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Quantum Topology

Spin modular categories

Anna Beliakova, Christian Blanchet, and Eva Contreras

Abstract. Modular categories are a well-known source of quantum 3-manifold invariants. In this paper we study structures on modular categories which allow to define refinements of quantum 3-manifold invariants involving cohomology classes or generalized spin and complex spin structures. A crucial role in our construction is played by objects which are invertible under tensor product. All known examples of cohomological or spin type refinements of the Witten–Reshetikhin–Turaev 3-manifold invariants are special cases of our construction. In addition, we establish a splitting formula for the refined invariants, generalizing the well-known product decomposition of quantum invariants into projective ones and those determined by the linking matrix.

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Contents

1. Introduction

In the late 80's, inspired by Witten's ideas [22], Reshetikhin and Turaev [18] came up with a construction of new 3-manifold invariants, known as WRT quantum invariants. Few years later, Turaev [20] formalized this construction by introducing the notion of *modular category*. His main result is that any modular category gives rise to a 3-dimensional Topological Quantum Field Theory (TQFT), and in particular, to quantum 3-manifold invariants.

A modular category over a field $\mathbf{k}$ is a special kind of $\mathbf{k}$-linear ribbon category which has a finite set of simple objects Γ, including the unit object $\mathbf{1} \in \Gamma$, satisfying domination and non-degeneracy axioms. If we replace the category by its additive closure, which we will always do here, then the domination axiom is equivalent to semisimplicity, with Γ representing all isomorphism classes of simple objects. A ribbon category is a monoidal category with braiding, twist and compatible duality. Ribbon categories are universal receivers for invariants of ribbon graphs [20]. Examples of modular categories arise from representation theory of quantum groups, when the quantum parameter q is a root of unity, or can be constructed skein theoretically [1, 3, 4, 10]. Many authors [2, 8, 19], observed independently that for some special values of q, the $\mathfrak{sl}_2$ WRT invariants admit spin and cohomological refinements. This was extended to $\mathfrak{sl}_n$ quantum invariants in [3] and [15]. Cohomological refinements give rise to Homotopy Quantum Field Theories (HQFT's), constructed by Turaev in [21]. However, spin refinements do not fit in the framework of HQFT's.

The main aim of this paper is to provide an algebraic setting for spin type refinements of quantum 3-manifold invariants. Although we restrict to closed 3-manifolds in this paper, we expect that all our results generalize to an appropriately chosen 3-cobordism category, and hence admit a TQFT version.

Before explaining our results let us recall few definitions.

Given a group G, a G-category was defined in [21, Section VI]. We slightly modify the definition to fit with the case where the category has direct sums. A G-category $\mathcal{C}$ is a monoidal $\mathbf{k}$-linear category equipped with a family of full subcategories $\{\mathcal{C}_\alpha\}_{\alpha\in G}$ such that

- each object of $\mathcal{C}$ is isomorphic to a direct sum over $\alpha \in G$ of objects in $\mathcal{C}_\alpha$;
- $\mathrm{Hom}_{\mathcal{C}}(U, V) = 0$ if $U \in \mathrm{Ob}(\mathcal{C}_\alpha)$, $V \in \mathrm{Ob}(\mathcal{C}_\beta)$, $\alpha \neq \beta$;
- for $U \in \mathrm{Ob}(\mathcal{C}_\alpha)$ and $V \in \mathrm{Ob}(\mathcal{C}_\beta)$, $U \otimes V \in \mathrm{Ob}(\mathcal{C}_{\alpha\beta})$.

Then the trivial object is in $\mathcal{C}_1$, and if $U \in \mathrm{Ob}(\mathcal{C}_\alpha)$ has a (left or right) dual U^*, then $U^* \in \mathrm{Ob}(\mathcal{C}_{\alpha-1})$.

Given a modular category with representative set of simple objects Γ, we call an object $t \in \Gamma$ *invertible* if there exists an object λ, such that $t \otimes \lambda \cong \mathbf{1}$. Let us denote by $G = G_{\mathcal{C}}$ the subset of invertible objects in the distinguished set Γ. Using tensor product and isomorphisms, we get on G the structure of a finite abelian group. In Section 5.2 we show that the braiding (or monodromy) coefficients of $\lambda \in \Gamma$ with $t \in G$ define a map Γ to the group of characters $\mathrm{Hom}(G, \mathbb{C}^*) = \widehat{G}$. This map induces on $\mathcal{C}$ the structure of a $\widehat{G}$-category.

Note the braiding matrix defines a bilinear form on G, and the twist coefficients extend it to a quadratic form. A special role in our approach will play a subgroup $H \subseteq G$, such that the bilinear form restricted to H is trivial while the quadratic extension is not. Clearly, in this case $\mathcal{C}$ also admits a structure of $\widehat{H}$-category induced by the characters $\mathrm{Hom}(H, \mathbb{C}^*) = \widehat{H}$, which is coarser than the $\widehat{G}$-structure for proper H, i.e. the $\widehat{H}$-subcategories are direct sums of the $\widehat{G}$-subcategories.

Let us state our main definition.

Definition. Let $\mathcal{C}$ be a modular category and $H \subseteq G$ be a subgroup in the group of invertible distinguished objects. The category $\mathcal{C}$ is called H-refinable if $H \subset \mathrm{Ob}(\mathcal{C}_1)$ for the $\widehat{H}$-structure on $\mathcal{C}$. Moreover, an H-refinable modular category $\mathcal{C}$ is *H-spin* if the twist quadratic form restricted to H is non–trivial, or equivalently if H has at least one element with twist coefficient -1. When H is cyclic of order d, we will use shorthand *d-spin* and *d-refinable*.

For example, the $\mathfrak{sl}_2$ modular category at the rth root of unity q is 2-refinable for $r = 0 \pmod 2$ and it is 2-spin if $r = 0 \pmod 4$. The group $G = H = \mathbb{Z}/2\mathbb{Z}$ is generated by the $(r-1)$-dimensional representation.

We say that $t \in G$ has order n if n is the minimal integer such that $t^n \cong 1$. We will see that the order of an element with twist -1 has to be even.

H-refinable modular categories which are non spin give rise to invariants of pairs (M, h) for any compact orientable 3-manifold M and a cohomology class $h \in H^1(M; \widehat{H})$. They also fit in the setting of modular group-categories introduced in [21]. In the spin case, the formalism of [21] does not apply since the non degeneracy condition required in trivial degree is not satisfied. Indeed the trivial degree subcategory is neither modular, nor modularizable.

Most results here could be applied to pre-modular categories for which the evaluation of the unknot with framing ± 1 and *Kirby color* are invertible. However, under a mild condition about positivity of the dimension of transparent objects which is satisfied in the examples, those pre-modular categories can be modularized [5, 14]. For this reason, and also because certain invertible objects are involved in the modularization, making a possible confusion with their role here, we choose to deal with modular categories.

Let us concentrate on the spin case. If we have a H-spin modular category, then the twist coefficients define an order 2 element $v \in \widehat{H}$, which we will call the spin character. We will extend the definition of generalized spin structures with modulo d coefficients given in [2] to this situation and define $(\widehat{H}, v)$ generalized spin structures. One of the results of this paper is the following.

Theorem 1. *Any H-spin modular category $\mathcal{C}$ with associated spin character $v \in \mathrm{Tor}_2(\widehat{H})$ provides a topological invariant $\tau_{\mathcal{C}}(M, \sigma)$ of a pair (M, σ), where σ is a $(\widehat{H}, v)$ generalized spin structure on M. Moreover,*

$$\tau_{\mathcal{C}}(M) = \sum_{\sigma} \tau_{\mathcal{C}}(M, \sigma).$$

We expect that Theorem 1 extends naturally to a spin type TQFT.

In Section 2 we define *d-complex* spin structures. Let us denote the set of such structures on M by $\mathrm{Spin}^c_d(M)$. Generalizing results of [6], we identify $\mathrm{Spin}^c_d(M)$ with the set of modulo d Chern vectors which are further used for constructing extensions of WRT invariants.

Theorem 2. *Suppose d is an even positive integer. For any $2d$-spin modular category $\mathcal{C}$, there exists a topological invariant $\tau_{\mathcal{C}}(M, \sigma)$ of a pair (M, σ), where $\sigma \in \mathrm{Spin}^c_d(M)$ is a d-complex spin structure on M.*

We call a modular category $\mathcal{C}$ *reduced* if it is $G_{\mathcal{C}}$-refinable. This means that all invertible objects have trivial characters.

Assume the group of invertible objects G of our modular category $\mathcal{C}$ is cyclic of order d, generated by t. For $0 < \delta \leq d$, let the character of t (induced by the braiding) be $\gamma_\delta(t) = e^{\frac{2\pi\sqrt{-1}\delta}{d}}$. Then there is a positive integer m, such that $\delta m = d$. Clearly, $\mathcal{C}$ is not reduced, but H-refinable, where H is the subgroup of order δ generated by $t^m \in \mathcal{C}_1$.

Given these data, one way to construct refined invariants is by using the δ-refinable structure on $\mathcal{C}$. In Section 7 we show that there is a more efficient way to compute this invariant. Namely, there always exists a smaller reduced category $\widetilde{\mathcal{C}}$, which leads to the same invariant up to a correction term fully determined by the linking matrix. If $\gcd(m,\delta)=1$, $\widetilde{\mathcal{C}}$ is particularly simple and coincides with $\mathcal{C}_1$.

Theorem 3. *Let $\mathcal{C}$ be a modular category in which the group G of invertible objects is cyclic and generated by t. Assume $d=|G|$, the character of t is γ_δ and $\delta m = d$. Then, there exists a reduced δ-refinable category $\widetilde{\mathcal{C}}$, a positive integer α and a root of unity ξ such that for any closed oriented 3-manifold M we have*

$$\tau_{\mathcal{C}}(M,\sigma) = \alpha^{-b_1(M)}\tau_{\widetilde{\mathcal{C}}}(M,\sigma)\tau_\xi^{\mathrm{MMO}}(M,\sigma)$$

where $\tau_\xi^{\mathrm{MMO}}(M,\sigma)$ is the refined Murakami–Ohtsuki–Okada invariant. We have either $\sigma \in H^1(M,\mathbb{Z}_\delta)$ or $\sigma \in \mathrm{Spin}_\delta(M)$, and $b_1(M)$ is the first Betti number.

In the particular case, when $\gcd(\delta,m)=1$, we have $\alpha=1$, ξ is a root of unity of order m and $\tau_\xi^{\mathrm{MMO}}(M,\sigma)$ does not depend on σ so that we have

$$\tau_{\mathcal{C}}(M) = \tau_{\widetilde{\mathcal{C}}}(M)\tau_\xi^{\mathrm{MMO}}(M).$$

The Murakami–Ohtsuki–Okada invariant defined in [16] depends only on the homological information which can be obtained from the linking matrix of the surgery link.

Theorem 3 generalizes the well-known decompositon results for quantum invariants stated in [2] and [9]. We expect that these decomposition results extend to refined TQFTs.

Organization of the paper. After recalling the basic definitions of Spin^c-structures, we give homotopy theoretical and also combinatorial descriptions of their reduction modulo d. Further we recall the definitions of ribbon and modular categories and of quantum 3-manifold invariants. Section 5 deals with modular categories containing invertible objects. The refined invariants are studied in Sections 6 and 7. The last section is devoted to the proof of Theorem 3.

2. Complex spin structures mod d

In this section, given a compact orientable 3-manifold M, we define the set $\mathrm{Spin}^c_d(M)$.

Throughout this paper all manifolds are assumed to be compact and oriented; all (co)homology groups are computed with integer coefficients, unless otherwise is specified; $\mathbb{Z}_x$ denotes the cyclic group of integers modulo x.

2.1. Spin^c-structures. Complex spin structures are additional structures some manifolds can be endowed with and just like the more common spin structures, they can be seen as a generalization of orientations. We recall basic facts and equivalent ways to define Spin^c-structures following the lines of [6] and [17].

Let $n \geq 1$ be an integer. The group $\mathrm{Spin}(n)$ is defined as the non-trivial double cover of the special orthogonal group $\mathrm{SO}(n)$:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\lambda} \mathrm{SO}(n) \longrightarrow 1.$$

Example. $\mathrm{Spin}(1) \cong \mathbb{Z}_2$, $\mathrm{Spin}(2) \cong S^1$ and $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

The complex spin group is defined as the quotient

$$\mathrm{Spin}^c(n) := \frac{\mathrm{Spin}(n) \times S^1}{\mathbb{Z}_2}$$

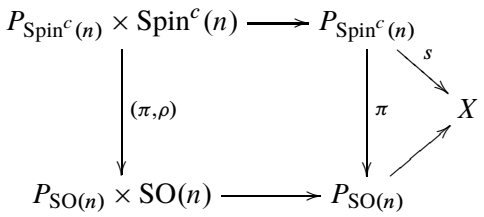
where $\mathbb{Z}_2$ is generated by $(-1,-1) \in \mathrm{Spin}(n) \times S^1$. It follows that the map $\rho\colon \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$, defined as $\rho([A,z]) = \lambda(A)$ is a principal S^1-fibration.

Example. $\mathrm{Spin}^c(3) \cong \mathrm{U}(2)$.

Let X be an n-dimensional Riemannian manifold and denote by $P_{\mathrm{SO}(n)} \to X$ the principal bundle of oriented orthonormal frames of X.

Definition ([17]). A Spin^c-structure on the manifold X is a principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c(n)} \xrightarrow{s} X$, together with a map $\pi\colon P_{\mathrm{Spin}^c(n)} \to P_{\mathrm{SO}(n)}$ that restricted to the

fibers is ρ, i.e., makes the following diagram commute, i.e.,



Since $\rho\colon \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ is the S^1-bundle of the unique non trivial line bundle over $\mathrm{SO}(n)$ (see [13]) and isomorphism classes of principal S^1-bundles over X are in one-to-one correspondence with elements of $H^2(X;\mathbb{Z})$, complex spin structures can be defined as cohomology classes:

Alternative definition ([17]). A Spin^c-structure on the manifold X is an element $\sigma \in H^2(P_{\mathrm{SO}(n)};\mathbb{Z})$ whose restriction to every fiber of $P_{\mathrm{SO}(n)} \to X$ is the unique non trivial element of $H^2(\mathrm{SO}(n);\mathbb{Z}) \cong \mathbb{Z}_2$, i.e.,

$$\mathrm{Spin}^c(X) = \{\sigma \in H^2(P_{\mathrm{SO}(n)};\mathbb{Z}) \mid \sigma_{|\mathrm{fiber}} \neq 0 \in H^2(\mathrm{SO}(n);\mathbb{Z})\}.$$

Let $\mathrm{BSO}(n)$ denote the Grassman manifold of oriented n-planes in $\mathbb{R}^\infty$ and let $\gamma_{\mathrm{SO}(n)}$ be the universal n-dimensional oriented vector bundle over $\mathrm{BSO}(n)$. Note that for discrete topological groups G, the classifying space BG is Eilenberg–MacLane of type $(G,1)$. Let h be the unique (up to homotopy) non-homotopically trivial map from $\mathrm{BSO}(n)$ to the Eilenberg–MacLane space $\mathrm{K}(\mathbb{Z},3)$. We fix h in its homotopy class and we define the fibration $\mathrm{BSpin}^c(n) \xrightarrow{\pi} \mathrm{BSO}(n)$ as the pull-back under h of the path-space fibration over $\mathrm{K}(\mathbb{Z},3)$. If we set $\gamma_{\mathrm{Spin}^c(n)} := \pi^*(\gamma_{\mathrm{SO}(n)})$, then:

Alternative definition. A Spin^c-structure on the manifold X is a homotopy class of bundle maps between the (stable) tangent bundle TX of X and $\gamma_{\mathrm{Spin}^c(n)}$.

