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Quantum Topology

Cable links and L-space surgeries

Eugene Gorsky[1] and Jennifer Hom[2]

Abstract. An L-space link is a link in S^3 on which all sufficiently large integral surgeries are L-spaces. We prove that for m, n relatively prime, the r-component cable link $K_{rm,rn}$ is an L-space link if and only if K is an L-space knot and $n/m \geq 2g(K) - 1$. We also compute HFL^- and $\widehat{\mathrm{HFL}}$ of an L-space cable link in terms of its Alexander polynomial. As an application, we confirm a conjecture of Licata [7] regarding the structure of $\widehat{\mathrm{HFL}}$ for (n, n) torus links.

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Contents

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1. Introduction

Heegaard Floer homology is a package of 3-manifold invariants defined by Ozsváth and Szabó [12, 13]. In its simplest form, it associates to a closed 3-manifold Y a graded vector space $\widehat{\mathrm{HF}}(Y)$. For a rational homology sphere Y, they show that

$$\dim \widehat{\mathrm{HF}}(Y) \geq |H_1(Y;\mathbb{Z})|.$$

If equality is achieved, then Y is called an *L-space*.

A knot $K \subset S^3$ is an *L-space knot* if K admits a positive L-space surgery. Let $S^3_{p/q}(K)$ denote p/q Dehn surgery along K. If K is an L-space knot, then $S^3_{p/q}(K)$ is an L-space for all $p/q \geq 2g(K) - 1$, where $g(K)$ denotes the Seifert genus of K [16, Corollary 1.4]. A link $L \subset S^3$ is an *L-space link* if all sufficiently large integral surgeries on L are L-spaces. In contrast to the knot case, if L admits a positive L-space surgery, it does not necessarily follow that all sufficiently large surgeries are also L-spaces; see [10, Example 2.3].

For relatively prime integers m and n, let $K_{m,n}$ denote the (m,n) cable of K, where m denotes the longitudinal winding. Without loss of generality, we will assume that $m > 0$. Work of Hedden [3] ("if" direction) and the second author [5] ("only if" direction) completely classifies L-space cable knots.

Theorem 1 ([3, 5]). *Let K be a knot in S^3, $m > 1$ and $\gcd(m,n) = 1$. The cable knot $K_{m,n}$ is an L-space knot if and only if K is an L-space knot and $n/m > 2g(K) - 1$.*

Remark 1.1. Note that when $m = 1$, we have that $K_{1,n} = K$ for all n.

We generalize this theorem to cable links with many components. Throughout the paper, we assume that each component of a cable link is oriented in the same direction.

Theorem 2. *Let K be a knot in S^3 and $\gcd(m,n) = 1$. The r-component cable link $K_{rm,rn}$ is an L-space link if and only if K is an L-space knot and $n/m \geq 2g(K) - 1$.*

In [14], Ozsváth and Szabó show that if K is an L-space knot, then $\widehat{\mathrm{HFK}}(K)$ is completely determined by $\Delta_K(t)$, the Alexander polynomial of K. Consequently, the Alexander polynomials of L-space knots are quite constrained (the non-zero coefficients are all ± 1 and alternate in sign) and the rank of $\widehat{\mathrm{HFK}}(K)$ is at most one in each Alexander grading. In [10, Theorem 1.15], Liu generalizes this result to give bounds on the rank of $HFL^-(L)$ in each Alexander multi-grading and on the coefficients of the multi-variable Alexander polynomial of an L-space link L in terms of the number of components of L. For L-space cable links, we have the following stronger result.

Definition 1.2. Define the $\mathbb{Z}$-valued functions $\mathbf{h}(k)$ and $\beta(k)$ by the equations

$$\sum_k \mathbf{h}(k)t^k = \frac{t^{-1}\Delta_{m,n}(t)(t^{mnr/2}-t^{-mnr/2})}{(1-t^{-1})^2(t^{mn/2}-t^{-mn/2})}, \qquad \beta(k) = \mathbf{h}(k-1)-\mathbf{h}(k)-1, \tag{1.1}$$

where $\Delta_{m,n}(t)$ is the Alexander polynomial of the cable knot $K_{m,n}$.

Throughout, we work with $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients. The following theorem gives a complete description of the homology groups $\widehat{HFL}$ for cable links with $n/m > 2g(K)-1$.

Theorem 3. *Let $K_{rm,rn}$ be a cable link with $n/m > 2g(K)-1$.*

(a) *If $\beta(k)+\beta(k+1) \leq r-2$, then*

$$\widehat{HFL}(K_{rm,rn}, k, \dots, k) \simeq \bigoplus_{i=0}^{\beta(k)} \binom{r-1}{i} \mathbb{F}_{-2\mathbf{h}(k)-i} \oplus \bigoplus_{i=0}^{\beta(k+1)} \binom{r-1}{i} \mathbb{F}_{-2\mathbf{h}(k)+2-r+i}.$$

(b) *If $\beta(k)+\beta(k+1) \geq r-2$, then*

$$\widehat{HFL}(K_{rm,rn}, k, \dots, k) \simeq \bigoplus_{i=0}^{r-2-\beta(k+1)} \binom{r-1}{i} \mathbb{F}_{-2\mathbf{h}(k)-i} \oplus \bigoplus_{i=0}^{r-2-\beta(k)} \binom{r-1}{i} \mathbb{F}_{-2\mathbf{h}(k)+2-r+i}.$$

(c) *If v has j coordinates equal to $k-1$ and $r-j$ coordinates equal to k for some k and $1 \leq j \leq r-1$, then*

$$\widehat{HFL}(K_{rm,rn}, (k-1)^j, k^{r-j}) \simeq \binom{r-2}{\beta(k)} \mathbb{F}_{-2\mathbf{h}(k)-\beta(k)-j}.$$

(d) *For all other Alexander gradings the groups $\widehat{HFL}$ vanish.*

We prove the parts of this theorem as separate Theorems 4.22, 4.24 and 4.25. We compute $\widehat{HFL}$ explicitly for several examples in Section 5. In particular, we use Theorem 3 to confirm a conjecture of Joan Licata [7, Conjecture 1] concerning $\widehat{HFL}$ for (n,n) torus links.

Theorem 4. *Suppose that* $0 \leq s \leq \frac{n-1}{2}$. *Then*

$$\widehat{\mathrm{HFL}}\left(T(n,n), \frac{n-1}{2} - s, \ldots, \frac{n-1}{2} - s\right)$$
$$= \bigoplus_{i=0}^{s} \binom{n-1}{i} \mathbb{F}_{(-s^2-s-i)} \oplus \bigoplus_{i=0}^{s-1} \binom{n-1}{i} \mathbb{F}_{(-s^2-s-n+2+i)}.$$

Combined with [7, Theorem 2], this completes the description of $\widehat{\mathrm{HFL}}(T(n,n))$.

The following theorem describes the homology groups HFL^- for cable links with $n/m > 2g(K) - 1$.

Theorem 5. *Let K be an L-space knot and $n/m > 2g(K) - 1$. Consider an Alexander grading $v = (v_1, \ldots, v_n)$. Suppose that among the coordinates v_i exactly λ are equal to k and all other coordinates are less than k. Let $|v| = v_1 + \cdots + v_n$. Then the Heegaard–Floer homology group* $\mathrm{HFL}^-(K_{rm,rn}, v)$ *can be described as follows.*

(*a*) *If $\beta(k) < r - \lambda$ then* $\mathrm{HFL}^-(K_{rm,rn}, v) = 0$.

(*b*) *If $\beta(k) \geq r - \lambda$ then*

$$\mathrm{HFL}^-(K_{rm,rn}, v) \simeq (\mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)})^{r-\lambda} \otimes \bigoplus_{i=0}^{\beta(k)-r+\lambda} \binom{\lambda-1}{i} \mathbb{F}_{(-2h(v)-i)},$$

where $h(v) = \mathbf{h}(k) + kr - |v|$.

We prove this theorem in Section 4.2. The structure of the homology for $n/m = 2g(K)-1$ (which is possible only if $m = 1$) is more subtle and is described in Theorem 4.26.

Finally, we describe HFL^- as an $\mathbb{F}[U_1, \ldots, U_r]$–module. We define a collection of $\mathbb{F}[U_1, \ldots, U_r]$–modules M_β for $0 \leq \beta \leq r-2$, $M_{r-1,k}$ for $k \geq 0$ and $M_{r-1,\infty}$. These modules can be defined combinatorially and do not depend on a link.

Theorem 6. *Let $R = \mathbb{F}[U_1, \ldots, U_r]$ and suppose that $n/m > 2g(K) - 1$. There exists a finite collection of diagonal lattice points $\mathbf{a}_i = (a_i, \ldots, a_i)$ (determined by m, n and the Alexander polynomial of K) such that* HFL^- *admits the following direct sum decomposition*:

$$\mathrm{HFL}^-(K_{rm,rn}) = \bigoplus_i R \cdot \mathrm{HFL}^-(K_{rm,rn}, \mathbf{a}_i).$$

Furthermore, for $\beta(a_i) \leq r-2$ one has $R \cdot \mathrm{HFL}^-(K_{rm,rn}, \mathbf{a}_i) \simeq M_{\beta(a_i)}$, and for $\beta(a_i) = r-1$ one has either $R \cdot \mathrm{HFL}^-(K_{rm,rn}, \mathbf{a}_i) \simeq M_{r-1,k}$ for some k or $R \cdot \mathrm{HFL}^-(K_{rm,rn}, \mathbf{a}_i) \simeq M_{r-1,\infty}$.

We compute HFL^- explicitly for several examples in Section 5.

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2. Dehn surgery and cable links

In this section, we prove Theorem 2. We begin with a result about Dehn surgery on cable links (cf. [4]).

Proposition 2.1. *The manifold obtained by* $(mn, p_2, \dots, p_r)$*–surgery on the* r*-component link* $K_{rm,rn}$ *is homeomorphic to*

$$S^3_{n/m}(K)\#L(m,n)\#L(p_2-mn,1)\#\cdots\#L(p_r-mn,1).$$

Proof. Recall (see, for example, [3, Section 2.4]) that mn-surgery on $K_{m,n}$ gives the manifold $S^3_{n/m}(K)\#L(m,n)$. Viewing $K_{m,n}$ as the image of $T_{m,n}$ on $\partial N(K)$, we have that the reducing sphere is given by the annulus $\partial N(K)\setminus N(T_{m,n})$ union two parallel copies of the meridional disk of the surgery solid torus; we obtain a sphere since the surgery slope coincides with the surface framing.

The link $K_{rm,rn}$ consists of r parallel copies of $K_{m,n}$ on $\partial N(K)$. Label these r copies $K^1_{m,n}$ through $K^r_{m,n}$. We perform mn-surgery on $K^1_{m,n}$ and consider the image $\widetilde{K}^i_{m,n}$ of $K^i_{m,n}$, $2\le i\le r$, in $S^3_{n/m}(K)\#L(m,n)$. Each $\widetilde{K}^i_{m,n}$ lies on $\partial N(K)\setminus N(T_{m,n})$ and thus on the reducing sphere. In particular, each $\widetilde{K}^i_{m,n}$ bounds a disk D^2_i in $S^3_{n/m}(K)\#L(m,n)$ such that the collection $\{D^2_2,\dots,D^2_r\}$ is disjoint. It follows that performing surgery on $\bigcup_{i=2}^r \widetilde{K}^i_{m,n}$ yields $r-1$ lens space summands. To see which lens spaces we obtain, note that the mn-framed longitude on $K^i_{m,n}\subset S^3$ coincides with the 0-framed longitude on $\widetilde{K}^i_{m,n}\subset S^3_{n/m}(K)\#L(m,n)$. Thus, p_i-surgery on $K^i_{m,n}$ corresponds to (p_i-mn)-surgery on $\widetilde{K}^i_{m,n}$, and the result follows. □

Let us recall that the linking number between each two components of $K_{rm,rn}$ equals $l := mn$. It is well-known that the cardinality of H_1 of the manifold obtained by $(p_1,p_2,\dots,p_r)$-surgery on $K_{rm,rn}$ equals $|\det\Lambda(p_1,\dots,p_r)|$, where

$$\Lambda_{ij}=\begin{cases} p_i, & \text{if } i=j,\\ l, & \text{if } i\neq j.\end{cases}$$

This cardinality can be computed using the following result.

Proposition 2.2. *One has the following identity*:

$$\det \Lambda(p_1, \dots, p_r) = (p_1 - l)\cdots(p_r - l) + l\sum_{i=1}^{r}(p_1 - l)\cdots\widehat{(p_i - l)}\cdots(p_r - l). \tag{2.1}$$

Proof. One can easily check that $\det \Lambda(l, p_2, \dots, p_r) = l(p_2 - l)\cdots(p_r - l)$. The expansion of the determinant in the first row yields a recursion relation

$$\begin{aligned}\det \Lambda(p_1, \dots, p_r) &= \det \Lambda(l, p_2 \dots, p_r) + (p_1 - l)\det \Lambda(p_2, \dots, p_r)\\ &= l(p_2 - l)\cdots(p_r - l) + (p_1 - l)\det \Lambda(p_2, \dots, p_r).\end{aligned}$$

Now (2.1) follows by induction in r. □

Corollary 2.3. *If $p_i \geq l$ for all i then* $\det \Lambda(p_1, \dots, p_r) \geq 0$.

In order to prove Theorem 2, we will need the following:

Theorem 2.4 ([10, Proposition 1.11]). *A link L is an L–space link if and only if there exists a surgery framing $\Lambda(p_1, \dots, p_r)$, such that for all sublinks $L' \subseteq L$,* $\det(\Lambda(p_1, \dots, p_r)|_{L'}) > 0$ *and $S^3_{\Lambda|_{L'}}(L')$ is an L–space.*

We will also need the following proposition, which we prove in Subsection 2.1 below.

