Non-formality in PIN(2)-monopole Floer homology

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Abstract. In previous work, we introduced a natural $\mathcal{A}_\infty$-structure on the Pin(2)-monopole Floer chain complex of a closed, oriented three-manifold Y, and showed that it is non-formal in the simplest case in which Y is the three-sphere S^3. In this paper, we explore further this non-formality phenomenon. Specifically, we provide explicit descriptions of several Massey products induced on homology, and discuss applications to the computation of the Pin(2)-monopole Floer homology of connected sums.

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Introduction

Starting with Manolescu's disproof of the longstanding Triangulation conjecture [29], the study of Pin(2)-symmetry in Seiberg–Witten theory, where

$$\mathrm{Pin}(2) = S^1 \cup j \cdot S^1 \subset \mathbb{H},$$

has spurred a lot of activity, especially in light of its applications to the study of the homology cobordism group Θ_H^3. The analogous theory of involutive Heegaard Floer homology [14] (which heuristically corresponds to a $\mathbb{Z}_4$-equivariant theory, where $\mathbb{Z}_4 = \langle j \rangle \subset \mathrm{Pin}(2)$) has also been very successful when addressing such problems. Despite all of this, still very little is known about Θ_H^3, and among the several natural questions one may ask, the following is particularly interesting.

Question 1. *Is there a torsion element in Θ_H^3 with Rokhlin invariant* 1?

The negative answer for 2-torsion elements was provided by Manolescu in [29], and is equivalent to the Triangulation conjecture being false by classic results of Galewski and Stern and Matumoto (see [28] for a nice survey). In a related fashion, the interest in Question 1 stems from the fact that a negative answer would imply the following criterion for triangulability: a closed orientable topological manifold M is triangulable if and only if its Kirby–Siebenmann invariant $\Delta(M) \in H^4(M; \mathbb{Z}/2\mathbb{Z})$ admits a lift to $H^4(M; \mathbb{Z})$. A partial negative answer to the question, when restricting the attention to connected sums of almost rational plumbed three-manifolds, was provided using involutive Heegaard Floer homology in [6]. On the other hand, as the problem involves the Rokhlin invariant, one could expect the full Pin(2)-symmetry, rather than $\mathbb{Z}_4$-symmetry, to play a central role in an approach to its answer.

With Question 1 as a motivation in mind, we study in this paper the more general problem of understanding the Pin(2)-monopole Floer homology of connected sums. The treatment of such a problem in the analogous setups of Pin(2)-equivariant Seiberg–Witten Floer homology and involutive Heegaard Floer homology can be found in [34], [15], [6], and [4]. Pin(2)-monopole Floer homology was introduced in [25] as a counterpart of Manolescu's invariants in the Morse-theoretic setting of Kronheimer–Mrowka's monopole Floer homology [17]; in particular, it can be used to provide an alternative disproof of the Triangulation conjecture. Throughout this paper we will denote by $\mathbb{F}$ the field with two elements. We will be mostly interested in the (completed) invariant $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ (pronounced *HS-to*) associated to a three-manifold equipped with a self-conjugate spin^c structure $\mathfrak{s} = \bar{\mathfrak{s}}$.

This is a graded module over the ring

$$\mathcal{R} = \mathbb{F}[[V]][Q]/Q^3,$$

where V and Q have degrees respectively -4 and -1, which is (up to grading shift) identified with $\widehat{\mathrm{HS}}_\bullet(S^3)$. It was shown in [23] that this package of invariants carries an extremely rich algebraic structure: namely, if we denote by $\hat{C}_\bullet^J(Y, \mathfrak{s})$ the chain complex underlying $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$, then $\hat{C}_\bullet^J(S^3)$ has a natural structure of $\mathcal{A}_\infty$-algebra, and $\hat{C}_\bullet^J(Y, \mathfrak{s})$ is naturally an $\mathcal{A}_\infty$-module over it. Furthermore, it was shown in [23] that the $\mathcal{A}_\infty$-algebra $\hat{C}_\bullet^J(S^3)$ is *not* formal (i.e. not quasi-isomorphic to its homology). The concept of non-formality has a very long history (see for example the celebrated results in [7] and [12]), and it has recently gained importance in understanding Floer theoretic invariants, especially those arising in symplectic geometry (see for example [1] and [20]).

The goal of the present paper is to explore these non-formality phenomena, and in particular their manifestation at the homology level as Massey products. Our interest in the study of these properties (especially towards Question 1) is that Massey products naturally appear when trying to explicitly understand the Pin(2)-monopole Floer homology of connected sums. Indeed, the main result of [23] described the Floer chain complex of a connected sum in terms of the $\mathcal{A}_\infty$-tensor product of the Floer complexes of the summands; this naturally leads to a spectral sequence, called the *Eilenberg–Moore spectral sequence*, whose E^2-page is

$$\mathrm{Tor}_{*,*}^{\mathcal{R}}(\widehat{\mathrm{HS}}_\bullet(Y_0, \mathfrak{s}_0), \widehat{\mathrm{HS}}_\bullet(Y_1, \mathfrak{s}_1))$$

and converges (up to grading shift) to $\widehat{\mathrm{HS}}_\bullet(Y_0 \# Y_1, \mathfrak{s}_0 \# \mathfrak{s}_1)$. Non-formality comes into play when studying the successive pages of this spectral sequence: both the higher differentials and the extension problems relating E^∞ to the actual group are naturally described in terms of certain Massey products of the two summands.

