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# Absolute gradings on ECH and Heegaard Floer homology

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**Abstract.** In joint work with Yang Huang, we defined a canonical absolute grading on Heegaard Floer homology by homotopy classes of oriented 2-plane fields. A similar grading was defined on embedded contact homology by Michael Hutchings. In this paper we show that the isomorphism between these homology theories defined by Colin, Ghiggini, and Honda preserves this grading.

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## 1. Introduction

Let $Y$ be a closed, connected and oriented three-manifold. The embedded contact homology (ECH) and the Heegaard Floer homology of $Y$ are invariants that have been studied and computed for many manifolds. ECH was defined by Hutchings using a contact form on $Y$, see [7], and Heegaard Floer homology was defined in [11] by Ozsváth and Szabó using a Heegaard decomposition of $Y$. These two homology theories have very distinct flavors, but they have recently been shown to be isomorphic by Colin, Ghiggini, and Honda [1, 2, 3]. More specifically, they construct an isomorphism $\Phi\colon \mathrm{HF}^+(-Y) \to \mathrm{ECH}(Y)$. Here $\mathrm{HF}^+(-Y)$ is a version of Heegaard Floer homology of $Y$ with the opposite orientation, which is isomorphic to the $-$ version of Heegaard Floer cohomology of $Y$ with its original orientation.

In joint work with Yang Huang [6], we defined a canonical absolute grading on Heegaard Floer homology by homotopy classes of oriented 2-plane fields on $Y$. A similar absolute grading had been defined in ECH by Hutchings [8]. Since these absolute gradings are defined in very different ways, it is not obvious that the isomorphism $\Phi$ would preserve them. On the other hand, when a contact form

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is given, it follows from Colin, Ghiggini and Honda's work that $\Phi$ maps what is called the contact invariant in one Floer homology to that of the other. It follows that in that particular spin-c structure, the absolute grading is preserved. The goal of this paper is to show that this holds in all generality as we now explain.

The orientation of $Y$ induces an isomorphism from this set to the set of homotopy classes of nonvanishing vector fields $\mathrm{Vect}(Y)$. So in this paper we will do all of our constructions with $\mathrm{Vect}(Y)$. For $\rho \in \mathrm{Vect}(Y)$, let $\mathrm{HF}^+_\rho(-Y)$ and $\mathrm{ECH}_\rho(Y)$ denote the submodules of $\mathrm{HF}^+(-Y)$ and $\mathrm{ECH}(Y)$, respectively, consisting of all elements of grading $\rho \in \mathrm{Vect}(Y)$. The main result of this paper is the following theorem.

**Theorem 1.1.** *Let* $\Phi\colon \mathrm{HF}^+(-Y) \to \mathrm{ECH}(Y)$ *be the isomorphism constructed by Colin, Ghiggini, and Honda. Then* $\Phi$ *maps* $\mathrm{HF}^+_\rho(-Y)$ *to* $\mathrm{ECH}_\rho(Y)$ *for all* $\rho \in \mathrm{Vect}(Y)$.

We recall that both $\mathrm{HF}^+(-Y)$ and $\mathrm{ECH}(Y)$ admit a map $U$ whose mapping cone is denoted by $\widehat{\mathrm{HF}}(-Y)$ and $\widehat{\mathrm{ECH}}(Y)$, respectively. In order to show that $\Phi$ is an isomorphism, Colin, Ghiggini and Honda first construct an isomorphism $\hat{\Phi}\colon \widehat{\mathrm{HF}}(-Y) \to \widehat{\mathrm{ECH}}(Y)$. They also show that the following diagram commutes.

$$\begin{array}{ccc} \widehat{\mathrm{HF}}(-Y) & \xrightarrow{\iota_*} & \mathrm{HF}(-Y) \\ \big\downarrow \hat{\Phi} & & \big\downarrow \Phi \\ \widehat{\mathrm{ECH}}(Y) & \xrightarrow{\iota_*} & \mathrm{ECH}(Y) \end{array} \tag{1}$$

Here the horizontal maps $\iota_*$ denote the natural maps given by the mapping cone construction. In order to show that $\Phi$ preserves the absolute grading, it is enough to prove that both maps $\iota_*$ and $\hat{\Phi}$ do.

The map $\hat{\Phi}$ is defined as a composition $\hat{\Phi} = \psi \circ \tilde{\Phi} \circ \psi'$ as follows.

$$\widehat{\mathrm{HF}}(-Y) \xrightarrow{\psi'} \widehat{\mathrm{HF}}(S, \mathbf{a}, \varphi(\mathbf{a})) \xrightarrow{\tilde{\Phi}} \mathrm{ECH}_{2g}(N_{(S,\varphi)}, \lambda) \xrightarrow{\psi} \widehat{\mathrm{ECH}}(Y).$$

Here $\widehat{\mathrm{HF}}(S, \mathbf{a}, \varphi(\mathbf{a}))$ is the homology of a chain group computed from the page of an open book decomposition $(S, \varphi)$ of $Y$ and $\mathrm{ECH}_{2g}(N_{(S,\varphi)}, \lambda)$ is the homology of a chain complex of generated by sets of Reeb orbits whose total intersection with a page is $2g$, where $\lambda$ is an appropriate contact form on $Y$. The maps $\psi'$, $\tilde{\Phi}$ and $\psi$ are all isomorphisms and we will show that all of them preserve the absolute grading.

This paper is organized as follows. In Section 2, we review the definitions of chain complexes of Heegaard Floer homology and ECH and the absolute grading on them. We explain how the chain complexes $\widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a}))$ and

$\mathrm{ECC}_{2g}(N_{(S,\varphi)}, \lambda)$ are obtained from an open book decomposition and how the absolute grading is defined on them. We also show that $\psi'$ preserves the grading. In Section 3, we recall some of the steps to construct the isomorphism $\widetilde{\Phi}$ and we prove that it preserves the absolute grading. This is the core of the proof of Theorem 1.1. Finally, in Section 4, we recall the construction of the map $\psi$ and we prove that it preserves the grading, finishing the proof of Theorem 1.1.

## 2. The absolute gradings

**2.1. The grading on Heegaard Floer homology.** A pointed Heegaard diagram is a quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, where $\Sigma$ is a closed oriented surface of genus $g$, the tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_g)$ are $g$-tuples of disjoint circles on $\Sigma$ which are linearly independent in $H_1(\Sigma)$ and $z$ is a point on $\Sigma$ in the complement of all of the circles $\alpha_i$ and $\beta_j$. Given a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, an intersection point is a $g$-tuple $\mathbf{x} = (x_1, \ldots, x_g)$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ and $\sigma$ is a permutation of $\{1, \ldots, g\}$. The chain complex $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is the $\mathbb{Z}$-module generated by the intersection points. One can define a differential $\partial$ on this complex and one can prove that $\partial^2 = 0$. The homology of this chain complex is denoted by $\widehat{\mathrm{HF}}(Y)$. It can be shown that the homology does not depend on the pointed Heegaard diagram and hence it is an invariant of $Y$. For details, see [11].

We now recall the definition of the other versions of Heegaard Floer homology. The complex $\mathrm{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is defined to be the $\mathbb{Z}$-module generated by $[\mathbf{x}, n]$, where $\mathbf{x}$ is an intersection point and $n \in \mathbb{Z}$. One can extend $\partial$ to $\mathrm{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ so that $\partial^2 = 0$. One can now define $\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to be the submodule of $\mathrm{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ generated by $[\mathbf{x}, n]$, for $n < 0$. One also defines $\mathrm{CF}^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to be the quotient of $\mathrm{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ by $\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. The homologies of these complexes are denoted by $\mathrm{HF}^\infty(Y)$, $\mathrm{HF}^-(Y)$ and $\mathrm{HF}^+(Y)$, respectively.

We will now recall the absolute grading on these homology groups. Let $(f, V)$ be a pair consisting of a self-indexing Morse function $f$ on $Y$ and a gradient-like vector field $V$, i.e. $df(V) > 0$, whenever $df \neq 0$. We assume that $f$ has exactly one index 0 and one index 3 critical points. We also assume that all stable and unstable manifolds intersect transversely. For each index 1 critical point $p_i$, let $U_i$ denote the unstable manifold containing $p_i$ and, for each index 2 critical point $q_j$, let $S_j$ denote the stable manifold containing $q_j$. The pair $(f, V)$ is said to be compatible with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ if

- $\Sigma = f^{-1}(3/2)$,
- $\alpha_i = U_i \cap \Sigma$ and $\beta_j = S_j \cap \Sigma$, for all $1 \le i, j \le g$.

