HOMFLY-PT and Alexander polynomials
from a doubled Schur algebra

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Abstract. We define a generalization of the Schur algebra which gives a unified setting for a quantum group presentation of the HOMFLY-PT polynomial, together with its specializations to the Alexander polynomial and to the $\mathfrak{sl}_m$ Reshetikhin–Turaev invariant.

1. Introduction

If two knot invariants were to be cited, they would surely be the Alexander and Jones polynomials. The latter gave rise to a whole family of quantum invariants [17], built from maps of representations of the quantum group $U_q(\mathfrak{g})$ attached to a semisimple Lie algebra $\mathfrak{g}$, and has been the corner stone of quantum topology. Although originating from a much more topological setting, the Alexander polynomial, whose developments are also still investigated, is known to admit a quantum group based description as well (see [20, 19] and references therein).

A unifying generalization of these two polynomials is provided by the two-variable HOMFLY-PT polynomial [5, 13]. Unfortunately, the quantum group description does not seem to extend very well to this context, mostly because the expected appearance of infinite-dimensional representations (for example of $U_q(\mathfrak{sl}_\infty)$) makes it impossible to define some of the basic intertwiners.

An alternative, or, more precisely, dual description of the $\mathfrak{sl}_m$ quantum invariants has been recently introduced by Cautis, Kamnitzer and Morrison [1] using

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skew Howe duality. In a Schur–Weyl flavored process, they described the intertwiners of the $U_q(\mathfrak{sl}_m)$–representations involved in the construction of the knot invariants as images of elements of a Howe dual quantum group $U_q(\mathfrak{gl}_k)$. As a consequence, they were able to give the first complete description by generators and relations of the monoidal subcategory of $U_q(\mathfrak{sl}_m)$ representations generated by exterior powers of the vector representation. A similar description using super skew Howe duality is also possible for the Lie superalgebra $\mathfrak{gl}_{1|1}$, which yields the Alexander polynomial, see [8] and [18]. In both cases, the intertwiners actually live in the so-called Schur algebra associated to the quantum group $U_q(\mathfrak{gl}_k)$. A major difference between the two cases, however, is that the dual of the vector representation of $\mathfrak{sl}_m$ is isomorphic to an exterior power of the vector representation itself, while this is not true for $\mathfrak{gl}_{1|1}$. In terms of link invariants, this means that, in the case of $\mathfrak{gl}_{1|1}$, using skew Howe duality one can only define a braid invariant. In order to obtain the actual link invariant (i.e. the Alexander polynomial), one needs to make sense of closing the braid. This closure process can be realized inside the Howe dual quantum group for the case of $\mathfrak{sl}_m$, but not in the case of $\mathfrak{gl}_{1|1}$. A similar story applies to the HOMFLY-PT polynomial: as a braid invariant, one can define it inside the Howe dual quantum group of $\mathfrak{sl}_M$ for $M \gg 0$ (ideally $M = \infty$), but one cannot perform the closure of the braid inside this quantum group.

In this paper, the first of a series of two [16], we introduce a doubling of the Schur algebra specifically designed to overcome this problem and provide a unified representation theoretical setting for the HOMFLY-PT polynomial and all its specializations. In particular, our doubled Schur algebra extends the Howe dual quantum group described above and makes it possible to perform internally the braid closure process for the $\mathfrak{sl}_m$ invariant as well as for the Alexander and the HOMFLY-PT polynomial. Indeed, the link invariants are defined directly inside the doubled Schur algebra.

We expect that our method could be helpful for further investigation of spectral sequences between $\mathfrak{sl}_m$ and triply graded link homology (see [7] and references therein). The dual approach we adopt here could more generally shed light on representation-theory-based categorifications of the HOMFLY-PT polynomial and of the Alexander polynomial. The latter could be particularly interesting for finding a connection with Heegaard–Floer link homology [12], a homology theory coming from symplectic geometry which categorifies the Alexander polynomial.

This first short paper is devoted to defining the doubled Schur algebra and the link invariants with values in such doubled Schur algebras. Let us quickly recall the definition of the link invariants we are interested in. We stress that we will always consider oriented framed links. First, the HOMFLY-PT polynomial is

the link invariant uniquely defined by the property of being multiplicative on the disjoint union of two links, by the skein formula

$$\text{(diagram)} - \text{(diagram)} = (q^{-1} - q)\,\text{(diagram)} \tag{†}$$

and by the following value for the unknot:

$$\text{(diagram)} = \frac{q^\beta - q^{-\beta}}{q - q^{-1}},$$

where q^β is a formal parameter. Note that this is a rescaled version of the usual definition. The $\mathfrak{sl}_m$ link invariant is obtained by replacing $q^\beta = q^m$. Reduced versions of these polynomials can be obtained by dividing by the value of the unknot.

Although this is not the original definition, the Alexander polynomial can be defined in a similar way by the skein relation (†) and by assigning the value 1 to the unknot. However, the Alexander polynomial is not multiplicative; its value on a split link is zero.

Our doubled Schur algebra is a quotient of the idempotented version of the quantized enveloping algebra of $\mathfrak{gl}_m$, and it depends on a variable β, which can be either an integer or a formal (*generic*) parameter. Our definition is similar to the definition of the so-called *generalized Schur algebras*, which are quotients of the idempotented version of $U(\mathfrak{gl}_m)$ where only a finite set of weights survive, and were introduced by Donkin [2] (for an arbitrary reductive Lie algebra) and studied extensively also by other authors (see for example [4] in the quantized case). Our doubled Schur algebra, however, is infinite dimensional, since an unbounded *infinite* set of weights is allowed. Indeed, while the generalized Schur algebra describes a finite set of irreducible finite dimensional representations, our doubled Schur algebra is the analogue for a set of *infinite dimensional* representations (see [16, §7]).

We develop a process to assign to the closure $\hat{\mathbf{b}}$ of a braid $\mathbf{b}$ an element $P_\beta(\hat{\mathbf{b}})$ of this doubled Schur algebra. Depending on the value of β, we recover either the HOMFLY-PT polynomial or the Reshetikhin–Turaev invariant attached to the Lie algebra $\mathfrak{sl}_m$:

Theorem 3.5. *Let L be a framed oriented link. If β is generic, then $P_\beta(L)$ is the HOMFLY-PT polynomial of L. If $\beta = m \in \mathbb{Z}$, with $m \geq 2$ then $P_m(L)$ is the $\mathfrak{sl}_m$ link invariant of L.*

The Alexander polynomial can also be obtained in the doubled Schur algebra. For a link L presented as the closure of a braid $\mathbf{b}$, we denote $\widetilde{\mathbf{b}}$ the closure of all but one of the strands of $\mathbf{b}$, and $\widetilde{P}_0(L)$ the associated element of the doubled Schur algebra for $\beta = 0$.

Corollary 4.3. *Let L be an oriented link. Then $\widetilde{P}_0(L)$ is the Alexander polynomial of L.*

Finally, the same process in the cases β generic or $\beta = m \geq 2$ yields the reduced version of the corresponding polynomials.

Proposition 4.4. *Let L be an oriented framed link.*

(1) *If β is generic, then $\widetilde{P}_\beta(L)$ is the reduced HOMFLY-PT polynomial of L, while*

(2) *if β is specialized to $m \in \mathbb{Z}_{>0}$, then $\widetilde{P}_m(L)$ is the reduced $\mathfrak{sl}_m$ invariant of L.*

We refer to the sequel [16] for the investigation of the representation-theoretical version of this story and its consequences, as well as the full generalization of the invariants to tangles, which best comes after exploring the formal properties of the doubled Schur algebra. However, in the last section of the present paper we briefly discuss how our main results extend to the case of colored invariants of knots and links. In addition, we go beyond the case of braid closures and consider any colored link diagrams. We thus obtain analogs of Theorem 3.5, Corollary 4.3 and Proposition 4.4 for general link diagrams in the colored case. The goal of this last part is to give a short and practical explanation of the process.

