues of two-bridge knots and observe that the subgroup of $G\widehat{\mathcal{K}}$ generated by them is stable under our Galois action.

We recall that a *two-bridge knot* (or link) is a topological knot (or link) which can be isotoped so that the natural height function given by the z -coordinate has only two maxima and two minima as critical points.

Definition A.1. A profinite knot (resp. link) is called a *two-bridge profinite knot* (resp. *link*) when it is isotopic to the presentation

$$a_{0,0}^{\epsilon_1} \cdot a_{0,2}^{\epsilon_2} \cdot b_4^{\epsilon_3} \cdot c_{1,1}^{\epsilon_4} \cdot c_{0,0}^{\epsilon_5} \quad \text{with } b_4 \in \widehat{B}_4,$$

which is depicted in Figure A.1 with its orientation ignored.

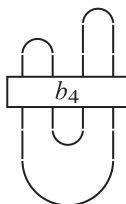


Figure A.1. Two-bridge profinite knot.

We note that each isotopy class of two-bridge (topological) knot naturally determines an isotopy class of two-bridge profinite knot.

We recall that a two-bridge (topological) knot is also called as a *rational knot* because each isotopy class of two-bridge knot (or link) is characterized by a rational number via its continued fraction expansion (cf. [25]). As its profinite analogue, it is interesting to see if the set of isotopy classes of two-bridge profinite knots can be also parametrized by any numbers or not.

We denote the subgroup of $G\widehat{\mathcal{K}}$ generated by two-bridge profinite knots by $Br(2)$.

Proposition A.2. *The subgroup $Br(2)$ is stable under our Galois action.*

Proof. It can be verified by showing that $\mathcal{Br}(2)$ is stable by our $\widehat{\text{GT}}$ -action. Let $\sigma = (\lambda, f) \in \widehat{\text{GT}}$ and let K be a two-bridge profinite knot with the above presentation. Then

$$\begin{aligned} \sigma\left(\frac{K}{\mathcal{O}}\right) &= \frac{\sigma(K)}{\sigma(\mathcal{O})} \\ &= \frac{\{\sigma(a_{0,0}^{\epsilon_1}) \cdot \sigma(a_{0,2}^{\epsilon_2}) \cdot \sigma(b_4^{\epsilon_3}) \cdot \sigma(c_{1,1}^{\epsilon_4}) \cdot \sigma(c_{0,0}^{\epsilon_5})\} \# \Lambda_f}{(\Lambda_f) \# 2} \\ &= \frac{a_{0,0}^{\epsilon_1} \cdot a_{0,2}^{\epsilon_2} \cdot (\sigma(b_4^{\epsilon_3}) \cdot f_{123}^{-1}) \cdot c_{1,1}^{\epsilon_4} \cdot c_{0,0}^{\epsilon_5}}{\Lambda_f}. \end{aligned}$$

Since both numerator and denominator of the last term are two-bridge knots and $\widehat{\text{GT}}$ acts on $G\widehat{\mathcal{K}}$ as a group automorphism, our claim is obtained. \square

Considering profinite analogues of other types of (topological) knots such as hyperbolic knots, torus knots, etc. and considering their Galois behaviors might be interesting problems.

B. Profinite framed knots

We will extend our definition of profinite analogue of knots into that of framed knots and show that the absolute Galois group acts on the group generated by them. The construction is given in a same way to our arguments in §2.

Definition B.1. The set $\widehat{\mathcal{FT}}$ of *profinite framed tangles* (isotopy classes of profinite framed tangle diagrams) is defined to be the set of profinite tangle diagrams divided by the *framed isotopy*, the equivalence given by a finite number of the moves (FT1)–(FT6). Here (FT1)–(FT5) are same to (T1)–(T5) given in §2 while (FT6) is given below. The set $\widehat{\mathcal{FK}}$ (resp. $\widehat{\mathcal{FL}}$) of *profinite framed knots* (resp. *links*) is the subset of $\widehat{\mathcal{FT}}$ which consists of framed isotopy classes of profinite knot (resp. link) diagrams.

(FT6) *First framed Reidemeister move:* for $c \in \widehat{\mathbb{Z}}$, $c_{k,l+2}^{\epsilon_1}, c_{k+2,l}^{\epsilon_2} \in C$, $a_{k+1,l+1}^{\epsilon_3} \in A$ and $\sigma_{k+1}^{\epsilon'_1}, \sigma_{k+3}^{\epsilon'_2} \in \widehat{B}$ representing $\sigma_{k+1} \in \widehat{B}_{k+l+2}$ and $\sigma_{k+3} \in \widehat{B}_{k+l+4}$ such that the sequence of the left hand side is consistent,

$$a_{k+1,l+1}^{\epsilon_3} \cdot (\sigma_{k+3}^{\epsilon'_2})^{-c} \cdot c_{k+2,l}^{\epsilon_2} \cdot (\sigma_{k+1}^{\epsilon'_1})^c \cdot c_{k,l}^{\epsilon_1} = c_{k,l}^{\epsilon'_3}. \tag{B.1}$$

Here ϵ'_3 is chosen to be $t(a_{k+1,l+1}^{\epsilon_3}) = t(c_{k,l}^{\epsilon'_3})$.

