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Quantum Topology

Heegaard Floer correction terms, with a twist

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Abstract. We use Heegaard Floer homology with twisted coefficients to define numerical invariants for arbitrary closed 3-manifolds equipped torsion spin^c structures, generalising the correction terms (or d-invariants) defined by Ozsváth and Szabó for rational homology 3-spheres and, more generally, for 3-manifolds with standard HF^∞. Our twisted correction terms share many properties with their untwisted analogues. In particular, they provide restrictions on the topology of 4-manifolds bounding a given 3-manifold.

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Contents

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1. Introduction

One of the most celebrated results in low dimensional topology is Donaldson's diagonalisability theorem for intersection forms of smooth 4-manifolds. It asserts that any negative definite intersection form of a closed smooth 4-manifold is diagonalisable over $\mathbb{Z}$. Both assumptions on the 4-manifold, being smooth and closed, are crucial. On the one hand, an equally fascinating result of Freedman shows that every unimodular symmetric bilinear form appears as the intersection form of some closed *topological* 4-manifold. On the other hand, an easy construction shows that any symmetric bilinear form is the intersection form of some smooth 4-manifold with boundary. Note, however, that one cannot control the topology of the boundary. In this paper we are interested in the possible intersection forms of smooth 4-manifolds bounding a fixed 3-manifold.

1.1. The main results. A purely algebraic result of Elkies [4] shows that Donaldson's theorem can be rephrased as a family of inequalities $c_1^2(\mathfrak{s}) + b_2(X) \leq 0$ where $\mathfrak{s}$ runs through all spin^c structures on a closed smooth 4-manifold X. It turns out that these inequalities admit generalisations to 4-manifolds with boundary. The first significant progress in this direction was made by Frøyshov [7] using Seiberg–Witten theory and later by Ozsváth and Szabó [16] in the context of Heegaard Floer homology. In this paper we will define a generalisation of the correction terms defined by Ozsváth and Szabó, using Heegaard Floer homology with twisted coefficients: to any spin^c 3-manifold $(Y, \mathfrak{t})$ with $c_1(\mathfrak{t})$ torsion we associate a rational number $\underline{d}(Y, \mathfrak{t})$, called the *twisted correction term* of $(Y, \mathfrak{t})$. One of the main goals of the paper is to prove the following general result.

Theorem 1.1. *Let $(Z, \mathfrak{s})$ be a smooth spin^c 4-manifold with connected spin^c boundary $(Y, \mathfrak{t})$. If, furthermore, $b_2^+(Z) = 0$ and $c_1(\mathfrak{t})$ is torsion, then*

$$c_1^2(\mathfrak{s}) + b_2^-(Z) \ \leq\ 4\underline{d}(Y, \mathfrak{t}) + 2b_1(Y). \tag{1.1}$$

Here $b_2^{\pm}(Z)$ are the dimensions of maximal positive and negative definite subspaces for the intersection form of Z. As indicated above, similar inequalities were obtained by Frøyshov [7], [8] for rational homology 3-spheres and by Ozsváth and Szabó [16] for 3-manifolds with "standard HF^∞" (see Section 3.3 below). Our approach is similar to the one taken by Ozsváth and Szabó, but it turns out that the use of twisted coefficients allows us to work with arbitrary 3-manifolds. The proof of Theorem 1.1 occupies Sections 2–4, including a brief review of Heegaard Floer homology with twisted coefficients and a discussion of the twisted correction terms and their properties. Starting with Section 5 we return to intersection forms of smooth 4-manifolds with boundary. As a sample result, we mention the following, although we actually prove a stronger statement in Corollary 5.4.

Theorem 1.2. *For any closed, oriented* 3*-manifold* Y *there are only finitely many isometry classes of even, semidefinite symmetric bilinear forms that can appear as intersection forms of smooth* 4*-manifolds bounded by* Y*.*

Note that Theorem 1.2 cannot hold for *topological* 4-manifolds. Indeed, using Freedman's result one can add arbitrary unimodular summands to the intersection form of any given 4-manifold by connect summing with suitable closed topological 4-manifolds. So the finiteness in Theorem 1.2 is an inherently smooth phenomenon.

In Section 6 we turn to some concrete examples and give some further applications. In particular, for a surface Σ_g of arbitrary genus g we compute the twisted correction terms of $\Sigma_g \times S^1$ – which has non-standard HF^∞ for $g \geq 1$ and therefore lies outside the scope of previously available techniques – and use Theorem 1.1 to deduce the following.

Theorem 1.3. *Let* Z *be a smooth* 4*-manifold with boundary* T^3 *or* $\Sigma_2 \times S^1$*. If the intersection form* Q_Z *is negative semidefinite and even, then its non-degenerate part is either trivial or isometric to* E_8*, and both of these occur.*

Again, we actually prove a slightly stronger statement (see Corollary 6.10).

1.2. Notation and terminology. By default, all manifolds are assumed to be smooth, compact, connected, and oriented. The letter Y will always indicate a closed 3-manifold. Similarly, we reserve Z for 4-manifolds with connected boundary, and W for cobordisms between non-empty 3-manifolds; and if $Y = \partial Z$, we refer to Z as a *filling* of Y. Spin^c structures on 3-manifolds will be denoted by $\mathfrak{t}$ and those on 4-manifolds by $\mathfrak{s}$. If $(Y, \mathfrak{t})$ is the spin^c boundary of $(Z, \mathfrak{s})$, then we call the latter a *spin*^{c} *filling*. Lastly, for 3- or 4-manifold with torsion-free second cohomology we write $\mathfrak{t}_0$ or $\mathfrak{s}_0$ for the unique spin^c structure with trivial first Chern class (provided that they exist, in the case of 4-manifolds).

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2. Review of Heegaard Floer homology

We recall some relevant definitions and facts about Heegaard Floer homology with twisted coefficients. The basic references for this material are [18, Section 8] and [10]. We will pay special attention to the role of ground rings.

2.1. Twisted coefficients. Fix a ground ring $\mathbb{F}$; usually $\mathbb{F} = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{F}_p$ for some prime p. Let Y be a closed, oriented 3-manifold equipped with a spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. The input for Heegaard Floer theory is a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ with some extra decorations (see [19] and [18] for details; for instance, we will suppress the basepoint from the notation). The output is a short exact sequence of chain complexes

$$0 \longrightarrow \underline{\mathrm{CF}}^-(Y,\mathfrak{t})_{\mathbb{F}} \xrightarrow{\iota} \underline{\mathrm{CF}}^\infty(Y,\mathfrak{t})_{\mathbb{F}} \xrightarrow{\pi} \underline{\mathrm{CF}}^+(Y,\mathfrak{t})_{\mathbb{F}} \longrightarrow 0 \tag{2.1}$$

over the ring $\mathbb{F}[U] \otimes_{\mathbb{F}} \mathbb{F}[H_2(Y)]$ whose homology groups, denoted by $\underline{\mathrm{HF}}^\circ(Y,\mathfrak{t})_{\mathbb{F}}$, are an invariant of $(Y,\mathfrak{t})$ known as *Heegaard Floer homology* with *fully twisted coefficients* over $\mathbb{F}$. Following [10] we write $R_Y = \mathbb{F}[H_2(Y)]$ so that $\mathbb{F}[U] \otimes_{\mathbb{F}} \mathbb{F}[H_2(Y)]$ becomes $R_Y[U]$; we also use the common shorthand notation

$$\mathcal{T}^- = U \cdot \mathbb{F}[U], \quad \mathcal{T}^\infty = \mathbb{F}[U,U^{-1}], \quad \text{and} \quad \mathcal{T}^+ = \mathbb{F}[U,U^{-1}]/U \cdot \mathbb{F}[U]$$

for the $\mathbb{F}[U]$-modules that have become known as *towers*. We think of them as relatively $\mathbb{Z}$-graded such that multiplication by U has degree -2 and a subscript $\mathcal{T}^\circ_d$ indicates that U^k lies in grading $d-2k$.

For any R_Y-module M one can further define Heegaard Floer homology groups with coefficients in M

$$\underline{\mathrm{HF}}^\circ(Y,\mathfrak{t};M)_{\mathbb{F}} = H_*(\underline{\mathrm{CF}}^\circ(Y,\mathfrak{t})_{\mathbb{F}} \otimes_{R_Y} M). \tag{2.2}$$

The most common choice for M is the ground ring $\mathbb{F}$ itself, endowed with the trivial R_Y-action. This yields the *untwisted* Heegaard Floer homology groups $\mathrm{HF}^\circ(Y,\mathfrak{t})_{\mathbb{F}}$. In all other cases it has become customary to speak of *twisted coefficients*. Note that for $M = R_Y$ one recovers the fully twisted homology groups $\underline{\mathrm{HF}}^\circ(Y,\mathfrak{t})_{\mathbb{F}}$. We will usually suppress the ground ring in the subscript from the notation whenever this does not cause confusion, but at times this more precise notation will be convenient.

It follows from general principles of homological algebra that (2.1) induces a long exact sequence of $R_Y[U]$-modules

$$\cdots \to \underline{\mathrm{HF}}^-(Y,\mathfrak{t};M)_{\mathbb{F}} \xrightarrow{\iota_*} \underline{\mathrm{HF}}^\infty(Y,\mathfrak{t};M)_{\mathbb{F}} \xrightarrow{\pi_*} \underline{\mathrm{HF}}^+(Y,\mathfrak{t};M)_{\mathbb{F}} \xrightarrow{\delta} \cdots \tag{2.3}$$

while (2.2) gives rise to a *universal coefficient spectral sequence*

$$E^2_{*,*} = \mathrm{Tor}^{R_Y}_*(\underline{\mathrm{HF}}^\circ_*(Y,\mathfrak{t})_{\mathbb{F}}, M) \implies \underline{\mathrm{HF}}^\circ(Y,\mathfrak{t};M)_{\mathbb{F}} \tag{2.4}$$

which highlights the universal role of $\underline{\mathrm{HF}}^\circ(Y,\mathfrak{t})_{\mathbb{F}}$.

We will mostly work with the fully twisted theory. As explained in [10, Section 3], the groups $\underline{\mathrm{HF}}^\circ(Y,\mathfrak{t})_{\mathbb{F}}$ carry a relative $\mathbb{Z}$-grading. Moreover, if $c_1(\mathfrak{t})$ is torsion, then the relative $\mathbb{Z}$-grading can be lifted to an absolute $\mathbb{Q}$-grading [20, Section 7]. We also recall the following result due to Ozsváth and Szabó which is of fundamental importance for our work.

Theorem 2.1 ([18, Theorem 10.12]). *If $c_1(\mathfrak{t})$ is torsion, then there is a unique equivalence class of orientation systems such that*

$$\underline{\mathrm{HF}}^\infty(Y,\mathfrak{t})_{\mathbb{F}} \cong \mathbb{F}[U,U^{-1}] = \mathcal{T}^\infty$$

as $R_Y[U]$-modules with a trivial R_Y-action on $\mathcal{T}^\infty$.

2.2. Cobordism maps. Now let $(W,\mathfrak{s})$ be a spin^c cobordism from $(Y,\mathfrak{t})$ to $(Y',\mathfrak{t}')$. It is well known that for any R_Y-module M there are induced *cobordism maps*

$$F^\circ_{W,\mathfrak{s};M}\colon \underline{\mathrm{HF}}^\circ(Y,\mathfrak{t};M) \longrightarrow \underline{\mathrm{HF}}^\circ(Y',\mathfrak{t}';M(W))$$

which have the peculiarity that the target generally depends on the cobordism through the coefficient module. Indeed, the $R_{Y'}$-module $M(W)$ can be described as follows. We consider the $(R_Y,R_{Y'})$-bimodule

$$B_W = \mathbb{F}[H_2(Y)_W + H_2(Y')_W] \subset \mathbb{F}[H_2(W)]$$

where $H_2(Y)_W$ denotes the image of the map $H_2(Y) \to H_2(W)$ induced by inclusion (and similarly for Y') and define

$$M(W) = M \otimes_{R_Y} B_W. \tag{2.5}$$

For example, in the fully twisted case $M = R_Y$ we have $R_Y(W) = B_W$ and we denote the cobordism map by

$$\underline{F}^\circ_{W,\mathfrak{s}}\colon \underline{\mathrm{HF}}^\circ(Y,\mathfrak{t}) \to \underline{\mathrm{HF}}^\circ(Y',\mathfrak{t}';B_W).$$

These cobordism maps will play an important role in the proof of Theorem 1.1. The dependence of the target on the cobordism is one of the extra complications compared to the untwisted situation.

Remark 2.2. There are slightly different definitions of $M(W)$ in the literature. Ours is essentially the same as in [20, Section 2.7] except that we work in the Poincaré dual picture (using H_2 instead of H^1). Another difference appears in [10, Section 2.2] where the the $R_{Y'}$-module $\bar{M} \otimes_{R_Y} B_W$ is used. Here $\bar{M}$ stands for M with the *conjugate* R_Y-module structure (for which $h \in H_2(Y)$ acts as $-h$). However, note that the conjugation only affects the R_Y-module structure and that we have $\bar{M} \otimes_{R_Y} B_W \cong M \otimes_{R_Y} B_W$ as $R_{Y'}$-modules.

2.3. A connected sum formula for fully twisted coefficients. The following is a generalisation of the connected sum formula in Heegaard Floer homology (see [18, Theorem 6.2]) to fully twisted coefficients. For technical reasons we work over a fixed ground *field* $\mathbb{F}$.