Proposition 2.5. *Let K be an L-space knot and $p_i > 0$, $i = 1, \dots, r$. If $n < 2g(K) - 1$, then the manifold obtained by $(p_1, \dots, p_r)$-surgery on the r-component link $K_{r,rn}$ is not an L-space.*

Proof of Theorem 2. If $K_{rm,rn}$ is an L-space link, then by [10, Lemma 1.10] all its components are L-space knots. On the other hand, its components are isotopic to $K_{m,n}$. Thus, if $m > 1$, then by Theorem 1, K is an L-space knot and $n/m > 2g(K) - 1$. If $m = 1$, then K must be an L-space knot and by Proposition 2.5, $n \geq 2g(K) - 1$.

Conversely, suppose that K is an L-space knot and $n/m \geq 2g(K) - 1$, i.e., $K_{m,n}$ is an L-space knot. Let us prove by induction on r that $(p_1, \dots, p_r)$-surgery on $K_{rm,rn}$ is an L-space if $p_i > l$ for all i. For $r = 1$ it is clear. By Proposition 2.1, the link $K_{rm,rn}$ admits an L-space surgery with parameters $l, p_2, \dots, p_r$. Let us apply Theorem 2.4. Indeed, by Corollary 2.3, one has $\det(\Lambda(l, p_2 \dots, p_r)|_{L'}) > 0$ and by the induction assumption $S^3_{\Lambda(l,p_2\dots,p_r)|_{L'}}(L')$ is an L–space for all sublinks L'. By [10, Lemma 2.5], $(p_1, \dots, p_r)$-surgery on $K_{rm,rn}$ is also an L-space for all $p_1 > l$. Therefore $K_{rm,rn}$ is an L-space link. □

2.1. Proof of Proposition 2.5. We will prove Proposition 2.5 using Lipshitz–Ozsváth–Thurston's bordered Floer homology [8], and specifically Hanselman–Watson's loop calculus [2]. That is, we will decompose the result of surgery on $K_{r,rn}$ into two pieces, one that is surgery on a torus link in the solid torus and the other the knot complement, and then apply a gluing result of Hanselman and Watson to conclude that the result of this surgery along $K_{r,rn}$ is not an L-space. The following was described to us by Jonathan Hanselman.

Let Y_1 denote the Seifert fibered space obtained by performing $(p_1, \ldots, p_r)$-surgery on the r-component $(r, 0)$-torus link in the solid torus. Consider the bordered manifold (Y_1, α_1, β_1), where α_1 is the fiber slope and β_1 lies in the base orbifold; that is, α_1 is the longitude and β_1 the meridian of the original solid torus. Let (Y_2, α_2, β_2) be the n-framed complement of K; that is, $Y_2 = S^3 \setminus N(K)$, α_2 is an n-framed longitude, and β_2 is a meridian. Let $(Y_1, \alpha_1, \beta_1) \cup (Y_2, \alpha_2, \beta_2)$ denote the result of gluing Y_1 to Y_2 by identifying α_1 with α_2 and β_1 with β_2. Note that $(Y_1, \alpha_1, \beta_1) \cup (Y_2, \alpha_2, \beta_2)$ is homeomorphic to $(p_1, \ldots, p_r)$-surgery along $K_{r,rn}$. We identify the slope $p\alpha_i + q\beta_i$ on ∂Y_i with the (extended) rational number $\frac{p}{q} \in \mathbb{Q} \cup \{\frac{1}{0}\}$.

The following lemma gives a description of $\widehat{\mathrm{CFD}}(Y_1, \alpha_1, \beta_1)$ in terms of the standard notation defined in [2, Section 3.2].

Lemma 2.6. *The invariant $\widehat{\mathrm{CFD}}(Y_1, \alpha_1, \beta_1)$ can be written in standard notation as a product of d_{k_i} where*

(1) $k_i \leq 0$ *for all* i*,*

(2) $k_i = 0$ *for at least one* i*,*

(3) $k_i = -r$ *for exactly one* i*.*

Proof. The computation is similar to the example in [2, Section 6.5]. A plumbing tree Γ for Y_1 is given in Figure 1. We first consider the plumbing tree Γ_i in Figure 2(a). We will build Γ by merging the Γ_i, $i = 1, \ldots, r$.

We proceed as in [2, Section 6.5]. Start with a loop (d_0) representing the tree Γ_0 in Figure 2(b). We have that $\Gamma_i = \mathcal{E}(\mathcal{T}^{p_i}(\Gamma_0))$ so by [2, Sections 3.3 and 6.3]:

$$\begin{aligned}\widehat{\mathrm{CFD}}(\Gamma_i) &= \mathrm{E}(\mathrm{T}^{p_i}((d_0))) \\ &= \mathrm{E}((d_{p_i})) \\ &= (d^*_{-p_i}) \\ &\sim (d_{-1}\underbrace{d_0 \ldots d_0}_{p_i}).\end{aligned}$$

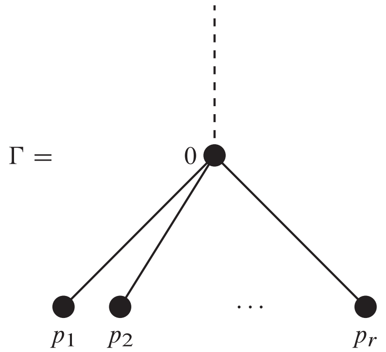

Figure 1. The plumbing tree Γ.

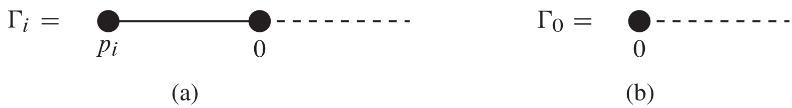

Figure 2. Left, the plumbing tree Γ_i. Right, the plumbing tree Γ_0.

We then have that $\Gamma = \mathcal{M}(\Gamma_2, \mathcal{M}(\Gamma_2, \dots, \mathcal{M}(\Gamma_{p_{r-1}}, \Gamma_{p_r})))$. By [2, Proposition 6.4], we have that $\widehat{\mathrm{CFD}}(\Gamma)$ is a represented by a product of d_{k_i} where $k_i \leq 0$ for all i and $k_i = 0$ for at least one i since each $p_i > 0$. Moreover, d_{-r} appears exactly once in the product, since we performed $r - 1$ merges. This completes the proof of the lemma. □

Lemma 2.7. *The slope* 1 *is not a strict L-space slope on* (Y_1, α_1, β_1).

Proof. We will apply a positive Dehn twist to (Y_1, α_1, β_1) to obtain $(Y_1, \alpha_1, \beta_1 + \alpha_1)$. We will show that 0 is not a strict L-space slope on $(Y_1, \alpha_1, \beta_1 + \alpha_1)$, and hence 1 is not a strict L-space slope on (Y_1, α_1, β_1).

By [2, Proposition 6.1], we have that $\widehat{\mathrm{CFD}}(Y_1, \alpha_1, \beta_1 + \alpha_1)$ can be obtained by applying τ to a loop representative of $\widehat{\mathrm{CFD}}(Y_1, \alpha_1, \beta_1)$. Since $\tau(d_k) = d_{k+1}$, it follows from Lemma 2.6 that $\widehat{\mathrm{CFD}}(Y_1, \alpha_1, \beta_1 + \alpha_1)$ can be written in standard notation as a product of d_{k_i} with $k_i \leq 1$ for all i, $k_i = 1$ for at least one i, and $k_i = 1 - r$ for exactly one i.

We claim that if a loop ℓ contains both positive and negative d_k segments (i.e., both $d_i, i > 0$ and $d_j, j < 0$), then in dual notation ℓ contains at least one a_i^* or b_j^* segment. Indeed, suppose by contradiction that ℓ has no a_i^* or b_j^*. Then ℓ consists of only d_i^* segments, $i \in \mathbb{Z}$. It is straightforward to see (for example, by considering the segments as drawn in [2, Figure 1]) that one cannot obtain a loop containing both positive and negative d_k segments from d_i^* segments, $i \in \mathbb{Z}$. This completes the proof of the claim.

Furthermore, note that $\widehat{\mathrm{CFD}}(Y_1, \alpha_1, \beta_1 + \alpha_1)$ consists of simple loops (see Definition 4.19 of [2]). Then by [2, Proposition 4.24], in dual notation ℓ has no a_k^* or b_k^* segments for $k < 0$. It now follows from Proposition 4.18 of [2] that 0 is not a strict L-space slope for $\widehat{\mathrm{CFD}}(Y_1, \alpha_1, \beta_1 + \alpha_1)$. Therefore, 1 is not a strict L-space slope on (Y_1, α_1, β_1), as desired. □

Remark 2.8. Note that by Proposition 4.18 of [2], we have that 0 and ∞ are strict L-space slopes on (Y_1, α_1, β_1). Since 1 is not a strict L-space slope, it follows from Corollary 4.5 of [2] that the interval of L-space slopes of (Y_1, α_1, β_1) contains the interval $[-\infty, 0]$.

Remark 2.9. An alternative proof of Lemma 2.7 follows from [9, Theorem 1.1]. Indeed, by setting $r_i = 1/p_i$ and $e_0 = -1$ in Figure 1 of [9], we see that $M(-1; 1/p_1, \dots, 1/p_r)$ is not an L-space, hence neither is $M(1; -1/p_1, \dots, -1/p_r)$, which is homeomorphic to filling (Y_1, α_1, β_1) along a curve of slope 1.

Lemma 2.10. *Let K be an L-space knot. If $n < 2g(K) - 1$, then 1 is not a strict L-space slope on the n-framed knot complement (Y_2, α_2, β_2).*

Proof. Since K is an L-space knot, we have that $S^3_K(p/q)$ is an L-space exactly when $p/q \geq 2g(K) - 1$. Since α_2 is an n-framed longitude, it follows that the interval of strict L-space slopes on (Y_2, α_2, β_2) is $(0, \frac{1}{2g(K)-1-n})$, that is, the reciprocal of the interval $(2g(K) - 1 - n, \infty)$. □

Proof of Proposition 2.5. The result now follows from [2, Theorem 1.3] combined with Lemmas 2.7 and 2.10; the slope 1 is not a strict L-space slope on either (Y_1, α_1, β_1) or (Y_2, α_2, β_2), and so the resulting manifold $(Y_1, \alpha_1, \beta_1) \cup (Y_2, \alpha_2, \beta_2)$, which is $(p_1, \dots, p_r)$-surgery on $K_{r,rn}$, is not an L-space. □

Remark 2.11. One can use similar methods to provide an alternate proof that $K_{r,rn}$ is an L-space link if K is an L-space knot and $n \geq 2g(K) - 1$. Indeed, if K is an L-space knot, then the interval of strict L-space slopes on the n-framed knot complement (Y_2, α_2, β_2) is $(0, \frac{1}{2g(K)-1-n})$ if $n \leq 2g(K) - 1$ and $(0, \infty] \cup [-\infty, \frac{1}{2g(K)-1-n})$ if $n > 2g(K) - 1$. Hence if $n \geq 2g(K) - 1$, then the interval of strict L-space slopes on (Y_2, α_2, β_2) contains the interval $(0, \infty)$. By Remark 2.8, we have that the interval of strict L-space slopes on (Y_1, α_1, β_1) contains $[-\infty, 0]$. Therefore, by [2, Theorem 1.4], if $n \geq 2g(K) = 1$, then the result of positive surgery (i.e., each surgery coefficient is positive) on $K_{r,rn}$ is an L-space.

3. A spectral sequence for L-space links

In this section we review some material from [1]. Given $u, v \in \mathbb{Z}^r$, we write $u \preceq v$ if $u_i \leq v_i$ for all i, and $u \prec v$ if $u \preceq v$ and $u \neq v$. Recall that we work with $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients.

Definition 3.1. Given a r-component oriented link L, we define an affine lattice over $\mathbb{Z}^r$:

$$\mathbb{H}(L) = \bigoplus_{i=1}^{r} \mathbb{H}_i(L), \qquad \mathbb{H}_i(L) = \mathbb{Z} + \frac{1}{2}\mathrm{lk}(L_i, L - L_i).$$

Let us recall that the Heegaard–Floer complex for a r-component link L is naturally filtered by the subcomplexes $A_L^-(L; v)$ of $\mathbb{F}[U_1, \ldots, U_r]$-modules for $v \in \mathbb{H}(L)$. Such a subcomplex is spanned by the generators in the Heegaard–Floer complex of Alexander filtration less than or equal to v in the natural partial order on $\mathbb{H}(L)$. The group $\mathrm{HFL}^-(L, v)$ can be defined as the homology of the associated graded complex:

$$\mathrm{HFL}^-(L, v) = H_*\Big(A^-(L; v) / \sum_{u \prec v} A^-(L; u)\Big). \tag{3.1}$$

One can forget a component L_r in L and consider the $(r-1)$-component link $L - L_r$. There is a natural forgetful map $\pi_r \colon \mathbb{H}(L) \to \mathbb{H}(L - L_r)$ defined by the equation:

$$\pi_r(v_1, \ldots, v_r) = (v_1 - \mathrm{lk}(L_1, L_r)/2, \ldots, v_{r-1} - \mathrm{lk}(L_{r-1}, L_r)/2)\,.$$

Similarly, one can define a map $\pi_{L'} \colon \mathbb{H}(L) \to \mathbb{H}(L')$ for every sublink $L' \subset L$. Furthermore, for large $v_r \gg 0$ the subcomplexes $A^-(L; v)$ stabilize, and by [15, Proposition 7.1] one has a natural homotopy equivalence $A^-(L; v) \sim A^-(L - L_r; \pi_r(v))$. More generally, for a sublink $L' = L_{i_1} \cup \cdots \cup L_{i_{r'}}$ one gets

$$A^-(L'; \pi_{L'}(v)) \sim A^-(L; v), \text{ if } v_i \gg 0 \text{ for } i \notin \{i_1, \ldots, i_{r'}\}. \tag{3.2}$$

We will use the "inversion theorem" of [1], expressing the h-function of a link in terms of the Alexander polynomials of its sublinks, or, equivalently, the Euler characteristics of their Heegaard–Floer homology. Define $\chi_{L,v} := \chi(\mathrm{HFL}^-(L, v))$. Then by [15]

$$\chi_L(t_1, \ldots, t_r) := \sum_{v \in \mathbb{H}(L)} \chi_{L,v} t_1^{v_1} \cdots t_r^{v_r} = \begin{cases} (t_1 \cdots t_r)^{1/2} \Delta(t_1, \ldots, t_r) & \text{if } r > 1, \\ \Delta(t)/(1 - t^{-1}) & \text{if } r = 1, \end{cases}$$

where $\Delta(t_1, \ldots, t_r)$ denotes the *symmetrized* Alexander polynomial.