2.1.1. The Chern map. Let us denote by $\alpha\colon \mathrm{Spin}^c(n) \to S^1$ the homomorphism $\alpha([A,z]) = z^2$. Then, to any Spin^c-structure σ on an n-dimensional manifold X (in the sense of Definition 2.1), one can associate a complex line bundle as follows: using the map α, the action of the group $\mathrm{Spin}^c(n)$ on the space $P_{\mathrm{Spin}^c(n)}$ extends to an action on the product $P_{\mathrm{Spin}^c(n)} \times \mathbb{C}$ and we consider its orbit space

$$\det(\sigma) := (P_{\mathrm{Spin}^c(n)} \times \mathbb{C})/\mathrm{Spin}^c(n),$$

called the *determinant line bundle* of σ. The Chern map

$$c\colon \mathrm{Spin}^c(X) \longrightarrow H^2(X)$$

is defined as $c(\sigma) := c_1(\det(\sigma))$, where c_1 is the first Chern class of the bundle $\det(\sigma)$, and it is affine over the doubling map $H^2(X) \xrightarrow{\cdot 2} H^2(X)$. See [6] for more details.

2.2. d-complex spin structures. In this subsection, for an even positive integer d, we define, d-complex spin structures (short Spin^c_d-structures) on n-dimensional manifolds and we describe some of their properties. Then, we focus on dimension three and we present a set of refined Kirby's moves for Spin^c_d-manifolds obtained by surgery along links in S^3.

Let $\beta\colon H^2(\mathrm{BSO}(n);\mathbb{Z}_2) \to H^3(\mathrm{BSO}(n);\mathbb{Z}_d)$ be the Bockstein homomorphism associated to the exact sequence of groups:

$$0 \longrightarrow \mathbb{Z}_d \xrightarrow{\cdot 2} \mathbb{Z}_{2d} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

The following lemma will help justify our construction of Spin^c_d-structures.

Lemma. *Let d be an even positive integer. For $n \geq 3$ the group $H^3(\mathrm{BSO}(n);\mathbb{Z}_d)$ is cyclic of order 2 generated by $\beta(w_2) \neq 0 \in H^3(\mathrm{BSO}(n);\mathbb{Z}_d)$, where w_2 is the second Stiefel–Whitney class.*

Since the elements of the group $H^3(\mathrm{BSO}(n);\mathbb{Z}_d)$ are in one-to-one correspondence with homotopy classes of maps in $[\mathrm{BSO}(n);\mathrm{K}(\mathbb{Z}_d,3)]$, there is a unique (up to homotopy) non-homotopically trivial map $g\colon \mathrm{BSO}(n) \to \mathrm{K}(\mathbb{Z}_d,3)$. We fix g in its homotopy class and define the fibration $\pi_d\colon \mathrm{BSpin}^c_d(n) \to \mathrm{BSO}(n)$ as the pull-back of the path space fibration of $\mathrm{K}(\mathbb{Z}_d,3)$ under the map g:

$$\begin{array}{ccc} \mathrm{BSpin}^c_d(n) & \longrightarrow & \mathrm{PK}(\mathbb{Z}_d,3) \\ \downarrow{\scriptstyle \pi_d} & & \downarrow \\ \mathrm{BSO}(n) & \xrightarrow{g} & \mathrm{K}(\mathbb{Z}_d,3) \end{array}$$

We set $\gamma_{\mathrm{Spin}^c_d(n)} = \pi_d^*(\gamma_{\mathrm{SO}(n)})$. Note that another choice of the map g in its homotopy class yields a different, but homotopy equivalent, space $\mathrm{BSpin}^c_d(n)$.

Let X be an n-dimensional Riemannian manifold.

Definition. A d-complex spin structure on X is a homotopy class of a bundle map between the (stable) tangent bundle TX of X and $\gamma_{\mathrm{Spin}^c_d(n)}$.

Alternative definition. A d-complex spin structure on X is a homotopy class of a lift $\bar{f}$ of f to $\mathrm{BSpin}^c_d(n)$, where $f\colon X \to \mathrm{BSO}(n)$ is a classifying map for the bundle TX.

$$\begin{array}{ccc} & & \mathrm{BSpin}^c_d(n) \\ & \overset{\bar{f}}{\nearrow} & \big\downarrow \pi_d \\ X & \xrightarrow{f} & \mathrm{BSO}(n) \end{array}$$

Since the fiber of π_d is the Eilenberg–MacLane space $\mathrm{K}(\mathbb{Z}_d, 2)$, there is a unique obstruction w_X to the existence of lifts $\bar{f}$ and this obstruction lies in the group $H^3(X; \mathbb{Z}_d)$.

Note that the universal obstruction $w \in H^3(\mathrm{BSO}(n); \mathbb{Z}_d)$ (obtained from w_X by setting $X = \mathrm{BSO}(n)$ and $f = \mathrm{id}_{\mathrm{BSO}(n)}$) is non-zero, therefore not all manifolds can admit Spin^c_d-structures. To see this, assume the contrary. Then, the fibration $\pi_d\colon \mathrm{BSpin}^c_d(n) \to \mathrm{BSO}(n)$ has a section s and the map $k \circ s$ lifts g to the contractible space $\mathrm{PK}(\mathbb{Z}_d, 3)$. Therefore g must be null-homotopic which contradicts the choice we made for g.

$$\begin{array}{ccc} \mathrm{BSpin}^c(\mathbb{Z}_d) & \xrightarrow{k} & \mathrm{PK}(\mathbb{Z}_d, 3) \\ {\scriptstyle s}\uparrow\big\downarrow{\scriptstyle \pi_d} & & \big\downarrow{\scriptstyle p} \\ \mathrm{BSO}(n) & \xrightarrow{g} & \mathrm{K}(\mathbb{Z}_d, 3) \end{array}$$

As a consequence, the universal obstruction w is the generator $\beta(w_2)$ of $H^3(\mathrm{BSO}(n); \mathbb{Z}_d)$.

Proposition. *The set* $\mathrm{Spin}^c_d(X)$ *of* d*-complex spin structures on a manifold* X *is non-empty, if and only if,* $\beta(w_2(X)) = 0 \in H^3(X; \mathbb{Z}_d)$. *If non-empty, the set* $\mathrm{Spin}^c_d(X)$ *is affine over* $H^2(X; \mathbb{Z}_d)$ ($w_2(X)$ *is the second Stiefel–Whitney class of* X).

Proof. Consider the commutative diagram

$$\begin{array}{ccc} & & \mathrm{BSpin}^c_d(n) \\ & \overset{\bar{f}}{\nearrow} & \big\downarrow \pi_d \\ X. & \xrightarrow{f} & \mathrm{BSO}(n) \end{array}$$

Let f be the classifying map of the (stable) tangent bundle of X. Then, the lifts $\bar{f}$ of f to $\mathrm{BSpin}^c(\mathbb{Z}_d)$ are in one-to-one correspondence with the sections of the bundle $f^*(\gamma_{\mathrm{Spin}^c_d(n)})$. The obstruction w_X to the existence of such sections is a characteristic cohomology class so by its naturality property we have

$$\begin{aligned} w_X(f^*(\gamma_{\mathrm{Spin}^c_d(n)})) &= f^*(w(\gamma_{\mathrm{Spin}^c_d(n)})) \\ &= f^*(\beta(w_2)). \end{aligned}$$

The above relation together with the naturality of Bockstein homomorphisms imply the result. The second part of the theorem follows by standard arguments of obstruction theory. □

2.2.1. Restriction to the boundary. Let us consider a manifold X with boundary. Then, any section of $TX|_{\partial X}$ transverse to ∂X and oriented outwards gives rise to a homotopy class of isomorphisms between the oriented vector bundles $\mathbb{R} \oplus T\partial X$ and $TX|_{\partial X}$. Therefore, there is a well-defined restriction map

$$\mathrm{Spin}^c_d(X) \to \mathrm{Spin}^c_d(\partial X)$$

affine over the map

$$i^*\colon H^2(X;\mathbb{Z}_d) \longrightarrow H^2(\partial X;\mathbb{Z}_d)$$

induced by the inclusion $i\colon \partial X \to X$.

2.2.2. From Spin^c to Spin^c_d. We have seen in Section 2.1 that the fibration

$$\pi\colon \mathrm{BSpin}^c(n) \longrightarrow \mathrm{BSO}(n)$$

is defined as the pull-back under h of the path-space fibration over $\mathrm{K}(\mathbb{Z},3)$. The modulo d restriction morphism $\xi\colon \mathbb{Z} \to \mathbb{Z}_d$ induces a map

$$\xi_*\colon \mathrm{K}(\mathbb{Z},3) \longrightarrow \mathrm{K}(\mathbb{Z}_d,3)$$

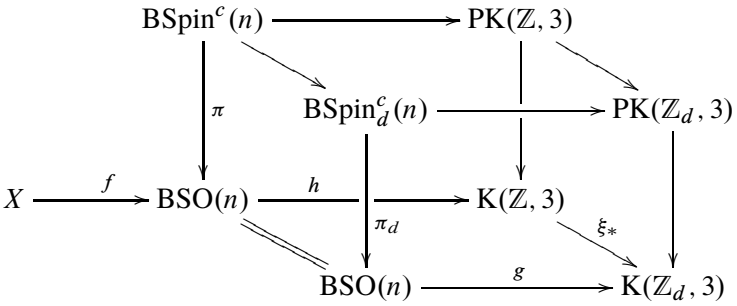
on the level of Eilenberg–MacLane spaces and further, via ξ_*, a natural map

$$\mathrm{BSpin}^c(n) \longrightarrow \mathrm{BSpin}^c_d(n).$$

As a result, there exists a well-defined natural map

$$\mathrm{Spin}^c(X) \longrightarrow \mathrm{Spin}^c_d(X)$$

affine over $H^2(X) \xrightarrow{\xi^*} H^2(X;\mathbb{Z}_d)$ induced by ξ.



2.3. Combinatorial description of Spin^c_d-structures. Let $L = (L_1, \ldots, L_n)$ be an oriented framed link in S^3 with linking matrix $(L_{ij})_{i,j=1,n}$ and denote by $S^3(L)$ the 3-manifold obtained by surgery. The manifold $S^3(L)$ is the boundary of a 4-manifold W_L, constructed from a 4-ball D^4 by attaching n 2-handles $(D^2 \times D^2)_i$ along embeddings of $-(S^1 \times D^2)_i$ in concordance with the orientation and framing of each component L_i. The 4-manifold W_L is sometimes called *trace of surgery*.

Let us define the set

$$\mathcal{S}^c_d(L) = \frac{\{(\sigma_1, \ldots, \sigma_n) \in (\mathbb{Z}_{2d})^n | \sigma_i = L_{ii} \pmod 2\}}{2\mathrm{Im}L}$$

whose elements will be called *modulo d Chern vectors*.

Theorem. *There is a canonical bijection*

$$\mathrm{Spin}^c_d(S^3(L)) \xrightarrow{\phi_L} \mathcal{S}^c_d(L).$$

Proof. We have seen in the previous section that Spin^c-structures induce d-complex spin structures. In particular, the map $r_d : \mathrm{Spin}^c(W_L) \to \mathrm{Spin}^c_d(W_L)$ is surjective since it is affine over the surjective map $H^2(W_L; \mathbb{Z}) \to H^2(W_L; \mathbb{Z}_d)$ induced by restriction modulo d of coefficients and similarly, the restriction map $r : \mathrm{Spin}^c_d(W_L) \to \mathrm{Spin}^c_d(S^3(L))$ is surjective since it is affine over the surjective map $H^2(W_L; \mathbb{Z}_d) \to H^2(S^3(L); \mathbb{Z}_d)$ induced by inclusion.

With the help of r and r_d, we define the map $\phi_L : \mathrm{Spin}^c_d(S^3(L)) \to \mathcal{S}^c_d(L)$ as follows: any $\sigma \in \mathrm{Spin}^c_d(S^3(L))$ is the image of an element $\tilde{\sigma} \in \mathrm{Spin}^c(W_L)$ under the composition $r \circ r_d$. The value $c(\tilde{\sigma}) \in H^2(W_L; \mathbb{Z}) \cong \mathbb{Z}^n$ is characteristic for L (see [6]), therefore $c(\tilde{\sigma}) = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ with $c_i = L_{ii} \pmod 2$. We set $\phi_L(\sigma)$ to be the image of $c(\tilde{\sigma}) \pmod{2d}$ in $\mathcal{S}^c_d(L)$.

- $\phi_L(\sigma)$ is well-defined: let us assume that $(r \circ r_d)^{-1}(\sigma)$ contains two different elements $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. Then, they differ by $y \in H^2(W_L) \cong \mathbb{Z}^n$ whose modulo d reduction belongs to $\mathrm{Im}L$ in $(\mathbb{Z}_d)^n$. Since $2y \pmod{2d}$ belongs to $2\mathrm{Im}L$ in $(\mathbb{Z}_{2d})^n$, we get that $[c(\tilde{\sigma}_2)] = [c(\tilde{\sigma}_1) + 2y] = [c(\tilde{\sigma}_1)]$ (the first equality follows from the fact that the Chern map is affine over the doubling map).
- ϕ_L is injective: let us assume that $\phi_L(\sigma_1) = \phi_L(\sigma_2)$. Then, the preimages $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ (of σ_1 and σ_2, respectively) under $r \circ r_d$ differ by $y \in H^2(W_L) \cong \mathbb{Z}^n$ such that $2y \pmod{2d}$ belongs to $2\mathrm{Im}L$ in $(\mathbb{Z}_{2d})^n$. This implies that $y \pmod{d}$ belongs to $\mathrm{Im}L$ in $(\mathbb{Z}_d)^n$ and therefore $\sigma_1 = \sigma_2$.