For $c \in \widehat{\mathbb{Z}}$, $c_{k+1,l+1}^{\epsilon_1} \in C$, $a_{k,l}^{\epsilon_2}, a_{k+2,l}^{\epsilon_3} \in A$ and $\sigma_{k+1}^{\epsilon'_2}, \sigma_{k+3}^{\epsilon'_3} \in \widehat{B}$ representing $\sigma_{k+1} \in \widehat{B}_{k+l+2}$ and $\sigma_{k+3} \in \widehat{B}_{k+l+4}$ such that the sequence of the left hand side is consistent,

$$a_{k,l}^{\epsilon_2} \cdot (\sigma_{k+1}^{\epsilon'_2})^{-c} \cdot a_{k+2,l}^{\epsilon_3} \cdot (\sigma_{k+3}^{\epsilon'_3})^c \cdot c_{k+1,l+1}^{\epsilon_1} = a_{k,l}^{\epsilon'_1}. \tag{B.2}$$

Here again ϵ'_1 is chosen to be $s(c_{k+1,l+1}^{\epsilon_3}) = s(a_{k,l}^{\epsilon'_1})$.

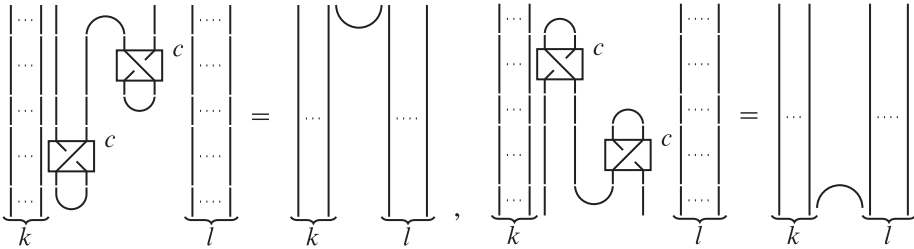


Figure B.1. (FT6): First framed Reidemeister move.

We note that, in the first (resp. second) equation, $c_{k,l}^{\epsilon_1} = c_{k,l}^{\epsilon'_3}$ (resp. $a_{k,l}^{\epsilon_2} = a_{k,l}^{\epsilon'_1}$) if and only if $c \equiv 0 \pmod{2}$. It is easy to see that our previous (T6) implies (FT6).

Remark B.2. There is a natural projection $\widehat{\mathcal{F}\mathcal{T}} \rightarrow \widehat{\mathcal{T}}$ and hence

$$\widehat{\mathcal{F}\mathcal{K}} \longrightarrow \widehat{\mathcal{K}}. \tag{B.3}$$

It is clear by definition.

Definition B.3. We introduce special tangles, *twists*, φ^\uparrow and φ^\downarrow by

$$\varphi^\uparrow := a_{1,0}^{\uparrow\curvearrowright} \cdot \sigma_1^{\uparrow\uparrow\downarrow} \cdot c_{1,0}^{\uparrow\curvearrowleft}, \quad \varphi^\downarrow := a_{1,0}^{\downarrow\curvearrowleft} \cdot \sigma_1^{\downarrow\downarrow\uparrow} \cdot c_{1,0}^{\downarrow\curvearrowright}.$$

The picture of φ^\uparrow is depicted in the first term of Figure B.2.

It is an easy exercise to derive the lemma below from (FT1)–(FT6).

Lemma B.4. (1) *The following equalities hold (cf. Figure B.2):*

$$\varphi^\uparrow = a_{0,1}^{\curvearrowright\uparrow} \cdot \sigma_2^{\downarrow\uparrow\uparrow} \cdot c_{0,1}^{\curvearrowleft\uparrow} = a_{0,1}^{\curvearrowright\uparrow} \cdot (\sigma_2^{-1})^{\uparrow\uparrow\downarrow} \cdot c_{1,0}^{\uparrow\curvearrowleft} = a_{0,1}^{\curvearrowright\uparrow} \cdot (\sigma_1^{-1})^{\uparrow\downarrow\uparrow} \cdot c_{1,0}^{\uparrow\curvearrowleft}.$$

(2) *We have the following equalities (cf. Figure B.3)*

$$(\varphi^\downarrow \otimes \uparrow) \cdot \curvearrowright = (\downarrow \otimes \varphi^\uparrow) \cdot \curvearrowright, \quad \curvearrowleft \cdot (\varphi^\uparrow \otimes \downarrow) = \curvearrowleft \cdot (\uparrow \otimes \varphi^\downarrow).$$

We also have the same equalities obtained by reversing all arrows.

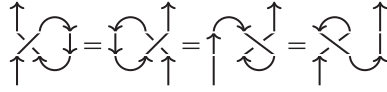


Figure B.2. twist φ^\uparrow .

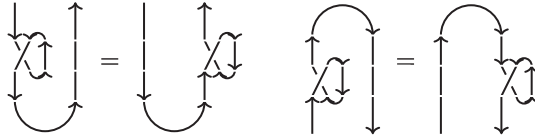


Figure B.3. Lemma B.4.(2).

The space $\widehat{\mathcal{FK}}$ (hence the subspace $\widehat{\mathcal{FK}}$) carries a structure of topological space by profinite topology on \widehat{B}_n ($n = 1, 2, \dots$) and the discrete topology on A and on C .

We have a framed version of our Theorem A in §0.

