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Itô's Formula for Non-Smooth Functions

By

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Abstract

Let us consider an application of forward local $C^{1,\varepsilon}$-semimartingale flows of C^1-diffeomorphisms. First a change of variable formula is derived and the existence of all moments of the appearing Jacobian is shown. Then as a consequence, Itô's formula holds for continuous functions which first and second order derivatives exist only in the sense of distributions.

§1. Introduction

Itô's formula, the mean-value theorem in stochastic calculus, is established in this paper for continuous functions having first and second order derivatives in the sense of distributions, evaluated along non-degenerate local Hölder-continuous space-time semi-martingales which are local diffeomorphisms w.r.t. a spatial parameter. Our result will be obtained by means of the theory of stochastic flows developed in Kunita [3]. In order to have a one-to-one correspondence between stochastic flows and (Itô's) stochastic differential equations, the latters have to be formulated in terms of continuous C-valued semimartingales.

Let us consider the investigated problem in the most familiar situation. Given a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ provided with the 1-dimensional standard Brownian motion $(B_t)_{0\leq t\leq T}$, where $0<T<\infty$. Then Itô's stochastic differential equation

$$\xi_t(x)=x+\int_0^t \sigma(\xi_s(x))\,dB_s+\int_0^t \beta(\xi_s(x))\,ds \tag{1}$$

has a unique solution $\xi_t(x)$, $0\leq t\leq T$, for any $x\in \boldsymbol{R}$, if the coefficients σ and β are Lipschitz-continuous functions on $\boldsymbol{R}$ of linear growth. Moreover, ξ_t is a continuous function of the spatial parameter $x\in\boldsymbol{R}$ for any $t\in[0, T]$, P-a.s.

In order to apply Itô's formula to a function F on $\boldsymbol{R}$ evaluated along the solution ξ_t, $0\leq t\leq T$, of (1), it conventionally has to be assumed that $F\in$

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$C^2(\boldsymbol{R})$. Then

$$F(\xi_t) - F(x) = \int_0^t F'(\xi_s)\sigma(\xi_s)dB_s + \int_0^t F'(\xi_s)\beta(\xi_s)ds + \frac{1}{2}\int_0^t F''(\xi_s)\sigma^2(\xi_s)ds \quad (2)$$

for $0 \le t \le T$, P-a.s. for any $x \in \boldsymbol{R}$.

If the first and second order derivatives $DF \in L^2(\lambda, \boldsymbol{R})$ and $D^2F \in L^1(\lambda, \boldsymbol{R})$, λ the Lebesque measure, of a continuous function F are given in the sense of distributions, i.e.

$$\int_{\boldsymbol{R}} DF(x)\gamma(x)dx = -\int_{\boldsymbol{R}} F(x)\gamma'(x)dx$$

for any test function γ which is smooth and has compact support, then our version of Itô's formula holds analogously to (2) in the case of a non-degenerate process $(\xi_t)_{0 \le t \le T}$, but now only for $\lambda \otimes P$-almost all $(x, \omega) \in \boldsymbol{R} \times \Omega$.

The proof is an approximation procedure in terms of a mollifier sequence, i.e. $J_\varepsilon F \to F$ uniformly on compact sets, $(J_\varepsilon F)' \to DF$ in $L^2_{loc}(\lambda, \boldsymbol{R})$ and $(J_\varepsilon F)'' \to D^2F$ in $L^1_{loc}(\lambda, \boldsymbol{R})$ as $\varepsilon \searrow 0$. The key to make use of these integrability properties is a change of variable formula: For any $F \in L^1(\lambda, \boldsymbol{R})$

$$\int_{\boldsymbol{R}} dx \int_0^T F(\xi_{0,t}(x))dt = \int_0^T dt \int_{\boldsymbol{R}} dz\, F(z) \det \partial_z \xi_{0,t}^{-1}(z)$$

