Quantum Topol. 8 (2017), 361–379
DOI 10.4171/QT/92

Quantum Topology

Fourier transform for quantum D-modules via the punctured torus mapping class group

Adrien Brochier and David Jordan[1]

Abstract. We construct a certain cross product of two copies of the braided dual $\widetilde{H}$ of a quasitriangular Hopf algebra H, which we call the elliptic double E_H, and which we use to construct representations of the punctured elliptic braid group extending the well-known representations of the planar braid group attached to H. We show that the elliptic double is the universal source of such representations. We recover the representations of the punctured torus braid group obtained in [15], and hence construct a homomorphism from E_H to the Heisenberg double D_H, which is an isomorphism if H is factorizable.

The universal property of E_H endows it with an action by algebra automorphisms of the mapping class group $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ of the punctured torus. One such automorphism we call the quantum Fourier transform; we show that when $H = U_q(\mathfrak{g})$, the quantum Fourier transform degenerates to the classical Fourier transform on $D(\mathfrak{g})$ as $q \to 1$.

Mathematics Subject Classification (2010). Primary: 16T05; Secondary: 16T25, 20F36.

Keywords. Quantum D-modules, elliptic braid group, mapping class groups.

1. Introduction

Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra, and let $\widetilde{H}$ denote the braided dual – also known as the reflection equation algebra – of H [8, 9, 10, 17]. This is the restricted dual vector space H°, but the multiplication is twisted from the standard one by the R-matrix (see Section 2 for details).

Let $\{e_i\}$ and $\{e^i\}$ denote dual bases of H and $\widetilde{H}$, respectively. Then the canonical element $X = \sum e^i \otimes e_i \in \widetilde{H} \otimes H$ is known to satisfy the following relation in $\widetilde{H} \otimes H^{\otimes 2}$:

$$X^{0,12} := (\mathrm{id} \otimes \Delta)(X) = (\mathcal{R}^{1,2})^{-1} X^{0,2} \mathcal{R}^{1,2} X^{0,1} \tag{1.1}$$

Here, $\widetilde{H}$ has index 0 in the tensor product, and Δ denotes the coproduct of H.

[1] We are grateful to D. Ben–Zvi, and to all three authors of [6], for their many helpful discussions and encouragement, and to P. Roche for bringing the article [2] to our attention.

There is a canonical action of the planar braid group $B_n(\mathbb{R}^2)$ on the nth tensor $V^{\otimes n}$ power of any H-module V. Given modules M for $\widetilde{H}$ and V for H, equation (1.1) allows one to define a similarly canonical action of the punctured planar braid group $B_n(\mathbb{R}^2\backslash\mathrm{disc})$ on $M \otimes V^{\otimes n}$, and moreover to show that $\widetilde{H}$ is universal for this action.

Theorem 1.1 ([8], Proposition 10). *Let B be an algebra, and suppose that $X_B \in B \otimes H$ satisfies relation (1.1). Then there is a unique homomorphism $\phi_B\colon \widetilde{H} \to B$ such that $(\phi_B \otimes \mathrm{id})(X) = X_B$.*

The main goal of this paper is to define elliptic analogs of the reflection equation algebra. The punctured elliptic braid group $B_n(T^2\backslash\mathrm{disc})$ is the free product of two copies of $B_n(\mathbb{R}^2\backslash\mathrm{disc})$, modulo certain relations. In Section 3 we construct an algebra E_H as a certain crossed product of two copies of $\widetilde{H}$, mimicking the cross relations of $B_n(T^2\backslash\mathrm{disc})$. We define canonical elements $X, Y \in E_H \otimes H$ by

$$X = \sum (e^i \otimes 1) \otimes e_i, \quad Y = \sum (1 \otimes e^i) \otimes e_i,$$

and characterize the cross relations on E_H as follows:

Theorem 1.2. *The cross relations of E_H are equivalent to the following commutation relation in $E_H \otimes H^{\otimes 2}$ for $X, Y, \mathcal{R}$:*

$$X^{0,1}\mathcal{R}^{2,1}Y^{0,2} = \mathcal{R}^{2,1}Y^{0,2}\mathcal{R}^{1,2}X^{0,1}\mathcal{R}^{2,1}. \tag{1.2}$$

We prove the following elliptic analog of Theorem 1.1.

Theorem 1.3. *Let B be an algebra, and $X_B, Y_B \in B \otimes H$ satisfying (1.1) individually, and (1.2) together. Then there exists a unique algebra morphism*

$$\phi_B\colon E_H \longrightarrow B$$

such that $X_B = (\phi_B \otimes \mathrm{id})(X)$ and $Y_B = (\phi_B \otimes \mathrm{id})(Y)$. Explicitly, ϕ_B is given by

$$\phi_B(x \otimes 1) = (\mathrm{id} \otimes x)(X_B) \quad \phi_B(1 \otimes x) = (\mathrm{id} \otimes x)(Y_B).$$

Equation (1.2) can be used to define representations of $B_n(T^2\backslash\mathrm{disc})$ in the same way as (1.1) is used for $B_n(\mathbb{R}^2\backslash\mathrm{disc})$; see Theorem 4.3. Recall that $B_n(T^2\backslash\mathrm{disc})$ carries a natural action of the punctured torus mapping class group, which is isomorphic to a certain central extension $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ of $\mathrm{SL}_2(\mathbb{Z})$. In the case H is a *ribbon* Hopf algebra, we show that this extends to an action of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ on E_H.

When $H = U_q(\mathfrak{g})$, we produce degenerations of E_H to the algebras of differential operators on G and, upon further degeneration, on $\mathfrak{g}$. Recall that the algebra of differential operators on an algebraic group G can be constructed as a semi-direct product

$$D(G) = U(\mathfrak{g}) \ltimes O(G),$$

where the action of $U(\mathfrak{g})$ on $O(G)$ is induced by that of $\mathfrak{g}$ on G by left invariant differential operators. This construction can be extended to any Hopf algebra and is known as the Heisenberg double [20]. This is a semi-direct product $D_H = H \ltimes H^\circ$, where H acts on its dual by the right coregular action.