The reason for us to restrict at first to the case of braid closures is because it allows for short and easy proofs of well-definedness and invariance, while the proofs of the last section rely on some more subtle arguments from [16]. Nonetheless, the process is very similar and allows to give constructions not only of the $\mathfrak{sl}_m$ link invariants, but also of the Alexander polynomial and of the HOMFLY-PT polynomial, starting from any diagram, in the general context of quantum groups. Once again, since the categorification of quantum groups has been widely studied, we believe that our construction can open new categorification perspectives for the Alexander and the HOMFLY-PT polynomial.

2. Doubled Schur algebra

In the following, we will denote by β either an integer or a formal parameter (in the last case, we will say that β is *generic*). We will work over the ring $\mathbf{k} = \mathbb{C}(q)[q^{\pm\beta}]$. If β is integer, this is just the field of rational functions $\mathbb{C}(q)$. If β is generic, then $\mathbb{C}(q)[q^{\pm\beta}]$ is the ring of Laurent polynomial over $\mathbb{C}(q)$ in the variable q^β.

We will allow the usual algebraic manipulations, for example we will write $q^{k+\beta}$ for $q^k q^\beta$.

For $x \in \mathbb{Z}\beta + \mathbb{Z}$ and $k \in \mathbb{N}$ we define

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2.1}$$

$$[k]! = [k][k-1]\cdots[1], \tag{2.2}$$

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{[x][x-1]\cdots[x-k+1]}{[k][k-1]\cdots[1]}. \tag{2.3}$$

Let $\mathsf{P} = \mathsf{P}_{k,l}^{\beta}$ be the set of sequences $\lambda = (\lambda_1, \ldots, \lambda_{k+l})$ with $\lambda_i \in \mathbb{N}$ for $i \leq k$ and $\lambda_i \in \beta - \mathbb{N}$ for $i \geq k+1$. We let also $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \in \mathbb{Z}^{k+l}$, the entry 1 being at position i.

Definition 2.1. We define the doubled Schur algebra $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta$ to be the $\mathbf{k}$-linear category with:

- **objects:** (formal finite direct sums of) symbols $\mathbf{1}_\lambda$ for $\lambda \in \mathsf{P}$;
- **morphisms:** generated by identity endomorphisms in $\mathrm{Hom}(\mathbf{1}_\lambda, \mathbf{1}_\lambda)$, which by a slight abuse of notation we also denote by $\mathbf{1}_\lambda$, and morphisms

$$\mathbf{1}_{\lambda+\alpha_i} E_i \mathbf{1}_\lambda \in \mathrm{Hom}(\mathbf{1}_\lambda, \mathbf{1}_{\lambda+\alpha_i}), \qquad \mathbf{1}_{\lambda-\alpha_i} F_i \mathbf{1}_\lambda \in \mathrm{Hom}(\mathbf{1}_\lambda, \mathbf{1}_{\lambda-\alpha_i}).$$

The morphisms are subject to the following relations:

(1) $[E_i, F_j]\mathbf{1}_\lambda = \delta_{i,j}[\lambda_i - \lambda_{i+1}]\mathbf{1}_\lambda$;

(2) $E_i E_j \mathbf{1}_\lambda = E_j E_i \mathbf{1}_\lambda$ and $F_i F_j \mathbf{1}_\lambda = F_j F_i \mathbf{1}_\lambda$ if $|i - j| > 1$;

(3) $E_i^2 E_j \mathbf{1}_\lambda - (q + q^{-1}) E_i E_j E_i \mathbf{1}_\lambda + E_j E_i^2 \mathbf{1}_\lambda = 0$ if $j = i \pm 1$,
    $F_i^2 F_j \mathbf{1}_\lambda - (q + q^{-1}) F_i F_j F_i \mathbf{1}_\lambda + F_j F_i^2 \mathbf{1}_\lambda = 0$ if $j = i \pm 1$.

We indifferently use $\mathbf{1}_{\lambda+\alpha_i} E_i \mathbf{1}_\lambda = E_i \mathbf{1}_\lambda = \mathbf{1}_{\lambda+\alpha_i} E_i$ (and similarly for F_i), since knowing the source of a morphism is enough to determine its target, and vice versa. In the equations above we set $\mathbf{1}_\lambda = 0$ if $\lambda \notin \mathsf{P}$ (and we say in this case that λ is not admissible), and this convention will be used throughout the paper. Moreover, we will denote $\mathrm{Hom}(\mathbf{1}_\lambda, \mathbf{1}_\mu)$ by $\mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1}_\lambda$. Notice that $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta$ can also be interpreted as an algebra $\bigoplus_{\lambda,\mu} \mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1}_\lambda$.

Remark 2.2. Let us denote by $U_q(\mathfrak{gl}_{k+l})_\eta$ the quantized enveloping algebra of $\mathfrak{gl}_{k+l}$ over **k**. Let also $\dot{U}_q(\mathfrak{gl}_{k+l})_\eta$ be its idempotented version, obtained roughly speaking by adjoining pairwise orthogonal idempotents $\mathbf{1}_\lambda$, where λ runs over all sequences $\lambda = (\lambda_1, \ldots, \lambda_{k+l})$ with $\lambda_i \in \mathbb{Z} \sqcup \beta + \mathbb{Z}$. The Cartan subalgebra acts on $\mathbf{1}_\lambda$ as usual by the weight λ. Then $\dot{U}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta$ is obtained as a quotient of $\dot{U}_q(\mathfrak{gl}_{k+l})_\beta$ by killing some of the weights. In particular, we can apply standard results about quantum enveloping algebras (like the PBW theorem or the triangular decomposition) to our doubled Schur algebra.

In case $l = 0$, we will use for the category $\dot{U}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta$ the usual notation $\dot{U}_q^{\geq 0}(\mathfrak{gl}_k)$.

Remark 2.3. Notice that $\dot{U}_q^{\geq 0}(\mathfrak{gl}_k)$ is just the quotient of the idempotented quantum group $\dot{U}_q(\mathfrak{gl}_k)$ modulo all weights with an entry smaller than zero. As well known, we have

$$\dot{U}_q^{\geq 0}(\mathfrak{gl}_k) = \bigoplus_{N \geq 0} \mathcal{S}(k, N), \tag{2.4}$$

where $\mathcal{S}(k, N)$ is the quantum Schur algebra of degree N associated to $\mathfrak{gl}_k$ (see [3]). We have a similar decomposition in the doubled case:

$$\dot{U}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta = \bigoplus_{N \in l\beta + \mathbb{Z}} \dot{U}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta^N, \tag{2.5}$$

where $\dot{U}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta^N$ is the full subcategory with objects $\mathbf{1}_\lambda$ such that $\sum_i \lambda_i = N$. Note that, in contrast to the usual case (2.4), in this decomposition each subcategory has infinitely many objects (provided $l \neq 0$). For example, $\dot{U}_q(\mathfrak{gl}_{\eta_{1,1}})_\beta^0$ has objects $(p, \beta - p)$ for all $p \in \mathbb{N}$.