An intersection point $\mathbf{x}$ determines $g$ flow lines $\gamma_1, \dots, \gamma_g$ connecting the points $p_i$ to the points $q_j$. The basepoint $z$ determines a flow line $\gamma_0$ from the index 0 critical point to the index 3 critical point. Outside the union of small neighborhoods of $\gamma_0, \dots, \gamma_g$, which we denote by $\nu(\gamma_0), \dots, \nu(\gamma_g)$, the vector field $V$ does not vanish. The absolute grading $\mathrm{gr}(\mathbf{x})$ is the homotopy class of an appropriate extension of $V$ to the union of all $\nu(\gamma_i)$, as we briefly explain. Figure 1(a) illustrates two transverse vertical sections of the vector field $V$ in $\nu(\gamma_i)$, for some $i \geq 1$ and Figure 1(b) illustrates a vertical section of $V$ in $\nu(\gamma_0)$. Now we substitute $V$ in these neighborhoods by the vector fields illustrated in Figure 2. We note that in $\nu(\gamma_0)$, the vector field in Figure 2(b) has a circle of zeros. We modify the vector field in a neighborhood of this circle so that it rotates clockwise on the $xy$-plane. Let $V^{\mathbf{x}}$ be the vector field obtained under this procedure. Then we define $\mathrm{gr}(\mathbf{x})$ to be the homotopy class of $V^{\mathbf{x}}$. For more details of this construction, see [6, §2.1].

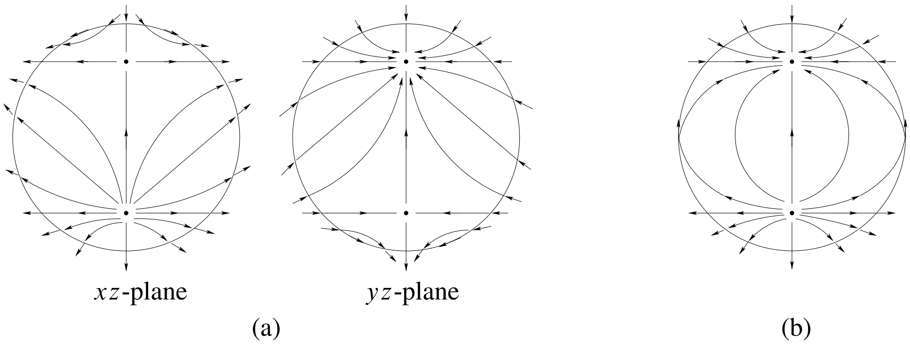


Figure 1. The vector field $V$

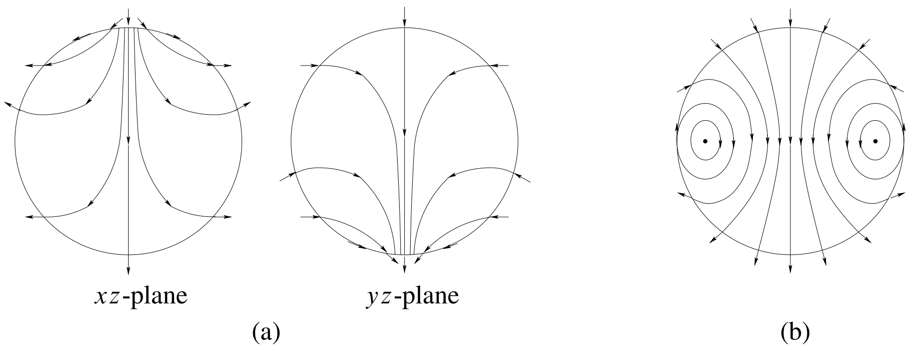


Figure 2. The modification of $V$

Two generators $\mathbf{x}$ and $\mathbf{y}$ of $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ are said to be in the same spin-c structure, if the vector fields $V^{\mathbf{x}}$ and $V^{\mathbf{y}}$ are homotopic in the complement of a 3-ball. For two such generators, one can define a relative grading $\mathrm{gr}(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}/d$, where $d$ is the divisibility of the first Chern class of the complex line bundle determined by a plane field transverse to $V^{\mathbf{x}}$ with the induced orientation. For details, we refer the reader again to [11].

We recall that for a given spin-c structure, the space of corresponding homotopy classes of nonvanishing vector fields is an affine space over $\mathbb{Z}/d$, for an appropriate $d$ as above. The main theorem of [6] says, in particular, that gr defines an absolute grading in $\widehat{\mathrm{HF}}(Y)$, i.e., if $\mathbf{x}$ and $\mathbf{y}$ are generators of $\widehat{\mathrm{HF}}(Y)$ in the same spin-c structure, then $\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\mathbf{y}) \in \mathbb{Z}/d$.

Now, for an intersection point $\mathbf{x}$ and $n \in \mathbb{Z}$, we define $\mathrm{gr}([\mathbf{x}, n]) = \mathrm{gr}(\mathbf{x}) + 2n$. The inclusion $\iota\colon \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \hookrightarrow \mathrm{CF}^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ mapping $x \mapsto [x, 0]$ induces the map $\iota_*$ in (1). It follows from the definition that this map preserves the absolute grading.

**2.2. Heegaard Floer homology and open book decompositions.** In this subsection, we will recall how to define the map $\psi'$ and we will show that it preserves the absolute grading.

An *open book* is a pair $(S, \varphi)$, where $S$ is a compact oriented surface with boundary and $\varphi$ is a diffeomorphism of $S$ which is the identity on $\partial S$. We will always assume that $\partial S$ is connected. We can construct a topological manifold by considering $S \times [0,1]/\sim$, where $(x, 1) \sim (\varphi(x), 0)$ for every $x \in S$ and $(x, t) \sim (x, t')$ for all $x \in \partial S$ and $t, t' \in [0, 1]$. Given an open book $(S, \varphi)$, let $\bar{S}$ be the surface obtained by gluing an annulus to $S$ and let $\bar{\varphi}$ be a diffeomorphism of $\bar{S}$ obtained by extending $\varphi$ such that $\bar{\varphi}$ is close to the identity in the annulus and equal to the identity in a neighborhood of $\bar{S}$. The quotients obtained by considering $(S, \varphi)$ and $(\bar{S}, \bar{\varphi})$ are homeomorphic and $\bar{S} \times [0,1]/\sim$ is actually a smooth manifold. We will say that $(S, \varphi)$ is an *open book decomposition* of $Y$ if $Y$ is diffeomorphic to $\bar{S} \times [0,1]/\sim$ where $(\bar{S}, \bar{\varphi})$ is constructed from $(S, \varphi)$ as above. The knot $\partial\bar{S} \times \{t\} \subset Y$ is called the *binding* and for each $t$ the surface $\bar{S} \times \{t\} \subset Y$ is called a *page*.

Let $(S, \varphi)$ be an open book decomposition of $Y$. Up to an isotopy of $\varphi$ relative to $\partial S$, we can assume that in a neighborhood $\nu(\partial S)$ of $\partial S$, we have $\varphi(y, \theta) = (y, \theta - y)$ where we identify $\nu(\partial S) \cong \partial S \times (-\varepsilon, 0]$. Then $(S, \varphi)$ gives rise to a Heegaard decomposition as follows. The Heegaard surface is $\Sigma := \bar{S} \times \{1/2\} \cup \bar{S} \times \{0\}$. If we denote the genus of $\bar{S}$ by $g$, then $\Sigma$ has genus $2g$. We choose a set of properly embedded arcs $\mathbf{a} = \{a_1, \ldots, a_{2g}\}$ of $\bar{S}$ such that

$\bar{S} \setminus \bigcup_i a_i$ is homeomorphic to a disk. Let $\alpha_i$ be the union of two copies of $a_i$ in $\bar{S} \times \{0\}$ and $\bar{S} \times \{1/2\}$. And let $\beta_i = b_i \cup h(a_i \cap S)$, where $b_i$ is an arc in $\Sigma \setminus (S \times \{0\})$ which is isotopic to $\alpha_i \cap (\Sigma \setminus (S \times \{0\}))$, extends $h(a_i \cap S)$ to a smooth curve in $\Sigma$ and has exactly one intersection with $\alpha_i$ in the interior of $\Sigma \setminus (S \times \{0\})$, see Figure 3(a). Hence $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram for $Y$. So $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is a Heegaard diagram[1] for $-Y$.

For each $i$, the circle $\alpha_i$ intersects $\beta_i$ in $\Sigma \setminus (\mathrm{int}(S) \times \{0\})$ at three points. We label them $y_i, y_i', y_i''$, as in Figure 3(a). We fix a basepoint $z \in S \times \{1/2\} \subset \Sigma$ away from neighborhoods of $\alpha_i \cap (S \times \{1/2\})$. One defines $\widehat{\mathrm{CF}}'(S, \mathbf{a}, \varphi(\mathbf{a}))$ to be the subcomplex of $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ generated by $2g$-tuples of intersection points contained in $S \times \{0\}$. One also defines $\widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a}))$ to be the quotient $\widehat{\mathrm{CF}}'(S, \mathbf{a}, \varphi(\mathbf{a}))/\sim$, where two $2g$-tuples of intersection points in $S \times \{0\}$ are equivalent if they differ by substituting $y_i$ by $y_i'$. There is an induced differential on $\widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a}))$ and the inclusion map induces a map $\widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a})) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ which is an isomorphism in homology by [1, Theorem 4.9.4]. The absolute grading on $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ clearly restricts to an absolute grading of $\widehat{\mathrm{CF}}'(S, \mathbf{a}, \varphi(\mathbf{a}))$. Let $\mathbf{x}$ be a generator of $\widehat{\mathrm{CF}}'(S, \mathbf{a}, \varphi(\mathbf{a}))$ containing $y_i$. Then

$$\mathrm{gr}(\mathbf{x}, \mathbf{x} \setminus \{y_i\} \cup \{y_i'\}) = 0.$$

So absolute grading on the complex $\widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a}))$ is well-defined. Moreover, by definition, the map $\widehat{\mathrm{CF}}'(S, \mathbf{a}, \varphi(\mathbf{a})) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ preserves the absolute grading. Therefore the isomorphism

$$\psi' \colon \widehat{\mathrm{HF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z) \longrightarrow \widehat{\mathrm{HF}}(S, \mathbf{a}, \varphi(\mathbf{a})) \tag{2}$$

preserves the absolute grading.