In [15], canonical functors are constructed from the category of modules over the Heisenberg double of a quasi-triangular Hopf algebra to the category of modules over the (unpunctured) torus braid group. This relies upon an alternate construction – due to Varagnolo and Vasserot [21] – of the Heisenberg double of a quasi-triangular Hopf algebra, which uses the braided dual $\widetilde{H}$ in place of H°. This presentation for the Heisenberg double also yields an isomorphism with the handle algebras introduced by Alekseev in [1] and studied further in [2, 3, 19] (see Remark 3.5).

Lifting the constructions of [15] to the unpunctured torus braid group, they can easily be re-interpreted as producing canonical elements X and Y in $D_H \otimes H$, satisfying equations (1.1) and (1.2). Hence, Theorem 1.3 yields a unique homomorphism $\Phi\colon E_H \to D_H$, compatible with the representations of the $B_n(T^2\backslash\text{disc})$ on both sides. The map Φ is an isomorphism if, and only if, H is *factorizable*. Since the quantum group $U_q(\mathfrak{g})$ is factorizable, we may identify the elliptic double $E_{U_q(\mathfrak{g})}$ with the algebra $D_q(G) := D_{U_q(\mathfrak{g})}$ of quantum differential operators on G.

In particular we obtain an $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ action on $D_q(G)$ by the above considerations. One such automorphism of $D_q(G)$ we call the *quantum Fourier transform*; its classical limit upon an appropriate degeneration is the classical Fourier transform on the Weyl algebra $D(\mathfrak{g})$. We expect that our quantum Fourier transform for $D_q(G)$ will be compatible with that on the braided dual of $U_q(\mathfrak{g})$ defined in [16], realizing the braided dual as an $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$-equivariant $D_q(G)$-module. Studying this category of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$-equivariant $D_q(G)$-modules more generally is an interesting direction of future research.

This paper is a companion to [5], in which we compute the value of a certain category valued 2-dimensional topological field theory attached to H-mod, and show that its value on a punctured torus is the category of H-equivariant E_H-modules.

2. The braided dual and its relatives

Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra, and denote by

- $H^e = H^{\mathrm{coop}} \otimes H$ where H^{coop} is H with opposite comultiplication
- $H^{[2]}$ the Hopf algebra which is $H \otimes H$ as an algebra, and with coproduct given by
$$\widetilde{\Delta}(x \otimes y) = (\mathcal{R}^{2,3})^{-1}(\tau^{2,3} \circ \Delta(x \otimes y))\mathcal{R}^{2,3}$$

where $\tau(a \otimes b) = b \otimes a$. Recall that the twist H^F of H by an invertible element $F \in H \otimes H$ is the Hopf algebra with the same multiplication, and with coproduct given by
$$\Delta^F(x) = F^{-1}\Delta(x)F.$$

In order for H^F to be co-associative, F must satisfy two conditions:

$$F^{12,3}F^{1,2} = F^{1,23}F^{2,3}, \quad (\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = 1.$$

Two twists F, F' are *equivalent* if there exists an invertible element $x \in H$, such that $\epsilon(x) = 1$ and
$$F' = \Delta(x)F(x^{-1} \otimes x^{-1}).$$

The following is standard (see [12]).

Proposition 2.1. *A twist induces a tensor equivalence H-mod $\to$ H^F-mod. Equivalent twists leads to isomorphic tensor functors.*

It is easily checked that $F = \mathcal{R}^{1,3}\mathcal{R}^{1,4} \in (H^e)^{\otimes 2}$ is a twist, and that

$$H^{[2],\mathrm{coop}} = (H^e)^F.$$

Let D be the "double braiding" $\mathcal{R}^{2,1}\mathcal{R}^{1,2}$. Since $D\Delta(x) = \Delta(x)D$ for all x, we have
$$H^D = H$$

as Hopf algebras. Similarly, $H^{[2],\mathrm{coop}}$ is in fact equal to $(H^e)^{F(D^{1,3})^k}$ for any $k \in \mathbb{Z}$, with F as above.

Let H° be the restricted Hopf algebra dual of H. It has a natural H-bimodule structure, hence a H^e left module structure given by:

$$(x \otimes y) \triangleright f := f(S^{-1}(x) \cdot y)$$

where S is the antipode of H and we use the fact that S^{-1} is a Hopf algebra isomorphism $H^{\mathrm{coop}} \to H_{\mathrm{op}}$. It turns H° into an algebra in H^e-mod.

Remark 2.2. Remember that the antipode of an Hopf algebra need not to be invertible in general, but this is implied by quasi-triangularity.

Remark 2.3. We use the inverse of the antipode rather than the antipode itself because it is convenient to consider the canonical element as an invariant element of $H^\circ \otimes H$, the image of $1 \in \mathbb{C}$ under the evaluation map $\mathbf{k} \to H^\circ \otimes H$, which means that H° really denotes the *left* dual of H in the rigid monoidal category of H-modules. This is slightly different from the convention used in [8, 15] but it allows us to label tensor factors from left to right.

Definition 2.4. *The kth twisted braided dual $\widetilde{H}_k$ is the algebra image of H° via the tensor functor H^e-mod $\to H^{[2],\mathrm{coop}}$-mod given by the twist $F(D^{1,3})^k$. Explicitly, this is H° as a vector space, with multiplication given by*

$$x \cdot y = m(\mathcal{R}^{1,3}\mathcal{R}^{1,4}(D^{1,3})^k \triangleright (x \otimes y))$$

where m is the multiplication of H°. This is an algebra in the category of $H^{[2],\mathrm{coop}}$-module with the same action as above, namely

$$(x \otimes y) \triangleright f = (u \longmapsto f(S^{-1}(x)uy)).$$

Remark 2.5. The algebra $\widetilde{H}_0$ is usually called the reflection dual, the braided dual or the reflection equation algebra in the literature.

Let X be the canonical element of $\widetilde{H}_k \otimes H$, that is the image of 1 under the coevaluation map $\mathbf{k} \to \widetilde{H}_k \otimes H$. If e_i is a basis of H and e^i the dual basis of $\widetilde{H}_k \cong H^\circ$, then $X = \sum e^i \otimes e_i$. If H is infinite dimensional then X lives in an appropriate completion of the tensor product.

