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Twisting, mutation and knot Floer homology

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Abstract. Let $\mathcal{L}$ be a knot with a fixed positive crossing and $\mathcal{L}_n$ the link obtained by replacing this crossing with n positive twists. We prove that the knot Floer homology $\widehat{HFK}(\mathcal{L}_n)$ 'stabilizes' as n goes to infinity. This categorifies a similar stabilization phenomenon of the Alexander polynomial. As an application, we construct an infinite family of prime, positive mutant knots with isomorphic bigraded knot Floer homology groups. Moreover, given any pair of positive mutants, we describe how to derive a corresponding infinite family of positive mutants with isomorphic bigraded $\widehat{HFK}$ groups, Seifert genera, and concordance invariant τ.

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1. Introduction

An interesting open question is the relationship between mutation and knot Floer homology. While many knot polynomials and homology theories are insensitive to mutation, the bigraded knot Floer homology groups can detect Conway mutation [19] and genus 2 mutation [14]. Conversely, explicit computations [2] and a combinatorial formulation [3] suggest that the δ-graded $\widehat{HFK}$ groups are invariant under Conway mutation. Recent work of the author has established invariance of δ-graded $\widehat{HFK}$ under Conway mutations on a large class of tangles [11].

In this paper, we construct an infinite family of prime, positive Conway mutants whose $\widehat{HFK}$ groups agree. Recall that a mutation is *positive* if the mutant links admit compatible orientations.

Theorem 1.1. *There exist an infinite family of positive Conway mutant knots* $\{KT_n, C_n\}_{n\in\mathbb{Z}}$ *such that*

(1) *for all n, the knots* KT_n *and* C_n *are not isotopic,*

(2) *for all* $|n| \gg 0$*, the knots* KT_n, C_n *are prime, and*

(3) *for all* $|n| \gg 0$*, there is a bigraded isomorphism*

$$\widehat{HFK}(KT_n) \cong \widehat{HFK}(C_n).$$

The families $\{KT_n\}$ and $\{C_n\}$ are constructed from the Kinoshita-Terasaka [10] and Conway knots, respectively, by adding n full twists to the knots just outside the mutation sphere. Note that while KT and C are distinguished by $\widehat{HFK}$ [19, 2] and their genera [8], adding sufficiently many twists forces their bigraded knot Floer groups (and therefore genera) to coincide. To distinguish KT_n and C_n, we use tangle invariants introduced by Cochran and Ruberman [7].

The bigraded invariance of $\widehat{HFK}$ for these mutant pairs follows from a more general fact. In a particular sense, 'most' positive Conway mutants have isomorphic knot Floer homology groups over $\mathbb{F}_2$. Specifically, each pair naturally lies in an infinite family of positive mutants whose pairwise, bigraded $\widehat{HFK}$ agree. Moreover, we can prove that 'most' positive Conway mutants have the same concordance invariant τ. It remains an compelling open question whether τ is preserved by Conway mutation.

Theorem 1.2. *Let $\mathcal{L}$ be an oriented knot with positive Conway mutant $\mathcal{L}'$. Let $\mathcal{L}_n, \mathcal{L}'_n$ denote the Conway mutants obtained by adding n half-twists along parallel-oriented strands of $\mathcal{L}$ just outside the mutation sphere* (*see Figure* 4). *Then for* $|n| \gg 0$ *there is a bigraded isomorphism*

$$\widehat{HFK}(\mathcal{L}_n) \cong \widehat{HFK}(\mathcal{L}'_n).$$

Moreover, for $|n| \gg 0$,

$$g(\mathcal{L}_{2n}) = g(\mathcal{L}'_{2n}),$$

and

$$\tau(\mathcal{L}_{2n}) = \tau(\mathcal{L}'_{2n}).$$

where g denotes the Seifert genus and τ denotes the knot Floer concordance invariant.

The geometric motivation for this construction is the *twist family* $\{K_{2n}\}$ obtained from a knot K by $-\frac{1}{n}$ surgery on an unknot that links two strands positively and geometrically twice, as in Figure 1. There has been recent interest in the relationship between this twisting operation and Heegaard Floer invariants. Motegi [13] and Baker and Motegi [4] investigated L-space surgeries and Seifert genera in the twist families obtained by the operation in Figure 1. Conversely, when the orientation on one of the strands is reversed, Hedden and Watson [9] found infinite families with isomorphic knot Floer homology.

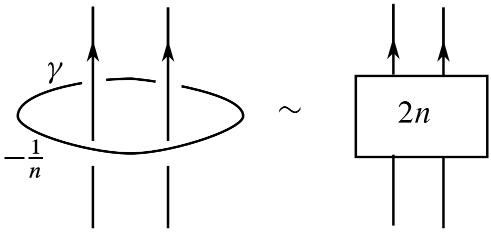

Figure 1. Surgery on an unknot γ introduces twisting.

The proof of Theorem 1.2 follows from a second result we prove in this paper. We show that knot Floer homology categorifies the following phenomenon for the Alexander polynomial of twist families. Let $\mathcal{L} \subset S^3$ be a link with a positive crossing. For a positive integer $n \in \mathbb{Z}$, let $\mathcal{L}_{2n}$ be the n^{th} link in the twist family, obtained by the surgery in Figure 1. As n goes to infinity, the Alexander polynomial of $\mathcal{L}_{2n}$ stabilizes to the form

$$\Delta_{\mathcal{L}_{2n}}(t) = t^{\frac{2n-k}{2}} \cdot f(t) + d \cdot \Delta_{T(2,2n-k)}(t) + t^{-\frac{2n-k}{2}} \cdot f(t^{-1})$$

for some integers k, d and some polynomial $f(t) \in \mathbb{Z}[\sqrt{t}]$ (Corollary 2.2). The Alexander polynomials of the twist family $\{\mathcal{L}_{2n-1}\}$, obtained as a twist family after resolving one of the new crossings, satisfy the same stabilization phenomenon.

In this paper, we show that the knot Floer homology of this extended twist family $\{\mathcal{L}_n\}$ stabilizes in a similar fashion as $|n|$ goes to infinity.

Theorem 1.3. *Let $\mathcal{L}$ be an oriented knot, γ an unknot as is Figure 1, and let $\{\mathcal{L}_n\}$ be the extended twist family obtained by surgery on γ and resolving a crossing.*

(1) *There exists some $k > 0$ such that for $|n|$ sufficiently large, the knot Floer homology of $\mathcal{L}_n$ satisfies*

$$\widehat{HFK}(\mathcal{L}_n, j) \cong \widehat{HFK}(\mathcal{L}_{n+2}, j+1) \qquad \text{for } j \geq -k,$$
$$\widehat{HFK}(\mathcal{L}_n, j) \cong \widehat{HFK}(\mathcal{L}_{n+2}, j-1)[2] \quad \text{for } j \leq k,$$

where $[i]$ denotes decreasing the Maslov grading by i.

(2) *There exists some $k > 0$ and bigraded vector spaces $\widehat{F}_\circ, \widehat{F}_\bullet, \widehat{A}, \widehat{B}$ such that for $|n|$ sufficiently large, there is a bigraded isomorphism*

$$\begin{aligned}\widehat{HFK}(\mathcal{L}_{2n}) \cong\ & \widehat{F}_\circ[2(n-k),(n-k)] \\ & \oplus \bigoplus_{i=k-n-1}^{0} \widehat{A}[2(n-k)+2i+1,(n-k)+2i+1] \\ & \oplus \bigoplus_{i=k-n}^{0} \widehat{B}[2(n-k)+2i,(n-k)+2i] \oplus \widehat{F}_\bullet[0,(k-n)]\end{aligned}$$

where $[i, j]$ *denotes decreasing the homological grading by* i *and Alexander grading by* j *and*

(a) $\widehat{F}_\bullet$ (*resp.* $\widehat{F}_\circ$) *is supported in positive* (*resp. negative*) *Alexander gradings,*

(b) $\widehat{A}, \widehat{B}$ *are supported in Alexander grading 0.*

(3) *For* n *sufficiently large, there is a bigraded isomorphism*

$$\widehat{HFK}(\mathcal{L}_{2n+1}) \cong \widehat{HFK}(\mathcal{L}_{2n}) \oplus \widehat{HFK}(\mathcal{L}_{2n+2})[1]$$

where the summands on the right are described in Part (2) *and* $[i]$ *denotes decreasing the Maslov grading by* i.

A key consequence (Lemma 2.10) of Theorem 1.3 is that, for $|n|$ sufficiently large, the skein exact triangle for the triple $(\mathcal{L}_{n+1}, \mathcal{L}_{n-1}, \mathcal{L}_n)$ splits. We apply this observation to prove Theorem 1.2.

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2. Twisting and knot Floer homology

Throughout this section, let $\mathcal{L}_1 \subset S^3$ be an oriented link with a distinguished positive crossing and for any $n \in \mathbb{Z}$ let $\mathcal{L}_n$ denote the oriented link obtained by replacing this crossing with n half twists.

2.1. Alexander polynomial. Let $\Delta_{\mathcal{L}}(t)$ denote the symmetrized Alexander polynomial of $\mathcal{L}$ and let $\Delta_k(t)$ denote the symmetrized Alexander polynomial of the $(2, k)$-torus link. For brevity, we may omit the variable t.