holds P-a.s. and it is in $L^1(P)$, where $\partial_z \xi_{0,t}^{-1}(z)$ denotes the Jacobian of the inverse of the flow $\xi_{s,t}$, $0 \le s \le t \le T$. We notice that this formula 'decomposes' the function F and the process ξ_t, i.e. its associated flow $\xi_{s,t}$. For the definition of a stochastic flow see the first section. There is a strong analogy to a flow obtained as the solution of a system of ordinary differential equations (dynamical system). But in the probabilistic case, the transport takes place along the solution of a stochastic differential equation, i.e. the paths have non-vanishing quadratic variation and are consequently nowhere differentiable. Nevertheless, the theory of stochastic flows reveals that under smoothness assumptions on the coefficients σ and β, the solution ξ_t, $0 \le t \le T$, of (1) depends smoothly on the initial value $x \in \boldsymbol{R}$ i.e. the spatial parameter. The composition $F(\xi_{0,t}(x))$, $0 \le t \le T$, is well defined as a stochastic process if ξ_t, $0 \le t \le T$, is non-degenerate which is satisfied for an elliptic σ. The change of variable formula is valid if σ as well as β are bounded and their derivatives σ', β' are bounded and Hölder-continuous. By the flow property, the Jacobian of the inverse of the stochastic flow is the same as the inverse of the Jacobian of the stochastic flow. Since the latter is a solution of an Itô's stochastic differential equation, all its moments exist and are uniformly bounded on compact sets of starting points.

The present approach is built up on the stochastic flow framework suggested by Kunita [3]. The underlying non-degenerate stochastic process with spatial parameter is a forward local $C^{1,\varepsilon}$-valued space-time semimartingale flow of

C^1-diffeomorphisms, where the required regularity properties are expressed in terms of its joint quadratic variation process and its drift process, respectively. The first and second order derivatives in the sense of distributions of a continuous space-time function F are only assumed to satisfy local integrability conditions, this because classical differentiability is a local property. Therefore, all our constructions are local, described by starting point dependent stopping times.

§2. A Change of Variable Formula

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{a\leq t\leq b}, P)$ be a probability space, where $-\infty<a<b<\infty$. Consider a continuous d-dimensional non-degenerate C-semimartingale

$$I(x, t) = M(x, t) + B(x, t), \tag{3}$$

$a\leq t\leq b$, $x\in \boldsymbol{R}^d$, i.e. $I(., t)(\omega)$ is continuous on $\boldsymbol{R}^d$ for any $t\in[a, b]$ and P-a.s. Moreover, $M(x, t)$, $a\leq t\leq b$, is a continuous localmartingale and $B(x, t)$, $a\leq t\leq b$, is a continuous process of bounded variation, for any value $x\in\boldsymbol{R}^d$ of the spatial parameter.

Following Kunita [3]'s §3, there exist a continuous increasing process A_t and a family of predictable processes $a(x, y, t)$, x, $y\in\boldsymbol{R}^d$, such that the joint quadratic variation of $I(x, t)$ can be expressed as

$$\langle I(x, t), I(y, t)\rangle = \langle M(x, t), M(y, t)\rangle = \int_a^t a(x, y, u)dA_u, \ P\text{-a.s.}$$

Consequently, the continuous bounded variation process $B(x, t)$, $a\leq t\leq b$, is considered to be given as

$$B(x, t) = \int_a^t b(x, u)dA_u, \ P\text{-a.s.}$$

by a family of predictable processes $b(t, x)$, $x\in\boldsymbol{R}^d$.

Assumption (A). $a(x, y, t)$, $b(x, t)$ and $\frac{\partial^2}{\partial x_i\partial y_j}a(x, y, t)$, $\frac{\partial}{\partial x_i}b(x, t)$, $1\leq i$, $j\leq d$, are uniformly in $\omega\in\Omega$ bounded on compact subsets of $(a, b)\times\boldsymbol{R}^d$ where $a(x, y, t)$ satisfies a uniform ellipticity condition. Moreover, $\frac{\partial^2}{\partial x_i\partial y_j}a(x, y, t)$ and $\frac{\partial}{\partial x_i}b(x, t)$, $1\leq i$, $j\leq d$, are locally δ-Hölder continuous w.r.t. (x, y) and x, respectively, for a $\delta\in(0, 1]$ with random Hölder coefficients in $L^1(A)$.