Remark 3.2. We choose the factor $(t_1 \cdots t_r)^{1/2}$ to match more established conventions on the gradings for the hat-version of link Floer homology. For example, the Alexander polynomial of the Hopf link equals 1, and one can check [15] that $\widehat{\mathrm{HFL}}$ is supported in Alexander degrees $(\pm\frac{1}{2}, \pm\frac{1}{2})$. Since the maximal Alexander degrees in $\widehat{\mathrm{HFL}}$ and HFL^- coincide, one gets $\chi_{T(2,2)}(t_1, t_2) = t_1^{1/2} t_2^{1/2}$.

The following "large surgery theorem" underlines the importance of $A^-(L; v)$.

Theorem 3.3 ([11]). *The homology of $A^-(L; v)$ is isomorphic to the Heegaard–Floer homology of a large surgery on L with $spin_c$-structure specified by v. In particular, if L is an L-space link, then $H_*(A^-(L, v)) \simeq \mathbb{F}[U]$ for all v and all U_i are homotopic to each other on the subcomplex $A^-(L; v)$.*

One can show that for L-space links the inclusion $h_v \colon A^-(L, v) \hookrightarrow A^-(S^3)$ is injective on homology, so it is multiplication by $U^{h_L(v)}$. Therefore the generator of $H_*(A^-(L, v)) \simeq \mathbb{F}[U]$ has homological degree $-2h_L(v)$. The function $h_L(v)$ will be called the *h–function* for an L–space link L. In [1] it was called an "HFL-weight function."

Furthermore, if L is an L-space link, then for large $N \in \mathbb{H}(L)$ one has

$$\chi(A^-(L; N)/A^-(L, v)) = h_L(v).$$

Hence, by (3.1) and the inclusion-exclusion formula one can write

$$\chi_{L,v} = \sum_{B \subset \{1,\ldots,r\}} (-1)^{|B|-1} h_L(v - e_B), \tag{3.3}$$

where e_B denotes the characteristic vector of the subset $B \subset \{1, \ldots, r\}$. Furthermore, by (3.2) for a sublink $L' = L_{i_1} \cup \cdots \cup L_{i_{r'}}$ one gets

$$h_{L'}(\pi_{L'}(v)) = h_L(v), \quad \text{if } v_i \gg 0 \text{ for } i \notin \{i_1, \ldots, i_{r'}\}. \tag{3.4}$$

For $r = 1$ equation (3.3) has the form $\chi_{L,v} = h(v-1) - h(v)$, so $h(v)$ can be easily reconstructed from the Alexander polynomial: $h_L(v) = \sum_{u \geq v+1} \chi_{L,v}$. For $r > 1$, one can also show that equation (3.3) (together with the boundary conditions (3.4)) has a unique solution, which is given by the following theorem.

Theorem 3.4 ([1]). *The h-function of an L-space link is determined by the Alexander polynomials of its sublinks as follow*:

$$h_L(v_1, \ldots, v_r) = \sum_{L' \subseteq L} (-1)^{r'-1} \sum_{u \succeq \pi_{L'}(v+\mathbf{1})} \chi_{L',u}, \tag{3.5}$$

where the sublink L' has r' components and $\mathbf{1} = (1, \ldots, 1)$.

Given an L-space link, we construct a spectral sequence whose E_2 page can be computed from the multi-variable Alexander polynomial by an explicit combinatorial procedure, and whose E_∞ page coincides with the group HFL⁻. The complex (3.1) is quasi-isomorphic to the iterated cone:

$$\mathcal{K}(v) = \bigoplus_{B\subset\{1,\ldots,r\}} A^-(L, v - e_B),$$

where the differential consists of two parts: the first acts in each summand and the second acts by inclusion maps between summands. There is a spectral sequence naturally associated to this construction. Its E_1 term equals

$$E_1(v) = \bigoplus_{B\subset\{1,\ldots,r\}} H_*(A^-(L, v - e_B)) = \bigoplus_{B\subset\{1,\ldots,r\}} \mathbb{F}[U]\langle z(v - e_B)\rangle,$$

where $z(u)$ is the generator of $H_*(A^-(L,u))$ of degree $-2h_L(u)$. The next differential ∂_1 is induced by inclusions and reads as

$$\partial_1(z(v - e_B)) = \sum_{i\in B} U^{h(v-e_B)-h(v-e_{B-i})} z(v - e_B + e_i). \tag{3.6}$$

We obtain the following result.

Theorem 3.5 ([1]). *Let L be an L-space link with r components and let $h_L(v)$ be the corresponding h-function. Then there is a spectral sequence with $E_2(v) = H_*(E_1, \partial_1)$ and $E_\infty \simeq \mathrm{HFL}^-(L, v)$.*

Remark 3.6. Let us write more precisely the bigrading on the E_2 page. The E_1 page is naturally bigraded as follows: a generator $U^m z(v - e_B)$ has *cube degree* $|B|$ and its homological degree *in* $A^-(L, v - e_B)$ equals $-2m - 2h(v - e_B)$. In short, we will write

$$\mathrm{bideg}\,(U^m z(v - e_B)) = (|B|, -2m - 2h(v - e_B)).$$

The homological degree of the same generator in $E_1(v)$ equals the sum of these two degrees. The differential ∂_1 has bidegree $(-1, 0)$, and, more generally, the differential ∂_k in the spectral sequence has bidegree $(-k, k-1)$.

In the next section we will compute the E_2 page for cable L-space links and show that $E_2 = E_\infty$. Let us discuss the action of the operators U_i on the E_2 page. Recall that U_i maps $A^-(L, v)$ to $A^-(L, v - e_i)$, and in homology one has

$$U_i z(v) = U^{1-h(v-e_i)+h(v)} z(v - e_i). \tag{3.7}$$

Since U_i commutes with the inclusions of various A^-, we get the following result.

Proposition 3.7. *Equation (3.7) defines a chain map from $\mathcal{K}(v)$ to $\mathcal{K}(v-e_i)$ commuting with the differential ∂_1, so we have a well-defined combinatorial map*

$$U_i\colon H_*(E_1(v),\partial_1)\longrightarrow H_*(E_1(v-e_i),\partial_1).$$

If $E_2=E_\infty$ then one obtains $U_i\colon \mathrm{HFL}^-(L,v)\to \mathrm{HFL}^-(L,v-e_i)$.

Furthermore, by the definition of $\widehat{\mathrm{HFL}}$ [15, Section 4] one gets

$$\widehat{\mathrm{HFL}}(L,v)=H_*\Big(A^-(L,v)/\Big[\sum_{i=1}^{r}A^-(v-e_i)\oplus\sum_{i=1}^{r}U_iA^-(v+e_i)\Big]\Big).$$

This implies the following result.

Proposition 3.8. *There is a spectral sequence with E_1 page*

$$\widehat{E}_1=\bigoplus_{B\subset\{1,\dots,r\}}\mathrm{HFL}^-(L,v+e_B)$$

and converging to $\widehat{E}_\infty=\widehat{\mathrm{HFL}}(L,v)$. The differential $\widehat{\partial}_1$ is given by the action of U_i induced by (3.7).

4. Heegaard–Floer homology for cable links

4.1. The Alexander polynomial and h–function. The Alexander polynomial of cable knots and links is given by the following well-known formula:

$$\Delta_{K_{rm,rn}}(t_1,\dots,t_r)=\Delta_K(t_1^m\cdots t_r^m)\cdot\Delta_{T(rm,rn)}(t_1,\dots,t_r), \tag{4.1}$$

where $T(rm,rn)$ denotes the (rm,rn) torus link. Throughout, let $\mathbf{t}=t_1\cdots t_r$ and $l=mn$.

Lemma 4.1. *The generating functions for the Euler characteristics of* HFL^- *for $K_{rm,rn}$ and $K_{m,n}$ are related by the following equation*:

$$\chi_{K_{rm,rn}}(t_1,\dots,t_r)=\chi_{K_{m,n}}(\mathbf{t})\cdot(\mathbf{t}^{l/2}-\mathbf{t}^{-l/2})^{r-1}. \tag{4.2}$$

Proof. The statement follows from the identity (4.1) and the expression for the Alexander polynomials of torus links:

$$\chi_{T(rm,rn)}(t_1,\dots,t_r)=\frac{(\mathbf{t}^{mn/2}-\mathbf{t}^{-mn/2})^r}{(\mathbf{t}^{m/2}-\mathbf{t}^{-m/2})(\mathbf{t}^{n/2}-\mathbf{t}^{-n/2})}.$$

□

Remark 4.2. The Alexander polynomial is determined up to a sign. By (4.2), the multivariable Alexander polynomial of a cable link is supported on the diagonal, so one can fix the sign by requiring its top coefficient to be positive.

From now on we will assume that K is an L-space knot and $n/m \geq 2g(K)-1$, so $K_{rm,rn}$ is an L-space link for all r. To simplify notation, we define $h_{rm,rn}(v) = h_{K_{rm,rn}}(v)$ and $\chi_{rm,rn}(v) = \chi_{K_{rm,rn},v}$. Let $c = l(r-1)/2$.

Theorem 4.3. *Suppose that* $v_1 \leq v_2 \leq \cdots \leq v_r$. *Then the following equation holds*:

$$\begin{aligned} &h_{rm,rn}(v_1,\ldots,v_r) \\ &\quad = h_{m,n}(v_1-c) + h_{m,n}(v_2-c+l) + \cdots + h_{m,n}(v_r-c+(r-1)l). \end{aligned} \tag{4.3}$$

Proof. We will use Theorem 3.4 to compute $h(v)$. Let L' be a sublink of $K_{rm,rn}$ with r' components, i.e., $L' = K_{r'm,r'n}$. By (4.2), one has

$$\chi_{K_{r'm,r'n}}(t_1,\ldots,t_{r'}) = \chi_{K_{m,n}}(\mathbf{t}) \cdot \mathbf{t}^{l(r'-1)/2} \sum_{j=0}^{r'-1} (-1)^j \binom{r'-1}{j} \mathbf{t}^{-lj},$$

hence $\chi_{L',u}$ does not vanish only if $u = (s,\ldots,s)$, and

$$\chi_{L',s,\ldots,s} = \sum_{j=0}^{r'-1} (-1)^j \binom{r'-1}{j} \chi_{m,n}(s - l(r'-1)/2 + lj).$$

Therefore

$$\begin{aligned} \sum_{u \succeq \pi_{L'}(v+\mathbf{1})} \chi_{L',u} &= \sum_{s > \max(\pi_{L'}(v))} \sum_{j=0}^{r'-1} (-1)^j \binom{r'-1}{j} \chi_{m,n}(s - l(r'-1)/2 + lj) \\ &= \sum_{j=0}^{r'-1} (-1)^j \binom{r'-1}{j} h_{m,n}(\max(\pi_{L'}(v)) - l(r'-1)/2 + lj). \end{aligned}$$

Furthermore, if $L' = L_{i_1} \cup \cdots \cup L_{i_{r'}}$ then

$$\pi_{L'}(v) = (v_{i_1} - l(r-r')/2, \ldots, v_{i_{r'}} - l(r-r')/2),$$

so

$$\max(\pi_{L'}(v)) = \max(v_{i_1},\ldots,v_{i'_r}) - l(r-r')/2 = \max(v_{L'}) - l(r-r')/2.$$

This means that (3.5) can be rewritten as follows:

$$\begin{aligned}
&h_{rm,rn}(v_1,\dots,v_r)\\
&\quad=\sum_{L',j}(-1)^{r'-1+j}\binom{r'-1}{j}h_{m,n}(\max(v_{L'})-l(r-1)/2+lj)\\
&\quad=\sum_{i,j}h_{m,n}(v_i-l(r-1)/2+lj)\sum_{L':v_i=\max(v_{L'})}(-1)^{r'-1+j}\binom{r'-1}{j}.
\end{aligned}$$

One can check that the inner sum vanishes unless $j=i-1$ (recall that $v_1\le v_2\le\cdots\le v_r$), so one gets

$$h_{rm,rn}(v_1,\dots,v_r)=\sum_i h_{m,n}(v_i-l(r-1)/2+l(i-1)).$$ □

Lemma 4.4. *The following identity holds*:

$$h_{rm,rn}(-v_1,\dots,-v_r)=h_{rm,rn}(v_1,\dots,v_r)+(v_1+\cdots+v_r).$$

Proof. Suppose that $v_1\le v_2\le\cdots\le v_r$. Then $-v_1\ge -v_2\ge\cdots\ge -v_r$. Therefore

$$\begin{aligned}
h_{rm,rn}(-v_1,\dots,-v_r)&=\sum_{i=1}^{r}h_{m,n}(-v_i-l(r-1)/2+l(r-i))\\
&=\sum_{i=1}^{r}h_{m,n}(-v_i+l(r-1)/2-l(i-1)).
\end{aligned}$$

It is known (e.g., [6]) that for all x,

$$h_{m,n}(-x)=h_{m,n}(x)+x,$$

hence

$$\begin{aligned}
&h_{m,n}(-v_i+l(r-1)/2-l(i-1))\\
&\quad=h_{m,n}(v_i-l(r-1)/2+l(i-1))+(v_i-l(r-1)/2+l(i-1)).
\end{aligned}$$

Finally, $\sum_{i=1}^{r}(-l(r-1)/2+l(i-1))=0$. □

Lemma 4.5. *One has* $h_{rm,rn}(k,k\dots,k) = \mathbf{h}(k)$, *where* $\mathbf{h}(k)$ *is defined by* (1.1).