Proposition 2.3. *Let* $(Y_1, \mathfrak{t}_1)$ *and* $(Y_2, \mathfrak{t}_2)$ *be spin*^{c} 3*-manifolds, and let* $\underline{\mathrm{CF}}^-(Y_i, \mathfrak{t}_i)$ *be the usual chain complex computing* $\underline{\mathrm{HF}}^-(Y_i, \mathfrak{t}_i)$*. Then there is an isomorphism*

$$\underline{\mathrm{HF}}^-(Y_1 \# Y_2, \mathfrak{t}_1 \# \mathfrak{t}_2) \cong H_*(\underline{\mathrm{CF}}^-(Y_1, \mathfrak{t}_1) \otimes_{\mathbb{F}[U]} \underline{\mathrm{CF}}^-(Y_2, \mathfrak{t}_2))[2] \tag{2.6}$$

where [2] *indicates a grading shift by* 2*.*

In the proof we will use the shorthand notation $\mathbb{S}_n$ for the 3-manifold $\#^n S^1 \times S^2$ which will also appear later on.

Proof. We argue as in the proof of [18, Theorem 6.2] to which we refer for further details and notation. As in the untwisted case, it is more convenient to study the complex $\underline{\mathrm{CF}}^{\leq 0}$ instead of $\underline{\mathrm{CF}}^-$. This explains the degree shift in (2.6): $\underline{\mathrm{CF}}^{\leq 0}$ is just $\underline{\mathrm{CF}}^-$ with a grading shift. The main difference between the twisted and untwisted cases lies in the definition of the twisted coefficients map

$$\underline{\Gamma}\colon \underline{\mathrm{CF}}^{\leq 0}(Y_1, \mathfrak{t}_1) \otimes_{\mathbb{F}[U]} \underline{\mathrm{CF}}^{\leq 0}(Y_2, \mathfrak{t}_2) \longrightarrow \underline{\mathrm{CF}}^{\leq 0}(Y_1 \# Y_2, \mathfrak{t}_1 \# \mathfrak{t}_2),$$

the analogue of the untwisted map Γ. The map $\underline{\Gamma}_0$ corresponding to Γ_0 in the proof of [18, Theorem 6.2] is defined in the same way, as the 'closest point map'. Once we have constructed $\underline{\Gamma}$, the rest of the argument follows verbatim, and we refer the reader to the original proof; we therefore focus only on constructing $\underline{\Gamma}$.

Choose a Heegaard diagram $(\Sigma_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)$ for Y_i, where Σ_i has genus g_i. As in the untwisted case, consider the triple Heegaard diagram

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = (\Sigma_1 \# \Sigma_2, \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2', \boldsymbol{\beta}_1 \cup \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1' \cup \boldsymbol{\beta}_2),$$

where the primes denote small Hamiltonian perturbations. It is immediate to check that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ represents $\widetilde{Y}_1 := Y_1 \# \mathbb{S}_{g_2}$, while $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$ represents $\widetilde{Y}_2 := Y_2 \# \mathbb{S}_{g_1}$, and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma})$ represents $Y_1 \# Y_2$. Let $R_i := R_{Y_i}$ for $i = 1, 2$. Using the canonical splitting $H_2(\widetilde{Y}_i) = H_2(Y_i) \oplus H_2(\mathbb{S}_{g_{3-i}})$ we can consider R_i as an $R_{\widetilde{Y}_i}$-module. Following [18, Section 8.2.2], we see that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ induces a map

$$f^{\leq 0}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \colon \underline{\mathrm{CF}}^{\leq 0}(\widetilde{Y}_1, \mathfrak{t}_1; R_1) \otimes_{\mathbb{F}[U]} \underline{\mathrm{CF}}^{\leq 0}(\widetilde{Y}_2, \mathfrak{t}_2; R_2) \to \underline{\mathrm{CF}}^{\leq 0}(Y_1 \# Y_2, \mathfrak{t}_1 \# \mathfrak{t}_2).$$

To conclude the proof, we need to find a map $\Phi_i \colon \underline{\mathrm{CF}}^{\leq 0}(Y_i, \mathfrak{t}_i) \to \underline{\mathrm{CF}}^{\leq 0}(\widetilde{Y}_1, \mathfrak{t}_i; R_i)$. In fact, it is an easy check that $\underline{\mathrm{CF}}^{\leq 0}(\widetilde{Y}_i, \mathfrak{t}_i; R_i) \cong \underline{\mathrm{CF}}^{\leq 0}(Y_i, \mathfrak{t}_i) \otimes_{\mathbb{F}[U]} \mathrm{CF}^{\leq 0}(\mathbb{S}_{g_{3-i}})$, and the latter factor has a canonical top degree generator Θ_i, as seen in the proof of [18, Theorem 6.2]. This gives the desired embeddings. □

2.4. Twisted surgery triangles. Let K be a knot in a 3-manifold Y. We write $Y_\lambda = Y_\lambda(K)$ for the λ-framed surgery on K and $W_\lambda = W_\lambda(K)$ for the corresponding surgery cobordism from Y to Y_λ. We write $E_K = Y \setminus \nu K$ for the knot exterior and consider the framing and the meridian of K as simple closed curves $\lambda, \mu \subset \partial E_K$ well defined up to isotopy. Since the map $H_1(\partial E_K) \to H_1(E_K)$ has rank 1, there is another essential simple closed curve $\lambda_0 \subset \partial E_K$, well defined up to isotopy, characterised by the property of having finite order in $H_1(E_K)$. By a slight abuse of terminology we refer to λ_0 as the 0-*framing* although it might not actually be a framing in general, as it may intersect the meridian more than once or not at all. Note that by definition of λ_0 we can find an oriented surface $S \subset E_K$ which bounds a number of parallel copies of λ_0 with the same orientation. An elementary exercise in homology acrobatics, which we leave to the reader, gives the following.

Lemma 2.4. *Let $\gamma \subset \partial E_K$ be an arbitrary essential simple closed curve and Y_γ the closed 3-manifold obtained by Dehn filling E_K along γ.*

(i) *If $\gamma \neq \lambda_0$, then $H_2(E_K) \cong H_2(Y_\gamma)$, induced by the inclusion.*

(ii) *If $\gamma = \lambda_0$, then $H_2(E_K) \cong H_2(Y_\gamma)/[\widehat{S}]$ where $\widehat{S} \subset Y_\lambda$ is obtained by capping off $S \subset E_K$ with spanning disks of γ in Y_γ.*

In particular, for a fixed choice of S, the ring $M_K = \mathbb{F}[H_2(E_K)]$ has canonical module structures over $R_Y = \mathbb{F}[H_2(Y)]$ and $R_{Y_\lambda} = \mathbb{F}[H_2(Y_\lambda)]$ where λ is an arbitrary framing of K.

With these remarks in place, the well-known exact triangle for surgeries on rationally null-homologous knots admits the following generalization to arbitrary knots in arbitrary 3-manifolds.

Proposition 2.5. *As before, let (K, λ) be a framed knot in a 3-manifold Y and let $M_K = \mathbb{F}[H_2(E_K)]$. Then there is an exact triangle of the form*

$$\begin{array}{ccc} \underline{\mathrm{HF}}^+(Y; M_K) & \xrightarrow{\quad F \quad} & \underline{\mathrm{HF}}^+(Y_\lambda; M_K) \\ & {\scriptstyle H} \nwarrow \quad \swarrow {\scriptstyle G} & \\ & \underline{\mathrm{HF}}^+(Y_{\lambda+\mu}; M_K) & \end{array}$$

where the maps F, G, and H are induced by surgery cobordisms and the relevant module structures of M_K are as in Lemma 2.4.

Proof. The proof of the twisted exact triangle [18, Theorem 9.21] works here with only minor modification; more precisely, one only needs observe that the proof of [18, Proposition 9.22] applies also with coefficients in M_K. □

We point out that if $\lambda_0 \notin \{\lambda, \mu, \lambda + \mu\}$, then Proposition 2.5 actually provides a surgery triangle for fully twisted coefficients. Indeed, let $\gamma \in \{\lambda, \mu, \lambda + \mu\}$ and write R_γ for $R_{Y_\gamma} = \mathbb{F}[H_2(Y_\gamma)]$ where $Y_\mu = Y$. Notice that when γ is not the 0-framing, then $M_K \cong R_\gamma$ by Lemma 2.4 so that $\underline{\mathrm{HF}}^\circ(Y_\gamma, \mathfrak{t}; M_K)$ is just $\underline{\mathrm{HF}}^\circ(Y_\gamma, \mathfrak{t})$.

However, if either λ, μ, or $\lambda + \mu$ agrees with λ_0, then some extra care has to be taken when using Proposition 2.5 to study fully twisted coefficients. For example, if λ is the 0-framing, in which case K is necessarily rationally null-homologous, then only Y and $Y_{\lambda+\mu}$ appear with fully twisted coefficients in (2.5). But we can give a fairly explicit computation of the group $\underline{\mathrm{HF}}^\infty(Y_\lambda, \mathfrak{t}; M_K)$ which will be useful in conjunction with Proposition 2.5. In fact, we have a free resolution of M_K as an R_λ-module

$$0 \longrightarrow R_\lambda \xrightarrow{\cdot(1-[\widehat{S}])} R_\lambda \longrightarrow M_K \longrightarrow 0$$

showing that $\mathrm{Tor}_*^{R_\lambda}(M_K, \mathcal{T}^\infty) = \mathcal{T}^\infty \oplus \mathcal{T}^\infty[1]$ whenever the R_λ-action on $\mathcal{T}^\infty$ is trivial (as it is in our case), and the universal coefficient spectral sequence collapses at the second page. We have thus proved the following.

Proposition 2.6. *When λ is the 0-framing, $\underline{\mathrm{HF}}^\infty(Y_\lambda, \mathfrak{t}; M_K) = \mathcal{T}^\infty \oplus \mathcal{T}^\infty[1]$.*

Of course, analogous considerations hold when $\lambda + \mu$ is the 0-framing and when K is homologically essential in Y (which is equivalent to $\mu = \lambda_0$).

3. Twisted correction terms

Let $(Y, \mathfrak{t})$ be a closed 3-manifold, equipped with a spin^c structure $\mathfrak{t}$ such that $c_1(\mathfrak{t})$ is torsion. We will refer to $(Y, \mathfrak{t})$ as a *torsion* spin^c *3-manifold*. In this section, we work over a ground field $\mathbb{F}$. Recall that, when Y is a rational homology sphere, the untwisted group $\mathrm{HF}^+(Y, \mathfrak{t})$ admits a U-equivariant splitting of the form

$$\mathrm{HF}^+(Y, \mathfrak{t}) \cong \mathcal{T}^+ \oplus \mathrm{HF}^+_{\mathrm{red}}(Y, \mathfrak{t}).$$

The *correction term* $d(Y, \mathfrak{t})$ is the degree of the element in $\mathrm{HF}^+(Y, \mathfrak{t})$ corresponding to $U^{-1} \in \mathcal{T}^+$. More generally, when $b_1(Y) > 0$ there is an action of the exterior algebra $\Lambda = \Lambda^*(H_1(Y)/\mathrm{Tor})$ on $\mathrm{HF}^\circ(Y, \mathfrak{t})$; Ozsváth and Szabó used this action to define a similar invariant for a restricted class of 3-manifolds, namely the ones with *standard* HF^∞. Recall that $(Y, \mathfrak{t})$ is said to have *standard* HF^∞ if $\mathrm{HF}^\infty(Y, \mathfrak{t})$ is isomorphic to $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}]$ as a Λ-module. Under this assumption, the kernel of the Λ-action on HF^∞ maps to a copy of $\mathcal{T}^+$ in $\mathrm{HF}^+(Y, \mathfrak{t})$ whose least degree is called the *bottom-most correction term* $d_b(Y, \mathfrak{t})$. It is clear that $d_b(Y, \mathfrak{t})$ generalizes $d(Y, \mathfrak{t})$ for rational homology spheres. We propose another generalisation that is available for *all* 3-manifolds.

Definition 3.1 (Twisted correction terms). Let $(Y, \mathfrak{t})$ be a torsion spin^c 3-manifold and let $\mathbb{F}$ be a field of characteristic p. We define the (*homological*) *twisted correction term* $\underline{d}_p(Y, \mathfrak{t}) \in \mathbb{Q}$ as the minimal grading among all non-zero elements in the image of $\pi_*\colon \underline{\mathrm{HF}}^\infty_*(Y, \mathfrak{t})_{\mathbb{F}} \to \underline{\mathrm{HF}}^+_*(Y, \mathfrak{t})_{\mathbb{F}}$. Similarly, there is a *cohomological* version $\underline{d}^*_p(Y, \mathfrak{t}) \in \mathbb{Q}$ defined using the map $\iota^*\colon \underline{\mathrm{HF}}^*_\infty(Y, \mathfrak{t})_{\mathbb{F}} \to \underline{\mathrm{HF}}^*_-(Y, \mathfrak{t})_{\mathbb{F}}$ on Heegaard Floer cohomology; since multiplication by U increases the degree by 2 in cohomology, $\underline{d}^*_p(Y, \mathfrak{t})$ is the maximal grading among all non-zero elements in the image of ι^*.

Remark 3.2. Using the universal coefficient theorem, it is easy to show that if $\mathbb{F}'$ is a field extension of $\mathbb{F}$, then the two corresponding correction terms coincide, hence the correction term only depends on the characteristic. In particular, this justifies the notational choice and shows that is suffices to consider $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$.

Remark 3.3. It is not known whether the twisted correction terms do in fact depend on p. To the best of our knowledge, there are no examples for which $\underline{d}_0(Y, \mathfrak{t})$ and $\underline{d}_p(Y, \mathfrak{t})$ are different for some $p > 0$ and our example computations in Section 6 give the same results for all values of p. Note that similar situations arise in the work of Frøyshov [8, p. 569] and Manolescu [12, Remark 3.12].