While the main result of [23] provides a general, yet not concretely applicable, connected sum formula, the main goal of this paper is to show that in many cases of interest the computations involving the $\mathcal{A}_\infty$-structure and the Eilenberg–Moore spectral sequence can be explicitly performed. Towards this end, our exposition will blend general results with concrete examples, and we will discuss how several results proved in the literature with different methods fit in our framework. Let us point out here two consequences of our computations. The first one involves linear independence in the homology cobordism group; while the first result of this kind was obtained in [10] using Yang-Mills theory, recently some Floer theoretic proofs have appeared [34] and [5]. We provide here a proof in our setting; the notion of *manifold of simple type M_n* appearing in the statement will be introduced

Section 3, and should be thought of as the analogue of the notion of manifold of projective type in [34]. For example, the Seifert space $-\Sigma(2, 4n - 1, 8n - 1)$ has simple type M_n.

Theorem 1. *Consider sequences of integers* $0 < n_1 < n_2 < \cdots$, *and suppose that, for each* i, Y_i *has simple type* M_{n_i}. *Then the* Y_i *are linearly independent in* Θ_H^3.

As in [34], the proof of this result only involves understanding connected sums of manifolds of simple type M_n with the given orientation. This is feasible in our setting as these manifolds have, as the name suggests, the simplest possible type of non-trivial Floer homology, and this makes the description of the Eilenberg–Moore spectral sequence feasible in this case. More challenging is the case in which we take connected sums with a manifold of simple type M_n with the opposite orientation. In this case many interesting Massey products arise, and understanding these will lead us to the following result about the Manolescu correction terms α, β, γ and the Frøyshov invariant δ (here $\delta = -h$ in the notation of [17]). This should be compared with the analogous one in involutive Heegaard Floer homology from [6].

Theorem 2. *Consider integers* A, B, C, D *such that*

- $C \le B \le A$ *and* $C \le D \le A$;

- A, B, C *have the same parity.*

Then there exists a homology sphere Y *with* $\alpha(Y) = A$, $\beta(Y) = B$, $\gamma(Y) = C$ *and* $\delta(Y) = D$.

Along the way, we will discuss how the U-action in the standard monopole Floer homology $\widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$ is related to the $\mathcal{A}_\infty$-structure on $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$. While it was shown in [35] that the $\mathcal{R}$-module structure of $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ does not recover the Frøyshov invariant δ, we obtain the following.

Theorem 3. *Let* $(Y, \mathfrak{s})$ *be a spin rational homology sphere. Then* $\delta(Y, \mathfrak{s})$ *is determined by the* $\mathcal{A}_\infty$-structure on $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$.

Let us discuss the content of the various sections. In Section 1, we begin by providing a review of the essential aspects of Pin(2)-monopole Floer homology needed in the rest of the paper. Given this background, we show in Section 2 that several natural Massey (bi)products (including for example $\langle \mathbf{x}, Q, Q^2 \rangle$, when

$Q \cdot \mathbf{x} = 0$ and $\langle V, \mathbf{x}, Q \rangle$, when $V \cdot \mathbf{x} = Q \cdot \mathbf{x} = 0$) can be described in terms of the Gysin exact triangle

$$\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s}) \xrightarrow{\;\;\cdot Q\;\;} \widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$$

$$\pi_* \searrow \qquad \swarrow \iota$$

$$\widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$$

relating $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ with the usual monopole Floer homology $\widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$. This will lead us to a proof of Theorem 3. The Gysin exact triangle can be explicitly understood in several cases including Seifert spaces and spaces obtained by surgery on L-space knots, as discussed in Section 3. In that section, we also introduce manifolds of simple type M_n, and discuss concrete examples. Given this, we turn our attention onto the study of the Eilenberg–Moore spectral sequence. In Section 4, we cover the relevant background in homological algebra over our ring $\mathcal{R}$ needed to describe concretely the E^2-page of the spectral sequence. This leads up to Section 5, where we study in detail the higher differentials and extension problems for connected sums with manifolds of simple type M_n. We will see how the Massey products described in Section 2 naturally arise when trying to understand connected sums with this kind of spaces. Finally in Section 6 we discuss examples involving connected sums of several manifolds of simple type M_n with either orientation, and use them to show Theorems 1 and 2.