Proof. Indeed, by (4.3) we have

$$h_{rm,rn}(k,\dots,k) = h_{m,n}(k - l(r-1)/2) + h_{m,n}(k - l(r-1)/2 + l) + \cdots \\ + h_{m,n}(k + l(r-1)/2),$$

so

$$\sum_k h_{rm,rn}(k,\dots,k)t^k = (t^{-l(r-1)/2} + \cdots + t^{l(r-1)/2}) \sum_k h_{m,n}(k)t^k \\ = \frac{(t^{lr/2} - t^{-lr/2})}{(t^{l/2} - t^{-l/2})} \cdot \frac{t^{-1}\Delta_{m,n}(t)}{(1-t^{-1})^2}.$$

□

For the rest of this section we will assume that $n/m > 2g(K) - 1$.

Lemma 4.6. *If* $v \le g(K_{m,n}) - l$, *then* $\mathrm{HFK}^-(K_{m,n}, v) \simeq \mathbb{F}$.

Proof. By [3, Theorem 1.10], $K_{m,n}$ is an L-space knot and hence by [14]

$$g(K_{m,n}) = \tau(K_{m,n}), \quad g(K) = \tau(K).$$

By [17], we have

$$g(K_{m,n}) = mg(K) + \frac{(m-1)(n-1)}{2},$$

so for $n/m > 2g(K) - 1$ we have

$$2g(K_{m,n}) = 2mg(K) + mn - m - n + 1 < mn + 1,$$

hence $l = mn \ge 2g(K_{m,n})$. On the other hand, it is well known that for $v \le -g(K_{m,n})$ one has $\mathrm{HFK}^-(K_{m,n}, v)) \simeq \mathbb{F}$. □

We will use the function β defined by (1.1).

Lemma 4.7. *If* $\beta(k) = -1$ *then* $\mathrm{HFK}^-(K_{m,n}, k - c) = 0$. *Otherwise*

$$\beta(k) = \max\{j : 0 \le j \le r - 1,\ \mathrm{HFK}^-(K_{m,n}, k - c + lj) \simeq \mathbb{F}\}. \tag{4.4}$$

Proof. By (1.1) and Lemma 4.5 we have

$$\beta(k)+1 = h_{rm,rn}(k-1,\dots,k-1) - h_{rm,rn}(k,\dots,k)$$
$$= \sum_{j=0}^{r-1}(h_{m,n}(k-1-c+lj) - h_{m,n}(k-c+lj)).$$

Note that $h_{m,n}(k-1-c+lj)-h_{m,n}(k-c+lj) = \dim \mathrm{HFK}^-(K_{m,n}, k-c+lj) \in \{0,1\}$. If $\mathrm{HFK}^-(K_{m,n}, k-c+lj) \simeq \mathbb{F}$ then $k-c+lj \le g(K_{m,n})$, so by Lemma 4.6 $\mathrm{HFK}^-(K_{m,n}, k-c+lj') \simeq \mathbb{F}$ for all $j' < j$. Therefore, if $\mathrm{HFK}^-(K_{m,n}, k-c) = 0$ then $\beta(k) = -1$, otherwise

$$\mathrm{HFK}^-(K_{m,n}, k-c+lj) = \begin{cases} \mathbb{F} & \text{if } j \le \beta(k), \\ 0 & \text{if } j > \beta(k). \end{cases}$$ □

Suppose that

$$v_1 = \cdots = v_{\lambda_1} = u_1,$$
$$v_{\lambda_1+1} = \cdots = v_{\lambda_1+\lambda_2} = u_2,$$
$$\vdots$$
$$v_{\lambda_1+\cdots+\lambda_{s-1}+1} = \cdots = v_r = u_s,$$

where $u_1 < u_2 < \cdots < u_s$ and $\lambda_1 + \cdots + \lambda_s = r$. We will abbreviate this as $v = (u_1^{\lambda_1}, \dots, u_s^{\lambda_s})$.

Lemma 4.8. *Suppose that $\beta(u_s) < r - \lambda_s$. Then for any subset $B \subset \{1,\dots,r-1\}$ one has $h_{rm,rn}(v-e_B) = h_{rm,rn}(v-e_B-e_r)$.*

Proof. To apply (4.3), one needs to reorder the components of the vectors $v-e_B$ and $v-e_B-e_r$. Note that in both cases the last (largest) λ_s components are equal either to u_s or to u_s-1, and the corresponding contributions to $h_{rm,rn}$ are equal to $h_{m,n}(u_s-c+l(r-\lambda_s)+lj)$ or to $h_{m,n}(u_s-c+l(r-\lambda_s)+lj-1)$, respectively $(j=0,\dots,\lambda_s-1)$. On the other hand, by (4.4) one has

$$\mathrm{HFK}^-(K_{m,n}, u_s-c+l(r-\lambda_s)+lj) = 0$$

and so

$$h_{m,n}(u_s-c+l(r-\lambda_s)+lj-1) = h_{m,n}(u_s-c+l(r-\lambda_s)+lj).$$ □

Lemma 4.9. *If $\beta(u_s) \geq r - \lambda_s$ then $h_{rm,rn}(v) = \mathbf{h}(u_s) + ru_s - |v|$.*

Proof. Since $\beta(u_s) \geq r - \lambda_s$, we have $\mathrm{HFK}^-(K_{m,n}, u_s - c + l(r - \lambda_s)) \simeq \mathbb{F}$, so

$$u_s - c + l(r - \lambda_s) \leq g(K_{m,n}).$$

For $i \leq r - \lambda_s$ we get

$$v_i - c + l(i-1) < u_s - c + l(i-1) \leq u_s - c + l(r - \lambda_s) - l \leq g(K_{m,n}) - l,$$

so by Lemma 4.6, $\mathrm{HFK}^-(K_{m,n}, w) \simeq \mathbb{F}$ for all $w \in [v_i - c + l(i-1), u_s - c + l(i-1)]$, and

$$h_{m,n}(v_i - c + l(i-1)) = h_{m,n}(u_s - c + l(i-1)) + (u_s - v_i).$$

Now the statement follows from Lemma 4.3. □

Lemma 4.10. *Suppose that $\beta(u_s) \geq r - \lambda_s$. Then for any subsets $B' \subset \{1, \dots, r - \lambda_s\}$ and $B'' \subset \{r - \lambda_s + 1, \dots, r\}$ one has*

$$h_{rm,rn}(v - e_{B'} - e_{B''}) = h_{rm,rn}(v) + |B'| + \min(|B''|, \beta(u_s) - r + \lambda_s + 1).$$

Proof. Since $\mathrm{HFK}^-(K_{m,n}, u_s - c + l(r - \lambda_s)) \simeq \mathbb{F}$, we have

$$u_s - c + l(r - \lambda_s) \leq g(K_{m,n}),$$

so for all $i \leq r - \lambda_s$ one has

$$v_i - c + l(i-1) < u_s - c + l(r - \lambda_s) - l \leq g(K_{m,n}) - l,$$

and by Lemma 4.6 $\mathrm{HFK}^-(K_{m,n}, v_i - c + l(i-1)) \simeq \mathbb{F}$, and

$$h_{m,n}(v_i - 1 - c + l(i-1)) = h_{m,n}(v_i - c + l(i-1)) + 1.$$

Therefore

$$h_{rm,rn}(v - e_{B'} - e_{B''}) = |B'| + h_{rm,rn}(v - e_{B''}).$$

Finally,

$$\begin{aligned} h_{rm,rn}(v - e_{B''}) - h_{rm,rn}(v) &= \sum_{j=0}^{|B''|} (h_{m,n}(u_s - 1 - c + l(r - \lambda_s) + lj) \\ &\quad - h_{m,n}(u_s - c + l(r - \lambda_s) + lj) \\ &= \min(|B''|, \beta(u_s) - r + \lambda_s + 1). \end{aligned}$$

□

4.2. Spectral sequence for HFL^-

Definition 4.11. Let $\mathcal{E}_r$ denote the exterior algebra over $\mathbb{F}$ with variables $z_1, \dots, z_r$. Let us define the *cube differential* on $\mathcal{E}_r$ by the equation

$$\partial(z_{\alpha_1} \wedge \cdots \wedge z_{\alpha_k}) = \sum_{j=1}^{k} z_{\alpha_1} \wedge \cdots \wedge \widehat{z}_{\alpha_j} \wedge \cdots \wedge z_{\alpha_k},$$

and the *b-truncated differential* on $\mathcal{E}_r[U]$ by the equation

$$\partial^{(b)}(z_{\alpha_1} \wedge \cdots \wedge z_{\alpha_k}) = \begin{cases} U\partial(z_{\alpha_1} \wedge \cdots \wedge z_{\alpha_k}) & \text{if } k \le b, \\ \partial(z_{\alpha_1} \wedge \cdots \wedge z_{\alpha_k}) & \text{if } k > b. \end{cases}$$

More invariantly, we define the *weight* of a monomial $z_\alpha = z_{\alpha_1} \wedge \cdots \wedge z_{\alpha_k}$ as $w(z_\alpha) = \min(|\alpha|, b)$, and the b-truncated differential is given by the equation

$$\partial^{(b)}(z_\alpha) = \sum_{i \in \alpha} U^{w(\alpha) - w(\alpha - \alpha_i)} z_{\alpha - \alpha_i}. \tag{4.5}$$

Indeed, $w(\alpha) - w(\alpha - \alpha_i) = 1$ for $|\alpha| \le b$ and $w(\alpha) - w(\alpha - \alpha_i) = 0$ for $|\alpha| > b$.

Definition 4.12. Let $\mathcal{E}_r^{\mathrm{red}} \subset \mathcal{E}_r$ be the subalgebra of $\mathcal{E}_r$ generated by the differences $z_i - z_j$ for all $i \neq j$.

Lemma 4.13. *The kernel of the cube differential ∂ on $\mathcal{E}_r$ coincides with $\mathcal{E}_r^{\mathrm{red}}$.*

Proof. It is clear that $\partial(z_i - z_j) = 0$, and Leibniz rule implies vanishing of ∂ on $\mathcal{E}_r^{\mathrm{red}}$. Let us prove that $\operatorname{Ker} \partial \subset \mathcal{E}_r^{\mathrm{red}}$. Since $(\mathcal{E}_r, \partial)$ is acyclic, it is sufficient to prove that the image of every monomial $z_{\alpha_1} \wedge \cdots \wedge z_{\alpha_k}$ is contained in $\mathcal{E}_r$. Indeed, one can check that

$$\partial(z_{\alpha_1} \wedge \cdots \wedge z_{\alpha_k}) = (z_{\alpha_2} - z_{\alpha_1}) \wedge \cdots \wedge (z_{\alpha_k} - z_{\alpha_{k-1}}). \qquad \square$$

Lemma 4.14. *The homology of $\partial^{(b)}$ is given by the following equation*:

$$\dim H_k(\mathcal{E}_r[U], \partial^{(b)}) = \begin{cases} \binom{r-1}{k} & \textit{if } k < b, \\ 0 & \textit{if } k \ge b. \end{cases}$$

Proof. Since ∂ is acyclic, one immediately gets $H_k(\mathcal{E}_r[U], \partial^{(b)}) = 0$ for $k \geq b$. For $k < b$, the homology is supported at the zeroth power of U and one has $H_k(\mathcal{E}_r[U]) \simeq \mathrm{Ker}(\partial|_{\wedge^k(z_1,\ldots,z_r)})$. The dimension of the latter kernel equals

$$\dim \mathrm{Ker}(\partial|_{\wedge^k(z_1,\ldots,z_r)}) = \dim \wedge^k(z_1 - z_2, \ldots, z_1 - z_r) = \binom{r-1}{k}. \qquad \square$$

Proof of Theorem 5. Let us compute $\mathrm{HFL}^-(K_{rm,rn}, v)$ using the spectral sequence constructed in Theorem 3.5. By Lemma 4.8, in case (a) it is easy to see that the complex (E_1, ∂_1) is contractible in the direction of e_r and $E_2 = H_*(E_1, \partial_1) = 0$.

In case (b) by Lemma 4.10 and (4.5) one can write $E_1 = \mathcal{E}_{r-\lambda_s}[U] \otimes_{\mathbb{F}[U]} \mathcal{E}_{\lambda_s}[U]$, a tensor product of chain complexes of $\mathbb{F}[U]$–modules, and ∂_1 acts as $U\partial$ on the first factor and as $\partial^{(\beta+1)}$ on the second one. This implies

$$E_2 = H_*(E_1, \partial_1) \simeq \mathcal{E}_{r-\lambda_s} \otimes_{\mathbb{F}} H_*(\mathcal{E}_{\lambda_s}[U], \partial^{(\beta+1)}). \tag{4.6}$$

Indeed, U acts trivially on $H_*\left(\mathcal{E}_{\lambda_s}[U], \partial^{(\beta+1)}\right)$, so one can take the homology of $\partial^{(\beta+1)}$ first and then observe that $U\partial$ vanishes on

$$\mathcal{E}_{r-\lambda_s}[U] \otimes_{\mathbb{F}[U]} H_*(\mathcal{E}_{\lambda_s}[U], \partial^{(\beta+1)}) \simeq \mathcal{E}_{r-\lambda_s} \otimes_{\mathbb{F}} H_*(\mathcal{E}_{\lambda_s}[U], \partial^{(\beta+1)}).$$

By Lemma 4.14, the E_2 page (4.6) agrees with the statement of the theorem, hence we need to prove that the spectral sequence collapses.

Indeed, the E_1 page is bigraded by the homological degree and $|B|$ (see Remark 3.6). By Lemma 4.14 any surviving homology class on the E_2 page of cube degree x has bidegree $(x, -2h_{rm,rn}(v) - 2x)$, so all bidegrees on the E_2 page belong to the same line of slope (-2). Therefore all higher differentials must vanish.

Finally, a simple formula for $h_{rm,rn}(v)$ in case (b) follows from Lemma 4.9. $\square$

4.3. Action of U_i. One can use Proposition 3.7 to compute the action of U_i on HFL^- for cable links. Recall that $R = \mathbb{F}[U_1, \ldots, U_r]$. Throughout this section we assume $n/m > 2g(K) - 1$. We start with a simple algebraic statement.