Proposition 3.4. *The correction term* $\underline{d}_0(Y, \mathfrak{t})$ *agrees with the minimal grading among all non-*$\mathbb{Z}$*-torsion elements in the image of* $\pi_*\colon \underline{\mathrm{HF}}^\infty_*(Y, \mathfrak{t})_\mathbb{Z} \to \underline{\mathrm{HF}}^+_*(Y, \mathfrak{t})_\mathbb{Z}$. *Furthermore, we have* $\underline{d}_0(Y, \mathfrak{t}) \geq \underline{d}_p(Y, \mathfrak{t})$ *for every prime* p.

Proof. The universal coefficient theorem shows that

$$\underline{\mathrm{HF}}^\circ(Y, \mathfrak{t})_\mathbb{Q} = \underline{\mathrm{HF}}^\circ(Y, \mathfrak{t})_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q},$$

and the first statement readily follows. The second statement follows from the universal coefficient theorem applied to the change of coefficients from $\mathbb{Z}$ to $\mathbb{F}_p$, together with the observation that $\underline{\mathrm{HF}}^\infty(Y, \mathfrak{t})_\mathbb{Z}$ has no $\mathbb{Z}$-torsion. □

In what follows, we will be sloppy and simply write $\underline{d}$ instead of $\underline{d}_p$ to signify that the results and computations will hold regardless of the characteristic.

If Y is a rational homology 3-sphere, then Proposition 3.4 shows that $\underline{d}(Y, \mathfrak{t})$ agrees with the usual correction term $d(Y, \mathfrak{t})$ as defined in [16, Definition 4.1]. As in that case, there is an alternative description. The long exact sequence (2.3) together with Theorem 2.1 gives rise to a (non-canonical) decomposition of $R_Y[U]$-modules

$$\underline{\mathrm{HF}}^+(Y, \mathfrak{t}) \cong \mathcal{T}^+ \oplus \underline{\mathrm{HF}}^+_{\mathrm{red}}(Y, \mathfrak{t}),$$

where $\underline{\mathrm{HF}}^+_{\mathrm{red}}(Y, \mathfrak{t})$ is defined as the cokernel of π_*. In such a decomposition $\underline{d}(Y, \mathfrak{t})$ appears as the minimal grading of non-zero elements in $\mathcal{T}^+$.

Example 3.5. By a direct computation of $\underline{\mathrm{HF}}^+$ one can check that

$$\underline{d}(S^1 \times S^2, \mathfrak{t}_0) = -\tfrac{1}{2}$$

and

$$\underline{d}(T^3, \mathfrak{t}_0) = \tfrac{1}{2}.$$

The computation for $S^1 \times S^2$ is easy; for T^3 see [16, Proposition 8.5]. Moreover, since $S^1 \times S^2$ and T^3 have orientation-reversing diffeomorphisms that preserve $\mathfrak{t}_0$ (up to conjugation), $\underline{d}^*$ agrees with $-\underline{d}$, as we will see in Proposition 3.6 below.

It turns out that many properties of the usual correction terms have analogues for $\underline{d}(Y, \mathfrak{t})$. In the rest of this section we describe the effects on $\underline{d}(Y, \mathfrak{t})$ of conjugation of spin^c structures, orientation reversal, and connected sums. In Section 4 we will study the behavior under negative semidefinite cobordisms.

3.1. Conjugation and orientation reversal. Recall that in Heegaard Floer theory one identifies spin^c structures with homology classes of nowhere vanishing vector fields. In particular, we have an on-the-nose equality $\mathrm{Spin}^c(Y) = \mathrm{Spin}^c(-Y)$ where $-Y$ denotes Y with the opposite orientation.[1] Moreover, if a spin^c structure $\mathfrak{t}$ is represented by a vector field v, then $-v$ represents the *conjugate* spin^c structure which we denote by $\bar{\mathfrak{t}}$.

Proposition 3.6. *The twisted correction terms of* $(Y, \mathfrak{t})$ *satisfy*

$$\underline{d}(Y, \bar{\mathfrak{t}}) = \underline{d}(Y, \mathfrak{t}) = -\underline{d}^*(-Y, \mathfrak{t}) = -\underline{d}^*(-Y, \bar{\mathfrak{t}}).$$

In particular, if Y has an orientation-reversing self-diffeomorphism that preserves $\mathfrak{t}$ up to conjugation, then $\underline{d}^*(Y, \mathfrak{t}) = -\underline{d}(Y, \mathfrak{t})$.

Proof. This follows exactly as in the proof of [16, Proposition 4.2] with some additional input for twisted coefficients from [10, Section 6]. □

It is interesting to note that the proof of [16, Proposition 4.2] also shows that for $b_1(Y) = 0$ we have $\underline{d}(Y, \mathfrak{t}) = \underline{d}^*(Y, \mathfrak{t})$ – both agreeing with $d(Y, \mathfrak{t})$ which therefore satisfies $d(-Y, \mathfrak{t}) = -d(Y, \mathfrak{t})$. However, according to Example 3.5 this argument has to fail for $b_1(Y) > 0$. In general, we do not see any obvious relation between $\underline{d}(Y, \mathfrak{t})$ and $\underline{d}^*(Y, \mathfrak{t})$.

3.2. Connected sums. Next we study the behavior of the twisted correction terms under the connected sum operation.

Proposition 3.7. *For torsion spin^c 3-manifolds $(Y_1, \mathfrak{t}_1)$ and $(Y_2, \mathfrak{t}_2)$ we have*

$$\underline{d}(Y_1 \# Y_2, \mathfrak{t}_1 \# \mathfrak{t}_2) = \underline{d}(Y_1, \mathfrak{t}_1) + \underline{d}(Y_2, \mathfrak{t}_2)$$

and

$$\underline{d}^*(Y_1 \# Y_2, \mathfrak{t}_1 \# \mathfrak{t}_2) = \underline{d}^*(Y_1, \mathfrak{t}_1) + \underline{d}^*(Y_2, \mathfrak{t}_2)$$

Proof. The idea is to show that, in the connected sum theorem for $\underline{\mathrm{HF}}^-$, the tensor product of the two towers is mapped surjectively onto the tower in the connected sum, and this immediately proves the statement.

[1] Note however that $c_1(Y, \mathfrak{t}) = -c_1(-Y, \mathfrak{t}) = c_1(-Y, \bar{\mathfrak{t}})$. So some caution is needed when working with the more common shortened notation $c_1(\mathfrak{t})$.

To see this, observe that from (2.6), the Künneth theorem yields a short exact sequence that splits:

$$0 \longrightarrow (\underline{\mathrm{HF}}^-(Y_1,\mathfrak{t}_1) \otimes_{\mathbb{F}[U]} \underline{\mathrm{HF}}^-(Y_2,\mathfrak{t}_2))[2] \stackrel{j}{\longrightarrow} \underline{\mathrm{HF}}^-(Y_1\#Y_2,\mathfrak{t}_1\#\mathfrak{t}_2) \\ \longrightarrow \mathrm{Tor}^{\mathbb{F}[U]}(\underline{\mathrm{HF}}^-(Y_1,\mathfrak{t}_1),\underline{\mathrm{HF}}^-(Y_2,\mathfrak{t}_2)) \longrightarrow 0.$$

Also, there is a splitting of $\mathbb{F}[U]$-modules $\underline{\mathrm{HF}}^-(Y_i,\mathfrak{t}_i) \cong \mathbb{F}[U]x_i \oplus \underline{\mathrm{HF}}^-_{\mathrm{red}}(Y_i,\mathfrak{t}_i)$, where each element in $\underline{\mathrm{HF}}^-_{\mathrm{red}}(Y_i,\mathfrak{t}_i)$ is U-torsion; this splitting is far from being unique, but, if we insist upon x_i being homogeneous, the degree of x_i is well-defined, and indeed $\deg x_i = \underline{d}(Y_i,\mathfrak{t}_i) - 2$.

The element $x := j(x_1 \otimes x_2)$ is homogeneous of degree $\underline{d}(Y_1,\mathfrak{t}_1)+\underline{d}(Y_2,\mathfrak{t}_2)-2$. Since the short exact sequence above splits, and since in the tensor product the only non-U-torsion summand is $\mathbb{F}[U]x$, we deduce that there is a decomposition of $\mathbb{F}[U]$-modules

$$\underline{\mathrm{HF}}^-(Y_1\#Y_2,\mathfrak{t}_1\#\mathfrak{t}_2) \cong \mathbb{F}[U]x \oplus T,$$

where T is the U-torsion summand. Since the decomposition above determines the degree of x, we obtain the desired equality. □

3.3. Manifolds with standard $\mathbf{HF}^\infty$. We now compare the twisted correction terms with the bottom-most correction terms that have been studied by Ozsváth and Szabó [16] and later by Levine and Ruberman [11].

Proposition 3.8. *If a torsion spin^c 3-manifold $(Y,\mathfrak{t})$ has standard HF^∞, then*

$$\underline{d}(Y,\mathfrak{t}) \le d_b(Y,\mathfrak{t}).$$

(It is understood that $\underline{d}$ and d_b are defined using the same coefficient field, and that the statement holds for all characteristics.)

Proof. Let H, Λ and R denote the group $H_2(Y)$, the exterior algebra $\Lambda^* H$ and the ring $\mathbb{Z}[H]$ respectively; endow $\mathbb{Z}$ with the trivial R-module structure, i.e. $\mathbb{Z} = R/(h-1 \mid h \in H)$. Take $\Lambda_R := R \otimes_{\mathbb{Z}} R$, endowed with the trivial differential and the obvious grading, as an R-module resolution of $\mathbb{Z}$, and consider R as the trivial R-module resolution of itself. The quotient map $R \to \mathbb{Z}$ induces a map between the two resolutions, that is an isomorphism of their degree-0 summands. This map, in turn, induces a map of (universal coefficient) spectral sequences from $\underline{\mathrm{HF}}^\circ(Y,\mathfrak{t})$ to $\mathrm{HF}^\circ(Y,\mathfrak{t})$.

If $(Y, \mathfrak{t})$ has standard HF^∞, the universal coefficient spectral sequence from $\underline{\mathrm{HF}}^\infty(Y, \mathfrak{t}) \otimes_R \Lambda_R$ to $\mathrm{HF}^\infty(Y, \mathfrak{t})$ collapses at the second page, and moreover the action of Λ on $\mathrm{HF}^\infty(Y, \mathfrak{t})$ is induced by the action of Λ on the first page of the spectral sequence. In particular, the bottom-most tower of $\mathrm{HF}^\infty(Y, \mathfrak{t})$ corresponds to the degree-0 component of Λ, and it follows that $\underline{\mathrm{HF}}^\infty(Y, \mathfrak{t})$ maps onto this tower under the map of spectral sequences described above.

Summing up, we have the following commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{HF}}^\infty(Y, \mathfrak{t}) & \longrightarrow & \mathrm{HF}^\infty(Y, \mathfrak{t}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{HF}}^+(Y, \mathfrak{t}) & \longrightarrow & \mathrm{HF}^+(Y, \mathfrak{t}) \end{array}$$

where the top horizontal map is an isomorphism of $\underline{\mathrm{HF}}^\infty(Y, \mathfrak{t})$ onto the kernel of the Λ-action on $\mathrm{HF}^\infty(Y, \mathfrak{t})$. It follows that $d_b(Y, \mathfrak{t}) \geq \underline{d}(Y, \mathfrak{t})$. □

While in general we do not expect equality of d_b and $\underline{d}$ to hold, there are families of examples where the two quantities agree; for instance, all rational homology spheres, and 0-surgeries along knots in the 3-sphere, as the following example shows.

Example 3.9. Let us consider a knot K in S^3; it follows from [16, Section 4.2] that $d_b(S^3_0(K)) = d_{-1/2}(S^3_0(K)) = d(S^3_{-1}(K)) - 1/2$. Let us now look at the twisted surgery exact triangle of [18, Theorem 9.14] associated to the framings $\infty, -1$ and 0 of K:

$$\cdots \longrightarrow \mathrm{HF}^+(S^3)[t, t^{-1}] \xrightarrow{F} \mathrm{HF}^+(S^3_{-1}(K))[t, t^{-1}] \xrightarrow{G} \underline{\mathrm{HF}}^+(S^3_0(K)) \xrightarrow{H} \cdots .$$

It is immediate to see that the map F is multiplication by $(1-t)$, that the restriction of G on the tower is modeled on the projection $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}]/(1-t) \cong \mathbb{Z}$, and that the map H vanishes on the tower; moreover, the map H has degree $-1/2$ in the spin^c structure with trivial Chern class. In particular,

$$\underline{d}(S^3_0(K)) \geq d(S^3_{-1}(K)) - 1/2 = d_b(S^3_0(K)).$$

Combined with the proposition above, this shows that $\underline{d}(S^3_0(K)) = d_b(S^3_0(K))$.

