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Quantum Topology

Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants

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Abstract. We consider the asymptotics of the Turaev–Viro and the Reshetikhin–Turaev invariants of a hyperbolic 3-manifold, evaluated at the root of unity $\exp(2\pi\sqrt{-1}/r)$ instead of the standard $\exp(\pi\sqrt{-1}/r)$. We present evidence that, as r tends to ∞, these invariants grow exponentially with growth rates respectively given by the hyperbolic and the complex volume of the manifold. This reveals an asymptotic behavior that is different from that of Witten's Asymptotic Expansion Conjecture, which predicts polynomial growth of these invariants when evaluated at the standard root of unity. This new phenomenon suggests that the Reshetikhin–Turaev invariants may have a geometric interpretation other than the original one via $SU(2)$ Chern–Simons gauge theory.

Mathematics Subject Classification (2010). Primary 57M27; Secondary 57M50, 57M25, 81R50.

Keywords. Reshetikhin–Turaev invariants, Turaev–Viro invariants, volume conjecture.

Contents

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1. Introduction

In [62], Witten provided a new interpretation of the Jones polynomial based on Chern–Simons gauge theory, and expanded on this idea to construct a sequence of complex valued 3-manifold invariants. This approach was formalized though the representation theory of quantum groups by Reshetikhin and Turaev [53, 54], who generalized the Jones polynomial to a sequence of polynomial invariants of a link, later called the colored Jones polynomials of that link. They also defined a sequence of 3-manifold invariants corresponding to Witten's invariants. The Reshetikhin–Turaev construction of 3-manifold invariants starts from a surgery description [33] of the manifold, and evaluates the colored Jones polynomials of the surgery data at certain roots of unity.

A different approach was developed by Turaev and Viro [60] who, from a triangulation of a closed 3-manifold, constructed real valued invariants of the manifold by using quantum $6j$-symbols [35]; these Turaev–Viro invariants turned out to be equal to the square of the norm of the Reshetikhin–Turaev invariants [55, 59, 61].

Using quantum dilogarithm functions, Kashaev [29, 30] used a different type of $6j$-symbols, involving the discrete quantum dilogarithm, to define for each integer n complex valued link invariants. He observed in a few examples, and conjectured in the general case, that the absolute value of these invariants grow exponentially with n, and that the growth rate is given by the hyperbolic volume of the complement of the link. In [44], Murakami and Murakami showed that Kashaev's invariants coincide with the values of the colored Jones polynomials at a certain root of unity, and reformulated Kashaev's conjecture as follows.

Volume conjecture ([30, 44]). *For a hyperbolic link L in S^3, let $J_n(L;q)$ be its n-th colored Jones polynomial. Then*

$$\lim_{n\to+\infty}\frac{2\pi}{n}\log|J_n(L;e^{\frac{2\pi\sqrt{-1}}{n}})| = \mathrm{Vol}(S^3\setminus L),$$

where $\mathrm{Vol}(S^3\setminus L)$ *is the hyperbolic volume of the complement of L.*

This conjecture has now been proved for a certain number of cases: the figure-eight knot [44]; all hyperbolic knots with at most six crossings [49, 51]; the Borromean rings [24]; the twisted Whitehead links [66]; the Whitehead chains [63]. Various extensions of this conjecture have been proposed, and proved for certain cases in [28, 46, 15, 18, 19, 47].

In the current paper we investigate the asymptotic behavior of the Reshetikhin–Turaev and the Turaev–Viro invariants, evaluated at the root of unity $q = e^{\frac{2\pi\sqrt{-1}}{r}}$. Supported by numerical evidence, we propose the following conjecture.

Conjecture 1.1. *For a hyperbolic* 3*-manifold* M, *let* $\mathrm{TV}_r(M;q)$ *be its Turaev–Viro invariant and let* $\mathrm{Vol}(M)$ *be its hyperbolic volume. Then for* r *running over all odd integers and for* $q = e^{\frac{2\pi\sqrt{-1}}{r}}$,

$$\lim_{r\to+\infty} \frac{2\pi}{r} \log(\mathrm{TV}_r(M;q)) = \mathrm{Vol}(M).$$

We here consider all types of hyperbolic 3-manifolds: closed, cusped or those with totally geodesic boundary. The Turaev–Viro invariant $\mathrm{TV}_r(M;q)$ is the original one defined in [60] when the manifold M is closed, and is its extension defined in [10] when M has non-empty boundary. See § 2.3 for details.

This conjecture should be contrasted with Witten's Asymptotic Expansion Conjecture (see [48]) which predicts that, when evaluated at $q = e^{\frac{\pi\sqrt{-1}}{r}}$, the Witten invariants of a 3-manifold (and therefore its Reshetikhin–Turaev and Turaev–Viro invariants) only grow polynomially, with a growth rate related to classical invariants of the manifold such as the Chern–Simons invariant and the Reidemeister torsion.

Conjecture 1.1 is motivated by the beautiful work of Costantino [16] relating the asymptotics of quantum $6j$-symbols to the volumes of truncated hyperideal tetrahedra. See also [20, 19].

We provide much supporting evidence for Conjecture 1.1. In § 3, we numerically calculate $\mathrm{TV}_r(M)$ for various hyperbolic 3-manifolds with cusps, including the figure-eight knot complement and its sister, the complements of the knots K_{5_2} and K_{6_1}, and the manifolds denoted by M_{3_6}, M_{3_8}, N_{1_1} and N_{2_1} in the Callahan–Hildebrand–Weeks census [14]. We also numerically calculate $\mathrm{TV}_r(M)$ for the smallest hyperbolic 3-manifolds with a totally geodesic boundary [23, 37].

Recently, Detcherry, Kalfagianni and the second author [22] provided a rigorous proof of Conjecture 1.1 for the figure-eight knot complement.

The Reshetikhin–Turaev invariants $\mathrm{RT}_r(M;q)$ are complex valued invariants of a closed oriented 3-manifold M, defined for all integers $r \geqslant 3$ and all primitive $2r$-th roots of unity q. For $q = e^{\frac{\pi\sqrt{-1}}{r}}$, these invariants provide a mathematical realization of Witten's invariants [62]. Following a skein theory approach pioneered by Lickorish [36, 38], Blanchet, Habegger, Masbaum, and Vogel [11] (see also Lickorish [39]) extended Reshetikhin–Turaev invariants to primitive r-th roots of unity q with r odd. In particular, $\mathrm{RT}_r(M;q)$ is defined at $q = e^{\frac{2\pi\sqrt{-1}}{r}}$

when r is odd. In § 4, we numerically compute Reshetikhin–Turaev invariants for various closed hyperbolic 3-manifolds obtained by integral Dehn surgery along the knots K_{4_1} and K_{5_2}. These calculations suggest the following conjecture.

Conjecture 1.2. *Let M be a closed oriented hyperbolic 3-manifold and let $\mathrm{RT}_r(M;q)$ be its Reshetikhin–Turaev invariants. Then for $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ with r odd and for a suitable choice of the arguments,*

$$\lim_{r\to+\infty} \frac{4\pi\sqrt{-1}}{r} \log(\mathrm{RT}_r(M;q)) = \mathrm{CS}(M) + \mathrm{Vol}(M)\sqrt{-1} \mod \pi^2\mathbb{Z},$$

where $\mathrm{CS}(M)$ denotes the Chern–Simons invariant of the hyperbolic metric of M multiplied by $2\pi^2$.

Ohtsuki [50] recently announced a proof of Conjecture 1.2 for the manifolds obtained by Dehn surgery along the knot K_{4_1}. By [55, 59, 61], Conjecture 1.2 implies Conjecture 1.1 for closed 3-manifolds.

Comparing Conjecture 1.2 with Witten's Asymptotic Expansion Conjecture, one sees a very different asymptotic behavior for the Reshetikhin–Turaev invariants evaluated at $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ and $q = e^{\frac{\pi\sqrt{-1}}{2r}}$. Our numerical calculations also suggest exponential growth at other roots of unity such as $q = e^{\frac{3\pi\sqrt{-1}}{r}}$. For these roots of unity, we expect a geometric interpretation of Reshetikhin–Turaev invariants that is different from the $SU(2)$ Chern–Simons gauge theory.

In § 5, we calculate $\mathrm{TV}_r(M)$ for the complements of the unknot, the Hopf link, the trefoil knot and the torus links $T_{(2,4)}$ and $T_{(2,6)}$. We also numerically calculate $\mathrm{TV}_r(M)$ for the complement of the torus knots $T_{(2,5)}$, $T_{(2,7)}$, $T_{(2,9)}$, $T_{(2,11)}$, $T_{(3,5)}$ and $T_{(3,7)}$. These computations suggest an Integrality Conjecture (Conjecture 5.1) which states that the Turaev–Viro invariants of torus link complement are integers independent of the roots of unity at which they are evaluated.

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2. Preliminaries

We recall the construction of Turaev–Viro invariants of 3-dimensional manifolds. In order to follow a uniform treatment for all cases, we extend their definition to pseudo 3-manifolds.

2.1. Pseudo 3-manifolds and triangulations. A *pseudo* 3-*manifold* is a topological space M such that each point p of M has a neighborhood U_p that is homeomorphic to a cone over a surface Σ_p. We call p a *singular point* and U_p a *singular neighborhood* if Σ_p is not a 2-sphere. In particular, a closed 3-manifold is a pseudo 3-manifold with no singular point, and every 3-manifold with boundary is homeomorpic to a pseudo 3-manifold with suitable singular neighborhoods of all singular points removed.

A *triangulation* $\mathcal{T}$ of a pseudo manifold M consists of a disjoint union $X = \bigsqcup \Delta_i$ of finitely many Euclidean tetrahedra Δ_i and of a collection of homeomorphisms Φ between pairs of faces in X such that the quotient space X/Φ is homeomorphic to M. The *vertices*, *edges*, *faces* and *tetrahedra* in $\mathcal{T}$ are respectively the quotients of the vertices, edges, faces and tetrahedra in X. From the definition, we see that a singular point of M must be a vertex of $\mathcal{T}$. We call the non-singular vertices of $\mathcal{T}$ the *inner vertices*. If M is a closed 3-manifold, then a triangulation of M is a triangulation of manifold in the usual sense; and if N is a 3-manifold with boundary obtained by removing all singular neighborhoods of a pseudo 3-manifold M, then a triangulation of M without inner vertices determines an ideal triangulation of N. (See Figure 1.)

In [41, 42, 52], it is proved that any two triangulations of a pseudo 3-manifold are related by a sequence of 0–2 and 2–3 Pachner moves. See the figure below, where in the $0-2$ move a new inner vertex is introduced.

2.2. Quantum $6j$-symbols. We now recall the definition and basic properties of the quantum $6j$-symbols. See [35, 32] for more details.