The symmetrized Alexander polynomial of $T(2,k)$ for $k \neq 0$ is either

$$\Delta_k(t) = \sum_{s=-\frac{k-1}{2}}^{\frac{k-1}{2}} (-t)^s \quad \text{or} \quad \Delta_k(t) = t^{\frac{1}{2}} \sum_{s=-\frac{k}{2}}^{\frac{k}{2}-1} (-t)^s, \tag{1}$$

according to whether k is odd or even. The unknot $T(2,1) \sim T(2,-1)$ has Alexander polynomial $\Delta_1(t) = \Delta_{-1}(t) = 1$; the unlink $T(2,0)$ is split and therefore $\Delta_0(t) = 0$; and the Hopf link $T(2,2)$ has Alexander polynomial $\Delta_2(t) = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. The Alexander polynomials of the $(2,k)$ torus knots satisfy the skein relation

$$\Delta_2\Delta_{n+1} = \Delta_{n-1} - \Delta_n \tag{2}$$

for all $n \in \mathbb{Z}$.

Proposition 2.1. *The Alexander polynomial of $\mathcal{L}_n$ satisfies the relation*

$$\Delta_{\mathcal{L}_n} = \Delta_{n+1}\Delta_{\mathcal{L}_0} + \Delta_n\Delta_{\mathcal{L}_{-1}}.$$

Proof. The formula is correct when $n = 0, -1$ since

$$\Delta_{\mathcal{L}_0} = \Delta_1\Delta_{\mathcal{L}_0} + \Delta_0\Delta_{\mathcal{L}_{-1}} = \Delta_{\mathcal{L}_0}$$

and

$$\Delta_{\mathcal{L}_{-1}} = \Delta_0\Delta_{\mathcal{L}_0} + \Delta_{-1}\Delta_{\mathcal{L}_{-1}} = \Delta_{\mathcal{L}_{-1}}.$$

For positive n, the formula follows inductively by using the oriented skein relation for the Alexander polynomial:

$$\begin{aligned}
\Delta_{\mathcal{L}_{n+1}} &= \Delta_2\Delta_{\mathcal{L}_n} + \Delta_{\mathcal{L}_{n-1}} \\
&= \Delta_2(\Delta_{n+1}\Delta_{\mathcal{L}_0} + \Delta_n\Delta_{\mathcal{L}_{-1}}) + \Delta_n\Delta_{\mathcal{L}_0} + \Delta_{n-1}\Delta_{\mathcal{L}_{-1}} \\
&= (\Delta_2\Delta_{n+1} + \Delta_n)\Delta_{\mathcal{L}_0} + (\Delta_2\Delta_n + \Delta_{n-1})\Delta_{\mathcal{L}_{-1}} \\
&= \Delta_{n+2}\Delta_{\mathcal{L}_0} + \Delta_{n+1}\Delta_{\mathcal{L}_{-1}}.
\end{aligned}$$

The first line is the oriented skein relation, the second is obtained by using the induction hypothesis, the third is obtained by rearranging the terms and the fourth is obtained by applying the oriented skein relation to the Alexander polynomials of torus knots. A similar inductive argument proves the formula for $n < -1$. □

Corollary 2.2. *There exists some $k > 0$, some $d \in \mathbb{Z}$, and some polynomial $f(t) \in \mathbb{Z}[\sqrt{t}]$ such that for n sufficiently large, the Alexander polynomial of $\mathcal{L}_n$ has the form*

$$\Delta_{\mathcal{L}_n}(t) = t^{\frac{n-k}{2}} \cdot f(t) + d \cdot \Delta_{T(2,n-k)}(t) + t^{-\frac{n-k}{2}} \cdot f(t^{-1}).$$

Proof. Without loss of generality, we assume that $\Delta_{\mathcal{L}_{-1}} \in \mathbb{Z}[t^{\pm 1}]$ and n is odd. Suppose that the Alexander polynomials of $\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_n, \mathcal{L}_{n+2}$ have the following forms:

$$\Delta_{\mathcal{L}_{-1}} = \sum_{s=-k_{-1}}^{k_{-1}} a_s t^s, \qquad \Delta_{\mathcal{L}_0} = t^{-\frac{1}{2}} \cdot \sum_{s=-k_0}^{k_0+1} b_s t^s,$$
$$\Delta_{\mathcal{L}_n} = \sum_{s=-k_n}^{k_n} c_s t^s, \quad \Delta_{\mathcal{L}_{n+2}} = \sum_{s=-k_{n+2}}^{k_{n+2}} d_s t^s.$$

The coefficients of Δ_n are ± 1. Setting $q = \frac{|n|-1}{2}$, the coefficients of

$$\Delta_n \cdot \Delta_{\mathcal{L}_{-1}} = (t^{-q} - t^{-q+1} + t^{-q+2} + \cdots \pm t^{q-2} \mp t^{q-1} \pm t^q)$$
$$\cdot (a_{k_{-1}} t^{-k_{-1}} + a_{k_{-1}-1} t^{-k_{-1}+1} + \cdots + a_{k_{-1}-1} t^{k_{-1}-1} + a_{k_{-1}} t^{-k_{-1}})$$

are an alternating sum of some subsets of the coefficients $\{a_s\}$. When $|n|$ is sufficiently large, the degree of $\Delta_n \Delta_{\mathcal{L}_{-1}}$ is $2(q + k_{-1}) + 1$, the leading coefficient is $\pm a_{k_{-1}}$, and the coefficient of t^0 in $\Delta_n \cdot \Delta_{\mathcal{L}_{-1}}$ is precisely the determinant $\Delta_{\mathcal{L}_{-1}}(-1)$, up to sign. An equivalent statement holds for the coefficients of $\Delta_{n+1}\Delta_{\mathcal{L}_0}$.

Applying Proposition 2.1, a straightforward computation shows that the coefficents $\{c_s\}$ are an alternating sum of some subsets of the coefficients $\{a_s\}$ and $\{b_s\}$. For n sufficiently large and $s \geq 0$ the coefficients of $\Delta_{\mathcal{L}_n}$ and $\Delta_{\mathcal{L}_{n+2}}$ satisfy the relation $c_s = d_{s+1}$. This implies that $\Delta_{\mathcal{L}_n}$ has the specified form. □

Corollary 2.3. *If $\mathcal{L}_1$ is a knot, then the degree of $\Delta_{\mathcal{L}_n}$ goes to infinity as $|n|$ goes to infinity.*

Proof. Since $\mathcal{L}_1$ is a knot, the determinant $|\Delta_{\mathcal{L}_n}(-1)|$ is nonzero when n is odd. Thus, in the formula from Corollary 2.2 the polynomial $f(t)$ and integer d cannot both be 0. As a result, the degree of $\Delta_{\mathcal{L}_n}$ is at least $n - k$ for some k independent of n. □

2.2. Knot Floer homology. In this subsection, we review the basic definitions of knot Floer homology. For a more thorough treatment, see [17, 20].

A multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, \mathbf{w})$ for an l-component link $L \subset S^3$ is a tuple consisting of a genus g Riemann surface Σ, two multicurves $\boldsymbol{\alpha} = \{\alpha_1 \cup \cdots \cup \alpha_{g+n}\}$ and $\boldsymbol{\beta} = \{\beta_1 \cup \cdots \cup \beta_{g+n}\}$, and two collections of basepoints $\mathbf{z} = \{z_1, \ldots, z_{n+1}\}$ and $\mathbf{w} = \{w_1, \ldots, w_{n+1}\}$ such that

(1) $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram for S^3,

(2) each component of $\Sigma \setminus \boldsymbol{\alpha}$ and $\Sigma \setminus \boldsymbol{\beta}$ contains exactly one **z**-basepoint and one **w**-basepoint,

(3) the basepoints $\mathbf{z}, \mathbf{w}$ determine the link L as follows: choose collections of embedded arcs $\{\gamma_1, \dots, \gamma_{n+1}\}$ in $\Sigma - \boldsymbol{\alpha}$ and $\{\delta_1, \dots, \delta_{n+1}\}$ in $\Sigma \setminus \boldsymbol{\beta}$ connecting the basepoints. Then after depressing the arcs $\{\gamma_i\}$ into the α-handlebody and the arcs $\{\delta_j\}$ into the β-handlebody, their union is L.

From a multipointed Heegaard diagram $\mathcal{H}$ we obtain a complex $\mathrm{CFK}^-(\mathcal{H})$. In the symmetric product $\mathrm{Sym}^{g+n}(\Sigma)$ of the Heegaard surface, the multicurves $\boldsymbol{\alpha}, \boldsymbol{\beta}$ determine $(g+n)$-dimensional tori $\mathbb{T}_\alpha = \alpha_1 \cup \cdots \cup \alpha_{g+n}$ and $\mathbb{T}_\beta = \beta_1 \cup \cdots \cup \beta_{g+n}$. The knot Floer complex $\mathrm{CFK}^-(\mathcal{H})$ is freely generated over $\mathbb{F}[U_1, \dots, U_{n+1}]$ by the intersection points $\mathfrak{G}(\mathcal{H}) = \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

The complex possesses two gradings, the Maslov grading M and the Alexander grading A. Given any Whitney disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ connecting the two generators, the relative gradings of two generators $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ satisfy the formulas

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2 \sum n_{w_i}(\phi)$$
$$A(\mathbf{x}) - A(\mathbf{y}) = \sum n_{z_i}(\phi) - \sum n_{w_i}(\phi)$$

where $\mu(\phi)$ is the Maslov index of ϕ and $n_{z_i}(\phi)$ and $n_{w_i}(\phi)$ denote the algebraic intersection numbers of ϕ with respect to the subvarieties $\{w_i\} \times \mathrm{Sym}^{g+n-1}(\Sigma)$ and $\{z_i\} \times \mathrm{Sym}^{g+n-1}(\Sigma)$. Note that while there are formulations of link Floer homology with an independent Alexander grading for each link component, we restrict to a single Alexander grading. In addition, multiplication by any formal variable U_{w_i} decrease the Maslov grading by 2 and the Alexander grading by 1. The graded Euler characteristic of the complex is a multiple of the Alexander polynomial of the link. Specifically, it satisfies the formula

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{l-1} \cdot \Delta_L(t) = \sum_{j \in \mathbb{Z}} t^j \cdot \Big(\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{F}} \widehat{CFK}(\mathcal{H})_{i,j} \Big),$$

where l is the number of components of L.