1. A quick review of Pin(2)-monopole Floer homology

In this section, we briefly review the fundamental aspects of Pin(2)-monopole Floer homology which will be needed in the paper, with a particular focus on the results of [23]. We refer the reader to [21] for a more detailed introduction to the subject, and to [25] for the details of the construction.

Formal properties.   To a closed, oriented three-manifold Y equipped with a self-conjugate spin^c structure $\mathfrak{s}$ we associated in [25] chain complexes

$$\check{C}_\bullet(Y, \mathfrak{s}), \quad \hat{C}_\bullet(Y, \mathfrak{s}), \quad \bar{C}_\bullet(Y, \mathfrak{s}) \tag{1}$$

equipped with a chain involution $\jmath$. The homology of the chain complexes recovers the monopole Floer homology groups

$$\widetilde{\mathrm{HM}}_\bullet(Y, \mathfrak{s}), \quad \widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s}), \quad \overline{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$$

of [17]. On the other hand, looking at the homology of the $\jmath$-invariant subcomplexes

$$\check{C}_\bullet^\jmath(Y, \mathfrak{s}), \quad \hat{C}_\bullet^\jmath(Y, \mathfrak{s}), \quad \bar{C}_\bullet^\jmath(Y, \mathfrak{s}),$$

one obtains the Pin(2)-monopole Floer homology groups fitting in a long exact sequence

$$\widetilde{\mathrm{HS}}_\bullet(Y, \mathfrak{s}) \xrightarrow{\quad j_* \quad} \widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$$
$$\overset{i_*}{\nwarrow} \qquad \overset{p_*}{\swarrow} \tag{2}$$
$$\overline{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$$

where the maps i_* and j_* preserve the grading, while p_* has degree -1. These are $\mathbb{Q}$-graded modules over $\mathcal{R}$, where the action can be described as follows: after identifying

$$\widehat{\mathrm{HS}}_\bullet(S^3) = \mathcal{R}\langle -1 \rangle,$$

the action is induced in homology by the multiplication map

$$\hat{m}_2 \colon \hat{C}_\bullet^\jmath(Y) \otimes \hat{C}_\bullet^\jmath(S^3) \longrightarrow \hat{C}_\bullet^\jmath(Y)$$

arising from the cobordism obtained by $([0,1] \times Y) \setminus \mathrm{int}(B^4)$ by attaching cylindrical ends. In [23], we introduced higher multiplications

$$\hat{m}_n \colon \hat{C}_\bullet^\jmath(Y) \otimes \hat{C}_\bullet^\jmath(S^3)^{\otimes n-1} \longrightarrow \hat{C}_\bullet^\jmath(Y)$$

obtained (in the spirit of Baldwin and Bloom's unpublished construction of a monopole category) by looking at an $(n-2)$-dimensional family of metrics and perturbations parameterized by the associahedron K_n. It is shown in [23] that in the simplest case in which Y is S^3, these operations (which we denote μ_n) define an $\mathcal{A}_\infty$-algebra structure on $\hat{C}_\bullet^\jmath(S^3)$, and for each Y the operations $\hat{m}_n$ on $\hat{C}_\bullet^\jmath(Y)$ define an $\mathcal{A}_\infty$-module structure over it (see [23] for the relevant background on $\mathcal{A}_\infty$-structures). For a fixed choice of data on S^3, such an $\mathcal{A}_\infty$-module structure on

$\hat{C}^J_\bullet(Y)$ is well defined up to $\mathcal{A}_\infty$-quasi-isomorphism. Indeed, it is shown in [23] that $\hat{C}^J_\bullet(Y)$ also admits $\mathcal{A}_\infty$-bimodule operations

$$\hat{m}_{i,j}: \hat{C}^J_\bullet(S^3)^{\otimes i-1} \otimes \hat{C}^J_\bullet(Y) \otimes \hat{C}^J_\bullet(S^3)^{\otimes j-1} \longrightarrow \hat{C}^J_\bullet(Y).$$

such that $\hat{m}_{1,n} = \hat{m}_n$. These will be relevant in the present paper when computing the $\mathcal{R}$-module structure on connected sums.

Remark 1.1. There are some technical subtleties involved in the construction of [23], as one needs to impose certain transversality conditions on the chains involved. In particular, the higher composition maps are only partially defined. On the other hand, for the content of this paper (which is mostly algebraic in nature), it will not be harmful to treat the structure constructed in [23] as genuine $\mathcal{A}_\infty$-structures.