Theorem B.5. (1) *The space $\widehat{\mathcal{FK}}$ carries a structure of a topological commutative monoid whose product structure*

$$\sharp^{\text{fr}}: \widehat{\mathcal{FK}} \times \widehat{\mathcal{FK}} \longrightarrow \widehat{\mathcal{FK}}.$$

is given by the connected sum (2.1) for $K_1 = \alpha_m \cdots \alpha_1$ and $K_2 = \beta_n \cdots \beta_1$.

(2) *Let \mathcal{FK} denote the set of framed isotopy classes of (topological) oriented framed knots.¹² Then there is a natural map*

$$h^{\text{fr}}: \mathcal{FK} \longrightarrow \widehat{\mathcal{FK}}.$$

The map is with dense image and is a monoid homomorphism with respect to the connected sum.

The map h^{fr} should be conjectured to be injective by the same reason to Conjecture 2.12.

Proof. The proof can be done in an almost same way to that of Theorems 2.11 and 2.16. We outline it by strengthening a slight difference.

¹² Framed knot means a knot equipped with a framing, that is, a nonzero normal vector field on it.

(1) Each framed isotopy class of profinite knot contains a profinite knot $K = \gamma_m \cdots \gamma_1$ for some m with $(\gamma_m, \gamma_1) = (\curvearrowright, \curvearrowleft)$ by (FT6) with $c \equiv 1 \pmod{2}$. Hence the connected sum could extend to $\widehat{\mathcal{FK}}$ once we have the well-definedness. To show the well-definedness, it is enough to show that $K_1 \# K_2$ is framed isotopic to $K'_1 \# K_2$ when K'_1 is obtained from K_1 by operating (FT3) or (FT6) on α_1 by a similar reason to the proof of Theorem 2.16. Consider the latter case (FT6). It suffices to show that $K_1 \# K_2$ is framed isotopic to $K_3 = \alpha_m \cdots \alpha_2 \cdot a_{1,1} \cdot \sigma_3^{-2c} \cdot c_{2,0} \cdot \sigma_1^{2c} \cdot \beta_{n-1} \cdots \beta_1$ for $c \in \widehat{\mathbb{Z}}$ (cf. Figure B.4). It can be shown in a similar way to Figure 2.17.

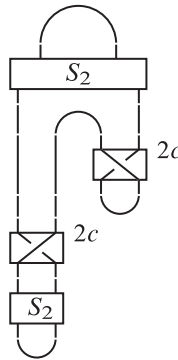


Figure B.4. K_3 .

Next consider the former case (FT3). Since K_1 is a profinite knot, m_1 and m_2 are both 0. By the above argument in case (FT6), we may assume that both T_1 and T_2 in Figure 2.11 should start from \curvearrowleft (i.e. $\alpha_1 = \beta_1 = \curvearrowleft$). Define T as in Figure 2.18. A successive application of commutativity of profinite braids with T shown in (FT4) and that of creations and annihilations with T shown in Lemma B.6 lead the framed isotopy equivalence shown in Figure 2.19. Hence we get the the well-defined product $\#^{\text{fir}}$.

The proofs of associativity, commutativity, continuity for $\#^{\text{fir}}$ can be done in a quite same way to the proof of Theorem 2.16.

(2) The result in [1] indicates that the set \mathcal{FT} of (framed) isotopy classes of framed tangles is described by the set of consistent finite sequences of fundamental tangles, elements of A , B and C modulo the (discrete) framed Turaev moves, which is equivalent to the moves replacing profinite tangles and braids by (discrete) tangles and braids in (FT1)–(FT6) and $c \in \widehat{\mathbb{Z}}$ by $c \in \mathbb{Z}$ in (FT6). Because we have a natural map $B_n \rightarrow \widehat{B}_n$ and the Turaev moves are special case of our 6 moves, we have a natural map $\mathcal{FT} \rightarrow \widehat{\mathcal{FT}}$, which yields our map h^{fir} . It is easy to see that it is homomorphic and is with dense image. \square

A framed analogue of Lemma 2.17 and 2.18 hold.

Lemma B.6. *Let T be a profinite tangle T with $s(T) = t(T) = \uparrow$.*

(1) *Its transpose \mathcal{L} (see Lemma 2.17) is equal to*

$$a_{1,0}^{\downarrow \curvearrowright} \cdot (e_1^{\downarrow} \otimes T \otimes e_1^{\downarrow}) \cdot c_{0,1}^{\curvearrowleft \downarrow}$$

(cf. Figure 2.21).

(2) *The equalities in Figure 2.23 hold for T .*

The same claim also holds for a profinite tangle T with $s(T) = t(T) = \downarrow$ by reversing all arrows.

Proof. (1) A proof is depicted in Figure B.5. We use (FT6) in the first equality. The second and the fourth equalities follow from Lemma B.4.(1). The third equality is obtained by the ‘commutativity’ of twists with annihilation, creation and profinite braids assured by Lemma B.4.(2) and (FT4). The fifth equation follows from (FT2) and (FT4).

