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A note on the Θ-invariant of 3-manifolds

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Abstract. In this note, we revisit the Θ-invariant as defined by R. Bott and the first author in [4]. The Θ-invariant is an invariant of rational homology 3-spheres with acyclic orthogonal local systems, which is a generalization of the 2-loop term of the Chern–Simons perturbation theory. The Θ-invariant can be defined when a cohomology group is vanishing. In this note, we give a slightly modified version of the Θ-invariant that we can define even if the cohomology group is not vanishing.

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1. Introduction

In 1998, R. Bott and the first author defined topological invariants of rational homology spheres with acyclic orthogonal local systems in [3] and [4]. These invariants were inspired by the Chern–Simons perturbation theory developed by M. Kontsevich in [6], S. Axelrod and M. I. Singer in [2]. The Chern–Simons perturbation theory gives invariants of 3-manifolds with flat connections of the trivial G-bundle over the 3-manifold, where G is a semi-simple Lie group. The composition of adjoint representation of G and the holonomy representation of the flat connection gives an orthogonal local system.

In [4], Bott and the first author constructed a real valued invariant, called Θ-invariant (In this note, we denote by Z_Θ the corresponding term), which is a generalization of a 2-loop term of Chern–Simons perturbation theory. The vanishing of a cohomology group (denoted by $H^*_-(\Delta; \pi_1^{-1}E \otimes \pi_2^{-1}E)$ in [4], $H^*_-(\Delta; E_\rho \boxtimes E_\rho)$ in this note) plays an important role in the construction of the Θ-invariant Z_Θ. There are few gaps in the proof of this vanishing (Lemma 1.2 of [4]). In this note, we show that a linear combination of Z_Θ and another term $Z_{O\text{-}O}$ is, however, a topological invariant of closed 3-manifolds with orthogonal acyclic local systems, when the local system is given by using a holonomy representation of a flat connection. The term $Z_{O\text{–}O}$ is also related to the 2-loop term of the Chern–Simons perturbation theory. We note that the second author proved that when $G = \mathrm{SU}(2)$, Z_Θ itself is an invariant of closed 3-manifolds with orthogonal local systems in [9].

The organization of this paper is as follows. In Section 2 we give a modified version of the Bott–Cattaneo Θ-invariant without proof. In Section 3 and Section 4 we prove a proposition and a theorem about consistency of the definition of Section 2. Both the invariant defined in Section 2 of this note and the Θ-invariant depend on the choice of a framing of the 3-manifold. In Section 5 we introduce a framing correction.

Orientation convention. In this note, all manifolds are oriented. Boundaries are oriented by the outward normal first convention. Products of oriented manifolds are oriented by the order of the factors. The interval $[0, 1] \subset \mathbb{R}$ is oriented from 0 to 1.

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2. The invariant

Let M be a closed oriented framed 3-manifold, namely a trivialization of the tangent bundle of M is fixed. We take a metric on M compatible with the framing. Let $\rho\colon \pi_1 \to G$ be a representation of the fundamental group into a semi-simple Lie group G. We denote by $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation of G, where $\mathfrak{g}$ is the Lie algebra of G. Since G is semi-simple, the Killing form of $\mathfrak{g}$ is non-degenerate. Since $\mathrm{Ad}(g)$ preserves the Killing form for any $g \in \mathfrak{g}$, the representation $\mathrm{Ad} \circ \rho$ is orthonormal with respect to the Killing form. A local system is a covariant functor from the fundamental groupoid of M to the category of finite dimensional vector spaces. Note that a representation of $\pi_1(M)$ gives a local system. We denote by E_ρ the local system given by $\mathrm{Ad} \circ \rho$. We assume that

E_ρ is acyclic, namely

$$H^*(M; E_\rho) = 0.$$

In this note, we say that such a representation ρ is *acyclic*.

2.1. A compactification of a configuration space. Let $\Delta = \{(x, x) : x \in M\} \subset M^2$ be the diagonal. We identify Δ with M by

$$\Delta \ni (x, x) \longrightarrow x \in M.$$

We orient Δ by using this identification. We denote by ν_Δ the normal bundle of Δ in M^2. We identify ν_Δ with the tangent bundle TM via the isomorphism defined by

$$TM \xrightarrow{\cong} \nu_\Delta, \quad (x, v) \longmapsto ((x, x), (-v, v))$$

where $x \in M$ and $v \in T_x M$. On the other hand, M is framed. Then TM is identified with $M \times \mathbb{R}^3$. Thus ν_Δ is identified with $M \times \mathbb{R}^3$.

Let $C_2(M) = B\ell(M^2, \Delta)$ be the compact 6-dimensional manifold with the boundary obtained by the real blowing up of M^2 along Δ. We denote by

$$q: C_2(M) \longrightarrow M^2$$

the blow-down map. As manifolds,

$$C_2(M) = (M^2 \setminus \Delta) \cup S\nu_\Delta$$

and $q(S\nu_\Delta) = \Delta$. Here $S\nu_\Delta$ is the unit sphere bundle of ν_Δ with respect to the metric on M. The manifold $C_2(M)$ is a compactification of the configuration space $M^2 \setminus \Delta$ of two distinct points. Obviously, $\partial C_2(M) = S\nu_\Delta$.

$S\nu_\Delta$ is identified with $\Delta \times S^2$. We denote by

$$p: \partial C_2(M) = \Delta \times S^2 \longrightarrow S^2$$

the projection. We use the same symbol q for the restriction map

$$q|_{\partial C_2(M)}: \partial C_2(M)(= \Delta \times S^2) \longrightarrow \Delta$$

of the blow-down map q.