4. Negative semidefinite cobordisms

In this section we prove the core technical result, Theorem 4.1 below, which will imply Theorem 1.1. We will work over the integers, but everything goes through for $\mathbb{Q}$ and $\mathbb{F}_p$ with obvious modifications. The proof is based on the strategy used in Ozsváth and Szabó's proof of Donaldson's theorem [16, Section 9]. Throughout this section $(W, \mathfrak{s})$ will be a spin^c cobordism between torsion spin^c 3-manifolds $(Y, \mathfrak{t})$ and $(Y', \mathfrak{t}')$. To obtain cleaner statements we renormalize the twisted correction terms to

$$\delta(Y, \mathfrak{t}) = 4\underline{d}(Y, \mathfrak{t}) + 2b_1(Y). \tag{4.1}$$

Theorem 4.1. *Let $(W, \mathfrak{s})$ be a negative semidefinite spin^c cobordism between torsion spin^c 3-manifolds $(Y, \mathfrak{t})$ and $(Y', \mathfrak{t}')$ such that the inclusion $Y \hookrightarrow W$ induces an injection $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$. Then*

$$\begin{aligned} c_1^2(\mathfrak{s}) + b_2^-(W) &\leq \delta(Y', \mathfrak{t}') - \delta(Y, \mathfrak{t}) \\ &= 4\underline{d}(Y', \mathfrak{t}') - 4\underline{d}(Y, \mathfrak{t}) + 2b_1(Y') - 2b_1(Y). \end{aligned} \tag{4.2}$$

Before going into the proof of the theorem we pause to derive some consequences of Theorem 4.1. To begin with, we show that it implies Theorem 1.1.

Proof of Theorem 1.1. Given a negative semidefinite filling $(Z, \mathfrak{s})$ of $(Y, \mathfrak{t})$ we consider the spin^c cobordism from $(S^3, \mathfrak{t}_0)$ to $(Y, \mathfrak{t})$ given by $W = Z \setminus B^4$ equipped with the restriction $\mathfrak{s}$. Since S^3 is simply connected, Theorem 4.1 applies and the desired inequality is immediate from the fact that $\delta(S^3, \mathfrak{t}_0) = 0$. □

Another consequence of Theorem 4.1 is that the twisted correction terms, like ordinary and generalised correction terms, are rational cobordism invariants.

Corollary 4.2. *If $(W, \mathfrak{s})$ is a rational homology cobordism between $(Y, \mathfrak{t})$ and $(Y', \mathfrak{t}')$, then $\underline{d}(Y, \mathfrak{t}) = \underline{d}(Y', \mathfrak{t}')$ and $\delta(Y, \mathfrak{t}) = \delta(Y', \mathfrak{t}')$.*

Proof. Both W and $-W$ are negative semidefinite, and both inclusions $Y, Y' \hookrightarrow W$ induce isomorphisms on rational homology by assumption. Hence applying Theorem 4.1 to W and $-W$ we get $\underline{d}(Y, \mathfrak{t}) \leq \underline{d}(Y', \mathfrak{t}')$ and $\underline{d}(Y', \mathfrak{t}') \leq \underline{d}(Y, \mathfrak{t})$. □

4.1. The proof of Theorem 4.1. As mentioned above our proof of Theorem 4.1 is modeled on Ozsváth and Szabó's proof of Donaldson's theorem in [16, Section 9]. The strategy is to equip the cobordism with a suitable handle decomposition and to investigate the behavior of the twisted correction terms under 1-, 2-, and 3-handle attachments. As usual, the 1- and 3-handles can be treated on an essentially formal level while the 2-handles require more sophisticated arguments – in this case establishing properties of cobordism maps on $\underline{\mathrm{HF}}^\infty$ with suitably twisted coefficients. We begin with the 1- and 3-handles.

Proposition 4.3. *If W consists of a single 1-or 3-handle attachment, then*

$$\underline{d}(Y', \mathfrak{t}') - \underline{d}(Y, \mathfrak{t}) = -\tfrac{1}{2}(b_1(Y') - b_1(Y)) = \begin{cases} -\frac{1}{2} & \text{for } 1\text{-handles,} \\ \frac{1}{2} & \text{for } 3\text{-handles,} \end{cases}$$

or, equivalently, $\delta(Y', \mathfrak{t}') = \delta(Y, \mathfrak{t})$.

Proof. In the case of a 1-handle attachment we have $Y' \cong Y\#(S^1 \times S^2)$ and the claim follows from Proposition 3.7, the computation of $\underline{d}(S^1 \times S^2, \mathfrak{t}_0) = -\frac{1}{2}$ in Example 3.5, and the fact that there is a unique spin^c structure on W extending $\mathfrak{t}$ whose restriction to Y' is torsion. Similarly, for 3-handles $Y \cong Y'\#(S^1 \times S^2)$. ☐

For the discussion of 2-handles we switch to a more fitting notation. We consider a framed knot (K, λ) in a 3-manifold Y and write $Y_\lambda = Y_\lambda(K)$ and $W_\lambda = W_\lambda(K)$ for the 3-manifold obtained by λ-framed surgery on K and the corresponding 2-handle cobordism. We have to discuss the cobordism maps induced by W_λ and it turns out that we have to distinguish two cases depending on whether K has infinite order in $H_1(Y)$ or it represents a torsion class. We begin with the former case, which requires some more subtle modifications of the standard arguments for untwisted coefficients.

We first introduce some terminology. For any subgroup $V \subset H_2(Y)$ we define

$$V^\perp = \{x \in H_1(Y) \,|\, x \cdot v = 0 \text{ for all } v \in V\} \subset H_1(Y).$$

Note that $V^\perp$ contains the torsion subgroup of $H_1(Y)$ and that the intersection pairing induces a canonical identification of $H_1(Y)/V^\perp$ with $V^* = \mathrm{Hom}(V, \mathbb{Z})$.

Definition 4.4. Let $(Y, \mathfrak{t})$ be a torsion spin^c 3-manifold and let V be a direct summand of $H_2(Y)$. Consider the coefficient module $M_V = \mathbb{Z}[H_2(Y)/V]$ with the obvious R_Y-action. We say that $(Y, \mathfrak{t})$ has *V-standard* $\underline{\mathrm{HF}}^\infty$ if there is an $R_Y[U]$-linear isomorphism

$$\underline{\mathrm{HF}}^\infty(Y, \mathfrak{t}; M_V) \cong \Lambda^* V \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}]$$

such that the action of $V^\perp \subset H_1(Y)$ is annihilating while $V^* = H_1(Y)/V^\perp$ acts by contraction on $\Lambda^* V$.

Example 4.5. (i) For $V = H_2(Y)$ the above definition agrees with the usual notion of "standard HF^∞" discussed in Section 3.3.

(ii) By Theorem 2.1 all 3-manifolds have standard $\underline{\mathrm{HF}}^\infty$ for $V = 0$ and according to Proposition 2.6 the same holds for any V of rank 1.

(iii) This example will be particularly relevant and has, in fact, already appeared in the proof of Proposition 2.3. Let $(Y, \mathfrak{t})$ be a spin^c 3-manifold; the proof of [18, Proposition 6.4] shows that for any R_Y-module M we have

$$\underline{\mathrm{HF}}^\circ(Y\#\mathbb{S}_n, \mathfrak{t}\#\mathfrak{t}_0; M) \cong \underline{\mathrm{HF}}^\circ(Y, \mathfrak{t}; M) \otimes \Lambda^* H_2(\mathbb{S}_n) \tag{4.3}$$

where M is considered as and module over $R_{Y\#\mathbb{S}_n} = R_Y \otimes_{\mathbb{Z}} R_{\mathbb{S}_n}$ with trivial $R_{\mathbb{S}_n}$ action. Moreover, the action of $H_1(Y\#\mathbb{S}_n)$ on the right-hand side is induced by the usual action of $H_1(Y)$ on the first factor, and by the contraction with elements of $H_1(\mathbb{S}_n)$ via the intersection product on the second factor. In particular, it is a matter of checking the definition to see that $(Y\#\mathbb{S}_n, \mathfrak{t}\#\mathfrak{t}_0)$ has standard $\underline{\mathrm{HF}}^\infty$ with respect to the subgroup of $H_2(Y\#\mathbb{S}_n)$ corresponding to $H_2(\mathbb{S}_n)$.

Proposition 4.6. *Let $(Y, \mathfrak{t})$ be a torsion spin^c 3-manifold and let (K, λ) be a framed knot in Y such that K has infinite order in $H_1(Y)$. Let V be a direct summand of $H_2(Y)$ such that $(Y, \mathfrak{t})$ has V-standard $\underline{\mathrm{HF}}^\infty$ and some $v \in V$ satisfies $[K]\cdot v \neq 0$. Then there is a subgroup V_K of $H_2(Y_\lambda)$ such that $M_{V_K} \cong M_V$. Moreover, Y_λ has V_K-standard $\underline{\mathrm{HF}}^\infty$ for any torsion spin^c structure $\mathfrak{t}'$ which is cobordant to $\mathfrak{t}$ via $(W_\lambda, \mathfrak{s})$; and the cobordism map induces an isomorphism*

$$\underline{\mathrm{HF}}^\infty(Y, \mathfrak{t}; M_V)/\ker[K] \xrightarrow{\cong} \underline{\mathrm{HF}}^\infty(Y_\lambda, \mathfrak{t}'; M_V)$$

where $\ker[K]$ is the kernel of the action of $[K]$.

Proof. One readily checks that the inclusion of Y in W_λ induces an isomorphism $H_2(W_\lambda) \cong H_2(Y)$. According to (2.5) we get maps

$$F^\infty_{W_\lambda,\mathfrak{s}}: \underline{\mathrm{HF}}^\infty(Y, \mathfrak{t}; M_V) \longrightarrow \underline{\mathrm{HF}}^\infty(Y_\lambda, \mathfrak{t}'; M_V)$$

for any $\mathfrak{s} \in \mathrm{Spin}^c(W_\lambda)$ and $\mathfrak{t}' = \mathfrak{s}|_{Y_\lambda}$. By a variation of Lemma 2.4 we see that $H_2(Y_\lambda) \cong H_2(E_K)$ and $H_2(Y) \cong H_2(E_K) \oplus \mathbb{Z}$ where the second summand is generated by a primitive element of $H_2(Y)$ that has non-trivial intersection with $[K]$. By assumption we can find such an element in V. Under the above identifications we can consider $V_K = \{v \in V \mid [K]\cdot v = 0\}$ as a subgroup of $H_2(E_K)$ and therefore of $H_2(Y_\lambda)$. Moreover, we have $H_2(Y)/V \cong H_2(Y_\lambda)/V_K$ and thus $M_V \cong M_{V_K}$.

Now suppose that $\mathfrak{t}'$ is torsion. We can put the maps induced by W_λ into a surgery triangle as before and argue as in the proof of [16, Proposition 9.3] that $F^\infty_{W_\lambda,\mathfrak{s}}$ vanishes on $\ker[K]$ and is injective on the quotient for all field coefficients. The only missing piece is a bound on the rank of $\underline{\mathrm{HF}}^\infty(Y_\lambda, \mathfrak{t}'; M_V)$ in each degree. To that end, we observe that the E_2-term of the relevant universal coefficient spectral sequence is given by

$$\begin{aligned}\operatorname{Tor}^{R_Y}(\underline{HF}^\infty(Y_\lambda,\mathfrak{t}'),M_V) &\cong \operatorname{Tor}^{\mathbb{Z}[H_2(Y_\lambda)]}(\mathbb{Z},\mathbb{Z}[H_2(Y)/V])\otimes_{\mathbb{Z}}\mathbb{Z}[U,U^{-1}]\\ &\cong \operatorname{Tor}^{\mathbb{Z}[V]}(\mathbb{Z},\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[U,U^{-1}]\\ &\cong \Lambda^* V\otimes_{\mathbb{Z}}\mathbb{Z}[U,U^{-1}]\end{aligned}$$

where the second isomorphism follows from Shapiro's lemma (see [1, p. 73], for example). The resulting rank bound can be used as a replacement of [16, Lemma 9.2] in the proof of [16, Proposition 9.3]. □

Remark 4.7. Note that in the above proof it was crucial for the action of K to have non-trivial image. Since any non-torsion element of $H_1(Y)$ annihilates $\underline{HF}^\circ(Y,\mathfrak{t})$ (see [10, Remark 5.2]), the proof does not work for $V=0$, that is, we cannot start with fully twisted coefficients for Y.

We now turn to the case when K has finite order in $H_1(Y)$.

Proposition 4.8. *Let $(Y,\mathfrak{t})$ and (K,λ) be as above and suppose that K has finite order in $H_1(K)$. If $b_2^+(W_\lambda)=0$, then W_λ induces an isomorphism*

$$\underline{HF}^\infty(Y,\mathfrak{t}) \xrightarrow{\cong} \underline{HF}^\infty(Y_\lambda,\mathfrak{t}')$$

where $\mathfrak{t}'$ is the restriction of an extension of $\mathfrak{t}$ to the surgery cobordism.

Remark 4.9. For those familiar with rational linking numbers we note that the b_2^+-condition is equivalent to $\operatorname{lk}_{\mathbb{Q}}(K,\lambda)\le 0$, so that the assumptions in the above propositions can be rephrased purely in 3-dimensional terms.

Proof of Proposition 4.8. The main idea is to study exact triangles relating suitable twisted Heegaard Floer homology groups of the manifolds Y, Y_λ, and $Y_{\lambda+\mu}$ where the latter is obtained by $\lambda+\mu$-framed surgery on K. There are three cases to consider according to the change of b_1 under the surgeries:

(1) $b_1(Y)=b_1(Y_\lambda)=b_1(Y_{\lambda+\mu})$,

(2) $b_1(Y)=b_1(Y_\lambda)<b_1(Y_{\lambda+\mu})$,

(3) $b_1(Y)=b_1(Y_{\lambda+\mu})<b_1(Y_\lambda)$.

Case (1) is an immediate adaptation of the proof of [16, Proposition 9.4]. In fact, all relevant cobordisms induce maps between the fully twisted Floer homology groups, and the proof proceeds exactly as in the untwisted case.