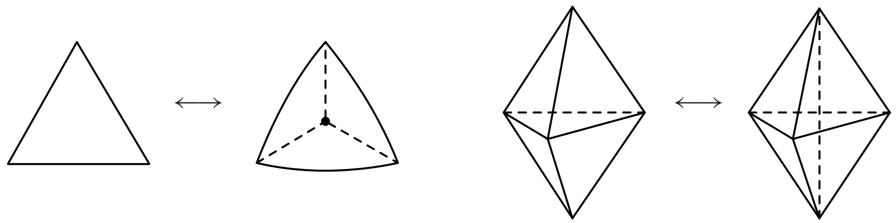

Figure 1

Throughout this subsection, we will fix an integer $r \geqslant 3$, and we let $I_r = \{0, 1/2, \dots, (r-2)/2\}$ be the set of non-negative half-integers less than or equal to $(r-2)/2$. The elements of I_r are traditionally called *colors*.

Let $q \in \mathbb{C}$ be a root of unity such that q^2 is a primitive root of unity of order r. For an integer n, the *quantum integer* $[n]$ is the real number defined by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$

and the associated *quantum factorial* is

$$[n]! = [n][n-1]\dots[1].$$

By convention, $[0]! = 1$.

A triple (i, j, k) of elements of I_r is called *admissible* if

(1) $i + j \geqslant k$, $j + k \geqslant i$ and $k + i \geqslant j$,

(2) $i + j + k \in \mathbb{Z}$,

(3) $i + j + k \leqslant r - 2$.

A 6-tuple (i, j, k, l, m, n) of elements of I_r is *admissible* if the triples (i, j, k), (j, l, n), (i, m, n), and (k, l, m) are admissible.

For an admissible triple (i, j, k), define

$$\Delta(i, j, k) = \sqrt{\frac{[i+j-k]![j+k-i]![k+i-j]!}{[i+j+k+1]!}}$$

with the convention that $\sqrt{x} = \sqrt{|x|}\sqrt{-1}$ when the real number x is negative.

Definition 2.1. The *quantum* $6j$*-symbol* of an admissible 6-tuple (i, j, k, l, m, n) is the number

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{-1}^{-2(i+j+k+l+m+n)} \Delta(i,j,k)\Delta(j,l,n)\Delta(i,m,n)\Delta(k,l,m) \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^z[z+1]!}{\mathfrak{N}},$$

where

$$\mathfrak{N} = [z-T_1]![z-T_2]![z-T_3]![z-T_4]![Q_1-z]![Q_2-z]![Q_3-z]!,$$

$$T_1 = i+j+k, \quad T_2 = j+l+n,$$

$$T_3 = i+m+n, \quad T_4 = k+l+m,$$

and

$$Q_1 = i+j+l+m,$$

$$Q_2 = i+k+l+n,$$

$$Q_3 = j+k+m+n.$$

A good way to memorize the definitions is to consider a tetrahedron as in Figure 2, and to attach the weights i, j, k, l, m, n to its edges as indicated in the figure. Then each of T_1, T_2, T_3, T_4 corresponds to a face of the tetrahedron, and each of Q_1, Q_2, Q_3 corresponds to a quadrilateral separating two pairs of the vertices.

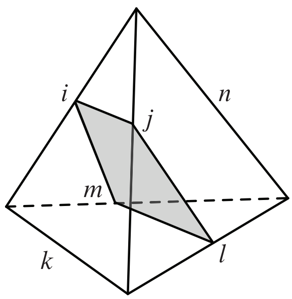

Figure 2. $6j$-symbols and the tetrahedron.

The following symmetries

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ l & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ l & j & k \end{vmatrix} = \begin{vmatrix} l & m & k \\ i & j & n \end{vmatrix} = \begin{vmatrix} l & j & n \\ i & m & k \end{vmatrix}$$

immediately follow from the definitions.

The quantum $6j$-symbols satisfy the following two important identities, which are crucial in the construction of the Turaev–Viro invariants. For $i \in I_r$, set

$$w_i = (-1)^{2i}[2i+1] \quad \text{and} \quad \eta = \sum_{i \in I_r} w_i^2.$$

Proposition 2.2 (orthogonality property). *For any admissible* 6*-tuple* (i, j, k, l, m, n)*,*

$$\sum_s w_s w_m \begin{vmatrix} i & j & m \\ k & l & s \end{vmatrix} \begin{vmatrix} i & j & n \\ k & l & s \end{vmatrix} = \delta_{mn}, \tag{2.1}$$

where δ is the Kronecker symbol, and where the sum is over all $s \in I_r$ such that the two 6*-tuples in the sum are admissible.* □

Corollary 2.3. *For any admissible triple* (i, j, k)*,*

$$\eta^{-1} \sum_{l,m,n} w_l w_m w_n \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = 1, \tag{2.2}$$

where the sum is over $l, m, n \in I_r$ such that the 6*-tuples (i, j, k, l, m, n) is admissible.* □

Proposition 2.4 (Biedenharn–Elliot identity). *For any $i, j, k, l, m, n, o, p, q \in I_r$ such that (o, p, q, i, j, k) and (o, p, q, l, m, n) are admissible,*

$$\sum_s w_s \begin{vmatrix} i & j & q \\ m & l & s \end{vmatrix} \begin{vmatrix} j & k & o \\ n & m & s \end{vmatrix} \begin{vmatrix} k & i & p \\ l & n & s \end{vmatrix} = \begin{vmatrix} o & p & q \\ i & j & k \end{vmatrix} \begin{vmatrix} o & p & q \\ l & m & n \end{vmatrix}, \tag{2.3}$$

where the sum is over $s \in I_r$ such that the three 6*-tuples in the sum are admissible.* □

2.3. Turaev–Viro invariants of pseudo 3-manifolds. Let q be a root of unity, and let r be such that q^2 is a primitive root of unity of order r. As in § 2.2, we consider the set $I_r = \{0, 1/2, 1, \ldots, (r-2)/2\}$ of colors, and the notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad w_i = (-1)^{2i}[2i+1], \quad \eta = \sum_{i \in I_r} w_i^2.$$

for every integer n and color $i \in I_r$.

For a triangulation $\mathcal{T}$ of a pseudo 3-manifold M, an *r-admissible coloring* of $(M,\mathcal{T})$ is a map

$$c\colon \{\text{edges of } \mathcal{T}\} \longrightarrow I_r$$

such that, for every 2-dimensional face F of $\mathcal{T}$, the colors $c(e_1), c(e_2), c(e_3) \in \mathcal{T}$ associated to the edges of F form an admissible triple. Such a coloring c associates to each edge e of $\mathcal{T}$ the number

$$|e|_c = w_{c(e)},$$

and to each tetrahedron Δ of $\mathcal{T}$ the $6j$-symbol

$$|\Delta|_c = \begin{vmatrix} c(e_{12}) & c(e_{13}) & c(e_{23}) \\ c(e_{34}) & c(e_{24}) & c(e_{14}) \end{vmatrix},$$

where the edges of Δ are indexed in such a way that, if v_1, v_2, v_3, v_4 denote the vertices of Δ, the edge e_{ij} connects v_i to v_j.

Definition 2.5. With the above definitions, the *Turaev–Viro invariant* of M associated to the root of unity q is defined as the sum

$$\mathrm{TV}_q(M,\mathcal{T}) = \eta^{-|V|} \sum_{c\in A_r} \prod_{e\in E} |e|_c \prod_{\Delta\in T} |\Delta|_c$$

where V, E, T, A_r respectively denote the sets of inner vertices, edges, tetrahedra and r-admissible colorings of the triangulation $\mathcal{T}$.

Theorem 2.6. *The above invariant* $\mathrm{TV}_q(M,\mathcal{T})$ *depends only on the pseudo-manifold M and on the root of unity q, not on the triangulation $\mathcal{T}$.*

Proof. Theorem 2.6 is proved by a straightforward extension to pseudo-3-manifolds of the original argument of Turaev and Viro in [60] for 3-manifolds.

The first ingredient is a purely topological statement, proved in [41, 42, 52], which says that any two triangulations of a pseudo 3-manifold are related by a sequence of the Pachner Moves 0–2 and 2–3 represented in Figures 3 and 4. The Pachner Move 0–2 replaces a 2-dimensional face of the triangulation by two tetrahedra meeting along 3 faces, and adds one vertex to the triangulation. The 2–3 Move replaces two tetrahedra meeting along one face by three tetrahedra sharing one edge.

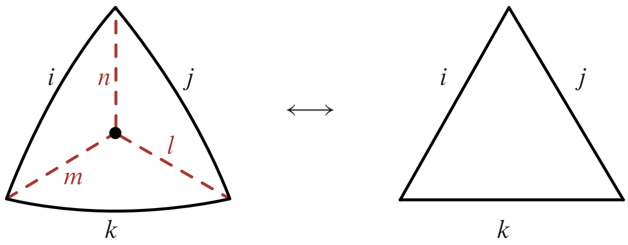

Figure 3. The Pachner Move 0–2.

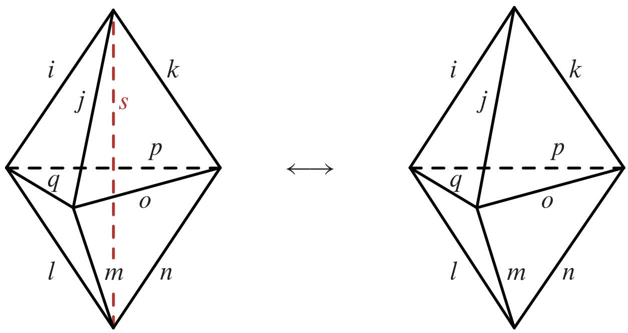

Figure 4. The Pachner Mover 2–3.

The second ingredient is algebraic, and is provided by the properties of $6j$-symbols given in § 2.2. Indeed, exactly as in [60], Corollary 2.3 of the Orthogonality Property of Proposition 2.1 shows that $\mathrm{TV}_r(M,\mathcal{T})$ is unchanged as we modify the triangulation $\mathcal{T}$ by a 0–2 move, and the Biedenharn–Elliot identity (2.3) guarantees the invariance under the 2–3 move. □

As mentioned in § 2.1, a triangulation of a pseudo 3-manifold without inner vertices determines an ideal triangulation of the 3-manifold with boundary obtained by removing all the singular neighborhoods. Hence for a 3-manifold M with boundary, one can define $\mathrm{TV}_r(M)$ using an ideal triangulation of M. Our invariant (and its construction) then coincides with the one defined in [10] using o-graphs.

Theorem 2.6 shows that, for any r and q as above, $\mathrm{TV}_q(M,\mathcal{T})$ is independent of the choice of of the triangulation $\mathcal{T}$. We will consequently omit the triangulation $\mathcal{T}$ and denote the invariant by $\mathrm{TV}_r(M)$ if $q = e^{\frac{2\pi\sqrt{-1}}{r}}$, or by $\mathrm{TV}_r(M;q)$ if we want to emphasize which root of unity q is being used.

3. Evidence for 3-manifolds with boundary

We now provide numerical evidence for Conjecture 1.1 for a few hyperbolic 3-manifolds with boundary. The closed manifold case will be considered in the next section. The reason for considering the two cases separately is that a hyperbolic 3-manifold with boundary often admits an ideal triangulation by a small number of tetrahedra, whereas a triangulation of a closed hyperbolic 3-manifold usually requires more tetrahedra. For example, it takes at least nine tetrahedra to triangulate the Weeks manifold, which is the smallest closed hyperbolic 3-manifold.