**2.3. The grading on embedded contact homology.** We will now recall the definition of the ECH chain complex and its absolute grading. Let $Y$ be a closed, oriented three-manifold, let $\lambda$ be a nondegenerate contact form on $Y$ and let $\xi = \ker(\lambda)$. The ECH chain complex $\mathrm{ECC}(Y, \lambda)$ is generated by finite orbit sets $\{(\gamma_i, m_i)\}$, where $\gamma_i$ are distinct single orbits of the Reeb vector field associated to $\lambda$, the numbers $m_i$ are positive integers, and $m_1 = 1$ whenever $\gamma_i$ is hyperbolic. After some extra choices, one can define a differential on $\mathrm{ECC}(Y, \lambda)$ that squares to 0. Its homology is independent of these choices and even of the contact form and is denoted by $\mathrm{ECH}(Y)$. For the details of this construction and the invariance, we refer the reader to [7].

[1] This construction is slightly more complicated than that in [1, §2.1] and it is not necessary for defining $\widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a}))$, but it will make it easier to choose an appropriate representative of $\mathrm{gr}(\mathbf{x})$ in §3.2.

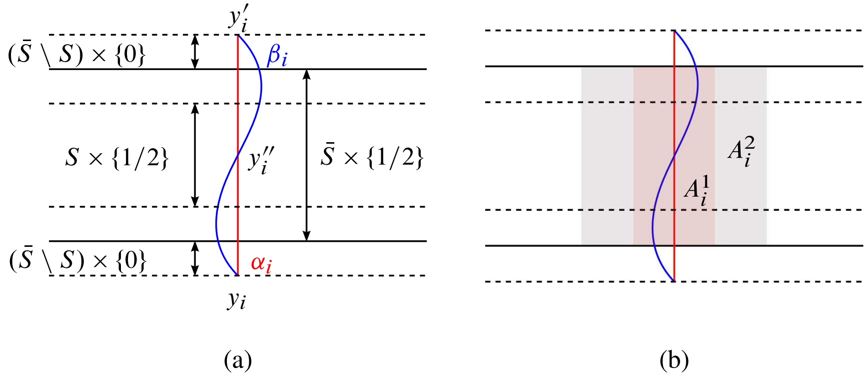


Figure 3. A neighborhood of the arcs $a_i$

The absolute grading on ECH is defined as follows. Let $\gamma = \{(\gamma_i, m_i)\}$ be an orbit set. The absolute grading $\mathrm{gr}(\gamma)$ is the homotopy class of the vector field obtained by modifying the Reeb vector field in disjoint neighborhoods of the Reeb orbits $\gamma_i$, as follows. For each $i$, fix a small tubular neighborhood of $\gamma_i$ and choose a braid $\zeta_i$ with $m_i$ strands in that neighborhood. Let $L$ be the union of the braids $\zeta_i$. A trivialization $\tau_i$ of $\xi$ over each $\gamma_i$, induces a framing on each $\zeta_i$. Let $\tau$ denote this framing on $L$. Now, for each component $K$ of $L$, let $N_K$ denote a small neighborhood of $K$ in $Y$. We can choose these neighborhoods so that $N_K$ and $N_{K'}$ do not intersect for different components $K$ and $K'$. The framing on $K$ induces a diffeomorphism $\phi_K \colon N_K \to S^1 \times D^2$ and a trivialization of $TN_K$, identifying $\xi = \{0\} \oplus \mathbb{R}^2$ and $R = (1,0,0)$. Using the previous identifications, one can define a vector field $P$ on $N_K$ as

$$\begin{aligned} P \colon S^1 \times D^2 &\longrightarrow \mathbb{R} \oplus \mathbb{R}^2, \\ (t, re^{i\theta}) &\longmapsto (-\cos(\pi r), \sin(\pi r) e^{-i\theta}). \end{aligned}$$

One now constructs a vector field on $Y$ by defining it to be given by $P$ in each neighborhood $N_K$ and to equal the Reeb vector field in the complement of the union of the neighborhoods $N_K$. Let $P_\tau(L)$ be the homotopy class of this vector field. Now define

$$\mathrm{gr}(\gamma) = P_\tau(L) - \sum_i w_{\tau_i}(\zeta_i) + CZ_\tau^I(\gamma), \tag{3}$$

Here $w_{\tau_i}(\zeta_i)$ denotes the writhe of $\zeta_i$ with respect to $\tau_i$ and

$$CZ_\tau^I(\gamma) = \sum_i \sum_{k=1}^{m_i} CZ_\tau(\gamma_i^k).$$

One can check that $\mathrm{gr}(\gamma)$ does not depend on the choice of $\tau$ or $L$. In [8], Hutchings proved that gr is an absolute grading on ECH, i.e., that if $\gamma$ and $\sigma$ are orbit sets with $[\gamma] = [\sigma] \in H_1(Y)$ then

$$\mathrm{gr}(\gamma) - \mathrm{gr}(\sigma) = I(\gamma, \sigma) \in \mathbb{Z}/d,$$

for an appropriate $d$ depending on $\ker(\lambda)$ and $[\gamma]$. Here $I(\gamma, \sigma)$ denotes the relative grading on ECH, i.e., the ECH index whose definition we shall not need to use.

**2.4. The module $\mathrm{ECC}_{2g}(N, \lambda)$.** We recall the definition of $\mathrm{ECC}_{2g}(N, \lambda)$ and explain the absolute grading on it.

Let $(S, \varphi)$ be an open book decomposition of $Y$ and let $\lambda$ be a contact form on $Y$ which is *adapted* to $(S, \varphi)$, i.e., the Reeb vector field $R_\lambda$ is a positively transverse to the interior of the pages and positively tangent to the binding. As in §2.2, we assume that $\varphi$ satisfies $\varphi(y, \theta) = (y, \theta - y)$ in a neighborhood of $\partial S$. It follows from [1, Lemma 2.1.1] that $\lambda$ and $\varphi$ can be chosen so that $\varphi$ is the return map of the Reeb vector field on $S \times \{0\}$. We recall from our construction in §2.2 that for each $t$ the surface $S \times \{t\}$ is a strict subset of a page. Let $N$ be the mapping torus of $\varphi$. Then we can write $Y = N \cup (S^1 \times D^2)$. The torus $\partial N$ is foliated by Reeb orbits. Up to a small isotopy of $\lambda$, we can assume that all the Reeb orbits in the complement of $\partial N$ are nondegenerate and that $\partial N$ is a negative Morse–Bott torus. Following [1], we define $\mathrm{ECC}_{2g}(N, \lambda)$ to be the $\mathbb{Z}/2$ vector space generated by orbit sets constructed from Reeb orbits in $\mathrm{int}(N)$ and two fixed orbits $\{e, h\}$ on $\partial N$, and whose total homology class intersects a page exactly $2g$ times. Here $e$ and $h$ play the roles of an elliptic and a hyperbolic orbit, respectively. The construction in §2.3 still works even though $\lambda$ is degenerate. So we obtain an absolute grading on $\mathrm{ECC}_{2g}(N, \lambda)$ taking values on $\mathrm{Vect}(Y)$.

## 3. The main isomorphism

In this section, we will prove the main part of Theorem 1.1, namely that the map $\widetilde{\Phi}$ preserves the absolute grading.

**3.1. The construction of $\widetilde{\Phi}$.** We now recall the construction of the map $\widetilde{\Phi}$ on the chain level

$$\widetilde{\Phi}\colon \widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a})) \longrightarrow \mathrm{ECC}_{2g}(N, \lambda).$$

This map is defined by counting rigid holomorphic curves with an ECH-type index equal to 0. We now review the relevant moduli spaces and this ECH-type index.

Throughout this section we fix an open book decomposition $(S, \varphi)$ of $Y$ satisfying the conditions given in §2.2 and we let $N$ be the mapping torus of $\varphi$. We denote by $g$ the genus of $S$ and we let $\lambda$ be a contact form on $Y$ which is adapted to $(S, \varphi)$. In order to prove that $\widetilde{\Phi}$ is an isomorphism, it is necessary to make a more specific choice of $\lambda$ as it is done in [1, §3], but this particular choice does not affect the absolute grading by Lemma 4.1.