Then, by Kunita [3]'s §4.7, Itô's stochastic differential equation

$$\xi_{s,t}(x) = x + \int_s^t I(\xi_{s,u}(x), du) \tag{4}$$

defines on every compact set $K\subset(a, b)\times\boldsymbol{R}^d$ a unique P-a.s. continuous forward

$C^{1,\varepsilon}$-semimartingale flow of local C^1-diffeomorphisms $\xi_{s,t}(x)$, $(s, x) \in K$, $s \leq t \leq \tau^K_{(s,x)}$, for $\varepsilon < \delta$, where $\tau^K_{(s,x)} = \inf\{t \geq s : (t, \xi_{s,t}(x)) \notin K\} \leq b$ for any $(s, x) \in (a, b) \times \boldsymbol{R}^d$. This means:

i) $\xi_{s,t}(x)$, $s \leq t \leq \tau^K_{(s,x)}$, is a continuous semimartingale.
ii) $\xi_{s,t} : \{x : (s, x) \in \mathrm{int}(K),\ t < \tau^K_{(s,x)}\} \to \boldsymbol{R}^d$ is once ε-Hölder continuously differentiable for any $t \in (s, b]$.
iii) $\xi_{s,t}(x) = \xi_{u,t}(\xi_{s,u}(x))$ on $\{x : (s, x) \in K,\ t \leq \tau^K_{(s,x)}\}$ for $s < u < t$, known as the (local) flow property.
iv) $\xi_{s,s}(x)$ is the identity for all $s \in (a, b)$ with $(s, x) \in K$.
v) $\xi_{s,t} : \{x : (s, x) \in \mathrm{int}(K),\ t < \tau^K_{(s,x)}\} \to \{\xi_{s,t}(x) : (s, x) \in \mathrm{int}(K),\ t < \tau^K_{(s,x)}\}$ is a C^1-diffeomorphism for any $t \in (s, b]$.

All the properties i)–v) depending on $\omega \in \Omega$ hold P-a.s. Cf Kunita [3]'s §4.

Lemma (a change of variable formula). *Let* (A) *be assumed and* $F \in L^1_{loc}((a, b) \times \boldsymbol{R}^d)$ *arbitrary. Then, for any compact* $K \subset (a, b) \times \boldsymbol{R}^d$,

$$\int_{\boldsymbol{R}^d} dx \int_s^{\tau^K_{(s,x)}} F(t, \xi_{s,t}(x))dt = \int_s^b dt \int_{\boldsymbol{R}^d} dz F(t, z) 1_{[s, \hat{\tau}^K_{(s,z)}]}(t) \det \partial_z \xi^{-1}_{s,t}(z) \tag{5}$$

holds P*-a.s. and it is in* $L^1(P)$, *where*

$$\hat{\tau}^K_{(s,z)} = \tau^K_{(s,x)} \quad \text{with} \quad x = \xi^{-1}_{s,t}(z)$$

and $\partial_z \xi^{-1}_{s,t}(z)$ *denotes the Jacobian of the inverse of the flow* $\xi_{s,t}(x)$.

Proof. The stochastic process $F(t, \xi_{s,t}(x))$, $s \leq t \leq b$, is well defined in the sense of Kunita [3]'s §1.2, since the 1-dimensional marginal distributions of any 1-point motion of $\xi_{s,t}(x)$, $s \leq t \leq b$, are absolutely continuous w.r.t. the Lebesgue measure λ. Let a compact set $K \subset (a, b) \times \boldsymbol{R}^d$ be fixed. Then $F \in L^1(\{z : (t, z) \in K\})$ for λ-almost all $t \in (a, b)$ and by the local C^1-diffeomorphic property of the flow $\xi_{s,t}$

$$\begin{aligned} &\int_{\{x:(s,x)\in K\}} dx F(t, \xi_{s,t}(x)) 1_{[s, \tau^K_{(s,x)}]}(t) \\ &= \int_{\xi_{s,t\wedge\tau^K_{(s,x)}}(\{x:(s,x)\in K\})} dz F(t, z) 1_{[s, \hat{\tau}^K_{(s,z)}]}(t) \det \partial_z \xi^{-1}_{s,t}(z) \end{aligned} \tag{6}$$

holds for λ-almost all $t \in (a, b)$ and P-a.s., where $\hat{\tau}^K_{(s,z)}$ in terms of $\xi_{s,t}(x)$ is well defined. Since $\xi_{s,t}(x)$ is differentiable in x, the flow property yields

$$\partial_z \xi^{-1}_{s,t}(z)|_{z=\xi_{s,t}(x)} = (\partial_x \xi_{s,t}(x))^{-1}$$

on $\{t < \tau^K_{(s,x)}\}$, P-a.s. The compact set

$$R_s = \{(t \wedge \tau^K_{(s,x)},\ \xi_{s,t\wedge\tau^K_{(s,x)}}(x)) : (s, x) \in K,\ s \leq t \leq b\}$$

is P-a.s. homeomorphic to

$$D_s=\{(t, x):(s, x)\in K,\ s\le t\le \tau^K_{(s,x)}\}$$

and therefore, by the space-time continuity of the Jacobian of the stochastic flow $\xi_{s,t}(x)$,

$$\max_{(t,z)\in R_s}|\det \partial_z \xi^{-1}_{s,t}(z)|=\max_{(t,x)\in D_s}|\det \partial_x \xi_{s,t}(x)|^{-1}<\infty,\ P\text{-a.s.}$$