The differential is defined by certain counts of pseudoholomorphic curves in $\mathrm{Sym}^{g+n}(\Sigma)$. We review some of the analytic details from [18]. Fix a complex structure j and Kahler form η on Σ and let J be the induced complex structure on $\mathrm{Sym}^{g+n}(\Sigma)$. The basepoints **z** determine a complex hypersurface $V_{\mathbf{z}}$ of $\mathrm{Sym}^{g+n}(\Sigma)$. These choices determine a set $\mathcal{J}(j, \eta, V_{\mathbf{z}})$ of almost-symmetric complex structures on $\mathrm{Sym}^{g+n}(\Sigma)$. Take $\mathbb{D}$ to be the infinite strip $[0,1] \times i\mathbb{R} \subset \mathbb{C}$. If j is generic, then we can choose a generic path J_s in a small neighborhood of

$J \in \mathcal{J}(j, \eta, V_{\mathbf{z}})$ such that for any pair $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, the moduli space of maps

$$\mathcal{M}_{J_s}(\phi) := \left\{ u\colon \mathbb{D} \longrightarrow \mathrm{Sym}^N(\Sigma) \,\middle|\, \begin{array}{llll} u(\{1\} \times i\mathbb{R}) & \subset \mathbb{T}_\alpha & \lim\limits_{t \to -\infty} u(s+it) & = \mathbf{x} \\ u(\{0\} \times i\mathbb{R}) & \subset \mathbb{T}_\beta & \lim\limits_{t \to \infty} u(s+it) & = \mathbf{y} \\ \frac{du}{ds} + J_s \frac{du}{dt} & = 0 & [u] & = \phi \end{array} \right\}$$

is transversely cut out. Translation in $\mathbb{D}$ induces an $\mathbb{R}$-action on $\mathcal{M}_{J_s}(\phi)$ and the unparametrized moduli space $\widehat{\mathcal{M}}_{J_s}(\phi)$ is the quotient space of this action. If $\mu(\phi) = 1$, then $\widehat{\mathcal{M}}_{J_s}(\phi)$ is a compact 0-manifold consisting of a finite number of points.

Let J_s be a generic path of almost-symmetric complex structures. Define

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}), \\ \mu(\phi)=1, \\ n_{\mathbf{z}}(\phi)=0}} \# \widehat{\mathcal{M}}_{J_s}(\phi) \cdot U_1^{n_{w_1}(\phi)} \dots U_{n+1}^{n_{w_{n+1}}(\phi)} \cdot \mathbf{y}.$$

This map preserves the Alexander grading, decreases the Maslov grading by 1, and satisfies $(\partial^-)^2 = 0$.

There are several versions of knot Floer homology we can define from the complex $\mathrm{CFK}^-(\mathcal{H})$:

(1) **Minus.** If L is a knot, the 'minus' version of knot Floer homology $\mathrm{HFK}^-(L)$ is the homology of the complex $\mathrm{CFK}^-(\mathcal{H})$: It is independent of the choice of Heegaard diagram $\mathcal{H}$ encoding L and the generic path J_s of almost-complex structures. Multiplication by U_i is a chain map on $\mathrm{CFK}^-(\mathcal{H})$ and the maps U_i and U_j are chain-homotopic for any i, j. Thus, $\mathrm{HFK}^-(L)$ is a $\mathbb{F}[U]$-module.

(2) **Collapsed minus**. When L is a link, we define the a collapsed version of the 'minus' version as follows. Given a collection of basepoints $w_{i_1}, \dots, w_{i_l}$, one on each component of the link L, define the quotient complex

$$\mathrm{CFK}^-_{\mathrm{C}}(\mathcal{H}) := \mathrm{CFK}^-(\mathcal{H}) / \langle \mathrm{U}_{\mathrm{i}_1} = \dots = \mathrm{U}_{\mathrm{i}_l} \rangle.$$

The homology of this complex $\mathrm{HFK}^-_{\mathrm{C}}(\mathrm{L})$ is well-defined, independent of the diagram $\mathcal{H}$ and the chosen basepoints, and is also an $\mathbb{F}[U]$-module.

(3) **Hat**. Choose a collection of basepoints $w_{i_1}, \dots, w_{i_l}$, one on each component of the link L, and define

$$\widehat{CFK}(\mathcal{H}) := \mathrm{CFK}^-(\mathcal{H}) / \langle U_{i_1} = \dots = U_{i_l} = 0 \rangle.$$

Let $\widehat{HFK}(L)$ denote the homology of this complex. It decomposes as the direct sum of its bigraded pieces:

$$\widehat{HFK}(L) \cong \bigoplus_{s,m\in\mathbb{Z}} \widehat{HFK}_m(L,s)$$

(4) **Tilde**. Let $\widetilde{\mathrm{CFK}}(\mathcal{H})$ be quotient complex of $\mathrm{CFK}^-(\mathcal{H})$ obtained by setting all formal U-variables equal to 0:

$$\widetilde{\mathrm{CFK}}(\mathcal{H}) := \mathrm{CFK}^-(\mathcal{H})/\langle U_1 = \cdots = U_{n+1} = 0\rangle.$$

The homology $\widetilde{HFK}$ is related to the 'hat' homology by a bigraded isomorphism

$$\widetilde{HFK} \cong \widehat{HFK} \otimes W^{\otimes(n+1-l)}$$

where W is a bigraded vector space supported in bigradings $(0,0)$ and $(-1,-1)$.

Knot Floer homology satifies a skein exact triangle that categorifies the skein relation for the Alexander polynomial.

Theorem 2.4 (skein exact triangle [17, 15, 21]). *Let $\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0$ be three links that differ at a single crossing. If the two strands of $\mathcal{L}_+$ meeting at the crossing lie in the same component, then there is an Alexander grading-preserving exact triangle*

$$\begin{array}{ccc} \widehat{HFK}_s(\mathcal{L}_0) & \xrightarrow{\hat{f}} & \widehat{HFK}_s(\mathcal{L}_+) \\ {\scriptstyle \hat{h}[1]}\nwarrow & & \swarrow{\scriptstyle \hat{g}} \\ & \widehat{HFK}_s(\mathcal{L}_-) & \end{array}$$

where $[i]$ denotes decreasing the Maslov grading by i.

If the two strands of $\mathcal{L}_+$ meeting at the crossing lie in two different components, then there is an Alexander grading-preserving exact triangle

$$\begin{array}{ccc} \widehat{HFK}_s(\mathcal{L}_0)\otimes V & \xrightarrow{\hat{f}} & \widehat{HFK}_s(\mathcal{L}_+) \\ {\scriptstyle \hat{h}[1]}\nwarrow & & \swarrow{\scriptstyle \hat{g}} \\ & \widehat{HFK}_s(\mathcal{L}_-) & \end{array}$$

where $[i]$ *denotes decreasing the Maslov grading by* i *and* V *is a bigraded vector space satisfying*

$$V_{m,s} = \begin{cases} \mathbb{F}^2 & \text{if } (m,s) = (-1,0), \\ \mathbb{F} & \text{if } (m,s) = (0,1) \text{ or } (-2,-1), \\ 0 & \text{otherwise.} \end{cases}$$

2.3. A multi-pointed Heegaard diagram for $\mathcal{L}_n$. Choose an $N+1$-bridge presentation for $\mathcal{L}_1$ so that near its distinguished crossing it has the form in the top right of Figure 2. The bridge presentation consists of $N+1$ bridges $\{a_0, \dots, a_N\}$ and $N+1$ overstrands $\{b_0, \dots, b_N\}$. Label the arcs in the bridge presentation so that a_1 and a_0 are the left and right bridges in Figure 2 and the overstrands b_1 and b_0 share endpoints with a_1 and a_0, respectively. We can also obtain a bridge presentation for $\mathcal{L}_0$, the link obtained by taking the 0-resolution of the distinguished crossing, in the top left of Figure 2. Let z_1 and z_0 denote the endpoints of the left and right bridges, respectively, in the local picture of the crossing.

From these initial bridge presentations, we can obtain a bridge presentation for $\mathcal{L}_n$. Let γ be a curve containing the points z_1 and z_0. Orient γ counter-clockwise, as the boundary of a disk containing the two points. If n is even, apply $\frac{n}{2}$ negative Dehn twists along γ to the arcs b_1 and b_0 to obtain a bridge presentation for $\mathcal{L}_n$. If n is odd, apply $\frac{n-1}{2}$ negative Dehn twists.