We define the divided powers

$$E_i^{(a)}\mathbf{1}_\lambda = \frac{1}{[a]!}E_i^a\mathbf{1}_\lambda, \quad F_i^{(a)}\mathbf{1}_\lambda = \frac{1}{[a]!}F_i^a\mathbf{1}_\lambda. \tag{2.6}$$

As in [11, §7.1.1], the following *higher quantum Serre relations* follow from 3 in Definition 2.1. For m, n with $m \geq n + 1$ we have

$$E_1^{(m)}E_2^{(n)}\mathbf{1}_\lambda = \sum_{\substack{r+s=m, \\ m-n \leq s \leq m}} \gamma_s E_1^{(r)}E_2^{(n)}E_1^{(s)}\mathbf{1}_\lambda, \tag{2.7}$$

$$F_1^{(m)}F_2^{(n)}\mathbf{1}_\lambda = \sum_{\substack{r+s=m, \\ m-n \leq s \leq m}} \gamma_s F_1^{(r)}F_2^{(n)}F_1^{(s)}\mathbf{1}_\lambda, \tag{2.8}$$

where

$$\gamma_s = \sum_{t=0}^{m-n-1} (-1)^{s+1+t} q^{-s(n-m+1+t)} \begin{bmatrix} s \\ t \end{bmatrix}. \tag{2.9}$$

Similarly, we can deduce from 1:

$$E_i^{(a)} F_i^{(b)} \mathbf{1}_\lambda = \sum_{t=0}^{\min\{a,b\}} \begin{bmatrix} a - b + \lambda_i - \lambda_{i+1} \\ t \end{bmatrix} F_i^{(b-t)} E_i^{(a-t)} \mathbf{1}_\lambda, \tag{2.10}$$

$$F_i^{(b)} E_i^{(a)} \mathbf{1}_\lambda = \sum_{t=0}^{\min\{a,b\}} \begin{bmatrix} -a + b - \lambda_i + \lambda_{i+1} \\ t \end{bmatrix} E_i^{(a-t)} F_i^{(b-t)} \mathbf{1}_\lambda. \tag{2.11}$$

Remark 2.4. As usual, one can define a version over $\mathbb{C}[q, q^{-1}, q^\beta, q^{-\beta}, [\beta]]$ generated by the divided powers by replacing 1 with (2.10) and (2.11) and 3 with (2.7) and (2.8).

Example 2.5. Let us denote by $\mathbf{0}$ the element $(0, \ldots, 0, \beta - 0, \ldots, \beta - 0) \in \mathsf{P}$, where the first $\beta - 0$ entry is at position $k + 1$. We have:

$$[E_k, F_k] \mathbf{1_0} = [0 - \beta + 0] \mathbf{1_0}, \tag{2.12}$$

and since $E_k F_k \mathbf{1_0} = 0$ (this follows by our convention on non-admissible weights) we get

$$F_k E_k \mathbf{1_0} = \frac{q^\beta - q^{-\beta}}{q - q^{-1}}. \tag{2.13}$$

In the case $\beta = m > 0$, this formula reminds us of the value of the unknot for the $\mathfrak{sl}_m$ link invariant, and in the case β generic its value for the HOMFLY-PT polynomial.

It is easy to check that we have a symmetry in our definition of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta$, which will correspond diagrammatically to a rotation of 180 degrees:

Lemma 2.6. *There is an algebra anti-isomorphism*

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \longrightarrow \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{l,k}})_\beta$$

given by

$$\begin{aligned}
\mathbf{1}_\lambda &\longmapsto \mathbf{1}_{(\beta - \lambda_{k+l}, \ldots, \beta - \lambda_{k+1}, \beta - \lambda_k, \ldots, \beta - \lambda_1)}, \\
E_i &\longmapsto F_{k+l-i}, \\
F_i &\longmapsto E_{k+l-i}.
\end{aligned} \tag{2.14}$$

Remark 2.7. In [1], Cautis, Kamnitzer and Morrison introduce the category $\dot{\mathbf{U}}_q^{\geq 0, \leq m}(\mathfrak{gl}_k)$ as the quotient of $\dot{\mathbf{U}}_q^{\geq 0}(\mathfrak{gl}_k)$ by the ideal generated by weights with an entry strictly bigger than m (that is, if a morphism factors through $\mathbf{1}_\lambda$ and λ has some entry strictly bigger than m, then we set this morphism equal to zero). This construction is closely related to ours, and it is easy to check that we can recover $\dot{\mathbf{U}}_q^{\geq 0, \leq m}(\mathfrak{gl}_k)$ as a quotient of our doubled Schur algebra for $\beta = m$.

For $k \leq k'$ and $l \leq l'$ there is an obvious map

$$\iota = \iota_{k,l}^{k',l'} : \dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta \longrightarrow \dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k',l'}})_\beta,$$

which can be constructed by composing iteratively the following elementary maps:

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta \longrightarrow \dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k+1,l}})_\beta, \tag{2.15a}$$

$$\mathbf{1}_{(\lambda_1,\ldots,\lambda_{k+l})} \longmapsto \mathbf{1}_{(0,\lambda_1,\ldots,\lambda_{k+l})}, \tag{2.15b}$$

$$E_i, F_i \longmapsto E_{i+1}, F_{i+1}, \tag{2.15c}$$

and

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta \longrightarrow \dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,l+1}})_\beta, \tag{2.16a}$$

$$\mathbf{1}_{(\lambda_1,\ldots,\lambda_{k+l})} \longmapsto \mathbf{1}_{(\lambda_1,\ldots,\lambda_{k+l},\beta)}, \tag{2.16b}$$

$$E_i, F_i \longmapsto E_i, F_i. \tag{2.16c}$$

Lemma 2.8. *The functor ι is fully faithful.*

We will give a self-contained proof of the fullness, while for the faithfulness we refer to [16] (indeed, the faithfulness follows from [16, Proposition 7.ll and Proposition 6.8]). However, we will not need faithfulness in the following.

Proof. It is sufficient to check the claim for the two functors (2.15) and (2.16). Let us consider the first functor (the proof for the second one being analogous). Let $\mathbf{1}_\lambda, \mathbf{1}_\mu$ be two objects of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k+1,l}})_\beta$ with $\lambda_1 = \mu_1 = 0$ and let $\varphi \in \mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k+1,l}})_\beta \mathbf{1}_\lambda$. By the PBW theorem, φ can be written as a linear combination of monomials $\mathbf{1}_\mu x y \mathbf{1}_\lambda$, where x is a composition of E_i's and y is a composition of F_j's. For weight reasons, it is immediate to see that if an F_1 or E_1 appears, then this monomial is zero in $\mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k+1,l}})_\beta \mathbf{1}_\lambda$ (notice that if both an E_1 and an F_1 appear, then the E_1 must be on the left!). Hence the monomial $\mathbf{1}_\mu x y \mathbf{1}_\lambda$ is the image of a monomial from $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta$. $\square$

As a consequence, we have the following result, which will be crucial for defining link invariants:

Proposition 2.9. (1) *The endomorphism space* $\mathbf{1_0}\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1_0}$ *is of dimension one.*

(2) *Let* $\lambda \in \mathsf{P}$ *be a sequence which differs only at one place from* $\mathbf{0}$. *Then the endomorphism space* $\mathbf{1}_\lambda \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1}_\lambda$ *is of dimension one.*

(3) *Let* $k \geq 2$, *and let* $\mu \in \mathsf{P}$ *be a sequence which differs from* $\mathbf{0}$ *only at* $\mu_1 = \mu_2 = a \in \mathbb{Z}_{>0}$. *Then the endomorphism space* $\mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1}_\mu$ *is of dimension* $(a+1)$, *generated by* $E_1^{(i)} F_1^{(i)} \mathbf{1}_\mu$ *for* $i = 0, \ldots, a$.