2.2. The natural transformations c and Tr. The Killing form gives an isomorphism $\mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g}^* \otimes \mathfrak{g}^*$. Let $\mathbf{1} \in \mathfrak{g} \otimes \mathfrak{g}$ the element corresponding to the Killing form in $\mathfrak{g}^* \otimes \mathfrak{g}^*$. By using an orthonormal basis $e_1, \ldots, e_{\dim \mathfrak{g}} \in \mathfrak{g}$ of $\mathfrak{g}$, $\mathbf{1}$ can be described as

$$\mathbf{1} = \sum_{i=1}^{\dim \mathfrak{g}} e_i \otimes e_i.$$

$\mathbf{1} \in \mathfrak{g} \otimes \mathfrak{g}$ is invariant under the diagonal action of $\pi_1(M)$. Thus we have a natural transformation

$$c \colon \underline{\mathbb{R}} \longrightarrow E_\rho \otimes E_\rho, \quad 1 \longmapsto \mathbf{1}.$$

Here $\underline{\mathbb{R}}$ is the trivial local system, namely a local system corresponding to the 1-dimensional trivial representation of $\pi_1(M)$.

We define a natural transformation

$$\mathrm{Tr} \colon E_\rho \otimes E_\rho \otimes E_\rho \longrightarrow \underline{\mathbb{R}}$$

as follows: for $x, y, z \in \mathfrak{g}$,

$$\mathrm{Tr}(x \otimes y \otimes z) = \langle [x, y], z \rangle$$

where $\langle , \rangle$ is the Killing form and $[,]$ is the Lie bracket.

Let $\pi_1, \pi_2 \colon M^2 \to M$ be the projections defined by

$$\pi_1(x_1, x_2) = x_1, \quad \pi_2(x_1, x_2) = x_2.$$

$\pi_1^* E_\rho \otimes \pi_2^* E_\rho$ is a local system on M^2. We denote

$$E_\rho \boxtimes E_\rho = \pi_1^* E_\rho \otimes \pi_2^* E_\rho.$$

We remark that $E_\rho \boxtimes E_\rho|_\Delta = E_\rho \otimes E_\rho$. The pull-back

$$F_\rho = q^*(E_\rho \boxtimes E_\rho)$$

is a local system on $C_2(M)$. Clearly, $F_\rho|_{\partial C_2(M)} = q^*(E_\rho \otimes E_\rho)$.

2.3. The involution T on $C_2(M)$. The involution $T_0 \colon M^2 \to M^2$ defined by $T_0(x_1, x_2) = (x_2, x_1)$ induces an involution $T \colon C_2(M) \to C_2(M)$. T_0, T induce homomorphisms T_0^*, T^* on the cohomology groups $H^*(M^2, E_\rho \boxtimes E_\rho)$, $H^*(C_2(M); F_\rho)$, and $H^*(\Delta; E_\rho \otimes E_\rho)$, and on the space of differential k-forms $\Omega^k(C_2(M); F_\rho)$. We denote by $H_+^*(M^2; E_\rho \boxtimes E_\rho)$ and $H_-^*(M^2; E_\rho \boxtimes E_\rho)$ the $+1, -1$ eigenspaces of the homomorphism T_0^* respectively. We use similar notations $H_+^*(C_2(M); F_\rho)$, $H_+^*(\Delta, E_\rho \otimes E_\rho)$, $\Omega_+^k(C_2(M); F_\rho), \ldots$ in the same manner.

Let $T_{S^2} \colon S^2 \to S^2$ be the involution defined as

$$T_{S^2}(x) = -x \quad \text{for any } x \in S^2.$$

We remark that $p \circ T|_{\partial C_2(M)} = T_{S^2} \circ p \colon \partial C_2(M) \to S^2$.

2.4. The invariant. Take a 2-form $\omega_{S^2} \in \Omega^2(S^2;\mathbb{R})$ on S^2 satisfying

$$\int_{S^2} \omega_{S^2} = 1$$

and

$$T_{S^2}^* \omega_{S^2} = -\omega_{S^2}.$$

The form $p^*\omega_{S^2}$ is a closed 2-form on $\partial C_2(M)$. Thus

$$c_*(p^*\omega_{S^2}) = p^*\omega_{S^2}\mathbf{1}$$

is a closed 2-form on $\partial C_2(M)$ such that $(T|_{C_2(M)})^* p^*\omega_{S^2}\mathbf{1} = -p^*\omega_{S^2}\mathbf{1}$. The closed 2-form $p^*\omega_{S^2}\mathbf{1}$ represents a cohomology class in $H^2_-(\partial C_2(M); F_\rho|_{\partial C_2(M)})$:

$$[p^*\omega_{S^2}\mathbf{1}] \in H^2_-(\partial C_2(M); F_\rho|_{\partial C_2(M)}).$$

Proposition 2.1. *There exist closed 2-forms*

$$\omega \in \Omega^2(C_2(M); F_\rho) \quad \textit{and} \quad \xi \in \Omega^2(\Delta; E_\rho \otimes E_\rho)$$

satisfying the following conditions:

(*1*) $\omega|_{\partial C_2(M)} = p^*\omega_{S^2}\mathbf{1} + q^*\xi$,

(2) $T^*\omega = -\omega$, $(T_0|_\Delta)^*\xi = -\xi$, *namely*

$$\omega \in \Omega^2_-(C_2(M); F_\rho) \quad \textit{and} \quad \xi \in \Omega^2_-(\Delta; E_\rho \otimes E_\rho).$$

Furthermore, the cohomology class $[\xi] \in H^2_-(\Delta; E_\rho \otimes E_\rho)$ *is independent of the choice of* ξ.

This proposition is proved in Section 3.