Let $\pi\colon \mathbb{R} \times N \to \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ be the map $(s, x, t) \mapsto (s, t)$ and let $B := (\mathbb{R} \times S^1) \setminus B^c$, where $B^c = (0, \infty) \times (1/2, 1)$. We also round the corners of $B$. Now define $W = \pi^{-1}(B)$ and $\Omega = ds \wedge dt + \omega$, where $\omega$ is a certain area form on $S$. Then $(W, \Omega)$ is a symplectic manifold with boundary. It has a positive end, which is diffeomorphic to $S \times [0, 1/2]$ and a negative end, which is diffeomorphic to $N$. The map $\pi$ restricts to a symplectic fibration $\pi_B\colon (W, \Omega) \to (B, ds \wedge dt)$ which admits a symplectic connection whose horizontal space is spanned by $\{\partial/\partial s, \partial/\partial t\}$. Now if we take a copy of $\mathbf{a} = (a_1, \ldots, a_{2g})$ on the fiber $\pi_B^{-1}(1, 1/2)$ and take its symplectic parallel transport along $\partial B$, we obtain a Lagrangian submanifold of $(W, \Omega)$, which is denoted by $L_{\mathbf{a}}$. For each $a_i \subset \mathbf{a}$ we denote by $L_{a_i}$ the corresponding component of $L_{\mathbf{a}}$.

We will consider $J$-holomorphic maps $u\colon (\dot{F}, j) \to (W, J)$ where $(\dot{F}, j)$ is a Riemann surface with boundary and punctures, both in the interior and on the boundary. A puncture $p$ is said to be positive or negative if the $s$-coordinate of $u(x)$ converges to $\infty$ or $-\infty$, respectively, as $x \to p$. Now to each generator $\mathbf{x}$ of $\widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a}))$ we can associate a subset of $S \times [0, 1/2]$ given by the union of $x_i \times [0, 1/2]$, for all $x_i \in \mathbf{x}$. Given $\mathbf{x}$, an orbit set $\gamma = \{(\gamma_i, m_i)\}$ in $\mathrm{ECC}_{2g}(N, \lambda)$ and an admissible almost-complex structure $J$, one defines $\mathcal{M}_J(\mathbf{x}, \gamma)$ to be the moduli space of immersed $J$-holomorphic maps $u\colon (\dot{F}, j) \to (W, J)$ satisfying the following conditions:

(a) $u(\partial \dot{F}) \subset L_{\mathbf{a}}$ and each component of $\partial \dot{F}$ is mapped to a different $L_{a_i}$.

(b) The boundary punctures are positive and the interior punctures are negative.

(c) At each boundary puncture, $u$ converges to a different chord $x_i \times [0, 1/2]$ and every chord $x_i \times [0, 1/2]$ is such an end of $u$.

(d) At an interior puncture, $u$ converges to an orbit $\gamma_i$ with some multiplicity. For each $i$, the total multiplicity of all ends converging to $\gamma_i$ is $m_i$.

(e) The energy of $u$ is bounded.

Let $\overline{W}$ denote the compactification of $W \subset \mathbb{R} \times N$ obtained by compactifying $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty, \infty\}$. A continuous map $u\colon \dot{F} \to W$ satifying (a)–(d) above can be compactified to a map $\bar{u}\colon \overline{F} \to \overline{W}$ mapping $\partial \overline{F}$ to

$$L_{\mathbf{x},\gamma} := L_{\mathbf{a}} \cup (\{\infty\} \times \mathbf{x} \times [0, 1/2]) \cup (\{-\infty\} \times \gamma) .$$

Two such maps $u$, $v$ are said to be homologous if the images of $\bar{u}$ and $\bar{v}$ are homologous in $H_2(\bar{W}, L_{\mathbf{x},\gamma})$. Let $H_2(W, \mathbf{x}, \gamma)$ denote the set of homology classes of such maps $u: \dot{F} \to W$.

For a homology class $A \in H_2(W, \mathbf{x}, \gamma)$, one defines its ECH-index $I(A)$ as follows. Let $u: \dot{F} \to W$ be a continuous map satisfying (a)–(d) above such that $[u] = A$ and let $\bar{u}: \bar{F} \to \bar{W}$ be its compactification. Now note that one can view $TS$ as a sub-bundle of $T\bar{W}$. We choose an orientation of the arcs $a_i$, which gives rise to a nonvanishing vector field along each $a_i$. This vector field induces a trivialization $\tau$ of $TS$ along $L_{\mathbf{a}} \subset \bar{W}$. We extend this trivialization arbitrarily along $\{\infty\} \times \mathbf{x} \times [0, 1/2]$ and along $\{-\infty\} \times \gamma$. Let $c_\tau(A)$ denote the first Chern class of $\bar{u}^*TS$ relative to $\tau$. Now let $C_1$ and $C_2$ be distinct embedded surfaces in $\bar{W}$ given by pushing $\bar{u}(\bar{F})$ off along vectors field which are transverse to it and trivial with respect to $\tau$ along the boundary. For more details see [1, §4]. Then $Q_\tau(A)$ is defined to be the signed count of intersections of $C_1$ and $C_2$. Now let $\mathcal{L}_0$ be a real, rank one subbundle of $TS$ along $\mathbf{x} \times [0, 1/2]$ defined as follows. At $\mathbf{x} \times \{0\}$, let $\mathcal{L}_0 = T\varphi(\mathbf{a})$ and at $\mathbf{x} \times \{1/2\}$, let $\mathcal{L}_0 = T\mathbf{a}$ in $TS$. Then $\mathcal{L}_0$ is defined by rotating counterclockwise by the minimum possible amount as we travel along $\mathbf{x} \times [0, 1/2]$. One defines $\mu_\tau(\mathbf{x})$ to be the sum of the Maslov indices of $\mathcal{L}_0$ along each $x_i \times [0, 1/2]$ with respect to $\tau$. The ECH-index is defined as

$$I(A) = c_\tau(A) + Q_\tau(A) + \mu_\tau(\mathbf{x}) - CZ_\tau^I(\gamma) - 2g.$$

Now $\widetilde{\Phi}(\mathbf{x})$ is defined as follows. The coefficient of an orbit set $\gamma$ in $\widetilde{\Phi}(\mathbf{x})$ is the count of maps $u$ in $\mathcal{M}_J(\mathbf{x}, \gamma)$ with $I([u]) = 0$. As explained in [1], for a generic $J$ this count is finite and all the maps that are counted are embeddings.

**3.2. The choice of an appropriate representative of gr(x).** Let $\mathbf{x}$ be a generator of $\widehat{\mathrm{CF}}'(S, \mathbf{a}, \varphi(\mathbf{a}))$. We will now explain how to choose a vector field in the equivalence class $\mathrm{gr}(\mathbf{x})$ that coincides with the Reeb vector field of a contact form on $Y$ in the complement of a small set in preparation for Proposition 3.1.

Let $(S, \varphi)$ be an open book decomposition of $Y$ and let $\lambda$ be a contact form on $Y$ which is *adapted* to $(S, \varphi)$ satisfying the conditions of §2.4. For each $i = 1, \dots, 2g$, let $A_i^1$ be a small closed neighborhood of $\alpha_i$ in $\bar{S}$ and let $A_i^2 \supset A_i^1$ be a small thickening of it in $\bar{S}$, as in Figure 3(b). The open book decomposition $(S, \varphi)$ gives rise to a Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ as in §2.2. Let $(f, V)$ be a pair which is compatible with $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$. In what follows when we take the Cartesian product of a subset of $\bar{S}$ and an interval in $\mathbb{R}$, we will always take the quotient by the equivalence relation generated by $(x, 1) \sim (\bar{\varphi}(x), 0)$ for all $x \in \bar{S}$ and $(x, t) \sim (x, t')$ for all $x \in \partial\bar{S}$. So we can see these products as subsets of $Y$. We can assume, without loss of generality, that:

- the critical points of $f$ belong to $(\bar{S} \times \{1/4\}) \cup (\bar{S} \times \{3/4\})$;
- every flow line corresponding to a point $y_i''$ as in Figure 3(a) belongs to $A_i^2 \times [1/4, 3/4]$ and along this flow line $V = -R_\lambda$. Moreover $V$ is not a positive multiple of $-R_\lambda$ elsewhere in $A_i^2 \times [1/4, 3/4]$;
- the flow line $\gamma_0$ corresponding to $z$ belongs to $(\bar{S} \setminus \bigcup_i A_i^2) \times [1/4, 3/4]$.

For $j = 1, 2$ we let

$$M^j = \nu\Big(\Big(\bar{S} \setminus \bigcup_{i=1}^{2g} A_i^j\Big) \times [1/4, 3/4]\Big) \subset Y.$$

Here $\nu(\cdot)$ denotes a small neighborhood in $Y$. We observe that $M^1$ and $M^2$ are 3-balls and $M^1 \supset M^2$. See Figure 4 for a picture of $M^1$ and $M^2$ in a neighborhood of $a_i \times \{1/2\}$. Let $\mathbf{x} = (x_1, \ldots, x_{2g})$ be a generator of $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$. It follows from the conditions on $(f, V)$ above that we can choose small enough neighborhoods $\nu(\gamma_i)$ for $i = 1, \ldots, 2g$ as in §2.1 which do not intersect $M^1$. We can also assume $\nu(\gamma_0) = M^1$ since $M^1$ contains no index one or two critical points. Under this identification, we require $(\bar{S} \times \{1/2\}) \cap M^1$ to be contained in the $xy$-plane. We can now modify $V|_{M^1}$ and define $V^{\mathbf{x}}|_{M^1}$ as in §2.1. Recall that $V^{\mathbf{x}}$ is transverse to the $xy$-plane except on a circle which we denote by $\Gamma$, see Figure 2(b). Up to a homotopy, we can assume that

$$\Gamma \cap M^2 = \partial \bar{S} \cap M^2.$$

Figure 4 shows $\Gamma$ in a neighborhood of $a_i \times \{1/2\}$. So $V^{\mathbf{x}}$ is positively tangent to the binding in $M^2$ and $V^{\mathbf{x}}$ is transverse to the interior of the pages in $M^2$.