Proof. In all the three cases, it follows immediately from the fullness part of Lemma 2.8 that the dimension is not higher than one, one and $(a + 1)$, respectively. For proving the other inequalities, one can either use the faithfulness part of Lemma 2.8 (hence using the results from [16]), or check them directly by exhibiting representations of the doubled Schur algebra. For case (1), the dimension has to be at least one since $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta$ possesses a trivial (one-dimensional) representation. In case (2), if the non-trivial entry of λ is equal to a or $\beta - a$, then one can construct a full functor from $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta$ to the category of representations of $U_q(\mathfrak{gl}_a)$. For case (3), one needs a full functor to the category of representations of $U_q(\mathfrak{gl}_{2a})$. We omit the lengthy but straightforward details (one defines the functor explicitly and checks that the defining relations are satisfied). $\square$

3. The HOMFLY-PT polynomial and the $\mathfrak{sl}_m$ polynomial

The HOMFLY-PT polynomial [5, 13] is a two-variable generalization of both the $\mathfrak{sl}_m$ polynomial and the Alexander polynomial. Although it appears as some generic version of the $\mathfrak{sl}_m$ polynomials, or maybe some limit version for $m \to \infty$, there is no convenient description of it as an intertwiner of $U_q(\mathfrak{sl}_\infty)$–representations. Our goal here is, by passing to the Howe dual side, to obtain a unified definition of the HOMFLY-PT polynomial and its specializations in a quantum group setting.

3.1. A glimpse of diagrammatic calculus: ladder webs. Inspired by the usual $\mathfrak{sl}_m$ case [1], it will be convenient to perform some computations diagrammatically. We define a diagrammatic version of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta$ to be the $\mathbf{k}$-linear category with

objects

$$\mathbf{1}_\lambda = \underset{\lambda_1}{\uparrow}\ \underset{\lambda_2}{\uparrow}\ \cdots\ \underset{\lambda_k}{\uparrow}\ \underset{\lambda_{k+1}}{\downarrow}\ \cdots\ \underset{\lambda_{k+l}}{\downarrow} \tag{3.1}$$

for $\lambda \in \mathsf{P}$ (and their formal finite direct sums), and morphisms generated by

$$\mathbf{1}_{\lambda+\alpha_i} E_i^{(a)} \mathbf{1}_\lambda = \begin{cases} \text{(diagram)} & \text{for } i < k, \\ \text{(diagram)} & \text{for } i > k, \\ \text{(diagram)} & \text{for } i = k, \end{cases} \tag{3.2}$$

$$\mathbf{1}_{\lambda+\alpha_i} F_i^{(a)} \mathbf{1}_\lambda = \begin{cases} \text{(diagram)} & \text{for } i < k, \\ \text{(diagram)} & \text{for } i > k, \\ \text{(diagram)} & \text{for } i = k. \end{cases} \tag{3.3}$$

subject to the diagrammatic versions of the relations 1, 2, and 3 from Definition 2.1. Here, diagrams are composed by stacking them on top of each other, and we always read our diagrams from the bottom to the top.

We will often depict the 0-labeled edges as dotted edges. Note that edges labeled with elements from $\beta - \mathbb{N}$ are oriented downward. We also often depict the $\beta-0$ edges dotted. Although these graphs have an interesting topological behavior [1, 14], we will here mostly use them just as a shorthand for actual computations in the doubled Schur algebra.
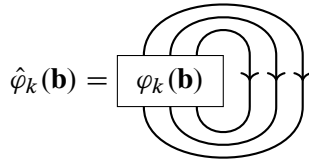
3.2. A link invariant. Let B_k denote the braid group of k strands, generated by $\sigma_1, \ldots, \sigma_{k-1}$. We can define a map $\varphi_k \colon B_k \to \dot{\mathbf{U}}_q^{\geq 0}(\mathfrak{gl}_k)$ using the following rules (which originate from [11, §5.2.1], see also [1, §6.1]):

$$\sigma = \begin{array}{c}\nwarrow\!\!\!\diagup\!\!\!\nearrow\end{array} \longmapsto q^{-1}\mathbf{1}_{(1,1)} - EF\mathbf{1}_{(1,1)}, \tag{3.4}$$

$$\sigma^{-1} = \begin{array}{c}\nwarrow\!\!\!\diagdown\!\!\!\nearrow\end{array} \longmapsto -EF\mathbf{1}_{(1,1)} + q\mathbf{1}_{(1,1)}. \tag{3.5}$$

These have to be understood as local rules: if on the left hand side one has the elementary braid σ_i, then one uses on the right hand side the generators E_i and F_i. Also for braids we read our pictures from the bottom to the top. Correspondingly, we compose the quantum group expression from right to left (as function composition). We will sometimes use the crossing diagrams on the left hand side of (3.4) and (3.5) to denote the corresponding morphism in the doubled Schur algebra.

The element $\varphi_k(\mathbf{b}) \in \dot{\mathbf{U}}_q^{\geq 0}(\mathfrak{gl}_k)$ can be "closed" in $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,k}})_\beta$ as follows:

$$\hat{\varphi}_k(\mathbf{b}) = \boxed{\varphi_k(\mathbf{b})} \tag{3.6}$$

where the caps are realized using the following elementary pattern

$$\tag{3.7}$$

which in formulas is $F_3 F_1 F_2 \mathbf{1}_{(1,1,\beta-1,\beta-1)}$, and similarly for the cups.

In this way, we associate to each braid $\mathbf{b}$ an element $\hat{\varphi}_k(\mathbf{b}) \in \mathbf{1_0}\dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,k}})_\beta\mathbf{1_0}$. By Proposition 2.9(1) this endomorphism space is naturally isomorphic to the base ring $\mathbf{k}$. Given a link L which is the closure of a braid $\mathbf{b} \in B_k$, we set then $P_\beta(L) = \hat{\varphi}_k(\mathbf{b})$. We can now state our first main result:

Theorem 3.1. *The element $P_\beta(L)$ is an invariant of the framed oriented link L.*

The proof of the theorem will consist of three lemmas. First, we need to check that φ_k respects the braid relations:

Lemma 3.2. *The map φ_k is well defined.*

Proof. This is a direct consequence of the general results from [11, Chapters 37 and 39], cf. also [1, Section 6]. Actually, the result can also easily be proven by direct computation. One has to check that (3.4) and (3.5) are inverse to each other, and that the braid relation holds. For example, we have

$$
\begin{aligned}
\mathord{\raisebox{-2pt}{\includegraphics[width=1em]{braid}}} \longmapsto \ & (-EF\mathbf{1}_{(1,1)} + q\mathbf{1}_{(1,1)})(q^{-1}\mathbf{1}_{(1,1)} - EF\mathbf{1}_{(1,1)}) && (3.8) \\
= \ & \mathbf{1}_{(1,1)} - (q + q^{-1})EF\mathbf{1}_{(1,1)} + EFEF\mathbf{1}_{(1,1)} \\
= \ & \mathbf{1}_{(1,1)} - (q + q^{-1})EF\mathbf{1}_{(1,1)} + [2]EF\mathbf{1}_{(1,1)} \\
= \ & \mathbf{1}_{(1,1)}.
\end{aligned}
$$

We leave it to the interested reader to check the braid relation. $\qquad\square$

Second, we need to check that the collection of maps $\{\hat{\varphi}_k\}$ yields an invariant of framed oriented links, i.e. that this family defines a (framed) Markov trace. We recall that the closures of two braids give isotopic links if and only if the two braids are related by a finite number of Markov moves, see for example [10, §2.5]. In our case, we need to consider a framed version of the second Markov move.