To simplify the notation, set

$$\mathrm{QV}_r(M) = \frac{2\pi}{r-2}\log(\mathrm{TV}_r(M; e^{\frac{2\pi\sqrt{-1}}{r}}))$$

for each odd integer $r \geqslant 3$. Similarly, write

$$\mathrm{TV}_r(L) = \mathrm{TV}_r(S^3 \setminus L) \quad \text{and} \quad \mathrm{QV}_r(L) = \mathrm{QV}_r(S^3 \setminus L)$$

when $M = S^3 \setminus L$ is a link complement.

3.1. The figure-eight knot complement and its sister. By Thurston's famous construction [56], the figure-eight knot complement $S^3 \setminus K_{4_1}$ has volume

$$\mathrm{Vol}(S^3 \setminus K_{4_1}) \approx 2.02988,$$

and has the ideal triangulation represented in Figure 5. In that figure, edges with the same labels (a or b) are glued together following the indicated orientations to form an edge of the ideal triangulation.

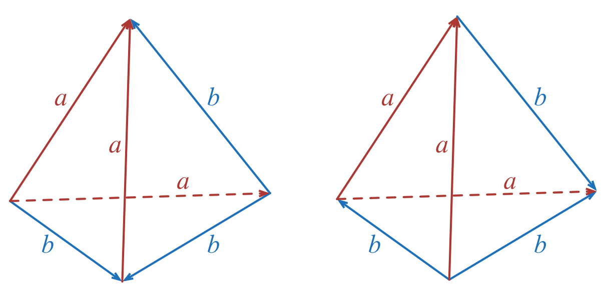

Figure 5

By Definition 2.5, we have

$$\mathrm{TV}_r(K_{4_1}) = \sum_{(a,b)\in A_r} w_a w_b \begin{vmatrix} a & a & b \\ a & b & b \end{vmatrix} \begin{vmatrix} a & a & b \\ a & b & b \end{vmatrix},$$

where A_r consists of the pairs (a,b) of elements of I_r such that (a,a,b) and (b,b,a) are admissible, i.e., $2a-b \geqslant 0$, $2b-a \geqslant 0$, $2a+b \leqslant r-2$, $2b+a \leqslant r-2$ and $2a+b$ and $2b+a$ are integers. From this formula, we have the following table of values of $\mathrm{QV}_r(K_{4_1})$.

r	11	13	15
$\mathrm{QV}_r(K_{4_1})$	2.40661	2.37755	2.34826
r	17	19	21
$\mathrm{QV}_r(K_{4_1})$	2.31907	2.29953	2.28227
r	23	25	31
$\mathrm{QV}_r(K_{4_1})$	2.26834	2.25634	2.22824
r	41	51	61
$\mathrm{QV}_r(K_{4_1})$	2.19685	2.17540	2.15953
r	71	81	91
$\mathrm{QV}_r(K_{4_1})$	2.14721	2.13731	2.12915
r	101	111	121
$\mathrm{QV}_r(K_{4_1})$	2.12230	2.11643	2.11136
r	131	141	151
$\mathrm{QV}_r(K_{4_1})$	2.10692	2.10299	2.09949
r	201	301	401
$\mathrm{QV}_r(K_{4_1})$	2.08641	2.07168	2.06344
r	501	701	1001
$\mathrm{QV}_r(K_{4_1})$	2.05810	2.05153	2.04614

Figure 6 below compares the values of the Turaev–Viro invariants $\mathrm{QV}_r(K_{4_1})$ and the Kashaev invariants $\langle K_{4_1}\rangle_r$ for various values of r. The dots represent the points $(r, \mathrm{QV}_r(K_{4_1}))$, the diamonds represent the points $(r, \frac{2\pi}{r}\log|\langle K_{4_1}\rangle_r|)$, and the squares represent the points $(r, \mathrm{Vol}(S^3 \setminus K_{4_1}))$. Note that the values of $\mathrm{QV}_r(K_{4_1})$ appear to converge to $\mathrm{Vol}(S^3 \setminus K_{4_1})$ much faster than $\langle K_{4_1}\rangle_r$ as r becomes large.

The manifold M_{2_2} in the Callahan–Hildebrand–Weeks census [14], also known as the "figure-eight sister," shares the same volume with the figure-eight knot complement, i.e.,

$$\mathrm{Vol}(M_{2_2}) = \mathrm{Vol}(S^3 \setminus K_{4_1}) \approx 2.02988.$$

It is also known that M_{2_2} is not the complement of any knot in S^3. According to Regina [12], M_{2_2} has the ideal triangulation represented in Figure 7.

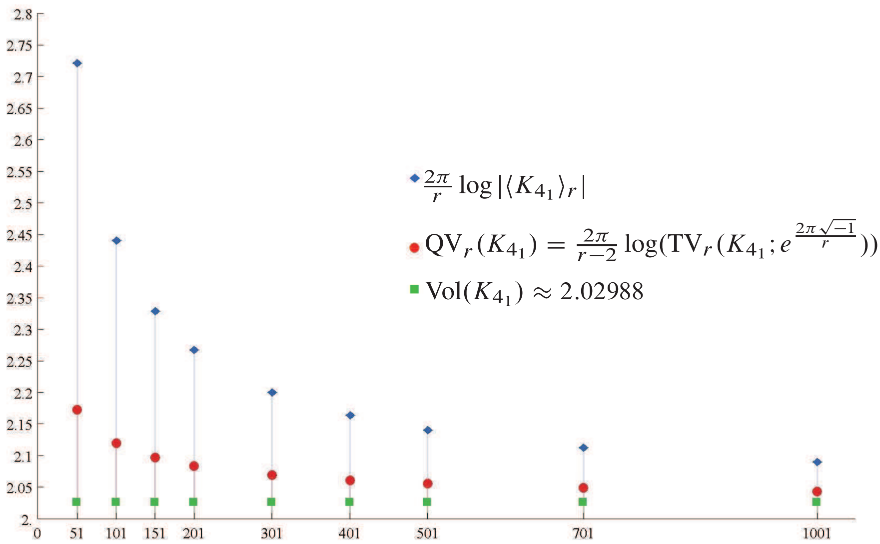

Figure 6. Comparison of different invariants for K_{4_1}.

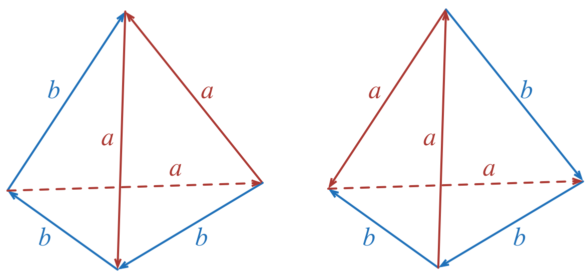

Figure 7

Since for each tetrahedron in this triangulation, the coloring is the same as that of $S^3 \setminus K_{4_1}$, the invariant $\mathrm{TV}_r(M_{2_2})$ has exactly the same formula as $\mathrm{TV}_r(K_{4_1})$. As a consequence, the Turaev–Viro invariants of these manifolds take the same values.

3.2. The K_{5_2} knot complement and its sisters. According to SnapPy [21] and Regina [12], the complement of the knot K_{5_2} has volume

$$\mathrm{Vol}(S^3 \setminus K_{5_2}) \approx 2.82812,$$

and admits the ideal triangulation represented in Figure 8. Since only the colors of the edges (according to which the edges are identified to form an edge of the triangulation) matters in the calculation of $\mathrm{TV}_r(M)$, we omit the arrows on the edges.

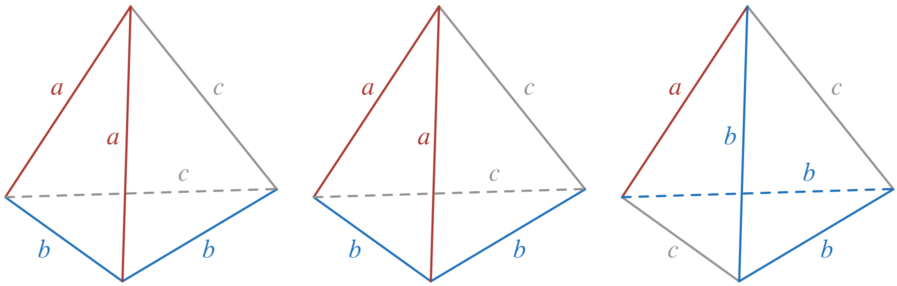

Figure 8

By Definition 2.5, we have

$$\mathrm{TV}_r(K_{5_2}) = \sum_{(a,b,c)\in A_r} w_a w_b w_c \begin{vmatrix} a & a & b \\ b & c & c \end{vmatrix} \begin{vmatrix} a & a & b \\ b & c & c \end{vmatrix} \begin{vmatrix} a & b & c \\ b & b & c \end{vmatrix},$$

where A_r consists of triples (a,b,c) of elements of I_r such that (a,a,b), (b,b,c), (c,c,a) and (a,b,c) are admissible. From this, we have the following table of values of $\mathrm{QV}_r(K_{5_2})$.

r	7	9	11	21
$\mathrm{QV}_r(K_{5_2})$	3.38531	3.32394	3.25282	3.09588
r	31	41	51	61
$\mathrm{QV}_r(K_{5_2})$	3.03657	3.00236	2.97925	2.96232
r	71	81	91	101
$\mathrm{QV}_r(K_{5_2})$	2.94927	2.93883	2.93027	2.92309
r	121	151	201	301
$\mathrm{QV}_r(K_{5_2})$	2.91169	2.89937	2.88586	2.87071

Figure 9 compares the values of $\mathrm{QV}_r(K_{5_2})$ with those of the Kashaev invariants $\langle K_{5_2}\rangle_r$. Again, the dots represent the points $(r, \mathrm{QV}_r(K_{5_2}))$, the diamonds represent the points $\left(r, \frac{2\pi}{r}\log|\langle K_{5_2}\rangle_r|\right)$, and the squares represent the points $(r, \mathrm{Vol}(S^3\setminus K_{5_2}))$.

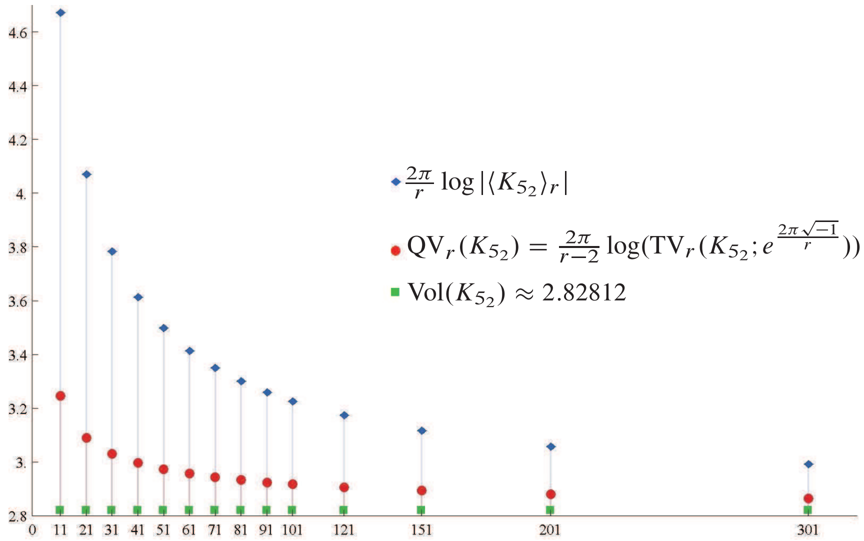

Figure 9. Comparison of different invariants for K_{5_2}.