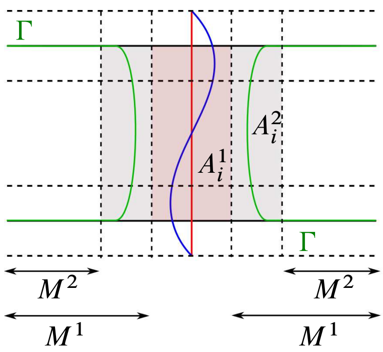


Figure 4. The curve $\Gamma$

We now let

$$Y^0 = Y \setminus \nu(\bar{S} \times [1/4, 3/4]).$$

The neighborhood above is chosen to be sufficiently small so that the complement of $Y^0 \cup M^2$ is a neighborhood of $\bigcup_i A_i^2 \times [1/4, 3/4]$. In $Y^0$ the vector field $V$ is nonvanishing and positively transverse to the pages. So using the construction of the paragraph above, we can assume that $V^{\mathbf{x}}$ equals the Reeb vector field $R_\lambda$ in $Y^0 \cup M^2$ except in the neighborhoods $\nu(\gamma_i)$ for $i = 1, \dots, 2g$. We identify each $\nu(\gamma_i)$ with a subset of $\mathbb{R}^2$ as in Figure 1(a). We can also assume that $V^{\mathbf{x}} = -R_\lambda$ along the $z$-axis in $\nu(\gamma_i)$ and that $V^{\mathbf{x}} \neq -R_\lambda$ in the complement of the $z$-axis in $\nu(\gamma_i)$ for $i = 1, \dots, 2g$, c.f. Figure 2(a).

### 3.3. The map $\tilde{\Phi}$ preserves the grading.

We will now prove a proposition, which is the main ingredient of the proof of Theorem 1.1.

**Proposition 3.1.** *Let $A \in H_2(W, \mathbf{x}, \gamma)$, where $\mathbf{x}$ is a generator of $\widehat{\mathrm{CF}}(S, \mathbf{a}, \varphi(\mathbf{a}))$ and $\gamma$ is a generator of $\mathrm{ECC}_{2g}(N, \lambda)$, respectively. Then*

$$\mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\gamma) = I(A). \tag{4}$$

We first recall a relative version of the Pontryagin–Thom construction. Let $v$ and $w$ be nonvanishing vector fields on a closed and oriented three-manifold $Y$. Assume that $v$ and $w$ coincide in $Y \setminus U$, where $U$ is an open set in $Y$. Let $\tau$ be a trivialization of $TY|_U$ and let $p \in S^2$ be a regular value of both $v$ and $w$ seen as maps $U \to S^2$. The one-manifolds $L_v := v^{-1}(p)$ and $L_w := w^{-1}(p)$ inherit framings by considering the isomorphisms of their normal bundles with $T_pS^2$ given by $v_*$ and $w_*$ along $L_v$ and $L_w$, respectively. Now if $L_v$ and $L_w$ are contained in the interior of $U$ and are homologous in $U$, there is a link cobordism $C \subset U \times [0, 1]$ from $L_v$ to $L_w$. That is, $C$ is a surface such that $\partial C = (L_v \times \{1\}) \cup (-L_w \times \{0\})$. The framings on $L_v$ and $L_w$ induce a framing on $C$ along $\partial C$ which we denote by $\nu$. The following lemma is a consequence of the classical Pontryagin–Thom construction and [6, Lemma 2.3].

**Lemma 3.2.** *Let $v$ and $w$ be nonvanishing vector fields and $L_v$ and $L_w$ the links as above. Let $C$ be an immersed cobordism from $L_v$ to $L_w$ and let $\delta(C)$ denote the number of self-intersections of $C$. Let $\tau$ denote the framing on $C$ along $\partial C$ which is induced by the framings on $L_v$ and $L_w$. Then*

$$[v] - [w] = c_1(NC, \tau) + 2\delta(C).$$

*Proof of Proposition* 3.1. In order to make it easier to visualize the construction below, we can apply a diffeomorphism of the base $\mathbb{R}/\mathbb{Z}$ of the fibration $\pi$ in §3.1 and change $W$ appropriately so that $\pi(\partial W) = [\varepsilon, 1-\varepsilon] \subset \mathbb{R}/\mathbb{Z}$, for small $\varepsilon > 0$. This is equivalent to substituting $B^c$ by $(0,\infty) \times (\varepsilon, 1-\varepsilon)$ in §3.1.

Let $u\colon \dot{F} \to W$ be an immersion such that $[u] = A$ and let $\bar{u}\colon \bar{F} \to \bar{W}$ denote its continuous compactification. We note that by rounding the corners of $\bar{W}$, we obtain a trivial cobordism from $N$ to itself which we denote by $[0,1] \times N$. In particular,

$$L_{\mathbf{a}} \subset \{1\} \times S \times [\varepsilon, 1-\varepsilon] \subset \{1\} \times N.$$

We now consider the intersection of the smoothing of $\bar{u}$ with $[\delta, 1] \times N$, for small $\delta > 0$. We obtain an immersed surface $C$ whose boundary is the union of $\{1\} \times (\mathbf{x} \times [-\varepsilon, \varepsilon])$, a curve on $L_{\mathbf{a}}$ which is transverse to $\{1\} \times S \times \{t\}$ for every $t$ and a link in $\{\delta\} \times N$ which is the union of braids about the Reeb orbits $\{\delta\} \times \gamma_i$ with $m_i$ strands. Here $\gamma = \{(\gamma_i, m_i)\}$. Let $L \subset N$ be the union of these braids under the identification $N \cong \{\delta\} \times N$.

Let $M^2 \subset Y$ and $V^{\mathbf{x}}$ be as in §3.2 and let $\tilde{N}$ be a small open neighborhood of $N \cup (Y \setminus M^2)$. We choose a trivialization of $T\tilde{N}$ as follows. We first orient the arcs $a_i \subset S$ so as to obtain a nonvanishing vector field on $S$ which is tangent to $a_i$. We can then extend this vector field to all of $S$. We choose a second vector field on $S$ such that these two vector fields form an oriented global frame of $S$. This frame induces a trivialization of $TS$ which gives rise to a trivialization of the pullback bundle of $TS$ over $[\varepsilon, 1-\varepsilon] \times S$. We extend it arbitrarily to a trivialization of the pullback bundle of $TS$ to all of $N$ and we denote this trivialization by $\tau$. Finally we let $R_\lambda$ be the third vector field on $N$ obtaining thus a global frame of $N$. We now extend this frame to a global frame of $\tilde{N}$ so that $R_\lambda$ is always the third vector field. This frame gives rise to a trivialization $T\tilde{N} \cong \tilde{N} \times \mathbb{R}^3$. Note that under this trivialization $R_\lambda$ is the constant vector field $(0,0,1)$.

Let $V_\tau^L$ be the vector field defined in §2.3 whose homotopy class is $P_\tau(L)$. Then it follows from §3.2 that we can assume that $V^{\mathbf{x}}$ and $V_\tau^L$ coincide in $Y \setminus \tilde{N}$. We shall use Lemma 3.2. We observe that $(V_\tau^L)^{-1}(0,0,-1) = L$. The framing can be calculated by considering the preimage of a vector near $(0,0,-1)$. The corresponding link gives a framing of the normal bundle $NL \cong \xi|_L$ which coincides with $\tau$.

Let $L^{\mathbf{x}} = (V^{\mathbf{x}})^{-1}(0,0,-1)$. It follows from the construction of $V^{\mathbf{x}}$ in §3.2 that $L^{\mathbf{x}}$ is a slight perturbation of

$$\bigcup_i \gamma_{x_i} \cup \bigcup_i \gamma_{y_i''}.$$

Here $\gamma_{x_i}$ denotes the flow line of $V$ going through $x_i \in \Sigma$. We note that $L^{\mathbf{x}}$ is transverse to the pages. So $\tau$ induces a framing of $L^{\mathbf{x}}$ as a link in $N$. Moreover $C \cap \{1\} \times N$ seen as a link in $N$ is isotopic to $L^{\mathbf{x}}$ through links that are transverse to the pages. Therefore $(C \cap \{1\} \times N, \tau)$ is framed isotopic to $(L^{\mathbf{x}}, \tau)$. By composing $C$ with this framed isotopy, we obtain an immersed cobordism $\widetilde{C}$ between $L$ and $L^{\mathbf{x}}$.