Lemma 3.3. *The map $\hat{\varphi}_k$ is invariant under the first Markov move, that is $\hat{\varphi}_k(\mathbf{b}_2\mathbf{b}_1) = \hat{\varphi}_k(\mathbf{b}_1\mathbf{b}_2)$ for any $\mathbf{b}_1, \mathbf{b}_2 \in B_k$.*

Proof. Diagrammatically, the first Markov move can be pictured as follows:
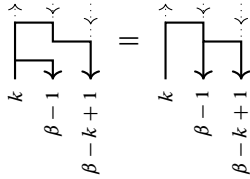

$$(3.9)$$
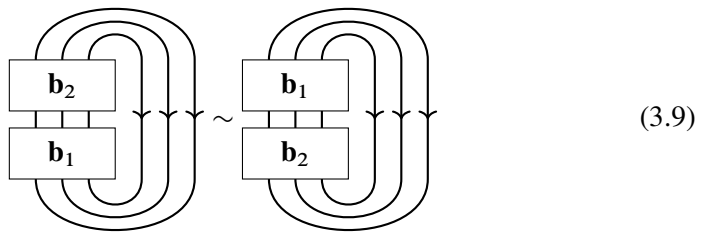
It is easy to see that checking this move boils down to showing the following equality in the doubled Schur algebra (and the symmetric one for cups):

We check this by showing the following chain of equalities:
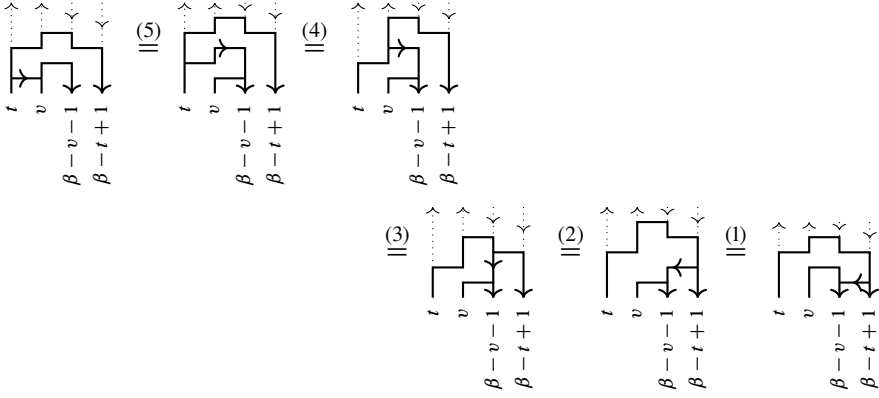
Equality (1) is a consequence of the relation $E_3 F_2^{(v)} = F_2^{(v)} E_3$, and (2) is a consequence of (2.10). Equality (3) follows from the following local move:
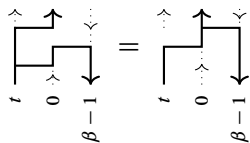

This corresponds to proving

$$F_1^{(k-1)} F_2^{(k-1)} F_1 \mathbf{1}_{(k,\beta-1,\beta-k+1)} = F_1^{(k)} F_2^{(k-1)} \mathbf{1}_{(k,\beta-1,\beta-k+1)}. \qquad (3.10)$$

From (2.7) we get

$$F_1^{(k)} F_2^{(k-1)} = \sum_{r+s=k} (-1)^{s+1} F_1^{(r)} F_2^{(k-1)} F_1^{(s)}. \qquad (3.11)$$

Unless $s \leq 1$, the corresponding term acting on $\mathbf{1}_{(k,\beta-1,\beta-k+1)}$ is zero by weight conventions, and we thus get (3.10) as desired.
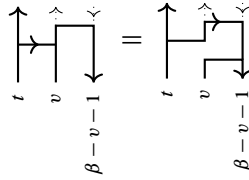
Equality (4) locally reduces to

This follows from

$$F_1^{(t-1)} F_2 F_1 \mathbf{1}_{(t,0,\beta-1)} = F_1^{(t-1)} F_2 F_1 E_1^{(t)} F_1^{(t)} \mathbf{1}_{(t,0,\beta-1)} \tag{3.12}$$

$$= F_1^{(t-1)} F_2 E_1^{(t-1)} F_1^{(t)} \mathbf{1}_{(t,0,\beta-1)}$$

$$= F_1^{(t-1)} E_1^{(t-1)} F_2 F_1^{(t)} \mathbf{1}_{(t,0,\beta-1)}$$

$$= F_2 F_1^{(t)} \mathbf{1}_{(t,0,\beta-1)}.$$

Finally, equality (5) reduces to

That is, we want to prove

$$F_2^{(v+1)} F_1 \mathbf{1}_{(t,v,\beta-v-1)} = F_2 F_1 F_2^{(v)} \mathbf{1}_{(t,v,\beta-v-1)} \tag{3.13}$$

We proceed by induction. The case $v = 0$ is immediate. For the general case, we use again a version of the higher quantum Serre relation (2.7):

$$F_2^{(v+1)} F_1 = \sum_{r=1}^{v+1} (-1)^{r-1} q^{-(v-1)r} F_2^{(v+1-r)} F_1 F_2^{(r)}. \tag{3.14}$$

Since $F_2^{(v+1)} \mathbf{1}_{(t,v,\beta-v-1)} = 0$, the sum becomes

$$F_2^{(v+1)} F_1 = \sum_{r=1}^{v} (-1)^{r-1} q^{-(v-1)r} F_2^{(v+1-r)} F_1 F_2^{(r)}. \tag{3.15}$$

We use the induction hypothesis

$$F_2^{(v+1-r)} F_1 \mathbf{1}_{(t,v-r,\beta-v+r-1)} = F_2 F_1 F_2^{(v-r)} \mathbf{1}_{(t,v-r,\beta-v+r-1)} \tag{3.16}$$

to obtain

$$F_2^{(v+1)} F_1 \mathbf{1}_{(t,v,\beta-v-1)} = \sum_{r=1}^{v} (-1)^{r-1} q^{-(v-1)r} F_2 F_1 F_2^{(v-r)} F_2^{(r)} \mathbf{1}_{(t,v,\beta-v-1)}$$

$$= \sum_{r=1}^{v} (-1)^{r-1} q^{-(v-1)r} \begin{bmatrix} v \\ r \end{bmatrix} F_2 F_1 F_2^{(v)} \mathbf{1}_{(t,v,\beta-v-1)}$$

$$= \left(1 - \sum_{r=0}^{v} (-1)^{r} q^{-(v-1)r} \begin{bmatrix} v \\ r \end{bmatrix} \right) F_2 F_1 F_2^{(v)} \mathbf{1}_{(t,v,\beta-v-1)}.$$

$$\tag{3.17}$$

As in [9, §0.2] we have $\sum_{r=0}^{v} (-1)^{r} q^{-(v-1)r} \begin{bmatrix} v \\ r \end{bmatrix} = 0$. This implies (3.13), and concludes the proof. □

The (unframed) second Markov move relates $\varphi_k(\mathbf{b})$ and $\varphi_{k+1}(\mathbf{b}\sigma_k)$; diagrammatically

$$\qquad\qquad (3.18)$$

Since we only want an invariant of framed links, it is enough to check the following:

Lemma 3.4. *We have* $\hat{\varphi}_{k+1}(\mathbf{b}\sigma_k) = q^{-\beta}\hat{\varphi}_k(\mathbf{b})$ *and* $\hat{\varphi}_{k+1}(\mathbf{b}\sigma_k^{-1}) = q^\beta\hat{\varphi}_k(\mathbf{b})$.