The sets $\mathrm{Spin}^c_d(S^3(L))$ and $\mathcal{S}^c_d(L)$ have the same cardinality, hence ϕ_L is bijective. □

From now on, we will refer to the set $\mathcal{S}^c_d(L)$ as the combinatorial description of d-complex spin structures on the surgered manifold $S^3(L)$.

2.4. Spin^c_d Kirby moves. A celebrated theorem of Kirby [7] states that two (oriented) framed links in S^3 produce the same manifold by surgery, if and only if, they are related by a finite sequence of local geometric transformations called *Kirby moves*. In what follows, we present a refined version of the original Kirby theorem for manifolds equipped with Spin^c_d-structures.

Theorem. *Let (L, σ) and (L', σ') be two oriented framed links with Chern vectors σ, σ'. Then, the manifolds $(S^3(L), \sigma)$ and $(S^3(L'), \sigma')$ are Spin^c_d-homeomorphic, if and only if, the pairs (L, σ) and (L', σ') are related by a finite sequence of the moves in Figure 2.1 or their inverses.*

Proof. We must check that for any of the Kirby moves (stabilization, handle slide and orientation reversal) $L \overset{K}{\to} L'$, the Chern vectors change under the homeomorphism $S^3(L) \to S^3(L')$ in the way described by Figure 2.1. To do this, note that the map $r \circ r_d \colon \mathrm{Spin}^c(W_L) \to \mathrm{Spin}^c_d(S^3(L))$ defined in the proof of Theorem 2.3 is surjective and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spin}^c(W_L) & \xrightarrow{K} & \mathrm{Spin}^c(W_{L'}) \\ \downarrow{\scriptstyle r\circ r_d} & & \downarrow{\scriptstyle r'\circ r'_d} \\ \mathrm{Spin}^c_d(S^3(L)) & \xrightarrow{K} & \mathrm{Spin}^c_d(S^3(L')) \end{array}$$

Since Spin^c-structures on W_L are combinatorially described as the elements of the set

$$\{(c_1, \ldots, c_n) \in \mathbb{Z}^n \mid c_i = L_{ii} \pmod 2\}$$

and their change under Kirby moves is known ([6]), the conclusion follows. □

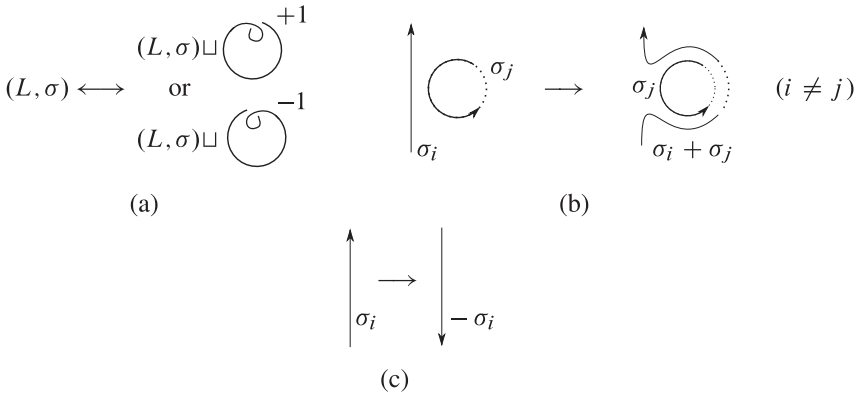


Figure 2.1. Refined Spin^c_d-Kirby moves (a) Stabilization (b) Handle slide (c) Orientation reversal. Note that we use the blackboard framing and the labels refer to modulo d Chern vectors.

3. Generalized spin structures

In [2] the second author introduced spin structures modulo an even integer d (short Spin$_d$-structures). In dimension three, he gave a combinatorial description of such structures as well as a refined set of Kirby moves. In this section we recall his results using the notations of Section 2.3 and extend the definition to a possibly non cyclic group of coefficients K with a distinguished order 2 element $v \in K$.

3.1. (K, v)-spin structures. Let K be a finite abelian group and v a non trivial element in the 2-torsion subgroup $\mathrm{Tor}_2(K)$, then we can define (K, v) generalized spin structures.

Definition. For $n \geq 3$, a (K, v) generalized spin structure on an SO(n) principal bundle $\mathrm{SO}(n) \hookrightarrow P \to M$ is a cohomology class $\sigma \in H^1(P, K)$ whose restriction to the fiber is equal to v in $H^1(\mathrm{SO}(n), K) \cong \mathrm{Tor}_2(K)$. A (K, v) generalized spin structure on an oriented manifold of dimension ≥ 3 is a (K, v) spin structure on its oriented framed bundle.

As usual this definition can be extended to dimensions less than 3 by using stabilization.

If K is decomposed as a product of cyclic groups, $K = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}$, then a (K, v) generalized spin structure is a sequence $(\sigma_1, \ldots, \sigma_k)$ where σ_j is either a d_j-spin structure or a mod d_j cohomology class, depending if the component v_j of the 2-torsion element v has order 2 or vanishes.

3.2. Combinatorial description of Spin_d-structures. If $K = \mathbb{Z}_d$, let us denote the set of $\left(K, v = \frac{d}{2}\right)$-structures on M by $\mathrm{Spin}_d(M)$. In the case when $M = S^3(L)$ (is obtained by surgery on a framed link in S^3) this set admits a nice combinatorial description.

Consider the set

$$\mathcal{S}_d(L) = \left\{(s_1, \ldots, s_n) \in (\mathbb{Z}_d)^n \;\middle|\; \sum_{j=1}^{n} L_{ij} s_j = \frac{d}{2} L_{ii} \pmod d\right\}.$$

The elements of $\mathcal{S}_d(L)$ are called *modulo d characteristic solutions of L*.

Lemma. *There is a canonical bijection*

$$\psi_L \colon \mathrm{Spin}_d(S^3(L)) \longrightarrow \mathcal{S}_d(L).$$

Proof. Given a Spin_d-structure σ on $S^3(L)$ it can be extended to W_L if and only if a certain cohomology class in $H^2(W_L, S^3(L); \mathbb{Z}_d)$ vanishes. We denote this class by $w(W_L; \sigma)$ and we call it relative obstruction. To any $\sigma \in \mathrm{Spin}_d(S^3(L))$ there is associated a relative obstruction $w(W_L; \sigma)$ in $H^2(W_L, S^3(L); \mathbb{Z}_d)$. Since the group $H^2(W_L, S^3(L); \mathbb{Z}_d)$ is free of rank n, taking the coefficients of the relative obstruction we obtain a map $\psi_L \colon \mathrm{Spin}_d(S^3(L)) \to (\mathbb{Z}_d)^n$. We will show that this map is injective and its image coincides with $\mathcal{S}_d(L)$. Let us consider the embedding $\eta \colon \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_d$. Then, the relative obstruction $w(W_L; \sigma) = \eta_*(w_2(W_L))$, where η_* is the induced map on the level of cohomology. Given an integral 2-cycle x, we denote by $x \cdot x$ its self-intersection number and by $[x]_m$ its modulo m restriction. The second Stiefel–Whitney class $w_2(W_L)$ is defined by the following equation:

$$\langle w_2(W_L), [x]_2 \rangle = x \cdot x \pmod 2, \quad \text{for all } x,$$

therefore, the relative obstruction is defined by

$$\langle w(W_L; \sigma), [x]_d \rangle = \frac{d}{2} x \cdot x \pmod d, \quad \text{for all } x.$$

Using functoriality and writing $w(W_L;\sigma)$ in the preferred basis of $H^2(W_L;\mathbb{Z}_d)$, the result follows. □

The set $\mathcal{S}_d(L)$ will be referred to as a combinatorial description of Spin$_d$-structures on the surgered manifold $S^3(L)$.

3.3. Spin$_d$ Kirby moves. The second author proved the following result:

Theorem ([2]). *Let (L,s) and (L',s') be two oriented framed links with characteristic solutions s,s'. Then, the manifolds $(S^3(L),s)$ and $(S^3(L'),s')$ are* Spin$_d$*-homeomorphic, if and only if, the pairs (L,s) and (L',s') are related by a finite sequence of the moves in Figure 3.1 or their inverses.*

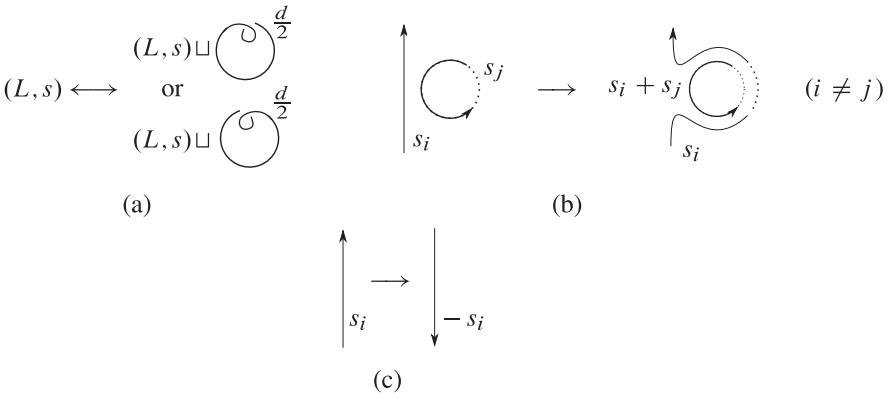


Figure 3.1. Refined Spin$_d$-Kirby moves (a) Stabilization (b) Handle slide (c) Orientation reversal. Note that we use the blackboard framing and the labels refer to modulo d characteristic solutions of L.

4. Modular categories

In this section we briefly recall the construction of WRT invariants.

4.1. Modular categories. We refer to [20] for a detailed treatment of this material.

A *ribbon* category is a monoidal category $\mathcal{C}$ equipped with braiding b, twist θ and duality satisfying compatibility conditions ensuring the existence of a

representation of the category of $\mathcal{C}$-colored tangle and more generally of $\mathcal{C}$-colored ribbon graphs, which is called Reshetikhin–Turaev ribbon functor and denoted by $F_{\mathcal{C}}$. Hence, in ribbon categories, a trace of an endomorphism f can be defined as this functor $F_{\mathcal{C}}(\hat{f})$ evaluated on the closure $\hat{f}$ of f. Graphically, we represent $f : x \to x$ by a rectangle with one outgoing and one incoming lines colored by x, and connecting these two lines in a circle we get $\hat{f}$. Then dimension of an object is the trace of its identity endomorphism.

Let $\mathbf{k}$ be a field of zero characteristic. A ribbon category is called $\mathbf{k}$-linear or pre-additive over $\mathbf{k}$ if the Hom sets are $\mathbf{k}$-vector spaces, composition and tensor product are bilinear and $\mathrm{End}(\mathbf{1}) = \mathbf{k}$. An object λ of the category is called *simple* if the map $u \mapsto u \cdot \mathrm{id}_\lambda$ from $\mathbf{k} = \mathrm{End}(\mathbf{1})$ to $\mathrm{End}(\lambda)$ is an isomorphism.

We will denote by $\mathcal{C}^{\oplus}$ the additive closure of a pre-additive ribbon category $\mathcal{C}$, which admits direct sums of objects and compositions of morphisms are modelled on the matrix multiplication ([12]).

Definition. A modular category over the field $\mathbf{k}$ is a $\mathbf{k}$-linear ribbon category $\mathcal{C}$ with a finite set of simple objects Γ that satisfies the following axioms.

(1) Normalization. The trivial object $\mathbf{1}$ is in Γ.

(2) Domination. For any object V of the category, there exists a finite decomposition

$$\mathrm{id}_V = \sum_i f_i \mathrm{id}_{\lambda_i} g_i$$

with $\lambda_i \in \Gamma$, for all i.

(3) Non-degeneracy. The matrix $S = (S_{\lambda\mu})_{\lambda,\mu\in\Gamma}$ is invertible over $\mathbf{k}$, where $S_{\lambda\mu} \in \mathbf{k}$ is the endomorphism of the trivial object associated with the (λ, μ)-colored, 0-framed Hopf link with linking $+1$.

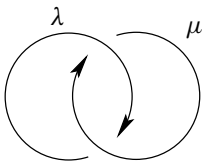


Figure 4.1. The Hopf link with linking number $+1$ and colors λ and μ

The category is called pre-modular if we remove the last axiom. Replacing $\mathcal{C}$ with its additive closure, we can reformulate the domination axiom as follows: for any $\lambda, \nu, \mu \in \Gamma$ there exist positive integers $C_{\lambda\nu}^{\mu}$ called *structure constants*, such that $\lambda \otimes \nu \simeq \bigoplus_{\mu\in\Gamma} C_{\lambda\nu}^{\mu}\ \mu$. The domination axiom says that any object decomposes into a direct sum of simple ones.

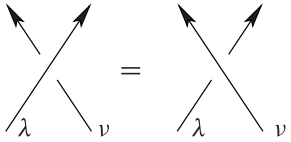
Example. Let $\mathfrak{g} = \mathfrak{sl}_N$ and q be an $(N + K)$th root of unity. Then, there exists an associated modular category $\mathcal{C}(N, K)$ $(N, K \geq 2)$ with simple objects given by partitions λ from

$$\Gamma = \{\lambda = (\lambda_1, \ldots, \lambda_s) \mid \lambda_1 \leq K, s < N\}.$$

See e.g. [3] for more details.

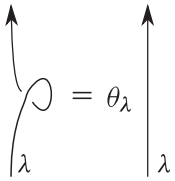
4.2. Properties of pre-modular categories. In what follows, we describe basic properties of pre-modular categories that will be relevant to the rest of the paper. We follow the lines of [1]. Unless otherwise stated, $\mathcal{C}$ is a pre-modular category over a field $\mathbf{k}$ with zero characteristic and Γ is the set of representatives of simple objects.

Definition. (a) An object λ of $\mathcal{C}$ is called *transparent* if for any object ν the following morphisms in $\mathcal{C}$ are equal

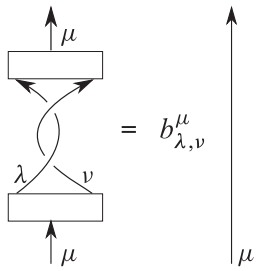


(b) A morphism $f \in \mathrm{Hom}(\lambda, \nu)$ is called *negligible* if $\mathrm{tr}(fg) = 0$ for any $g \in \mathrm{Hom}(\nu, \lambda)$.