Now, we have the following 2-forms:

$$q^*\pi_1^*\xi \in \Omega^2(C_2(M); q^*(E_\rho^{\otimes 2} \boxtimes \underline{\mathbb{R}})),$$
$$q^*\pi_2^*\xi \in \Omega^2(C_2(M); q^*(\underline{\mathbb{R}} \boxtimes E_\rho^{\otimes 2})).$$

Then we obtain closed 6-forms

$$\omega^3 \in \Omega^6(C_2(M); F_\rho^{\otimes 3}) \quad \text{and} \quad (q^*\pi_1^*\xi)(q^*\pi_2^*\xi)\omega \in \Omega^6(C_2(M); F_\rho^{\otimes 3}).$$

Since $F_\rho^{\otimes 3} = q^*(E_\rho^{\otimes 3} \boxtimes E_\rho^{\otimes 3})$, the natural transformation $\mathrm{Tr}\colon E_\rho^{\otimes 3} \to \underline{\mathbb{R}}$ induces a natural transformation

$$\mathrm{Tr}^{\boxtimes 2}\colon F_\rho^{\otimes 3} \longrightarrow (\underline{\mathbb{R}} \boxtimes \underline{\mathbb{R}} =)\underline{\mathbb{R}}.$$

Therefore we get closed 6-forms

$$\mathrm{Tr}^{\boxtimes 2}\,\omega^3, \mathrm{Tr}^{\boxtimes 2}((q^*\pi_1^*\xi)(q^*\pi_2^*\xi)\omega) \in \Omega^6(C_2(M);\mathbb{R}).$$

Definition 2.2. We set

$$Z_\Theta(\omega) = \int_{C_2(M)} \mathrm{Tr}^{\boxtimes 2}\, \omega^3,\ Z_{O\text{-}O}(\omega, \xi) = \int_{C_2(M)} \mathrm{Tr}^{\boxtimes 2}((q^*\pi_1^*\xi)(q^*\pi_2^*\xi)\omega),$$

$$Z_1(M, \rho) = Z_\Theta(\omega) - 3Z_{O\text{-}O}(\omega, \xi).$$

Theorem 2.3. *$Z_1(M, \rho)$ is an invariant of M, ρ (independent of the choices of ω and ξ). Furthermore, $Z_1(M, \rho)$ is invariant under homotopy of the framing.*

This theorem is proved in Section 4.

Remark 2.4. When we can take $\xi = 0$, obviously $Z_{O\text{-}O}(\omega, \xi) = 0$ and then $Z_1(M, \rho)$ coincides with the Θ-invariant $I_{(\Theta,\mathrm{tr},\mathrm{tr})}(M)$ of the framed 3-manifold M given in Theorem 2.5 in [4].

3. Proof of Proposition 2.1

In the following commutative diagram, the top horizontal line is a part of the long exact sequence of the pair $(C_2(M), \partial C_2(M))$ and the bottom line is that of (M^2, Δ). Thanks to the excision theorem, the right vertical homomorphism q^* is an isomorphism:

$$\begin{array}{ccc}
H^2_-(\partial C_2(M); q^*(E_\rho \otimes E_\rho)) & \xrightarrow{\delta^*_{C_2(M)}} & H^3_-(C_2(M), \partial C_2(M); F_\rho) \\
\uparrow{\scriptstyle (q|_{\partial C_2(M)})^*} & \circlearrowleft & \uparrow{\scriptstyle q^*}\ \cong \\
H^2_-(\Delta; E_\rho \otimes E_\rho) & \xrightarrow[\cong]{\delta^*_{M^2}} & H^3_-(M^2, \Delta; E_\rho \boxtimes E_\rho)
\end{array}$$

Since $H^2_-(M^2; E_\rho \boxtimes E_\rho) = H^3_-(M^2; E_\rho \boxtimes E_\rho) = 0$, the homomorphism $\delta^*_{M^2}$ on the bottom line is an isomorphism. Set

$$\Phi = (\delta^*_{M^2})^{-1} \circ (q^*)^{-1} \circ \delta^*_{C_2(M)} : H^2_-(\partial C_2(M); q^*(E_\rho \otimes E_\rho)) \longrightarrow H^2_-(\Delta; E_\rho \otimes E_\rho).$$

We take a closed 2-form $\xi \in \Omega^2_-(\Delta; E_\rho \otimes E_\rho)$ such that

$$\Phi([p^*\omega_{S^2}\mathbf{1}]) = -[\xi] \in H^2_-(\Delta; E_\rho \otimes E_\rho).$$

The above diagram implies that $\Phi(q^*[\xi]) = [\xi]$. Then $\Phi(p^*\omega_{S^2}\mathbf{1} + q^*\xi) = 0$. Thus $\delta^*_{C_2(M)}(p^*\omega_{S^2}\mathbf{1} + q^*\xi) = 0$. Therefore there exists a closed 2-form $\omega \in \Omega^2_-(C_2(M); F_\rho)$ such that

$$\omega|_{\partial C_2(M)} = p^*\omega_{S^2}\mathbf{1} + q^*\xi.$$

Conversely, if there exists a closed 2-form $\omega \in \Omega^2_-(C_2(M); F_\rho)$ such that $\omega|_{\partial C_2(M)} = p^*\omega_{S^2}\mathbf{1} + q^*\xi$, then $\Phi(\omega|_{\partial C_2(M)}) = 0$ so that $[\xi] = -\Phi([p^*\omega_{S^2}\mathbf{1}])$.

4. Proof of Theorem 2.3

The proof is reduced to the following two propositions:

Proposition 4.1. *Let $\omega, \omega' \in \Omega^2_-(C_2(M); F_\rho)$ be closed 2-forms such that*

$$\omega|_{\partial C_2(M)} = \omega'|_{\partial C_2(M)} = p^*\omega_{S^2}\mathbf{1} + q^*\xi.$$

Then $Z_\Theta(\omega) = Z_\Theta(\omega')$ and $Z_{O\text{-}O}(\omega, \xi) = Z_{O\text{-}O}(\omega', \xi)$ hold.