Proof. The claim reduces to computing

$$\qquad\qquad (3.19)$$

Since the two computations are similar, we will only write out the first one. This corresponds to

$$q^{-1}F_2E_2\mathbf{1}_{(1,0,\beta-0)} - F_2E_1F_1E_2\mathbf{1}_{(1,0,\beta-0)}. \qquad (3.20)$$

Since

$$F_2E_2\mathbf{1}_{(1,0,\beta-0)} = [\beta]\mathbf{1}_{(1,0,\beta-0)} \qquad (3.21)$$

and

$$\begin{aligned}
F_2E_1F_1E_2\mathbf{1}_{(1,0,\beta-0)} &= E_1F_2E_2\mathbf{1}_{(0,1,\beta-0)}F_1\mathbf{1}_{(1,0,\beta-0)} \\
&= [\beta-1]E_1F_1\mathbf{1}_{(1,0,\beta-0)} \qquad (3.22) \\
&= [\beta-1]\mathbf{1}_{(1,0,\beta-0)},
\end{aligned}$$

we obtain

$$\begin{aligned}
q^{-1}F_2E_2\mathbf{1}_{(1,0,\beta-0)} - F_2E_1F_1E_2\mathbf{1}_{(1,0,\beta-0)} &= q^{-1}[\beta]\mathbf{1}_{(1,0,\beta-0)} - [\beta-1]\mathbf{1}_{(1,0,\beta-0)} \\
&= q^{-\beta}\mathbf{1}_{(1,0,\beta-0)}. \qquad \square
\end{aligned}$$

This concludes the proof of Theorem 3.1. We notice that our construction contains the HOMFLY-PT polynomial and the $\mathfrak{sl}_m$ link invariant:

Theorem 3.5. *Let L be a framed oriented link. If β is generic, then $P_\beta(L)$ is the HOMFLY-PT polynomial of L. If $\beta = m \in \mathbb{Z}$, with $m \geq 2$ then $P_m(L)$ is the $\mathfrak{sl}_m$ link invariant of L.*

Proof. It is enough to check that our link invariant verifies the same skein and normalization relations as the HOMFLY-PT polynomial (respectively, the $\mathfrak{sl}_m$ link invariant). The skein relation can be checked as follows:

$$P_\beta \left(\text{⤬} \right) - P_\beta \left(\text{⤬} \right) = q^{-1}\mathbf{1}_{(1,1)} - EF\mathbf{1}_{(1,1)} + EF\mathbf{1}_{(1,1)} - q\mathbf{1}_{(1,1)}$$

$$= (q^{-1} - q)\mathbf{1}_{(1,1)} \tag{3.23}$$

$$= (q^{-1} - q)P_\beta \left(\text{)(} \right).$$

The unknot value is provided by Example 2.5. $\qquad\qquad\square$

4. The Alexander polynomial

We denote by $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_0$ the $\beta = 0$ specialization of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta$.

The link invariant P_0 respects the skein relation (3.23), which is, up to renormalization, the skein relation of the Alexander polynomial. However, for the unknot we have $P_\beta(\bigcirc) = 0$ (since now $\beta = 0$). Actually, one sees as in [19] that $P_\beta(L) = 0$ for any link L. The usual trick [6, 19] for obtaining a non-trivial link invariant is to cut open one of the strands. We explain now how this is possible, by lifting the argument for the ribbon category of $U_q(\mathfrak{gl}(1|1))$–representations (see for example [19]) to $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_0$. We will first do this in the generic case, and specialize later to $\beta = 0$.

Let $\mu \in \mathsf{P}$ be a weight which differs from $\mathbf{0}$ only at $\mu_1 = 1$ and $\mu_2 = 1$. By Proposition 2.9(3) the space $\mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1}_\mu$ is 2-dimensional, generated by $\mathbf{1}_\mu$ and $E_1 F_1 \mathbf{1}_\mu$. The usual decomposition of the tensor product of representations $\mathbb{C}^m \otimes \mathbb{C}^m$ can be lifted to $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta$: setting $e_1 = \frac{1}{[2]}E_1 F_1 \mathbf{1}_\mu$ and $e_2 = \mathbf{1}_\mu - e_1$, we obtain a decomposition of $\mathbf{1}_\mu$ into primitive orthogonal idempotents. Notice that e_1 and e_2 are central idempotents, since $\mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1}_\mu$ is generated by e_1 and e_2.

Proposition 4.1. *Let* $\mathbf{w} \in \mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1}_\mu$. *Then*

$$
\begin{array}{c}
\text{[diagram]}
\end{array}
\qquad (4.1)
$$

Proof. This proof is similar to the proof of [19, Proposition 4.5]. The main point is that, as we explained above, $\mathbf{1}_\mu \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{1,1}})_0 \mathbf{1}_\mu$ is a two-dimensional semisimple algebra, and e_1, e_2 are primitive orthogonal central idempotents. Hence we have diagrammatically

$$
\begin{array}{c}
\text{[diagram]}
\end{array}
$$

and the two middle terms are zero.

Now, let R denote the braiding. Then there exist $a, b \in \mathbf{k}$ with $R = ae_1 + be_2$. In particular, we have $Re_1 = ae_1$, $Re_2 = be_2$ and $R^{-1}e_1 = a^{-1}e_1$, $R^{-1}e_2 = b^{-1}e_2$. We can thus insert braidings in the above pictures and obtain

$$
\begin{array}{c}
\text{[diagram]}
\end{array}
$$

which concludes the proof. $\qquad\square$

Let now $\lambda \in \mathsf{P}$ be a weight which differs from $\mathbf{0}$ only at $\lambda_1 = 1$. Given a braid $\mathbf{b} \in B_k$, we denote by $\tilde{\varphi}_k(\mathbf{b}) \in \mathbf{1}_\lambda \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,k-1}})_\beta \mathbf{1}_\lambda = \mathbf{k}$ the closure of all but the first strand of $\varphi_k(\mathbf{b})$ in $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,k-1}})_\beta$. Given a link L which is the closure of a braid $\mathbf{b} \in B_k$, we set then $\widetilde{P}_\beta(L) = \tilde{\varphi}_k(\mathbf{b})$ and we have then the following counterpart of Theorem 3.1:

Theorem 4.2. *The element* $\widetilde{P}_\beta(L)$ *is an invariant of the framed oriented link L.*

Proof. As for Theorem 3.1, we need to prove that the maps $\{\tilde{\varphi}_k\}$ define a framed Markov trace. The proof of the invariance under the second (framed) Markov move is exactly the same as for Lemma 3.4. Let us discuss invariance under the first Markov move. We want to show $\tilde{\varphi}_k(\mathbf{b}_2\mathbf{b}_1) = \tilde{\varphi}_k(\mathbf{b}_1\mathbf{b}_2)$, or in pictures:

$$\tag{4.2}$$

By induction we can assume that $\mathbf{b}_2 = \sigma_i^{\pm 1}$ is just a crossing between two strands. Unless $i = 1$, Lemma 3.3 (or rather the argument in its proof) applies and gives us the claim. If $\mathbf{b}_2 = \sigma_1^{\pm 1}$ then by Proposition 4.1 we can write $\tilde{\varphi}_k(\mathbf{b}_2\mathbf{b}_1) = \tilde{\varphi}_k(\sigma_1^{\mp 1}\mathbf{b}_2\mathbf{b}_1\sigma_1^{\pm 1}) = \tilde{\varphi}_k(\mathbf{b}_1\sigma_1^{\pm 1})$ and we are done. $\qquad\square$