(2) It is a direct consequence of (1). □

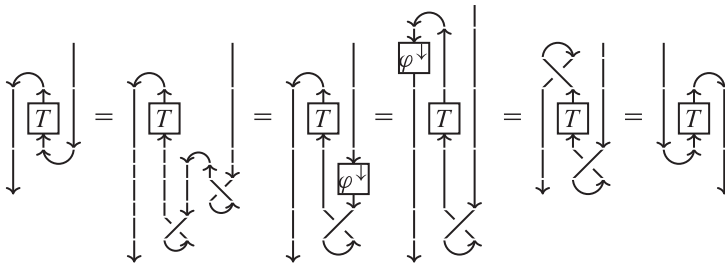


Figure B.5. Proof of Lemma B.6.(1).

Definition B.7. *The group of profinite framed knots $\text{Frac } \widehat{\mathcal{FK}}$ is defined to be the group of fraction of the monoid $\widehat{\mathcal{FK}}$ as in the same way to Definition 2.35.*

We also have a framed version of our Theorem B in §0.

Theorem B.8. *Fix an embedding from the algebraic closure $\overline{\mathbb{Q}}$ of the rational number field \mathbb{Q} into the complex number field \mathbb{C} . Then the group $\text{Frac } \widehat{\mathcal{FK}}$ of profinite knots admits a non-trivial topological $G_{\mathbb{Q}}$ -module structure. Namely, there is a non-trivial continuous Galois representation on profinite knots*

$$\rho_0^{\text{fr}}: G_{\mathbb{Q}} \longrightarrow \text{Aut } \text{Frac } \widehat{\mathcal{FK}}.$$

Proof. This is achieved by establishing a consistent continuous action of the profinite Grothendieck–Teichmüller group $\widehat{\text{GT}}$ on $\text{Frac } \widehat{\mathcal{FK}}$ and using the inclusion from $G_{\mathbb{Q}}$ into $\widehat{\text{GT}}$.

We introduce this $\widehat{\text{GT}}$ -action on $\text{Frac } \widehat{\mathcal{FK}}$ by using the action on each profinite fundamental tangle given in Definition 2.39. The proof of its well-definedness, i.e. to check that the action preserves (FT1)–(FT6), can be done in the same way to that of Theorem 2.41 but for (FT6), which is proved below. Let $\sigma = (\lambda, f) \in \widehat{\text{GT}}$. Let r_1 and r_2 be two profinite knots. Assume that r_1 is obtained from r_2 by a single operation of the move (FT6). Then by Proposition 1.16 and Proposition 2.42, we have

$$\begin{aligned}
& \sigma(a_{k,l}^{\epsilon_2}) \cdot \sigma((\sigma_{k+1}^{\epsilon_2'})^{-c}) \cdot \sigma(a_{k+2,l}^{\epsilon_3}) \cdot \sigma((\sigma_{k+3}^{\epsilon_3'})^c) \cdot \sigma(c_{k+1,l+1}^{\epsilon_1}) \\
&= a_{k,l}^{\epsilon_2} \cdot f_{1\dots k,k+1,k+2}^{s(a_{k,l}^{\epsilon_2})} \cdot f_{1\dots k,k+1,k+2}^{-1,s(a_{k,l}^{\epsilon_2})} \cdot (\sigma_{k+1}^{\epsilon_2'})^{-c\lambda} \cdot f_{1\dots k,k+1,k+2}^{t(a_{k+2,l}^{\epsilon_3})} \\
&\quad \cdot a_{k+2,l}^{\epsilon_3} \cdot f_{1\dots k+2,k+3,k+4}^{s(a_{k+2,l}^{\epsilon_3})} \cdot f_{1\dots k+2,k+3,k+4}^{-1,s(a_{k+2,l}^{\epsilon_3})} \cdot (\sigma_{k+3}^{\epsilon_3'})^{c\lambda} \\
&\quad \cdot f_{1\dots k+2,k+3,k+4}^{t(c_{k+1,l+1}^{\epsilon_1})} \cdot f_{1\dots k+1,k+2,k+3}^{-1,t(c_{k+1,l+1}^{\epsilon_1})} \cdot c_{k+1,l+1}^{\epsilon_1} \\
&= a_{k,l}^{\epsilon_2} \cdot (\sigma_{k+1}^{\epsilon_2'})^{-c\lambda} \cdot f_{1\dots k,k+1,k+2}^{t(a_{k+2,l}^{\epsilon_3})} \cdot a_{k+2,l}^{\epsilon_3} \cdot (\sigma_{k+3}^{\epsilon_3'})^{c\lambda} \cdot f_{1\dots k+2,k+3,k+4}^{t(c_{k+1,l+1}^{\epsilon_1})} \\
&\quad \cdot f_{1\dots k+1,k+2,k+3}^{-1,t(c_{k+1,l+1}^{\epsilon_1})} \cdot c_{k+1,l+1}^{\epsilon_1} \\
&= a_{k,l}^{\epsilon_2} \cdot (\sigma_{k+1}^{\epsilon_2'})^{-c\lambda} \cdot a_{k+2,l}^{\epsilon_3} \cdot (\sigma_{k+3}^{\epsilon_3'})^{c\lambda} \\
&\quad \cdot f_{1\dots k+2,k+3,k+4}^{t(c_{k+1,l+1}^{\epsilon_1})} \cdot f_{1\dots k,k+1,k+2}^{t(c_{k+1,l+1}^{\epsilon_1})} \cdot f_{1\dots k+1,k+2,k+3}^{-1,t(c_{k+1,l+1}^{\epsilon_1})} \cdot c_{k+1,l+1}^{\epsilon_1}.
\end{aligned}$$