Formality and connected sums. Recall that an $\mathcal{A}_\infty$-algebra $\mathcal{A}$ is called *formal* if it is quasi-isomorphic to its homology (see for example [12] for the special case of dgas). A classical obstruction to formality is provided by Massey products: given the homology classes $[a], [b]$ and $[c]$ in $H_*(\mathcal{A})$ such that $[a] \cdot [b] = [b] \cdot [c] = 0$, after choosing r, s such that $\partial r = ab$ and $\partial s = bc$, we define their *triple Massey product* to be the homology class

$$\langle [a], [b], [c] \rangle = [rc + as + \mu_3(a, b, c)].$$

The product $\langle [a], [b], [c] \rangle$ is well defined in a suitable quotient of $H_*(\mathcal{A})$. Inductively, one can define the n-fold Massey products for n-tuples of homology classes such that all lower Massey products vanish in a consistent way. In the present paper, we will mostly focus on triple and four-fold Massey products. Let us review the definition of the latter, as it will be relevant in the sequel. Suppose we are given homology classes $[a_i]$ in $H_*(\mathcal{A})$ for $i = 1, \ldots, 4$ such that

$$[a_1] \cdot [a_2] = [a_2] \cdot [a_3] = [a_3] \cdot [a_4] = 0.$$

Choose b_i for $i = 1, 3$ such that $\partial b_i = a_i a_{i+1}$. Suppose that the triple Massey products (defined in terms of these choices of b_i) vanish, so that we have c_1, c_2 such that

$$\partial c_1 = b_1 a_3 + a_1 b_2 + \mu_3(a_1, a_2, a_3), \quad \partial c_2 = b_2 a_4 + a_2 b_3 + \mu_3(a_2, a_3, a_4).$$

The four-fold Massey product $\langle [a_1], [a_2], [a_3], [a_4] \rangle$ is defined to be

$$[a_1 c_2 + c_1 a_4 + \mu_3(a_1, a_2, b_3) + \mu_3(a_1, b_2, a_4) + \mu_3(b_1, a_3, a_4) + \mu_4(a_1, a_2, a_3, a_4)].$$
(3)

Again, this is well defined in a suitable quotient of $H_*(\mathcal{A})$. The analogous definitions carry over when defining the Massey products for an $\mathcal{A}_\infty$-(bi)module $\mathcal{M}$ over $\mathcal{A}$.

It was shown in [23] the $\mathcal{A}_\infty$-structure on $\hat{C}_\bullet^J(S^3)$ is *not* formal: while the relevant triple Massey products are zero, we have

$$\langle Q, Q^2, Q, Q^2 \rangle = V.$$

Intuitively speaking, this is a cohomological manifestation of the non-triviality of the fiber bundle

$$\mathbb{R}P^2 \hookrightarrow B\operatorname{Pin}(2) \longrightarrow \mathbb{H}P^\infty.$$

The goal of this paper is to explore the non-formality properties of the $\mathcal{A}_\infty$-module $\hat{C}_\bullet^J(Y)$. This is particularly interesting in light of the main theorem of [23], which we now recall.

Theorem 1.2. *There exists a quasi-isomorphism of $\mathcal{A}_\infty$-bimodules*

$$\hat{C}_\bullet^J(Y_0, \mathfrak{s}_0) \widetilde{\otimes}_{\hat{C}_\bullet(S^3)} (\hat{C}_\bullet^J(Y_1, \mathfrak{s}_1))^{\mathrm{opp}} \cong \hat{C}_\bullet^J(Y_0 \# Y_1, \mathfrak{s}_0 \# \mathfrak{s}_1)\langle -1 \rangle.$$

where opp *denotes the opposite bimodule.*

Here by $\langle n \rangle$ we denote grading shift *downwards* by n, i.e.

$$(M\langle n \rangle)_d = M_{d-n},$$

while $\widetilde{\otimes}$ denotes the $\mathcal{A}_\infty$-tensor product, whose definition we now recall. Let $\mathcal{N}$ and $\mathcal{M}$ be (respectively a right and left) $\mathcal{A}_\infty$-modules over $\mathcal{A}$, their $\mathcal{A}_\infty$-tensor product is defined to be the vector space

$$\mathcal{N}\widetilde{\otimes}\mathcal{M} = \bigoplus_{n \geq 0} N \otimes A^n \otimes M$$

equipped with the differential

$$\begin{aligned}
\partial(\mathbf{x}|a_1|\cdots|a_n|\mathbf{y}) &= \sum_{i=0}^n m_{i+1}(\mathbf{x}|a_1|\cdots|a_i)|a_{i+1}|\cdots|a_n|\mathbf{y} \\
&+ \sum_{i=1}^n \sum_{j=0}^{n-i} \mathbf{x}|a_1|\cdots|a_{i-1}|\mu_{j-i+1}(a_i|\cdots|a_j)|a_{j+1}|\cdots|a_n|\mathbf{y} \\
&+ \sum_{i=1}^n \mathbf{x}|a_1|\cdots|a_{i-1}|m_{n-i-1}(a_i|\cdots|a_n|\mathbf{y}).
\end{aligned}$$

Here M, A and N denote the underlying $\mathbb{F}$-vector spaces of $\mathcal{M}, \mathcal{A}$ and $\mathcal{N}$ and, for simplicity, we will always denote elements of tensor products with bars | instead of $\otimes$s. By considering the natural filtration given by

$$F_k = \bigoplus_{n \leq k} N \otimes A^n \otimes M,$$

we obtain the following.