Now let $\tau'$ denote the framings on $L^{\mathbf{x}}$ and $L$ induced from the Pontryagin–Thom construction. It follows from Lemma 3.2 that

$$\mathrm{gr}(\mathbf{x}) - P_\tau(L) = c_1(N\widetilde{C}, \tau') + 2\delta(\widetilde{C}). \tag{5}$$

We claim that

$$c_1(N\widetilde{C}, \tau') = c_1(N\widetilde{C}, \tau) + \mu_\tau(\mathbf{x}) - 2g. \tag{6}$$

To prove the claim, we will compute the difference $c_1(N\widetilde{C}, \tau') - c_1(N\widetilde{C}, \tau)$. This difference is given by $\tau'|_{L^{\mathbf{x}}} - \tau|_{L^{\mathbf{x}}}$, since $\tau'|_L = \tau|_L$. We orient $L^{\mathbf{x}}$ so that it intersects the pages positively, i.e., the orientation follows the flow of $V$ along $\gamma_{x_i}$ and of $-V$ along $\gamma_{y_i''}$. Under the trivialization $(\tau, R_\lambda)$, we have $L^{\mathbf{x}} = (V^{\mathbf{x}})^{-1}(0, 0, -1)$. The framing $\tau'|_{L^{\mathbf{x}}}$ is determined by the projection of the vector field $\frac{d}{d\varepsilon}|_{\varepsilon=0}(V^{\mathbf{x}})^{-1}(\varepsilon, 0, -1)$ to $TS|_{L^{\mathbf{x}}}$. Let $v_{\tau'}$ denote this projection. Let $v_\tau$ be the constant vector field $(1, 0, 0)$ of $T\tilde{N}$. So $v_\tau$ is tangent to $L_{\mathbf{a}}$ along $\bigcup_i \gamma_{y_i''}$. So the difference $\tau'|_{L^{\mathbf{x}}} - \tau|_{L^{\mathbf{x}}}$ is the signed count of turns of $v_{\tau'}$ with respect to $v_\tau$ as we travel along $L^{\mathbf{x}}$. We observe that since $\varepsilon$ is small, we can assume that $L_{\mathbf{a}}$ is tangent to the unstable surfaces corresponding to each $\beta_i$ near $\{\varepsilon\} \times S$ and to the stable surfaces corresponding to each $\alpha_i$ near $\{1 - \varepsilon\} \times S$. So the vector field $v_\tau$ rotates a quarter of a turn positively about $\gamma_{y_i''}$ for each $i$ with respect to a reference frame in which the stable and unstable surfaces of $V$ are contained in the coordinate axes of $\mathbb{R}^2$, c.f. [6, §2]. Hence $v_{\tau'}$ rotates a quarter of a turn negatively about $\gamma_{y_i''}$ with respect to the same reference frame. So along each $\gamma_{y_i''}$ we obtain a contribution of $-1/2$ to $\tau'|_{L^{\mathbf{x}}} - \tau|_{L_{\mathbf{x}}}$. Now we compute the difference $\tau'|_{L^{\mathbf{x}}} - \tau|_{L^{\mathbf{x}}}$ along each $\gamma_{x_i}$. If $v_\tau$ makes $1/2 + n$ positive half-turns about $\gamma_{x_i}$, we obtain a contribution of $-1/2 - n$ to $\tau'|_{L^{\mathbf{x}}} - \tau|_{L^{\mathbf{x}}}$. In that case, this component will contribute by $-n$ to $\mu_\tau(\mathbf{x})$. Since there are $2g$ segments $\gamma_{x_i}$ and $\gamma_{y_i''}$, the total difference $\tau'|_{L^{\mathbf{x}}} - \tau|_{L^{\mathbf{x}}}$ is $\mu_\tau(\mathbf{x}) - 2g$ and we have proven (6).

It remains to compute $c_1(N\widetilde{C}, \tau)$. We first note that $c_1(N\widetilde{C}, \tau) = c_1(NC, \tau)$ since $\widetilde{C}$ is obtained from $C$ by adding a trivial framed cobordism. We will now use a classical construction in topology. Consider a generic section of $NC$ which is trivial with respect to $\tau$ along $\partial C$. We move $C$ in the direction of this section and we obtain a surface $C'$ which intersects $C$ transversely. Then

$$c_1(NC, \tau) = C \cdot C' - 2\delta(C), \tag{7}$$

where $C \cdot C'$ denotes the signed count of intersections of $C$ and $C'$. But these surfaces are not necessarily $\tau$-trivial. In fact, the linking number of $\partial C$ and $\partial C'$ is $-\sum_i w_\tau(\zeta_i)$ in $\{0\} \times \tilde{N}$ and 0 in $\{1\} \times \tilde{N}$. Following a standard calculation in ECH, see e.g. [8, §2.7], we obtain

$$C \cdot C' = Q_\tau(A) + \sum_i w_\tau(\zeta_i). \tag{8}$$

Combining (3), (5), (6), (7), and (8), we obtain (4). □

Now if $\gamma$ is a term in $\tilde{\Phi}(\mathbf{x})$, it follows from Proposition 3.1 that $\mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\gamma) = 0$. So $\tilde{\Phi}$ preserves the grading on the chain level, and therefore the isomorphism $\tilde{\Phi}\colon \widehat{\mathrm{HF}}(S, \mathbf{a}, \varphi(\mathbf{a})) \to \mathrm{ECH}_{2g}(N, \lambda)$ preserves the grading.

## 4. ECH and open book decompositions

In this section, we will recall the definition of the map $\psi$ and we will prove that it preserves the absolute grading, which is the last step in the proof of Theorem 1.1.

**4.1. The hat version of ECH.** The $U$ map is a degree $-2$ chain map

$$U\colon \mathrm{ECC}(Y, \lambda) \longrightarrow \mathrm{ECC}(Y, \lambda).$$

The chain complex $\widehat{\mathrm{ECC}}(Y, \lambda)$ is defined to be the mapping cone of $U$. The homology of $\widehat{\mathrm{ECC}}(Y, \lambda)$ is denoted by $\widehat{\mathrm{ECH}}(Y, \lambda)$. Again, it follows from [12] that the $U$ map in homology does not depend on any choices so we can write $\widehat{\mathrm{ECH}}(Y)$. We obtain an exact triangle, as follows.

$$\begin{array}{ccc} \mathrm{ECH}(Y) & \xrightarrow{\quad U \quad} & \mathrm{ECH}(Y) \\ & \nwarrow \quad \swarrow & \\ & \widehat{\mathrm{ECH}}(Y) & \end{array} \tag{9}$$

We define the absolute grading on $\widehat{\mathrm{ECC}}(Y, \lambda, J)$ so that $\widehat{\mathrm{ECC}}(Y, \lambda) \to \mathrm{ECC}(Y, \lambda)$ has degree 0. Hence for $\rho \in \mathrm{Vect}(Y)$, we can write $\widehat{\mathrm{ECH}}_\rho(Y)$. We note that the map $\mathrm{ECH}(Y, \lambda) \to \widehat{\mathrm{ECH}}(Y, \lambda)$ has degree 1.

**4.2. Cobordism maps in ECH.** In this subsection, we will show that the cobordisms maps in ECH defined by Hutchings and Taubes in [9] preserve the absolute grading. This fact will be used in the next subsection.

Let $\lambda$ be a contact form on $Y$. The symplectic action of an orbit set $\gamma = \{(m_i, \gamma_i)\}$ is defined to be $\mathcal{A}_\lambda(\gamma) := \sum_i m_i \int_{\gamma_i} \lambda$. For $L > 0$, the filtered ECH chain complex $\mathrm{ECC}^L(Y, \lambda)$ is defined to be the subcomplex of $\mathrm{ECC}(Y, \lambda)$ generated by all orbit sets $\gamma$ with $\mathcal{A}_\lambda(\gamma) < L$. Since the differential decreases the action, the subgroup $\mathrm{ECC}^L(Y, \lambda)$ is indeed a subcomplex. Its homology is denoted by $\mathrm{ECH}^L(Y, \lambda)$ and it is independent of the almost-complex structure by [9, Theorem 1.3(a)].

For $i = 1, 2$, let $(Y_i, \lambda_i)$ be a @3@-manifold with contact form $\lambda_i$. An exact symplectic cobordism from $(Y_1, \lambda_1)$ to $(Y_2, \lambda_2)$ is a pair $(W, d\lambda)$, where $W$ is a compact 4-manifold, $d\lambda$ is a symplectic form, $\partial W = Y_1 \cup (-Y_2)$ and $\lambda|_{Y_i} = \lambda_i$ for $i = 1, 2$. According to [9, Theorem 1.9], such cobordisms induce maps

$$\Phi^L(X, \lambda) \colon \mathrm{ECH}^L(Y_1, \lambda_1) \longrightarrow \mathrm{ECH}^L(Y_2, \lambda_2).$$

The maps $\Phi^L$ are constructed by taking the composition of the corresponding map in Seiberg–Witten Floer homology and the isomorphism from ECH to Seiberg–Witten Floer homology.

**Lemma 4.1.** *Let* $([0,1] \times Y, d\lambda)$ *be an exact cobordism from* $(Y, \lambda_1)$ *to* $(Y, \lambda_0)$. *Then, for every* $L > 0$, *the map* $\Phi^L([0,1] \times Y, \lambda)$ *preserves the absolute grading, i.e.* $\Phi^L([0,1] \times Y, \lambda)$ *maps* $\mathrm{ECH}^L_\rho(Y_1, \lambda_1)$ *to* $\mathrm{ECH}^L_\rho(Y_0, \lambda_0)$ *for every* $\rho \in \mathrm{Vect}(Y)$.