Proposition 4.15. *Let $\mathcal{C}$ be an $\mathbb{F}$-algebra. Given a finite collection of elements $c_\alpha \in \mathcal{C}$ and vectors $v^{(\alpha)} \in \mathbb{Z}^r$, consider the ideal $\mathcal{I} \subset \mathcal{C} \otimes_{\mathbb{F}} R$ generated by $c_\alpha \otimes U_1^{v_1^{(\alpha)}} \cdots U_r^{v_r^{(\alpha)}}$. Then the following statements hold:*

(a) *the quotient $(\mathcal{C} \otimes_{\mathbb{F}} R)/\mathcal{I}$ can be equipped with a $\mathbb{Z}^r$–grading, with U_i of grading $(-e_i)$ and $\mathcal{C}$ of grading* 0;

(b) *the subspace of* $(\mathcal{C} \otimes_{\mathbb{F}} R)/\mathcal{I}$ *with grading* v *is isomorphic to*

$$[(\mathcal{C} \otimes_{\mathbb{F}} R)/\mathcal{I}](v) \simeq \mathcal{C}/(c_\alpha : v^{(\alpha)} \preceq -v).$$

Proof. Straightforward. □

Definition 4.16. We define $\mathcal{A}_r = \mathcal{E}_r \otimes_{\mathbb{F}} R$ and $\mathcal{A}_r^{\mathrm{red}} = \mathcal{E}_r^{\mathrm{red}} \otimes_{\mathbb{F}} R$. Let $\mathcal{I}'_\beta$ denote the ideal in $\mathcal{A}_r$ generated by the monomials $(z_{i_1} \wedge \cdots \wedge z_{i_s}) \otimes U_{i_{s+1}} \cdots U_{i_{\beta+1}}$ for all $s \leq \beta + 1$ and all tuples of pairwise distinct $i_1, \ldots, i_{\beta+1}$. Let $\mathcal{I}_\beta := \mathcal{I}'_\beta \cap \mathcal{A}_r^{\mathrm{red}}$ be the corresponding ideal in $\mathcal{A}_r^{\mathrm{red}}$.

The algebras $\mathcal{A}_r$ and $\mathcal{A}_r^{\mathrm{red}}$ are naturally $\mathbb{Z}^{r+1}$–graded: the generators z_i have Alexander grading 0 and homological grading (-1), the generators U_i have Alexander grading $(-e_i)$ and homological grading (-2).

Definition 4.17. We define $\mathcal{H}(k) := \bigoplus_{\max(v) \leq k} \mathrm{HFL}^-(K_{rm,rn}, v)$. Since U_i decreases the Alexander grading, $\mathcal{H}(k)$ is naturally an R–module.

The following theorem clarifies the algebraic structure of Theorem 5.

Theorem 4.18. *The following graded* R*–modules are isomorphic*:

$$\mathcal{H}(k)/\mathcal{H}(k-1) \simeq \mathcal{A}_r^{\mathrm{red}}/\mathcal{I}_{\beta(k)}[-2\mathbf{h}(k)]\{k, \ldots, k\},$$

where $[\cdot]$ *and* $\{\cdot\}$ *denote the shifts of the homological grading and the Alexander grading, respectively.*

Proof. By definition, $\mathcal{H}(k)/\mathcal{H}(k-1)$ is supported on the set of Alexander gradings v such that $\max(v) = k$. The monomial $U_1 \cdots U_r$ belongs to the ideal $\mathcal{I}_{\beta(k)}$, so $\mathcal{A}_r^{\mathrm{red}}/\mathcal{I}_{\beta(k)}$ is supported on the set of Alexander gradings u with $\max(u) = 0$.

Suppose that exactly λ components of v are equal to k. Without loss of generality we can assume $v_1, \ldots, v_{r-\lambda} < k$ and $v_{r-\lambda+1} = \cdots = v_r = k$. It follows from Lemma 4.13 and the proof of Theorem 5 that $\mathrm{HFL}^-(K_{rm,rn}, v)$ is isomorphic to the quotient of $\mathcal{E}_r^{\mathrm{red}}$ by the ideal generated by degree $\beta - r + \lambda + 1$ monomials in $(z_i - z_j)$ for $i, j > r - \lambda$.

Consider the subspace of $\mathcal{A}_r/\mathcal{I}'_\beta$ of Alexander grading $(v_1 - k, \ldots, v_r - k)$. By Proposition 4.15 it is isomorphic to a quotient of $\mathcal{E}_r$ modulo the following relations. For each subset $B \subset \{1, \ldots, r - \lambda\}$ and each degree $\beta + 1 - |B|$ monomial m' in variables z_i for $i \notin B$ there is a relation $m' \otimes \prod_{b \in B} U_b \in \mathcal{I}'_\beta$. All these relations can be multiplied by an appropriate monomial in R to have Alexander grading $(v_1 - k, \ldots, v_r - k)$.

Note that such m' should contain at most $r-\lambda-|B|$ factors with indices in $\{1,\dots,r-\lambda\}\setminus B$, hence it contains at least $\beta-r+\lambda+1$ factors with indices in $\{r-\lambda+1,\dots,r\}$. Therefore $[\mathcal{A}_r/\mathcal{I}'_\beta](v_1-k,\dots,v_r-k)$ is naturally isomorphic to the quotient of $\mathcal{E}_r$ by the ideal generated by degree $\beta-r+\lambda+1$ monomials in z_i for $i>r-\lambda$.

We conclude that the space $[\mathcal{A}_r^{\mathrm{red}}/\mathcal{I}_{\beta(k)}](v_1-k,\dots,v_r-k)$ is isomorphic to $\mathrm{HFL}^-(K_{rm,rn},v)$. The action of U_i on $\mathcal{H}(k)$ is described by Proposition 3.7. One can check that it commutes with the above isomorphisms for different v, so we get the isomorphism of R–modules. □

We illustrate the above theorem with the following example (cf. Example 5.8).

Example 4.19. Let us describe the subspaces of $\mathcal{A}_3^{\mathrm{red}}/\mathcal{I}_1$ with various Alexander gradings. The ideal $\mathcal{I}_1$ equals:

$$\mathcal{I}_1=((z_1-z_2)(z_2-z_3),(z_1-z_2)U_3,\\ (z_1-z_3)U_2,(z_2-z_3)U_1,U_1U_2,U_1U_3,U_2U_3)\subset\mathcal{A}_3^{\mathrm{red}}.$$

In the Alexander grading $(0,0,0)$ one gets

$$[\mathcal{A}_3^{\mathrm{red}}/\mathcal{I}_1](0,0,0)\simeq\mathcal{E}_3^{\mathrm{red}}/((z_1-z_2)(z_2-z_3))=\langle 1,z_1-z_2,z_2-z_3\rangle,$$

in the Alexander grading $(k,0,0)$ (for $k>0$) one gets two relations

$$U_1^k(z_1-z_2)(z_2-z_3),U_1^{k-1}(z_2-z_3)\in\mathcal{I}_1.$$

Since the latter implies the former, we get

$$[\mathcal{A}_3^{\mathrm{red}}/\mathcal{I}_1](k,0,0)\simeq\mathcal{E}_3^{\mathrm{red}}/(z_2-z_3)=\langle 1,z_1-z_2\rangle.$$

The map

$$U_1\colon[\mathcal{A}_3^{\mathrm{red}}/\mathcal{I}_1](0,0,0)\longrightarrow[\mathcal{A}_3^{\mathrm{red}}/\mathcal{I}_1](1,0,0)$$

is a natural projection

$$\mathcal{E}_3^{\mathrm{red}}/((z_1-z_2)(z_2-z_3))\longrightarrow\mathcal{E}_3^{\mathrm{red}}/(z_2-z_3),$$

while the map

$$U_1\colon[\mathcal{A}_3^{\mathrm{red}}/\mathcal{I}_1](k,0,0)\longrightarrow[\mathcal{A}_3^{\mathrm{red}}/\mathcal{I}_1](k+1,0,0)$$

is an isomorphism for $k>0$.

The gradings $(0,k,0)$ and $(0,0,k)$ can be treated similarly. Furthermore, $U_iU_j\in\mathcal{I}_1$ for $i\neq j$, so all other graded subspaces of $\mathcal{A}_3^{\mathrm{red}}/\mathcal{I}_1$ vanish.

Since the multiplication by U_i preserves the ideal $\mathcal{I}_\beta$, we get the following useful result.

Corollary 4.20. *If* $\max(v) = \max(v - e_i)$*, then the map*

$$U_i\colon \mathrm{HFL}^-(K_{rm,rn}, v) \longrightarrow \mathrm{HFL}^-(K_{rm,rn}, v - e_i)$$

is surjective.

Lemma 4.21. *Suppose that* $\max(v) = k$ *and* $\max(v - e_i) = k - 1$*, and the homology group* $\mathrm{HFL}^-(K_{rm,rn}, v)$ *does not vanish. Then* $\beta(k) = r - 1$*,* $\beta(k-1) \geq r - 2$ *and the map*

$$U_i\colon \mathrm{HFL}^-(K_{rm,rn}, v) \longrightarrow \mathrm{HFL}^-(K_{rm,rn}, v - e_i)$$

is surjective.

Proof. Since $\max(v) = k$ and $\max(v - e_i) = k - 1$, the multiplicity of k in v equals 1, so by Theorem 5 $\beta(k) \geq r - 1$, hence $\beta(k) = r - 1$. Therefore $\mathrm{HFL}^-(K_{rm,rn}, v) \simeq \mathcal{E}_r^{\mathrm{red}}$, so U_i is surjective. Indeed, by Theorem 5 $\mathrm{HFL}^-(K_{rm,rn}, v - e_i)$ is naturally isomorphic to a quotient of $\mathcal{E}_r^{\mathrm{red}}$, and by Proposition 3.7 U_i coincides with a natural quotient map. Finally, by (4.4)

$$\mathrm{HFK}^-(K_{m,n}, k - c + l(r-1)) \simeq \mathbb{F},$$

and by Lemma 4.6

$$\mathrm{HFK}^-(K_{m,n}, k - 1 - c + l(r-2)) \simeq \mathbb{F},$$

so $\beta(k-1) \geq r - 2$. □

Proof of Theorem 6. Let us prove that the homology classes with diagonal Alexander gradings generate HFL^- over R. Indeed, given $v = (v_1 \leq \cdots \leq v_r)$ with $\mathrm{HFL}^-(K_{rm,rn}, v) \neq 0$, by Theorems 5 and 4.18 one can check that

$$\mathrm{HFL}^-(K_{rm,rn}, v_r, \ldots, v_r) \neq 0$$

and by Corollary 4.20 the map

$$U_1^{v_r - v_1} \cdots U_{r-1}^{v_r - v_{r-1}}\colon \mathrm{HFL}^-(K_{rm,rn}, v_r, \ldots, v_r) \to \mathrm{HFL}^-(K_{rm,rn}, v)$$

is surjective.

Let us describe the R-modules generated by the diagonal classes in degree $(k, \dots, k)$. If $\beta(k) = -1$ then $\mathrm{HFL}^{-}(K_{rm,rn}, k, \dots, k) = 0$. If $0 \le \beta(k) \le r-2$ then by Lemma 4.21 the submodule $R \cdot \mathrm{HFL}^{-}(K_{rm,rn}, k, \dots, k)$ does not contain any classes with maximal Alexander degree less than k, so by Theorem 4.18

$$R \cdot \mathrm{HFL}^{-}(K_{rm,rn}, k, \dots, k) \simeq \mathcal{A}_r^{\mathrm{red}}/\mathcal{I}_{\beta(k)} =: M_{\beta(k)}$$

Suppose that $\beta(k) = r-1$, and consider minimal a and maximal b such that $a \le k \le b$ and $\beta(i) = r-1$ for $i \in [a,b]$. If there is no minimal a, we set $a = -\infty$. By Lemma 4.21, $\beta(a-1) = r-2$ and all the maps

$$\mathrm{HFL}^{-}(K_{rm,rn}, b, \dots, b) \xrightarrow{U_1 \cdots U_r} \mathrm{HFL}^{-}(K_{rm,rn}, b-1, \dots, b-1)$$
$$\cdots \longrightarrow \mathrm{HFL}^{-}(K_{rm,rn}, a, \dots, a) \xrightarrow{U_1 \cdots U_r} \mathrm{HFL}^{-}(K_{rm,rn}, a-1, \dots, a-1)$$

are surjective. Therefore

$$R \cdot \mathrm{HFL}^{-}(K_{rm,rn}, b, \dots, b) \simeq \mathcal{A}_r^{\mathrm{red}}/(U_1 \cdots U_r)^{b-a}\mathcal{I}_{r-2} =: M_{r-1,b-a+1}$$

is supported in all Alexander degrees with maximal coordinates in $[a,b]$ and in Alexander degrees with maximal coordinate $(a-1)$ which appears with multiplicity at least 2.

Finally, we get the following decomposition of HFL^{-} as an R–module:

$$\mathrm{HFL}^{-}(K_{rm,rn}) = \bigoplus_{\substack{k:0 \le \beta(k) < r-1 \\ \beta(k+1) < r-1}} M_{\beta(k)} \oplus \bigoplus_{\substack{a,b:\beta(a-1)=r-2 \\ \beta(b+1)<r-1 \\ \beta([a,b])=r-1}} M_{r-1,b-a+1} \oplus M_{r-1,\infty}. \qquad \square$$

Note that for $r = 1$ we get $M_{0,l} \simeq \mathbb{F}[U_1]/(U_1^l)$ and $M_{0,+\infty} \simeq \mathbb{F}[U]$.