(by a successive application of (1.4))

$$\begin{aligned}
&= a_{k,l}^{\epsilon_2} \cdot (\sigma_{k+1}^{\epsilon_2'})^{-c\lambda} \cdot a_{k+2,l}^{\epsilon_3} \cdot (\sigma_{k+3}^{\epsilon_3'})^{c\lambda} \\
&\quad \cdot f_{1\dots k,k+1,k+2}^{t(c_{k+1,l+1}^{\epsilon_1})} \cdot f_{1\dots k+1,k+2,k+3}^{-1,t(c_{k+1,l+1}^{\epsilon_1})} \cdot f_{k+2,k+3,k+4}^{t(c_{k+1,l+1}^{\epsilon_1})} \cdot c_{k+1,l+1}^{\epsilon_1}
\end{aligned}$$

(By (FT4) and Lemma 1.8)

$$\begin{aligned}
&= a_{k,l}^{\epsilon_2} \cdot (\sigma_{k+1}^{\epsilon_2'})^{-c\lambda} \\
&\quad \cdot f_{1\dots k,k+1,k+2}^{t(a_{k+2,l}^{\epsilon_3})} \cdot a_{k+2,l}^{\epsilon_3} \cdot (\sigma_{k+3}^{\epsilon_3'})^{c\lambda} \cdot f_{k+2,k+3,k+4}^{t(c_{k+1,l+1}^{\epsilon_1})} \cdot c_{k+1,l+1}^{\epsilon_1}.
\end{aligned}$$

By Lemma B.10, twists $(\varphi^\uparrow)^c$ and $(\varphi^\downarrow)^c$ ($c \in \widehat{\mathbb{Z}}$) ‘commute’ with profinite tangles. Thus the previous formula equals

$$a_{k,l}^{\epsilon_2} \cdot (\sigma_{k+1}^{\epsilon'_2})^{-c\lambda} \cdot a_{k+2,l}^{\epsilon_3} \cdot (\sigma_{k+3}^{\epsilon'_3})^{c\lambda} \cdot c_{k+1,l+1}^{\epsilon_1} \cdot f_{1\dots k,k+1,k+2}^{s(a_{k,l}^{\epsilon'_1})} \cdot a_{k+2,l}^{\epsilon_0} \cdot f_{k+2,k+3,k+4}^{s(a_{k+2,l}^{\epsilon_0})} \cdot c_{k+1,l+1}^{\epsilon'_0}$$

with appropriate orientations ϵ_0 and ϵ'_0 . Hence it equals

$$a_{k,l}^{\epsilon'_1} \cdot f_{1\dots k,k+1,k+2}^{s(a_{k,l}^{\epsilon'_1})} \cdot a_{k+2,l}^{\epsilon_0} \cdot f_{k+2,k+3,k+4}^{s(a_{k+2,l}^{\epsilon_0})} \cdot c_{k+1,l+1}^{\epsilon'_0} = \sigma(a_{k,l}^{\epsilon'_1}) \cdot a_{k+2,l}^{\epsilon_0} \cdot f_{k+2,k+3,k+4}^{s(a_{k+2,l}^{\epsilon_0})} \cdot c_{k+1,l+1}^{\epsilon'_0}$$

Therefore we have $\sigma(r_1) = \sigma(r_2) \# \Lambda_f$. Because $\alpha(r_1) = \alpha(r_2) + 1$, we may say that (B.2) is preserved by the $\widehat{\text{GT}}$ -action. The proof for (B.1) can be done in the same way. Thus the first framed Reidemeister move (FT6) is preserved by the $\widehat{\text{GT}}$ -action.

The rest part of our claim; continuity, non-triviality, etc, was shown in the proof of Theorem 2.41.(2)–(5). □

Lemma B.9. *The map sending $c \in \mathbb{N}$ to the c -th power $(\varphi^\uparrow)^c \in \widehat{\mathcal{FT}}$ of φ^\uparrow extends additively and continuously into $c \in \widehat{\mathbb{Z}}$ and it is given by*

$$(\varphi^\uparrow)^c = a_{0,1} \cdot (\sigma_1^{-c})^{\uparrow\downarrow\uparrow} \cdot c_{1,0}^{\uparrow\curvearrowright}. \tag{B.4}$$

with $a_{0,1} = a_{0,1}^{\curvearrowright\uparrow}$ or $a_{0,1}^{\curvearrowleft\uparrow}$ according to $c \equiv 1$ or $0 \pmod{2}$ (cf. Figure B.6).

The same statement also holds for φ^\downarrow .

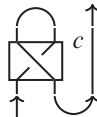


Figure B.6. $(\varphi^\uparrow)^c$.