For any object $\lambda \in \Gamma$ the *twist coefficient* θ_λ is defined by the equality:



(c) The *braiding coefficients* between two objects $\lambda, \nu \in \Gamma$ are defined as a collection $\{b^{\mu}_{\lambda,\nu}\}$ for all μ which is a summand in $\lambda \otimes \nu$ such that



where the boxes represent a morphism $i\colon \mu \to \lambda \otimes \nu$ and a morphism $p\colon \lambda \otimes \nu \to \mu$ such that $p \circ i = \mathrm{id}_{\mu}$. Note that $b^{\nu}_{\lambda,\mu} = \theta_{\mu}(\theta_{\lambda}\theta_{\nu})^{-1}$.

We discuss in detail a formula that will be used extensively throughout the paper. Let us start by fixing some notations. Given objects $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \Gamma$, we denote by $H^{\beta_1,\ldots,\beta_m}_{\alpha_1,\ldots,\alpha_n}$ the $\mathbf{k}$-module of morphisms $\mathrm{Hom}_{\mathcal{C}}(\alpha_1 \otimes \cdots \otimes \alpha_n, \beta_1 \otimes \cdots \otimes \beta_m)$.

The modules $H^{\lambda\nu}_{\mu}$, $H^{\lambda}_{\mu\nu^*}$, $H^{\mu^*\lambda}_{\nu^*}$, $H^{\mu^*}_{\nu^*\lambda^*}$, $H^{\nu\mu^*}_{\lambda^*}$ and $H^{\nu}_{\lambda^*\mu}$ are mutually isomorphic, as well as the modules $H_{\mu\nu^*\lambda^*}$, $H^{\lambda\nu\mu^*}$ and all obtained from them by a cyclic permutation of colors. Identifying these modules along the isomorphisms, we get a symmetrized multiplicity module $\widetilde{H}^{\lambda\nu\mu^*}$ for which only the cyclic order of colors is important. The elements of $\widetilde{H}^{\lambda\nu\mu^*}$ are represented by a round coupon with one incoming line (colored with μ) and two outgoing ones (colored by λ and ν).

The pairing

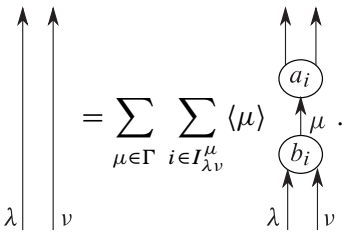
$$\langle\ ,\ \rangle\colon \widetilde{H}^{\lambda\nu\mu^*} \otimes \widetilde{H}^{\mu\lambda^*\nu^*} \longrightarrow \mathbf{k}$$

defined as

$$\langle f, g\rangle = \mathrm{tr}(fg)$$

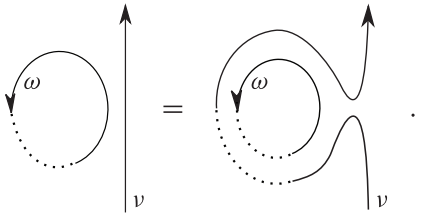
is non-degenerate since the category $\mathcal{C}$ can be assumed without negligible morphisms (if any, they can be quotiened out). The symmetrized modules $\widetilde{H}^{\lambda\nu\mu^*}$ and $\widetilde{H}^{\mu\lambda^*\nu^*}$ are dual to each other, therefore we can choose bases $(a_i)_{i\in I^{\mu}_{\lambda\nu}}$ for $\widetilde{H}^{\mu\lambda^*\nu^*}$ and $(b_i)_{i\in I^{\mu}_{\lambda\nu}}$ for the module $\widetilde{H}^{\lambda\nu\mu^*}$ that are dual with respect to $\langle\ ,\ \rangle$. Note that the composition $b_j \circ a_i$ is an endomorphism of the simple object μ, so it is of the form $c_{ij} \cdot \mathrm{id}_{\mu}$, $c_{ij} \in \mathbf{k}$. Comparing the traces gives $\delta_{ij} = c_{ij}\langle\mu\rangle$ so $b_j \circ a_i = \delta_{ij}\langle\mu\rangle^{-1}\mathrm{id}_{\mu}$. For any simple objects $\lambda, \nu \in \Gamma$, the domination axiom

applied to $\mathrm{id}_{\lambda\otimes\nu}$ yields the following relation, known as the *fusion formula*:

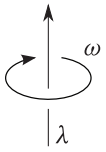


For any pre-modular category $\mathcal{C}$, let us define the *Kirby color* $\omega = \sum_{\lambda\in\Gamma}\langle\lambda\rangle\lambda$ as an element of $K_0(\mathcal{C}^{\oplus}) \otimes \mathbf{k}$, where $K_0(\mathcal{C}^{\oplus})$ denotes the Grothendieck ring of the category $\mathcal{C}^{\oplus}$.

Proposition (sliding property, [1]). *In any pre-modular category the following handle sliding identity holds*

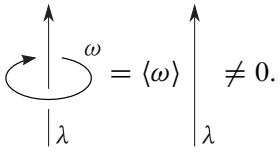


Lemma (killing property, [1]). *Let $\mathcal{C}$ be a pre-modular category with $\langle\omega\rangle \neq 0$ and $\lambda \in \Gamma$. Then, the morphism*

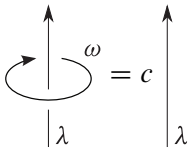


is non-trivial if and only if λ is transparent.

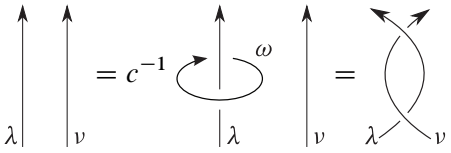
Proof. If λ is transparent, then the map



Conversely, if we assume that for some $c \in \mathbf{k}$, $c \neq 0$ we have



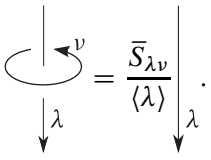
then, by sliding any ν-colored strand along the ω-colored one, we obtain the following equalities of morphisms in $\mathcal{C}$



and therefore, λ is transparent. □

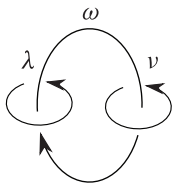
Proposition ([1]). *A pre-modular category with $\langle\omega\rangle \neq 0$ and with no non-trivial transparent objects is modular.*

Proof. Let $\overline{S} = (\overline{S}_{\lambda\nu})_{\lambda\nu}$ be a matrix with entries given by the relation

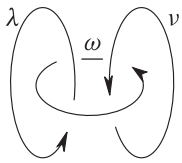


We want to prove that the product $S \cdot \overline{S} = \langle\omega\rangle\,\mathrm{I}$, where I is the identity matrix of size $|\Gamma|$. We have

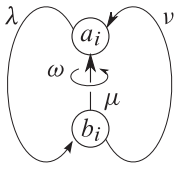
$$(S \cdot \overline{S})_{\lambda\nu} =$$



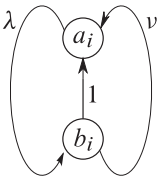
$$=$$



$$= \sum_{\mu\in\Gamma} \sum_{i\in I^{\mu}_{\lambda\nu *}} \langle\mu\rangle$$



$$= \langle\omega\rangle \sum_{i \in I^1_{\lambda\nu^*}}$$



$$= \langle\omega\rangle \cdot \delta_{\lambda\nu},$$

where $\delta_{\lambda\nu}$ is the Kronecker index. The second equality holds by isotopy, the third equality holds by the *fusion formula*, while the fourth equality is a consequence of the *killing property*. The last equality can be proved using the structure of the modules $\mathrm{Hom}(\mathbf{1}, \lambda \otimes \nu^*)$ and $\mathrm{Hom}(\lambda \otimes \nu^*, \mathbf{1})$. □

Note. If $\mathcal{C}$ is a modular category, then $\langle\omega\rangle$ is invertible in $\mathbf{k}$, and hence $\langle\omega\rangle \neq 0$.

4.3. WRT invariants. Given a modular category $\mathcal{C}$ and a closed 3-manifold $M = S^3(L)$ obtained by surgery on S^3 along a framed link L, whose linking matrix has b_+ positive and b_- negative eigenvalues, we define

$$\tau_{\mathcal{C}}(M) = \frac{F_{\mathcal{C}}(L(\omega, \ldots, \omega))}{F_{\mathcal{C}}(U_1(\omega))^{b_+} F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}} \tag{1}$$

where $U_{\pm 1}$ denotes the ± 1-framed unknot and $F_{\mathcal{C}}\colon \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C}$ is the natural ribbon functor, $\mathrm{Rib}_{\mathcal{C}}$ is the category of ribbon graphs.

Theorem ([20]). *For any modular category $\mathcal{C}$, $\tau_{\mathcal{C}}(M)$ defines a topological invariant of M, independent on the choice of the link L.*

Proof. We need to show that $\tau_{\mathcal{C}}(M)$ is well-defined and does not change under Kirby moves.

The fact that $F_{\mathcal{C}}(U_{\pm 1}(\omega))$ are non-zero follows from the properties of the ribbon functor $F_{\mathcal{C}}$ and the non–degeneracy axiom for $\mathcal{C}$.

The first Kirby move is easy to establish. The invariance under the second Kirby move is provided by Proposition 4.2. In order to prove invariance under orientation reversal, let us consider the link L' obtained from L by reversing the orientation of a component L_k. Without loss of generality, we may assume $k = 1$.

Thus we have

$$\begin{aligned} F_{\mathcal{C}}(L'(\omega,\dots,\omega)) &= \sum_{\lambda\in\Gamma}\langle\lambda\rangle F_{\mathcal{C}}(L'(\lambda,\omega,\dots,\omega)) \\ &= \sum_{\lambda\in\Gamma}\langle\lambda^*\rangle F_{\mathcal{C}}(L(\lambda^*,\omega,\dots,\omega)) \\ &= \sum_{\mu\in\Gamma}\langle\mu\rangle F_{\mathcal{C}}(L(\mu,\omega,\dots,\omega)) \\ &= F_{\mathcal{C}}(L(\omega,\dots,\omega)). \end{aligned}$$

The linking matrix $(L'_{ij}) = {}^tS(L_{ij})S$ with $S = \mathrm{diag}(-1,1,\dots,1)$ hence the matrices (L'_{ij}) and (L_{ij}) have the same eigenvalues. In particular, $b'_+ = b_+$ and $b'_- = b_-$ so $\tau_{\mathcal{C}}(M,L') = \tau_{\mathcal{C}}(M,L)$. This concludes the proof. □

5. Group-categories from invertible objects

In this section, we show that for any modular category $\mathcal{C}$, its group of invertible distinguished objects $G \subset \Gamma$ defines the structure of $\widehat{G}$-category on $\mathcal{C}$.

5.1. Invertible objects. An object λ in Γ is called *invertible* if there exists another object $\nu \in \Gamma$ such that $\lambda \otimes \nu \cong 1$. Invertible objects in Γ form a finite abelian group under tensor multiplication. Here isomorphisms may be necessary to identify a tensor product or dual with an object in Γ. Let us denote this group by $G \subset \Gamma$ and let $\widehat{G} = \mathrm{Hom}(G,\mathbb{C}^*)$ be the group of its characters. We say that the *tensor order* of λ is d if $\lambda^{\otimes d} \cong 1$ with d minimal.

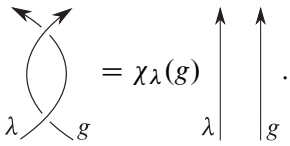
Example. In Example 4.1 the object K (the longest row) is invertible of order N.

The following lemma shows that the group G acts on Γ.

Lemma. *For any $\lambda \in \Gamma$ and any invertible $g \in G$, $\lambda \otimes g$ is isomorphic to an object in Γ.*

Proof. By the domination axiom we have $\lambda \otimes g \cong \bigoplus_i \mu_i$. Multiplying this identity by the inverse of g, we get $\lambda = \bigoplus_i (\mu_i \otimes g^{-1})$. Thus, the right hand is simple, and so $\lambda \otimes g$ has to be simple too. □

5.2. Group-category. Assume $\mathcal{C}$ is a modular category with $G \subset \Gamma$ the group of invertible distinguished objects. We can define a $\widehat{G}$-structure on $\mathcal{C}$ as follows: Given $\lambda \in \Gamma$, the braiding coefficient of λ with elements of G defines a map $\Gamma \to \widehat{G}$ which associates to each λ a character $\chi_\lambda \in \widehat{G}$ defined by the equality:



Indeed, the previous lemma implies that the braiding operator acts on λ and g as a multiplication by $S_{\lambda g}/\langle\lambda\rangle\langle g\rangle$, i.e. only one braiding coefficient is non-zero. Using the fact that g is of finite order, we deduce that this coefficient is a root of unity of that order. Observe that $\chi_{\lambda\otimes\mu} = \chi_\lambda\chi_\mu$.

For any $\alpha \in \widehat{G}$, let $\mathcal{C}_\alpha$ be the full subcategory whose objects are isomorphic to direct sums of simple object in $\{\lambda \in \Gamma \mid \chi_\lambda = \alpha\}$. Using the semisimple decompositionof objects, we have that

- each object of $\mathcal{C}$ is isomorphic to a direct sum over $\alpha \in \widehat{G}$ of objects in $\mathcal{C}_\alpha$;
- $\mathrm{Hom}_{\mathcal{C}}(U, V) = 0$ if $U \in \mathrm{Ob}(\mathcal{C}_\alpha)$, $V \in \mathrm{Ob}(\mathcal{C}_\beta)$, $\alpha \neq \beta$;
- for $U \in \mathrm{Ob}(\mathcal{C}_\alpha)$ and $V \in \mathrm{Ob}(\mathcal{C}_\beta)$, $U \otimes V \in \mathrm{Ob}(\mathcal{C}_{\alpha\beta})$.

This proves the following.

Proposition. *A modular category $\mathcal{C}$ with group G of invertible distinguished objects is $\widehat{G}$-category.*

If $\lambda \in \mathcal{C}_\alpha$, for some α, we say that λ is an homogenous object of *degree* $\alpha = \deg(\lambda)$.

Additive notation in cyclic case. If G is cyclic of order d, we call $\mathcal{C}$ a *modular d-category*. Fixing a generator t of G and a primitive dth root of unity e_d, we have the decomposition

$$\Gamma = \Gamma_0 \sqcup \Gamma_1 \sqcup \cdots \sqcup \Gamma_{d-1},$$

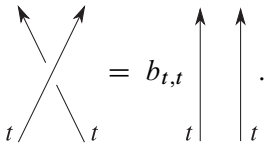
where $\Gamma_i = \{\lambda \in \Gamma | \chi_\lambda(t) = e_d^i\}$. In this case we identify $\widehat{G}$ with $\mathbb{Z}_d$ and use additive notation.