Proposition 4.2. *Let $\omega_{S^2,0}, \omega_{S^2,1} \in \Omega^2(S^2; \mathbb{R})$ be closed 2-forms satisfying*

$$\int_{S^2} \omega_{S^2,0} = \int_{S^2} \omega_{S^2,1} = 1,$$

$$T^*_{S^2}\omega_{S^2,0} = -\omega_{S^2,0} \quad \text{and} \quad T^*_{S^2}\omega_{S^2,1} = -\omega_{S^2,1}.$$

Let $\{p_t \colon \Delta \times S^2 \to S^2\}_{t\in[0,1]}$ be a homotopy such that $p_0 = p$ and $p_t \circ T|_{\partial C_2(M)} = T_{S^2} \circ p_t$ for $t = 0, 1$. Let $\omega_0, \omega_1 \in \Omega^2_-(C_2(M); F_\rho)$ and $\xi_0, \xi_1 \in \Omega^2_-(\Delta; E_\rho \otimes E_\rho)$ be closed 2-forms satisfying

$$\omega_0|_{\partial C_2(M)} = p_0^*\omega_{S^2,0}\mathbf{1} + q^*\xi_0, \omega_1|_{\partial C_2(M)} = p_1^*\omega_{S^2,1}\mathbf{1} + q^*\xi_1.$$

Then

$$Z_\Theta(\omega_0) - 3Z_{O\text{-}O}(\omega_0, \xi_0) = Z_\Theta(\omega_1) - 3Z_{O\text{-}O}(\omega_1, \xi_1)$$

holds.

4.1. Proof of Proposition 4.1

Lemma 4.3. *There exists a 1-form $\eta \in \Omega^1_-(M^2; E_\rho \boxtimes E_\rho)$ such that*

$$\omega - \omega' = d(q^*\eta).$$

Proof. In the following diagram, the top horizontal line is a part of the long exact sequence of the pair $(C_2(M), \partial C_2(M))$ and the bottom line is that of (M^2, Δ). The left vertical homomorphism q^* is an isomorphism because of the excision theorem:

$$\begin{array}{ccc} H^2_-(C_2(M), \partial C_2(M); F_\rho) & \longrightarrow & H^2_-(C_2(M); F_\rho) \\ q^* \uparrow \cong & \circlearrowleft & q^* \uparrow \\ H^2_-(M^2, \Delta; E_\rho \boxtimes E_\rho) & \longrightarrow & H^2_-(M^2; E_\rho \boxtimes E_\rho) \end{array}$$

The closed 2-form $\omega - \omega'$ gives a cohomology class in $H^2_-(C_2(M), \partial C_2(M); F_\rho)$ and then $((q^*)^{-1}(\omega - \omega'))|_{M^2}$ gives a cohomology class in $H^2_-(M^2; E_\rho \boxtimes E_\rho)$. Since $H^2_-(M^2; E_\rho \boxtimes E_\rho) = 0$, there exists a 1-form $\eta \in \Omega^1_-(M^2; E_\rho \boxtimes E_\rho)$ such that

$$d\eta = ((q^*)^{-1}(\omega - \omega'))|_{M^2}.$$

Thus we have $d(q^*\eta) = \omega - \omega'$. $\square$

Thanks to Lemma 4.3 and Stokes's theorem,

$$\begin{aligned}
Z_\Theta(\omega) - Z_\Theta(\omega') &= \int_{C_2(M)} \mathrm{Tr}^{\boxtimes 2}((\omega - \omega')(\omega^2 + \omega\omega' + \omega'^2)) \\
&= \int_{C_2(M)} \mathrm{Tr}^{\boxtimes 2}(d(q^*\eta)(\omega^2 + \omega\omega' + \omega'^2)) \\
&= \int_{\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}((q^*\eta)|_{\partial C_2(M)}(\omega^2 + \omega\omega' + \omega'^2)|_{\partial C_2(M)}) \\
&= 3\int_{\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}((q^*\eta)|_{\partial C_2(M)}(p^*\omega_{S^2}\mathbf{1} + q^*\xi)^2) \\
&= 6\int_{\Delta\times S^2} \mathrm{Tr}^{\boxtimes 2}(q^*(\eta|_\Delta)p^*\omega_{S^2}\mathbf{1}q^*\xi) \\
&= 6\int_{\Delta} \mathrm{Tr}^{\boxtimes 2}(\eta|_\Delta\xi\mathbf{1}).
\end{aligned}$$

To simplify the notation, we set $\bar{\eta} = \eta|_\Delta$.

Let $l\colon E_\rho \otimes E_\rho \to E_\rho$ be a natural transformation induced from the Lie bracket $[,]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. We have $l_*(\bar{\eta}) \in \Omega^1(\Delta; E_\rho)$, $l_*(\xi) \in \Omega^2(\Delta; E_\rho)$. Let $I\colon E_\rho \otimes E_\rho \to \underline{\mathbb{R}}$ be a natural transformation induced from the inner product of $\mathfrak{g}$. Then $I_*(l_*(\bar{\eta})l_*(\xi))$ is a 3-form in $\Omega^3(\Delta; \mathbb{R})$.

Lemma 4.4. $\mathrm{Tr}^{\boxtimes 2}(\bar{\eta}\xi\mathbf{1}) = \frac{1}{2}I_*(l_*(\bar{\eta})l_*(\xi))$.