Corollary 1.3. *There is a spectral sequence whose E^2-page is*

$$\mathrm{Tor}^{\mathcal{R}}_{*,*}(\widehat{\mathrm{HS}}_{\bullet}(Y_0, \mathfrak{s}_0), \widehat{\mathrm{HS}}_{\bullet}(Y_1, \mathfrak{s}_1))$$

and which converges to $\widehat{\mathrm{HS}}_{\bullet}(Y_0 \# Y_1, \mathfrak{s}_0 \# \mathfrak{s}_1)\langle -1 \rangle$.

We will refer to this as the *Eilenberg–Moore spectral sequence*, see [23] for its heuristic motivation. Here $\mathrm{Tor}^{\mathcal{R}}_{*,*}$ is taken in the category of graded $\mathcal{R}$-modules, and is therefore a bigraded object. It can be computed by taking a graded projective resolution of M, tensoring it with N and taking the homology of the resulting complex. Recall, as a general fact, that given modules M_0, M_1 over $\mathcal{R}$, the identity

$$\mathrm{Tor}^{\mathcal{R}}_{0,*}(M_0, M_1) = M_0 \otimes_{\mathcal{R}} M_1 \tag{4}$$

holds. Corollary 1.3 follows from the fact that the E^1-page of the spectral sequence associated to the filtration $\{F_N\}$ on $\mathcal{N} \widetilde{\otimes} \mathcal{M}$ is naturally identified with the tensor product of $H_*(\mathcal{M})$ with the bar resolution of $H_*(\mathcal{N})$; here the key point is that $\widehat{C}^J_{\bullet}(S^3)$ is cohomologically unital. Of course, $\mathrm{Tor}^{\mathcal{R}}_{*,*}$ is independent of the choice of resolution; in Section 4, we will discuss some convenient resolutions to compute $\mathrm{Tor}^{\mathcal{R}}_{*,*}$ efficiently for our purposes.

While the computation of the E^2-page only depends on the module structure, the key observation is that the higher differentials in the Eilenberg–Moore spectral sequence are determined by the Massey products of the two summands. We have, for example, the following consequence of the standard staircase argument (see for example Section 8 of [23]).

Lemma 1.4. *Suppose we are given* $\mathbf{x} \in H_*(\mathcal{M})$, $r_1, \ldots, r_n \in H_*(\mathcal{A})$ *and* $\mathbf{y} \in H_*(\mathcal{N})$ *such that*

$$\mathbf{x}r_1 = r_1 r_2 = \cdots = r_{n-1} r_n = r_n \mathbf{y} = 0,$$

so that $\mathbf{x}|r_1|\ldots|r_n|\mathbf{y}$ *defines a class in* $(E^2_{n,*}, d_2)$. *Then,*

$$d_2(\mathbf{x}|r_1|\ldots|r_n|\mathbf{y}) = \langle \mathbf{x}, r_1, r_2 \rangle |r_3|\ldots|r_n|\mathbf{y} + \mathbf{x}|\langle r_1, r_2, r_3 \rangle|r_4|\ldots|r_n|\mathbf{y}$$
$$+ \mathbf{x}|r_1|\ldots|\langle r_{n-2}, r_{n-1}, r_n \rangle|\mathbf{y} + \mathbf{x}|r_1|\ldots|\langle r_{n-1}, r_n, \mathbf{y} \rangle$$

as an element of $E^2_{n-2,*+1}$.

In general, there are classes in E^2 that cannot be described as a simple tensor. In Section 5, we will discuss the differentials of some of these more complicated classes in terms of certain generalized Massey products.

Manolescu correction terms.   From the $\mathcal{R}$-module structure of Pin(2)-monopole Floer homology, taking as inspiration Frøyshov's invariant [9][17], one can extract plenty of information regarding cobordisms between manifolds. For simplicity, let $(Y, \mathfrak{s})$ be a rational homology sphere Y equipped with a self-conjugate spin^c structure $\mathfrak{s}$ or, equivalently, a spin structure (as $b_1 = 0$). We can fix an identification, up to grading shift, of graded $\mathcal{R}$-modules

$$\overline{\mathrm{HS}}_\bullet(Y, \mathfrak{s}) \equiv \widetilde{\mathcal{R}}$$

where we set

$$\widetilde{\mathcal{R}} = \mathbb{F}[V^{-1}, V]][Q]/(Q^3),$$

where $\mathbb{F}[V^{-1}, V]]$ (which we denote by $\mathcal{V}$) denotes Laurent power series. We have the direct sum of $\mathbb{F}[[V]]$-modules

$$\widetilde{\mathcal{R}} = \mathcal{V} \oplus Q \cdot \mathcal{V} \oplus Q^2 \cdot \mathcal{V}.$$

Recall that $\widetilde{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ and $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ vanish in degrees respectively low and high enough, so that

$$i_*\colon \overline{\mathrm{HS}}_\bullet(Y, \mathfrak{s}) \equiv \widetilde{\mathcal{R}} \to \widetilde{\mathrm{HS}}_\bullet(Y, \mathfrak{s}),$$

$$p_*\colon \widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s}) \longrightarrow \widetilde{\mathcal{R}} \equiv \overline{\mathrm{HS}}_\bullet(Y, \mathfrak{s}).$$

are isomorphism is degrees respectively high and low enough.