4.4. Spectral sequence for $\widehat{\mathrm{HFL}}$

Theorem 4.22. *If $\beta(k) + \beta(k+1) \le r-2$ then the spectral sequence for $\widehat{\mathrm{HFL}}(K_{rm,rn}, k, \dots, k)$ degenerates at the $\hat{E}_2$ page and*

$$\widehat{\mathrm{HFL}}(K_{rm,rn}, k, \dots, k) \simeq \bigoplus_{i=0}^{\beta(k)} \binom{r-1}{i} \mathbb{F}_{-2\mathbf{h}(k)-i} \oplus \bigoplus_{i=0}^{\beta(k+1)} \binom{r-1}{i} \mathbb{F}_{-2\mathbf{h}(k)+2-r+i}.$$

Proof. By Proposition 3.8, for a given v there is a spectral sequence with $\widehat{E}_1$ page

$$\widehat{E}_1 = \bigoplus_{B\subset\{1,\dots,r\}} \mathrm{HFL}^-(L, v + e_B)$$

and converging to $\widehat{E}_\infty = \widehat{\mathrm{HFL}}(L, v)$. If $v = (k, \dots, k)$ then (for $B \neq \emptyset$) the maximal coordinate of $v + e_B$ equals $k + 1$ and appears with multiplicity $\lambda = |B|$. Therefore, by Theorem 5 $\mathrm{HFL}^-(L, v + e_B)$ does not vanish if and only if either $B = \emptyset$ or $|B| \geq r - \beta(k + 1)$, and it is given by Theorem 5. By (1.1) we have $\mathbf{h}(k + 1) = \mathbf{h}(k) - \beta(k + 1) - 1$.

The spectral sequence is bigraded by the homological (Maslov) grading at each vertex of the cube and the "cube grading" $|B|$. The differential $\widehat{\partial}_1$ acts along the edges of the cube, and decreases the Maslov grading by 2 and the cube grading by 1.

One can check using Theorem 4.18 that its homology $\widehat{E}_2$ does not vanish in cube degrees 0 and $r - \beta(k + 1)$, so one can write

$$\widehat{E}_2 = \widehat{E}_2^0 \oplus \widehat{E}_2^{r-\beta(k+1)},$$

and

$$\widehat{E}_2^0 \simeq \bigoplus_{i=0}^{\beta(k)} \binom{r-1}{i} \mathbb{F}_{-2\mathbf{h}(k)-i},$$

$$\widehat{E}_2^{r-\beta(k+1)} \simeq \bigoplus_{i=0}^{\beta(k+1)} \binom{r-1}{i} \mathbb{F}_{-2\mathbf{h}(k+1)-3\beta(k+1)+i}.$$

By (1.1) we have

$$\mathbf{h}(k + 1) = \mathbf{h}(k) - \beta(k + 1) - 1,$$

so

$$-2\mathbf{h}(k + 1) - 3\beta(k + 1) + i = -2\mathbf{h}(k) + 2 - \beta(k + 1) + i.$$

A higher differential $\widehat{\partial}_s$ decreases the cube grading by s and decreases the Maslov grading by $s + 1$. Therefore the only nontrivial higher differential is $\widehat{\partial}_{r-\beta(k+1)}$ which vanishes by degree reasons too. Indeed, the maximal Maslov grading in $\widehat{E}_2^{r-\beta(k+1)}$ equals $-2\mathbf{h}(k) + 2$ while the minimal Maslov grading in $\widehat{E}_2^0$ equals $-2\mathbf{h}(k) - \beta(k)$, so the differential can decrease the Maslov grading at most by $\beta(k) + 2$. On the other hand, $\widehat{\partial}_{r-\beta(k+1)}$ drops it by $r - \beta(k + 1) + 1$, and for $\beta(k) + \beta(k + 1) < r - 1$ one has $r - \beta(k + 1) + 1 > \beta(k) + 2$. Therefore $\widehat{\partial}_{r-\beta(k+1)} = 0$ and the spectral sequence vanishes at the $\widehat{E}_2$ page. □

We illustrate the proof of Theorem 4.22 by Examples 5.4 and 5.5

Lemma 4.23. *The following identity holds*:

$$\beta(1-k)+\beta(k)=r-2.$$

Proof. By (1.1) and Lemma 4.5,

$$\beta(k)=h(k-1,\dots,k-1)-h(k,\dots,k)-1,$$

$$\beta(1-k)=h(-k,\dots,-k)-h(1-k,\dots,1-k)-1.$$

By Lemma 4.4,

$$h(-k,\dots,-k)=h(k,\dots,k)+kr,$$

$$h(1-k,\dots,1-k)=h(k-1,\dots,k-1)+r(k-1).$$

These two identities imply the desired statement. □

Theorem 4.24. *If $\beta(k)+\beta(k+1)\geq r-2$, then*

$$\widehat{\mathrm{HFL}}(K_{rm,rn},k,\dots,k) \simeq \bigoplus_{i=0}^{r-2-\beta(k+1)} \binom{r-1}{i}\mathbb{F}_{-2\mathbf{h}(k)-i} \oplus \bigoplus_{i=0}^{r-2-\beta(k)} \binom{r-1}{i}\mathbb{F}_{-2\mathbf{h}(k)+2-r+i}.$$

Proof. By Lemma 4.23 we get $\beta(-k)=r-2-\beta(k+1)$ and $\beta(1-k)=r-2-\beta(k)$, so

$$\beta(k)+\beta(k+1)+\beta(-k)+\beta(1-k)=2(r-2),$$

so $\beta(-k)+\beta(1-k)\leq r-2$. By Theorem 4.22 the spectral sequence degenerates for $\widehat{\mathrm{HFL}}(-k,\dots,-k)$ and

$$\widehat{\mathrm{HFL}}(K_{rm,rn},-k,\dots,-k) \simeq \bigoplus_{i=0}^{r-2-\beta(k+1)} \binom{r-1}{i}\mathbb{F}_{-2\mathbf{h}(-k)-i} \oplus \bigoplus_{i=0}^{r-2-\beta(k)} \binom{r-1}{i}\mathbb{F}_{-2\mathbf{h}(-k)+2-r+i}.$$

Finally, by [15, Proposition 8.2] we have

$$\widehat{\mathrm{HFL}}_{\bullet}(K_{rm,rn},k,\dots,k)=\widehat{\mathrm{HFL}}_{\bullet-2kr}(K_{rm,rn},-k,\dots,-k)$$

and by Lemma 4.4 $\mathbf{h}(k)=\mathbf{h}(-k)-kr$. □

Theorem 4.25. *Off-diagonal homology groups are supported on the union of the unit cubes along the diagonal. In such a cube with corners $(k,\dots,k)$ and $(k+1,\dots,k+1)$ one has*

$$\widehat{\mathrm{HFL}}(K_{rm,rn},(k-1)^j,k^{r-j}) \simeq \binom{r-2}{\beta(k)}\mathbb{F}_{-2\mathbf{h}(k)-\beta(k)-j}.$$

Proof. We use the spectral sequence from HFL^- to $\widehat{\mathrm{HFL}}$. By Theorem 4.18, all the $\widehat{E}_2$ homology outside the union of these cubes vanish (since some U_i would provide an isomorphism between $\mathrm{HFL}^-(K_{rm,rn},v)$ and $\mathrm{HFL}^-(K_{rm,rn},v-e_i)$). Furthermore, if $\beta(k)=r-1$ then the homology in the cube vanish too, so we can focus on the case $\beta(k)\le r-2$.

One can check that $\widehat{E}_2$ does not vanish in cube degrees $j-\beta(k),\dots,j$ and

$$\widehat{E}_2^{j-c} \simeq \binom{j-1}{c}\binom{r-1-j}{\beta(k)-c}\mathbb{F}_{-2\mathbf{h}(k)-\beta(k)-c}.$$

Note that the *total* homological degree on $\widehat{E}_2^{j-c}$ equals $-2\mathbf{h}(k)-\beta(k)-j$ and does not depend on c. Therefore all higher differentials in the spectral sequence must vanish and the rank of $\widehat{\mathrm{HFL}}$ equals:

$$\sum_{c=0}^{\beta}\binom{j-1}{c}\binom{r-1-j}{\beta(k)-c}=\binom{r-2}{\beta(k)}. \qquad \square$$

We illustrate this proof by Example 5.6.

4.5. Special case: $m=1$, $n=2g(K)-1$. The case $m=1, n=2g(K)-1$ is special since Lemma 4.6 is not always true. Indeed, $K_{m,n}=K$ and $l=n=2g(K)-1$, but for $v=g(K)-l=1-g(K)$ we have $\mathrm{HFL}^-(K,v)=0$. However, it is clear that in all other cases Lemma 4.6 is true, so for generic v Lemmas 4.8 and 4.10 hold true. This allows one to prove an analogue of Theorem 5.

Theorem 4.26. *Assume that $m=1, n=2g(K)-1$ (so $l=2g(K)-1$) and suppose that $v=(u_1^{\lambda_1},u_2^{\lambda_2},\dots,u_s^{\lambda_s})$ where $u_1<\dots<u_s$. Then the Heegaard–Floer homology group $\mathrm{HFL}^-(K_{rm,rn},v)$ can be described as follow.*

(a) *Assume that $u_s-c+l(r-\lambda_s)=g(K)-\nu l$ with $1\le\nu\le\lambda_s$. Then*

$$\mathrm{HFL}^-(K_{rm,rn},v)\simeq(\mathbb{F}_{(0)}\oplus\mathbb{F}_{(-1)})^{r-\lambda_s}\otimes\left[\bigoplus_{j=0}^{\nu-2}\binom{\lambda_s-1}{j}\mathbb{F}_{(-2h(v)-j)} \oplus\binom{\lambda_s-1}{\nu}\mathbb{F}_{(-2h(v)+2-\nu)}\right]$$

(b) *In all other cases, the homology is given by Theorem 5.*

Proof. One can check that the proof of Lemma 4.8 fails if $u_s - c + l(r - \lambda_s) = g(K) - l$, and remains true in all other cases. Similarly, the proof of Lemma 4.10 fails only if $u_s - c + l(r - \lambda_s) + lj = g(K) - l$ for $1 \le j \le \lambda_s - 1$, which is equivalent to $u_s - c + l(r - \lambda_s) = g(K) - (j + 1)$. This proves (b).

Let us consider the special case (a). Note that

$$\begin{aligned} &h_{m,n}(u_s - c + l(r - \lambda_s) + lj - 1) - h_{m,n}(u_s - c + l(r - \lambda_s) + lj) \\ &\quad = \chi(\mathrm{HFK}^-(K, g(K) + l(j - \nu)) \\ &\quad = \begin{cases} 1 & \text{if } j < \nu - 1, \\ 0 & \text{if } j = \nu - 1, \\ 1 & \text{if } j = \nu, \\ 0 & \text{if } j > \nu. \end{cases} \end{aligned}$$

Given a pair of subsets $B' \subset \{1, \dots, r - \lambda_s\}$ and $B'' \subset \{r - \lambda_s + 1, \dots, r\}$, one can write, analogously to Lemma 4.10:

$$h_{rm,rn}(v - e_{B'} - e_{B''}) = h_{rm,rn}(v) + |B'| + w(B''),$$

where

$$w(B'') = \begin{cases} |B''| & \text{if } |B''| \le \nu - 1, \\ \nu - 1 & \text{if } |B''| = \nu, \\ \nu & \text{if } |B''| > \nu. \end{cases}$$

By the Künneth formula, the E_2 page of the spectral sequence is determined by the "deformed cube homology" with the weight function $w(B'')$, as in (4.5). If ∂, as above, denotes the standard cube differential, then, similarly to Lemma 4.14, the homology of ∂_U^w is isomorphic to the kernel of ∂ in cube degrees $0, \dots \nu - 2$ and ν.

Finally, we need to prove that all higher differentials vanish. For a homology generator α on the E_2 page of cube degree x, its bidegree is equal either to $(x, -2h(v) - 2x)$ or to $(x, -2h(v) - 2x + 2)$. The differential ∂_k has bidegree $(-k, k - 1)$ (see Remark 3.6), so the bidegree of $\partial_k(\alpha)$ is equal either to $(x - k, -2h(v) - 2x + k - 1)$ or to $(x - k, -2h(v) - 2x + k + 1)$. Since $-2x + k + 1 < -2(x - k)$ for $k > 1$, we have $\partial_k(\alpha) = 0$. □

The action of U_i in this special case can be described similarly to Theorem 4.18. However, it is not true that U_i is surjective whenever it does not obviously vanish. In particular, the following example shows that HFL^- may be not generated by diagonal classes, so Theorem 6 does not hold. We leave the appropriate adjustment of Theorem 6 as an exercise to a reader.

Example 4.27. Consider $T_{2,2}$, the $(2,2)$ cable of the trefoil. We have $g(K)=l=1$ and $c=1/2$, so by Theorem 4.26

$$\mathrm{HFL}^-(T_{2,2},1/2,1/2)\simeq \mathbb{F}_{(-1)},\quad \mathrm{HFL}^-(T_{2,2},-1/2,1/2)\simeq \mathbb{F}_{(-2)}\oplus\mathbb{F}_{(-3)}.$$

Therefore U_1 is not surjective. Furthermore, the class in $\mathrm{HFL}^-(T_{2,2},-1/2,1/2)$ of homological degree (-2) is not in the image of any diagonal class under the R–action.

5. Examples

5.1. (n,n) torus links. The symmetrized multi-variable Alexander polynomial of the (n,n) torus link equals (for $n>1$):

$$\Delta_{T_{n,n}}(t_1,\dots,t_n)=((t_1\cdots t_n)^{1/2}-(t_1\cdots t_n)^{-1/2})^{n-2}.$$

Each pair of components has linking number 1, so $c=(n-1)/2$. The homology groups $\mathrm{HFL}^-(T(n,n),v)$ are described by the following theorem, which is a special case of Theorem 5.

Theorem 5.1. *Consider the (n,n) torus link, and an Alexander grading $v=(v_1,\dots,v_n)$. Suppose that among the coordinates v_i exactly λ are equal to k and all other coordinates are less than k. Let $|v|=v_1+\cdots+v_n$. Then*

$$\mathrm{HFL}^-(T(n,n),v)=\begin{cases}0 & \text{if } k>\lambda-\frac{n+1}{2},\\ (\mathbb{F}_{(0)}\oplus\mathbb{F}_{(-1)})^{n-1}\otimes\mathbb{F}_{2|v|} & \text{if } k<-\frac{n-1}{2},\\ (\mathbb{F}_{(0)}\oplus\mathbb{F}_{(-1)})^{n-\lambda}\otimes\bigoplus_{i=0}^{\lambda-\frac{n+1}{2}-k}\binom{\lambda-1}{i}\mathbb{F}_{(-2h(v)-i)} & \text{if } -\frac{n-1}{2}\le k\le\lambda-\frac{n+1}{2},\end{cases}$$

where $h(v)=\frac{1}{2}\big(\frac{n-1}{2}-k\big)\big(\frac{n-1}{2}-k+1\big)+kn-|v|$ in the last case.