The manifold M_{3_6} in the Callahan–Hildebrand–Weeks census [14] is also the complement of the $(-2, 3, 7)$-pretzel knot of Figure 10. It has the same volume as $S^3 \setminus K_{5_2}$, namely

$$\mathrm{Vol}(M_{3_6}) = \mathrm{Vol}(S^3 \setminus K_{5_2}) \approx 2.82812.$$

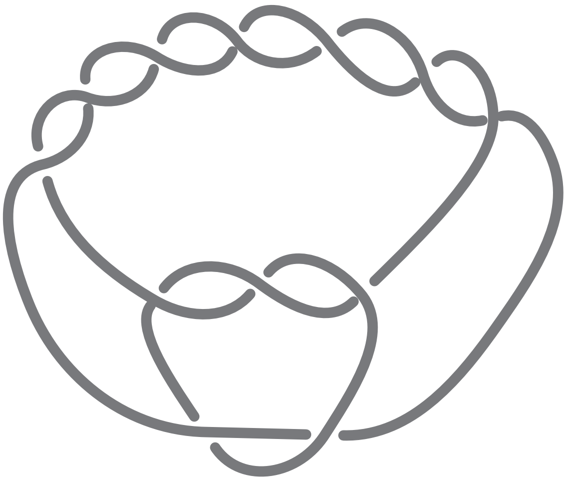

Figure 10. The $(-2, 3, 7)$–pretzel knot.

According to Regina, M_{3_6} can be represented by the ideal triangulation of Figure 11.

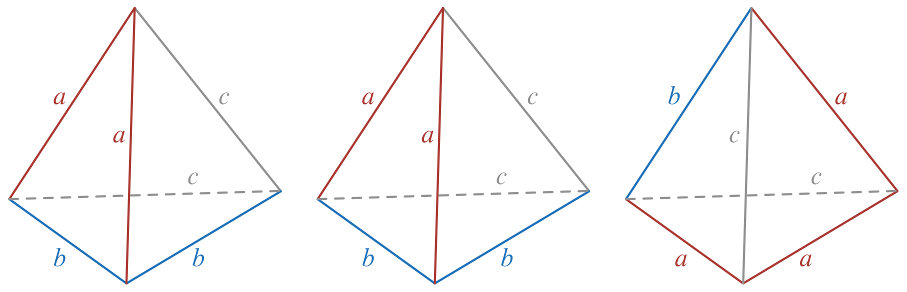

Figure 11

Then for $r \geqslant 3$, we have

$$\mathrm{TV}_r(M_{3_6}) = \sum_{(a,b,c)\in A_r} w_a w_b w_c \begin{vmatrix} a & a & b \\ b & c & c \end{vmatrix} \begin{vmatrix} a & a & b \\ b & c & c \end{vmatrix} \begin{vmatrix} a & b & c \\ a & a & c \end{vmatrix},$$

where A_r consists of all triples (a, b, c) of elements of I_r such that (a, a, b), (b, b, c), (c, c, a), (a, b, c) and (a, a, c) are admissible.

The table below shows a few values of $\mathrm{QV}_r(K_{5_2})$ and $\mathrm{QV}_r(M_{3_6})$. We observe that $\mathrm{QV}_r(K_{5_2})$ and $\mathrm{QV}_r(M_{3_6})$ are distinct, but are getting closer to each other and seem to converge to 2.82812 as r grows.

r	9	11
$\mathrm{QV}_r(K_{5_2})$	3.3239396087031623282	3.2528240712684816477
$\mathrm{QV}_r(M_{3_6})$	3.2936286562299185780	3.2291939333749922011
r	21	31
$\mathrm{QV}_r(K_{5_2})$	3.0958786489268195966	3.0365668215995635907
$\mathrm{QV}_r(M_{3_6})$	3.0954357480831343159	3.0365081953458580040
r	51	101
$\mathrm{QV}_r(K_{5_2})$	2.9792536251826401549	2.9230944207585713174
$\mathrm{QV}_r(M_{3_6})$	2.9792532229139281449	2.9230944207610719723

The manifold M_{3_8} in the Callahan–Hildebrand–Weeks census [14], which is not the complement of any knot in S^3, also has the same volume as $S^3 \setminus K_{5_2}$. According to Regina, M_{3_8} has an ideal triangulation that has the same colors as that of $S^3 \setminus K_{5_2}$ drawn above. As a consequence, $\mathrm{QV}_r(M_{3_8})$ coincides with $\mathrm{QV}_r(K_{5_2})$ for all $r \geqslant 3$.

3.3. The K_{6_1} knot complement. According to SnapPy [21] and Regina [12], the complement of the knot K_{6_1} has volume

$$\mathrm{Vol}(S^3 \setminus K_{6_1}) \approx 3.16396,$$

and can be described by the ideal triangulation of Figure 12.

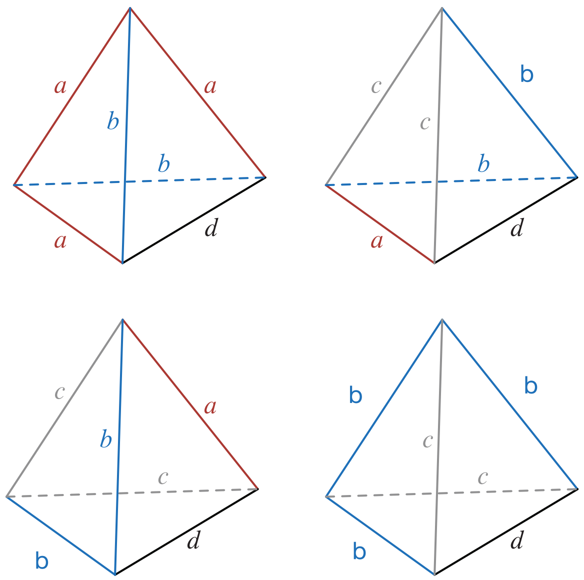

Figure 12

This gives

$$\mathrm{TV}_r(K_{6_1}) = \sum_{(a,b,c,d)\in A_r} w_a w_b w_c w_d \begin{vmatrix} a & a & b \\ a & d & b \end{vmatrix} \begin{vmatrix} a & c & c \\ b & b & d \end{vmatrix} \begin{vmatrix} b & b & c \\ a & c & d \end{vmatrix} \begin{vmatrix} b & b & c \\ b & d & c \end{vmatrix},$$

where A_r consists of quadruples (a, b, c, d) of elements of I_r such that all the triples involved are admissible. From this, we have the following table of values of $\mathrm{QV}_r(K_{6_1})$.

r	21	31	41	51
$\mathrm{QV}_r(K_{6_1})$	3.34732	3.31699	3.29688	3.28214
r	5	7	9	11
$\mathrm{QV}_r(K_{6_1})$	3.34732	3.31699	3.29688	3.28214
r	61	71	81	91
$\mathrm{QV}_r(K_{6_1})$	3.27076	3.26165	3.25417	3.24790
r	101	121	151	201
$\mathrm{QV}_r(K_{6_1})$	3.24255	3.23390	3.22431	3.21353

Figure 13 compares a few values of $\mathrm{QV}_r(K_{6_1})$ with those of the Kashev invariant $\langle K_{6_1}\rangle_r$ and with the volume $\mathrm{Vol}(S^3 \setminus K_{6_1})$.

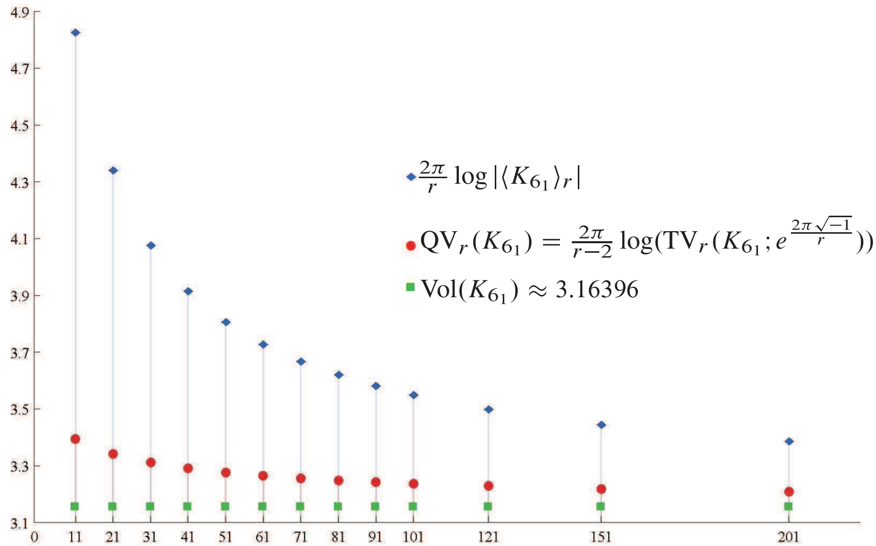

Figure 13. Comparison of different invariants for K_{6_1}.

3.4. Some non-orientable cusped 3-manifolds

3.4.1. The Gieseking manifold. The manifold N_{1_1} in the Callahan–Hildebrand–Weeks census [14], also known as the Gieseking manifold, is the smallest non-orientable cusped 3-manifold. It has an ideal triangulation with a single tetrahedron which, by an Euler characteristic calculation, has only one edge. According to SnapPy [21] and Regina [12], the Gieseking manifold has volume

$$\mathrm{Vol}(N_{1_1}) \approx 1.01494.$$

By Definition 2.5, we have

$$\mathrm{TV}_r(N_{1_1}) = \sum_{a\in A_r} w_a \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix},$$

where A_r consist of integers a such that $0 \leqslant a \leqslant \lfloor (r-2)/3 \rfloor$. Here $\lfloor\ \rfloor$ is the floor function that $\lfloor x \rfloor$ equals the greatest integer less than or equal to x. From this, we have the following table of values of $\mathrm{QV}_r(N_{1_1})$.

r	7	9	11	21
$QV_r(N_{1_1})$	1.81736	1.66782	1.62276	1.43255
r	31	41	51	61
$QV_r(N_{1_1})$	1.33012	1.27064	1.23174	1.20411
r	71	81	91	101
$QV_r(N_{1_1})$	1.18335	1.16711	1.15401	1.14319
r	201	301	401	501
$QV_r(N_{1_1})$	1.08943	1.06872	1.05748	1.05035

3.4.2. Manifold N_{2_1}. According to SnapPy [21] and Regina [12], the manifold N_{2_1} in Callahan–Hildebrand–Weeks census [14] has volume

$$\mathrm{Vol}(N_{2_1}) \approx 1.83193,$$

and has the following ideal triangulation in Figure 14.