Proof. Equation (B.4) for $c \in \mathbb{N}$ can be shown by induction. Assume the validity for c with $c \equiv 1 \pmod{2}$. Then

$$\begin{aligned} (\varphi^\uparrow)^{c+1} &= (\varphi^\uparrow)^c \cdot \varphi^\uparrow \\ &= a_{0,1}^{\curvearrowright \uparrow} \cdot (\sigma_1^{-c})^{\uparrow \downarrow \uparrow} \cdot c_{1,0}^{\uparrow \cup} \cdot \varphi^\uparrow, \end{aligned}$$

(by Lemma B.4.(2))

$$= a_{0,1}^{\curvearrowright \uparrow} \cdot (e_\downarrow \otimes \varphi^\uparrow \otimes e_\uparrow) \cdot (\sigma_1^{-c})^{\uparrow \downarrow \uparrow} \cdot c_{1,0}^{\uparrow \cup}$$

(by Lemma B.4.(1))

$$= a_{0,1}^{\curvearrowright \uparrow} \cdot (\sigma_1^{-c-1})^{\uparrow \downarrow \uparrow} \cdot c_{1,0}^{\uparrow \cup}.$$

The case for c with $c \equiv 0 \pmod{2}$ can be done in the same way. By the expression of (B.4), it is immediate to see that $(\varphi^\uparrow)^c$ consistently extends to the case for $c \in \widehat{\mathbb{Z}}$. □

As an analogue of Lemma B.4 (2), we have

Lemma B.10. *We have the following equalities*

$$((\varphi^\downarrow)^c \otimes \uparrow) \cdot \cup = (\downarrow \otimes (\varphi^\uparrow)^c) \cdot \cup, \quad \curvearrowright \cdot ((\varphi^\uparrow)^c \otimes \downarrow) = \curvearrowright \cdot (\uparrow \otimes (\varphi^\downarrow)^c)$$

for $c \in \widehat{\mathbb{Z}}$. We also have the same equalities obtained by reversing all arrows.

Proof. A proof is depicted in Figure B.7. We note that we use Lemma B.4.(1) and (FT6) to deduce the second and the third equality respectively. □

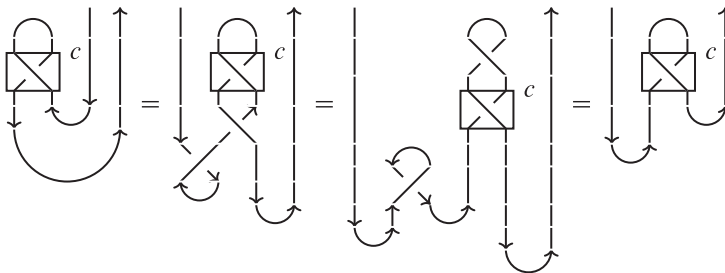


Figure B.7. Proof of Lemma B.10.

We note that the $G_{\mathbb{Q}}$ -module structure on the group $\text{Frac } \widehat{\mathcal{K}}$ of profinite knots given in Theorem B in §0 is induced from that on the group $\text{Frac } \widehat{\mathcal{FK}}$ of profinite framed knots by (B.3).

Acknowledgements. The author would like to express his profound gratitude to Masanori Morishita who persistently encouraged this research. His thanks are also directed to Hiroshi Fujiwara who patiently helped him to draw many graphics in this paper, Hitoshi Murakami, Toshitake Kohno and Tetsuya Ito who generously explained him lots of valuable issues in low dimensional topology related to this paper. Part of the paper was written at Max Planck Institute for Mathematics. He also thanks the institute for its hospitality. This work was supported by Grant-in-Aid for Young Scientists (A) 24684001 and Daiko Foundation. The efforts of the referees to make this paper better are gratefully acknowledged.

References

- [1] D. Bar-Natan, Non-associative tangles. In W. H. Kazez (ed.), *Geometric topology*. Proceedings of the 1993 Georgia International Topology Conference held at the University of Georgia, Athens, GA, August 2–13, 1993. AMS/IP Studies in Advanced Mathematics, 2.1. American Mathematical Society, Providence, R.I., and International Press, Cambridge, MA, 1997, 139–183. [MR 1470726](#) [Zbl 1146.22009](#)
- [2] D. Bar-Natan, Vassiliev homotopy string link invariants. *J. Knot Theory Ramifications* **4** (1995), no. 1, 13–32. [MR 1321289](#) [Zbl 0878.57003](#)
- [3] G. V. Belyĭ, Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), no. 2, 267–276, 479. [MR 0534593](#) [Zbl 0409.12012](#)
- [4] J. S. Birman, *Braids, links, and mapping class groups*. Annals of Mathematics Studies, 82. Princeton University Press, Princeton, N.J., and University of Tokyo Press, Tokyo, 1974. [MR 0375281](#) [Zbl 0305.57013](#)
- [5] S. Chmutov, S. Duzhin, J. and Mostovoy, *Introduction to Vassiliev knot invariants*. Cambridge University Press, Cambridge, 2012. [MR 2962302](#) [Zbl 1245.57003](#)
- [6] P. Deligne, Le groupe fondamental de la droite projective moins trois points. In Y. Ihara, K. Ribet, and J.-P. Serre (eds.), *Galois groups over \mathbf{Q}* . Papers from the Workshop on Galois Groups over \mathbf{Q} and Related Topics held in Berkeley, California, March 23–27, 1987. Mathematical Sciences Research Institute Publications, 16. Springer-Verlag, New York etc., 1989, 79–297. [MR 1012168](#) [Zbl 0742.14022](#)
- [7] V. G. Drinfel'd, Quasi-Hopf algebras. *Algebra i Analiz* **1** (1989), no. 6, 114–148. In Russian. English translation, *Leningrad Math. J.* **1** (1990), no. 6, 1419–1457. [MR 1047964](#) [Zbl 0718.16033](#)
- [8] V. G. Drinfel'd, On quasitriangular quasi-Hopf algebras and a group closely connected with $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. *Algebra i Analiz* **2** (1990), no. 4, 149–181. In Russian. English translation, *Leningrad Math. J.* **2** (1991), no. 4, 829–860. [MR 1080203](#) [Zbl 0728.16021](#)