Proof. Since $T_0|_\Delta = \mathrm{id}$, $\Omega^*_-(\Delta; E \otimes E)) = \Omega^*(\Delta; (E \otimes E)_-)$. Then we only need to check the claim on $\mathfrak{g}^{\otimes 3} \otimes \mathfrak{g}^{\otimes 3}$. Let $e_1, \ldots, e_{\dim \mathfrak{g}} \in \mathfrak{g}$ be an orthonormal basis of $\mathfrak{g}$. Then $\{e_i \otimes e_j - e_j \otimes e_i : i < j\}$ is a basis of $(\mathfrak{g} \otimes \mathfrak{g})^-$. It is enough to show the claim for this basis:

$$
\begin{aligned}
&\mathrm{Tr}^{\boxtimes 2}\Big((e_i\otimes e_j - e_j\otimes e_i)\otimes(e_k\otimes e_l - e_l\otimes e_k)\otimes\Big(\sum_n e_n\otimes e_n\Big)\Big)\\
&= 2(\langle [e_i,e_k],[e_j,e_l]\rangle - \langle [e_i,e_l],[e_j,e_k]\rangle)\\
&= 2(\langle e_i,[e_k,[e_j,e_l]]\rangle + \langle e_i,[e_l,[e_k,e_j]]\rangle)\\
&= 2((-\langle e_i,[e_j,[e_l,e_k]]\rangle - \langle e_i,[e_l,[e_k,e_j]]\rangle) + \langle e_i,[e_l,[e_k,e_j]]\rangle)\\
&= 2\langle e_i,[e_j,[e_k,e_l]]\rangle\\
&= 2\langle [e_i,e_j],[e_k,e_l]\rangle\\
&= \frac{1}{2}\langle 2[e_i,e_j],2[e_k,e_l]\rangle\\
&= \frac{1}{2}\langle l(e_i\otimes e_j - e_j\otimes e_i) l(e_k\otimes e_l - e_l\otimes e_k)\rangle. \qquad \square
\end{aligned}
$$

Corollary 4.5. $\int_\Delta \mathrm{Tr}^{\boxtimes 2}(\bar{\eta}\xi\mathbf{1}) = 0$.

Proof. Thanks to the above lemma,

$$\int_\Delta \mathrm{Tr}^{\boxtimes 2}(\bar{\eta}\xi\mathbf{1}) = \frac{1}{2}\int_\Delta I_*(l_*(\bar{\eta})l_*(\xi)).$$

Since E_ρ is acyclic, $[l_*(\xi)] = 0 \in H^2(\Delta; E_\rho) = 0$. Thus there exists a 1-form $\zeta \in \Omega^1(\Delta; E_\rho)$ such that $d\zeta = l_*(\xi)$. Therefore

$$
\begin{aligned}
\int_\Delta I_*(l_*(\bar{\eta})l_*(\xi)) &= \int_\Delta I_*(l_*(\bar{\eta})d\zeta)\\
&= \int_\Delta I_*(dl_*(\bar{\eta})\zeta) - \int_\Delta dI_*(l_*(\bar{\eta})\zeta).
\end{aligned}
$$

The first term of the last line is vanishing because $dl_*(\bar{\eta}) = l_*(d\eta|_\Delta)$ and $d\eta = 0$ on Δ. Thus we have

$$\int_\Delta I_*(l_*(\bar{\eta})l_*(\xi)) = -\int_\Delta dI_*(l_*(\bar{\eta})\zeta) = 0. \qquad \square$$

Thanks to the above lemma, we have

$$Z_\Theta(\omega) - Z_\Theta(\omega') = 0.$$

Similarly,

$$\begin{aligned} Z_{O\text{-}O}(\omega,\xi) - Z_{O\text{-}O}(\omega',\xi) &= \int_{C_2(M)} \mathrm{Tr}^{\boxtimes 2}((q^*\pi_1^*\xi)(q^*\pi_2^*\xi)(\omega-\omega')) \\ &= \int_{C_2(M)} \mathrm{Tr}^{\boxtimes 2}((q^*\pi_1^*\xi)(q^*\pi_2^*\xi)dq^*\eta) \\ &= \int_{\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}(q^*((\pi_1|_\Delta)^*\xi(\pi_2|_\Delta)^*\xi\bar{\eta})). \end{aligned}$$

Since $(\pi_1|_\Delta)^*\xi(\pi_2|_\Delta)^*\xi\bar{\eta}$ is a 5-form on the 3-dimensional manifold Δ, the last term is vanishing. This completes the proof of Proposition 4.1.

4.2. Proof of Proposition 4.2. Since $[\omega_{S^2,0}] = [\omega_{S^2,1}] \in H^2(S^2;\mathbb{R})$, there exists a closed 2-form $\widetilde{\omega_{S_2}} \in \Omega^2([0,1]\times S^2;\mathbb{R})$ such that $\widetilde{\omega_{S^2}}|_{\{t\}\times S^2} = \omega_{S^2,t}$ for $t = 0, 1$.

Since $[\xi_0] = [\xi_1]$(Proposition 2.1), there exists a closed 1-form

$$\tilde{\xi} \in \Omega^1([0,1]\times\Delta, \pi_\Delta^*(E_\rho\otimes E_\rho))$$

such that $\tilde{\xi}|_{\{0\}\times\Delta} = \xi_0$ and $\tilde{\xi}|_{\{1\}\times\Delta} = \xi_1$. Here $\pi_\Delta\colon [0,1]\times\Delta \to \Delta$ is the projection.

Let $\pi_{C_2(M)}\colon [0,1]\times C_2(M) \to C_2(M)$ be the projection. Let

$$\tilde{q} = \mathrm{id}_{[0,1]}\times q\colon [0,1]\times C_2(M) \longrightarrow [0,1]\times M^2$$

and we also denote the restriction map

$$\tilde{q}|_{[0,1]\times\partial C_2(M)}\colon [0,1]\times\partial C_2(M) \longrightarrow [0,1]\times\Delta$$

as $\tilde{q}$. By a similar argument as in Proposition 2.1, we can take a closed 2-form

$$\tilde{\omega} \in \Omega^2([0,1]\times C_2(M), \pi_{C_2(M)}^* F_\rho)$$

such that

$$\tilde{\omega}|_{[0,1]\times\partial C_2(M)} = \tilde{p}^*\widetilde{\omega_{S^2}}\mathbf{1} + \tilde{q}^*\tilde{\xi}.$$

Here

$$\tilde{p} = \{p_t\}_t\colon ([0,1]\times\partial C_2(M) =)[0,1]\times\Delta\times S^2 \longrightarrow S^2$$

is the homotopy between p_0 and p_1.