Proposition 2.6. *The element X satisfies*

$$X^{0,12} = D^k(\mathcal{R}^{1,2})^{-1}X^{0,2}\mathcal{R}^{1,2}X^{0,1} \tag{2.1}$$

in $\widetilde{H}_k \otimes H^{\otimes 2}$. This implies that X satisfies the reflection equation

$$\mathcal{R}^{2,1}X^{0,2}\mathcal{R}^{1,2}X^{0,1} = X^{0,1}\mathcal{R}^{2,1}X^{0,2}\mathcal{R}^{1,2}$$

in $\widetilde{H}_k \otimes H^{\otimes 2}$.

The braided dual is in fact universal for this property in the following sense.

Proposition 2.7. *Let B be an algebra and $X_B \in B \otimes H$ satisfying equation (2.1) in $B \otimes H^{\otimes 2}$ for some $k \in \mathbb{Z}$. Then there exists a unique algebra morphism*

$$\phi_B : \widetilde{H}_k \longrightarrow B$$

such that $(\phi_B \otimes \mathrm{id})(X) = X_B$. Explicitly, ϕ_B is given by

$$H^\circ \cong \widetilde{H} \ni f \longmapsto (f \otimes \mathrm{id})(X).$$

Propositions 2.6 and 2.7 are proved in [8] in the case $k = 0$. The general proof is similar. Note that the fact that these axioms all leads to the same reflection equation, regardless of the value of k, essentially follows from the fact that the left hand side of (2.1) is invariant under conjugation by D.

Let $u = m((S \otimes \mathrm{id})(R^{2,1}))$ where m is the multiplication of H. Then $\nu = uS(u)$ is central and satisfies

$$\Delta(\nu) = D^{-2}(\nu \otimes \nu)$$

implying that

$$D^{k-2} = \Delta(\nu) D^k (\nu^{-1} \otimes \nu^{-1})$$

meaning that D^{k-2} and D^k are equivalent. Therefore, they lead to isomorphic tensor functors, from which follows Proposition 2.8.

Proposition 2.8. *For any $k \in \mathbb{Z}$, the algebras $\widetilde{H}_k$ and $\widetilde{H}_{k+2}$ are isomorphic.*

Therefore, it is enough to consider $\widetilde{H}_0$ and $\widetilde{H}_1$. Moreover, if H is a ribbon Hopf algebra, then by definition ν admits a central square root implying by a similar argument.

Proposition 2.9. *If H is a ribbon Hopf algebra then all the $\widetilde{H}_k$ are isomorphic.*

Remark 2.10. For any k, equation (2.1) plays the same role in the reflection equation, as the hexagon axiom in the Yang-Baxter equation, encoding some kind of compatibility with the tensor product of H-modules. Topologically, it corresponds to a "strand doubling" operation for the additional generator of the braid group of the punctured plane. Formally, such an operation depends on the choice of a framing, while a ribbon element removes the dependence on the framing.

3. The elliptic double

Let T denote the following element in $(H^{[2],\mathrm{coop}})^{\otimes 2}$, which we identify as a vector space with $H^{\otimes 4}$:

$$T = (\mathcal{R}^{3,2})^{-1}(\mathcal{R}^{3,1})^{-1}(\mathcal{R}^{4,2})^{-1}\mathcal{R}^{1,4}.$$

Proposition 3.1. *The element T satisfies the hexagon axioms*

$$(\mathrm{id} \otimes \Delta_{H^{[2],\mathrm{coop}}})T = T^{1,3}T^{1,2} \qquad (\Delta_{H^{[2],\mathrm{coop}}} \otimes \mathrm{id})T = T^{1,3}T^{2,3}$$

in $(H^{[2],\mathrm{coop}})^{\otimes 3}$.

Proof. This is a straightforward computation with the Yang-Baxter equation. The computation is depicted in braids in Figure 1. □

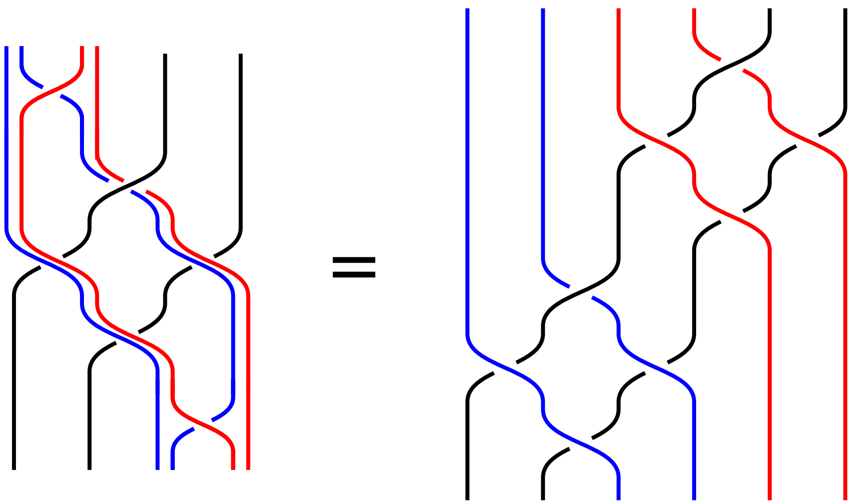

Figure 1. A braid diagram proof of $(\mathrm{id} \otimes \Delta)(T) = T_{1,3}T_{1,2}$.

Corollary 3.2. *The vector space $\widetilde{H}_k^{\otimes 2}$ carries an associative multiplication, in which $\widetilde{H}_k \otimes 1$ and $1 \otimes \widetilde{H}_k$ are sub-algebras, and the cross relations are given by*

$$(1 \otimes g)(f \otimes 1) = T \rhd (f \otimes g).$$

While this is well known, we include a proof here for the reader's convenience.