- [9] H. Furusho, Pentagon and hexagon equations. *Ann. of Math.* (2) **171** (2010), no. 1, 545–556. [MR 2630048](#) [Zbl 1257.17019](#)
- [10] H. Furusho, Galois action on knots II: Proalgebraic string links and knots. To appear in *Periods in quantum field theory and arithmetic*. Preprint 2014. [arXiv:1405.4575](#) [math.QA]
- [11] T. Gannon and M. A. Walton, Galois relations on knot invariants. *Lett. Math. Phys.* **38** (1996), no. 2, 185–194. [Zbl 1403070](#) [MR 0857.57003](#)
- [12] A. Grothendieck (ed.), *Revêtements étales et groupe fondamental*. Séminaire de géométrie algébrique du Bois Marie 1960/61 (SGA 1). Exposés I à XIII. Augmenté de deux exposés de M. Raynaud. Lecture Notes in Mathematics. 224. Springer-Verlag, Berlin etc., 1971. [Zbl 0234.14002](#)
- [13] A. Grothendieck, Esquisse d’un programme. In L. Schneps and P. Lochak (eds.), *Geometric Galois actions*. 1. Around Grothendieck’s “Esquisse d’un programme.” London Mathematical Society Lecture Note Series, 242. Cambridge University Press, Cambridge, 1997, 5–48; English translation on *ibid.*, 243–283. [MR 1483107](#) [Zbl 0901.14001](#)
- [14] N. Habegger and X.-S. Lin, The classification of links up to link-homotopy. *J. Amer. Math. Soc.* **3** (1990), no. 2, 389–419. [MR 1026062](#) [Zbl 0704.57016](#)
- [15] R. Hain, The Hodge de Rham theory of relative Malcev completion. *Ann. Sci. École Norm. Sup.* (4) **31** (1998), no. 1, 47–92. [MR 1604294](#) [Zbl 0911.14008](#)
- [16] Y. Ihara, Braids, Galois groups, and some arithmetic functions. In I. Satake (ed.), *Proceedings of the International Congress of Mathematicians*. Vol. I. Held in Kyoto, August 21–29, 1990. Mathematical Society of Japan, Tokyo, and Springer-Verlag, Tokyo etc., 1991, 99–120. [MR 1159208](#) [Zbl 0757.20007](#)
- [17] Y. Ihara, On the embedding of $\text{Gal}(\widehat{\mathbf{Q}}/\mathbf{Q})$ into $\widehat{\text{GT}}$. With an appendix: “The action of the absolute Galois group on the moduli space of spheres with four marked points” by M. Emsalem and P. Lochak. In L. Schneps (ed.), *The Grothendieck theory of dessins d’enfants*. Papers from the Conference on Dessins d’Enfant held in Luminy, April 19–24, 1993. London Mathematical Society Lecture Note Series, 200. Cambridge University Press, Cambridge, 1994, 289–321. [MR 1305402](#) [Zbl 0849.12005](#)
- [18] Y. Ihara and M. Matsumoto, On Galois actions on profinite completions of braid groups. In M. D. Fried, Sh. S. Abhyankar, W. Feit, Y. Ihara, and H. Voelklein (eds.) *Recent developments in the inverse Galois problem*. Contemporary Mathematics, 186. American Mathematical Society, Providence, R.I., 1995, 173–200. [MR 1352271](#) [Zbl 1352262](#)
- [19] A. Joyal and R. Street, Braided tensor categories. *Adv. Math.* **102** (1993), no. 1, 20–78. [MR 1250465](#) [MR 0817.18007](#)
- [20] M. M. Kapranov, Analogies between the Langlands correspondence and topological quantum field theory. In S. Gindikin, J. Lepowsky, and R. L. Wilson (eds.), *Functional analysis on the eve of the 21st century*. Vol. I. In honor of the eightieth birthday of I. M. Gel’fand. Papers from the conference held at Rutgers University, New Brunswick, New Jersey, October 24–27, 1993. Progress in Mathematics, 131. Birkhäuser Boston, Boston, MA, 1995, 119–151. [MR 1373001](#) [Zbl 0858.11062](#)