Now it may be concluded that

$$\int_s^b dt\int_{\mathbf{R}^d} dx|F(t, \xi_{s,t}(x))|1_{[s,\tau^K_{(s,x)}]}(t)$$

$$\le \max_{(t,x)\in D_s}|\det \partial_x \xi_{s,t}(x)|^{-1}\int_K\int|F(t, z)|dtdz<\infty,\ P\text{-a.s.}$$

Then Fubini's theorem yields the first part of the assertion, from which the second part follows, if

$$\sup_{(t,z)\in K} P[1_{[s,\tau^K_{(s,\cdot)}]}(t)\,|\det \partial_z \xi^{-1}_{s,t}(z)|]<\infty$$

is proved, where this time Fubini's theorem w.r.t. P has been applied.

It is sufficient to show that

$$\sup_{\{(t,x):(s,x)\in K, s\le t\le b\}} P[\|(\partial_x \xi_{s,t\wedge\tau^K_{(s,x)}}(x))^{-1}\|^r]<\infty \quad \text{for} \quad 1\le r<\infty. \tag{7}$$

$(\partial_x \xi_{s,t}(x))^{-1}=U(x, t)$ satisfies on K, by means of Kunita [3]'s theorem 3.3.3,

$$U_{ij}(x, t)=\delta_{ij}-\int_s^t\sum_{l=1}^d U_{il}(x, u)\frac{\partial}{\partial x_j}I^l(\xi_{s,u}(x), du)$$

$$+\int_s^t\sum_{l=1}^d U_{il}(x, u)\sum_{m=1}^d\frac{\partial^2}{\partial x_j\partial y_m}a^{ml}(\xi_{s,u}(x), \xi_{s,u}(x), u)dA_u,$$

P-a.s. for $1\le i, j\le d$, which forms together with (4), the defining equation for $\xi_{s,t}(x)$, a system of d^2+d Itô's stochastic differential equations. Under the assumption (A) Kunita [3]'s theorem 3.4.6, adapted to our local situation, can be applied to this system. Consequently, its solutions have finite moments of any order, depending continuously on the starting point; therefore (7) holds.

§3. A Version of Itô's Formula

This is a consequence of the change of variable formula (5). Let D be an open subset of $(a, b)\times\mathbf{R}^d$. A function $F\in C(D, \mathbf{R})$ is assumed to possess finite

$$D_sF\in L^1_{loc}(D),\ D_iF\in L^2_{loc}(D) \text{ and } D_iD_jF\in L^1_{loc}(D) \quad \text{for} \quad 1\le i, j\le d$$

where $D_s = \frac{\partial}{\partial s}$ and $D_i = \frac{\partial}{\partial x_i}$, $1 \le i \le d$, are to be understood in the sense of distributions with respect to the space of test functions $C^\infty_{comp}(D) = \{\gamma : \gamma$ has a compact support contained in D and its derivatives of any order exist and are continuous.$\}$.

Under the same assumption (A) as for the change of variable formula (5), a continuous C-semimartingale $I(x, t)$ like (3) and the stochastic flow $\xi_{s,t}(x)$, being the solution of Itô's stochastic differential equation (4) driven by $I(x, t)$, are considered. Then

Theorem (a version of Itô's formula).

$$
\begin{aligned}
&F(t \wedge \tau_{(s,x)}, \xi_{s,t\wedge\tau_{(s,x)}}(x)) - F(s, x) \qquad (8)\\
&= \int_s^{t\wedge\tau_{(s,x)}} D_s F(u, \xi_{s,u}(x))du + \sum_{i=1}^{d} \int_s^{t\wedge\tau_{(s,x)}} D_i F(u, \xi_{s,u}(x)) I^i(\xi_{s,u}(x), du)\\
&+ \frac{1}{2}\sum_{i,j=1}^{d} \int_s^{t\wedge\tau_{(s,x)}} D_i D_j F(u, \xi_{s,u}(x)) a^{ij}(\xi_{s,u}(x), \xi_{s,u}(x), u) dA_u
\end{aligned}
$$

for $t \in [s, b)$, $\lambda \otimes P$-*a.s. where*

$$\tau_{(s,x)} = \inf\{u \in [s, b) : (u, \xi_{s,u}(x)) \notin D\}$$

and $s \in (a, b)$.