*Proof.* The maps $\Phi^L([0,1] \times Y, \lambda)$ are defined as a composition of maps

$$\mathrm{ECH}^L(Y_1, \lambda_1) \to \widehat{HM}_L(Y, \lambda_1) \to \widehat{HM}_L(Y, \lambda_0) \longrightarrow \mathrm{ECH}^L(Y, \lambda_0). \quad (10)$$

Here $\widehat{HM}_L(Y, \lambda_1)$ and $\widehat{HM}_L(Y, \lambda_0)$ are appropriate filtered Seiberg–Witten Floer cohomology groups, as explained in [9]. The second map in (10) is a filtered version of the cobordism maps defined in [10, §25]. Now it follows from the definition of these maps that if an element of $\widehat{HM}_L(Y, \lambda_1)$ has grading $\rho_1 = [v_1] \in \mathrm{Vect}(Y)$, then its image in $\widehat{HM}_L(Y, \lambda_2)$ is the sum of elements of (possibly different) gradings $\rho_0 = [v_0]$ such that for each such $\rho_0$ there exists an almost-complex structure $J$ on $[0,1] \times Y$ satisfying

$$v_i^\perp = T(\{i\} \times Y) \cap J(T(\{i\} \times Y)), \quad i = 0, 1.$$

Now, for $t \in [0,1]$, we let $\xi_t = T(\{t\} \times Y) \cap J(T(\{t\} \times Y))$. Since $T(\{t\} \times Y)$ cannot be invariant under $J$, it follows that $\xi_t$ is a 2-plane field for every $t$. Therefore $\{\xi_t\}$ is a homotopy between $v_0^\perp$ and $v_1^\perp$. Hence $\rho_0 = \rho_1$. So the second map in (10) preserves the absolute grading.

Now, the first and third maps in (10) preserve the grading by [5]. Therefore $\Phi^L([0,1] \times Y, \lambda)$ preserves the grading. ◻

**4.3. The map $\psi$.** Let $\lambda$ be a contact form adapted to the open book $(S, \varphi)$ with the compatibility conditions required in §2.4. The map

$$\psi\colon \mathrm{ECH}_{2g}(N, \lambda) \longrightarrow \widehat{\mathrm{ECH}}(Y)$$

is defined to be the composition $\psi = \hat{\Psi}_1 \circ \hat{\Psi}_2$ as follows:

$$\mathrm{ECH}_{2g}(N, \lambda) \xrightarrow{\hat{\Psi}_2} \widehat{\mathrm{ECH}}(N, \partial N, \lambda) \xrightarrow{\hat{\Psi}_1} \widehat{\mathrm{ECH}}(Y).$$

We will now recall the definition of $\widehat{\mathrm{ECH}}(N, \partial N, \lambda)$, show how to extend the absolute grading to it and prove that $\hat{\Psi}_1$ and $\hat{\Psi}_2$ preserve the grading.

We let $\mathrm{ECC}(N, \lambda)$ denote the chain complex generated by orbit sets contructed from Reeb orbits in the interior of $N$ and the orbits $\{e, h\}$ where $e$ and $h$ are seen are elliptic and hyperbolic orbits, respectively. The differential counts Morse–Bott buildings of ECH index 1, as explained in [4]. Then $\mathrm{ECC}_{2g}(N, \lambda)$ is a subcomplex of $\mathrm{ECC}(N, \lambda)$ and the construction of §2.3 endows $\mathrm{ECC}(N, \lambda)$ with an absolute grading taking values in $\mathrm{Vect}(Y)$. Let $\mathrm{ECH}(N, \lambda)$ denote its homology. The inclusion induces a map $\iota_*\colon \mathrm{ECH}_{2g}(N, \lambda) \to \mathrm{ECH}(N, \lambda)$. Following the notation in [4], we define $\widehat{\mathrm{ECH}}(N, \partial N, \lambda)$ to be the quotient of $\mathrm{ECH}(N, \lambda)$ by the equivalence relation generated by $[\gamma] \sim [e\gamma]$ where $\gamma = \prod_i \gamma_i^{m_i}$ is written multiplicatively. The map $\hat{\Psi}_2$ is the composition of $\iota_*$ with the quotient map, which can be shown to be an isomorphism. It follows from Lemma 4.2 below that $\mathrm{gr}(\gamma) = \mathrm{gr}(e\gamma)$. So the absolute grading on $\mathrm{ECH}(N, \lambda)$ descends to the quotient $\widehat{\mathrm{ECH}}(N, \partial N, \lambda)$. Therefore the map $\hat{\Psi}_2$ preserves the grading.

The definition of $\hat{\Psi}_1$ is much more complicated and the proof that it preserves the grading will be the goal of the rest of this paper. Let $\mathrm{ECC}^\flat(N, \lambda)$ be the chain complex generated by orbit sets contructed from Reeb orbits in the interior of $N$ and $\{e\}$ and let $\mathrm{ECH}^\flat(N, \lambda)$ denote its the homology. Now let $\mathrm{ECH}(N, \partial N, \lambda)$ denote the quotient of $\mathrm{ECH}^\flat(N, \lambda)$ by the equivalence relations generated by $[\gamma] \sim [e\gamma]$. Similarly to the paragraph above, the quotient map induces an absolute grading on $\mathrm{ECH}(N, \partial N, \lambda)$ taking values on $\mathrm{Vect}(Y)$.

In [4], Colin, Ghiggini and Honda also constructed an isomorphism

$$\Psi_1\colon \mathrm{ECH}(N, \partial N, \lambda) \longrightarrow \mathrm{ECH}(Y).$$

We will now recall the construction of $\Psi_1$, show that it preserves the grading and explain why this implies that $\hat{Psi}_1$ also preserves the grading. Recall that $Y = N \cup (S^1 \times D^2)$. We write the solid torus $S^1 \times D^2$ as $V \cup (T^2 \times [0, 1])$ where $V$ is a smaller tubular neighborhood of the binding $S^1 \times \{0\}$, which is again a solid torus. Let $\lambda_V$ be a contact form on $V$ which is nondegenerate in the interior of $V$ such that the Reeb vector field of $\lambda_V$ is positively transverse to the interior of the pages and positively tangent to the binding and such that $\partial V$ is a *positive* Morse–Bott torus. The precise construction of $\lambda_V$ will not be necessary here and we refer the reader to [4, §8.1]. We denote by $e'$ and $h'$ the elliptic and hyperbolic orbits obtained after a Morse–Bott perturbation near $\partial V$. Let $\{L_k\}$ be an increasing sequence such that $\lim_{k\to\infty} L_k = \infty$. Following [4, §9.3], we can choose a family of contact forms $\{\lambda_k\}$ on $Y$ which equal $\lambda$ in a neighborhood of $N$ and a positive multiple of $\lambda_V$ in a neighborhood of $V$ such that $\lambda_k$ is a Morse–Bott contact form and all Reeb orbits in $T^2 \times [0,1]$ have action larger than $L_k$. So as in [4, §9.2], we can perturb $\{\lambda_k\}$ to a sequence of contact forms $\{\lambda'_k\}$ satisfying, in particular, the following conditions:

- $\lambda'_k$ coincides with $\lambda_k$ outside neighborhoods of the Morse–Bott tori.
- The Reeb orbits of $\lambda_k$ of action less than $L_k$ are nondegenerate and are either the Reeb orbits of $\lambda$ and $\lambda_V$ in the interior of $N$ and $V$, respectively, or one of the orbits $e$, $h$, $e'$ or $h'$.

Hence $\mathrm{ECC}^{L_k}(Y, \lambda'_k)$ is generated by elements of the form $\gamma_V \cdot \gamma_N$, where $\gamma_V$ is an orbit set contructed from Reeb orbits in the interior of $V$ and $\{e', h'\}$, and $\gamma_N$ is a generator of $\mathrm{ECC}(N, \lambda)$. For $L > 0$, let $\mathrm{ECC}^{\flat, L}_{\leq k}(N, \lambda)$ be the subcomplex of $\mathrm{ECC}^{\flat}(N, \lambda)$ generated by orbit sets $\gamma$ with action $\int_\gamma \lambda < L$ and whose total homology class intersects a page up to $k$ times. Following [4, §9.7], we can define another increasing sequence $\{L'_k\}$ with $\lim_{k\to\infty} L'_k = \infty$ such that the maps $\sigma_k$ below are well-defined.

$$\begin{aligned}\sigma_k \colon \mathrm{ECC}^{\flat, L'_k}_{\leq k}(N, \lambda) &\longrightarrow \mathrm{ECC}^{L_k}(Y, \lambda'_k)\\ \gamma &\longmapsto \sum_{i=0}^{\infty} (e')^i \cdot (\partial'_N)^i \gamma.\end{aligned}$$

Here $\partial'_N \gamma$ is defined by the equation $\partial_N \gamma = \partial^\flat_N \gamma + h \partial'_N \gamma$, where $\partial_N$ and $\partial^\flat_N$ are the differentials in $\mathrm{ECC}(N, \lambda)$ and $\mathrm{ECC}^\flat(N, \lambda)$, respectively. It follows from [4, Lemma 9.7.2] that the maps $\sigma_k$ are chain maps so they induce maps

$$\sigma_k \colon \mathrm{ECH}^{\flat, L'_k}_{\leq k}(N, \lambda) \longrightarrow \mathrm{ECH}^{L_k}(Y, \lambda'_k).$$