Definition 1.5. Given a nonzero $r \in \widetilde{\mathcal{R}}$, we say that $\mathbf{x} \in \widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ is *based of type r* if $p_*(\mathbf{x}) = r$. If $p_*(\mathbf{x}) = 0$, we say that $\mathbf{x}$ is *unbased*. We will say that $\mathbf{x}$ is $\mathcal{V}$, $Q \cdot \mathcal{V}$ or $Q^2 \cdot \mathcal{V}$-based according to where $p_*(\mathbf{x})$ belongs to.

We also call the images of $\mathcal{V}$, $Q \cdot \mathcal{V}$ and $Q^2 \cdot \mathcal{V}$ in $\widetilde{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ under i_* respectively the α, β and γ-tower. The Manolescu correction terms (defined first in the setting on Pin(2)-equivariant Seiberg–Witten Floer homology [29]) are the numerical invariants defined as

$$\alpha = \frac{1}{2} \min\{\deg(\mathbf{x}) \mid \mathbf{x} \in \alpha\text{-tower}\},$$

$$\beta = \frac{1}{2}(\min\{\deg(\mathbf{x}) \mid \mathbf{x} \in \beta\text{-tower}\} - 1),$$

$$\gamma = \frac{1}{2}(\min\{\deg(\mathbf{x}) \mid \mathbf{x} \in \gamma\text{-tower}\} - 2).$$

Using the long exact sequence relating the three Floer groups, these numerical invariants can also be described in terms of based elements of $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ as follows

$$\alpha = -\frac{1}{2}(\max\{\deg(\mathbf{x}) \mid \text{there exists a } Q^2 \cdot \mathcal{V}\text{-based element } \mathbf{x}\} + 4),$$

$$\beta = -\frac{1}{2}(\max\{\deg(\mathbf{x}) \mid \text{there exists a } Q \cdot \mathcal{V}\text{-based element } \mathbf{x}\} + 3),$$

$$\gamma = -\frac{1}{2}(\max\{\deg(\mathbf{x}) \mid \text{there exists a } \mathcal{V}\text{-based element } \mathbf{x}\} + 2).$$

These invariants are rational lifts of $-\mu(Y, \mathfrak{s})$, where μ denotes the Rokhlin invariant, and they provide obstructions to the existence of spin cobordisms with $b_2^+ = 0, 1, 2$, see [25][24]. As a consequence, they are invariant under homology cobordism. These corresponding numerical invariants for $-Y$ (the manifold obtained from Y by orientation reversal) can be obtained as follows:

$$\alpha(-Y, \mathfrak{s}) = -\gamma(Y, \mathfrak{s}),$$
$$\beta(-Y, \mathfrak{s}) = -\beta(Y, \mathfrak{s}),$$
$$\gamma(-Y, \mathfrak{s}) = -\alpha(Y, \mathfrak{s}).$$

The key point behind these identities is Poincaré duality, which is the isomorphism of $\mathcal{R}$-modules

$$\widetilde{\mathrm{HS}}^\bullet(-Y, \mathfrak{s}) \cong \widehat{\mathrm{HS}}_{-1-\bullet}(Y, \mathfrak{s}), \tag{5}$$

together with the fact that $\widetilde{\mathrm{HS}}^\bullet(-Y, \mathfrak{s})$ is the dual $\mathcal{R}$-module of $\widetilde{\mathrm{HS}}_\bullet(-Y, \mathfrak{s})$. In fact, (5) holds at the level of $\mathcal{A}_\infty$-modules. Here the $\mathcal{A}_\infty$-structure on $\widetilde{\mathrm{HS}}^\bullet(-Y, \mathfrak{s})$ has cohomological grading, consistent with the fact that the cohomological action of Q and V have degrees respectively 1 and 4. Furthermore, $\widetilde{\mathrm{HS}}^\bullet(-Y, \mathfrak{s})$ is the dual $\mathcal{A}_\infty$-module of $\widetilde{\mathrm{HS}}_\bullet(-Y, \mathfrak{s})$. While the latter is a general notion of duality, we will not discuss it in detail here as in all our examples it will admit a much more concrete and computable manifestation (see for example the proof of Lemma 2.7).