Proof. It suffices to check associativity on pure tensors in $\widetilde{H}_k$, as these span all of $E_H^{(k)}$. Since $\widetilde{H}_k$ is an associative algebra, the only types of expressions on which it remains to check associativity are of the form $(1 \otimes g) \otimes (1 \otimes h) \otimes (f \otimes 1)$ and $(1 \otimes g) \otimes (h \otimes 1) \otimes (f \otimes 1)$. Write $T = \sum t_i \otimes t_i'$. For the first case, we have

$$
\begin{aligned}
&(m \circ (m \otimes \mathrm{id}))((1 \otimes g) \otimes (1 \otimes h) \otimes (f \otimes 1)) \\
&\quad = \sum_i \sum_j ((t_i t_j) \rhd f) \otimes (t_i' \rhd g)(t_j' \rhd h) \\
&\quad = \sum ((t_i \rhd f) \otimes \Delta(t_i) \rhd gh),
\end{aligned}
$$

so that associativity follows from the second equation in Proposition 3.1. The second case follows similarly. □

Definition 3.3. *We denote by $E_H^{(k)}$ the algebra given by Corollary 3.2.*

Choose a basis $(e_i)_{i \in I}$ of H and define $X, Y \in E_H^{(k)} \otimes H$ by

$$X = \sum e^i \otimes 1 \otimes e_i, \quad Y = \sum 1 \otimes e^i \otimes e_i,$$

where we use the vector space identification $E_H^{(k)} \cong \widetilde{H}^{\otimes 2}$. The main result of this section is the following theorem.

Theorem 3.4. *The cross relations of E_H are equivalent to the commutation relation in $E_H \otimes H^{\otimes 2}$ for $X, Y, \mathcal{R}$:*

$$X^{0,1}\mathcal{R}^{2,1}Y^{0,2} = \mathcal{R}^{2,1}Y^{0,2}\mathcal{R}^{1,2}X^{0,1}\mathcal{R}^{2,1}.$$

Proof. By definition every element $f \in \widetilde{H}_k$ can be written as

$$f = \sum e^i f(e_i)$$

hence the product gf in $E_H^{(k)}$ is obtained by applying $(\mathrm{id}_{E_H^{(k)}} \otimes f \otimes g)$ to

$$Y^{0,2}X^{0,1}$$

and fg by applying the same element to

$$X^{0,1}Y^{0,2}.$$

Therefore all commutations relation can be gathered into a "matrix" equation

$$Y^{0,2}X^{0,1} = T \triangleright_0 X^{0,1}Y^{0,2} \tag{3.1}$$

where T acts on the $E_H^{(k)}$ (i.e. 0th) component. We recall the following identities:

$$\mathcal{R}^{-1} = (S \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S^{-1})(\mathcal{R}). \tag{3.2}$$

Applying S^{-1} to the first factor of the relation $(S \otimes \mathrm{id})(R)R = 1$, setting $\mathcal{R} = \sum r_1 \otimes r_2 = \sum r_1' \otimes r_2'$ – using apostrophes to distinguish between copies of $\mathcal{R}$ – one has the following useful identity (note the order of the terms):

$$\sum S^{-1}(r_1)r_1' \otimes r_2'r_2 = 1. \tag{3.3}$$

Then equation (3.1) reads, in coordinates,

$$\begin{aligned} &((1 \otimes e^j)(e^i \otimes 1)) \otimes e_i \otimes e_j \\ &\quad = ((r_2r_1' \otimes r_2''''r_2'' \otimes S(r_1'''')S(r_1) \otimes S(r_1'')r_2') \triangleright e^i \otimes e^j) \otimes e_i \otimes e_j. \end{aligned} \tag{3.4}$$

The left $H^{[2]}$ action on $\widetilde{H}_k$ is by definition dual to the right $H^{[2]}$ action on H, therefore

$$\sum((x \otimes y) \triangleright e^i) \otimes e_i = \sum e^i \otimes S^{-1}(x)e_i y$$

Using this, equation (3.4) can be rewritten

$$((1 \otimes e^j)(e^i \otimes 1)) \otimes e_i \otimes e_j = e^i \otimes e^j \otimes S^{-1}(r_1')S^{-1}(r_2)e_i r_2''''r_2'' \otimes r_1 r_1'''' e_j S(r_1'')r_2'.$$

Then, using the R-matrix relations (3.2) and (3.3) to move elements from the right hand side to the left hand side (and reassigning apostrophes for the sake of clarity) we obtain

$$\big((1 \otimes e^j)(e^i \otimes 1)\big) \otimes r_2 r_1' e_i r_2'' \otimes r_1 e_j r_2' r_1'' = e^i \otimes e^j \otimes e_i r_2 \otimes r_1 e_j$$

which is exactly (1.2). □

Remark 3.5. If H is semi-simple, then as a vector space $\widetilde{H}_k \cong H^\circ$ has a Peter-Weyl decomposition

$$\widetilde{H}_k = \bigoplus V^* \otimes V$$

where the sum is over representatives of finite dimensional simple H-modules. Under this identification, the relations of Theorem 3.4 coincide with those of the graph algebra of the punctured torus of [1, Def. 12].

Equation (1.2) is a defining relation for $E_H^{(k)}$, in the following sense.

Corollary 3.6. *Let B be an algebra, and $X_B, Y_B \in B \otimes H$ satisfying both the axiom (2.1) and equation (1.2) (with X and Y replaced by X_B and Y_B). Then there exists a unique algebra morphism*

$$\phi_B \colon E_H^{(k)} \longrightarrow B$$

such that $X_B = (\phi_B \otimes \mathrm{id})(X)$ and $Y_B = (\phi_B \otimes \mathrm{id})(Y)$. Explicitly, ϕ_B is given by

$$\phi_B(x \otimes 1) = (\mathrm{id} \otimes x)(X_B), \quad \phi_B(1 \otimes x) = (\mathrm{id} \otimes x)(Y_B).$$

4. Braid group and mapping class group actions

In this section we construct representations of the punctured torus braid group from $E_H^{(k)}$. First, we have

Definition 4.1. *The punctured elliptic braid group $B_n(T^2 \backslash \mathrm{disc})$ is the fundamental group of the configuration space of n points in $T^2 \backslash \mathrm{disc}$.*

Proposition 4.2. *The group $B_n(T^2 \backslash \mathrm{disc})$ is generated by*

$$X_1, \dots, X_n, \quad Y_1, \dots, Y_n, \quad \sigma_1, \dots, \sigma_{n-1},$$

with relations

- *the X_i's (resp. Y_i's) pairwise commute,*
- *the planar braid relation for the σ_i's,*
- *the following cross relations:*

$$X_{i+1} = \sigma_i X_i \sigma_i \quad Y_{i+1} = \sigma_i Y_i \sigma_i, \tag{4.1}$$

$$X_1 Y_2 = Y_2 X_1 \sigma_1^2. \tag{4.2}$$

The results of the previous section easily imply the following theorem.