Example. Let us assume that $N = (N, K)$ in Examples 4.1 and 5.1. Then $G = \mathbb{Z}_N$ is generated by K. Hence, the category $\mathcal{C}(N, K)$ is a modular N-category with $\deg(\lambda) = \sum_i \lambda_i \pmod N$.

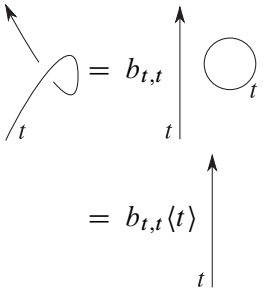
We will need the following lemma.

Lemma. *Let $\mathcal{C}$ be a modular d-category such that the group $G \cong \mathbb{Z}_d$ is generated by t. Then $\langle t \rangle = 1$ if d is odd, and $\langle t \rangle = \pm 1$ if d is even.*

Proof. Let $b_{t,t}$ be the braiding coefficient, such that



Using



we have $\theta_t = b_{t,t}\langle t \rangle$. Similarly $\theta_t^{-1} = b_{t,t}^{-1}\langle t \rangle$. We get that $\langle t \rangle = \pm 1$. The object t has order d so we have $\theta_{t^d} = 1$. On the other hand using that $b_{t,t}^d = 1$, we compute $\theta_{t^d} = b_{t,t}^{d^2}\theta_t^d = \langle t \rangle^d$. The statement follows. □

Let us recall the following definition which will play a central role in our exposition.

Definition. Let $H \subseteq G$ be a subgroup of the group G of distinguished objects in a modular category $\mathcal{C}$. The modular category $\mathcal{C}$ is called *H-refinable* if all objects of H have trivial $\widehat{H}$-degree. If, in addition, H has at least one element with twist coefficient -1, we call the category *H-spin*.

In the case H is cyclic of order d, we will use the shorthand d-refinable and d-spin.

Example. Given an even integer $d = (N, K) = \alpha\beta$, such that $(\alpha, N') = (\beta, K') = 1$ for $N' = \frac{N}{d}$ and $K' = \frac{K}{d}$, The second author constructed in [3] a category $\dot{\mathcal{C}}(N, K)$ with

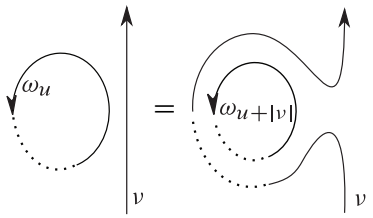
$$\dot{\Gamma} = \{(1^N)^{\otimes i} \otimes \lambda \mid 0 \leq i < \alpha, \lambda_1 \leq K, \lambda_1^\vee < N\}$$

and $G = \mathbb{Z}_d$ generated by $1^N \otimes K$. Here $\lambda^\vee$ denotes the transpose partition. The second author proved that the category $\dot{\mathcal{C}}(N, K)$ is d-spin modular.

5.3. Sliding identities in pre-modular group-categories. Assume $\mathcal{C}$ is a pre-modular d-category. For any $d \in \mathbb{Z}_d$, let us define the refined Kirby colors as follows:

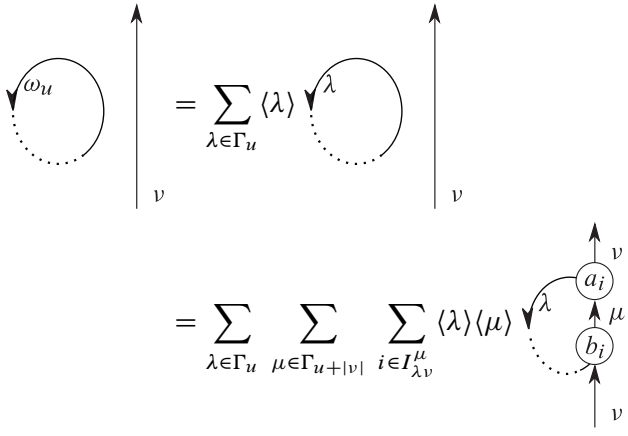
$$\omega_u = \sum_{\lambda \in \Gamma_u} \langle \lambda \rangle \lambda.$$

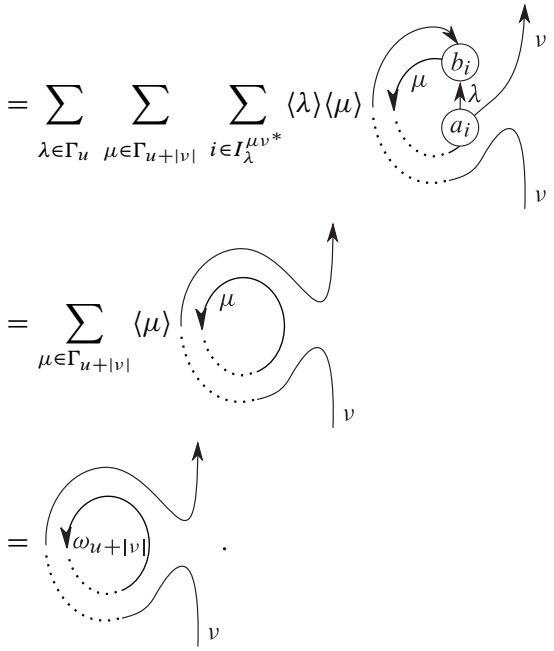
Lemma (graded sliding property). *In any pre-modular d-category $\mathcal{C}$ we have the following equality of morphisms*:



with $|\nu| = \deg(\nu)$.

Proof. The proof is the same as in the non-graded case, using the fact that $\mathrm{Hom}(\lambda \otimes \nu, \mu)$ is zero unless $\deg(\mu) = \deg(\lambda) + \deg(\nu)$. Hence, the sum over μ can be restricted to $\Gamma_{u+|\nu|}$.





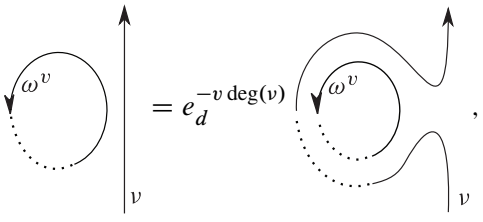
In the second and the fourth equalities we use the *fusion formula*, the third equality holds by isotopy. □

For any $0 \leq v < d$, and a primitive dth root of unity e_d, we define

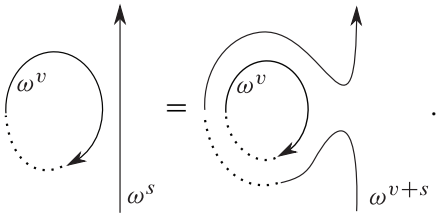
$$\omega^v = \sum_{\lambda\in\Gamma} e_d^{v\deg(\lambda)} \langle\lambda\rangle\, \lambda$$

the dual refined Kirby color. Note that $\omega^0 = \omega$, however other dual Kirby colors depend on the choice of e_d. We use the graded sliding identity to prove the following lemma.

Proposition (dual sliding property). *In any pre-modular d-category $\mathcal{C}$ we have the following equality of morphisms*:

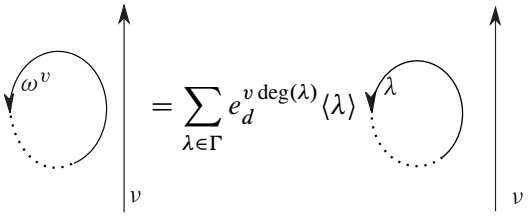


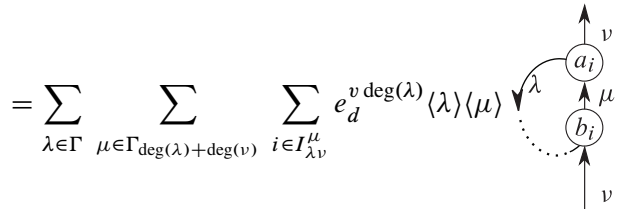
(a)

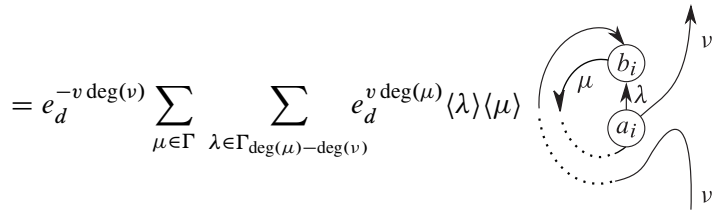


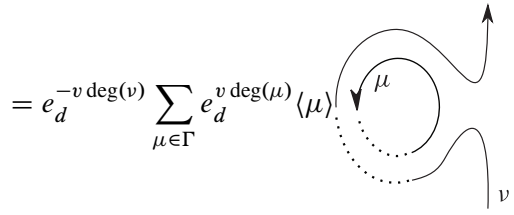
(b)

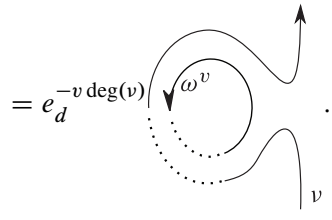
Proof. (a) We have the following equalities:





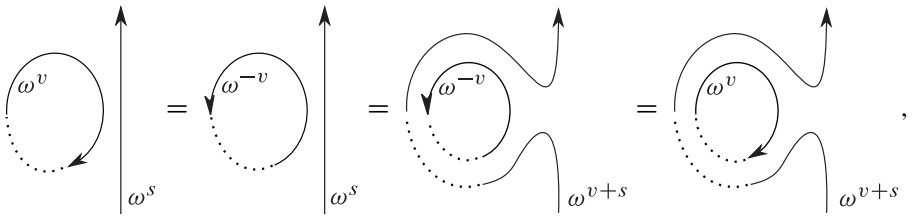






The second and fourth equalities follow from the *fusion formula*, the third equality holds by isotopy.

(b) For a $\mathcal{C}$-colored ribbon graph Ω with an annulus component a colored with the dual Kirby color ω^s, and the ribbon graph Ω' obtained from Ω by reversing the orientation of a and changing its color to ω^{-s}, $F_{\mathcal{C}}(\Omega) = F_{\mathcal{C}}(\Omega')$. In particular, we have



where the second equality is a straightforward application of (a). □

6. Spin modular categories

This section is devoted to the proof of Theorem 1. We give first the proof in the cyclic case and then deduce the general statement. For d-spin modular categories we obtain invariants of 3-manifolds equipped with Spin_d structure. In cohomological case we get refined invariants for 3-manifolds with modulo d 1-dimensional cohomology classes.

For the rest of this section $M = S^3(L)$ is a closed 3-manifold obtained by surgery on S^3 along a framed link $L = (L_1, \dots, L_n)$ whose linking matrix (L_{ij}) has b_+ positive and b_- negative eigenvalues.

6.1. Spin refinements, cyclic case. Let d be an even, positive integer and $\mathcal{C}$ be a d-spin modular category. For any $s = (s_1, \dots, s_n) \in \mathrm{Spin}_d(M)$ let us define

$$\tau_{\mathcal{C}}(M, s) = \frac{F_{\mathcal{C}}(L(\omega_{s_1}, \dots, \omega_{s_n}))}{F_{\mathcal{C}}(U_1(\omega))^{b_+} F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}}. \tag{2}$$

Theorem. *For any d-spin modular category $\mathcal{C}$, $\tau_{\mathcal{C}}(M, s)$ is a topological invariant of the pair (M, s). Moreover,*

$$\tau_{\mathcal{C}}(M) = \sum_{s \in \mathrm{Spin}_d(M)} \tau_{\mathcal{C}}(M, s)\,.$$

Proof. To prove the first statement we need to show that $\tau(M,s)$ is well-defined and does not change under refined Kirby moves.

Invariance under the first two Spin_d Kirby moves follows immediately from the *graded sliding property* and the next lemma which implies that $F_{\mathcal{C}}(U_\epsilon(w)) = F_{\mathcal{C}}(U_\epsilon(w_{\frac{d}{2}}))$ is invertible for $\epsilon = \pm 1$. In order to prove invariance under orientation reversal, let us consider the link L' obtained from L by reversing the orientation of a component L_k. Without loss of generality, we may assume that $k = 1$. We have seen in the proof of Theorem 4.3 that the linking matrices (L'_{ij}) and (L_{ij}) have the same eigenvalues so $b'_+ = b_+$, $b'_- = b_-$.
Since $s' = (-s_1, s_2, \ldots, s_n)$ we have

$$\begin{aligned} F_{\mathcal{C}}(L'(\omega_{-s_1}, \omega_{s_2}, \ldots, \omega_{s_n})) &= \sum_{\lambda \in \Gamma_{-s_1}} \langle \lambda \rangle F_{\mathcal{C}}(L'(\lambda, \omega_{s_2}, \ldots, \omega_{s_n})) \\ &= \sum_{\lambda \in \Gamma_{-s_1}} \langle \lambda^* \rangle F_{\mathcal{C}}(L(\lambda^*, \omega_{s_2}, \ldots, \omega_{s_n})) \\ &= \sum_{\mu \in \Gamma_{s_1}} \langle \mu \rangle F_{\mathcal{C}}(L(\mu, \omega_{s_2}, \ldots, \omega_{s_n})) \\ &= F_{\mathcal{C}}(L(\omega_{s_1}, \omega_{s_2}, \ldots, \omega_{s_n})) \end{aligned}$$

and therefore $\tau_{\mathcal{C}}(M, L'; s') = \tau_{\mathcal{C}}(M, L; s)$. It remains to prove that

$$F_{\mathcal{C}}(L(\omega_{s_1}, \ldots, \omega_{s_n})) = 0$$

if s_i do not solve $\sum_{j=1}^n L_{ij} s_j = \frac{d}{2} L_{ii} \pmod d$. The proof is in three steps.

Assume that the first component L_1 of our link is the ± 1-framed unknot. Then it can be unlinked from the rest of L by applying the Fenn–Rourke move. The graded sliding identity and Lemma 6.1 tell us that s_1 should solve the above equations.

Assume L_1 is the a-framed unknot. Then we add a ± 1-framed unknot to our link (with an invertible invariant) and slide it along L_1 (perform the inverse Fenn–Rourke move). This changes the framing on L_1 by ∓ 1 and allows to reduce this case to the previous one.