We can obtain a multipointed Heegaard diagram encoding $\mathcal{L}_n$ from its bridge presentation. For $i = 1, \dots, N$, let α_i be the boundary of a tubular neighborhood of the arc a_i and let β_i be the boundary of a tubular neighborhood of b_i. Label the endpoints of the bridges as z- and w-basepoints so that the oriented boundary of a_i is $w_i - z_i$. Set $\mathbf{z} = (z_0, z_1, \dots, z_N)$; $\mathbf{w} = (w_0, w_1, \dots, w_N)$; $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$; and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_N)$. Let $\mathcal{H}_n := (S^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, \mathbf{w})$ denote this multipointed Heegaard diagram. Note that locally, the diagrams $\mathcal{H}_0$ and $\mathcal{H}_1$ are identical. In addition, the diagram $\mathcal{H}_{n+2}$ can be obtained from the diagram $\mathcal{H}_n$ by applying a negative Dehn twist along γ to the multicurve $\boldsymbol{\beta}$.

Let $T_\gamma : S^2 \to S^2$ denote the positive Dehn twist along γ and let $T^*_\gamma : \mathrm{Sym}^N(S^2) \to \mathrm{Sym}^N(S^2)$ be the induced map. If $\mathcal{H}_n = (S^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}, \mathbf{w})$ is a multipointed Heegaard diagram for $\mathcal{L}_n$ then $\mathcal{H}_{n+2} = (S^2, \boldsymbol{\alpha}, (T_\gamma)^{-1}\boldsymbol{\beta}, \mathbf{z}, \mathbf{w})$ is a multipointed Heegaard diagram for $\mathcal{L}_{n+2}$. The generators $\mathfrak{G}(\mathcal{H}_n)$ of $\widetilde{\mathrm{CFK}}(\mathcal{H}_n)$ are $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and the generators $\mathfrak{G}(\mathcal{H}_{n+2})$ of $\widetilde{\mathrm{CFK}}(\mathcal{H}_{n+2})$ are $\mathbb{T}_\alpha \cap (T^*_\gamma)^{-1}\mathbb{T}_\beta$. The negative Dehn twist introduces four new intersection points between α_1 and β_1 and does not destroy any. Thus, there is a set injection from $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to $\mathbb{T}_\alpha \cap (T^*_\gamma)^{-1}\mathbb{T}_\beta$. If $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a generator, let $\mathbf{x}'$ be the corresponding generator in $\mathbb{T}_\alpha \cap (T_\gamma^*)^{-1}\mathbb{T}_\beta$. The map $(T_\gamma^*)^{-1}$ also induces a bijective map on Whitney disks from $\pi_2(\mathbf{x},\mathbf{y})$ to $\pi_2(\mathbf{x}',\mathbf{y}')$. If $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ is a Whitney disk, let ϕ' denote the corresponding Whitney disk.

For n sufficiently large, we partition the generators $\mathfrak{G}(\mathcal{H}_n)$ into three sets according to their vertex along α_1. The curves α_1, β_1 intersect twice near the basepoint z_1 and we label these points C, D as in Figure 3. The i^{th} negative Dehn twist along γ introduces four new intersection points, which we label a_i, b_i, c_i, d_i. See Figure 3. Let $\mathbf{x} = (v_1, \dots, v_N)$ be a generator where $v_i \in \alpha_i$. Define three sets:

$$\mathfrak{G}(\mathcal{H}_n)_+ := \{\mathbf{x} = (v_1, \dots, v_N) \mid v_1 = C \text{ or } D\},$$

$$\mathfrak{G}(\mathcal{H}_n)_{\text{twist}} := \{\mathbf{x} = (v_1, \dots, v_N) \mid v_1 \in \{a_i, b_i, c_i, d_i\} \text{ for some } i\},$$

$$\mathfrak{G}(\mathcal{H}_n)_- := \mathfrak{G}(\mathcal{H}_n) \setminus (\mathfrak{G}(\mathcal{H}_n)_+ \cup \mathfrak{G}(\mathcal{H}_n)_{\text{twist}}).$$

Lemma 2.5. *Choose* $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H}_n)$ *and* $\phi \in \pi_2(\mathbf{x},\mathbf{y})$. *Let* $\mathbf{x}', \mathbf{y}'$ *be the corresponding generators in* $\mathfrak{G}(\mathcal{H}_{n+2})$ *and* ϕ' *the corresponding Whitney disk in* $\pi_2(\mathbf{x}',\mathbf{y}')$.

(1) *If* $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H}_n)_- \cup \mathfrak{G}(\mathcal{H}_n)_{\text{twist}}$ *then*

$$n_z(\phi') = n_z(\phi), \quad n_w(\phi') = n_w(\phi), \quad \mu(\phi') = \mu(\phi).$$

(2) *if* $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H}_n)_+$ *then*

$$n_z(\phi') = n_z(\phi), \quad n_w(\phi') = n_w(\phi), \quad \mu(\phi') = \mu(\phi).$$

(3) *If* $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)_+$ *and* $\mathbf{y} \in \mathfrak{G}(\mathcal{H}_n)_- \cup \mathfrak{G}(\mathcal{H}_n)_{\text{twist}}$ *then*

$$n_z(\phi') = n_z(\phi) - 2, \quad n_w(\phi') = n_w(\phi), \quad \mu(\phi') = \mu(\phi) - 2.$$

Proof. Let $D(\phi)$ be the domain in Σ corresponding to ϕ. Orient γ counterclockwise, as the boundary of the disk containing the basepoints z_1, z_0. In the first two cases, the algebraic intersection of the β-components of $\partial D(\phi)$ with γ is 0. Thus, the intersection numbers n_z and n_w and the Maslov index μ are unchanged by the Dehn twist.

To prove the third case, choose some $\mathbf{x} = (C, v_2, \dots, v_N) \in \mathfrak{G}(\mathcal{H}_n)_-$ and let $\mathbf{x}_t = (c_k, v_2, \dots, v_N)$ where $k = \lfloor \frac{n}{2} \rfloor$. There is a Whitney disk $\phi \in \pi_2(\mathbf{x}_t, \mathbf{x})$ satisfying

$$n_z(\phi) = 1, \quad n_w(\phi) = 0, \quad \mu(\phi) = 1.$$

The corresponding disk $\phi' \in \pi_2(\mathbf{x}_t', \mathbf{x}')$ satisfies

$$n_z(\phi') = 3, \quad n_w(\phi') = 0, \quad \mu(\phi') = 3.$$

The final case now follows from the first two and this observation since n_z, n_w, and μ are additive under the composition

$$\pi_2(\mathbf{x}_1, \mathbf{x}_2) \times \pi_2(\mathbf{x}_2, \mathbf{x}_3) \longrightarrow \pi_2(\mathbf{x}_1, \mathbf{x}_3). \qquad \square$$

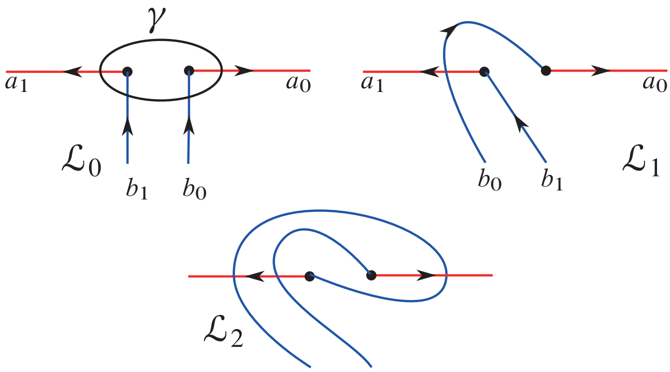

Figure 2. Local pictures for $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ in bridge position. The diagram for $\mathcal{L}_2$ can be obtained from the diagram for $\mathcal{L}_0$ by applying a negative Dehn twist along γ to the overstrands.

Lemma 2.6. *Let* $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)$. *For n sufficiently large, if* $A(\mathbf{x}) \leq 1$ *then* $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)_- \cup \mathfrak{G}(\mathcal{H}_n)_{\text{twist}}$.

Proof. Choose $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H}_n)_+$. By abuse of notation, let $\mathbf{x}, \mathbf{y}$ denote the corresponding generators in $\mathfrak{G}(\mathcal{H}_{n'})_+$ for any $n' \geq 0$. From Lemma 2.5, we can conclude that $A(\mathbf{x}) - A(\mathbf{y})$ is independent of n. Moreover, since $\mathfrak{G}(\mathcal{H}_n)_+$ is finite, there is some constant K, independent of n, such that $|A(\mathbf{x}) - A(\mathbf{y})| < K$ for all n. Similarly, there is some constant L such that if $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H}_n)_-$, then $|A(\mathbf{x}) - A(\mathbf{y})| < L$ for any n. However, if $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)_+$ and $\mathbf{y} \in \mathfrak{G}(\mathcal{H}_n)_-$, then Lemma 2.5 also implies that $A(\mathbf{x}) - A(\mathbf{y})$ grows without bound as n limits to infinity. In particular, $A(\mathbf{x}) - A(\mathbf{y}) > 0$ when n is sufficiently large.

Let $A_{\max}$ denote the maximal Alexander grading of any generator $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)$. We claim that if n is sufficiently large, then $A(\mathbf{x}) = A_{\max}$ implies $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)_+$. To prove this claim, suppose that $\mathbf{x}$ satisfies $A(\mathbf{x}) = A_{\max}$. If n is sufficiently large, then $\mathbf{x}$ must be in $\mathfrak{G}(\mathcal{H}_n)_{\text{twist}} \cup \mathfrak{G}(\mathcal{H}_n)_+$ by the above argument. Thus $\mathbf{x}$ has the form

$$\mathbf{x} = (V, v_2, \dots, v_n)$$

where $V \in \{C, D, a_1, b_1, c_1, d_1, \dots, a_k, b_k, c_k, d_k\}$ and $k = \lfloor \frac{n}{2} \rfloor$. However, if $V \neq D$ define

$$\mathbf{y} := (D, v_2, \dots, v_n).$$

There is a domain $\phi \in \pi_2(\mathbf{y}, \mathbf{x})$ with $n_z(\phi) > 0$ and $n_w(\phi) = 0$. This contradicts the assumption that $A(\mathbf{x}) = A_{\max}$. Consequently, $V = D$ and $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)_+$.