Theorem 4.3. *There exists a unique group morphism*

$$\phi \colon B_n(T^2 \backslash \mathrm{disc}) \longrightarrow (E_H^{(k)} \otimes H^{\otimes n})^\times \rtimes S_n$$

given by

$$X_1 \longmapsto X^{0,1}, \quad Y_1 \longmapsto Y^{0,1}, \quad \sigma_i \longmapsto (i, i+1)\mathcal{R}^{i,i+1}.$$

Proof. The first two set of cross relations can obviously be taken as a definition of X_i, Y_i for $i > 1$. That these operators pairwise commute follows from the reflection equation and the Yang-Baxter equation. The last cross relation is nothing but the defining equation (1.2) of $E_H^{(k)}$. □

Let $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ denote the group generated by A, B, Z with relations

$$A^4 = (AB)^3 = Z, \quad (A^2, B) = 1.$$

Clearly, Z is central, so this is a central extension,

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{SL}_2(\mathbb{Z})} \longrightarrow \mathrm{SL}_2(\mathbb{Z}) \longrightarrow 1.$$

Proposition 4.4. *The group* $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ *acts on* $B_n(T^2 \backslash \mathrm{disc})$ *in the following way*:

$$\begin{aligned} A \cdot \sigma_i &= \sigma_i, & B \cdot \sigma_i &= \sigma_i, \\ A \cdot X_1 &= Y_1, & A \cdot Y_1 &= Y_1 X_1^{-1} Y_1^{-1}, \\ B \cdot X_1 &= X_1, & B \cdot Y_1 &= Y_1 X_1^{-1}. \end{aligned}$$

Proposition 4.5. *Let B be an algebra and $(X_B, Y_B) \in B \otimes H$ satisfying equation (1.2) and axioms (2.1) with $k = 1$. Then, so does $(X_B, Y_B X_B^{-1})$ and $(Y_B, Y_B X_B^{-1} Y_B^{-1})$.*

Proof. Equation (1.2) is exactly one of the defining relation of $B_{1,n}^1$ so that it is satisfied follows from the previous proposition. So we just have to check that $Y_B X_B^{-1}$ and $Y_B X_B^{-1} Y_B^{-1}$ satisfies (2.1) with $k = 1$. This is a direct computation:

$$\begin{aligned} &(Y_B X_B^{-1})^{0,12} \\ &\quad = \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} Y_B^{0,1} (X_B^{0,1})^{-1} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} (\mathcal{R}^{2,1})^{-1} \\ &\quad = \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} Y_B^{0,1} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} (\mathcal{R}^{2,1})^{-1} (X_B^{0,1})^{-1} \mathcal{R}^{2,1} (\mathcal{R}^{2,1})^{-1} \\ &\quad = \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} \mathcal{R}^{1,2} Y_B^{0,1} \mathcal{R}^{2,1} (\mathcal{R}^{2,1})^{-1} (X_B^{0,1})^{-1} \\ &\quad = \mathcal{R}^{2,1} Y_B^{0,2} (X_B^{0,2})^{-1} \mathcal{R}^{1,2} Y_B^{0,1} (X_B^{0,1})^{-1}, \end{aligned}$$

where at lines 2 and 3 we use the reflection equation and the elliptic commutation relation respectively. The second part is proved by doing the exact same computation replacing Y_B by $Y_B X_B^{-1}$ and X_B by Y_B. □

Corollary 4.6. *There is an action of* $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ *on* $E_H^{(1)}$, *uniquely determined by its action on canonical elements* X, Y *as follows*:

$$A \cdot X = Y, \quad A \cdot Y = YX^{-1}Y^{-1},$$
$$B \cdot X = X, \quad B \cdot Y = YX^{-1}.$$

Moreover, the action is compatible with the $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$*-action on* $B_n(T^2 \backslash \mathrm{disc})$,

Proof. This follows from Proposition 4.5 together with the universal property stated in Corollary 3.6. □

5. Relation with the Heisenberg double and quantum Fourier transform

Since $\widetilde{H}_0$ is a $H^{[2],\mathrm{coop}}$-module algebra, one can form the semi-direct product $\widetilde{H} \rtimes H^{[2],\mathrm{coop}}$. It is easily checked that $H \otimes 1 \subset H^{[2],\mathrm{coop}}$ is a coideal subalgebra, hence the following definition makes sense.

Definition 5.1. *The Heisenberg double* D_H *is the subalgebra* $\widetilde{H}_0 \rtimes (H \otimes 1)$.

Remark 5.2. The standard definition of the Heisenberg double involves H^e and the usual dual, instead of $H^{[2]}$ and the braided dual. However, it is shown in [21] that these two algebras are isomorphic.

Clearly, the double braiding $\mathcal{R}^{2,1}\mathcal{R}^{1,2}$ satisfies axiom (2.1) with $k = 0$. This is a manifestation of the embedding of the cylinder braid group on n strands into the ordinary braid group on $n+1$ strands. Let ϕ_H be the factorization map

$$\begin{aligned} \phi_H \colon \widetilde{H}_0 &\longrightarrow H, \\ f &\longmapsto (f \otimes \mathrm{id})(\mathcal{R}^{2,1}\mathcal{R}^{1,2}). \end{aligned}$$

.

Theorem 5.3 ([15]). *The canonical element* $X \in D_H \otimes H$ *together with the image of the double braiding under the inclusion* $H \otimes H \to D_H \otimes H$ *satisfy the commutation relation* (1.2).

Corollary 5.4. *There exists a canonical algebra map from the elliptic double to the Heisenberg double, given by the identity on the first* $\widetilde{H}_0$ *component and defined on the second component by the factorization map* ϕ_H.

Proof. It follows from the universal property of Corollary 3.6. □

Definition 5.5. *A quasi-triangular Hopf algebra is called factorizable if* ϕ_H *is injective.*

Let I_H be the image of ϕ_H and let D'_H be the subalgebra $\widetilde{H} \rtimes (I_H \otimes 1)$ of D_H.

Theorem 5.6. *If* H *is a factorizable Hopf algebra, then* D'_H *is isomorphic as an algebra to* $E_H^{(0)}$.