Case (2) also follows from an adaptation of the same proof, but with more substantial modifications. In this case, in fact, there is a surgery exact triangle that reads as follows (see [10, Theorem 9.1]):

$$\begin{array}{ccc} \underline{HF}^+(Y)[t,t^{-1}] & \xrightarrow{F} & \underline{HF}^+(Y_\lambda)[t,t^{-1}] \\ {\scriptstyle H}\nwarrow & & \swarrow{\scriptstyle G} \\ & \underline{HF}^+(Y_{\lambda+\mu}) & \end{array}$$

Here, F is t-equivariant and is, in fact, the map $\underline{F} \otimes \mathbf{1}$, where $\underline{F}$ is the map induced by the surgery cobordism between the twisted Floer homology groups. Moreover, t acts as the class of the capped-off surface $T \in H_2(Y_{\lambda+\mu})$. Since T acts as the identity on $\underline{HF}^\infty(Y_{\lambda+\mu})$, for all sufficiently large degrees the map F is multiplication by $(1-t)$, and in particular it induces a surjection on the towers in $\underline{HF}^+(Y,\mathfrak{t})$ for each torsion spin^c structure $\mathfrak{t}$ on Y. Now the argument runs as in the untwisted case to show the desired inequality; compare with [18, Theorem 9.1].

In case (3), we use the surgery triangle of Proposition 2.5:

$$\begin{array}{ccc} \underline{HF}^+(Y) & \xrightarrow{F} & \underline{HF}^+(Y_\lambda; M_K) \\ {\scriptstyle H}\nwarrow & & \swarrow{\scriptstyle G} \\ & \underline{HF}^+(Y_{\lambda+\mu}) & \end{array}$$

As in the proof of Proposition 4.6 we show that the infinity version of G has the same kernel as the action of the dual knot of K, say $K' \subset Y_\lambda$. Moreover, the usual argument shows that the infinity version of F, which is just $F^\infty_{W_\lambda,\mathfrak{s}}$, is injective; and by exactness it injects into $\ker[K']$ which, according to Proposition 2.6, is graded isomorphic to $\underline{HF}^\infty(Y_\lambda,\mathfrak{t}')$. Again observing that the argument goes through with arbitrary field coefficients, we see that $F^\infty_{W_\lambda,\mathfrak{s}}$ maps isomorphically onto $\ker[K']$. □

Proof of Theorem 4.1. The key is the standard observation that whenever we have a cobordism $(W,\mathfrak{s})$ between torsion spin^c 3-manifolds $(Y,\mathfrak{t})$ and $(Y',\mathfrak{t}')$ inducing an isomorphism $F^\infty_{W,\mathfrak{s}}\colon \underline{HF}^\infty(Y,\mathfrak{t}) \to \underline{HF}^\infty(Y',\mathfrak{t}')$, then $\underline{d}(Y,\mathfrak{t}) + \deg F^+_{W,\mathfrak{s}} \le \underline{d}(Y',\mathfrak{t}')$, as an easy diagram chase shows. Unfortunately, we cannot apply this argument directly because in general the target of the cobordism maps will not have fully twisted coefficients.

To circumvent this problem, we observe that the left-hand side of the inequality (4.2) is additive while the right-hand side behaves telescopically when two negative semidefinite cobordisms are composed. Conversely, one can also show that the left-hand side splits appropriately when W is cut along a separating 3-manifold in its interior. It would therefore be enough to prove Theorem 4.1 for cobordisms consisting of single handle attachments. In fact, this strategy works quite well since Proposition 4.3 covers 1- and 3-handles, while Proposition 4.8 allows us to run the standard argument mentioned above. What remains are 2-handle attachments along knots in essential homology classes. It turns out that these actually cannot be treated separately but have to be paired with 1-handles. It is at this point that the assumption on the map $H_1(Y;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ becomes relevant and we are forced to work with the coefficient systems that appear in Proposition 4.6.

As a last preparatory remark, we can restrict our attention to the case when $H_1(Y;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is not only injective but actually an isomorphism. Indeed, if it is not surjective, say of corank k, then we perform surgery on an embedded circle $C \subset W \setminus \partial W$ which represents a non-zero class in $H_1(W;\mathbb{Q})$ not contained in the image of $H_1(Y;\mathbb{Q})$. The resulting cobordism W' has the same boundary as W and is easily seen to satisfy $b_2^{\pm}(W') = b_2^{\pm}(W)$ and the map $H_1(Y;\mathbb{Q}) \to H_1(W';\mathbb{Q})$ has corank $k-1$. Moreover, the restriction of $\mathfrak{s}$ to $W \setminus \nu C$ extends to W' and any such extension $\mathfrak{s}'$ satisfies $c_1^2(\mathfrak{s}') = c_1^2(\mathfrak{s})$. In particular, the left-hand side of (4.2) is the same for $(W,\mathfrak{s})$ and $(W',\mathfrak{s}')$. By successive surgeries we can therefore cut down $H_1(W;\mathbb{Q})$ to the image of $H_1(Y;\mathbb{Q})$.

We now begin the actual proof. We choose a handle decomposition of W and put it in *standard ordering* as defined by Ozsváth and Szabó (see [16, p. 243]). This means that the handles are attached in order of increasing index and, moreover, the 2-handle attachments are ordered such that b_1 of the intermediate 3-manifolds first decreases, then stays constant, and finally increases. For the existence of such a handle decomposition, see [16, p. 244]. We cut W into two pieces $W_{12} \cup_N W_{23}$ such that W_{12} contains all 1-handles and the decreasing 2-handles while W_{23} contains the remaining 2- and 3-handles. Observe that the b_1-decreasing 2-handles are exactly those that are attached along essential knots. So by the above remarks Theorem 4.1 holds for W_{23} and we can restrict our attention to W_{12}. Since we are assuming that $H_1(Y;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is an isomorphism, there must be exactly as many b_1-decreasing 2-handles as there are 1-handles, say we have n each. Our goal is to show that $(W_{12},\mathfrak{s})$ induces an isomorphism between $\underline{\mathrm{HF}}^\infty(Y,\mathfrak{t})$ and $\underline{\mathrm{HF}}^\infty(N,\mathfrak{t}_N)$ where $\mathfrak{t}_N = \mathfrak{s}|_N$ is easily seen to be torsion. We further decompose W_{12} into pieces V_1 and V_2 along $Y\#\mathbb{S}_n$ where V_i contains all i-handles. Note that the attaching circles $K_1,\dots,K_n \subset Y\#\mathbb{S}_n$ of the 2-handles span the subspace $H_1(\mathbb{S}_n;\mathbb{Q}) \subset H_1(Y\#\mathbb{S}_n;\mathbb{Q})$. In particular, W_{12} is a rational homology cobordism which, in turn, implies that the twisted cobordism map has the correct functoriality. To show that it is an isomorphism we invoke the composition law for twisted coefficients [10, Section 2.3]. On the one hand, we observe that in the identification of Example 4.5 (iii) we have

$$F^{\infty}_{V_1,\mathfrak{s}}(\underline{\mathrm{HF}}^{\infty}(Y,\mathfrak{t})) \cong \underline{\mathrm{HF}}^{\infty}(Y,\mathfrak{t}) \otimes \Lambda^n H_2(\mathbb{S}_n),$$

which follows from the definition of the maps induces by 1-handles, see [20, Section 4.3]. On the other hand, Proposition 4.6 applies to the 2-handles with $V = H_2(\mathbb{S}_n)$ and shows that $F^{\infty}_{V_2,\mathfrak{s}}$ maps the image of $F^{\infty}_{V_1,\mathfrak{s}}$ isomorphically onto $\underline{\mathrm{HF}}^{\infty}(N,\mathfrak{t}_N)$. We can therefore conclude that we have an isomorphism $F^{\infty}\colon \underline{\mathrm{HF}}^{\infty}(Y,\mathfrak{t}) \xrightarrow{\cong} \underline{\mathrm{HF}}^{\infty}(N,\mathfrak{t}_N)$, which finishes the proof. □

5. Intersection forms of smooth fillings

We already mentioned that Theorem 1.1 imposes restrictions on the possible intersection forms of smooth 4-manifolds with fixed boundary. We will now make the nature of these restrictions more precise. We begin with some general remarks about non-degenerate symmetric bilinear forms over the integers. Let L be a free Abelian group of rank n, equipped with an integer-valued symmetric bilinear form S and let $d = |\det S|$. Recall that S is called *non-degenerate* if $d \neq 0$ and *unimodular* if $d = 1$. We will refer to the expressions of the form $S(x,x)$, $x \in L$, as *squares* of S. We say that S is *semidefinite* (or simply *definite* in the non-degenerate case) if all non-zero squares have the same sign, and *indefinite* otherwise. Furthermore, S is called *even* if all squares are even, and *odd* otherwise. If S is non-degenerate then L canonically embeds into the *dual group* $L^* = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ as a subgroup of index d. Consequently, we can identify L with its image in L^* and extend S to a rational-valued form on L^* as follows. For any $\lambda \in L^*$ we have $d\lambda \in L$ and we set

$$S^*(\lambda,\mu) = \tfrac{1}{d^2} S(d\lambda, d\mu) = \tfrac{1}{d}\lambda(d\mu) \in \tfrac{1}{d}\mathbb{Z} \subset \mathbb{Q}.$$

for any pair $\lambda, \mu \in L^*$.

Remark 5.1. A less intrinsic but more geometric picture emerges when we embed L as a *lattice* in $\mathbb{R}^n$ in such a way that S corresponds to the standard inner product with the same signature as S (which is possible by Sylvester's law of inertia). After fixing such an embedding $L \subset \mathbb{R}^n$ one can conveniently think of L^* as the *dual lattice* $\{y \in \mathbb{R}^n \mid x \cdot y \in \mathbb{Z} \text{ for all } x \in L\}$ leading to a chain of inclusions $L \subset L^* \subset \mathbb{R}^n$ and both S and S^* are given by the relevant inner product on $\mathbb{R}^n$.

The main purpose for introducing L^* is that it serves as a host for the *characteristic covectors* of S which form the set

$$\chi^*(S) = \{\kappa \in L^* \mid \kappa(x) \equiv S(x,x) \mod 2 \text{ for all } x \in L\}.$$

From these we extract a numerical invariant

$$m(S) = \min\{|S^*(\kappa,\kappa)| \mid \kappa \in \chi^*(S)\} \in \mathbb{Q}$$

which, in the case of a definite lattice in $\mathbb{R}^n$, measures the length of the shortest characteristic covector of S. As we will see, this is a rather powerful invariant of S. For mainly cosmetic reasons we will consider the equivalent invariant

$$d(S) = n - m(S) \in \mathbb{Q}. \tag{5.1}$$

that we call the *defect* of S.[2] To the best of our knowledge these invariants first appeared implicitly in the work of Elkies [4] and [5], which was inspired by Donaldson's theorem. We will say more about their algebraic significance after explaining the relation to Theorem 1.1.

Now let Z be a smooth filling of a fixed 3-manifold Y and let $\ker(Q_Z)$ be the kernel of the intersection form on $H_2(Z)$. The quotient $L_Z = H_2(Z)/\ker(Q_Z)$ is easily seen to be free Abelian of rank $b_2^+(Z) + b_2^-(Z)$ and Q_Z descends to a non-degenerate form on L_Z, henceforth denoted by S_Z, which we will refer to as the *non-degenerate intersection form* of Z. Together with the observation that $\ker(Q_Z)$ contains the image of $H_2(Y)$ as a subgroup of full rank, the universal coefficient theorem gives identifications

$$\begin{aligned} L_Z^* &\cong \left\{\xi \in H^2(Z) \,\middle|\, \langle \xi, x\rangle = 0 \text{ for all } x \in \ker(Q_Z)\right\}/\text{torsion} \\ &= \left\{\xi \in H^2(Z) \,\middle|\, \xi|_Y \in H^2(Y) \text{ is torsion}\right\}/\text{torsion}. \end{aligned}$$

Moreover, an inspection of the homology sequence of the pair shows that L_Z^*/L_Z injects into the torsion subgroup of $H_1(Y)$ so that $|\det(S_Z)|$ is bounded by the order of the torsion subgroup of $H_1(Y)$. In order to state a more algebraic reformulation of Theorem 1.1 we introduce the notation

$$\delta(Y) = \max\{\delta(Y,\mathfrak{t}) \mid \mathfrak{t} \in \mathrm{Spin}^c(Y) \text{ torsion}\}$$

which gives an invariant that does not depend on any spin^c structure but only on Y.

[2] While defects of lattices tend to show up quite frequently in relation questions surrounding Donaldson's theorem, there does not seem to be a standard terminology. We follow [6, p. 8].

Theorem 5.2. *Let Z be a smooth filling of Y. If $b_2^+(Z) = 0$, then any characteristic covector $\kappa \in \chi^*(S_Z)$ satisfies*

$$b_2^-(Z) + \kappa^2 \leq \delta(Y).$$

In particular, the defect of S_Z satisfies $d(S_Z) \leq \delta(Y)$.