Finally, assume L_1 is arbitrary. Then we can unknot L_1 by adding ± 1-framed unknots to our link in such a way, that their linking number with L_1 is zero. This again reduces the situation to the previous case. □

Lemma. *For any d-spin modular category $\mathcal{C}$, $F_{\mathcal{C}}(U_{\pm 1}(w_u))$ is zero unless $u = \frac{d}{2}$.*

Proof. Recall that invertible objects form an abelian group under tensor multiplication, which acts on Γ. In particular, its cyclic subgroup $H = \langle t \rangle \cong \mathbb{Z}_d$ acts on each Γ_u. Let us denote by $\widetilde{\Gamma}_u$ the set of orbits under this action and by $\tilde{\omega}_u$ the corresponding reduced Kirby color. Note that $\omega_u = \sum_{i=0}^{d-1} \langle t \rangle^i t^i \tilde{\omega}_u$. Let $H_{a,b}$ be the (a, b)-framed Hopf link with linking matrix -1.

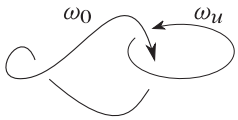


Figure 6.1. The Hopf link $H_{1,0}(\omega_0, \omega_u)$.

After sliding the second component of $H_{(1,0)}(\omega_0, \omega_u)$ along the first one we get

$$\begin{aligned} F_{\mathcal{C}}(U_1(w_u))F_{\mathcal{C}}(U_{-1}(w_u)) &= F_{\mathcal{C}}(H_{1,0}(\omega_0, \omega_u)) \\ &= \sum_{i,j=0}^{d-1} \langle t \rangle^{i+j} F_{\mathcal{C}}(H_{1,0}(t^i \tilde{\omega}_0, t^j \tilde{\omega}_u)) \\ &= \sum_{i,j=0}^{d-1} \langle t \rangle^{i+j} F_{\mathcal{C}}(H_{1,0}(\tilde{\omega}_0, \tilde{\omega}_u)) e_d^{ui} \theta_t^i \langle t \rangle^{i+j} \\ &= d F_{\mathcal{C}}(H_{1,0}(\tilde{\omega}_0, \tilde{\omega}_u)) \sum_{i=0}^{d-1} e_d^{ui} (-1)^i \end{aligned}$$

which is zero unless $u = \frac{d}{2} \pmod d$. Here we used that $\theta_t = -1$ and $\langle t \rangle = \pm 1$, by Lemma 5.2. Since $F_{\mathcal{C}}(U_1(w_u))$ and $F_{\mathcal{C}}(U_{-1}(w_u))$ are complex conjugate to each other, the result follows. □

6.2. Cohomological refinements. In this subsection we assume that $\mathcal{C}$ is a non-spin d-refinable modular category.

The elements $h \in H^1(M; \mathbb{Z}_d)$ are combinatorially given by solutions of $\sum_{j=1}^n L_{ij} h_j = 0 \pmod d$, for all $i = 1, \ldots, n$. Let L be an oriented framed link and $h \in H^1(S^3(L); \mathbb{Z}_d)$. The usual Kirby moves admit refinements for manifolds equipped with such structures as follows.

- STABILIZATION. $(L,h) \to (L \sqcup U_{\pm 1}, (h,0))$.
- HANDLE SLIDE. $(L,h) \to (L',h')$ where L' is obtained from L by sliding component L_i along L_j and $h'_k = h_k$ if $k \neq j$ and $h'_j = h_j \mp h_i$. Here the sign depends on whether the orientations of L_i and L_j match or not, respectively.
- ORIENTATION REVERSAL. $(L,h) \to (L',h')$ where L' is obtained from L by reversing the orientation of component L_i and $h'_j = h_j$ if $j \neq i$ and $h'_i = -h_i$.

For any $h = (h_1, \ldots, h_n) \in H^1(M; \mathbb{Z}_d)$ let us define

$$\tau_{\mathcal{C}}(M,h) = \frac{F_{\mathcal{C}}(L(\omega_{h_1}, \ldots, \omega_{h_n}))}{F_{\mathcal{C}}(U_1(\omega))^{b_+} F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}}. \tag{3}$$

Theorem. *For any non-spin d-refinable modular category, $\tau_{\mathcal{C}}(M,h)$ is a topological invariant of the pair (M,h). Moreover,*

$$\tau_{\mathcal{C}}(M) = \sum_{h \in H^1(M;\mathbb{Z}_d)} \tau_{\mathcal{C}}(M,h).$$

The proof is based on the following lemma.

Lemma. *For any non-spin d-refinable modular category $\mathcal{C}$, $F_{\mathcal{C}}(U_{\pm 1}(w_u))$ is zero unless $u = 0$.*

Proof. Just as in the proof of Lemma 6.1, we consider the Hopf link $H_{1,0}(\omega_0, \omega_u)$ with linking number -1 and we slide the second component along the first one. We get

$$\begin{aligned} F_{\mathcal{C}}(U_1(w_u)) F_{\mathcal{C}}(U_{-1}(w_u)) &= F_{\mathcal{C}}(H_{1,0}(\omega_0, \omega_u)) \\ &= d F_{\mathcal{C}}(H_{1,0}(\tilde{\omega}_0, \tilde{\omega}_u)) \sum_{j=0}^{d-1} e_d^{uj} \end{aligned}$$

which is zero, unless $u = 0$. □

6.3. Spin refinements, general case. The proof of Theorem 1 follows from the two previous cases. An $(\widehat{H}, v)$ generalized spin structure σ on $M = S^3(L)$ is described by a sequence of coefficients $(s_1, \ldots, s_n) \in \widehat{H}^n$ satisfying a characteristic equation:

$$(s_1, \ldots, s_n)(\widehat{H} \otimes L) = v \otimes (L_{11}, \ldots, L_{nn})$$

Indeed, the Kirby element decomposes using $\widehat{H}$-degree, and the formula for the refined invariant given in (2) still holds. The condition for non vanishing in Lemma 6.1 is $u = v$.

The solution of these equations is a sequence of σ_i where the index i runs over cyclic components $H_i \cong \mathbb{Z}_{d_i}$ of H. Moreover, either $\sigma_i \in \mathrm{Spin}_{d_i}(M)$ or $\sigma_i \in H^1(M, \mathbb{Z}_{d_i})$, depending on the twist coefficient of the corresponding generator. Combining the two previous theorems we get

$$\tau_{\mathcal{C}}(M) = \sum_{\sigma} \tau_{\mathcal{C}}(M, \sigma).$$

7. Complex spin refinements

This section is devoted to the proof of Theorem 2. Again, according to the twist coefficients, we will either get an extension of WRT invariants for 3-manifolds equipped with modulo d complex spin structures or with 2-dimensional cohomology classes.

Throughout this section $\mathcal{C}$ is a $2d$-spin modular category with d even. For any $\sigma \in \mathcal{S}^c_d(M)$, let us define

$$\tau_{\mathcal{C}}(M, \sigma) = (-d)^{-n} \sum_{(\epsilon_1, \dots, \epsilon_n) \in \sigma} \frac{F_{\mathcal{C}}(L(\omega^{\epsilon_1}, \dots, \omega^{\epsilon_n}))}{F_{\mathcal{C}}(U_1(\omega))^{b_+} F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}} \tag{4}$$

where the shorthand $(\epsilon_1, \dots, \epsilon_n) \in \sigma$ means that the summation is taken over all elements of $(\mathbb{Z}_{2d})^n$ in the equivalence class of σ.

Theorem. *Let d be an even integer. For any $(2d)$-spin modular category $\mathcal{C}$, $\tau_{\mathcal{C}}(M, \sigma)$ is a topological invariant of the pair (M, σ).*

Proof. In order to prove that $\tau_{\mathcal{C}}(M, \sigma)$ is a topological invariant of the Spin^c_d-manifold (M, σ), we have to check invariance under the Spin^c_d Kirby moves of Theorem 2.4.

We start by checking invariance under the first Kirby move. Let (L', σ') be obtained from (L, σ) by a positive stabilization. We have that

$$\begin{aligned}\tau_{\mathcal{C}}(M, L'; \sigma') &= (-d)^{-(n+1)} \sum_{(\epsilon_1,\dots,\epsilon_{n+1})\in\sigma'} \frac{F_{\mathcal{C}}(L(\omega^{\epsilon_1},\dots,\omega^{\epsilon_n}))F_{\mathcal{C}}(U_1(\omega^{\epsilon_{n+1}}))}{F_{\mathcal{C}}(U_1(\omega))^{b_+ +1}F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}} \\ &= (-d)^{-n} \sum_{(\epsilon_1,\dots,\epsilon_n)\in\sigma} \frac{F_{\mathcal{C}}(L(\omega^{\epsilon_1},\dots,\omega^{\epsilon_n}))}{F_{\mathcal{C}}(U_1(\omega))^{b_+}F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}} \\ &\qquad \sum_{\epsilon_{n+1}\in 1+2\mathbb{Z}_{2d}} \frac{F_{\mathcal{C}}(U_1(\omega^{\epsilon_{n+1}}))}{(-d)F_{\mathcal{C}}(U_1(\omega))} \\ &= \tau_{\mathcal{C}}(M, L; \sigma) \sum_{\substack{x=0 \\ x \text{ even}}}^{2d-1} \frac{F_{\mathcal{C}}(U_1(\omega^{1+x}))}{(-d)F_{\mathcal{C}}(U_1(\omega))}.\end{aligned}$$

To compute the sum

$$\sum_{\substack{x=0 \\ x \text{ even}}}^{2d-1} F_{\mathcal{C}}(U_1(\omega^{1+x})),$$

we write the dual Kirby color ω^{1+x} in terms of the refined (graded) Kirby colors ω_i as follows:

$$\omega^{1+x} = \sum_{i=0}^{2d-1} \sum_{\lambda\in\Gamma_i} e_{2d}^{(1+x)i} \langle\lambda\rangle\lambda = \sum_{i=0}^{2d-1} e_{2d}^{(1+x)i}\omega_i.$$

Lemma 6.1 together with the identity

$$\sum_{\substack{x=0 \\ x \text{ even}}}^{2d-1} e_{2d}^{(1+x)i} = \begin{cases} 0 & \text{if } i \neq d, \\ -d & \text{if } i = d, \end{cases}$$

gives

$$\begin{aligned}\sum_{\substack{x=0 \\ x \text{ even}}}^{2d-1} F_{\mathcal{C}}(U_1(\omega^{1+x})) &= \sum_{i=0}^{2d-1} \sum_{\substack{x=0 \\ x \text{ even}}}^{2d-1} e_{2d}^{(1+x)i} F_{\mathcal{C}}(U_1(\omega_i)) \\ &= -dF_{\mathcal{C}}(U_1(\omega))\end{aligned}$$

so $\tau_{\mathcal{C}}(M, L'; \sigma') = \tau_{\mathcal{C}}(M, L; \sigma)$.

Analogously, $\tau_{\mathcal{C}}(M,\sigma)$ is invariant under a negative stabilization. The invariance under the second Kirby move is provided by the *dual sliding property*. Finally, we must check invariance under orientation reversal. For that, let (L',σ') be obtained from (L,σ) by changing the orientation of a component L_k. Without any loss of generality, we may assume that $k=1$ and, just like in the proof of Theorem 4.3, we get $b'_+ = b_+$, $b'_- = b_-$. We have that

$$\begin{aligned}
&\sum_{(\epsilon'_1,\dots,\epsilon'_n)\in\sigma'} F_{\mathcal{C}}(L'(\omega^{\epsilon'_1},\dots,\omega^{\epsilon'_n}))\\
&\quad= \sum_{(\epsilon_1,\dots,\epsilon_n)\in\sigma} F_{\mathcal{C}}(L'(\omega^{-\epsilon_1},\omega^{\epsilon_2},\dots,\omega^{\epsilon_n}))\\
&\quad= \sum_{(\epsilon_1,\dots,\epsilon_n)\in\sigma}\ \sum_{\lambda\in\Gamma} e_{2d}^{-\epsilon_1\deg(\lambda)}\langle\lambda\rangle F_{\mathcal{C}}(L'(\lambda,\omega^{\epsilon_2},\dots,\omega^{\epsilon_n}))\\
&\quad= \sum_{(\epsilon_1,\dots,\epsilon_n)\in\sigma}\ \sum_{\lambda\in\Gamma} e_{2d}^{\epsilon_1\deg(\lambda^*)}\langle\lambda^*\rangle F_{\mathcal{C}}(L(\lambda^*,\omega^{\epsilon_2},\dots,\omega^{\epsilon_n}))\\
&\quad= \sum_{(\epsilon_1,\dots,\epsilon_n)\in\sigma}\ \sum_{\mu\in\Gamma} e_{2d}^{\epsilon_1\deg(\mu)}\langle\mu\rangle F_{\mathcal{C}}(L(\mu,\omega^{\epsilon_2},\dots,\omega^{\epsilon_n}))\\
&\quad= \sum_{(\epsilon_1,\dots,\epsilon_n)\in\sigma} F_{\mathcal{C}}(L(\omega^{\epsilon_1},\dots,\omega^{\epsilon_n})).
\end{aligned}$$

The first equality above is a consequence of the fact that $(L'_{ij}) = {}^tS(L_{ij})S$, for $S=\mathrm{diag}(-1,1,\dots,1)$. This concludes the proof. □

7.1. Homological refinements. Let d be a positive integer and $\mathcal{C}$ be a non–spin d-refinable modular category.

The group $H_1(M;\mathbb{Z}_d)$ is described combinatorially as the set $(\mathbb{Z}_d)^n/\mathrm{Im}L$. The Kirby moves for the pair (M,h) where M is obtained by surgery on a link L and $h\in H_1(M;\mathbb{Z}_d)$ can be described as follows.

- Stabilization. $(L,h)\to(L\sqcup U_{\pm1},(h,0))$;
- Handle slide. $(L,h)\to(L',h')$ where L' is obtained from L by sliding component L_i along L_j and $h'_k = h_k$ if $k\neq i$ and $h'_i = h_i \pm h_j$. Here the sign depends on whether the orientations of L_i and L_j match or not, respectively;
- Orientation reversal. $(L,h)\to(L',h')$ where L' is obtained from L by changing the orientation of component L_i and $h'_j = h_j$ if $j\neq i$ and $h'_i = -h_i$.

For any $h \in H_1(M;\mathbb{Z}_d)$ let us define

$$\tau_{\mathcal{C}}(M,h) = d^{-n} \sum_{(\epsilon_1,\ldots,\epsilon_n)\in h} \frac{F_{\mathcal{C}}(L(\omega^{\epsilon_1},\ldots,\omega^{\epsilon_n}))}{F_{\mathcal{C}}(U_1(\omega))^{b_+}F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}} \tag{5}$$

where the shorthand $(\epsilon_1,\ldots,\epsilon_n) \in h$ means that the summation is taken over all elements of $(\mathbb{Z}_d)^n$ in the equivalence class of h.