Proof. The algebra map $E_H^{(0)} \to D_H$ is given by $\mathrm{id} \otimes \phi_H$. Since H is factorizable this map is injective, and its image is D'_H by definition. □

Let G be a reductive algebraic group, $\mathfrak{g}$ its Lie algebra and $U = U_q(\mathfrak{g})$ the corresponding quantum group. Recall (see e.g. [7, Chapter 9]) that this is a quasi-triangular Hopf algebra[1] over $\mathbb{C}(q)$ for q a variable which deform the enveloping algebra of $\mathfrak{g}$. Denote by $U' = U_q(\mathfrak{g})'$ its ad-locally finite part.

Theorem 5.7 ([4, 18]). *U is a factorizable ribbon Hopf algebra, and the image of the factorization map* $(U^*) \to U$ *is* U'.

Let $D_q(G)$ be the subalgebra $\widetilde{U} \rtimes U'$ of the Heisenberg double of U. It is a deformation of the algebra of differential operators on G. Thanks to the above theorem, $D_q(G)$ is isomorphic to $E_U^{(0)}$ which is itself isomorphic to $E_U^{(1)}$. Altogether this implies the following result.

Corollary 5.8. *The isomorphism* $D_q(G) \cong E_U^{(1)}$ *together with the formulas of Corollary 4.6 yield an action of* $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ *on* $D_q(G)$ *by algebra automorphism.*

6. Relation to classical Fourier transform

In this section we show how the Weyl algebra of $\mathfrak{g}$ and the classical Fourier transform can be obtained both directly as the elliptic double of a certain Hopf algebra and via an appropriate degeneration of the elliptic double of the corresponding quantum group. Let $U_\hbar(\mathfrak{g})$ be the "formal" version of the quantum group. This a topological quasi-triangular Hopf algebra over $\mathbb{C}[[\hbar]]$, where $\hbar$ is a formal variable, deforming the enveloping algebra of $\mathfrak{g}$ and whose definition can be found, e.g., in [7, Chapter 6]. Since directly taking the classical (i.e. $\hbar = 0$) limit of the elliptic commutation relation gives the commutative algebra $S(\mathfrak{g})^{\otimes 2}$ we will have to consider a slightly more complicated degeneration.

[1] This is not quite true since the R-matrix does not belong to $U_q(\mathfrak{g})^{\otimes 2}$ but only to a certain completion of it, but it is still enough for our purposes.

Let $S(\mathfrak{g})$ denote the symmetric algebra on $\mathfrak{g}$, equipped with its standard coproduct $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in \mathfrak{g}$, making it a commutative, cocommutative Hopf algebra. Let $r \in \mathfrak{g}^{\otimes 2}$ denote the quasi-classical limit of the R-matrix of $U_\hbar(\mathfrak{g})$, i.e.,

$$\mathcal{R} = 1 + \hbar r + O(\hbar^2).$$

Then, in a straightforward way, the completion of the symmetric algebra $(\widehat{S}(\mathfrak{g}), \mathcal{R}_0 = \exp(r))$ is a quasi-triangular, factorizable Hopf algebra[2]. Let $t = r + r^{2,1} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and let C denote the corresponding Casimir element, i.e. $C = m(t)$ where m is the multiplication of $S(\mathfrak{g})$. Then $\nu_0 = \exp(-C/2)$ is a ribbon element. Since $\mathcal{R}_0 \notin S(\mathfrak{g})^{\otimes 2}$, $S(\mathfrak{g})$ is not strictly speaking a ribbon Hopf algebra, but the construction of the elliptic double is still well defined in this situation.

Let $D(\mathfrak{g})$ be the algebra of differential operators on $\mathfrak{g}$, i.e. the Weyl algebra. As a vector space it is $S(\mathfrak{g}^*)^{\otimes 2}$, the two copies of $S(\mathfrak{g}^*)$ are subalgebras and the cross relations are

$$[f \otimes 1, 1 \otimes g] = \langle f, g \rangle \quad \text{for all } f, g \in \mathfrak{g}^*, \tag{6.1}$$

where $\langle\ ,\ \rangle$ is the pairing on $\mathfrak{g}^*$ induced by t. The first result of this section is the following proposition.

Proposition 6.1. *The 0th elliptic double of* $(S(\mathfrak{g}), \mathcal{R}_0)$ *is isomorphic to the Weyl algebra* $D(\mathfrak{g})$ *and the action of the generator* A *of* $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ *coincides with the classical Fourier transform. That is, on generators* $(f, g) \in \mathfrak{g}^* \times \mathfrak{g}^* \subset D(\mathfrak{g})$, *we have,*

$$A(f, g) = (-g, f).$$

The operator B *acts by*

$$B(f, g) = (f - g, g).$$

Proof. Let E be the 0th elliptic double of $(S(\mathfrak{g}), \mathcal{R}_0)$. Let e_i be a basis of $\mathfrak{g}$, e^i the dual basis of $\mathfrak{g}^*$ and define $x, y \in E \otimes U(\mathfrak{g})$ by

$$x = \sum e^i \otimes 1 \otimes e_i, \quad y = \sum 1 \otimes e^i \otimes e_i.$$

The restricted dual of $S(\mathfrak{g})$ is $S(\mathfrak{g}^*)$ and the images of the corresponding canonical elements in $E \otimes S(\mathfrak{g})$ are $X = \exp(x)$ and $Y = \exp(y)$ respectively. Since $S(\mathfrak{g})$ is commutative, equation (2.1) reduces to the standard relation,

$$(\mathrm{id} \otimes \Delta)(X) = X^{0,1} X^{0,2}$$

[2] Here the tensor product is the topological one, i.e. $\widehat{S}(\mathfrak{g})^{\otimes 2} := \widehat{S}(\mathfrak{g} \times \mathfrak{g})$

in $(S(\mathfrak{g}^*) \otimes 1) \otimes S(\mathfrak{g})^{\otimes 2} \subset E \otimes S(\mathfrak{g})^{\otimes 2}$, hence the braided dual and the restricted dual coincide. Likewise, the defining equation of the elliptic double reduces to

$$(X^{0,1}, Y^{0,2}) = \mathcal{R}_0^{2,1}\mathcal{R}_0^{1,2}$$

in $E \otimes S(\mathfrak{g})^{\otimes 2}$, where $(a,b) = aba^{-1}b^{-1}$. Since

$$[x^{0,1}, t^{1,2}] = [y^{0,2}, t^{1,2}] = 0,$$

this equation is equivalent to

$$[x^{0,1}, y^{0,2}] = t^{1,2}.$$

Applying f and g to the first and second components, respectively, of the above equation gives the defining relations (6.1) of $D(\mathfrak{g})$.