Corollary. *If the set* $\bar{D}$ *is assumed to be compact then both sides of Itô's formula* (8) *integrated w.r.t.* $\varphi(x)\,dx$ *on* $\boldsymbol{R}^d$ *for any bounded measurable function* φ *are equal P-a.s. and they are in* $L^1(P)$.

Proof. There exists an increasing sequence $(D^{(m)})_{m\in N}$ of open sets with compact closure $\overline{D^{(m)}}$ contained in D such that

$$D = \bigcup_{m \in N} D^{(m)}. \qquad (9)$$

Denote by $\tau^{(m)}_{(s,x)} = \inf\{u \in [s, b) : (u, \xi_{s,u}(x)) \notin D^{(m)}\}$ the corresponding increasing sequence of stopping times with $\lim_{m\to\infty} \tau^{(m)}_{(s,x)} = \tau_{(s,x)}$, P-a.s. The mollifier theory, cf Friedman [1] Chap. X, provides for every $\overline{D^{(m)}}$, $m \in N$, a sequence $(J^{(m)}_\varepsilon)_{\varepsilon>0}$ of smoothing operators such that

$$
\begin{aligned}
J^{(m)}_\varepsilon F &\to F \text{ uniformly on } \overline{D^{(m)}},\\
D_s J^{(m)}_\varepsilon F = J^{(m)}_\varepsilon D_s F &\to D_s F \text{ in } L^1(\overline{D^{(m)}}),\\
D_i J^{(m)}_\varepsilon F = J^{(m)}_\varepsilon D_i F &\to D_i F \text{ in } L^2(\overline{D^{(m)}}) \text{ and}\\
D_i D_j J^{(m)}_\varepsilon F = J^{(m)}_\varepsilon D_i D_j F &\to D_i D_j F \text{ in } L^1(\overline{D^{(m)}})
\end{aligned}
$$

as $\varepsilon \searrow 0$. Following Kunita [3]'s theorem 4.7.2, Itô's formula for smooth functions evaluated along stochastic flows may be applied locally on $\overline{D^{(m)}}$:

$$\begin{aligned} &J_\varepsilon^{(m)}F(t\wedge\tau_{(s,x)}^{(m)},\ \xi_{s,t\wedge\tau_{(s,x)}^{(m)}}(x))-J_\varepsilon^{(m)}F(s,\ x) \\ &=\int_s^{t\wedge\tau_{(s,x)}^{(m)}}\frac{\partial}{\partial s}J_\varepsilon^{(m)}F(u,\ \xi_{s,u}(x))du \\ &+\sum_{i=1}^d\int_s^{t\wedge\tau_{(s,x)}^{(m)}}\frac{\partial}{\partial x_i}J_\varepsilon^{(m)}F(u,\ \xi_{s,u}(x))I^i(\xi_{s,u}(x),\ du) \\ &+\frac{1}{2}\sum_{i,j=1}^d\int_s^{t\wedge\tau_{(s,x)}^{(m)}}\frac{\partial^2}{\partial x_i\partial x_j}J_\varepsilon^{(m)}F(u,\ \xi_{s,u}(x))a^{ij}(\xi_{s,u}(x),\ \xi_{s,u}(x),\ u)dA_u \end{aligned} \tag{10}$$

for $t\in[s,\ b)$, P-a.s.