Thanks to Proposition 4.1, both $Z_\Theta(\omega)$ and $Z_{O\text{-}O}(\omega,\xi)$ depend only on $\omega|_{\Delta\times S^2}$ and ξ. Thus we have

$$Z_\Theta(\omega_0) = Z_\Theta(\tilde{\omega}|_{\{0\}\times C_2(M)}),$$

$$Z_\Theta(\omega_1) = Z_\Theta(\tilde{\omega}|_{\{1\}\times C_2(M)}),$$

$$Z_{O\text{-}O}(\omega_0,\xi_0) = Z_{O\text{-}O}(\tilde{\omega}|_{\{0\}\times C_2(M)},\xi_0),$$

$$Z_{O\text{-}O}(\omega_1,\xi_1) = Z_{O\text{-}O}(\tilde{\omega}|_{\{1\}\times C_2(M)},\xi_1).$$

We note that, with our orientation convention,

$$\partial([0,1]\times C_2(M)) = \{1\}\times C_2(M) - \{0\}\times C_2(M) - [0,1]\times\partial C_2(M).$$

Therefore, by using Stokes' theorem,

$$\begin{aligned}
0 &= \int_{[0,1]\times C_2(M)} d\,\mathrm{Tr}^{\boxtimes 2}\,\tilde{\omega}^3 \\
&= \int_{\{1\}\times C_2(M)} \mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}|^3_{\{1\}\times C_2(M)}) - \int_{\{0\}\times C_2(M)} \mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}|^3_{\{0\}\times C_2(M)}) \\
&\quad - \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}|^3_{[0,1]\times\partial C_2(M)}) \\
&= Z_\Theta(\tilde{\omega}|_{\{1\}\times C_2(M)}) - Z_\Theta(\tilde{\omega}|_{\{0\}\times C_2(M)}) - \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}(\tilde{p}^*\widetilde{\omega_{S^2}}\mathbf{1} + \tilde{q}^*\tilde{\xi})^3 \\
&= Z_\Theta(\omega_1) - Z_\Theta(\omega_0) - \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}(3\tilde{p}^*\widetilde{\omega_{S^2}}\mathbf{1}\tilde{q}^*\tilde{\xi}^2)
\end{aligned}$$

We denote

$$\widetilde{\pi_i} = \mathrm{id}_{[0,1]}\times\pi_i\colon [0,1]\times M^2 \longrightarrow [0,1]\times M \quad \text{for } i = 1,2.$$

We have

$$\begin{aligned}
0 &= \int_{[0,1]\times C_2(M)} d\,\mathrm{Tr}^{\boxtimes 2}((\tilde{q}^*\tilde{\pi}_1{}^*\tilde{\xi})(\tilde{q}^*\tilde{\pi}_2{}^*\tilde{\xi})\tilde{\omega}) \\
&= Z_{O\text{-}O}(\omega_1,\xi_1) - Z_{O\text{-}O}(\omega_0,\xi_0) \\
&\quad - \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}((\tilde{q}^*(\widetilde{\pi_1}|_{[0,1]\times\Delta})^*\tilde{\xi}(\widetilde{\pi_2}|_{[0,1]\times\Delta})^*\tilde{\xi})\tilde{\omega}|_{[0,1]\times\partial C_2(M)}).
\end{aligned}$$

Here,

$$\widetilde{\pi_1}|_{[0,1]\times\Delta} = \widetilde{\pi_2}|_{[0,1]\times\Delta} : [0,1]\times\Delta \longrightarrow M.$$

Thus

$$(\widetilde{\pi_1}|_{[0,1]\times\Delta})^*\tilde{\xi}(\widetilde{\pi_2}|_{[0,1]\times\Delta})^*\tilde{\xi} = \tilde{\xi}^2$$

under the identification $\Delta = M$. We have

$$Z_{\mathcal{O\!-\!O}}(\omega_1,\xi_1) - Z_{\mathcal{O\!-\!O}}(\omega_0,\xi_0) = \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}(\tilde{p}^*\widetilde{\omega_{S^2}}\mathbf{1}\tilde{q}^*\tilde{\xi}^2).$$

Then,

$$Z_\Theta(\omega_1) - Z_\Theta(\omega_0) = 3(Z_{\mathcal{O\!-\!O}}(\omega_1,\xi_1) - Z_{\mathcal{O\!-\!O}}(\omega_0,\xi_0)).$$

This completes the proof of Proposition 4.2.

5. A framing correction

In this section, we introduce a correction term for framings to give an invariant of closed 3-manifolds equipped with acyclic representations. Let M be a closed oriented 3-manifold (without framings). Recall that $\partial C_2(M)$ is identified with the unit sphere bundle STM (see Section 2.1). Take a framing $f: TM \to M\times\mathbb{R}^3$ of M. Then (M,f) is a framed 3-manifold. Let $p: (\partial C_2(M) =)STM \to S^2$ be the projection defined by the framing f. Let $\delta(f)\in\mathbb{Z}$ be the signature defect (or Hirzebruch defect, see [1] or [5] for the details) of a framing f. For the convenience of the reader, we give a short review of the construction of $\delta(f)$ in the next section. Let $\rho: \pi_1(M)\to G$ be an acyclic representation as in Section 2.1.

Theorem 5.1. *$Z_1((M,f),\rho) - \mathrm{Tr}^{\boxtimes 2}(\mathbf{1}^{\otimes 3})\delta(f)$ is a topological invariant of M, ρ.*

5.1. The signature defect $\delta(p)$. Let W be a compact 4-manifold such that $\partial W = M$ and its Euler characteristic is zero. Then there exists an $\mathbb{R}^3$ subbundle T^vW of TW satisfying $T^vW|_M = TM$. Let $ST^vW\to W$ be the unit sphere bundle of $T^vW\to W$. Thus ST^vW is a 6-dimensional manifold with $\partial ST^vW = STM$. We denote by $F_W\to ST^vW$ the tangent bundle along the fiber of the S^2 bundle $\pi: ST^vW\to W$.