Proof. Indeed, $\beta(k) = \frac{n-1}{2} - k$ for $k > -\frac{n-1}{2}$ and $\beta(k) = n-1$ for $k \le -\frac{n-1}{2}$. By Theorem 5, the homology group $\mathrm{HFL}^-(T(n,n),v)$ does not vanish if and only if

$$k \le \lambda - \frac{n+1}{2}. \tag{5.1}$$

If $k \ge -\frac{n-1}{2}$, equation (4.3) implies

$$h_{n,n}(v) = \frac{1}{2}\Big(\frac{n-1}{2} - k\Big)\Big(\frac{n-1}{2} - k + 1\Big) + kn - |v|.$$

If $k \le -\frac{n-1}{2}$, equation (4.3) implies $h_{n,n}(v) = -|v|$. Furthermore, for all v satisfying (5.1) one has

$$\mathrm{HFL}^-(T(n,n),v) = (\mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)})^{n-\lambda} \otimes \bigoplus_{j=0}^{\lambda-\frac{n+1}{2}-k} \binom{\lambda-1}{j} \mathbb{F}_{(-2h_{n,n}(v)-j)}.$$

Finally, if $k - \frac{n-1}{2}$, then (5.1) holds for all λ and $\lambda - \frac{n+1}{2} - k > \lambda - 1$, hence

$$\begin{aligned}\mathrm{HFL}^-(T(n,n),v) &= (\mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)})^{n-\lambda} \otimes \bigoplus_{j=0}^{\lambda-1} \binom{\lambda-1}{j} \mathbb{F}_{(-2h_{n,n}(v)-j)} \\ &= (\mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)})^{n-1} \otimes \mathbb{F}_{(-2h_{n,n}(v))}. \qquad \square\end{aligned}$$

Remark 5.2. One can check that, in agreement with [1], the condition (5.1) defines the multi-dimensional semigroup of the plane curve singularity $x^n = y^n$.

Corollary 5.3. *We have the following decomposition of* HFL^- *as an* R*-module*:

$$\mathrm{HFL}^-(T(n,n)) = M_0 \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_{n-2} \oplus M_{n-1,+\infty}.$$

To prove Theorem 4, we use Theorem 3.

Proof of Theorem 4. We have $\beta(\frac{n-1}{2} - s) = s$ for $s < n-1$, and

$$\beta(\frac{n-1}{2} - s) + \beta(\frac{n-1}{2} - s + 1) = 2s - 1 \le n-2 \ \le s \le \frac{n-1}{2}.$$

Therefore for $s \le \frac{n-1}{2}$ Theorem 4.22 implies the degeneration of the spectral sequence from HFL^- to $\widehat{\mathrm{HFL}}$, and

$$\begin{aligned}&\widehat{\mathrm{HFL}}\Big(T(n,n), \frac{n-1}{2} - s, \ldots, \frac{n-1}{2} - s\Big) \\ &= \bigoplus_{i=0}^{s} \binom{n-1}{i} \mathbb{F}_{(-s^2-s-i)} \oplus \bigoplus_{i=0}^{s-1} \binom{n-1}{i} \mathbb{F}_{(-s^2-s-n+2+i)}. \qquad \square\end{aligned}$$

Let us illustrate the degeneration of the spectral sequence from HFL^- to $\widehat{\mathrm{HFL}}$ in some examples.

Example 5.4. For $s = 0$ we have $\widehat{E}_1 = \widehat{E}_2 = \mathbb{F}_{(0)}$. For $s = 1$ the $\widehat{E}_1$ page has nonzero entries in cube degree 0 where one gets

$$\mathrm{HFL}^-\left(T(n,n), \frac{n-1}{2} - 1, \ldots, \frac{n-1}{2} - 1\right) \simeq \mathbb{F}_{(-2)} \oplus (n-1)\mathbb{F}_{(-3)},$$

and in cube degree n where one gets $\mathbb{F}_{(0)}$. Indeed, the differential $\widehat{\partial}_1$ vanishes, so for $n > 2$

$$\widehat{\mathrm{HFL}}\left(T(n,n), \frac{n-1}{2} - 1, \ldots, \frac{n-1}{2} - 1\right) \simeq \mathbb{F}_{(-2)} \oplus (n-1)\mathbb{F}_{(-3)} \oplus \mathbb{F}_{(-n)}.$$

Note that for $n = 2$ the differential $\widehat{\partial}_2$ does not vanish, so the bound $s \leq \frac{n-1}{2}$ is indeed necessary for the spectral sequence to collapse at $\widehat{E}_2$ page.

Example 5.5. The case $s = 2$ is more interesting. The $\widehat{E}_1$ page has nonzero entries in cube degree 0, $n-1$ (where we have n vertices) and n, where one has

$$\begin{aligned}
\widehat{E}_1^0 &= \mathbb{F}_{(-6)} \oplus (n-1)\mathbb{F}_{(-7)} \oplus \binom{n-1}{2}\mathbb{F}_{(-8)}, \\
\widehat{E}_1^{n-1} &= n(\mathbb{F}_{(-4)} \oplus \mathbb{F}_{(-5)}), \\
\widehat{E}_1^n &= \mathbb{F}_{(-2)} \oplus (n-1)\mathbb{F}_{(-3)}.
\end{aligned}$$

The differential $\widehat{\partial}_1$ cancels some summands in $\widehat{E}_1^{n-1}$ and $\widehat{E}_1^n$:

$$\begin{aligned}
\widehat{E}_2^0 &= \mathbb{F}_{(-6)} \oplus (n-1)\mathbb{F}_{(-7)} \oplus \binom{n-1}{2}\mathbb{F}_{(-8)}, \\
\widehat{E}_2^{n-1} &= (n-1)\mathbb{F}_{(-4)} + \mathbb{F}_{(-5)}.
\end{aligned}$$

For $n > 4$ all higher differentials vanish and

$$\begin{aligned}
&\widehat{\mathrm{HFL}}\left(T(n,n), \frac{n-1}{2} - 2, \ldots, \frac{n-1}{2} - 2\right) \\
&\quad \simeq \mathbb{F}_{(-6)} \oplus (n-1)\mathbb{F}_{(-7)} \oplus \binom{n-1}{2}\mathbb{F}_{(-8)} \oplus (n-1)\mathbb{F}_{(-3-n)} + \mathbb{F}_{(-4-n)}.
\end{aligned}$$

The following example illustrates the computation of $\widehat{\mathrm{HFL}}$ for the off-diagonal Alexander gradings.

Example 5.6. Let us compute the homology $\widehat{\mathrm{HFL}}(T(n,n),v)$ for

$$v = \Big(\frac{n-1}{2}-2\Big)^j \Big(\frac{n-1}{2}-1\Big)^{n-j} \quad (1 \le j \le n-1)$$

using the spectral sequence from HFL^-. In the n dimensional cube $(v+e_B)$ almost all all vertices have vanishing HFL^-, except for the vertex $\big(\frac{n-1}{2}-1,\dots,\frac{n-1}{2}-1\big)$

$$\mathrm{HFL}^-\Big(\frac{n-1}{2}-1,\dots,\frac{n-1}{2}-1\Big) = F_{(-2)} \oplus (n-1)\mathbb{F}_{(-3)}$$

and j of its neighbors with homology $\mathbb{F}_{(-4)} \oplus \mathbb{F}_{(-5)}$. Clearly, $\widehat{E}_2$ is concentrated in degrees j (with homology $(n-1-j)\mathbb{F}_{(-3)}$) and $(j-1)$ (with homology $(j-1)\mathbb{F}_{(-4)}$). Note that both parts contribute to the total degree $(-3-j)$, so

$$\widehat{\mathrm{HFL}}(T(n,n),v) = (n-1-j)\mathbb{F}_{(-3-j)} \oplus (j-1)\mathbb{F}_{(-3-j)} = (n-2)\mathbb{F}_{(-3-j)}.$$

Finally, we draw all the homology groups HFL^- for $(2,2)$ and $(3,3)$ torus links.

Example 5.7. For the Hopf link, one has two cases. If $v_1 < v_2$, then the condition (5.1) implies $v_2 \le -1/2$. If $v_1 = v_2$, then (5.1) implies $v_2 \ge 1/2$.

The nonzero homology of the Hopf link is shown in Figure 3 and Table 1

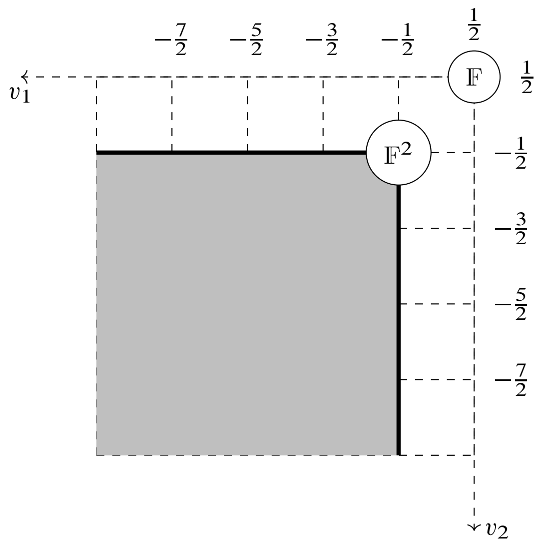

Figure 3. HFL^- for the (2,2) torus link: $\mathbb{F}^2$ on thick lines and in the grey region.

Table 1. Maslov gradings for the $(2,2)$ torus link.

Alexander grading	Homology
$(1/2,1/2)$	$\mathbb{F}_{(0)}$
$(a,b), a,b \le -1/2$	$\mathbb{F}_{(2a+2b)} \oplus \mathbb{F}_{(2a+2b-1)}$

Example 5.8. For the $(3,3)$ torus link, one has two cases. If $v_1 \le v_2 < v_3$, then the condition (5.1) implies $v_3 \le 1$. If $v_1 < v_2 = v_3$, then (5.1) implies $v_3 \le 0$. Finally, if $v_1 = v_2 = v_3$, then (5.1) implies $v_3 \le 1$. In other words, nonzero homology appears at the point $(1,1,1)$, at three lines $(0,0,k), (0,k,0), (k,0,0)$ $(k \le 0)$ and at the octant $\max(v_1,v_2,v_3) \le -1$.

This homology is shown in Figure 4 and Table 2.

5.2. More general torus links. The HFL^- homology of the $(4,6)$ torus link is shown in Figure 5 and Table 3. Note that as an $\mathbb{F}[U_1,U_2]$ module it can be decomposed into 5 copies of $M_0 \simeq \mathbb{F}$, a copy of $M_{1,1}$ and a copy of $M_{1,+\infty}$. In particular, the map $U_1U_2\colon HFL^-(-2,-2) \to HFL^-(-3,-3)$ is surjective with one-dimensional kernel.

5.3. Non-algebraic example. In this subsection we compute the Heegaard–Floer homology for the $(4,6)$-cable of the trefoil. Its components are $(2,3)$-cables of the trefoil, which are known to be L-space knots (cf. [3]), but not algebraic knots. By Theorem 2, the $(4,6)$-cable of the trefoil is an L-space link, but its homology is not covered by [1].

The Alexander polynomial of the $(2,3)$-cable of the trefoil equals:

$$\Delta_{T_{2,3}}(t) = \frac{(t^6 - t^{-6})(t^{1/2} - t^{-1/2})}{(t^{3/2} - t^{-3/2})(t^2 - t^{-2})},$$

hence the Euler characteristic of its Heegaard–Floer homology equals

$$\chi_{2,3}(t) = \frac{\Delta_{T_{2,3}}(t)}{1 - t^{-1}} = t^3 + 1 + t^{-1} + t^{-3} + t^{-4} + \cdots.$$

By (4.1), the bivariate Alexander polynomial of the $(4,6)$-cable equals:

$$\begin{aligned}\chi_{4,6}(t_1,t_2) &= \chi_{2,3}(t_1 \cdot t_2)((t_1t_2)^3 - (t_1t_2)^{-3})\\ &= (t_1t_2)^6 + (t_1t_2)^3 + (t_1t_2)^2 + (t_1t_2)^{-1} + (t_1t_2)^{-2} + (t_1t_2)^{-5}.\end{aligned}$$

The nonzero Heegaard–Floer homology are shown in Figure 6 and the corresponding Maslov gradings are given in Table 4. Note that as $\mathbb{F}[U_1,U_2]$ module it can be decomposed in the following way:

$$HFL^- \simeq 4M_0 \oplus M_{1,1} \oplus M_{1,2} \oplus M_{1,+\infty}.$$

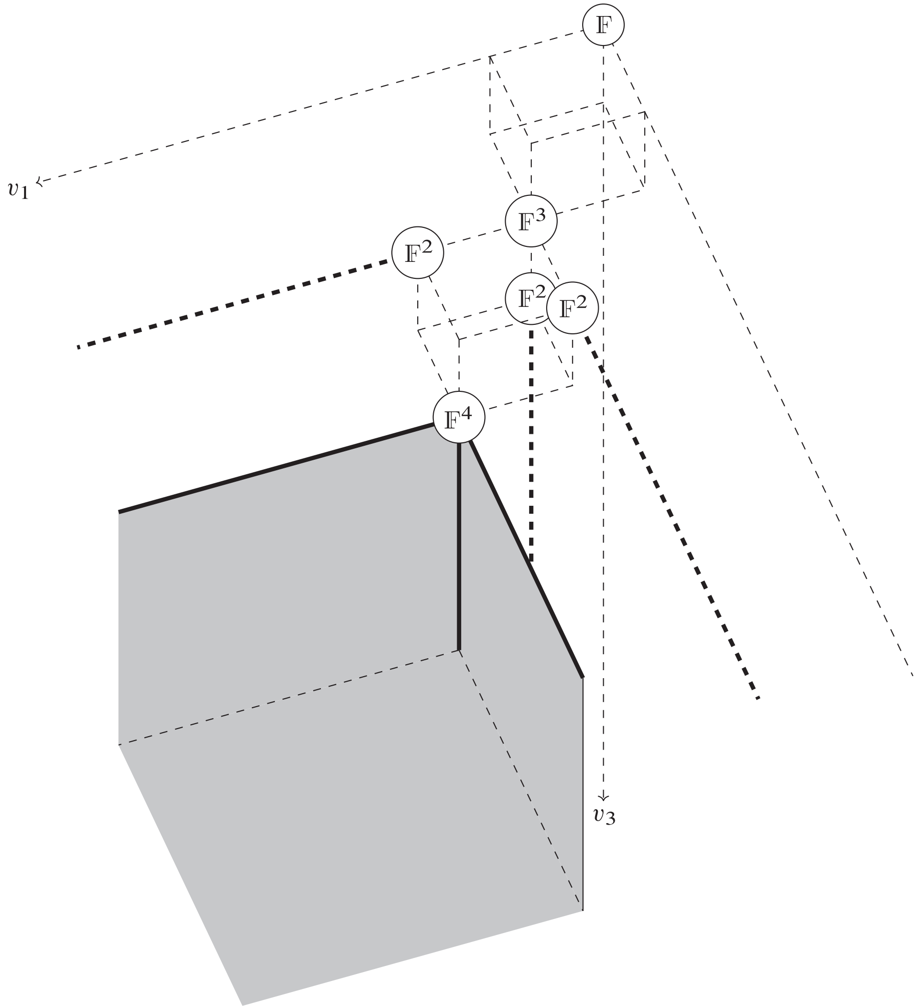

Figure 4. HFL⁻ for the (3,3) torus link: $\mathbb{F}^2$ on dashed thick lines; $\mathbb{F}^4$ on solid thick lines and in the shaded region. Top Alexander grading is $(1, 1, 1)$.