Corollary 2.3 implies that there exists some $k > 0$ such that $A_{\max} \geq n - k$ for n sufficiently large. This further implies that if $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)_+$ then $A(\mathbf{x}) > n - k - K$. Thus, for n sufficiently large, no generator $\mathbf{x}$ with $A(\mathbf{x}) \leq 1$ can live in $\mathfrak{G}(\mathcal{H}_n)_+$. □

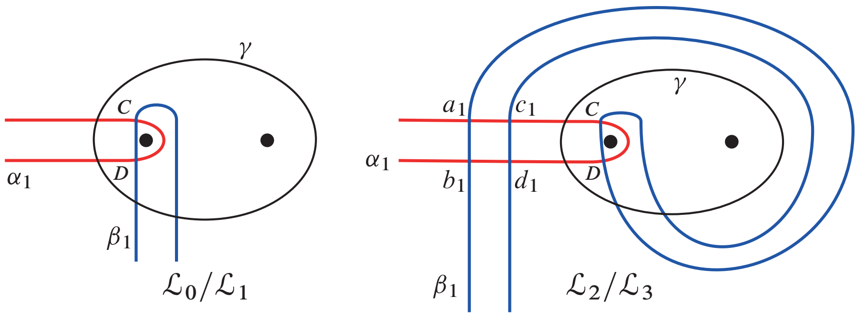

Figure 3. Local pictures of the Heegaard diagram for $\mathcal{L}_0$ or $\mathcal{L}_1$ (on left) and $\mathcal{L}_2$ or $\mathcal{L}_3$ on right. The Heegaard diagram on the right can be obtained from the left by applying a negative Dehn twist along γ to $\boldsymbol{\beta}$. Both basepoints are z-basepoints.

Define a linear map

$$\psi : \bigoplus_{s \leq 1} \widetilde{\mathrm{CFK}}_s(\mathcal{H}_n) \longrightarrow \widetilde{\mathrm{CFK}}(\mathcal{H}_{n+2})$$

by setting $\psi(\mathbf{x}) = \mathbf{x}'$ for each generator in $\mathfrak{G}(\mathcal{H}_n)$.

Lemma 2.7. *Suppose* $\mathbf{x} \in \mathfrak{G}(\mathcal{H}_n)_- \cup \mathfrak{G}(\mathcal{H}_n)_{\mathrm{twist}}$. *Then* $A(\psi\mathbf{x}) = A(\mathbf{x}) - 1$.

Proof. From Lemmas 2.5 and 2.6 we can conclude that ψ preserves relative gradings. Thus for all $\mathbf{x}$, the Alexander gradings satisfy $A(\psi\mathbf{x}) = A(\mathbf{x}) + a$ for some $a \in \mathbb{Z}$.

The Alexander grading shift follows from the computation of the Alexander polynomial in Corollary 2.2. Let s be the minimal Alexander grading in which the Euler characteristic of $\widetilde{\mathrm{CFK}}(\mathcal{H}_n)$ is nonzero. The Euler characteristics of the $\widetilde{\mathrm{CFK}}(\mathcal{H}_n)_{*,j}$ and $\widetilde{\mathrm{CFK}}(\mathcal{H}_{n+2})_{*,j+a}$ agree if $j \leq 1$. Thus $s + a$ is the minimal Alexander grading for which the Euler characteristic of $\widetilde{\mathrm{CFK}}(\mathcal{H}_{n+2})$ is nonzero. The Alexander polynomial computation implies that $s + a = s - 1$ and thus $a = -1$. □

Proposition 2.8. *For n sufficiently large, the map*

$$\psi : \bigoplus_{s \leq 1} \widetilde{\mathrm{CFK}}(\mathcal{H}_n)_s \longrightarrow \bigoplus_{s \leq 0} \widetilde{\mathrm{CFK}}(\mathcal{H}_{n+2})_s$$

is a bijection of chain complexes.

Proof. From Lemmas 2.5 and 2.6 we can conclude that ψ is an isomorphism of bigraded $\mathbb{F}$-vector spaces and that it preserves relative gradings. Thus, we just need to check that ψ is a chain map.

Fix $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H}_n)_- \cup \mathfrak{G}(\mathcal{H}_n)_{\text{twist}}$ and some $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with $n_{\mathbf{z}}(\phi) = 0$ and $\mu(\phi) = 1$. We can choose an open neighborhood W of $\overline{D(\phi)}$ to be disjoint from the curve γ in Figure 3. Thus, we can assume that the support of T_γ is disjoint from W and that the support of T_γ^* is disjoint from $\mathrm{Sym}^N(W)$. Genericity of paths of almost-complex structures is an open and dense condition. Thus we can choose some path J_s such that the moduli spaces $\mathcal{M}_{J_s}(\phi_x)$ and $\mathcal{M}_{J_s}(\phi_y)$ are transversely cut out for all choices of $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$; $\phi_x \in \pi_2(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{T}_\alpha \cap T_\gamma^* \mathbb{T}_\beta$, and $\phi_y \in \pi_2(\mathbf{y}_1, \mathbf{y}_2)$.

Choose some $u \in \mathcal{M}_{J_s}(\phi)$. The Localization Principle [22, Lemma 9.9] states that the image of u is contained in $\mathrm{Sym}^N(W) \subset \mathrm{Sym}^N(S^2)$. Since T_γ^* is the identity on $\mathrm{Sym}^N(W)$, it is clear that $u \in \mathcal{M}_{J_s}(\phi')$ as well. Conversely, all maps $u' \in \mathcal{M}_{J_s}(\phi')$ also lie in $\mathcal{M}_{J_s}(\phi)$. After quotienting by the $\mathbb{R}$-action, this implies that $\#\widehat{\mathcal{M}}_{J_s}(\phi) = \#\widehat{\mathcal{M}}_{J_s}(\phi')$. It is now clear that $\psi(\widehat{\partial}\mathbf{x}) = \widehat{\partial}\psi\mathbf{x}$ for any $\mathbf{x} \in \widetilde{\mathrm{CFK}}(\mathcal{H}_n)_{\leq 1}$. □

2.4. $\widehat{HFK}$ computations

Proposition 2.9. *There exists some $k > 0$ such that for n sufficiently large, the knot Floer homology of $\mathcal{L}_n$ satisfies*

$$\widehat{HFK}_i(\mathcal{L}_n, j) \cong \widehat{HFK}_i(\mathcal{L}_{n+2}, j+1) \qquad \text{for } j \geq -k,$$

and

$$\widehat{HFK}_i(\mathcal{L}_n, j) \cong \widehat{HFK}_i(\mathcal{L}_{n+2}, j-1)[2] \quad \text{for } j \leq k,$$

where $[i]$ denotes decreasing the homological grading by i.

Proof. Let ψ be the map from Proposition 2.8 and let ψ_* be the induced map on homology. Proposition 2.8 implies that for some $m \in \mathbb{Z}$, the map

$$\psi_* : \widetilde{HFK}_i(\mathcal{L}_n, j) \longrightarrow \widetilde{HFK}_{i+m}(\mathcal{L}_{n+2}, j-1)$$

is an isomorphism for all $i \in \mathbb{Z}$ and $j \leq 1$. If $|n|$ is much greater than the number of basepoints, then a corresponding isomorphism

$$\widehat{HFK}_i(\mathcal{L}_n, j) \cong \widehat{HFK}_{i+m}(\mathcal{L}_{n+2}, j-1)$$

holds for the 'hat' version of knot Floer homology for all $i \in \mathbb{Z}$ and $j \leq 1$. To prove the second isomorphism of the proposition, we need to show that $m = -2$.

To compute the Maslov grading shift, we apply the skein exact triangle. Fix n sufficiently large. Let s be the minimal Alexander grading in which $\widehat{HFK}(\mathcal{L}_n)$ is supported. Thus $s-1$ is the minimal Alexander grading in which $\widehat{HFK}(\mathcal{L}_{n+2})$ is supported. Applying the skein exact sequence to the triple $(\mathcal{L}_{n+2}, \mathcal{L}_n, \mathcal{L}_{n+1})$, we can see that

$$\hat{f}\colon \widehat{HFK}(\mathcal{L}_{n+1}, s-1) \longrightarrow \widehat{HFK}(\mathcal{L}_{n+2}, s-1)$$

is a graded isomorphism. Moreover, since $\widehat{HFK}(\mathcal{L}_{n+2}, j) \cong \widehat{HFK}(\mathcal{L}_n, j) \cong 0$ for $j < s-1$, exactness implies that $s-1$ is also the minimal Alexander grading in which $\widehat{HFK}(\mathcal{L}_{n+1})$ is supported.