Take a closed 2-form $\alpha_W\in\Omega^2(ST^vW;\mathbb{R})$ such that $\alpha_W|_{STM} = p^*\omega_{S^2}$ and $[\alpha_W] = e(F_W)/2\in H^2(ST^vW;\mathbb{R})$, where $e(F_W)$ is the Euler class of $F_W\to ST^vW$.

Lemma 5.2. *When* $\partial W = M = \emptyset$*,*

$$\int_{ST^vW} \alpha_W^3 = \frac{3}{4}\,\mathrm{Sign}\,W.$$

Here Sign W *is the signature of* W.

Proof. We give an outline of the proof. See Appendix of [8] or Proposition 2.45 of [7], for the details of the proof.

Since W is closed, $\int_{ST^vW} \alpha_W^3 = \int_{ST^vW} \left(\frac{1}{2}e(F_W)\right)^3$. We denote by $p_1(F_W) \in H^4(ST^vW;\mathbb{R})$ the first Pontrjagin class of the bundle F_W. We remark that $\underline{\mathbb{R}} \oplus F_W = \pi^*T^vW$ and $\underline{\mathbb{R}} \oplus T^vW = TW$. Here $\underline{\mathbb{R}}$ is the trivial $\mathbb{R}$ bundle over an appropriate manifold. Therefore,

$$\begin{aligned}
\int_{ST^vW} \alpha_W^3 &= \frac{1}{8}\int_{ST^vW} e(F_W)^3 \\
&= \frac{1}{8}\int_{ST^vW} e(F_W)p_1(F_W) \\
&= \frac{1}{8}\int_{ST^vW} e(F_W)\pi^*p_1(T^vW) \\
&= \frac{1}{4}\int_{W} p_1(TW) \\
&= \frac{3}{4}\,\mathrm{Sign}\,W.
\end{aligned}$$

□

Thanks to the Novikov additivity for the signature, the following corollary holds.

Corollary 5.3. *The signature defect* $\delta(f)$*, defined as*

$$\delta(f) = \int_{ST^vW} \alpha_W^3 - \frac{3}{4}\,\mathrm{Sign}\,W,$$

is independent of the choices of W *and* α_W.

5.2. Proof of Theorem 5.1. Let $f_0, f_1\colon TM \to M \times \mathbb{R}^3$ be framings and let $p_0, p_1\colon \partial C_2(M) \to S^2$ be the projections given by framings f_0, f_1 respectively. Since $[p_0^*\omega_{S^2}]$ and $[p_1^*\omega_{S^2}]$ are in $H^2_-(\Delta \times S^2;\mathbb{R}) = H^2(S^2;\mathbb{R}) = \mathbb{R}$, $[p_0^*\omega_{S^2}] = [p_1^*\omega_{S^2}]$. Thus there exists a closed 2-form

$$\tilde{\omega}_\partial \in \Omega^2_-([0,1] \times \partial C_2(M);\mathbb{R})$$

such that

$$\tilde{\omega}_\partial|_{\{0\}\times\partial C_2(M)} = p_0^*\omega_{S^2} \quad \text{and} \quad \tilde{\omega}_\partial|_{\{1\}\times\partial C_2(M)} = p_1^*\omega_{S^2}.$$

We recall that $(-\xi) \in \Omega^2_-(\Delta; E_\rho \otimes E_\rho)$ is a closed 2-form representing

$$\Phi([p^*\omega_{S^2}\mathbf{1}]) = \Phi \circ c_*([p^*\omega_{S^2}])$$

when we take a projection $p\colon \partial C_2(M) \to S^2$ given by a framing f. The homomorphism $\Phi \circ c_*$ is independent from the choice of a framing. Then we can use the same $\xi \in \Omega^2_-(\Delta; E_\rho \otimes E_\rho)$ for any framing.

By a similar argument as in proof of Proposition 2.1, we can take a closed 2-form

$$\tilde{\omega} \in \Omega^2([0,1] \times C_2(M); \pi^*_{C_2(M)}F_\rho)$$

such that

$$\tilde{\omega}|_{[0,1]\times\partial C_2(M)} = \tilde{\omega}_\partial \mathbf{1} + Q^*\xi.$$

Here, $\pi_{C_2(M)}\colon [0,1] \times C_2(M) \to C_2(M)$ and $Q\colon [0,1] \times \partial C_2(M) \to \Delta$ are the projections. We denote by

$$\omega_0 = \tilde{\omega}|_{\{0\}\times C_2(M)},$$

$$\omega_1 = \tilde{\omega}|_{\{1\}\times C_2(M)}.$$

Then,

$$Z_1((M, f_0), \rho) = Z_\Theta(\omega_0) - 3Z_{O\!-\!O}(\omega_0, \xi),$$

$$Z_1((M, f_1), \rho) = Z_\Theta(\omega_1) - 3Z_{O\!-\!O}(\omega_1, \xi).$$

Thanks to Stokes' theorem,

$$\begin{aligned}
0 &= \int_{[0,1]\times C_2(M)} d\,\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}^3) \\
&= Z_\Theta(\omega_1) - Z_\Theta(\omega_0) - \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}|^3_{[0,1]\times\partial C_2(M)}) \\
&= Z_\Theta(\omega_1) - Z_\Theta(\omega_0) - \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}^3_\partial \mathbf{1}^{\otimes 3}) - \int_{[0,1]\times\partial C_2(M)} 3\,\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}^2_\partial \mathbf{1}^{\otimes 2} Q^*\xi) \\
&= Z_\Theta(\omega_1) - Z_\Theta(\omega_0) - \int_{[0,1]\times\partial C_2(M)} \tilde{\omega}^3_\partial\, \mathrm{Tr}^{\boxtimes 2}(\mathbf{1}^{\otimes 3}) - \int_{[0,1]\times\partial C_2(M)} 3\,\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}^2_\partial \mathbf{1}^{\otimes 2} Q^*\xi) \\
&= Z_\Theta(\omega_1) - Z_\Theta(\omega_0) - \mathrm{Tr}^{\boxtimes 2}(\mathbf{1}^{\otimes 3}) \int_{[0,1]\times\partial C_2(M)} \tilde{\omega}^3_\partial - \int_{[0,1]\times\partial C_2(M)} 3\,\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}^2_\partial \mathbf{1}^{\otimes 2} Q^*\xi).
\end{aligned}$$