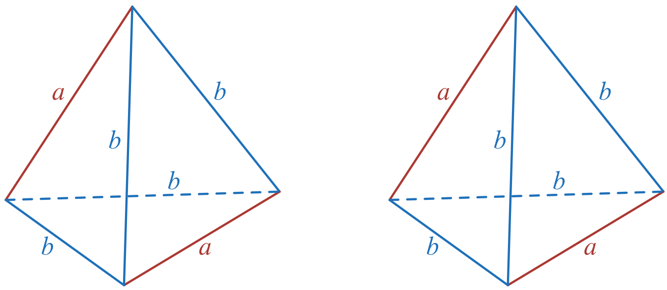

Figure 14

By Definition 2.5, we have

$$\mathrm{TV}_r(N_{2_1}) = \sum_{(a,b)\in A_r} w_a w_b \begin{vmatrix} a & b & b \\ a & b & b \end{vmatrix} \begin{vmatrix} a & b & b \\ a & b & b \end{vmatrix},$$

where A_r consist of the pairs (a,b) of elements of I_r such that (a,b,b) is admissible. From this, we have the following table of values of $QV_r(N_{2_1})$.

r	5	7	9	11
$QV_r(N_{2_1})$	2.90345	2.54929	2.46119	2.42036
r	21	31	41	51
$QV_r(N_{2_1})$	2.20099	2.11235	2.06163	2.02810
r	61	71	81	91
$QV_r(N_{2_1})$	2.00403	1.98578	1.97140	1.95974
r	101	121	151	201
$QV_r(N_{2_1})$	1.95006	1.93489	1.91876	1.90140

3.5. Smallest hyperbolic 3-manifolds with totally geodesic boundary. By [37], any orientable hyperbolic 3-manifold $M_{\min}$ with non-empty totally geodesic boundary that has minimum volume admits a tetrahedral decomposition by two regular truncated hyperideal tetrahedra of dihedral angles $\pi/6$. As a consequence, such a hyperbolic manifold has volume

$$\mathrm{Vol}(M_{\min}) \approx 6.452.$$

Such minimums are not unique and are classified in [23]. In particular, the boundary of each of them is a connected surface of genus 2, and an Euler characteristic calculation shows that each ideal triangulation of $M_{\min}$ with two tetrahedra has only one edge, as in Figure 15.

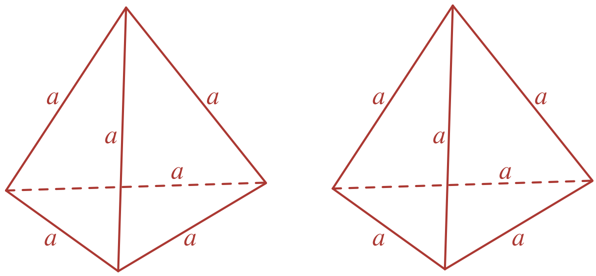

Figure 15

Therefore, for $r \geqslant 3$,

$$\mathrm{TV}_r(M_{\min}) = \sum_{a \in A_r} w_a \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix} \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix},$$

where A_r consists of all integers a with $0 \leqslant a \leqslant \lfloor (r-2)/3 \rfloor$. Here $\mathrm{TV}_r(M_{\min})$ is negative for some values of r. In this case, we require the argument of the logarithm to be in the interval $[0, 2\pi)$, so that the imaginary part of $\mathrm{QV}_r(M_{\min}) = \frac{2\pi}{r-2}\log(\mathrm{TV}_r(M_{\min}))$ is either 0 or $2\pi^2/(r-2)$. As a consequence, this imaginary part converges to 0 and it suffices to consider the real part to test the convergence of $\mathrm{QV}_r(M_{\min})$.

We have the following table of the values of the real part $\Re(\mathrm{QV}_r(M_{\min}))$ of $\mathrm{QV}_r(M_{\min})$.

r	11	21	31	41]
$\Re(\mathrm{QV}_r(M_{\min}))$	4.39782	5.12434	5.44590	5.63235
r	51	61	71	81
$\Re(\mathrm{QV}_r(M_{\min}))$	5.75566	5.84395	5.91063	5.96297
r	91	101	201	301
$\Re(\mathrm{QV}_r(M_{\min}))$	6.00526	6.04022	6.21400	6.28075
r	401	501	1001	2001
$\Re(\mathrm{QV}_r(M_{\min}))$	6.31684	6.33970	6.38935	6.41741

Figure 16 below illustrates the asymptotic behavior of $\mathrm{QV}_r(M_{\min})$, where the dots represent the points $(r, \mathrm{QV}_r(M_{\min}))$ and the squares represent the points $(r, \mathrm{Vol}(M_{\min}))$.

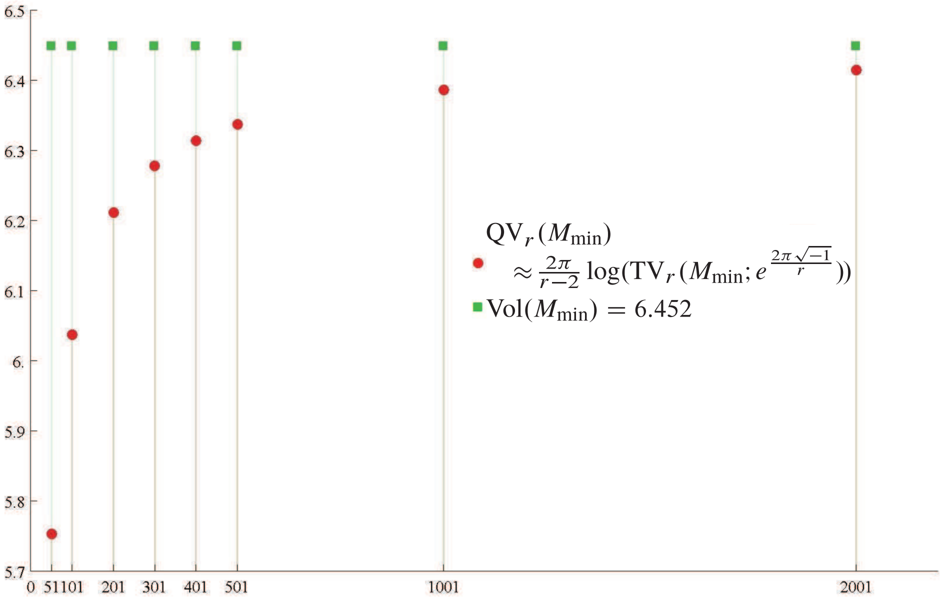

Figure 16. Asymptotics of $\mathrm{QV}_r(M_{\min})$.

4. Evidence for closed 3-manifolds

We now consider Conjectures 1.1 and 1.2 in the case of closed manifolds ([55, 59, 61]), and provide evidence for these conjectures by numerically calculating the Reshetikhin–Turaev invariants of a few closed 3-manifolds obtained by doing Dehn surgeries along the knots K_{4_1} and K_{5_2}.

According to [39], if M is obtained from S^3 by doing a p-surgery along a knot K, then for an odd $r \geqslant 3$ the Reshetikhin–Turaev invariant $\mathrm{RT}_r(M;q)$ of M at $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ is calculated as

$$\mathrm{RT}_r(M;q) = \frac{2}{r} e^{-\epsilon(p)\left(-\frac{3}{r}-\frac{r+1}{4}\right)\pi\sqrt{-1}} \Big(\sum_{n=0}^{r-2} \Big(\sin \frac{2(n+1)\pi}{r}\Big)^2 (-e^{\frac{\pi\sqrt{-1}}{r}})^{p(n^2+2n)} J_{n+1}(K; e^{\frac{4\pi\sqrt{-1}}{r}}) \Big), \tag{4.1}$$

where $\epsilon(p)$ is the sign of p, and $J_n(K; e^{\frac{4\pi\sqrt{-1}}{r}})$ is the value of the n-th colored Jones polynomial $J_n(K;t)$ of K at $t = e^{\frac{4\pi\sqrt{-1}}{r}}$, normalized in such a way that $J_n(\text{unknot}) = 1$.

Remark 4.1. The conventions in skein theory ([32, 11, 39]) make use of a variable A that is, either a primitive $2r$-th root of unity for an integer r, or a primitive r-th root of unity for an odd integer r. The root of unity q in the definition of the Turaev–Viro invariant then corresponds to A^2, while the variable t of the colored Jones polynomial corresponds to A^4. Formula (4.1) deals with the case where $A = e^{\frac{\pi\sqrt{-1}}{r}}$ for r odd, in which case $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ and $t = e^{\frac{4\pi\sqrt{-1}}{r}}$.

Remark 4.2. Formula (4.1) is directly derived from [39, § 4.1]. In Lickorish's notation and letting $A = e^{\frac{\pi\sqrt{-1}}{r}}$, one has

$$\mu = \frac{1}{\sqrt{r}} \sin\frac{2\pi}{r},$$

$$\langle\mu\omega\rangle^{\sigma}_{U_-} = e^{-\epsilon(p)\left(-\frac{3}{r}-\frac{r+1}{4}\right)\pi\sqrt{-1}}, \quad \langle\mu\omega\rangle^{-1}_{U} = \frac{2}{\sqrt{r}}\sin\frac{2\pi}{r},$$

and for K_p the knot K with framing p,

$$\langle\omega\rangle_{K_p} = \sum_{n=0}^{r-2}\Big(\frac{\sin\frac{2(n+1)\pi}{r}}{\sin\frac{2\pi}{r}}\Big)^2 (-e^{\frac{\pi\sqrt{-1}}{r}})^{p(n^2+2n)} J_{n+1}(K; e^{\frac{4\pi\sqrt{-1}}{r}}).$$

Multiplying the above terms together, one gets formula (4.1).

To calculate the growth rate of $\mathrm{RT}_r(M;q)$ as r approaches infinity, it is equivalent to calculate the limit of the following quantity:

$$Q_r(M) = 2\pi\sqrt{-1}\log\Big(\frac{\mathrm{RT}_r(M; e^{\frac{2\pi\sqrt{-1}}{r}})}{\mathrm{RT}_{r-2}(M; e^{\frac{2\pi\sqrt{-1}}{r-2}})}\Big),$$

where the logarithm log is chosen so that its imaginary part lies in the interval $(-\pi, \pi)$.

4.1. Surgeries along the figure-eight knot. In this subsection, we denote by M_p the manifold obtained from S^3 by doing a p-surgery along the figure-eight knot K_{4_1}. Recall from [56] that M_p is hyperbolic if and only if $|p| \geqslant 5$. By [43], the n-th colored Jones polynomial of K_{4_1} equals

$$J_n(K_{4_1}, t) = \sum_{k=0}^{n-1}\prod_{i=1}^{k}(t^{\frac{n-i}{2}} - t^{-\frac{n-i}{2}})(t^{\frac{n+i}{2}} - t^{-\frac{n+i}{2}}). \tag{4.2}$$

In the tables below, we list the values of $Q_r(M_p)$ modulo $\pi^2\mathbb{Z}$ for $p = -6, -5, 5, 6, 7, 8$ and for $r = 51$, 101, 151, 201, 301 and 501.