Now apply the skein exact triangle to the triple $(\mathcal{L}_{n+1}, \mathcal{L}_{n-1}, \mathcal{L}_n)$. The modules $\widehat{HFK}(\mathcal{L}_{n+1})$ and $\widehat{HFK}(\mathcal{L}_n) \otimes V$ are supported in Alexander grading $s-1$ but $\widehat{HFK}(\mathcal{L}_{n-1}, s-1) \cong 0$. Thus

$$\hat{f}\colon (\widehat{HFK}(\mathcal{L}_n) \otimes V)_{*,s-1} \longrightarrow \widehat{HFK}_*(\mathcal{L}_{n+1}, s-1)$$

is an isomorphism. Combining the above two steps, this implies that there is a bigraded isomorphism $\widehat{HFK}(\mathcal{L}_n, s)[2] \cong \widehat{HFK}(\mathcal{L}_{n+2}, s-1)$. Thus, the grading shift must be $m = -2$. This proves the second formula.

The first statement follows from the second using the symmetry

$$\widehat{HFK}_i(\mathcal{L}_n, j) \cong \widehat{HFK}_{i-2j}(\mathcal{L}_n, -j). \qquad \square$$

Lemma 2.10. *For $|n|$ sufficiently large, the skein exact triangle for the triple $(\mathcal{L}_{n+1}, \mathcal{L}_{n-1}, \mathcal{L}_n)$ is a split short exact sequence. Consequently, either*

$$\widehat{HFK}(\mathcal{L}_n) \simeq \widehat{HFK}(\mathcal{L}_{n-1})[1] \oplus \widehat{HFK}(\mathcal{L}_{n+1}), \tag{3}$$

or

$$\widehat{HFK}(\mathcal{L}_n) \otimes V \simeq \widehat{HFK}(\mathcal{L}_{n-1})[1] \oplus \widehat{HFK}(\mathcal{L}_{n+1}), \tag{4}$$

depending on whether the two strands through the twist region of $\mathcal{L}_n$ lie in distinct components or the same component of the link.

Proof. Without loss of generality, we assume that n is chosen so that in $\mathcal{L}_n$, the two strands through the twist region lie in different components. For any $j \in \mathbb{Z}$, the triangle inequality applied to the skein exact triangle proves that

$$\text{rk } \widehat{HFK}_i(\mathcal{L}_n, j) \leq \text{rk } \widehat{HFK}_{i+1}(\mathcal{L}_{n-1}, j) + \text{rk } \widehat{HFK}_i(\mathcal{L}_{n+1}, j), \quad (5)$$

$$\text{rk } (\widehat{HFK}(\mathcal{L}_{n+1}) \otimes V)_{i-1,j} \leq \text{rk } \widehat{HFK}_i(\mathcal{L}_n, j) + \text{rk } \widehat{HFK}_{i-1}(\mathcal{L}_{n+2}, j). \quad (6)$$

Suppose that $j \leq 0$. Then applying Proposition 2.9 to $|n|$ sufficiently large, we can conclude that

$$\begin{aligned}
\text{rk } (\widehat{HFK}(\mathcal{L}_{n+1}) \otimes V)_{i-1,j} &= \text{rk } \widehat{HFK}_{i+1}(\mathcal{L}_{n+1}, j+1) && (7)\\
&\quad + 2 \cdot \text{rk } \widehat{HFK}_i(\mathcal{L}_{n+1}, j) \\
&\quad + \text{rk } \widehat{HFK}_{i-1}(\mathcal{L}_{n+1}, j-1) \\
&= \text{rk } \widehat{HFK}_{i-1}(\mathcal{L}_{n+3}, j) + \text{rk } \widehat{HFK}_i(\mathcal{L}_{n+1}, j) && (8)\\
&\quad + \text{rk } \widehat{HFK}_i(\mathcal{L}_{n+1}, j) + \text{rk } \widehat{HFK}_{i+1}(\mathcal{L}_{n-1}, j) \\
&\geq \text{rk } \widehat{HFK}_{i-1}(\mathcal{L}_{n+2}, j) + \text{rk } \widehat{HFK}_i(\mathcal{L}_n, j). && (9)
\end{aligned}$$

Equation 7 follows from the definition of V and Equation 8 can be obtained from Equation 7 using Proposition 2.9. Finally, applying Inequality 5 twice yields Inequality 9. Combining Inequalities 6 and 9 proves that

$$\text{rk } \widehat{HFK}_i(\mathcal{L}_n, j) + \text{rk } \widehat{HFK}_{i-1}(\mathcal{L}_{n+2}, j) = \text{rk } (\widehat{HFK}(\mathcal{L}_{n+1}) \otimes V)_{i-1,j}. \quad (10)$$

Furthermore, the symmetry $\text{rk } \widehat{HFK}_{i-2j}(\mathcal{L}_n, -j) = \text{rk } \widehat{HFK}_i(\mathcal{L}_n, j)$ implies that Equation 10 holds for all $j \in \mathbb{Z}$. This proves that the rank of $\hat{g}$ is 0 in the skein exact triangle for the triple $(\mathcal{L}_{n+2}, \mathcal{L}_n, \mathcal{L}_{n+1})$.

Now we consider the triple $(\mathcal{L}_{n+1}, \mathcal{L}_{n-1}, \mathcal{L}_n)$. For $j \leq 0$, we can conclude that

$$\begin{aligned}
(V \otimes \widehat{HFK}(\mathcal{L}_n))_{i,j} &\simeq \widehat{HFK}_i(\mathcal{L}_n, j+1) \oplus \widehat{HFK}_{i-1}(\mathcal{L}_n, j) && (11)\\
&\quad \oplus \widehat{HFK}_{i-1}(\mathcal{L}_n, j) \oplus \widehat{HFK}_{i-2}(\mathcal{L}_n, j-1) \\
&\simeq \widehat{HFK}_{i-2}(\mathcal{L}_{n-2}, j) \oplus \widehat{HFK}_{i-1}(\mathcal{L}_n, j) && (12)\\
&\quad \oplus \widehat{HFK}_{i-1}(\mathcal{L}_n, j) \oplus \widehat{HFK}_i(\mathcal{L}_{n+2}, j) \\
&\cong V \otimes (\widehat{HFK}(\mathcal{L}_{n-1})[1] \oplus \widehat{HFK}(\mathcal{L}_{n+1}))_{i,j}. && (13)
\end{aligned}$$

The isomorphism in Line 11 follows from the definition of V; Line 12 is obtained by applying Proposition 2.9; and the isomorphism in Line 13 follows from the isomorphism in Line 4, which has already been proven. Thus

$$V \otimes (\widehat{HFK}(\mathcal{L}_{n-1})[1] \oplus \widehat{HFK}(\mathcal{L}_{n+1})) \cong V \otimes \widehat{HFK}(\mathcal{L}_n).$$

Removing the V factors on both sides proves the statement for $j \leq 0$ and the statement for $j \geq 0$ follows from symmetry. □

Proof of Theorem 1.3. The first statement is Proposition 2.9.

To prove the second, fix some n_0 sufficiently large so that Part (1) applies. Set $k = n_0$ and

$$\widehat{F}_\circ := \bigoplus_{j=-\infty}^{-2} \widehat{HFK}(\mathcal{L}_{n_0}, j) \quad \widehat{A} := \widehat{HFK}(\mathcal{L}_{n_0}, -1)[-1],$$
$$\widehat{F}_\bullet := \bigoplus_{j=2}^{\infty} \widehat{HFK}(\mathcal{L}_{n_0}, j) \quad \widehat{B} := \widehat{HFK}(\mathcal{L}_{n_0}, 0).$$

Then the statement holds for $n = n_0$. Applying Part (1) inductively for $n > n_0$ proves Part (2).

Finally, Part (3) follows from Part (2) and Lemma 2.10. □

3. Positive mutants

Throughout this section, let $\mathcal{L}$ be an oriented link with an essential Conway sphere as in Figure 4. Let $\mathcal{L}_{k,l}$ denote the link obtained by adding k half-twists above and l half-twists below. Note that applying two flypes along the horizontal axis to the tangle $\mathcal{T}$ is an isotopy between $\mathcal{L}_{k,l}$ and $\mathcal{L}_{k+2,l-2}$. Thus, for any $k, l, i \in \mathbb{Z}$, the link $\mathcal{L}_{k+2i,l-2i}$ is isotopic to $\mathcal{L}_{k,l}$. Furthermore, $\mathcal{L}_{k+1,l-1}$ can be obtained from $\mathcal{L}_{k,l}$ by the positive mutation and then a flype.

3.1. Bigraded invariance. We prove the first part of Theorem 1.2 in this subsection and leave the second piece to the following subsection. In addition, we then use Theorem 1.2 to prove Theorem 1.1.

Theorem 3.1. *For $|n|$ sufficiently large, the mutants $\mathcal{L}_{n,0}$ and $\mathcal{L}_{n+1,-1}$ have isomorphic knot Floer homology*

$$\widehat{HFK}(\mathcal{L}_{n,0}) \simeq \widehat{HFK}(\mathcal{L}_{n+1,-1}).$$

Proof. Let $|n|$ be sufficiently large. Suppose that at the n^{th}-crossing above the mutation sphere, the two strands of $\mathcal{L}_{n,0}$ lie in different components. By Lemma 2.10, the knot Floer homology for the two mutant links is given by

$$\widehat{HFK}(\mathcal{L}_{n,0}) \simeq \widehat{HFK}(\mathcal{L}_{n-1,0})[1] \oplus \widehat{HFK}(\mathcal{L}_{n+1,0}),$$
$$\widehat{HFK}(\mathcal{L}_{n+1,-1}) \simeq \widehat{HFK}(\mathcal{L}_{n+1,-2})[1] \oplus \widehat{HFK}(\mathcal{L}_{n+1,0}).$$

The statements now follows from the fact that $\mathcal{L}_{n-1,0}$ and $\mathcal{L}_{n+1,-2}$ are isotopic.