We denote $\bar{\pi}_i\colon [0,1] \times M^2 \to M, (t, x_1, x_2) \mapsto x_i$ for $i = 1, 2$. We have,

$$\begin{aligned}
&\int_{[0,1]\times C_2(M)} d\,\mathrm{Tr}^{\boxtimes 2}((\tilde{q}^*\bar{\pi}_1^*\xi)(\tilde{q}^*\bar{\pi}_2^*\xi)\tilde{\omega}) \\
&\quad = Z_{O\text{-}O}(\omega_1,\xi) - Z_{O\text{-}O}(\omega_0,\xi) \\
&\qquad - \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}((\tilde{q}^*(\bar{\pi}_1|_{[0,1]\times\Delta})^*\xi(\bar{\pi}_2|_{[0,1]\times\Delta})^*\xi)\tilde{\omega}_\partial \mathbf{1}) \\
&\quad = Z_{O\text{-}O}(\omega_1,\xi) - Z_{O\text{-}O}(\omega_0,\xi) - \int_{[0,1]\times\partial C_2(M)} \mathrm{Tr}^{\boxtimes 2}(Q^*\xi^2\tilde{\omega}_\partial \mathbf{1}) \\
&\quad = Z_{O\text{-}O}(\omega_1,\xi) - Z_{O\text{-}O}(\omega_0,\xi).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&Z_1((M,f_0),\rho) - Z_1((M,f_1),\rho) \\
&\quad = \mathrm{Tr}^{\boxtimes 2}(\mathbf{1}^{\otimes 3}) \int_{[0,1]\times\partial C_2(M)} \tilde{\omega}_\partial^3 \quad + \int_{[0,1]\times\partial C_2(M)} 3\,\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}_\partial^2 \mathbf{1}^{\otimes 2} Q^*\xi).
\end{aligned}$$

Lemma 5.4. $\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}_\partial \mathbf{1}^{\otimes 2} Q^*\xi) = 0$.

Proof. Let

$$T_E\colon E_\rho \otimes E_\rho \longrightarrow E_\rho \otimes E_\rho$$

be the involution induced by

$$\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}, \quad x \otimes y \longmapsto y \otimes x.$$

Clearly,

$$\mathrm{Tr}^{\boxtimes 2} \circ T_E^{\otimes 3} = \mathrm{Tr}^{\boxtimes 2}\colon E^{\otimes 3} \otimes E^{\otimes 3} \longrightarrow \underline{\mathbb{R}}.$$

Since $T_E(\mathbf{1}) = \mathbf{1}$ and $T_E^* = (T_0|_\Delta)^*$ on $\Omega^1(\Delta; E_\rho \otimes E_\rho)$,

$$\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}_\partial \mathbf{1}^{\otimes 2} Q^*\xi) = \mathrm{Tr}^{\boxtimes 2}(T_E^{\otimes 3}(\tilde{\omega}_\partial \mathbf{1}^{\otimes 2} Q^*\xi)) = -\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}_\partial \mathbf{1}^{\otimes 2} Q^*\xi).$$

Thus $\mathrm{Tr}^{\boxtimes 2}(\tilde{\omega}_\partial \mathbf{1}^{\otimes 2} Q^*\xi) = 0$. □

Lemma 5.5. *We have*

$$\delta(f_1) - \delta(f_0) = \int_{[0,1]\times\partial C_2(M)} \tilde{\omega}_\partial^3.$$

Proof. We take a compact 4-manifold W with $\partial W = M$ and its Euler characteristic is zero. Take a collar neighborhood $[0,1] \times \partial M$ of $M = \partial W$ in W such that $\{1\} \times M = \partial W$. Set

$$W_0 = W \setminus ([0,1] \times M).$$

We can take $T^v W$ as $T^v W|_{[0,1]\times M} = [0,1] \times TM$. Thus $ST^v W|_{[0,1]\times M}$ is identified with $[0,1] \times \partial C_2(M)$. Take a closed 2-form $\alpha_W \in \Omega^2(ST^v W; \mathbb{R})$ satisfying $\alpha_W|_{[0,1]\times STM} = \tilde{\omega}_\partial$ and $[\alpha_W] = \frac{1}{2} e(F_W)$. Then we have

$$\begin{aligned}
\delta(f_1) - \delta(f_0) &= \Big(\int_{ST^v W} \alpha_W^3 - \frac{3}{4} \operatorname{Sign} W \Big) - \Big(\int_{ST^v W_0} (\alpha_W|_{ST^v W_0})^3 - \frac{3}{4} \operatorname{Sign} W_0 \Big) \\
&= \int_{[0,1]\times STM} (\alpha_W|_{[0,1]\times STM})^3 \\
&= \int_{[0,1]\times \partial C_2(M)} \tilde{\omega}_\partial^3.
\end{aligned}$$

□

By the above two lemmas,

$$Z_1((M, f_0), \rho) - \operatorname{Tr}^{\boxtimes 2}(\mathbf{1}^{\otimes 3})\delta(f_0) = Z_1((M, f_1), \rho) - \operatorname{Tr}^{\boxtimes 2}(\mathbf{1}^{\otimes 3})\delta(f_1).$$

Namely, $Z_1((M, f), \rho) - \operatorname{Tr}^{\boxtimes 2}(\mathbf{1}^{\otimes 3})\delta(f)$ is independent of the choice of a framing f.

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