4.1.1. $p = -6$. According to SnapPy [21],

$$\mathrm{CS}(M_{-6}) + \mathrm{Vol}(M_{-6})\sqrt{-1} \approx -1.34092 + 1.28449\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.2), we have

r	51	101
$Q_r(M_{-6})$	$-1.34241 + 1.22717\sqrt{-1}$	$-1.32879 + 1.28425\sqrt{-1}$
r	151	201
$Q_r(M_{-6})$	$-1.33549 + 1.28440\sqrt{-1}$	$-1.33786 + 1.28443\sqrt{-1}$
r	301	501
$Q_r(M_{-6})$	$-1.33956 + 1.28446\sqrt{-1}$	$-1.34043 + 1.28448\sqrt{-1}$

4.1.2. $p = -5$. According to SnapPy [21],

$$\mathrm{CS}(M_{-5}) + \mathrm{Vol}(M_{-5})\sqrt{-1} \approx -1.52067 + 0.98137\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.2), we have

r	51	101
$Q_r(M_{-5})$	$-1.50445 + 0.87410\sqrt{-1}$	$-1.51521 + 0.98003\sqrt{-1}$
r	151	201
$Q_r(M_{-5})$	$-1.51712 + 0.98130\sqrt{-1}$	$-1.51865 + 0.98131\sqrt{-1}$
r	301	501
$Q_r(M_{-5})$	$-1.51977 + 0.98134\sqrt{-1}$	$-1.52035 + 0.98136\sqrt{-1}$

4.1.3. $p = 5$. According to SnapPy [21],

$$\mathrm{CS}(M_{5}) + \mathrm{Vol}(M_{5})\sqrt{-1} \approx 1.52067 + 0.98137\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.2), we have

r	51	101
$Q_r(M_{5})$	$1.50445 + 0.87410\sqrt{-1}$	$1.51521 + 0.98003\sqrt{-1}$
r	151	201
$Q_r(M_{5})$	$1.51712 + 0.98130\sqrt{-1}$	$1.51865 + 0.98131\sqrt{-1}$
r	301	501
$Q_r(M_{5})$	$1.51977 + 0.98134\sqrt{-1}$	$1.52035 + 0.98136\sqrt{-1}$

4.1.4. $p = 6$. According to SnapPy [21],

$$\mathrm{CS}(M_6) + \mathrm{Vol}(M_{-6})\sqrt{-1} \approx 1.34092 + 1.28449\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.2), we have

r	51	101
$Q_r(M_6)$	$1.34241 + 1.22717\sqrt{-1}$	$1.32879 + 1.28425\sqrt{-1}$
r	151	201
$Q_r(M_6)$	$1.33549 + 1.28440\sqrt{-1}$	$1.33786 + 1.28443\sqrt{-1}$
r	301	501
$Q_r(M_6)$	$1.33956 + 1.28446\sqrt{-1}$	$1.34043 + 1.28448\sqrt{-1}$

4.1.5. $p = 7$. According to SnapPy [21],

$$\mathrm{CS}(M_7) + \mathrm{Vol}(M_7)\sqrt{-1} \approx 1.19653 + 1.46378\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.2), we have

r	51	101
$Q_r(M_7)$	$1.10084 + 1.43670\sqrt{-1}$	$1.18016 + 1.46354\sqrt{-1}$
r	151	201
$Q_r(M_7)$	$1.18930 + 1.46367\sqrt{-1}$	$1.19246 + 1.46372\sqrt{-1}$
r	301	501
$Q_r(M_7)$	$1.19472 + 1.46375\sqrt{-1}$	$1.19588 + 1.46377\sqrt{-1}$

4.1.6. $p = 8$. According to SnapPy [21],

$$\mathrm{CS}(M_8) + \mathrm{Vol}(M_8)\sqrt{-1} \approx 1.07850 + 1.58317\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.2), we have

r	51	101
$Q_r(M_8)$	$0.96311 + 1.57167\sqrt{-1}$	$1.05821 + 1.58282\sqrt{-1}$
r	151	201
$Q_r(M_8)$	$1.06949 + 1.58304\sqrt{-1}$	$1.07343 + 1.58309\sqrt{-1}$
r	301	501
$Q_r(M_8)$	$1.07625 + 1.58313\sqrt{-1}$	$1.07769 + 1.58315\sqrt{-1}$

4.2. Surgeries along K_{5_2}. In this subsection, we let M_p be the manifold obtained from S^3 by doing a p-surgery along the knot K_{5_2}. Recall that M_p is hyperbolic if and only if $p \leqslant -1$ or $p \geqslant 5$. By [40], the n-th colored Jones polynomial of K_{5_2} is equal to

$$J_n(K_{5_2}, t) = \sum_{k=0}^{n-1} t^{-\frac{k(k+3)}{4}} c_k \prod_{i=1}^{k} (t^{\frac{n-i}{2}} - t^{-\frac{n-i}{2}})(t^{\frac{n+i}{2}} - t^{-\frac{n+i}{2}}), \tag{4.3}$$

where

$$c_k = (-1)^k t^{-\frac{5k^2+7k}{4}} \sum_{i=0}^{k} t^{-\frac{i^2-2i-3ki}{2}} \frac{[k]!}{[i]![k-i]!}.$$

Here the formula differs from that of [40] by replacing t with t^{-1}. This comes from the chirality of K_{5_2}. Here we stick to the convention that is used in SnapPy [21], which is the mirror image of the one used in [40].

In the tables below, we list the values of $Q_r(M_p)$ modulo $\pi^2\mathbb{Z}$ for $p = -3, -2, -1, 5, 6, 7$ and for $r = 51, 75, 101, 125, 151$ and 201.

4.2.1. $p = -3$. According to SnapPy [21],

$$\mathrm{CS}(M_{-3}) + \mathrm{Vol}(M_{-3})\sqrt{-1} \approx -4.45132 + 2.10310\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.3), we have

r	51	75
$Q_r(M_{-3})$	$-4.37951 + 2.10038\sqrt{-1}$	$-4.41819 + 2.10200\sqrt{-1}$
r	101	125
$Q_r(M_{-3})$	$-4.43323 + 2.10247\sqrt{-1}$	$-4.43957 + 2.10268\sqrt{-1}$
r	151	201
$Q_r(M_{-3})$	$-4.44329 + 2.10281\sqrt{-1}$	$-4.44681 + 2.10293\sqrt{-1}$

4.2.2. $p = -2$. According to SnapPy [21],

$$\mathrm{CS}(M_{-2}) + \mathrm{Vol}(M_{-2})\sqrt{-1} \approx -4.63884 + 1.84359\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.3), we have

r	51	75
$Q_r(M_{-2})$	$-4.59073 + 1.84822\sqrt{-1}$	$-4.61357 + 1.84289\sqrt{-1}$
r	101	125
$Q_r(M_{-2})$	$-4.62490 + 1.84317\sqrt{-1}$	$-4.62978 + 1.84331\sqrt{-1}$
r	151	201
$Q_r(M_{-2})$	$-4.63265 + 1.84339\sqrt{-1}$	$-4.63536 + 1.84348\sqrt{-1}$

4.2.3. $p = -1$. According to SnapPy [21],

$$\mathrm{CS}(M_{-1}) + \mathrm{Vol}(M_{-1})\sqrt{-1} \approx -4.86783 + 1.39851\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.3), we have

r	51	75
$Q_r(M_{-1})$	$-4.85045 + 1.39808\sqrt{-1}$	$-4.85817 + 1.39817\sqrt{-1}$
r	101	125
$Q_r(M_{-1})$	$-4.84865 + 1.40943\sqrt{-1}$	$-4.86157 + 1.39827\sqrt{-1}$
r	151	201
$Q_r(M_{-1})$	$-4.86355 + 1.39834\sqrt{-1}$	$-4.86542 + 1.39841\sqrt{-1}$

4.2.4. $p = 5$. According to SnapPy [21],

$$\mathrm{CS}(M_5) + \mathrm{Vol}(M_5)\sqrt{-1} \approx -1.52067 + 0.98137\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.3), we have

r	51	75
$Q_r(M_5)$	$-1.50445 + 0.87410\sqrt{-1}$	$-1.48899 + 0.96890\sqrt{-1}$
r	101	125
$Q_r(M_5)$	$-1.51521 + 0.98003\sqrt{-1}$	$-1.51539 + 0.98098\sqrt{-1}$
r	151	201
$Q_r(M_5)$	$-1.51712 + 0.98130\sqrt{-1}$	$-1.51865 + 0.98131\sqrt{-1}$

4.2.5. $p = 6$. According to SnapPy [21],

$$\mathrm{CS}(M_6) + \mathrm{Vol}(M_6)\sqrt{-1} \approx -1.51206 + 1.41406\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.3), we have

r	51	75
$Q_r(M_6)$	$-1.46756 + 1.40044\sqrt{-1}$	$-1.50631 + 1.41501\sqrt{-1}$
r	101	125
$Q_r(M_6)$	$-1.50836 + 1.41339\sqrt{-1}$	$-1.50968 + 1.41356\sqrt{-1}$
r	151	201
$Q_r(M_6)$	$-1.51042 + 1.41372\sqrt{-1}$	$-1.51113 + 1.41386\sqrt{-1}$

4.2.6. $p = 7$. According to SnapPy [21],

$$\mathrm{CS}(M_7) + \mathrm{Vol}(M_7)\sqrt{-1} \approx -1.55255 + 1.75713\sqrt{-1} \mod \pi^2\mathbb{Z},$$

and by (4.1) and (4.3), we have

r	51	75
$Q_r(M_7)$	$-1.53822 + 1.75178\sqrt{-1}$	$-1.55297 + 1.75315\sqrt{-1}$
r	101	125
$Q_r(M_7)$	$-1.55265 + 1.75507\sqrt{-1}$	$-1.55257 + 1.75582\sqrt{-1}$
r	151	201
$Q_r(M_7)$	$-1.55255 + 1.75625\sqrt{-1}$	$-1.55254 + 1.75664\sqrt{-1}$

5. An integrality conjecture for torus link complements

In this section, we study the Turaev–Viro invariants for torus link complements. We propose the following Integrality Conjecture 5.1, and provide evidence by both rigorous (§ 5.1) and numerical (§ 5.2) calculations.

Conjecture 5.1. *Let $T_{(m,n)}$ be the (m,n)-torus link in S^3. If r is relatively prime to m and n, then $\mathrm{TV}_r(S^3 \setminus T_{(m,n)})$ is an integer independent of the choice of the roots of unity q.*

5.1. Calculations for some torus links. In this subsection, we will rigorously calculate $\mathrm{TV}_r(M)$ for the complements of the unknot, the trefoil knot, the Hopf link and the torus links $T_{(2,4)}$ and $T_{(2,6)}$. As in the previous sections, for a link L in S^3 we let

$$\mathrm{TV}_r(L) = \mathrm{TV}_r(S^3 \setminus L).$$

All the ideal triangulations used in this section are obtained by using Regina [12] and SnapPy [21], and for simplicity, we will omit the arrows on the edges and keep only the colors.