Theorem. *For any non–spin d-refinable modular category $\mathcal{C}$, $\tau_{\mathcal{C}}(M,h)$ is a topological invariant of the pair (M,h).*

Proof. In order to prove that $\tau_{\mathcal{C}}(M,h)$ is a topological invariant of the manifold (M,h), we have to check invariance under the Kirby moves listed above.

We start by checking invariance under the first Kirby move. Let (L',h') be obtained from (L,h) by a positive stabilization.

$$\begin{aligned}
\tau_{\mathcal{C}}(M,L';h') &= d^{-(n+1)} \sum_{(\epsilon_1,\ldots,\epsilon_{n+1})\in h'} \frac{F_{\mathcal{C}}(L(\omega^{\epsilon_1},\ldots,\omega^{\epsilon_n}))F_{\mathcal{C}}(U_1(\omega^{\epsilon_{n+1}}))}{F_{\mathcal{C}}(U_1(\omega))^{b_++1}F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}} \\
&= d^{-n} \sum_{(\epsilon_1,\ldots,\epsilon_n)\in h} \frac{F_{\mathcal{C}}(L(\omega^{\epsilon_1},\ldots,\omega^{\epsilon_n}))}{F_{\mathcal{C}}(U_1(\omega))^{b_+}F_{\mathcal{C}}(U_{-1}(\omega))^{b_-}} \sum_{i=0}^{d-1} \frac{F_{\mathcal{C}}(U_1(\omega^i))}{dF_{\mathcal{C}}(U_1(\omega))} \\
&= \tau_{\mathcal{C}}(M,L;h) \frac{F_{\mathcal{C}}(U_1(\sum_{i=0}^{d-1}\omega^i))}{dF_{\mathcal{C}}(U_1(\omega))}.
\end{aligned}$$

We compute

$$\begin{aligned}
\sum_{i=0}^{d-1} \omega^i &= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1}\sum_{\lambda\in\Gamma_j} e_d^{ij}\langle\lambda\rangle\lambda \\
&= \sum_{j=0}^{d-1}\sum_{\lambda\in\Gamma_j} (\sum_{i=0}^{d-1} e_d^{ij})\langle\lambda\rangle\lambda \\
&= d\cdot\omega_0
\end{aligned}$$

since $\sum_{i=0}^{d-1} e_d^{ij} = 0$ unless $j=0$. Using Lemma 6.2 we get that $F_{\mathcal{C}}(U_1(\omega)) = F_{\mathcal{C}}(U_1(\omega_0))$ and therefore $\tau_{\mathcal{C}}(M,L';h') = \tau_{\mathcal{C}}(M,L;h)$.

Analogously, $\tau_{\mathcal{C}}(M,h)$ is invariant under a negative stabilization. The invariance under the second Kirby move is provided by the *dual sliding property*. Finally, we must check invariance under orientation reversal. For that, let (L',h') be obtained from (L,h) by changing the orientation of a component L_k. Without

any loss of generality, we may assume that $k = 1$ and, just like in the proof of Theorem 4.3, we get $b'_+ = b_+$, $b'_- = b_-$. Since $h' = (-h_1, h_2, \dots, h_n)$, it follows that

$$\begin{aligned}
&\sum_{(\epsilon'_1,\dots,\epsilon'_n)\in h'} F_{\mathcal{C}}(L'(\omega^{\epsilon'_1},\dots,\omega^{\epsilon'_n}))\\
&= \sum_{(\epsilon_1,\dots,\epsilon_n)\in h} F_{\mathcal{C}}(L'(\omega^{-\epsilon_1},\omega^{\epsilon_2},\dots,\omega^{\epsilon_n}))\\
&= \sum_{(\epsilon_1,\dots,\epsilon_n)\in h}\ \sum_{\lambda\in\Gamma} e_d^{-\epsilon_1\deg(\lambda)}\langle\lambda\rangle F_{\mathcal{C}}(L'(\lambda,\omega^{\epsilon_2},\dots,\omega^{\epsilon_n}))\\
&= \sum_{(\epsilon_1,\dots,\epsilon_n)\in h}\ \sum_{\lambda\in\Gamma} e_d^{\epsilon_1\deg(\lambda^*)}\langle\lambda^*\rangle F_{\mathcal{C}}(L(\lambda^*,\omega^{\epsilon_2},\dots,\omega^{\epsilon_n}))\\
&= \sum_{(\epsilon_1,\dots,\epsilon_n)\in h}\ \sum_{\mu\in\Gamma} e_d^{\epsilon_1\deg(\mu)}\langle\mu\rangle F_{\mathcal{C}}(L(\mu,\omega^{\epsilon_2},\dots,\omega^{\epsilon_n}))\\
&= \sum_{(\epsilon_1,\dots,\epsilon_n)\in h} F_{\mathcal{C}}(L(\omega^{\epsilon_1},\dots,\omega^{\epsilon_n})).
\end{aligned}$$

The first equality above is a consequence of the fact that $(L'_{ij}) = {}^tS(L_{ij})S$, for $S = \mathrm{diag}(-1, 1, \dots, 1)$. This concludes the proof. □

8. Decomposition formula

This section is devoted to the proof of Theorem 3. Reader interested in the case $\gcd(m, \delta) = 1$ only can skip this section and consult an easy direct argument in Appendix.

Throughout this section d is any positive integer, it needs not to be even.

8.1. Strategy of the proof. Let us recall the setting. We assume that $\mathcal{C}$ be a modular category with cyclic group of invertible objects G generated by t. Let $\deg(t) = \delta \in \mathbb{Z}_d \cong \widehat{G}$ and $m\delta = d$. Moreover, let us split $\delta = \alpha\beta$, such that $\gcd(\beta, \alpha m) = 1, \alpha \equiv m \pmod 2$.

For any $V \in \mathrm{Ob}(\mathcal{C})$ we can choose $\chi_V(t) = \kappa^{\deg(V)}$ where κ is a primitive d-th root of unity. Let us fix the generator t, so that $\kappa = e^{\frac{\sqrt{-1}2\pi}{d}}$. The twist coefficient for t is $\theta_t = b_{t,t}\langle t\rangle$ and satisfies $\theta_t^2 = \kappa^\delta = e^{\frac{\sqrt{-1}2\pi}{m}}$. We consider the subgroup of invertible objects $H \cong \mathbb{Z}_\delta$ generated by t^m. Clearly, $H \subset \mathcal{C}_0$, so the modular category $\mathcal{C}$ is δ-refinable. The twist coefficient for the generator t^m

is $\theta_{t^m} = b_{t,t}^{m^2}\langle t\rangle^m$. It is equal to -1 if δ is even, m is odd and either $\langle t\rangle = 1$ and $b_{t,t} = \theta_t = e^{\frac{\sqrt{-1}\pi}{m}}$, or $\langle t\rangle = -1$ and $b_{t,t} = -\theta_t = -e^{\frac{\sqrt{-1}\pi}{m}}$; it is equal to 1 in all other cases. The modular category $\mathcal{C}$ is δ-spin if δ is even, m is odd and θ_t has order $2m$, and δ-cohomological in all other cases.

We now present the idea of the proof of the decomposition statement. We define a tensor category $\mathcal{C}'$ with simple objects represented by $\Gamma' = \Gamma \times \mathbb{Z}_\alpha$. The tensor product in $\mathcal{C}'$ mimics central extension of groups using 2-cocycles. We lift the map $\deg\colon \Gamma \to \mathbb{Z}_d$ into a map $f\colon \Gamma \to \mathbb{Z}_{d\alpha}$, which plays the role of a section. Further, we extend f into a map $\mathrm{Ob}(\mathcal{C}') \to \mathbb{Z}_{\alpha d}$, such that

$$f(V,k) = f(V) + dk \quad \text{for } (V,k) \in \Gamma'.$$

Given two elements (V,k) and (W,l) of Γ' we define their tensor product as $(V \otimes W, k+l)$. We allow in $\mathcal{C}'$ direct sums of objects with homogenous f value.

For $X = (V,k) \in \mathrm{Ob}(\mathcal{C}')$ and $Y = (W,l) \in \mathrm{Ob}(\mathcal{C}')$ we set

$$\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \begin{cases} 0 & \text{if } f(X) \neq f(Y), \\ \mathrm{Hom}_{\mathcal{C}}(V,W) & \text{else.} \end{cases}$$

The category $\mathcal{C}'$ is a tensor category over $\mathbf{k}$ with unit object $(\mathbf{1},0)$ and compatible duality. Note that (V^*,l) is a left and right dual for (V,k) if l is choosen so that $f(V^*,l) = -f(V,k)$.

Proposition. *The category $\mathcal{C}'$ is semisimple with Γ' as representative set of simple objects. The group of invertible objects is $G' \cong \mathbb{Z}_d \times \mathbb{Z}_\alpha$, generated by $(t,0)$ and $(\mathbf{1},1)$.*

Proof. The object (V,j) is invertible in $\mathcal{C}'$ if and only if V is invertible in $\mathcal{C}$. We deduce the last statement. To prove semisimplicity, it is enough to decompose the tensor product of two objects objects (V,j), (V',j') in Γ'. We have in the category $\mathcal{C}$ a decomposition

$$\mathrm{id}_V \otimes \mathrm{id}_{V'} = \sum_W \sum_{\nu \in I_{V,V'}^W} b_\nu \,\mathrm{id}_W\, a_\nu.$$

We set $\chi_{V,V'}^W = (f(V) + f(V') - f(W))/d$. Then we have in the category $\mathcal{C}'$ the following decomposition

$$\mathrm{id}_{(V,j)} \otimes \mathrm{id}_{(V',j')} = \sum_W \sum_{\nu \in I_{V,V'}^W} b_\nu \,\mathrm{id}_{(W,j+j'+\chi_{V,V'}^W)}\, a_\nu. \qquad \square$$

Further, let us give $\mathcal{C}'$ a ribbon structure which twists the one given on $\mathcal{C}$. The braiding is given by a formula

$$b'_{(V,k),(W,l)} = \xi^{-f(V,k)f(W,l)} b_{V,W},$$

with appropriate choice of a root of unity ξ whose order $2\alpha d$ if d is even and αd if d is odd. Using duality, the twist is then given by

$$\theta'_{(V,k)} = \xi^{-f(V,k)^2} \theta_V.$$

The corresponding colored link invariants $F = F_{\mathcal{C}}$ and $F' = F_{\mathcal{C}'}$ are equal up to a power of ξ which is computed from map f and linking numbers. Note that ξ is choosen such that αm elements of G' become transparent.

The G'-category $\mathcal{C}'$ is premodular and can be modularized as described in [1, 5]. Simple objects in the modularization $\widetilde{\mathcal{C}}$ are obtained from those of $\mathcal{C}'$ quotienting by a free action. The set $\widetilde{\Gamma}$ of simple objects in $\widetilde{\mathcal{C}}$ has cardinality $|\Gamma|/m$. Below we give a detailed proof of the decomposition formula in the spin case, the cohomological cases can be proven similarly.

Proof of Theorem 3, spin case. We consider here the spin case, which means that d is even, m is odd and the twist θ_t has order $2m$. The generator t can be choosen so that $\theta_t = e^{\frac{\sqrt{-1}\pi}{m}}$. Let $\xi = e^{\frac{\sqrt{-1}\pi l}{\alpha^2 m}}$ with $\beta^2 l \equiv 1 + \alpha^2 m \bmod 2\alpha^2 m$. Note that $\xi^{\alpha d} = 1$ so that the modified braiding is well defined.

The braiding coefficients for the generators of G' are

$$b'_{X,(t,0)} = (\kappa \xi^{-2\delta})^{f(X)} = e^{\frac{2\sqrt{-1}\pi(1-l\beta^2)f(X)}{d}},$$

$$b'_{X,(\mathbf{1},1)} = (\xi^{-2d})^{f(X)} = e^{\frac{2\sqrt{-1}\pi l f(X)\beta}{\alpha}}.$$

The twist coefficients are

$$\theta'_{(t,0)} = e^{\frac{\sqrt{-1}\pi}{m}} e^{-\frac{\sqrt{-1}\pi l\beta^2}{m}} = -1,$$

$$\theta'_{(\mathbf{1},1)} = e^{-\frac{\sqrt{-1}\pi l d^2}{\alpha^2 m}} = 1.$$

It follows that the group of transparent objects is generated by $(t,0)^\beta$ which has trivial twist and quantum dimension 1. Applying the results of [1, 5] we see that the category $\mathcal{C}'$ is modularizable, i.e. that there exists a modular category $\widetilde{\mathcal{C}}$ and a dominant ribbon functor $\mathcal{C}' \to \widetilde{\mathcal{C}}$. Here the group of transparent objects acts

freely on the set Γ' of simple objects in $\mathcal{C}'$. This is proved using the map f and the fact that $f((t,0)^\beta) = \beta\delta$ has order αm which is the order of $(t,0)^\beta$. Hence, the simple objects $\widetilde{\Gamma}$ in $\widetilde{\mathcal{C}}$ are represented by cosets in Γ' under this free action.

In the category $\widetilde{\mathcal{C}}$ the group of invertible objects is $\widetilde{G} = \langle (t,0), (\mathbf{1},1)\rangle \cong \mathbb{Z}_\beta \times \mathbb{Z}_\alpha \cong \mathbb{Z}_\delta$. The twist coefficient for $(t,0)$ is equal to -1, so the category $\widetilde{\mathcal{C}}$ is δ-spin. Denote by ω' the Kirby element in $\mathcal{C}'$ which represents αm times the Kirby element $\tilde{\omega}$ in $\widetilde{\mathcal{C}}$. We write the graded decomposition

$$\omega' = \sum_{c\in\mathbb{Z}_\delta} \omega'_c = \alpha \sum_{c\in\mathbb{Z}_\delta} \omega_c,$$

so that

$$\omega'_c = \sum_{\substack{\gamma\in\mathbb{Z}_{\alpha d}\\ \gamma\equiv c \mod \delta}} \sum_{\substack{X\in\Gamma'\\ f(X)=\gamma}} \langle X\rangle X = \sum_{\gamma\equiv c \mod \delta} \omega'_{|f=\gamma}.$$

Moreover, for any $\mathbb{Z}_{\alpha d} \ni \gamma \equiv c \bmod \delta$, the Kirby color $\tilde{\omega}_c$ in $\widetilde{\mathcal{C}}$ is represented by $\omega'_{|f=\gamma}$, i.e.

$$\tilde{\omega}_c = \omega'_{|f=\gamma} = \frac{1}{\alpha m}\,\omega'_c.$$

Indeed, the set $\omega'_{|f=\gamma}$ consists of all $(V,k) \in \Gamma'$ such that $f(V) = \gamma - kd \bmod \alpha d$. There are $|\Gamma|/d$ such elements. Acting with $(t^\beta,0)$ we can shift the degree of solutions by δ. In this way we obtain all $(V,k) \in \Gamma'$ with $\deg(V) = c \bmod \delta$. Taking the quotient by this action we get $\tilde{\omega}_c$.