Proof. This is an immediate consequence of Theorem 1.1 once we understand the relationship between spin^c structures on Z and characteristic covectors of S_Z. Since this is common folklore, we shall be brief. Using the identification of L_Z^* in (5.2), each spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Z)$ with $c_1(\mathfrak{s}|_Y)$ torsion gives rise to an element $\kappa_{\mathfrak{s}} \in L_Z^*$. Moreover, we have $\kappa_{\mathfrak{s}}(x) = \langle c_1(\mathfrak{s}), \bar{x}\rangle$ for any $\bar{x} \in H_2(Z)$ representing $x \in L_Z$ and, since $c_1(\mathfrak{s})$ reduces to $w_2(Z)$ which evaluates as the mod 2 self-intersections, it follows that $\kappa_{\mathfrak{s}} \in \chi^*(S_Z)$. One readily checks that $\chi^*(S_Z)$ has a free and transitive action of $2L_Z^*$ which can be realised by the action of $H^2(Z)$ on $\mathrm{Spin}^c(Z)$. Hence, all characteristic covectors have the form $\kappa_{\mathfrak{s}}$ for some $\mathfrak{s}$. What is left to check is that $\kappa_{\mathfrak{s}}^2 = c_1^2(\mathfrak{s})$ which, in essence, follows from Poincaré duality after unraveling the definition. Theorem 1.1 then yields the inequality $b_2^-(Z) + \kappa_{\mathfrak{s}}^2 \leq \delta(Y, \mathfrak{t})$ where $\mathfrak{t} = \mathfrak{s}|_Y$. Taking maxima on both sides leads to the desired inequalities. □

Having identified the defect of the non-degenerate intersection form as the algebraic invariant of semi-definite fillings obstructed by the δ-invariant, and thus by the twisted correction terms, we now take a closer look from an algebraic perspective. We restrict our attention to a negative definite form S of rank n. An important feature is that the defect $d(S)$ a priori lies in the bounded range

$$0 \leq d(S) \leq n. \tag{5.2}$$

The right inequality holds by definition with equality precisely when S is even (both conditions are equivalent to $0 \in \chi^*(S)$). The left inequality was first proved by Elkies [4] for unimodular forms and was extended by Owens and Strle [15] to the general case. More interestingly, their results also show that the equality $d(S) = 0$ characterises the trivial lattice $S \cong \boldsymbol{I}_n = n\langle -1\rangle$. This already shows that the defect is a powerful invariant. Theorem 5.2 together with (5.2) yields the following.

Corollary 5.3. *If Y has a smooth, negative semidefinite filling, then $\delta(Y) \geq 0$.*

Recall that any negative definite form can be decomposed as $S = S_0 \oplus \boldsymbol{I}_r$ where S_0 is *minimal* in the sense that it has no element of square -1. The number r

and the isometry class of S_0 are uniquely determined by S. Since the defect is clearly additive under orthogonal sums and $d(\boldsymbol{I}_r) = 0$, we see that $d(S) = d(S_0)$. An immediate consequence of this is that an upper bound on the defect does not imply an upper bound for the rank. However, if S happens to be even (and thus minimal), then we have $d(S) = \mathrm{rk}(S)$ as noted above. Using this we get the following refinement of Theorem 1.2.

Corollary 5.4. *Let Z be a smooth filling of Y. Suppose that S_Z is negative definite and splits as $S_Z \cong S_0 \oplus \boldsymbol{I}_r$ with S_0 minimal and even. Then $\mathrm{rk}(S_0) \leq \delta(Y)$. In particular, this leaves a finite list of possible isometry classes for S_0.*

Proof. Since S_0 is even, its rank agrees with $d(S_0) = d(S_Z)$ and the bound follows from Theorem 5.2. Moreover, possibly up to a sign the determinant of S_0 agrees with that of S_Z, which is bounded in absolute value by the order of the torsion subgroup of $H_1(Y)$. Since there are only finitely many isomorphism classes of definite forms with given rank and determinant (see [13, p. 18], for example), the result follows. □

It is an interesting question to what extent the assumption that S_0 is even is necessary in Corollary 5.4. In essence, this was already asked by Elkies [5, p. 650].

Question 5.5 (Elkies). *Let S_0 be a minimal (unimodular) lattice. Does an upper bound on $d(S_0)$ imply an upper bound on the rank of S_0?*

As far as we know, this question is still open. Some evidence for an affirmative answer is available in the unimodular case. It should be noted that the defect of a unimodular lattice, as we have defined it, is divisible by 8 as a consequence of van der Blij's lemma [13, p. 24], so the first possible defects are 0, 8, 16, and 24. Elkies showed that there are exactly 14 non-trivial minimal unimodular lattices with $d(L_0) \leq 8$, see [4, 5]; in addition, rank bounds are known for $d(L_0) \leq 24$, see [9] and [14]. Curiously, the lattices with low defect are at the opposite end of the spectrum as even lattices, which realize the highest possible defect for a given rank, for which the rank bound is obvious. Lastly, we remark that the question appears to be completely uncharted territory in the non-unimodular case.

6. Computations and applications

After the abstract algebraic considerations in Section 5 we now turn to more concrete problems. We begin by giving a computation of the twisted correction terms of $\Sigma_g \times S^1$ for a surface Σ_g of arbitrary genus g. For $g \geq 1$ these are arguably the simplest examples of 3-manifolds with non-standard HF^∞ and as such they are not accessible to the previously available (untwisted) correction terms.

6.1. A surface times a circle. Recall from Example 3.5 that $\underline{d}(S^1 \times S^2, \mathfrak{t}_0) = -\frac{1}{2}$ and $\underline{d}(T^3, \mathfrak{t}_0) = \frac{1}{2}$. It turns out that this pattern continues as follows.

Theorem 6.1. *Let Σ_g be a closed, oriented surface of genus g. Then the unique torsion spin^c structure $\mathfrak{t}_0$ on the product $\Sigma_g \times S^1$ satisfies*

$$\underline{d}(\Sigma_g \times S^1, \mathfrak{t}_0) = \begin{cases} -\frac{1}{2} & g \text{ even,} \\ +\frac{1}{2} & g \text{ odd.} \end{cases}$$

In other words, we have

$$\delta(\Sigma_g \times S^1) = \delta(\Sigma_g \times S^1, \mathfrak{t}_0) = 8 \left\lceil \tfrac{g}{2} \right\rceil$$

where $\lceil \cdot \rceil$ is the ceiling function.

We split the proof into two parts. We first exhibit an explicit filling that realises the lower bound $\delta(\Sigma_g \times S^1)$. The second part is an inductive argument based on a computation of $\underline{d}(\Sigma_2 \times S^1, \mathfrak{t}_0)$ which will occupy most of the present section.

Proposition 6.2. *$\Sigma_g \times S^1$ has a smooth filling Z_g with even, negative semidefinite intersection form of rank $b_2^-(Z) = 8 \left\lceil \frac{g}{2} \right\rceil$. In particular, we have $\delta(\Sigma_g \times S^1) \geq 8 \left\lceil \frac{g}{2} \right\rceil$.*

In Lemma 6.9 below we will also determine the intersection form of the 4-manifold Z_g constructed below.

Proof. We first construct a 4-manifold Z'_g as the complement of a (symplectic) genus-g surface of self-intersection 0 in a blow-up of $\mathbb{CP}^2$. We start with a configuration of $g+1$ complex curves of which g are smooth generic conics in a pencil, and the remaining one is a generic line. This configuration has $2g$ double points and four points of multiplicity g. One can resolve the double points in the symplectic category, hence obtaining a symplectic curve with four points of multiplicity $2g$. We now blow up $\mathbb{CP}^2$ at these points, and at $4g+1$ generic points of the curve. Taking the proper transform gives a smooth symplectic curve C of self-intersection 0 in $X = \mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$ in the homology class

$$[C] = (2g+1)h - g(e_1 + \cdots + e_4) - (e_5 + \cdots + e_{4g+5}).$$

The canonical divisor K_X of X is Poincaré dual to $e_1 + \cdots + e_{4g+5} - 3h$, hence the adjunction formula reads

$$0 = \langle K_X, C\rangle + C^2 + \chi(C) = 4g - (4g+1) - (2g+1) + 2 - 2g(C),$$

showing that C has genus $g(C) = g$. In particular, the complement Z'_g of an open, regular neighbourhood of C in X is a filling of $\Sigma_g \times S^1$ and we claim that for odd g it has all the required properties. In fact, it is negative semidefinite, since $C^2 = 0$ and $b_2^+(X) = 1$; moreover, since $b_2^-(X) = 4g+5$, we have that

$$b_2^-(Z'_g) = b_2^-(X) - 1 = 4g + 4.$$

Finally, $[C]$ is easily seen to be characteristic in $H_2(X)$ if g is odd, hence the intersection form on the complement is even: in fact, if $x \in [C]^\perp$, then $x^2 \equiv x \cdot [C] = 0$.

For odd g we can therefore take $Z_g = Z'_g$. For even g use the following trick. Let V_g be a cobordism from Σ_g to Σ_{g+1} obtained by attaching a 3-dimensional 1-handle and let $W_g = V_g \times S^1$. Then the intersection form on $H_2(W_g)$ is trivial and $H_1(\Sigma_g \times S^1)$ injects into $H_1(W_g)$. In particular, a Mayer–Vietoris argument shows that if Z is a filling of $\Sigma_g \times S^1$ with $b_2^+(Z) = 0$, then $Z \cup W_g$ is any filling of $\Sigma_{g+1} \times S^1$ with $b_2^+(Z \cup W_g) = 0$ and $b_2^-(Z \cup W_g) = b_2^-(Z)$. Moreover, if Z has an even intersection form, then so does $Z \cup W_g$. So for g even and positive we let $Z_g = Z_{g-1} \cup W_{g-1}$. □

The second ingredient for our proof of Theorem 6.1 is the following special case.

Proposition 6.3. *The correction term of $\Sigma_2 \times S^1$ with its unique torsion spin^c structure $\mathfrak{t}_0$ is $\underline{d}(\Sigma_2 \times S^1, \mathfrak{t}_0) = -\frac{1}{2}$.*

The computation is lengthy and technical and we postpone it until Section 6.1.1. We first explain how it fits into the proof of Theorem 6.1.

Proof of Theorem 6.1. For brevity we write $Y_g = \Sigma_g \times S^1$ and omit the unique torsion spin^c structure from the notation. We proceed by induction on g. As mentioned in Example 3.5, the computations of $\underline{d}(Y_g)$ for $g = 0$ or 1 are covered in the literature, and the case $g = 2$ is obtained in Proposition 6.3 above.

Suppose now that $g > 2$. There is a cobordism from Y_g to $Y_{g-2} \# Y_2$ obtained by attaching a single 2-handle along a null-homologous knot with framing 0. This is shown in Figure 1: in the top picture, the dashed curve represents the attaching curve of the 2-handle, and the other curves give a surgery presentation for Y_g; the bottom picture is obtained from the one on top by a handleslide, and it shows that the positive boundary of the cobordism is $Y_{g-2}\#Y_2$.

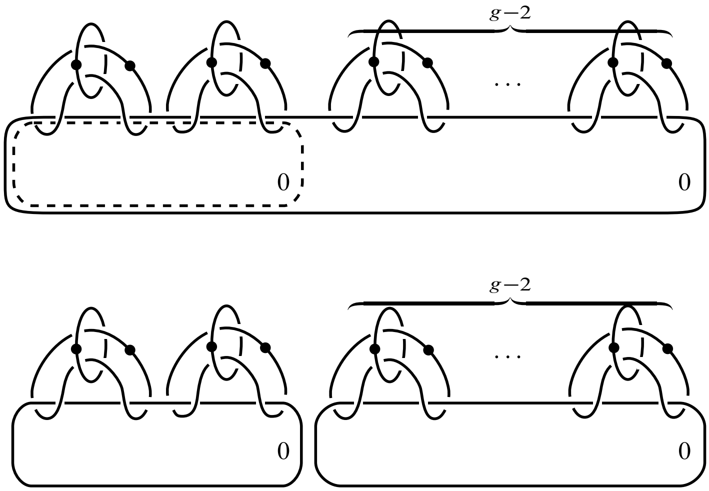


Figure 1. The handleslide.

In particular, the assumptions of Theorem 4.1 are satisfied by this cobordism, and applying additivity we get

$$\begin{aligned} 2\underline{d}(Y_g) + b_1(Y_g) &\leq 2\underline{d}(Y_{g-2}\#Y_2) + b_1(Y_{g-2}\#Y_2) \\ &= 2\underline{d}(Y_{g-2}) + 2\underline{d}(Y_2) + b_1(Y_{g-2}\#Y_2), \end{aligned}$$

showing that $\underline{d}(Y_g) \leq \underline{d}(Y_{g-2})$. On the other hand, Proposition 6.2 ensures that Y bounds an even, negative semidefinite 4-manifold Z with $b_2^-(Z) = 4g+4$ if g is odd and $b_2^-(Z) = 4g$ if g is even. The fact that Z is even implies that we can find a spin^c strucutre $\mathfrak{s} \in \mathrm{Spin}^c(Z)$ with $c_1(\mathfrak{s})$ torsion; hence, applying Theorem 1.1 to $(Z, \mathfrak{s})$, we obtain

$$0 + b_2^-(Z) \leq 4\underline{d}(Y_g) + 2b_1(Y_g),$$

from which we get $\underline{d}(Y_g) \geq \frac{1}{2}$ for g odd, and $\underline{d}(Y_g) \geq -\frac{1}{2}$ for g even. □

6.1.1. Computation of $\underline{d}(\Sigma_2 \times S^1, \mathfrak{t}_0)$. This subsection is devoted to the proof of Proposition 6.3. In what follows, we will denote by K the right-handed trefoil $T_{2,3}$, by K^2 the connected sum of two copies of K, i.e. $K^2 = T_{2,3}\#T_{2,3}$. Also, we will denote by $M(a,b,c,d)$ the manifold obtained by doing surgery along the framed link $\mathbf{L}$ in Figure 2. Notice that the 0-framed component of $\mathbf{L}$ is distinguished, since it is the only component of Seifert genus 2 in the complement of the other components.