Table 2. Maslov gradings for the $(3, 3)$ torus link.

Alexander grading	Homology
$(1, 1, 1)$	$\mathbb{F}_{(0)}$
$(0, 0, 0)$	$\mathbb{F}_{(-2)} \oplus 2\mathbb{F}_{(-3)}$
$(0, 0, k)$, $(0, k, 0)$ and $(k, 0, 0)$ $(k < 0)$	$\mathbb{F}_{(2k-2)} \oplus \mathbb{F}_{(2k-3)}$
(a, b, c), $a, b, c \leq -1$	$\mathbb{F}_{(2a+2b+2c)} \oplus 2\mathbb{F}_{(2a+2b+2c-1)}$ $\oplus\mathbb{F}_{(2a+2b+2c-2)}$

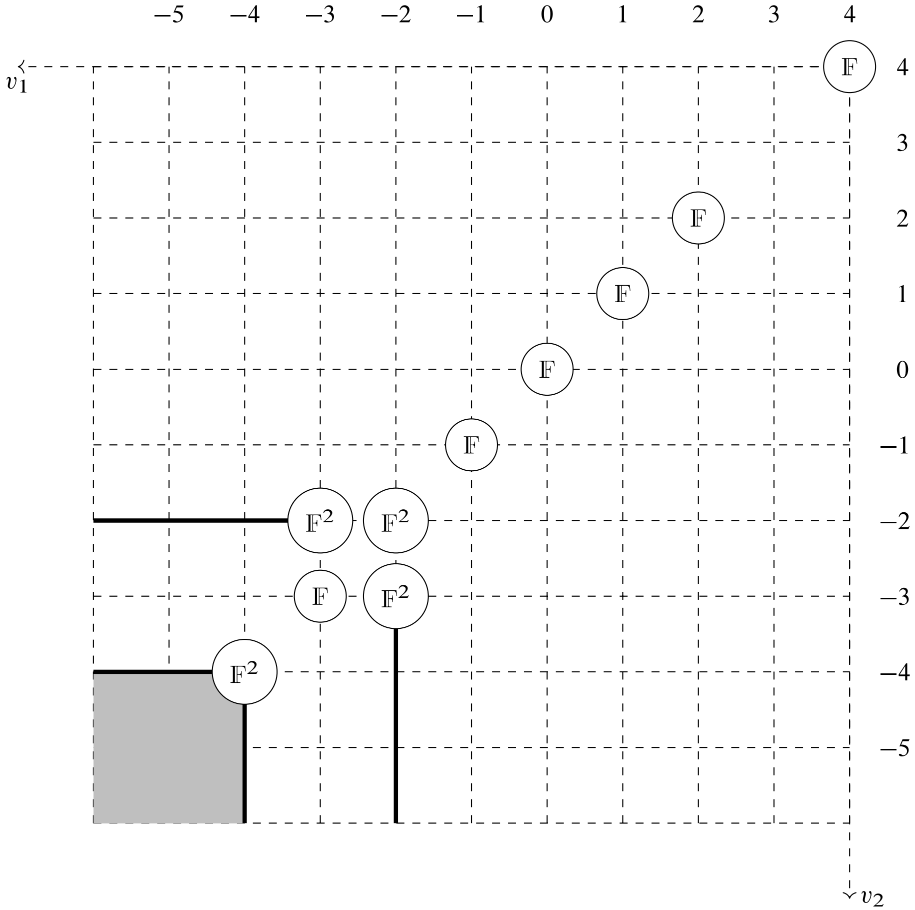

Figure 5. HFL^{-} for the (4,6) torus link: $\mathbb{F}^2$ on thick lines and in the grey region.

Table 3. Maslov gradings for the $(4, 6)$ torus link.

Alexander grading	Homology
$(4, 4)$	$\mathbb{F}_{(0)}$
$(2, 2)$	$\mathbb{F}_{(-2)}$
$(1, 1)$	$\mathbb{F}_{(-4)}$
$(0, 0)$	$\mathbb{F}_{(-6)}$
$(-1, -1)$	$\mathbb{F}_{(-8)}$
$(-2, k)$ and $(k, -2)$, $k \leq -2$	$\mathbb{F}_{(2k-6)} \oplus \mathbb{F}_{(2k-7)}$
$(-3, -3)$	$\mathbb{F}_{(-12)}$
(a, b), $a, b \leq -4$	$\mathbb{F}_{(2a+2b)} \oplus \mathbb{F}_{(2a+2b-1)}$

Table 4. Maslov gradings for the (4,6) cable of the trefoil.

Alexander grading	Homology
$(6,6)$	$\mathbb{F}_{(0)}$
$(3,3)$	$\mathbb{F}_{(-2)}$
$(2,2)$	$\mathbb{F}_{(-4)}$
$(0,k)$ and $(k,0)$, $k \geq 0$	$\mathbb{F}_{(2k-6)} \oplus \mathbb{F}_{(2k-7)}$
$(-1,-1)$	$\mathbb{F}_{(-10)}$
$(-2,-2)$	$\mathbb{F}_{(-12)}$
$(-3,k)$ and $(k,-3)$, $k \geq -3$	$\mathbb{F}_{(2k-8)} \oplus \mathbb{F}_{(2k-9)}$
$(-4,k)$ and $(k,-4)$, $k \geq 10$	$\mathbb{F}_{(2k-10)} \oplus \mathbb{F}_{(2k-11)}$
$(-5,-5)$	$\mathbb{F}_{(-22)}$
(a,b), $a,b \leq -6$	$\mathbb{F}_{(2a+2b)} \oplus \mathbb{F}_{(2a+2b-1)}$

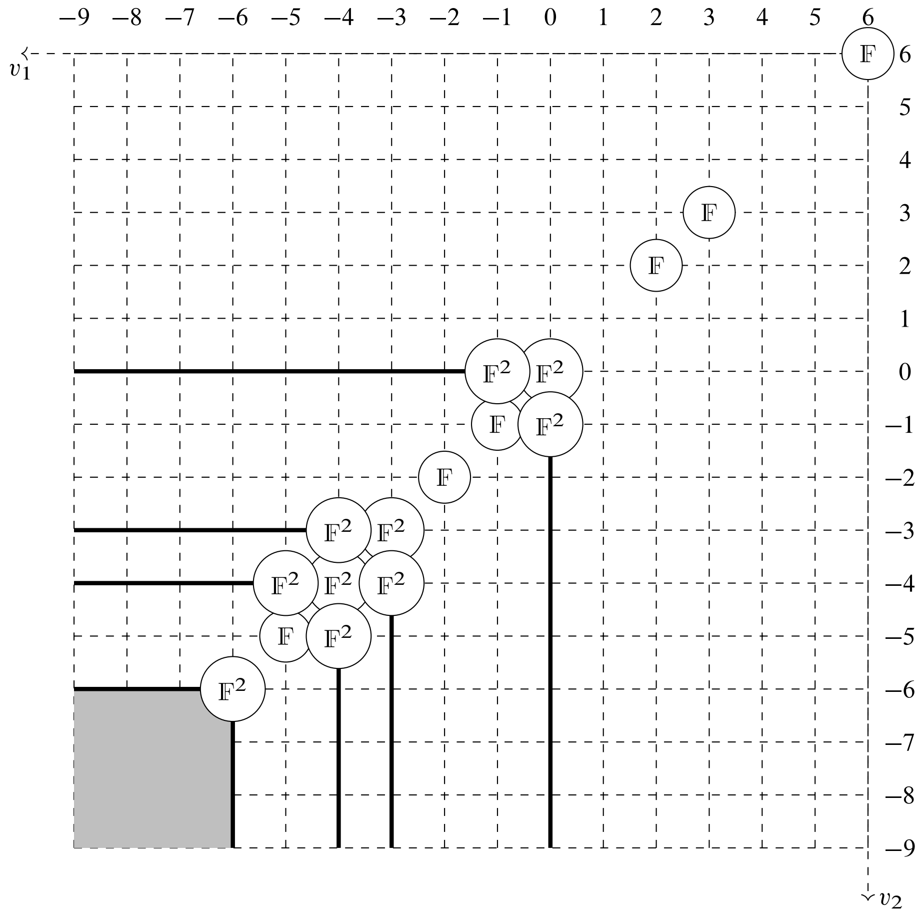

Figure 6. HFL^- for the (4,6) cable of the trefoil: $\mathbb{F}^2$ on thick lines and in the grey region.

References

[1] E. Gorsky and A. Némethi, Lattice and Heegaard–Floer homologies of algebraic links. *Int. Math. Res. Not. IMRN* **2015**, no. 23, 12737–12780. MR 3431635 Zbl 1342.57005

[2] J. Hanselman and L. Watson, A calculus for bordered Floer homology. Preprint 2015. arXiv:1508.05445 [math.GT]

[3] M. Hedden, On knot Floer homology and cabling II. *Int. Math. Res. Not. IMRN* **2009**, no. 12, 2248–2274. MR 2511910 Zbl 1172.57008

[4] W. Heil, Elementary surgery on Seifert fiber spaces. *Yokohama Math. J.* **22** (1974), 135–139. MR 0375320 Zbl 0297.57006

[5] J. Hom, A note on cabling and L-space surgeries. *Algebr. Geom. Topol.* **11** (2011), no. 1, 219–223. MR 2764041 Zbl 1221.57019

[6] J. Hom, T. Lidman, and N. Zufelt, Reducible surgeries and Heegaard Floer homology. *Math. Res. Lett.* **22** (2015), no. 3, 763–788. MR 3350104 Zbl 1323.57006

[7] J. Licata, Heegaard Floer homology of (n, n)-torus links: computations and questions. Preprint 2012. arXiv:1208.0394 [math.GT]

[8] R. Lipshitz, P. Ozsváth, and D. Thurston, Bordered Heegaard Floer homology: Invariance and pairing. Preprint 2008. arXiv:0810.0687 [math.GT]

[9] P. Lisca and A. I. Stipsicz, Ozsváth–Szabó invariants and tight contact 3-manifolds. III. *J. Symplectic Geom.* **5** (2007), no. 4, 357–384. MR 2413308 Zbl 1149.57037

[10] Y. Liu, L-space surgeries on links. *Quantum Topol.* **8** (2017), no. 3, 505–570. MR 3692910 Zbl 06784952

[11] C. Manolescu and P. Ozsváth, Heegaard Floer homology and integer surgeries on links. Preprint, arXiv:1011.1317v1 [math.GT]

[12] P. Ozsváth and Z. Szabó, Holomorphic disks and three-manifold invariants: properties and applications. *Ann. of Math.* (2) **159** (2004), no. 3, 1159–1245. MR 2113020 Zbl 1081.57013

[13] P. Ozsváth and Z. Szabó, Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math.* (2) **159** (2004), no. 3, 1027–1158. MR 2113019 Zbl 1073.57009

[14] P. Ozsváth and Z. Szabó, On knot Floer homology and lens space surgeries. *Topology* **44** (2005), no. 6, 1281–1300. MR 2168576 Zbl 1077.57012

[15] P. Ozsváth and Z. Szabó, Holomorphic disks, link invariants and the multi-variable Alexander polynomial. *Algebr. Geom. Topol.* **8** (2008), no. 2, 615–692. MR 2443092 Zbl 1144.57011

[16] P. Ozsváth and Z. Szabó, Knot Floer homology and rational surgeries. *Algebr. Geom. Topol.* **11** (2011), no. 1, 1–68. MR 2764036 Zbl 1226.57044

[17] T. Shibuya, On the genus of torus links. *Kobe J. Math.* **2** (1985), no. 2, 123–125. MR 0847178 Zbl 0598.57004

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Eugene Gorsky, Department of Mathematics, UC Davis, One Shields Ave, Davis, CA 95616, USA

International Laboratory of Representation Theory and Mathematical Physics, NRU-HSE, 7 Vavilova St., 117312 Moscow, Russia

e-mail: egorskiy@math.ucdavis.edu

Jennifer Hom, School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA 30332-0160, USA

e-mail: jhom6@math.gatech.edu