Remark 1.6. More generally, using these duality relations together with the long exact sequence (2), we can extract information about $\widehat{\mathrm{HS}}_\bullet(-Y, \mathfrak{s})$ from $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$. On the other hand, unlike the case of usual monopole Floer homology, the $\mathcal{R}$-module structure of $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ does not determine the $\mathcal{R}$-module structure of $\widehat{\mathrm{HS}}_\bullet(-Y, \mathfrak{s})$. In fact, we will see in Lemma 2.7 and in the proof of Corollary 5.4 that certain non-trivial multiplications between elements of $\mathcal{R}$ and $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$ correspond to the existence of non-trivial *Massey products* on $\widehat{\mathrm{HS}}_\bullet(-Y, \mathfrak{s})$.

Remark 1.7. Of course, one can define analogues of the correction terms also in cases in which $b_1 > 0$, depending on the structure of $\overline{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$. In fact, one can define a correction term for each $\mathbb{F}[V^{-1}, V]]$ summand of $\overline{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$. It is shown in [26] that the number of such summands only depends on the triple cup product of Y, together with the Rokhlin invariants of the $2^{b_1(Y)}$ spin structures inducing $\mathfrak{s}$. For example, there are two cases when $b_1(Y) = 1$: when the two spin structures have the same Rokhlin invariant, one obtains six correction terms, while in the case the two spin structures have different Rokhlin invariants one obtains four correction terms (see [24] and also Section 3).

Definition of the Floer chain complexes.   Let us now review the main features of the Floer chain complexes introduced in [25] that will be needed in the rest of the paper. The key input of Pin(2)-symmetry is a natural involution on the moduli space of configurations

$$\jmath \colon \mathcal{B}(Y, \mathfrak{s}) \longrightarrow \mathcal{B}(Y, \mathfrak{s}),$$

whose fixed points are the reducible configurations $[B, 0]$ where B is the spin connection of one of the $2^{b_1(Y)}$ spin structures inducing $\mathfrak{s}$. One would like to perform the construction of the Floer chain complexes from [17] in a way such that this symmetry is preserved. The main complication is that one needs to work with Morse–Bott singularities. For a generic $\jmath$-equivariant perturbation, the critical set in the blown-up moduli space of configurations $\mathcal{B}^\sigma(Y, \mathfrak{s})$ consists of

- a finite number of pairs of irreducible solutions, acted freely by the action of $\jmath$;

- for each non spin reducible critical point, an infinite tower of critical points as in the Morse setting (the free action of $\jmath$ of non-spin reducible critical points lifts to an action of the towers);

- for each spin reducible critical point, an infinite tower of reducible submanifolds, each a copy of S^2 (the involution $\jmath$ acts as the antipodal map on each critical submanifold).

The chain complexes with involution (1) arise as some version of Morse–Bott chain complexes. The underlying vector spaces are generated over $\mathbb{F}$ by suitable geometric chains with values in the critical submanifolds, i.e. smooth maps

$$f \colon \sigma \longrightarrow C$$

where σ is chosen among a suitable generalization of manifolds with boundary

and C is a critical submanifold. The differential of such a chain σ combines the singular boundary within C together with fibered products with moduli spaces of flows $M(C, C')$ from C to another critical submanifold C', which we consider as singular chains with values in C'. In our case, we are naturally lead to deal with δ-chains and the key modification (inspired from [27]) is that we consider chains which are non-degenerate, namely both $f(\sigma)$ and $f(\partial\sigma)$ are not contained in the image of smaller dimensional chains. For our purposes, we will only need that 3-cycles in the critical submanifolds which are copies of S^2 are zero at the chain level.

Example 1.8. Consider the classes $Q, Q^2 \in \mathcal{R} = \widehat{\mathrm{HS}}_\bullet(S^3)$. These are represented respectively by generator in the one and zero dimensional homology of C_{-1}, the first unstable critical submanifold (where we consider the round metric on S^3, and a small perturbation). Of course, we know $Q \cdot Q^2 = 0$. In fact, such a product is zero at the chain level: for dimensional reasons, it is a 3-chain in the second unstable critical submanifold C_{-2}, and because it is closed, it vanishes. For a similar reason, the triple Massey product $\langle Q, Q^2, Q \rangle$ also vanishes at the chain level.

2. Description of certain Massey products

In general, the determination of the Massey products of an $\mathcal{A}_\infty$-module over an $\mathcal{A}_\infty$-algebra is a rather involved process, as it requires the understanding of higher compositions. Our goal in the present section is to show that in the case of Pin(2)-monopole Floer homology, many natural Massey (bi)products can be described very explicitly in terms of the relation with the U-action in usual monopole Floer homology. While we will work in the setting of $\widehat{\mathrm{HS}}_\bullet$, all results carry over for $\widetilde{\mathrm{HS}}_\bullet$ and $\overline{\mathrm{HS}}_\bullet$. Before stating the main results of the section, let us recall the Gysin exact sequence

$$\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s}) \xrightarrow{\cdot Q} \widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$$
$$\overset{\pi_*}{\swarrow} \qquad \overset{\iota_*}{\swarrow}$$
$$\widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$$