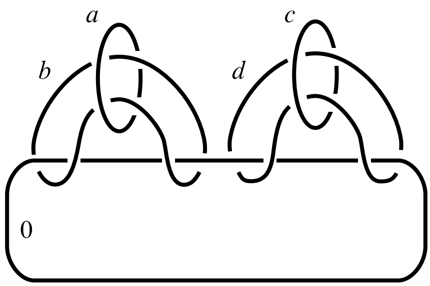


Figure 2. A surgery diagram for $M(a, b, c, d)$.

We note here the following identifications:

$$\begin{aligned} M(\infty, 1, 1, 1) &\cong S^3_0(K), & M(0, \infty, 1, 1) &\cong S^3_0(K)\#(S^2 \times S^1), \\ M(1, 1, 1, 1) &\cong S^3_0(K^2), & M(0, 0, 0, \infty) &\cong T^3\#(S^2 \times S^1), \\ M(0, 0, \infty, 1) &\cong T^3, & M(0, 0, 0, 0) &\cong \Sigma_2 \times S^1. \end{aligned}$$

When a 3-manifold admits a unique torsion spin^c structure (and this is the case for all manifolds in this section, except for one, in the proof of Lemma 6.6), we suppress the spin^c structure from the notation.

Remark 6.4. Note that the connected sum formula for Heegaard Floer homology with twisted coefficients implies that taking a connected sum with a (twisted coefficients) L-space $(Y, \mathfrak{t})$ (i.e. $\underline{HF}^+_{\mathrm{red}}(Y, \mathfrak{t}) = 0$) corresponds to a degree-shift by $\underline{d}(Y, \mathfrak{t})$; in particular, since $S^2 \times S^1$ is a twisted coefficients L-space with correction term $-\frac{1}{2}$, the groups $\underline{HF}^+(M(0, \infty, 1, 1))$ and $\underline{HF}^+(M(0, 0, 0, \infty))$ are easily computed from the corresponding groups $\underline{HF}^+(S^3_0(K))$ and $\underline{HF}^+(T^3)$, respectively. These two latter groups have in fact been computed in [16, Lemma 8.6 and Proposition 8.5].

In what follows, we denote by $\mathbb{F}(s)_d$ the ring $\mathbb{F}[s, s^{-1}]$ of Laurent polynomials in the variable s over the field $\mathbb{F}$, supported in degree d; we denote by $\mathbb{F}(s, t)_d$ the ring $\mathbb{F}[s, s^{-1}, t, t^{-1}]$, supported in degree d. More generally, given a module M over a ring R it will be convenient to write $M(s)$ for the module $M \otimes_R R[s, s^{-1}]$. Also, given an element $r \in R$, we denote by π_r the projection $M \to M/(r-1)M$.

Lemma 6.5. *Identify* $\mathbb{F}[H_2(S^3_0(K))]$ *with* $\mathbb{F}(s)$*. The plus-hat long exact sequence for the twisted Heegaard Floer homology of* $S^3_0(K)$ *reads*:

$$\begin{array}{ccccc}
\underline{\mathrm{HF}}^+(S^3_0(K)) & \longrightarrow & \underline{\widehat{\mathrm{HF}}}(S^3_0(K)) & \longrightarrow & \underline{\mathrm{HF}}^+(S^3_0(K)) \\
\big\downarrow = & & \big\downarrow = & & \big\downarrow = \\
\mathcal{T}^+_{-\frac{1}{2}} \oplus \mathbb{F}(s)_{-\frac{3}{2}} & \xrightarrow{\left(\begin{smallmatrix} 0 & 1-s \\ 0 & 0 \end{smallmatrix}\right)} & \mathbb{F}(s)_{-\frac{1}{2}} \oplus \mathbb{F}(s)_{-\frac{3}{2}} & \xrightarrow{\left(\begin{smallmatrix} \pi_s & 0 \\ 0 & 1 \end{smallmatrix}\right)} & \mathcal{T}^+_{-\frac{1}{2}} \oplus \mathbb{F}(s)_{-\frac{3}{2}}
\end{array}$$

Proof. Recall that K is an L-space knot, and that in fact $S^3_1(K)$ is an L-space with $d(S^3_1(K)) = -2$. It follows that $\mathrm{HF}^+(S^3_1(K)) = \underline{\mathrm{HF}}^+(S^3_1(K)) = \mathcal{T}^+_{-2}$ and $\widehat{\mathrm{HF}}(S^3_1(K)) = \underline{\widehat{\mathrm{HF}}}(S^3_1(K)) = \mathbb{F}_{-2}$.

Consider the surgery exact triangle associated to 0-surgery along K; since the hat-plus long exact sequence is natural with respect to cobordisms, this triangle fits into a long exact sequence of triangles as follows:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \underline{\mathrm{HF}}^+(S^3)(s) & \longrightarrow & \underline{\widehat{\mathrm{HF}}}(S^3)(s) & \longrightarrow & \underline{\mathrm{HF}}^+(S^3)(s) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \underline{\mathrm{HF}}^+(S^3_0(K)) & \longrightarrow & \underline{\widehat{\mathrm{HF}}}(S^3_0(K)) & \longrightarrow & \underline{\mathrm{HF}}^+(S^3_0(K)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \underline{\mathrm{HF}}^+(S^3_1(K))(s) & \longrightarrow & \underline{\widehat{\mathrm{HF}}}(S^3_1(K))(s) & \longrightarrow & \underline{\mathrm{HF}}^+(S^3_1(K))(s) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Observe that the horizontal maps from plus to hat have degree $+1$, the next ones have degree 0 and the following ones have degree -2; that the vertical maps from $S^3_1(K)$ to S^3 are a sum of maps of non-negative degree, while all the other ones involving the torsion spin^c structure on $S^3_0(K)$ have degree $-\frac{1}{2}$. Finally, since $\underline{\mathrm{HF}}^\infty(S^3_0(K)) = \mathcal{T}^\infty$, we also obtain that $\underline{\mathrm{HF}}^+(S^3_0(K))$ contains a single tower. It follows that the vertical map $\underline{\mathrm{HF}}^+(S^3_1(K))(s) \to \underline{\mathrm{HF}}^+(S^3)$ is (up to a sign) multiplication by $U(1-t)$. An easy diagram chase completes the proof. □

Lemma 6.6. *Identify* $\mathbb{F}[H_2(S^3_0(K^2))]$ *with* $\mathbb{F} = \mathbb{F}(s)$*. The plus-hat long exact sequence for the twisted Heegaard Floer homology of* $S^3_0(K^2)$ *in the torsion spin*^{c} *structure reads*:

$$\begin{array}{ccccc}
\underline{\mathrm{HF}}^+(S^3_0(K^2)) & \longrightarrow & \underline{\widehat{\mathrm{HF}}}(S^3_0(K^2)) & \longrightarrow & \underline{\mathrm{HF}}^+(S^3_0(K^2)) \\
\Big\downarrow = & & \Big\downarrow = & & \Big\downarrow = \\
\mathcal{T}^+_{-\frac{1}{2}} \oplus \mathbb{F}_{-\frac{3}{2}} \oplus \mathbb{F}_{-\frac{5}{2}} & \xrightarrow{\begin{pmatrix} 0 & 1-s & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} & \mathbb{F}_{-\frac{1}{2}} \oplus \mathbb{F}^2_{-\frac{3}{2}} \oplus \mathbb{F}_{-\frac{5}{2}} & \xrightarrow{\begin{pmatrix} \pi_s & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} & \mathcal{T}^+_{-\frac{1}{2}} \oplus \mathbb{F}_{-\frac{3}{2}} \oplus \mathbb{F}_{-\frac{5}{2}}
\end{array}$$

Proof. This is analogous to the proof of Lemma 6.5, hence here we only outline the differences. K^2 is not an L-space knot, but the Heegaard Floer homology of $S^3_{12}(K^2)$ was computed in [17, Lemma 4.1], at least in the spin^c structure which is relevant for the computation of $\mathrm{HF}^+(S^3_0(K^2))$, and which is relevant to us (called $Q(0)$ in loc. cit.). Namely:

$$\mathrm{HF}^+(S^3_{12}(K^2), Q(0)) = \mathbb{F}_{-\frac{3}{4}} \oplus \mathcal{T}^+_{-\frac{3}{4}}$$

Instead of using the surgery exact triangle for $+1$-surgery, we use the triangle for twisted $+12$-surgery, where the degrees of the vertical maps are $-\frac{9}{4}$ (from 0-surgery to 12-surgery), $-\frac{11}{4}$ (from 12-surgery to S^3) and $-\frac{1}{2}$ (from S^3 to the 0-surgery). A diagram chase as above proves the lemma. □

Lemma 6.7. $\underline{\mathrm{HF}}^+_{\mathrm{red}}(M(0,1,1,1))$ *is supported in degrees at most* -2*, and*

$$\underline{d}(M(0,1,1,1)) = -1.$$

Proof. We need to set up some notation. Let

$$Y = M(0,1,1,1);$$

$H_2(Y)$ is generated by classes s and t, where s is represented by a capped-off Seifert surface for the marked component of $\mathbf{L}$, and t is represented by a capped-off Seifert surface for the first component of $\mathbf{L}$ (i.e. the one with framing 0 in this surgery). This identifies $\mathbb{F}[H_2(Y)]$ with $\mathbb{F}(s,t)$.

Since Y fits into an surgery triangle with $S^3_0(K) = M(\infty,1,1,1)$ and $S^3_0(K^2) = M(1,1,1,1)$, we have the following long exact sequence of exact triangles, as in the proof of Lemma 6.5:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \underline{\mathrm{HF}}^{+}(S^3_0(K))(t) & \xrightarrow{\alpha} & \underline{\widehat{\mathrm{HF}}}(S^3_0(K))(t) & \xrightarrow{\beta} & \underline{\mathrm{HF}}^{+}(S^3_0(K))(t) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \underline{\mathrm{HF}}^{+}(Y) & \longrightarrow & \underline{\widehat{\mathrm{HF}}}(Y) & \longrightarrow & \underline{\mathrm{HF}}^{+}(Y) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \underline{\mathrm{HF}}^{+}(S^3_0(K^2))(t) & \xrightarrow{\alpha^2} & \underline{\widehat{\mathrm{HF}}}(S^3_0(K^2))(t) & \xrightarrow{\beta^2} & \underline{\mathrm{HF}}^{+}(S^3_0(K^2))(t) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^+} & & \downarrow{\scriptstyle \widehat{F}} & & \downarrow{\scriptstyle F^+} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Notice that the "new" variable t is the one associated with the *second* Seifert surface, since the first Seifert surface generates the homology of $S^3_0(K)$ and $S^3_0(K^2)$.

Notice that, since $\underline{\mathrm{HF}}^{+}_{\mathrm{red}}(S^3_0(K))$ is supported in degree $-\frac{3}{2}$ and the map from $\underline{\mathrm{HF}}^{+}(S^3_0(K))$ to $\underline{\mathrm{HF}}^{+}(Y)$ has degree $-\frac{1}{2}$, the image of $\underline{\mathrm{HF}}^{+}(S^3_0(K))(t)$ in $\underline{\mathrm{HF}}^{+}_{\mathrm{red}}(Y)$ is supported in degrees at most -2. In order to prove the statement, it is therefore enough to prove that the image of $\underline{\mathrm{HF}}^{+}_{\mathrm{red}}(Y)$ in $\underline{\mathrm{HF}}^{+}(S^3_0(K^2))(t)$ is supported in degrees at most $-\frac{5}{2}$ since the map $\underline{\mathrm{HF}}^{+}(Y) \to \underline{\mathrm{HF}}^{+}(S^3_0(K^2))(t)$, too, has degree $-\frac{1}{2}$. This is in turn equivalent to showing that the vertical map starting from $\mathbb{F}(s,t)_{-\frac{3}{2}} \subset \underline{\mathrm{HF}}^{+}(S^3_0(K^2))$ is nonzero. Let $x_0 := 1 \in \mathbb{F}(s,t)_{-\frac{3}{2}}$.

For degree reasons, the image $F^+(x_0)$ of x_0 in $\underline{\mathrm{HF}}^{+}(S^3_0(K))$ lies in the reduced part, which is a copy of $\mathbb{F}(s,t)$. Hence, it is torsion if and only if it vanishes; our assumption becomes that $F^+(x_0) = 0$.

Observe that the map F^+ restricts to multiplication by ± 1 on the tower $\mathcal{T}^+(t)$, as in the proof of Lemma 6.5. Since the bottom-most element of the tower is in the image of β^2, by commutativity of the diagram, it follows that the restriction of $\widehat{F}$ to the subspace $\mathbb{F}(s,t)_{-\frac{1}{2}}$ of $\underline{\widehat{\mathrm{HF}}}(S^3_0(K^2))$ does not vanish.

For the same reason, since $\alpha^2(x_0)$ lies in the same subspace, we obtain that $\alpha(F^+(x_0))$ does not vanish either, hence $F^+(x_0) \neq 0$, as required.