5.1.1. The unknot

Proposition 5.2. *Let U be the unknot in S^3. Then*

$$\mathrm{TV}_r(U) = 1$$

for all $r \geqslant 3$ and for all $q \in \mathbb{C}$ such that q^2 is a primitive root of unity of degree r.

Proof. The complement of the unknot admits the ideal triangulation represented in Figure 17.

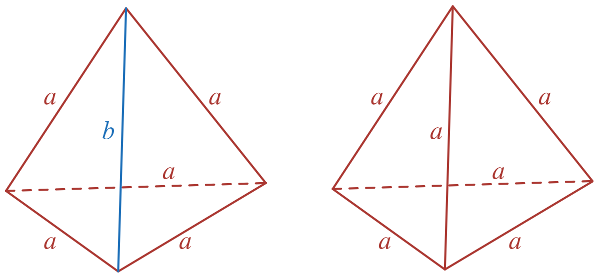

Figure 17

Therefore, for each $r \geqslant 3$, we have

$$\mathrm{TV}_r(U) = \sum_{a,b} w_a w_b \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix}$$
$$= \sum_{a} w_a \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix} \Big(\sum_b w_b \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} \Big),$$

where in the first row $(a,b) \in I_r \times I_r$ runs over all the admissible colorings at level r, and in the second row a is over all the elements of I_r such that (a,a,a) is admissible and b is over all elements of I_r such that (a,a,b) is admissible. Then the result follows from the following identity

$$\sum_b w_b \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} = \delta_{0,a}. \tag{5.1}$$

To prove (5.1), we use the Orthogonality Property. Letting $m = 0$, $s = b$ and $i = j = k = l = n = a$ in (2.1), we have

$$\sum_b w_b w_0 \begin{vmatrix} a & a & 0 \\ a & a & b \end{vmatrix} \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} = \delta_{0,a},$$

where b is over all elements of I_r such that (a,b) is admissible at level r. Since $w_0 = 1$ and

$$\begin{vmatrix} a & a & 0 \\ a & a & b \end{vmatrix} = \frac{1}{[2a+1]},$$

we have

$$\sum_b w_b \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} = [2a+1] \cdot \delta_{0,a} = \delta_{0,a}. \qquad \square$$

Conjecture 5.3. *Let K be a knot in S^3. Then $\mathrm{TV}_r(K) = 1$ for all $r \geqslant 3$ and for all $q \in \mathbb{C}$ such that q^2 is a primitive root of unity of degree r if and only if $K = U$.*

Remark 5.4. It is interesting to know whether there is an $M \neq S^3 \setminus U$, not necessarily a link complement, such that $\mathrm{TV}_r(M) = 1$ for all r and q.

5.1.2. The trefoil knot

Proposition 5.5. *Let $T_{(2,3)}$ be the trefoil knot in S^3. Then*

$$\mathrm{TV}_r(T_{(2,3)}) = \left\lfloor \frac{r-2}{3} \right\rfloor + 1$$

for all $r \geqslant 3$ and for all $q \in \mathbb{C}$ such that q^2 is a primitive root of unity of degree r.

Proof. The complement of trefoil knot $T_{(2,3)}$ admits the ideal triangulation represented in Figure 18.

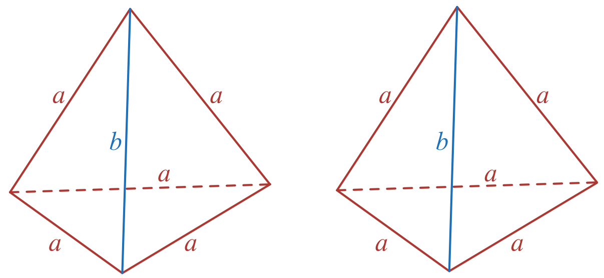

Figure 18

Therefore, for each $r \geqslant 3$, we have

$$\mathrm{TV}_r(T_{(2,3)}) = \sum_{a,b} w_a w_b \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix},$$

where $(a, b) \in I_r \times I_r$ runs over all the admissible colorings at level r. The triple (a, a, a) being admissible implies that $a \in \mathbb{Z}$ and $a \leqslant (r-2)/3$. Hence the right hand side equals

$$\sum_{0 \leqslant a \leqslant \frac{r-2}{3}} \left(\sum_b w_b w_a \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} \right),$$

where a is over all the integers in that range and b is over all elements of I_r such that (a, a, b) is admissible. Letting $i = j = k = l = m = n = a$ and $s = b$ in the Orthogonality Property (2.1), we have

$$\sum_b w_b w_a \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} \begin{vmatrix} a & a & a \\ a & a & b \end{vmatrix} = 1,$$

where b is over all elements of I_r such that (a, a, b) is admissible. As a consequence,

$$\mathrm{TV}_r(T_{(2,3)}) = \sum_{0 \leqslant a \leqslant \frac{r-2}{3}} 1 = \left\lfloor \frac{r-2}{3} \right\rfloor + 1. \qquad \square$$

5.1.3. The Hopf link and torus links $T_{(2,4)}$ and $T_{(2,6)}$

Proposition 5.6. *Let $T_{(2,2)}$ be the Hopf link in S^3. Then*

$$\mathrm{TV}_r(T_{(2,2)}) = r - 1$$

for all $r \geqslant 3$ and for all $q \in \mathbb{C}$ such that q^2 is a primitive root of unity of degree r.

Proof. The complement of the Hopf link admits the ideal triangulation represented in Figure 19.

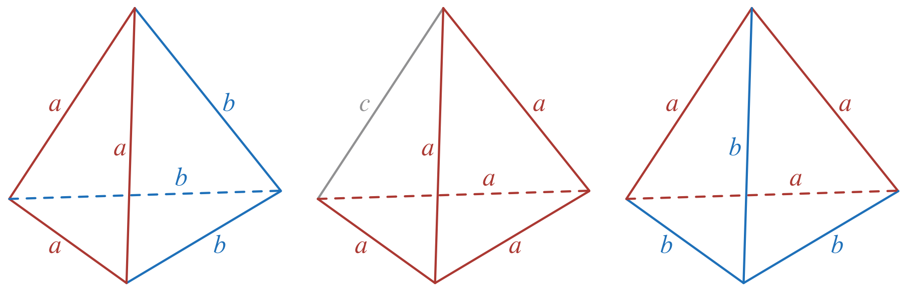

Figure 19

Therefore, for each $r \geqslant 3$, we have

$$\begin{aligned} \mathrm{TV}_r(T_{(2,2)}) &= \sum_{a,b,c} w_a w_b w_c \begin{vmatrix} a & a & a \\ a & a & c \end{vmatrix} \begin{vmatrix} a & a & a \\ b & b & b \end{vmatrix} \begin{vmatrix} a & a & a \\ b & b & b \end{vmatrix} \\ &= \sum_{a,b} w_a w_b \begin{vmatrix} a & a & a \\ b & b & b \end{vmatrix} \begin{vmatrix} a & a & a \\ b & b & b \end{vmatrix} \left(\sum_c w_c \begin{vmatrix} a & a & a \\ a & a & c \end{vmatrix} \right), \end{aligned}$$

where in the first row (a, b, c) runs over all the admissible colorings at level r, and in the second row c runs over all elements of I_r such that all the involved quantum

$6j$-symbols are admissible. By (5.1), we have

$$\sum_c w_c \begin{vmatrix} a & a & a \\ a & a & c \end{vmatrix} = \delta_{0,a}.$$

Therefore,

$$\mathrm{TV}_r(T_{(2,2)}) = \sum_b w_0 w_b \begin{vmatrix} 0 & 0 & 0 \\ b & b & b \end{vmatrix} \begin{vmatrix} 0 & 0 & 0 \\ b & b & b \end{vmatrix} = \sum_b 1,$$

where b is over all the elements in I_r such that $(0, b, b)$ is admissible. Since this holds for all elements b in I_r,

$$\mathrm{TV}_r(T_{(2,2)}) = |I_r| = r - 1. \qquad \square$$

Proposition 5.7. *Let $T_{(2,4)}$ be the $(2,4)$-torus link in S^3. Then*

$$\mathrm{TV}_r(T_{(2,4)}) = \Big(\Big\lfloor \frac{r-2}{2} \Big\rfloor + 1\Big)\Big(\Big\lfloor \frac{r-1}{2} \Big\rfloor + 1\Big)$$

for all $r \geqslant 3$ and for all $q \in \mathbb{C}$ such that q^2 is a primitive root of unity of degree r.

Proof. The complement of the torus link $T_{(2,4)}$ has the following ideal triangulation represented in Figure 20.

Therefore, for each $r \geqslant 3$, we have

$$\begin{aligned} \mathrm{TV}_r(T_{(2,4)}) &= \sum_{(a,b,c,d) \in A_r} w_a w_b w_c w_d \begin{vmatrix} a & a & b \\ c & c & c \end{vmatrix} \begin{vmatrix} a & a & b \\ c & c & c \end{vmatrix} \begin{vmatrix} b & b & b \\ a & a & d \end{vmatrix} \begin{vmatrix} b & b & b \\ a & a & a \end{vmatrix} \\ &= \sum_{a,b,c} w_a w_c \begin{vmatrix} a & a & b \\ c & c & c \end{vmatrix} \begin{vmatrix} a & a & b \\ c & c & c \end{vmatrix} \begin{vmatrix} b & b & b \\ a & a & a \end{vmatrix} \Big(\sum_d w_d w_b \begin{vmatrix} b & b & b \\ a & a & d \end{vmatrix} \Big), \end{aligned}$$

where in the second row a, b, c run over elements of I_r such that all the involved triples are admissible. We claim that

$$\sum_d w_d w_b \begin{vmatrix} b & b & b \\ a & a & d \end{vmatrix} = \sqrt{-1}^{\,2a+2b} \sqrt{[2a+1][2b+1]} \cdot \delta_{0,b}.$$

Indeed, letting $m = 0$, $i = j = a$, $k = l = n = b$ and $s = d$ in the Orthogonality Property (2.1), we have

$$\sum_d w_d w_b \begin{vmatrix} b & b & b \\ a & a & 0 \end{vmatrix} \begin{vmatrix} b & b & b \\ a & a & d \end{vmatrix} = \delta_{0,b}.$$

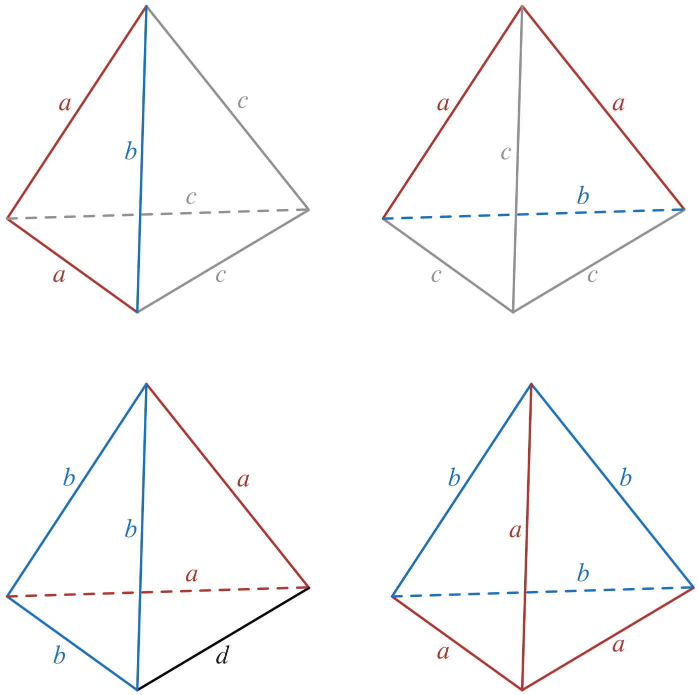

Figure 20

Then the claim follows from the fact that

$$\begin{vmatrix} b & b & b \\ a & a & 0 \end{vmatrix} = \frac{\sqrt{-1}^{2a+2b}}{\sqrt{[2a+1][2b+1]}}.$$