- [21] C. Kassel, *Quantum groups*. Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995. [MR 1321145](#) [Zbl 0808.17003](#)
- [22] C. Kassel and V. G. Turaev, Chord diagram invariants of tangles and graphs. *Duke Math. J.* **92** (1998), no. 3, 497–552. [MR 1620522](#) [Zbl 0947.57010](#)
- [23] M. Kontsevich, Vassiliev’s knot invariants. In S. M. Gel’fand and S. Gindikin (ed.), *I. M. Gelfand Seminar*. Advances in Soviet Mathematics, 16, Part 2. American Mathematical Society, Providence, R.I., 1993, 137–150. [MR 1237836](#) [Zbl 0839.57006](#)
- [24] T. T. Q. Le and J. Murakami, The universal Vassiliev–Kontsevich invariant for framed oriented links. *Compositio Math.* **102** (1996), no. 1, 41–64. [MR 1394520](#) [Zbl 0851.57007](#)
- [25] W. B. R. Lickorish, *An introduction to knot theory*. Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997. [MR 1472978](#) [Zbl 0886.57001](#)
- [26] X.-S. Lin, Power series expansions and invariants of links. In W. H. Kazez (ed.), *Geometric topology*. Proceedings of the 1993 Georgia International Topology Conference held at the University of Georgia, Athens, GA, August 2–13, 1993. AMS/IP Studies in Advanced Mathematics, 2.1. American Mathematical Society, Providence, R.I., and International Press, Cambridge, MA, 1997, 184–202. [MR 1470727](#) [Zbl 0897.57006](#)
- [27] P. Lochak and L. Schneps, The Grothendieck–Teichmüller group and automorphisms of braid groups. In L. Schneps (ed.), *The Grothendieck theory of dessins d’enfants*. Papers from the Conference on Dessins d’Enfant held in Luminy, April 19–24, 1993. London Mathematical Society Lecture Note Series, 200. Cambridge University Press, Cambridge, 1994, 323–358. [MR 1305403](#) [Zbl 0827.20050](#)
- [28] B. Mazur, Primes, knots and Po. Lecture note from the conference “Geometry, Topology, and Group Theory” in honor of the 80th birthday of V. Poenaru. Autrans, July 1–6, 2012. <http://www.math.harvard.edu/~mazur/papers/Po8.pdf>
- [29] B. Milnor, Link groups. *Ann. of Math.* (2) **59**, (1954). 177–195. [MR 0071020](#) [Zbl 0055.16901](#)
- [30] B. Milnor, Isotopy of links. In Fox, R.H. (ed.) D. C. Spencer and A. W. Tucker (eds.), *Algebraic geometry and topology*. A symposium in honor of S. Lefschetz. Princeton University Press, Princeton, N.J., 1957, 280–306. [MR 0092150](#) [Zbl 0080.16901](#)
- [31] M. Morishita, *Knots and primes*. An introduction to arithmetic topology. Based on the Japanese original (Springer, 2009). Universitext. Springer, Berlin etc., 2012. [MR 2905431](#) [Zbl 1267.57001](#)
- [32] T. Ohtsuki, *Quantum invariants*. A study of knots, 3-manifolds, and their sets. Series on Knots and Everything, 29. World Scientific Publishing Co., River Edge, N.J., 2002. [MR 1881401](#) [Zbl 0991.57001](#)
- [33] Ohtsuki, Problems on invariants of knots and 3-manifolds. With an introduction by J. Roberts. In T. Ohtsuki, T. Kohno, T. Le, J. Murakami, J. Roberts, and V. G. Turaev, (eds.), *Invariants of knots and 3-manifolds*. Proceedings of the workshop, Kyoto, Japan, September 17–21, 2001. Geometry and Topology Monographs. 4. Geometry and Topology Publications, Coventry, 2002, 377–572. [MR 2065029](#) [Zbl 1163.57302](#)

- [34] S. Piunikhin, Combinatorial expression for universal Vassiliev link invariant. *Comm. Math. Phys.* **168** (1995), no. 1, 1–22. [MR 1324388](#) [Zbl 0828.57007](#)
- [35] A. Reznikov, Three-manifolds class field theory (homology of coverings for a nonvirtually b_1 -positive manifold). *Selecta Math. (N.S.)* **3** (1997), no. 3, 361–399. [MR 1481134](#) [Zbl 0892.57012](#)
- [36] L. Ribes and P. Zalesskii, *Profinite groups*. Second edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 40*. Springer-Verlag, Berlin, 2010. [MR 2599132](#) [Zbl 1197.20022](#)
- [37] L. Schneps, The Grothendieck–Teichmüller group \widehat{GT} : a survey. In L. Schneps and P. Lochak (eds.), *Geometric Galois actions. 1. Around Grothendieck’s “Esquisse d’un programme.”* London Mathematical Society Lecture Note Series, 242. Cambridge University Press, Cambridge, 1997, 183–203. [MR 1483118](#) [Zbl 0910.20019](#)
- [38] P. Traczyk, A new proof of Markov’s braid theorem. In V. F. R. Jones, J. Kania-Bartoszyńska, J. H. Przytycki, P. Traczyk, and V. G. Turaev (eds.), *Banach Center Publications, 42*. Polish Academy of Sciences, Institute of Mathematics, Warsaw, 1998, 409–419. [MR 1634469](#) [Zbl 0901.57018](#)
- [39] V. G. Turaev, Operator invariants of tangles, and R -matrices. *Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989), no. 5, 1073–1107, 1135. In Russian. English translation, *Math. USSR-Izv.* **35** (1990), no. 2, 411–444 [MR 1024455](#) [Zbl 0707.57003](#)
- [40] P. Vogel, Representation of links by braids: a new algorithm. *Comment. Math. Helv.* **65** (1990), no. 1, 104–113. [MR 1036132](#) [Zbl 0703.57004](#)
- [41] S. Yamada, The minimal number of Seifert circles equals the braid index of a link. *Invent. Math.* **89** (1987), no. 2, 347–356. [MR 0894383](#) [Zbl 0634.57004](#)

Received October 26, 2014

Hidekazu Furusho, Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan

e-mail: furusho@math.nagoya-u.ac.jp