By Proposition 2.9(2), we have $\mathbf{1}_\lambda \dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}_{k,l}})_\beta \mathbf{1}_\lambda = \mathbf{k}$. We can then state the main result of this section:

Corollary 4.3. *Let L be an oriented link. Then $\widetilde{P}_0(L)$ is the Alexander polynomial of L.*

Proof. The scalar $P_0(L)$ is an invariant of oriented framed links by Theorem 4.2. Actually, since $\beta = 0$, it follows immediately by Lemma 3.4 that this is even an invariant of unframed links. By (3.23), this link invariant satisfies the skein formula of the Alexander polynomial. Since its value on the unknot is 1, it has to coincide with the Alexander polynomial. $\qquad\square$

Note that the process we developed to recover the Alexander polynomial is not *stricto sensu* the specialization of the process yielding the HOMFLY-PT polynomial: there is the extra step of cutting open one strand. In fact, it is very easy to identify the new invariants obtained by this process also for other values of β, and in this setting the Alexander polynomial is a specialization of the reduced HOMFLY-PT polynomial:

Proposition 4.4. *Let L be an oriented framed link. Then*
(1) *if β is generic, $\widetilde{P}_\beta(L)$ is the reduced HOMFLY-PT polynomial of L, while*
(2) *if β is specialized to $m \in \mathbb{Z}_{>0}$, $\widetilde{P}_m(L)$ is the reduced $\mathfrak{sl}_m$ invariant of L.*

Here reduced means normalized to 1 for the unknot (see [15] for our conventions on link invariants).

5. General links and colored case

We have seen how to construct link invariants with values in $\dot{\mathbf{U}}_q(\mathfrak{gl}_{k+l})_\beta$ in the case where L is the closure of a braid. This simplified setup has the advantage of making the exposition very straightforward, but our analysis can also be pushed beyond the case of braid closures. Most of the results and studies allowing for such an extension have been detailed in a second paper [16].

In this last section we want to explain, using the doubled Schur algebra, how to assign an invariant to a link diagram with all strands colored by positive integers k, corresponding to labeling the strands with the k-th exterior powers of the fundamental representation. We restrict here to the case of knots and links for simplicity of exposition, but it is possible to generalize this to the case of tangles (see [16]). We will not recall all the technicalities and we will mostly refer to [16] for the proofs. Here we only intend to focus on the explicit construction yielding the link invariants. This approach, we believe, will be central in future categorification processes.

5.1. The doubled Schur algebra associated to a sign sequence. The first step consists of extending the definition of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta$ so that it encompasses the cases where the strands are not necessarily upward-directed on the left and downward-directed on the right.

Let us fix a sequence $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_h) \in \{\pm 1\}^h$ of signs. Let $\mathsf{P}_{\boldsymbol{\eta}}$ be the set of sequences $\lambda = (\lambda_1, \ldots, \lambda_h)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$ if $\eta_i = 1$ and $\lambda_i \in \beta - \mathbb{Z}_{\geq 0}$ if $\eta_i = -1$. We let also $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \in \mathbb{Z}^h$, the entry 1 being at position i.

Definition 5.1 (See also [16, Definition 7.1]). We define $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta$ to be the additive **k**-linear category with objects:

- **objects:** formal direct sums of symbols $\mathbf{1}_\lambda$ for $\lambda \in \mathsf{P}_{\boldsymbol{\eta}}$;

- **morphisms:** generated by identity endomorphisms in $\mathrm{Hom}(\mathbf{1}_\lambda, \mathbf{1}_\lambda)$, which by a slight abuse of notation we also denote by $\mathbf{1}_\lambda$, and morphisms

$$\mathbf{1}_{\lambda+\alpha_i} E_i \mathbf{1}_\lambda \in \mathrm{Hom}(\mathbf{1}_\lambda, \mathbf{1}_{\lambda+\alpha_i}), \quad \mathbf{1}_{\lambda-\alpha_i} F_i \mathbf{1}_\lambda \in \mathrm{Hom}(\mathbf{1}_\lambda, \mathbf{1}_{\lambda-\alpha_i}).$$

The morphisms are subject to relations 1, 2, and 3 from Definition 2.1.

For $k, l \geq 0$ let

$$\boldsymbol{\eta}_{k,l} = (\underbrace{1, \ldots, 1}_{k}, \underbrace{-1, \ldots, -1}_{l}). \tag{5.1}$$

Then one recovers Definition 2.1 as a special case. Actually, we showed in [16, Corollary 7.4] that if η is a permutation of $\eta_{k,l}$, then $\dot{\mathbf{U}}_q(\mathfrak{gl}_\eta)_\beta$ is isomorphic to $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\eta_{k,l}})_\beta$. Thus, in principle one gets nothing more by considering the category $\dot{\mathbf{U}}_q(\mathfrak{gl}_\eta)_\beta$ for a general sequence of signs η. However, the definition of these more general doubled Schur algebras allows for an easier process to obtain link invariants as a special case of tangle invariants.

5.2. Turning the diagram into a ladder diagram. Generalizing what we did in Section 3, we can assign to an oriented framed link L equipped with a labeling ℓ of its strands by positive integers an element of $\dot{\mathbf{U}}_q(\mathfrak{gl}_\eta)_\beta$, for some appropriate choice β. This procedure is described in full detail in [16], but we recall it quickly. First, we can represent $\dot{\mathbf{U}}_q(\mathfrak{gl}_\eta)_\beta$ diagrammatically. In particular, we assign to the entry $\lambda_k \in \mathbb{Z}_{\geq 0}$ of an object of $\dot{\mathbf{U}}_q(\mathfrak{gl}_\eta)_\beta$ an upward λ_k-labeled strand, and to the entry $\lambda_k \in \beta - \mathbb{Z}_{\geq 0}$ a downward λ_k-labeled. To morphisms we assigned trivalent graphs as in (3.1), (3.2), and (3.3), with the only difference that now strands can be alternatively upward or downward oriented.

Now, the idea is to assign to a strand labeled by a of the link L also a strand labeled by a in the doubled Schur algebra. To upward crossings we want to assign the maps

$$
\overset{\nwarrow\quad\nearrow}{\underset{a\quad\;\;b}{\diagdown\!\!\!\!\diagup}} \mapsto T^{\nwarrow\!\!\!\nearrow}\mathbf{1}_{(a,b)} =
\begin{cases}
q^{-a}\displaystyle\sum_{s\geq 0}(-q)^s E^{(b-a+s)}F^{(s)}\mathbf{1}_{(a,b)} & \text{if } a-b\leq 0,\\[3mm]
q^{-b}\displaystyle\sum_{s\geq 0}(-q)^s F^{(a-b+s)}E^{(s)}\mathbf{1}_{(a,b)} & \text{if } a-b\geq 0,
\end{cases}
\tag{5.2}
$$

and

$$
\overset{\nwarrow\quad\nearrow}{\underset{a\quad\;\;b}{\diagup\!\!\!\!\diagdown}} \mapsto T^{\nwarrow\!\!\!\nearrow}\mathbf{1}_{(a,b)} =
\begin{cases}
q^{a}\displaystyle\sum_{s\geq 0}(-q)^{-s} F^{(s)}E^{(b-a+s)}\mathbf{1}_{(a,b)} & \text{if } a-b\leq 0,\\[3mm]
q^{b}\displaystyle\sum_{s\geq 0}(-q)^{-s} E^{(s)}F^{(a-b+s)}\mathbf{1}_{(a,b)} & \text{if } a-b\geq 0,
\end{cases}
\tag{5.3}
$$

which generalize Equations (3.4) and (3.5) for arbitrary coloring a, b of the strands (see the end of Section 5 in [16]). Similarly, we want to assign the following maps to downward crossings:

$$
\overset{\;}{\underset{\beta-a\;\;\beta-b}{\diagdown\!\!\!\!\diagdown}} \mapsto T^{\diagdown\!\!\!\diagdown} =
\begin{cases}
q^{-a}\displaystyle\sum_{s\geq 0}(-q)^s E^{(s)}F^{(b-a+s)}\mathbf{1}_{(a,b)} & \text{if } a-b\leq 0,\\[3mm]
q^{-b}\displaystyle\sum_{s\geq 0}(-q)^s F^{(s)}E^{(a-b+s)}\mathbf{1}_{(a,b)} & \text{if } a-b\geq 0,
\end{cases}
\tag{5.4}
$$

and

$$
\begin{array}{c}
\diagdown\!\!\!\!\diagup \\[-4pt]
\diagup\!\!\!\!\diagdown \\[-6pt]
{\scriptstyle \beta-a}\;\;{\scriptstyle \beta-b}
\end{array}
\;\mapsto\; T^{\times} =
\begin{cases}
q^a \sum_{s\geq 0}(-q)^{-s}\,F^{(b-a+s)}\,E^{(s)}\mathbf{1}_{(a,b)} & \text{if } a-b\leq 0, \\[12pt]
q^b \sum_{s\geq 0}(-q)^{-s}\,E^{(a-b+s)}\,F^{(s)}\mathbf{1}_{(a,b)} & \text{if } a-b\geq 0.
\end{cases}
\tag{5.5}
$$

The only little catch is that, starting from a particular diagram of the link L, we need to make sure that we can translate it into an element of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta$ for some choice of $\boldsymbol{\eta}$. In other words, we need first to perform a sequence of isotopies after which our link diagram can be written as a composition of upward- or downward-pointing crossings. Moreover, we need to make our link diagram rigid, so that at each level every strands occupies only one of the positions $1, \ldots, h$, and at each position either all strands are upward-pointing or downward-pointing (but not both!).

To do so, first we give to the diagram the shape of a ladder. This can be achieved by decomposing the original diagram into a vertical composition of caps, cups and crossings (together with vertical strands around), and restrict (at the cost of adding caps and cups) to the case where all crossings are either upward or downward oriented. The caps and cups are easily expressed in terms of elementary trivalent graphs where some of the edges carry trivial labels.

Now, we want to label the strands using non-negative integers for upward strands and labels in $\beta - \mathbb{N}$ for the downward strands. It could happen, however, that a vertical trivial strand connects a downward and an upward strand, which we want to avoid (since the sign sequence defining $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta$ is fixed). Up to increasing the number of uprights, it is easy to perform isotopies so that all segments on a vertical line have the same orientation – which then allows for a consistent labeling. We illustrate this process in Example 5.3.

Once we have performed these operations to our link diagram, we can associate to it a morphism in the doubled Schur algebra using (5.2), (5.3), (5.4), and (5.5) for crossings and (3.2) and (3.3) for caps and cups, as explained above, and we get an element $P_\beta^\ell(L) \in \mathrm{End}_{\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta}(\mathbf{0}) = \mathbf{k}$ for some sign sequence $\boldsymbol{\eta}$.

Theorem 5.2. *Let (L, ℓ) be a labeled oriented framed link. The element $P_\beta^\ell(L)$ is an invariant of (L, ℓ). Furthermore, for β generic $P_\beta^\ell(L)$ is the colored HOMFLY-PT polynomial associated to L, while specializing $\beta = m$ yields the colored $\mathfrak{sl}_m$ Reshetikhin–Turaev polynomial of L.*

We refer to [15] for the definition (and our normalization) of the colored HOMFLY-PT polynomial.

Proof. This follows from the results in [16], and in particular from Proposition 6.15 and Corollary 7.12. □

Example 5.3. We explain with an example the process of defining $P_\beta^\ell(L)$ for the figure-8 knot L labeled by $a \in \mathbb{Z}_{>0}$. First, we can choose $\boldsymbol{\eta} = (1, 1, -1, -1, -1, 1)$ and perform isotopies so that

 becomes
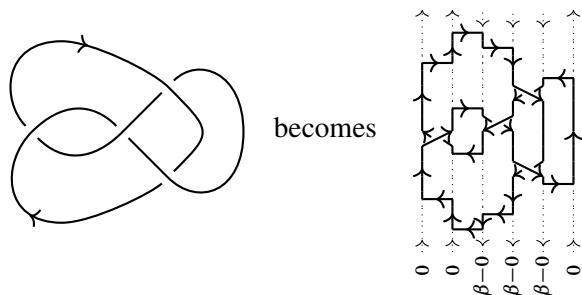
Since the strand is colored by the label a, then we get the following element of $\mathbf{1_0}\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta\mathbf{1_0}$:

$$F_2^{(a)} F_3^{(a)} F_1^{(a)} E_5^{(a)} T_4^{\rtimes} F_2^{(a)} T_3^{\rtimes} T_1^{\rtimes} E_2^{(a)} T_4^{\rtimes} F_5^{(a)} E_1^{(a)} E_3^{(a)} E_2^{(a)} \mathbf{1_0}.$$

Using the defining relations of $\dot{\mathbf{U}}_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta$, one can then reduce the above element to a scalar, and compute in this way the HOMFLY-PT polynomial of the figure-8 knot. Similarly we can specialize β to m and compute the $\mathfrak{sl}_m$ link invariant.

Example 5.4. It might be inspiring, also with the aim of exploring a potential categorification, to check the Reidemeister moves directly in the doubled Schur algebra, avoiding the results from [16]. For example, checking the mixed second Reidemeister move amounts to consider the picture

(5.6)

(which corresponds to fixing $\boldsymbol{\eta} = (-1, +1, +1, -1)$) and check in $U_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta$ that

$$E_1 T_2^{\rtimes} E_3 F_3 T_2^{\rtimes} F_1 \mathbf{1}_{(\beta-0,0,1,\beta-1)} = \mathbf{1}_{(\beta-0,0,1,\beta-1)}. \tag{5.7}$$

5.3. Cutting open and the Alexander polynomial. Let L be an oriented framed link and ℓ a labeling of its strands by positive integers. Let a be one of the colors of the strands of L, and denote by $\widetilde{L}$ the $(1,1)$–tangle obtained from L by cutting open one of the strands labeled by a. Similarly as before, we can assign to L an element of $\mathbf{1}_\lambda U_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta \mathbf{1}_\lambda$ for some sequence $\boldsymbol{\eta}$. Notice now that λ has only one nontrivial entry, which is equal to a. By Proposition 2.9, the space $\mathbf{1}_\lambda U_q(\mathfrak{gl}_{\boldsymbol{\eta}})_\beta \mathbf{1}_\lambda$ is naturally isomorphic to $\mathbf{k}$, and we denote the scalar obtained in this way by $\widetilde{P}_\beta^{\ell,a}(L)$. Following the same path as in Section 4, we have

Proposition 5.5. *Let L be an oriented framed link labeled by ℓ. The element $\widetilde{P}_\beta^{\ell,a}(L)$ is an invariant of L, and it is equal to the reduced colored HOMFLY-PT polynomial when β is generic, and to the reduced colored $\mathfrak{sl}_m$ polynomial for $\beta = m \in \mathbb{Z}_{>0}$.*

Proof. The proof is the same as for Theorem 4.2 after one proves the following variant of Proposition 4.1:

$$(5.8)$$

This can be shown with the same argument of Proposition 4.1, or alternatively by explicit computation (see [16, Proposition 8.3]). □

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