Since $(S(\mathfrak{g}), \mathcal{R}_0)$ is ribbon, $E^{(0)}_{S(\mathfrak{g})}$ is isomorphic to $E^{(1)}_{S(\mathfrak{g})}$. Pulling back the action of the A generator of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ through this isomorphism, we find

$$x \longmapsto y, \quad y \longmapsto Y^{-1}(-x + (1 \otimes C))Y.$$

It is easily seen that the cross relations of $D(\mathfrak{g})$ implies

$$Y^{-1}xY = x + (1 \otimes C).$$

Hence A maps x to y and y to $-x$.

Pulling back the B action through this isomorphism one get

$$x \longmapsto x, \quad y \longmapsto \log(e^y e^{-x} e^{1\otimes C/2}).$$

Since

$$[x, y] = 1 \otimes C$$

and since $1 \otimes C$ commutes with x and y, the Baker–Campbell–Hausdorff formula implies that

$$\log(e^y e^{-x} e^{1\otimes C/2}) = y - x$$

as required. □

Remark 6.2. Since A^4 acts as the identity, the above action of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ on $D(\mathfrak{g})$ factors through an action of $\mathrm{SL}_2(\mathbb{Z})$. It coincides with the one coming from an homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to Sp(\mathfrak{g} \oplus \mathfrak{g})$, the latter being the group of linear symplectomorphisms of the vector space $\mathfrak{g} \oplus \mathfrak{g}$, equipped with the symplectic form coming from the Killing form.

Remark 6.3. It is interesting to ask whether the action of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ on $D_q(G)$ can be degenerated to an action on $D(G)$, not just to $D(\mathfrak{g})$. The degeneration procedure for obtaining $D(G)$ from $D_q(G)$ is not compatible, however, with the $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$-action; hence, a naïve attempt at re-creating the procedure for $D(\mathfrak{g})$ will not work. This is not surprising, as there is not a good notion of Fourier transform on $D(G)$, essentially because the cotangent bundle $T^*G = G \times \mathfrak{g}^*$ has fewer symplectomorphisms than $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^* \cong \mathfrak{g} \oplus \mathfrak{g}$.

Let $U_{\hbar^2}(\mathfrak{g})$ be the $\mathbb{C}[[\hbar]]$-Hopf algebra obtained by formally replacing $\hbar$ by $\hbar^2$ in the definition of the product, the coproduct and the R-matrix of $U_\hbar(\mathfrak{g})$. Denote by δ_n the map $(\mathrm{id} - \epsilon)^{\otimes n} \circ \Delta^n$ where ϵ is the counit of $U_{\hbar^2}(\mathfrak{g})$. Denote by $\widehat{U}$ the quantum formal series Hopf algebra (QFSHA) attached to $U_{\hbar^2}(\mathfrak{g})$, i.e. the subalgebra

$$\widehat{U} = \{x \in U_{\hbar^2}(\mathfrak{g}),\ \delta_n(x) \in \hbar^n U_{\hbar^2}(\mathfrak{g}),\ \text{for all } n \geq 0\}$$

It is known [11, 14] that $\widehat{U}$ is a flat deformation of $\widehat{S}(\mathfrak{g})$. Hence, choose a $\mathbb{C}[[\hbar]]$-module identification

$$\psi \colon \widehat{U} \longrightarrow \widehat{S}(\mathfrak{g})[[\hbar]]$$

which is the identity modulo $\hbar$, and let $U \subset \widehat{U}$ be the preimage under ψ of $S(\mathfrak{g})[[\hbar]]$.

Proposition 6.4. *We have the following*:

(a) *U is a Hopf algebra*;

(b) *there is a canonical bialgebra isomorphisms*:

$$\widehat{U}/(\hbar) \cong \widehat{S}(\mathfrak{g}), \quad U/(\hbar) \cong S(\mathfrak{g});$$

(c) *the R-matrix of $U_{\hbar^2}(\mathfrak{g})$ belongs to $\widehat{U}^{\otimes 2}$ and its image in $\widehat{S}(\mathfrak{g})^{\otimes 2}$ is $\mathcal{R}_0$.*

One can therefore consider the 0th elliptic double of U. A direct consequence of the above proposition is then the following corollary.

Corollary 6.5. *The algebra E_U is a flat deformation of the Weyl algebra $D(\mathfrak{g})$, and the $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$-action on E_U degenerates to the $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$-action on $D(\mathfrak{g})$. In particular, the quantum Fourier transform degenerates to the classical one.*

Proof of Proposition 6.4. All of this can be checked explicitly. A more conceptual argument is as follows: recall that $(\mathfrak{g}, \mu, \delta, r)$ is a quasi-triangular Lie bialgebra, where we denote by μ its bracket and by δ its co-bracket. The quantum group $U_{\hbar^2}(\mathfrak{g})$ is obtained by applying an Etingof–Kazhdan quantization functor [13] to the $\mathbb{C}[[\hbar]]$-quasi-triangular Lie bialgebra $(\mathfrak{g}[[\hbar]], \mu, \hbar^2\delta, \hbar^2 r)$. On the other hand, $\widehat{U}$ is the quasi-triangular Hopf algebra obtained by applying the same functor to the quasi-triangular Lie bialgebra $(\mathfrak{g}[[\hbar]], \hbar\mu, \hbar\delta, r)$. The QFSHA construction is the lift of the inclusion,

$$(\mathfrak{g}[[\hbar]], \hbar\mu, \hbar\delta, r) \longrightarrow (\mathfrak{g}[[\hbar]], \mu, \hbar^2\delta, \hbar^2 r),$$

given by $x \mapsto \hbar x$ (since $r \in \mathfrak{g}^{\otimes 2}$, its image is indeed $\hbar^2 r$).