Therefore,

$$\begin{aligned} \mathrm{TV}_r(T_{(2,4)}) &= \sum_{a,c} w_a w_c \begin{vmatrix} a & a & 0 \\ c & c & c \end{vmatrix} \begin{vmatrix} a & a & 0 \\ c & c & c \end{vmatrix} \begin{vmatrix} 0 & 0 & 0 \\ a & a & a \end{vmatrix} \sqrt{-1}^{2a} \sqrt{[2a+1]} \\ &= \sum_{a,c} (-1)^{2a}[2a+1](-1)^{2c}[2c+1] \frac{(-1)^{2a+2c}}{[2a+1][2c+1]} \\ &\qquad \frac{\sqrt{-1}^{2a}}{\sqrt{[2a+1]}} \sqrt{-1}^{2a} \sqrt{[2a+1]} \\ &= \sum_{a,c} 1, \end{aligned}$$

where a, c run over all the elements of I_r such that (c, c, a) and $(a, a, 0)$ are admissible. Counting the number of such pairs (a, c), we have

$$\mathrm{TV}_r(T_{(2,4)}) = \left(\left\lfloor \frac{r-2}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{r-1}{2} \right\rfloor + 1\right). \qquad \square$$

Proposition 5.8. *Let $T_{(2,6)}$ be the $(2,6)$-torus link in S^3. Then*

$$\mathrm{TV}_r(S^3 \setminus T_{(2,6)}) = \left(\left\lfloor \frac{r-2}{3} \right\rfloor + 1\right)\left(\left\lfloor \frac{2r-2}{3} \right\rfloor + 1\right)$$

for all $r \geqslant 3$ and for all $q \in \mathbb{C}$ such that q^2 is a primitive root of unity of degree r.

Proof. The complement of the torus link $T_{(2,6)}$ has the following ideal triangulation represented in Figure 21.

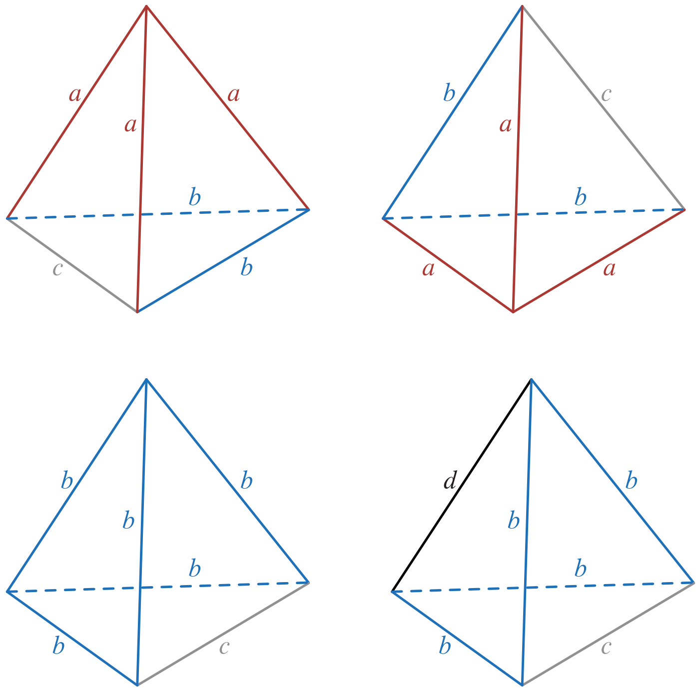

Figure 21

Therefore, for each $r \geqslant 3$, we have

$$\begin{aligned}\mathrm{TV}_r(T_{(2,6)}) &= \sum_{(a,b,c,d)\in A_r} w_a w_b w_c w_d \begin{vmatrix} a & a & c \\ b & b & a \end{vmatrix}\begin{vmatrix} a & a & c \\ b & b & a \end{vmatrix}\begin{vmatrix} b & b & c \\ b & b & b \end{vmatrix}\begin{vmatrix} b & b & d \\ b & b & c \end{vmatrix} \\ &= \sum_{a,b,c} w_a w_b \begin{vmatrix} a & a & c \\ b & b & a \end{vmatrix}\begin{vmatrix} a & a & c \\ b & b & a \end{vmatrix}\begin{vmatrix} b & b & c \\ b & b & b \end{vmatrix}\left(\sum_d w_d w_c \begin{vmatrix} b & b & d \\ b & b & c \end{vmatrix}\right),\end{aligned}$$

where in the second row a, b, c run over elements of I_r such that all the involved triples are admissible. We claim that

$$\sum_d w_d w_c \begin{vmatrix} b & b & d \\ b & b & c \end{vmatrix} = (-1)^{2b}[2b+1] \cdot \delta_{0,c}.$$

Indeed, letting $m = 0$, $i = j = k = l = b$, $n = c$ and $s = d$ in the Orthogonality Property (2.1), we have

$$\sum_d w_d w_c \begin{vmatrix} b & b & d \\ b & b & 0 \end{vmatrix} \begin{vmatrix} b & b & d \\ b & b & c \end{vmatrix} = \delta_{0,c}.$$

Then the claim follows from the fact that

$$\begin{vmatrix} b & b & d \\ b & b & 0 \end{vmatrix} = \frac{(-1)^{2b}}{[2b+1]}.$$

Therefore,

$$\begin{aligned} \mathrm{TV}_r(T_{(2,6)}) &= \sum_{a,b} w_a w_b \begin{vmatrix} a & a & 0 \\ b & b & a \end{vmatrix} \begin{vmatrix} a & a & 0 \\ b & b & a \end{vmatrix} \begin{vmatrix} b & b & 0 \\ b & b & b \end{vmatrix} (-1)^{2b}[2b+1] \\ &= \sum_{a,b} (-1)^{2a}[2a+1](-1)^{2b}[2b+1] \\ &\qquad \frac{(-1)^{2a+2b}}{[2a+1][2b+1]} \frac{(-1)^{2b}}{[2b+1]} (-1)^{2b}[2b+1] \\ &= \sum_{a,b} 1, \end{aligned}$$

where a, b run over all the elements of I_r such that (a, a, b) and (b, b, b) are admissible. Counting the number of such pairs (a, b), we have

$$\mathrm{TV}_r(T_{(2,6)}) = \Big(\Big\lfloor \frac{r-2}{3} \Big\rfloor + 1\Big)\Big(\Big\lfloor \frac{2r-2}{3} \Big\rfloor + 1\Big). \qquad \square$$

Remark 5.9. Conjecture 1.1 can be generalized to non-hyperbolic 3-manifolds by considering the Gromov norm, and Propositions 5.2, 5.5, 5.6, 5.7, 5.8 prove that for the corresponding cases.

5.2. Numerical evidence for Conjecture 5.1. In this subsection, we provide further evidence for Conjecture 5.1 by numerically calculating the Turaev–Viro invariants for the complements of the torus knots $T_{(2,5)}$, $T_{(3,5)}$, $T_{(2,7)}$, $T_{(3,7)}$, $T_{(2,9)}$ and $T_{(2,11)}$.

5.2.1. Knot $T_{(2,5)}$. Table 1 contains the values of $\mathrm{TV}_r(T_{(2,5)}; e^{\frac{k\pi\sqrt{-1}}{r}})$ for $k = 1, 2, 3$ and $r \leqslant 20$.

5.2.2. Knot $T_{(3,5)}$. Table 2 contains the values of $\mathrm{TV}_r(T_{(3,5)}; e^{\frac{k\pi\sqrt{-1}}{r}})$ for $k = 1, 2, 3$ and $r \leqslant 20$.

5.2.3. Knot $T_{(2,7)}$. Table 3 contains the values of $\mathrm{TV}_r(T_{(2,7)}; e^{\frac{k\pi\sqrt{-1}}{r}})$ for $k = 1, 2, 3$ and $r \leqslant 21$.

5.2.4. Knot $T_{(3,7)}$. Table 4 contains the values of $\mathrm{TV}_r\left(T_{(3,7)}; e^{\frac{k\pi\sqrt{-1}}{r}}\right)$ for $k = 1, 2, 3$ and $r \leqslant 21$.

5.2.5. Knot $T_{(2,9)}$. Table 5 contains the values of $\mathrm{TV}_r\left(T_{(2,9)}; e^{\frac{k\pi\sqrt{-1}}{r}}\right)$ for $k = 1, 2, 3$ and $r \leqslant 22$.

5.2.6. Knot $T_{(2,11)}$. Table 6 below contains the values of $\mathrm{TV}_r\left(T_{(2,11)}; e^{\frac{k\pi\sqrt{-1}}{r}}\right)$ for $k = 1, 2, 3$ and $r \leqslant 22$.

Table 1

$k \setminus r$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	0.381966	1	3	3	2	0.763932	2	5	5	3	1.14590	3	7	7	4	1.52786
2	1		2.61803		3		2		2		5		7.85410		7		4	
3		1	2.61803		3	3		5.23607	2		5	3		3	7		4	10.4721

Table 2

$k \setminus r$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	0.381966	1	2	3	2	1.38197	2	4	4	3	1.76393	3	6	6	4	2.14590
2	1		2.61803		2		2		2		4		6.23607		6		4	
3		1	2.61803		2	3		3.61803	2		4	3		3	6		4	8.85410

Table 3

$k \setminus r$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	2	1	0.307979	1	4	3	3	5	2	0.615957	2	7	5	5	8	3	0.923936
2	1		2		0.643104		4		3		2		2		5		8		1.92931
3		1	2		5.04892	1		3	3		2	10.0978		7	5		8	3	

Table 4

$k \setminus r$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	2	1	0.198062	1	3	3	3	4	2	0.841166	2	2	5	5	6	3	1.03923
2	1		2		3.24698		3		3		2		2		5		6		11.5429
3		1	2		1.55496	1		3	3		2	1.86294		2	5		6	3	

Table 5

$k \setminus r$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	1	1	2	3	1	0.283119	1	5	4	3	3	5	7	2	0.566237	2	9	7	5
2	1		1		3		0.426022		5		3		5		2		2		7	
3		1	1		3	1		1	5		3	3		7	2		2	9		5

Table 6

$k \setminus r$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	1	1	1	2	3	4	1	0.271554	1	6	5	4	3	3	5	7	9	2	0.543108
2	1		1		2		4		0.353253		6		4		3		7		2	
3		1	1		2	3		1	0.582964		6	5		3	3		7	9		1.16593

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