It makes sense to evaluate both Reshetikhin–Turaev ribbon functors $F = F_{\mathcal{C}}$, and $F' = F_{\mathcal{C}'}$ on $\mathcal{C}'$ colored links. Let $M = S^3(L)$ be a 3-manifold given by surgery on the n-component link L with signature (b_+, b_-), and $\sigma \in \mathrm{Spin}_\delta(M)$ represented by coefficients $c_j \in \mathbb{Z}_\delta$, $1 \le j \le n$. For objects $X_j \in \Gamma'$, $1 \le j \le n$, we have

$$F(L(X_1,\dots,X_n)) = \xi^{{}^t f(X) L f(X)} F'(L(X_1,\dots,X_n))$$

where ${}^t f(X) L f(X) = \sum_{i,j} L_{ij} f(X_i) f(X_j)$. Note that the left hand side is invariant under action of $(t,0)^\beta$ on objects and can be used for the evaluation of the reduced invariant $\tau_{\widetilde{\mathcal{C}}}(M,\sigma)$ which we want to compare with $\tau_{\mathcal{C}}(M,\sigma)$.

We have

$$\tau_{\mathcal{C}}(M,\sigma) = \frac{F(L(\omega_{c_1},\dots,\omega_{c_n}))}{(F(U_1(\omega_{\delta/2})))^{b_+}(F(U_{-1}(\omega_{\delta/2})))^{b_-}} = \frac{\alpha^{-n+b_++b_-}F(L(\omega'_{c_1},\dots,\omega'_{c_n}))}{(F(U_1(\omega'_{\delta/2})))^{b_+}(F(U_{-1}(\omega'_{\delta/2})))^{b_-}},$$

$$\begin{aligned} F(L(\omega'_{c_1},\dots,\omega'_{c_n})) &= \sum_{\substack{\gamma\in(\mathbb{Z}_{\alpha d})^n \\ \gamma_i\equiv c_i \mod \delta}} F(L(\omega'_{|f=\gamma_1},\dots,\omega'_{|f=\gamma_n})) \\ &= \sum_{\substack{\gamma\in(\mathbb{Z}_{\alpha d})^n \\ \gamma_i\equiv c_i \mod \delta}} \xi^{t\gamma L\gamma} F'(L(\omega'_{|f=\gamma_1},\dots,\omega'_{|f=\gamma_n})) \\ &= \sum_{\substack{\gamma\in(\mathbb{Z}_{\alpha d})^n \\ \gamma_i\equiv c_i \mod \delta}} \xi^{t\gamma L\gamma} F'(L(\tilde{\omega}_{c_1},\dots,\tilde{\omega}_{c_n})) \\ &= F'(L(\tilde{\omega}_{c_1},\dots,\tilde{\omega}_{c_n})) \sum_{\substack{\gamma\in(\mathbb{Z}_{\alpha d})^n \\ \gamma_i\equiv c_i \mod \delta}} \xi^{t\gamma L\gamma}. \end{aligned}$$

After normalization we get

$$\tau_{\mathcal{C}}(M,\sigma) = \alpha^{-b_1(M)}\tau_{\widetilde{\mathcal{C}}}(M,\sigma)\, g^{-b_+}(\bar{g})^{-b_-} \sum_{\substack{\gamma\in(\mathbb{Z}_{\alpha d})^n \\ \gamma\equiv c \mod \delta}} \xi^{t\gamma L\gamma}$$

where

$$g = \sum_{\substack{\gamma\in\mathbb{Z}_{\alpha d} \\ \gamma\equiv\delta/2 \mod \delta}} \xi^{\gamma^2}$$

and $\bar{g}$ is the complex conjugate. One can check, following the graded construction in Section 6 that the formula

$$\tau_{\xi}^{\mathrm{MOO}}(M,\sigma) = g^{-b_+}(\bar{g})^{-b_-} \sum_{\gamma\equiv c \mod \delta} \xi^{t\gamma L\gamma}$$

defines an invariant of (M,σ). We conclude

$$\tau_{\mathcal{C}}(M,\sigma) = \alpha^{-b_1(M)}\tau_{\widetilde{\mathcal{C}}}(M,\sigma)\tau_{\xi}^{\mathrm{MOO}}(M,\sigma).$$

In the case $\gcd(m,\delta)=1$, we have $\alpha=1$ and

$$\sum_j L_{ij}\gamma_j = \frac{\delta}{2}L_{ii} \mod \delta.$$

We can assume that our surgery presentation has even linking matrix (the obstruction given by the spin cobordism group vanishes), so that $c \in \mathrm{Ker}(L \otimes \mathbb{Z}_\delta)$. Decomposing $\gamma = \gamma_0 + \delta x$ with $\gamma_0 \in \mathrm{Ker}\, L$ and $x \in \mathbb{Z}_m$ we see that

$$\begin{aligned}\sum_{\gamma \equiv c \mod \delta} \xi^{t\gamma L\gamma} &= \sum_{x \in (\mathbb{Z}_m)^n} \xi^{t(\gamma_0+\delta x)L(\gamma_0+\delta x)} \\ &= \sum_{x \in (\mathbb{Z}_m)^n} \xi^{t(\delta x)L(\delta x)} \\ &= \sum_{x \in (\mathbb{Z}_m)^n} \xi^{txLx}\end{aligned}$$

does not depend on c. Summing over σ we get

$$\tau_{\mathcal{C}}(M) = \tau_{\widetilde{\mathcal{C}}}(M)\tau_\xi^{\mathrm{MOO}}(M). \qquad \square$$

Appendix

Here we give a simple direct proof of Theorem 3 in the case when $\gcd(m, \delta) = 1$. For readers convenience, we repeat the statement.

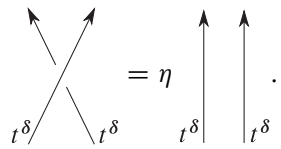
Theorem. *Let $\mathcal{C}$ be a modular d-category with the cyclic group G of invertible objects, generated by t. Assume* $\deg t = \delta$, $\gcd(\delta, d/\delta) = 1$. *Then there exist a subcategory $\widetilde{\mathcal{C}} \subset \mathcal{C}$ and a root of unity ξ, such that for any closed orientable 3-manifold M*

$$\tau_{\mathcal{C}}(M) = \tau_{\widetilde{\mathcal{C}}}(M)\tau_\xi^{\mathrm{MMO}}(M).$$

Let m be such that $\delta m = d$ and we set

$$\widetilde{\Gamma} := \{\lambda \in \Gamma \mid \deg(\lambda) = 0 \pmod m\}.$$

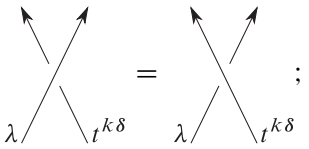
Let $\widetilde{\mathcal{C}}$ be the full ribbon subcategory of $\mathcal{C}$ generated by $\widetilde{\Gamma}$ and $\tilde{\omega}$ be the corresponding Kirby color. Let K be a subgroup generated by t^m. In this situation K has order δ and $\widetilde{\Gamma} = \Gamma_0 \amalg \Gamma_m \amalg \cdots \amalg \Gamma_{(\delta-1)m}$. Moreover, consider $\eta \in \mathbb{C}$ such that



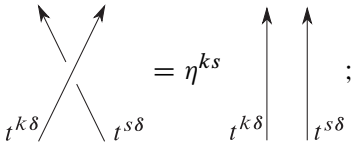
We will need the following proposition.

Proposition. *Let $\xi = \eta\langle t\rangle^{\delta}$. Then we have the following equalities of morphisms*:

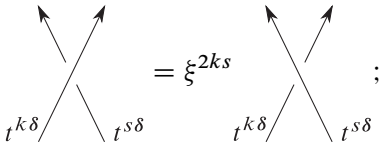
(a) *for any $\lambda \in \widetilde{\Gamma}$ and $k \in \{0, \dots, m-1\}$*



(b) *for any $k, s \in \{0, \dots, m-1\}$*



(c) *for any $k, s \in \{0, \dots, m-1\}$*



(d) *for any $k \in \{0, \dots, m-1\}$, the twist coefficient $\theta_{t^{k\delta}} = \xi^{k^2}$.*

The proof is a straightforward computation using $\langle t\rangle = \pm 1$ and $\xi^2 = \eta^2 = e_d^{\delta^3}$.

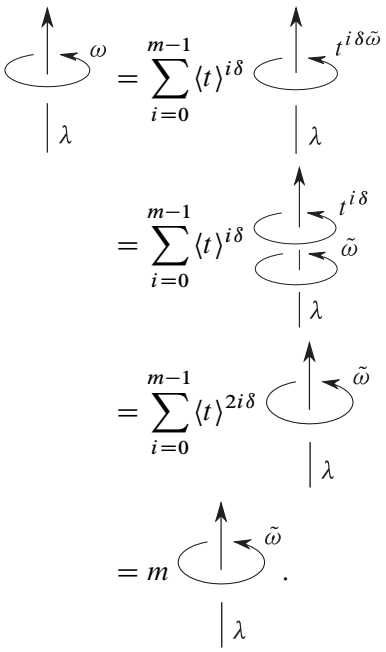
Lemma. *The category $\widetilde{\mathcal{C}}$ is modular and reduced, in the sense that it is $\widetilde{G}$-refinable.*

Proof. Clearly, $1 \in \widetilde{\Gamma}$ since $\deg(1) = 0$. Duality axiom holds since for $\lambda \in \widetilde{\Gamma}$, $\deg(\lambda^*) = -\deg(\lambda)$ and hence $\lambda^* \in \widetilde{\Gamma}$. Domination follows trivially from the same property of $\mathcal{C}$. It remains to prove the non-degeneracy.

Observe that the Kirby color ω decomposes as the sum

$$\omega = \sum_{i=0}^{m-1} \langle t\rangle^{i\delta} t^{i\delta} \tilde{\omega}$$

since $(\delta, m) = 1$. Decomposing ω as above and using Proposition 8.1 (a) we get for any $\lambda \in \widetilde{\Gamma}$ the following equalities



The *killing property* combined with Proposition 4.2 implies non-degeneracy. □

Proof of Theorem. Let M be presented by an oriented framed link

$$L = (L_1, \ldots, L_n)$$

with all components L_i unknotted (such a link always exists, see [Il]). Then, the invariant

$$\tau_{\mathcal{C}}(M) = \frac{F_{\mathcal{C}}(L(\omega, \ldots, \omega))}{(F_{\mathcal{C}}(U_1(\omega)))^{b_+}(F_{\mathcal{C}}(U_{-1}(\omega)))^{b_-}}.$$

If we replace

$$\omega = \sum_{i=0}^{m-1} \langle t \rangle^{i\delta} t^{i\delta} \tilde{\omega}$$

in $F_{\mathcal{C}}(L(\omega, \ldots, \omega))$ and apply Proposition (a) we obtain

$$F_{\mathcal{C}}(L(\omega, \ldots, \omega)) = F_{\widetilde{\mathcal{C}}}(L(\tilde{\omega}, \ldots, \tilde{\omega})) \sum_{i_1, \ldots, i_n = 0}^{m-1} \langle t \rangle^{(i_1 + \cdots + i_n)\delta} F_{\mathcal{C}}(L(t^{i_1\delta}, \ldots, t^{i_n\delta})).$$

In particular, for $\epsilon = \pm 1$, we have

$$F_{\mathcal{C}}(U_\epsilon(\omega)) = F_{\widetilde{\mathcal{C}}}(U_\epsilon(\tilde{\omega})) \sum_{i=0}^{m-1} \langle t \rangle^{i\delta} F_{\mathcal{C}}(U_\epsilon(t^{i\delta})).$$

Given the link L with components colored by $t^{i_1\delta}, \dots, t^{i_n\delta}$, we can compute $F_{\mathcal{C}}(L(t^{i_1\delta}, \dots, t^{i_n\delta}))$ as follows: first we make each component of L zero framed; then we unlink the components (using Proposition (c) and (d) as many times as necessary). Finally, we obtain n disjoint and unlinked copies of the zero framed unknot with colors $t^{i_1\delta}, \dots, t^{i_n\delta}$ and the relation:

$$F_{\mathcal{C}}(L(t^{i_1\delta}, \dots, t^{i_n\delta})) = \langle t \rangle^{(i_1+\dots+i_n)\delta} \cdot \xi^{(i_1,\dots,i_n)L(i_1,\dots,i_n)^t},$$

where (L_{ij}) is the linking matrix of L. Similarly

$$F_{\mathcal{C}}(U_\epsilon(t^{i\delta})) = \langle t \rangle^{i\delta} \cdot \xi^{\epsilon i^2}$$

and the Reshetikin–Turaev invariant decomposes as

$$\tau_{\mathcal{C}}(M) = \tau_{\widetilde{\mathcal{C}}}(M) \frac{\displaystyle\sum_{l \in (\mathbb{Z}_m)^n} \xi^{t l L l}}{\Big(\displaystyle\sum_{i \in \mathbb{Z}_m} \xi^{i^2} \Big)^{b_+} \Big(\displaystyle\sum_{i \in \mathbb{Z}_m} \xi^{-i^2} \Big)^{b_-}}.$$

Note that for d even, ξ is an mth root of unity if m is odd and a $2m$th root of unity if m is even and according to [16],

$$\tau_\xi^{\mathrm{MMO}}(M) = \frac{\displaystyle\sum_{l \in (\mathbb{Z}_m)^n} \xi^{t l L l}}{\Big(\displaystyle\sum_{i \in \mathbb{Z}_m} \xi^{i^2} \Big)^{b_+} \Big(\displaystyle\sum_{i \in \mathbb{Z}_m} \xi^{-i^2} \Big)^{b_-}}$$

is a topological invariant of M independent on the choice of L, known as the Murakami–Ohtsuki–Okada invariant. □

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Anna Beliakova, Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

e-mail: anna@math.uzh.ch

Christian Blanchet, Université Paris Diderot, Sorbonne Paris Cité, IMJ-PRG, UMR 7586 CNRS, F-75013, Paris, France

e-mail: christian.blanchet@imj-prg.fr

Eva Contreras, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

e-mail: eva_tudor@yahoo.com