One can show that the product, the coproduct and the antipode on $\widehat{U}$ restrict to a well-defined Hopf algebra structure on U. By construction, the reduction modulo $\hbar$ of $\widehat{U}$ is the quantization of the $\mathbb{C}$-quasi-triangular Lie bialgebra,

$$(\mathfrak{g}[[\hbar]], \hbar\mu, \hbar\delta, r)/(\hbar) \cong (\mathfrak{g}, 0, 0, r),$$

which is easily seen to be $(\widehat{S}(\mathfrak{g}), \mathcal{R}_0)$. □

References

[1] A. Yu. Alekseev, Integrability in Hamiltonian Chern–Simons theory. *Algebra i Analiz* **6** (1994), no. 2, 53–66. In Russian. English translation, *St. Petersburg Math. J.* **6** (1995), no. 2, 241–253. MR 1290818 Zbl 0819.58011 Zbl 0824.58026 (transl.)

[2] A. Yu. Alekseev, H. Grosse, and V. Schomerus, Combinatorial quantization of the Hamiltonian Chern–Simons theory II. *Comm. Math. Phys.* **174** (1996), no. 3, 561–604. MR 1370081 Zbl 0842.58038

[3] A. Yu. Alekseev and V. Schomerus, Representation theory of Chern–Simons observables. *Duke Math. J.* **85** (1996), no. 2, 447–510. MR 1417624 Zbl 0869.58005

[4] P. Baumann and F. Schmitt, Classification of bicovariant differential calculi on quantum groups (a representation-theoretic approach). *Comm. Math. Phys.* **194** (1998), no. 1, 71–86. MR 1628278 Zbl 0926.17012

[5] D. Ben-Zvi, A. Brochier, and D. Jordan, Integrating quantum groups over surfaces. Preprint 2015. arXiv:1501.04652 [math.QA]

[6] D. Calaque, B. Enriquez, and P. Etingof, Universal KZB equations: the elliptic case. In Y. Tschinkel and Y. Zarhin (eds.), *Algebra, arithmetic, and geometry.* Vol. I. In honor of Yu. I. Manin on the occasion of his 70th birthday. Progress in Mathematics, 269. Birkhäuser Boston, Boston, MA, 2009, 165–266. MR 2641173 Zbl 1241.32011

[7] V. Chari and A. Pressley, *A guide to quantum groups.* Cambridge University Press, Cambridge, 1994. MR 1300632 Zbl 0839.17009 Zbl 0839.17010 (paperback ed., 1995)

[8] J. Donin, P. P. Kulish, and A. I. Mudrov, On a universal solution to the reflection equation. *Lett. Math. Phys.* **63** (2003), no. 3, 179–194. MR 1992884 Zbl 1042.17011

[9] J. Donin and A. Mudrov, Reflection equation- and FRT-type algebras. *Czechoslovak J. Phys.* **52** (2002), no. 11, 1201–1206. Quantum groups and integrable systems (Prague, 2002). MR 1966929 Zbl 1064.17007

[10] J. Donin and A. Mudrov, Reflection equation, twist, and equivariant quantization. *Israel J. Math.* **136** (2003), 11–28. MR 1998103 Zbl 1046.17005

[11] V. G. Drinfel′d, Quantum groups. In A. M. Gleason (ed.), *Proceedings of the International Congress of Mathematicians.* Vol. 1. Held at the University of California, Berkeley, California, August 3–11, 1986. American Mathematical Society, Providence, R.I., 1987, 798–820. MR 0934283 Zbl 0667.16003

[12] V. G. Drinfel′d, Quasi-Hopf algebras. *Algebra i Analiz* **1** (1989), no. 6, 114–148. In Russian. English translation, *Leningrad Math. J.* **1** (1990), no. 6, 1419–1457. MR 1047964 Zbl 0718.16033

[13] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras. I. *Selecta Math.* (*N.S.*) **2** (1996), no. 1, 1–41. MR 1403351 Zbl 0863.17008

[14] F. Gavarini, The quantum duality principle. *Ann. Inst. Fourier* (*Grenoble*) **52** (2002), no. 3, 809–834. MR 1907388 Zbl 1054.17011

[15] D. Jordan, Quantum D-modules, elliptic braid groups, and double affine Hecke algebras. *Int. Math. Res. Not. IMRN* **2009** (2009), no. 11, 2081–2105. MR 2507111 Zbl 1185.17013

[16] V. Lyubashenko and S. Majid, Braided groups and quantum Fourier transform. *J. Algebra* **166** (1994), no. 3, 506–528. MR 1280590 Zbl 0810.17006

[17] S. Majid, *Foundations of quantum group theory.* Cambridge University Press, Cambridge, 1995. MR 1381692 Zbl 0857.17009

[18] N. Y. Reshetikhin and M. A. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems. *J. Geom. Phys.* **5** (1988), no. 4, 533–550. MR 1075721 Zbl 0711.17008

[19] P. Roche and A. Szenes, Trace functionals on noncommutative deformations of moduli spaces of flat connections. *Adv. Math.* **168** (2002), no. 2, 133–192. MR 1912130 Zbl 1053.53062

[20] M. A. Semenov-Tian-Shansky, Poisson Lie groups, quantum duality principle, and the quantum double. In P. J. Sally, Jr., M. Flato, J. Lepowsky, N. Reshetikhin, and G. J. Zuckerman (eds.), *Mathematical aspects of conformal and topological field theories and quantum groups.* Papers from the AMS-IMS-SIAM Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups held at Mount Holyoke College, South Hadley, Massachusetts, June 13–19, 1992. Contemporary Mathematics, 175. American Mathematical Society, Providence, R.I., 1994, 219–248. MR 1302020 Zbl 0841.17008

[21] M. Varagnolo and E. Vasserot, Double affine Hecke algebras at roots of unity. *Represent. Theory* **14** (2010), 510–600. MR 2672950 Zbl 1280.20005

Received July 26, 2014

Adrien Brochier, Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

e-mail: adrien.brochier@uni-hamburg.de

David Jordan,
School of Mathematics and Maxwell Institute for Mathematical Sciences,
University of Edinburgh, James Clerk Maxwell Building, Kings Buildings,
Mayfield Road, Edinburgh EH9 3JZ, UK

e-mail: d.jordan@ed.ac.uk