Finally, notice that the image of F^+ cannot contain the bottom-most element of the tower of $\underline{\mathrm{HF}}^{+}(S^3_0(K))$: the restriction of F^+ onto the tower of $\underline{\mathrm{HF}}^{+}(S^3_0(K^2))$ certainly does not, and the reduced part is supported in lower degrees. □

Proof of Proposition 6.3. The argument here will be similar to the one seen in the proofs of the three lemmas above. We first claim that $\underline{\mathrm{HF}}^+_{\mathrm{red}}(M(0,0,1,1))$ is supported in degrees at most $-\frac{3}{2}$ and that $\underline{d}(M(0,0,1,1)) = -\frac{3}{2}$: in fact, $M(0,0,1,1)$ fits into a surgery triple with $M(0,1,1,1)$ and $M(0,\infty,1,1) \cong S^3_0(K)\#(S^2 \times S^1)$, which gives the following long exact sequence in Heegaard Floer homology:

$$\cdots \longrightarrow \underline{\mathrm{HF}}^+(M(0,\infty,1,1))(t) \longrightarrow \underline{\mathrm{HF}}^+(M(0,0,1,1)) \\ \longrightarrow \underline{\mathrm{HF}}^+(M(0,1,1,1))(t) \longrightarrow \cdots.$$

Here t is the new generator corresponding to the new class, and needs not be confused with the t used above.

However, combining Lemma 6.5 and Remark 6.4 we obtain that the reduced part of the leftmost group is supported in degrees at most -2 and the same holds for the rightmost group, thanks to Lemma 6.7. The same argument as above shows that $\underline{\mathrm{HF}}^+_{\mathrm{red}}(M(0,0,1,1))$ is supported in degrees at most $-\frac{3}{2}$ and allows for the computation of $\underline{d}$.

An analogous computation, combined with [16, Proposition 8.5], shows that $\underline{\mathrm{HF}}^+_{\mathrm{red}}(M(0,0,0,1))$ is supported in negative degrees and that $\underline{d}(M(0,0,0,1)) = 0$; this time, however, the map from the tower of $\underline{\mathrm{HF}}^+(M(0,0,1,1))(t)$ is no longer injective, so one needs to be more careful. Let us now look at the surgery triple involving $M(0,0,0,0)$, $M(0,0,0,1)$ and $M(0,0,0,\infty)$: we have

$$\cdots \longrightarrow \underline{\mathrm{HF}}^+(M(0,0,0,\infty))(t) \longrightarrow \underline{\mathrm{HF}}^+(M(0,0,0,0)) \\ \longrightarrow \underline{\mathrm{HF}}^+(M(0,0,0,1))(t) \longrightarrow \cdots.$$

and, as above, using Remark 6.4 and the computation of $\underline{\mathrm{HF}}^+(T^3)$, we conclude the proof of the proposition. □

6.1.2. Intersection forms of fillings of $\Sigma_g \times S^1$. With Theorems 5.2 and 6.1 at our disposal, we have a concrete restriction for intersection forms of smooth fillings of $\Sigma_g \times S^1$ on which we now elaborate. Note that since $H_1(\Sigma_g \times S^1)$ is torsion-free, the non-degenerate intersection form S_Z of any filling Z is automatically unimodular. Furthermore, if Z is smooth and S_Z is negative definite, which we henceforth assume, then its defect is bounded by $d(S_Z) \le 8\left\lceil \frac{g}{2} \right\rceil$. As before we write S_Z as $S_0 \oplus \boldsymbol{I}_r$ with S_0 minimal and ask and ask what the possibilities for S_0 are. Obviously, the trivial form is realised by $\Sigma_g \times D^2$. In the proof of Theorem 6.1 we constructed a filling Z_g with a more interesting intersection form which we now determine.

Example 6.8. Recall that the vectors of the form $e_i + e_j$ and $\frac{1}{2}(e_1 + \cdots + e_n)$ in $\mathbb{R}^n$ generate a lattice $\Gamma_n \subset \mathbb{R}^n$ which is unimodular for $n = 4k$, even for $n = 8k$, and odd for $n = 8k + 4$. Moreover, one can show that Γ_{4k} is irreducible (hence minimal) and satisfies $d(\Gamma_{8k}) = d(\Gamma_{8k+4}) = 8k$. Recall that Z_g was obtained as the complement of a genus g surface in $\mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}}^2$ in the homology class $x = (2g+1)h - g(e_1 + \cdots + e_4) - (e_5 + \cdots + e_{4g+5})$. By Lemma 6.9 below S_{Z_g} is isomorphic to Γ_{4g+4}. Moreover, for any $h < g$ we can extend Z_h to a filling of Z_g by adding a cobordism as in the proof of Theorem 6.1. By blowing up these fillings we can realise the forms $\Gamma_{4h+4} \oplus \boldsymbol{I}_r$ with $h \leq g$ and $r \geq 0$ by smooth fillings of $\Sigma_g \times S^1$.

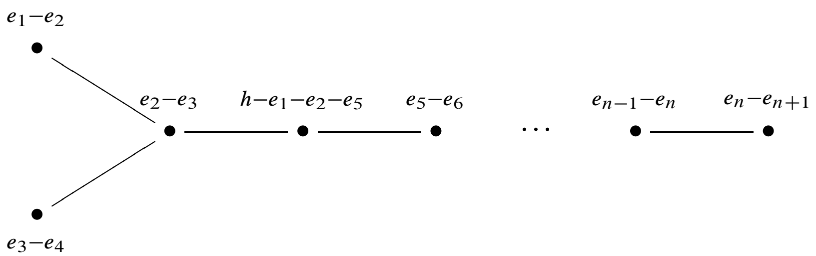
Lemma 6.9. *Fix a positive integer* g *(not necessarily odd), and let* $\langle x \rangle$ *be the sublattice of* $H_2(\mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}}^2)$ *generated by the class*

$$x = (2g+1)h - g(e_1 + e_2 + e_3 + e_4) - (e_5 + \cdots + e_{4g+5}).$$

The lattice $Q = \langle x \rangle^{\perp}/\langle x \rangle$ *is of type* Γ_{4g+4}.

Proof. For brevity we let $n = 4g + 4$. As seen in Example 6.8, Q is the non-degenerate intersection form of the 4-manifold Z_g whose boundary has torsion-free homology. Hence, Q is unimodular. Moreover, Q is a *root lattice*, since the associated vector space is generated by the set $\mathcal{R}$ of *roots* of Q (i.e. elements of square -2)

$$\mathcal{R} = \{e_1 - e_2,\, e_2 - e_3,\, e_3 - e_4,\, e_5 - e_6, \ldots, e_n - e_{n+1},\, h - e_1 - e_2 - e_5\}.$$



The roots in $\mathcal{R}$ intersect according to the Dynkin diagram D_n shown above; therefore, $\mathcal{R}$ generates a sublattice $Q' \subset Q$ of index 2 isomorphic to the lattice D_n. We think of D_n as sitting in $\mathbb{R}^n$ (with orthonormal basis $\{f_1, \ldots, f_n\}$), generated by the roots $\{f_1 + f_2, f_1 - f_2, \ldots, f_{n-1} - f_n\}$.

In particular, the two "short legs" of the Dynkin diagram are f_1+f_2 and f_1-f_2. Recall that there are only two unimodular overlattices of D_n up to isomorphism: $\mathbb{Z}^n$ and Γ_n, both sitting in $\mathbb{R}^n$; see [3, Section 1.4]. The overlattice is $\mathbb{Z}^n$ if and only if it contains f_1, i.e. only if it contains half of the sum of the two "short legs" of the Dynkin diagram, and it is Γ_n otherwise. Since the action of the Weyl group on the set of fundamental sets of roots is transitive, we may assume that the isomorphism between Q' and D_n identifies the two chosen bases. The two short legs of $\mathcal{R}$ are e_1-e_2 and e_3-e_4, and their sum y is represented by vectors $y+X$ none of which divisible by 2 in Q: if k is odd $\langle h, y+kx\rangle$ is odd, and if k is even $\langle e_1, y+kh\rangle$ is odd. Hence Q is isomorphic to Γ_n. □

For fillings of T^3 and $\Sigma_2 \times S^1$ we have $d(S_Z) = d(S_0) \leq 8$. As mentioned at the end of Section 5, this leaves 14 possibilities for S_0, assuming that it is non-trivial (see Elkies' list in [4, p. 326]). Among these we find $\Gamma_8 \cong E_8$ and Γ_{12} from Example 6.8 above. It is well-known that E_8 is the only non-trivial even lattice of rank at most 8. As an immediate consequence we get a slightly stronger version of Theorem 1.3.

Corollary 6.10. *Let Z be a smooth filling of T^3 or $\Sigma_2 \times S^1$ with S_Z negative definite of the form $S_0 \oplus \boldsymbol{I}_r$ with S_0 even. Then S_0 is either trivial or isomorphic to E_8, and both cases occur.*

All other lattices in Elkies' list are odd of rank at least 12. We have seen in Example 6.8 that Γ_{12} is realised by a smooth filling of $\Sigma_2 \times S^1$. However, we do not know whether it also appears for T^3; Theorem 5.2 does not provide any obstruction in this case. We therefore ask:

Question 6.11. *Can Γ_{12} be realised by smooth fillings of T^3?*

Of course, there is the more general question of which odd lattices in Elkies' list, if any, appear for T^3. This should be compared to Frøyshov's work on fillings of the Poincaré sphere [7, Proposition 2].

Our findings about T^3 and $\Sigma_2 \times S^1$ leave the possibility that Theorem 5.2 is the only obstruction for realising even lattices. In order to test this we consider $\Sigma_3 \times S^1$ and $\Sigma_4 \times S^1$ for which $d(S_Z) \leq 16$. The only non-trivial even lattices of rank at most 16 are E_8, $E_8 \oplus E_8$, and Γ_{16} (see [2, Table 16.7]). In Example 6.8 we have realised all but $E_8 \oplus E_8$.

Question 6.12. *Can $E_8 \oplus E_8$ be realised by a smooth filling of $\Sigma_3 \times S^1$ or $\Sigma_4 \times S^1$?*

In the same spirit one may wonder which of the 24 even lattices of rank 24 (see [2, Chapter 18] for a list) are realised by fillings of $\Sigma_5 \times S^1$ or Σ_6. For example:

Question 6.13. *Can the Leech lattice be realised by a smooth filling of* $\Sigma_5 \times S^1$?

Another way to look at this question is which is the simplest 3-manifold that can appear on the boundary of a smooth 4-manifold whose non-degenerate intersection form is a given lattice, for example the Leech lattice. In this special case $\Sigma_5 \times S^1$ might be considered a satisfactory answer.

If we increase the genus further, thereby further weakening our bound on $d(S_Z)$, we soon enter uncharted algebraic territory. As mention above, up to defect 24 there is a manageable list of even unimodular lattices and the number of minimal lattices is known to be finite but possibly large [9, Chapter 5]. Beyond this range the number of lattices allowed by Theorem 5.2 explodes, rendering any attempt of an enumeration extremely difficult if not impossible. As a consequence, without further obstructions there is little hope for a classification of all lattices that can appear as non-degenerate intersection forms of fillings of 3-manifolds Y for which $\delta(Y)$ is large.

6.2. Embeddings into closed 4-manifolds. In this section, P will be a fixed integral homology sphere such that $d := d(P, \mathfrak{t}_0) \neq 0$ for the unique spin^c structure $\mathfrak{t}_0$ on P. In what follows, the spin^c structure will be omitted from the notation whenever possible; also, nP will denote the connected sum of $|n|$ copies of P or $-P$ (i.e. P with the reversed orientation) depending on the sign of n. For example, the Poincaré sphere satisfies these requirements, since $d(S^3_{+1}(T_{2,3})) = -2$.

Proposition 6.14. *Fix a* 3*-manifold* Y *and a closed, smooth* 4*-manifold* X *with definite intersection form. There exists an integer* N*, depending only on* Y*, such that* $Y_n = Y \# nP$ *does not embed in* X *as a separating hypersurface for* $|n| > N$.

Proof. Notice that the statement is independent of the orientation of P, thus we can pick the orientation of P for which $d > 0$. In particular, as P is an integral homology sphere, d is an even integer, hence $d \geq 2$. Similarly, we can assume that X is negative definite.

For each n there is an isomorphism $\mathrm{Spin}^c(Y) \to \mathrm{Spin}^c(Y_n)$ defined by $\mathfrak{t} \mapsto \mathfrak{t}\#\mathfrak{t}_n$ that carries torsion spin^c structures to torsion spin^c structures, where $\mathfrak{t}_n$ is the unique spin^c structure on nP.

Now assume that there is a separating embedding $Y_n \hookrightarrow X$. Let Z and Z' be the closures of the connected components of $X \setminus Y_n$, labelled so that $\partial Z = -\partial Z' = Y_n$. Notice that both Z and Z' are negative semidefinite so that Corollary 5.3 yields $\delta(Y_n) \geq 0$ and also $\delta(-Y_n) \geq 0$. Since correction terms are additive, so is δ, hence

$$0 \leq \delta(Y_n) = \delta(Y) + 4nd \quad \text{and} \quad 0 \leq \delta(-Y_n) = \delta(-Y) - 4nd.$$

Since $d \geq 2$, it follows that $-\delta(Y)/4d \leq n \leq \delta(-Y)/4d$. We can now choose

$$N = \left\lfloor \max\left\{\frac{\delta(Y)}{8}, \frac{\delta(-Y)}{8}\right\}\right\rfloor. \qquad \square$$

Note that if $\delta(Y)$ and $\delta(-Y)$ are both negative, then N can be chosen to be -1, hence Y_n never embeds in a closed definite 4-manifold.

Example 6.15. Let $Y = \Sigma_g \times S^1$ and P the Poincaré homology sphere; since $\delta(Y) = 8\lceil\frac{g}{2}\rceil$ and $d(P) = -2$, we get that $Y\#nP$ does not embed in a negative definite 4-manifold as a separating hypersurface if $n > \lceil\frac{g}{2}\rceil$. In particular, when Y is either the 3-torus T^3 or $\Sigma_2 \times S^1$, this shows that $Y\#nP$ cannot be embedded in a negative definite 4-manifold X whenever $n \geq 2$.

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