In this proof, $A_t=t$ is assumed. The general case follows by change of time scale, which is routine, cf Kunita [3]'s §3.2. First the relation (10) is considered as $\varepsilon\searrow 0$ and then as $m\to\infty$. For the stochastic integral appearing in (10)

$$\begin{aligned} &\int_{\mathbf{R}^d}dx\,P\Big[\int_s^{t\wedge\tau_{(s,x)}^{(m)}}|D_iF(u,\ \xi_{s,u}(x))|^2a^{ii}(\xi_{s,u}(x),\ \xi_{s,u}(x),\ u)du\Big] \\ &\le\sup_{(u,z)\in\overline{D^{(m)}}}a^{ij}(z,\ z,\ u)P\Big[\int_s^t du\int_{\mathbf{R}^d}dz|D_iF(u,\ z)|^2 1_{[s,\hat\tau_{(s,z)}^{(m)}]}(u)|\det\,\partial_z\xi_{s,u}^{-1}(z)|\Big]<\infty \end{aligned}$$

holds by the change of variable formula (5), where

$$\hat\tau_{(s,z)}^{(m)}=\tau_{(s,x)}^{(m)}\quad\text{with}\quad x=\xi_{s,u}^{-1}(z),\ P\text{-a.s.}$$

By the definition of stochastic integrals in terms of L^2-isomorphisms

$$\begin{aligned} &L^2(\lambda\otimes P)-\lim_{\varepsilon\searrow 0}\int_s^{t\wedge\tau_{(s,x)}^{(m)}}\frac{\partial}{\partial x_i}J_\varepsilon^{(m)}F(u,\ \xi_{s,u}(x))M^i(\xi_{s,u}(x),\ du) \\ &=\int_s^{t\wedge\tau_{(s,x)}^{(m)}}D_iF(u,\ \xi_{s,u}(x))M^i(\xi_{s,u}(x),\ du)\in L^2(\lambda\otimes P) \end{aligned}$$

for $t\in[s,\ b)$ follows. Then Chebyshev's inequality and the Borel-Cantelli lemma yield a subsequence $(\varepsilon_{1,n})_{n\in N}$ for which this convergence holds $\lambda\otimes P$-almost surely for every $1\le i\le d$. The three bounded variation terms on the RHS of (10) are treated similarly. Their $L^1(\lambda\otimes P)$-convergence follows by means of the change of variable formula (5). In case of the third term this means

$$\begin{aligned} &\int dx\,P\Big[\int_s^{t\wedge\tau_{(s,x)}^{(m)}}\Big|\frac{\partial^2}{\partial x_i\partial x_j}J_\varepsilon^{(m)}F(u,\ \xi_{s,u}(x))- \\ &\qquad D_iD_jF(u,\ \xi_{s,u}(x))\Big|\,|a^{ij}(\xi_{s,u}(x),\ \xi_{s,u}(x),\ u)|du\Big] \\ &\le\sup_{(u,z)\in\overline{D^{(m)}}}(|a^{ij}(z,\ z,\ u)|P[1_{[s,\hat\tau_{(s,z)}^{(m)}]}(u)|\det\,\partial_z\xi_{s,u}^{-1}(z)|]) \\ &\qquad\cdot\Big\|\frac{\partial^2}{\partial x_i\partial x_j}J_\varepsilon^{(m)}F-D_iD_jF\Big\|_{L^1(\overline{D^{(m)}})}\to 0 \end{aligned}$$

as $\varepsilon\searrow 0$ for any $1\le i,\ j\le d$.

With the same method as used to extract $(\varepsilon_{1,n})_{n\in N}$, a subsequence $(\varepsilon_{2,n})_{n\in N} \subset (\varepsilon_{1,n})_{n\in \mathbf{N}}$ can be found along which the convergence on the RHS of (10) for $t \in [s, b)$ takes place $\lambda \otimes P$-almost surely. This follows for the LHS of (10) from the uniform convergence of $J_\varepsilon^{(m)} F$ to F on $\overline{D^{(m)}}$. Consequently, having (10) which holds P-a.s. for all $x \in \boldsymbol{R}^d$, (8) considered on $\overline{D^{(m)}}$ holds $\lambda \otimes P$-a.s. Since $\lim_{m\to\infty} \tau_{(s,x)}^{(m)} = \tau_{(s,x)}$, P-a.s. and F is assumed to be continuous on D, (8) finally follows by (9), noticing that the set of exceptional (x, ω) is contained in a countable union of $\lambda \otimes P$-zero sets.

Remark. Krylov [2] develops a version of Itô's formula for continuous functions F with generalized derivatives. Given a bounded state space Q, he assumes that $F \in L^p(Q)$ where p has to be greater or equal to the dimension of Q. Moreover, the coefficients of the underlying diffusion process are related to each other. Then Krylov's version of Itô's formula holds for all starting points and it is proved by geometrical arguments as well as substantial results from the theory of parabolic differential equations.

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