Similarly, if at the n^{th}-crossing the two strands lie in the same component, then

$$\widehat{HFK}(\mathcal{L}_{n,0}) \otimes V \simeq \widehat{HFK}(\mathcal{L}_{n-1,0})[1] \oplus \widehat{HFK}(\mathcal{L}_{n+1,0}),$$
$$\widehat{HFK}(\mathcal{L}_{n+1,-1}) \otimes V \simeq \widehat{HFK}(\mathcal{L}_{n+1,-2})[1] \oplus \widehat{HFK}(\mathcal{L}_{n+1,0}),$$

and it is clear that $\widehat{HFK}(\mathcal{L}_{n,0})$ and $\widehat{HFK}(\mathcal{L}_{n+1,-1})$ are isomorphic. □

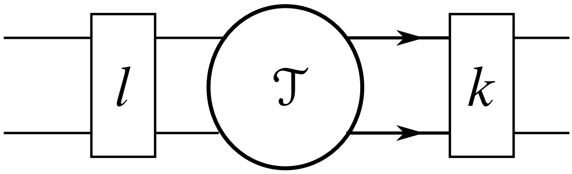

Figure 4. The link $\mathcal{L}_{k,l}$ near the Conway sphere containing the tangle $\mathcal{T}$.

We can now use the Kinoshita-Terasaka and Conway knots to prove Theorem 1.1. Let KT denote the Kinoshita-Terasaka knot ($11n42$) and let C denote the Conway knot ($11n34$). Figure 5 contains a diagram of KT. The knots KT and C are positive mutants and thus Theorem 3.1 applies. Let γ be the curve on the Conway sphere for KT that is fixed by the involution of S^2 corresponding to the positive mutation. Let $\widetilde{KT}$ and $\widetilde{C}$ denote links given by the unions of KT and C, respectively, with γ.

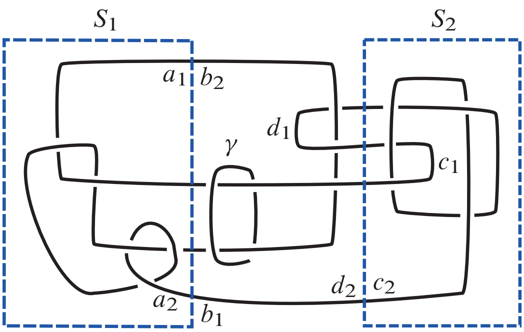

Figure 5. The Kinoshita-Terasaka knot 11n42 linked with an unknotted curve γ. The two Conway spheres are marked in dotted outline and the arcs of KT cut out by S_1 are labeled a_1, a_2, b_1, b_2.

Let S_1, S_2 denote the Conway spheres in Figure 5. Then S_1 decomposes KT into the tangles T_1, T_2 while S_2 decomposes KT into R_1, R_2. The tangle T_1 consists of two arcs a_1, a_2 with a_1 unknotted and a_2 an arc with a right-handed trefoil tied in. The tangle T_2 consists of two arcs b_1, b_2, each unknotted. Similarly, the tangle R_1 consists of two arcs c_1, c_2, with c_1 unknotted and c_2 an arc with a left-handed trefoil tied in, while R_2 consists of two unknotted arcs d_1, d_2. We can obtain C from KT by mutating the tangle T_2. Orient KT so that its are are, in order, a_1, b_1, a_2, b_2. With this orientation, after the positive mutation, the ordering on the arcs of C is a_1, b_2, a_2, b_1.

Lemma 3.2. *There exist exactly 2 essential Conway spheres for KT and C.*

Proof. Suppose there is a third essential Conway sphere S. The tangles T_1, T_2 are prime, so we can assume that either (a) S lies completely within T_1 or T_2, or (b) it intersects both T_1 and T_2 along essential 2-punctured disks.

It is useful to think of the Kinoshita-Terasaka knot as the union of five rational tangles. This is not obvious from Figure 5 but is clear in [19, Figure 1]. To describe the composition of rational tangles, let $[p]$ denotes the tangle with p positive horizontal twists, let $[-\frac{1}{q}]$ denote the tangle with q positive vertical twists; let addition refer to horizontal composition and let multiplication refers to vertical compositions. Then, for example, Figure 6 depicts the tangle T_2 as $[2] * \left(\left[\frac{1}{3}\right] + \left[-\frac{1}{2}\right]\right)$. Its complement T_1 in KT can be expressed as $\left[-\frac{1}{3}\right] + \left[\frac{1}{2}\right]$.

Rational tangles have no essential disks or spheres. Therefore the tangle T_1 contains a unique essential disk, where the rational tangles $[-\frac{1}{3}]$ and $[\frac{1}{2}]$ were joined, and no essential Conway spheres. Similarly, the tangle T_2 contains a unique essential disk where the tangles [2] and R_1 were joined and no essential spheres. However, the boundaries of these two disks are not isotopic. Specifically, the boundary of the disk in T_1 separates the endpoints of a_1 from the endpoints of b_1. The boundary of the disk in T_2 separates the endpoints of b_1 from the endpoints of b_2.

Finally, there are the same two corresponding disks in C with the same boundaries up to isotopy. □

Let KT_n be the knot obtained by performing $-\frac{1}{n}$-surgery to γ and let C_n denote the corresponding positive mutant. Diagrammatically, this surgery corresponds to applying $2n$ half-twists just outside the mutation sphere. The mutants KT_n and C_n have the same Conway spheres. In particular, each knot KT_n is obtained from the union of T_1 and T_2, however the gluing map that identifies S^2 with S^2 is modified by n Dehn twists along γ:

$$KT_n = T_1 \cup_{\phi_n} T_2 \qquad\qquad C_n = T_1 \cup_{\phi_n \circ \tau} T_2. \tag{14}$$

Lemma 3.3. *There exist exactly 2 essential Conway spheres for KT_n and C_n for all $n \in \mathbb{Z}$.*

Proof. KT_n and C_n are comprised of the same pair of tangles T_1, T_2 as KT and C. Thus, there are unique essential disks with boundaries in the mutation sphere. However, the boundaries of the disks determine different partitions of the 4 points where the knot intersects S_1. Thus, the Dehn twists along γ cannot match up the boundaries to give a closed sphere. □

To distinguish KT_n from C_n, we will adopt a strategy similar to the one in [7].[1] Let L be an ordered, oriented 2-component link and label the components x and y. When L has linking number 0, Cochran [6] defined a sequence of higher-order linking invariants β^i_x, β^i_y for $i \geq 0$. The invariants $\beta^1_x(L) = \beta^1_y(L)$ are the Sato–Levine invariant. The higher invariants are defined inductively by taking 'derivatives' of the original link L as follows. Since $lk(x, y) = 0$, we can choose a Seifert surface F for x disjoint from y and similarly a Seifert surface G for y disjoint from x. We can further assume that F and G intersect transversely along a knot K. The 'partial derivatives' of L are the links

$$D_x(L) := x \cup K \quad \text{and} \quad D_y(L) := y \cup K. \tag{15}$$

The derived links also have linking number 0 and therefore the process can be iterated. Cochran's higher order linking invariants are then defined inductively by

$$\beta^{i+1}_x(L) := \beta^i_x(D_x(L)) \quad \text{and} \quad \beta^{i+1}_y(L) := \beta^i_y(D_y(L)). \tag{16}$$

In general, these higher order invariants are manifestly non-symmetric in x and y.

In [7], Cochran and Ruberman apply the higher-order linking invariants to define invariants of tangles. Let T be a tangle consisting of an ordered pair of two disjoint arcs in B^3. Let $C(T)$ be any 2-component rational closure of T. That is, $C(T)$ is the union of T and (B^3, T_0), the tangle consisting of two boundary-parallel arcs. The difference

$$I^i(T) := \beta^i_x(C(T)) - \beta^i_y(C(T)) \tag{17}$$

is well defined and independent of the choice of rational closure $C(T)$ [7, Theorem 4.1]. Therefore it defines an invariant of the tangle T for each $i \geq 0$. Reversing the ordering of the arcs of T changes the sign of $I^i(T)$ for all i. Thus, it follows that if there is a diffeomorphism of T that exchanges the two arcs, then $I^i(T) = 0$ for all $i \geq 0$ [7, Lemma 4.3].

Lemma 3.4. *Let T_2 and R_2 be the tangles for the KT knots. Then $I^2(T_2)$ and $I^2(R_2)$ are nonzero.*

[1] I would also like to thank Chuck Livingston for providing an unpublished draft of [7].

Proof. R_2 is the mirror of T_2, so it suffices to prove the statement for T_2. Take the rational closure L of T_2 in Figure 6 and let x and y denote the components labeled in the figure. Both components are unknots and the linking number is 0. Let F and G be the Seifert surfaces of x and y, respectively, obtained by Seifert's algorithm. Then F intersects y in two points. We can remove this intersection by tubing between the intersection points along an arc of y connecting them. Let F' denote this new surface. Similarly, G intersects x in two points and by adding a tube we can choose G' disjoint from x. Then $K = F' \pitchfork G'$ is the union of 0-framed pushoffs of the arcs in x and y corresponding to the tubes. See Figure 6.