Following [4, Cor. 3.2.3], there are chain maps

$$\Phi_k \colon \mathrm{ECC}^{L_k}(Y, \lambda'_k) \longrightarrow \mathrm{ECC}^{L_{k+1}}(Y, \lambda'_{k+1})$$

which are given by cobordism maps as in §4.2. So we obtain a directed system

$$\begin{array}{ccc} \mathrm{ECC}^{\flat, L'_k}_{\leq k}(N,\lambda) & \xrightarrow{\sigma_k} & \mathrm{ECC}^{L_k}(Y,\lambda'_k) \\ \big\downarrow \iota_k & & \big\downarrow \Phi_k \\ \mathrm{ECC}^{\flat, L'_{k+1}}_{\leq k+1}(N,\lambda) & \xrightarrow{\sigma_k} & \mathrm{ECC}^{L_{k+1}}(Y,\lambda'_{k+1}) \end{array} \tag{11}$$

where $\iota_k$ denotes the inclusion. The maps $\Phi_k$ induce maps in homology with respect to which one can take the direct limit $\lim_{k\to\infty} \mathrm{ECH}^{L_k}(Y,\lambda'_k)$. There is also a nondegenerate contact form $\lambda_0$ and cobordism maps $\mathrm{ECH}^{L_k}(Y,\lambda_0) \to \mathrm{ECH}^{L_k}(Y,\lambda'_k)$. It is shown in [4, Cor. 3.2.3] that the direct limit of these maps is an isomorphism

$$\mathrm{ECH}(Y,\lambda_0) \cong \lim_{k\to\infty} \mathrm{ECH}^{L_k}(Y,\lambda'_k). \tag{12}$$

Now we note that $\mathrm{ECH}^{\flat}(N,\lambda) = \lim_{k\to\infty} \mathrm{ECH}^{\flat, L'_k}_{\leq k}(N,\lambda)$. Therefore the maps $\sigma_k$ give rise to a map

$$\bar{\sigma}\colon \mathrm{ECH}^{\flat}(N,\lambda) \longrightarrow \lim_{k\to\infty} \mathrm{ECH}^{L_k}(Y,\lambda'_k) \cong \mathrm{ECH}(Y,\lambda_0).$$

The calculations in [4, §9.7] imply that $\bar{\sigma}([\gamma]) = \bar{\sigma}([e\gamma])$. Hence we obtain a map

$$\Psi_1\colon \mathrm{ECH}(N,\partial N,\lambda) \longrightarrow \mathrm{ECH}(Y).$$

It is shown in [4, Theorem 9.8.3] that $\Psi_1$ is an isomorphism.

We will now prove a useful lemma.

**Lemma 4.2.** *Let $\gamma$ be an orbit set obtained from the Reeb orbits of $\lambda$ in the interior of $N$, respectively, and the orbits $e$, $h$, $e'$ or $h'$. Then $\mathrm{gr}(\gamma) \in \mathrm{Vect}(Y)$ is well-defined. Moreover,*

$$\mathrm{gr}(e\gamma) = \mathrm{gr}(\gamma), \qquad \mathrm{gr}(h\gamma) = \mathrm{gr}(\gamma) + 1, \tag{13a}$$

$$\mathrm{gr}(e'\gamma) = \mathrm{gr}(\gamma) + 2, \quad \mathrm{gr}(h'\gamma) = \mathrm{gr}(\gamma) + 1. \tag{13b}$$

*Proof.* To see that $\gamma$ has a well-defined grading, first note that there exists $k_0$ such that $\gamma \in \mathrm{ECC}^{L_k}(Y,\lambda'_k)$ for every $k \geq k_0$. So we define $\mathrm{gr}(\gamma)$ using the contact form $\lambda'_k$ for some $k \geq k_0$. It follows from Lemma 4.1 that the maps $\Phi_k$ preserve the grading. So $\mathrm{gr}(\gamma) \in \mathrm{Vect}(Y)$ is well-defined.

To prove (13), we can restrict to the case when $\gamma$ does not contain $e$, $h$, $e'$ or $h'$. The general case is a straightforward consequence of this case. Let $\tau$

be a trivialization of $\xi$ over $\gamma$ and let $L$ be a link as in §2.3 so that $\operatorname{gr}(\gamma) = P_\tau(L) - w_\tau(L) + CZ^I_\tau(\gamma)$, where $w_\tau(L)$ denotes the sum of the writhes of all components of $L$. Let $x \in \{e, h, e', h'\}$. The tangent bundle of the Morse–Bott torus containing $x$ determines a trivialization of $\xi|_x$ which we denote by $\eta$. Let $\zeta$ be a knot obtained by pushing $x$ in a direction which is transverse to the Morse–Bott torus containing $x$ such that $\zeta$ is in the interior of $Y \setminus N$. Then $w_\eta(\zeta) = 0$. Now let $D$ be a the disk in $Y \setminus N$ bounding $x$. It follows from [8, Lemma 3.4(d)] that

$$P_{(\tau,\eta)}(L \cup \zeta) - P_\tau(L) = c_1(\xi|_D, \eta) = 1.$$

Moreover,

$$\begin{aligned} CZ_\eta(x) &= -1, && \text{if } x = e,\\ CZ_\eta(x) &= 0, && \text{if } x = h, h',\\ CZ_\eta(x) &= 1, && \text{if } x = e'. \end{aligned}$$

Therefore it follows from (3) that (13) holds. □

**Proposition 4.3.** *The isomorphism* $\Psi_1 \colon \mathrm{ECH}(N, \partial N, \lambda) \to \mathrm{ECH}(Y)$ *preserves the grading.*

*Proof.* Let $\gamma$ be an orbit set in $\mathrm{ECC}^{\flat, L'_k}_{\leq k}(N, \lambda)$ for some $k$. Since $\partial_N$ decreases the grading by 1, it follows that $\operatorname{gr}(h\partial'_N\gamma) = \operatorname{gr}(\gamma) - 1$. Now, by Lemma 4.2, $\operatorname{gr}(\partial'_N\gamma) = \operatorname{gr}(\gamma) - 2$. Hence for all $0 \geq i \geq k$,

$$\operatorname{gr}((e')^i \cdot (\partial'_N)^i\gamma) = \operatorname{gr}(\gamma) - 2i + 2i = \operatorname{gr}(\gamma).$$

So $\sigma_k$ preserves the grading. Now, it is tautological that the inclusion $\iota_k$ in (11) preserves the grading. Moreover, by Lemma 4.1, the maps $\Phi_k$ and the isomorphism (12) preserve the grading. Hence after passing to homology and taking the direct limit we conclude that $\bar{\sigma}$, and hence $\Psi_1$, preserve the grading. □

We now define two chain maps as follows.

$$\begin{array}{cccccc} \iota\colon \mathrm{ECC}^\flat(N,\lambda) & \longrightarrow & \mathrm{ECC}(N,\lambda), & \pi\colon \mathrm{ECC}(N,\lambda) & \longrightarrow & \mathrm{ECC}^\flat(N,\lambda),\\ \gamma & \longmapsto & h\gamma, & \gamma_1 + h\gamma_2 & \longmapsto & \gamma_1. \end{array} \tag{14}$$

Here $\gamma_1$ and $\gamma_2$ do not contain $h$. These maps descend to homology and to the quotients $\widehat{\mathrm{ECH}}(N, \partial N, \lambda)$ and $\mathrm{ECH}(N, \partial N, \lambda)$. It follows from [4, §9.9] that these maps fit into an exact triangle

$$\begin{array}{ccc} \mathrm{ECH}(N, \partial N, \lambda) & \xrightarrow{\qquad\qquad} & \mathrm{ECH}(N, \partial N, \lambda) \\ {\scriptstyle \pi_*}\nwarrow & & \swarrow{\scriptstyle \iota_*} \\ & \widehat{\mathrm{ECH}}(N, \partial N, \lambda) & \end{array} \tag{15}$$

where the map $\mathrm{ECH}(N, \partial N, \lambda) \to \mathrm{ECH}(N, \partial N, \lambda)$ is a version of the $U$ map. Moreover there exists an isomorphism $\widehat{\Psi}_1 \colon \mathrm{ECH}(N, \partial N, \lambda) \to \widehat{\mathrm{ECH}}(Y)$ such that $\Psi_1$ and $\widehat{\Psi}_1$ give an isomorphism from (15) to (9). It follows from (14) that $\iota_*$ increases the grading by 1 and that $\pi_*$ preserves the grading. Hence we obtain the following diagram.

$$\begin{CD} \cdots @>>> \mathrm{ECH}_{\rho-1}(N, \partial N, \lambda) @>>> \widehat{\mathrm{ECH}}_{\rho}(N, \partial N, \lambda) @>>> \mathrm{ECH}_{\rho}(N, \partial N, \lambda) @>>> \cdots \\ @. @VV{\Psi_1}V @VV{\widehat{\Psi}_1}V @VV{\Psi_1}V @. \\ \cdots @>{U}>> \mathrm{ECH}_{\rho-1}(Y) @>>> \widehat{\mathrm{ECH}}_{\rho}(Y) @>>> \mathrm{ECH}_{\rho}(Y) @>{U}>> \cdots \end{CD}$$

Therefore $\widehat{\Psi}_1$ preserves the grading.

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