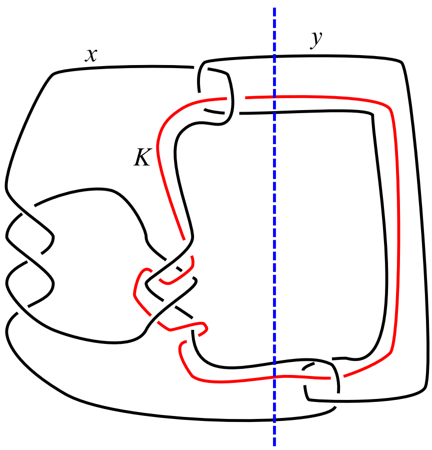

Figure 6. A rational closure L of the tangle T_2.

The link $D_x(L) = x \cup K$ is isotopic to the Whitehead link, while the link $D_y(L)$ is the unlink. Let W denote $D_x(L)$ and fix an orientation on W. Let V be obtained by changing one positive crossing of W to a negative crossing and let Z be the oriented 0-resolution of W at this crossing. The oriented resolution splits x into two components x_1, x_2. Let Z_1, Z_2 be the 2-component sublinks of Z consisting of x_1 and K or x_2 and K, respectively. Then the crossing change formula for the Sato-Levine invariant (see e.g.[12]) implies that

$$\beta^1(V) - \beta^1(W) = \mathrm{lk}(Z_1) \cdot \mathrm{lk}(Z_2).$$

The linking numbers of Z_1 and Z_2 are nonzero while V is the unlink, so $\beta^1(W) = \beta^2_x(L) \neq 0$. However, $D_y(L)$ is the unlink and so $\beta^2_Y(L) = 0$. Thus $I^2(T_2) = \beta^2_x(L) - \beta^2_y(L) \neq 0$. $\square$

Proposition 3.5. *For all n, the knots KT_n and C_n are not isotopic.*

Proof. Suppose there is an isotopy of KT_n to C_n. We can assume that this isotopy acts transitively on the set of Conway spheres. According to Lemma 3.3, there are two such spheres and so the isotopy either fixes them or swaps them.

If the isotopy swaps the Conway spheres, then the arc a_2 must be sent to one of the arcs c_1, c_2, d_1, d_2. However, the arc a_2 contains a right-handed trefoil and none of the latter four arcs do. Thus, it is impossible for an orientation-preserving diffeomorphism to send S_1 to S_2 and the isotopy must fix the Conway spheres.

Of the four arcs a_1, a_2, b_1, b_2, only a_2 is knotted so the isotopy must send a_2 to itself and therefore a_1 to itself as well. Consequently the isotopy must send the tangles T_1, T_2 to themselves. However, the isotopy now must exchange b_1 and b_2 since the mutation exchanged these arcs. This implies there is a diffeomorphism from T_2 to itself that exchanges the arcs. This is a contradiction since $I^2(T_2) \neq 0$. □

Remark 3.6. Proposition 3.5 also shows that the standard Kinoshita-Terasaka and Conway knots are not isotopic.

Proof of Theorem 1.1. Let $\widehat{KT}$ be the union of KT with γ and let $\widehat{C}$ be the union of C with γ. A computation in SnapPy [5] shows that the links $\widehat{KT}$ and $\widehat{C}$ are hyperbolic with volume ~ 23.975. Thus, by the hyperbolic Dehn surgery theorem, surgery on γ with slope $-\frac{1}{n}$ is hyperbolic for all but finitely many values of n. Consequently, for $|n|$ sufficiently large the knot KT_n is hyperbolic and thus prime. Again for $|n|$ sufficiently large, Theorem 3.1 states that there is a bigraded isomorphism

$$\widehat{HFK}(KT_n) \cong \widehat{HFK}(C_n).$$

Thus $\{(KT_n, C_n)\}$ for $|n| \gg 0$ is the required family of prime mutants. □

Remark 3.7. It is possible that $|n| = 1$ is sufficiently large for the family in Theorem 1.1. A computation with the *py_hfk* Python module [1] shows that all four of the knots $KT_1, C_1, KT_{-1}, C_{-1}$ are $\widehat{HFK}$ homologically thin. Thus, the Alexander polynomial determines the knot Floer homology and

$$\widehat{HFK}(KT_1) \cong \widehat{HFK}(C_1) \quad \text{and} \quad \widehat{HFK}(KT_{-1}) \cong \widehat{HFK}(C_{-1}).$$

3.2. Concordance invariants. The knot Floer group $\mathrm{HFK}^-(K)$ contains a well-known concordance invariant $\tau(K)$ [16]. The free part $\mathrm{HFK}^-(K)/\mathrm{Tors}$ is isomorphic to the polynomial ring $\mathbb{F}[U]$ and $-\tau(K)$ is the maximal grading of a nontorsion element. More specifically,

$$\tau(K) := -\max\{A(\mathbf{x})) \colon \mathbf{x} \text{ is not } U^k\text{-torsion for any } k > 0\}.$$

More generally, if L is a 2-component link, we can choose a pair of elements x_1, x_2 with homogeneous bigradings that generate $\mathrm{HFK}^-(L)/\mathrm{Tors} \cong \mathbb{F}[U]^2$. The *$\tau$-set* of L is the set $\tau(L) = \{-A(\mathbf{x}_1), -A(\mathbf{x}_2)\}$.

Lemma 3.8. *Let K be a knot, let L a 2-component link, and suppose K can be obtained from L by an elementary merge cobordism. Then*

$$\tau(K) \in \tau(L).$$

Proof. The statement follows easily from the following three inequalities, proven in [21, Chapter 8]. Since L and K are related by an elementary saddle move, their τ values satisfy the inequalities

$$\tau(K) - 1 \leq \tau_{\min}(L) \leq \tau(K) \quad \text{and} \quad \tau(K) \leq \tau_{\max}(L) \leq \tau(K) + 1.$$

In addition, since L has 2 components, the maximum and minimum values satisfy

$$\tau_{\max}(L) - \tau_{\min}(L) \leq 2 - 1 = 1. \qquad \square$$

Theorem 3.9. *Let K be a knot with an essential Conway sphere as in Figure 4 and let $K' = K_{1,-1}$ be its positive mutant. For all $n \in \mathbb{Z}$, the knots $K_{2n,0}$ and $K'_{2n,0} = K_{2n+1,-1}$ are positive mutants and for $|n| \gg 0$, we have*

$$\tau(K_{2n,0}) = \tau(K'_{2n,0}).$$

Proof. From Theorem 1.3, we can conclude that there exist integers a, b such that for $n \gg 0$, if $\widehat{HFK}_0(K_{2n,0}, s)$ is nontrivial, then s lies in the interval $[a+n, b+n]$. Moreover, Theorem 3.1 implies that $\widehat{HFK}_0(K'_{2n,0}, s)$ is supported in the same interval of Alexander gradings.

Set $c_n := \tau(K_{2n,0}) - n$ and consider the sequence $\{c_n\}$ for $n \gg 0$. Since $\tau(K_{2n+2,0}) - \tau(K_{2n,0}) \leq 1$, the sequence is monotonically decreasing. It is also bounded from below by a and thus $\lim_{n\to\infty} c_n = C$ exists. Define the sequence $\{c'_n\}$ and limit C' similarly for the family $\{K'_{2n,0}\}$.

Choose n_0 so that $\tau(K_{2n,0}) = C + n$ and $\tau(K'_{2n,0}) = C' + n$ for all $n \geq n_0$. Set $L_n = K_{2n+1,0}$. There are elementary merge cobordisms from L_n to $K_{2n,0}$ and $K_{2n+2,0}$ given by resolving and introducing a positive crossing, respectively. Thus $\tau(L_n) = \{C+n, C+n+1\}$. There are also elementary merge cobordisms from K_n to $K_{2n+1,1} = K'_{2n+2,0}$ and $K_{2n+1,-1} = K'_{2n}$ given by introducing a positive and negative crossing, respectively. Thus, $\tau(L_n) = \{C'+n, C'+n+1\}$. Clearly $C = C'$ and the statement for $n > 0$ follows immediately. The statement for $n < 0$ follows by an identical argument. $\square$

Proof of Theorem 1.2. The theorem is a combination of Theorems 3.1 and 3.9, along with the easy corollary that $\mathcal{L}_n$ and $\mathcal{L}'_n$ have the same genera because the Seifert genus is exactly the highest Alexander grading supporting $\widehat{HFK}$. □

The following proposition is also an easy consequence of Lemma 3.8. We do not need it to prove Theorem 3.9 but it may be interesting in its own right.

Proposition 3.10. *Let K be a knot with an essential Conway sphere as in Figure 4 and let $K' = K_{1,-1}$ be its positive mutant. Furthermore, set $L = K_{1,0} = K'_{0,1}$ and $L' = K_{0,1} = K'_{1,0}$. Then either*

$$\tau(K) = \tau(K')$$

or

$$\tau(L) = \tau(L').$$

Proof. Resolving a positive crossing gives an elementary merge cobordism from $L = K_{1,0}$ to $K_{2,0}$. Introducing a negative crossing gives an elementary merge cobordism from $L_n = K_{1,0}$ to $K_{1,-1} = K'$. There are similar elementary merge cobordisms from L' to K and K'. Thus, by Lemma 3.8,

$$\tau(K), \tau(K') \in \tau(L) \quad \text{and} \quad \tau(K), \tau(K') \in \tau(L').$$

Moreover, the sets $\tau(L)$ and $\tau(L')$ have at most 2 elements. Thus, if $\tau(K) \neq \tau(K')$ then $\tau(L) = \tau(L') = \{\tau(K), \tau(K')\}$. □

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