introduced in [25]. Here the maps ι_* and π_* preserve the degree, while multiplication by Q has degree -1. It is an exact triangle of $\mathcal{R}$-modules where on $\widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$

we have that Q acts as 0 and V acts as U^2. In the case of a homology sphere of Rokhlin invariant 0, in degrees between $-4k$ and $-4k-3$ with $k \gg 0$ the sequence looks like
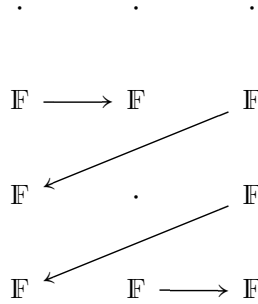
where the side columns represents $\widehat{\mathrm{HS}}_\bullet$ and the middle column represents $\widehat{\mathrm{HM}}_\bullet$. Let us record the following general observation.

Lemma 2.1. *If* $\mathbf{x} \in \widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$, *then* $Q^2 \cdot \mathbf{x} = \pi_*(U \cdot \iota_*(\mathbf{x}))$.

We will prove this result later. Let us define the following Massey operations:

- if $Q \cdot \mathbf{x} = V \cdot \mathbf{x} = 0$, $\langle Q, \mathbf{x}, V \rangle$, which is well defined up to $\operatorname{Im} Q + \operatorname{Im} V$;

- if $Q \cdot \mathbf{x} = 0$, $\langle \mathbf{x}, Q, Q^2 \rangle$, well defined up to $\operatorname{Im} Q^2$;

- if $Q^2 \cdot \mathbf{x} = 0$, $\langle \mathbf{x}, Q^2, Q \rangle$, well defined up to $\operatorname{Im} Q$;

- if $Q \cdot \mathbf{x} = 0$ and $\langle \mathbf{x}, Q, Q^2 \rangle = 0$, $\langle \mathbf{x}, Q, Q^2, Q \rangle$ well defined up to $\operatorname{Im} Q$ (recall that $\langle Q, Q^2, Q \rangle$ vanishes at the chain level, see Example 1.8).

On the other hand, using the Gysin exact sequence, we can define the following four operations.

(1) Suppose $Q \cdot \mathbf{x} = V \cdot \mathbf{x} = 0$. As $Q \cdot \mathbf{x} = 0$, $\mathbf{x} = \pi_*(\mathbf{y})$ for some $\mathbf{y}$. Then

$$\pi_*(U^2 \cdot \mathbf{y}) = V \cdot \pi_*(\mathbf{y}) = V \cdot \mathbf{x} = 0,$$

so that there exists $\mathbf{z}$ such that $\iota_*(\mathbf{z}) = U^2 \cdot \mathbf{y}$. We define $\Phi_1(\mathbf{x}) = \mathbf{z}$. It is readily checked that such an element is well defined up to elements in $\operatorname{Im} V + \operatorname{Im} Q$.

(2) Suppose $Q \cdot \mathbf{x}=0$. Then again $\mathbf{x} = \pi_*(\mathbf{y})$ for some $\mathbf{y}$. We then define $\Phi_2(\mathbf{x}) = \pi_*(U \cdot \mathbf{y})$. This is well defined up to $\operatorname{Im} Q^2$ in light of Lemma 2.1.

(3) Suppose $Q^2 \cdot \mathbf{x} = 0$. By Lemma 2.1, $\pi_*(U \cdot \iota_*(\mathbf{x})) = 0$, hence $U \cdot \iota_*(\mathbf{x}) = \iota_*(\mathbf{y})$ for some $\mathbf{y}$. Then we set $\Phi_3(\mathbf{x}) = \mathbf{y}$. This is well defined up to $\operatorname{Im} Q$.

(4) Suppose $\langle \mathbf{x}, Q, Q^2 \rangle = 0$. By the second bullet of Theorem 2.2 below, we have $\Phi_2(\mathbf{x}) = 0$. Then, up to choosing a different $\mathbf{y}$ in bullet (2) above,

$$\Phi_2(\mathbf{x}) = \pi_*(U \cdot \mathbf{y}) = 0,$$

and therefore $U \cdot \mathbf{y} = \iota_*(\mathbf{w})$ for some $\mathbf{w}$. Finally, we set $\Phi_4(\mathbf{x}) = \mathbf{w}$.

We will show in Section 3 that these four operations Φ_i are explicitly computable in many cases. Their importance for our purposes is the following result.

Theorem 2.2. *Let* $\mathbf{x}$ *be an element in* $\widehat{\mathrm{HS}}_\bullet(Y, \mathfrak{s})$. *We have the following identities*:

(1) *if* $Q \cdot \mathbf{x} = V \cdot \mathbf{x} = 0$, $\langle Q, \mathbf{x}, V \rangle = \Phi_1(\mathbf{x})$;
(2) *if* $Q \cdot \mathbf{x} = 0$, $\langle \mathbf{x}, Q, Q^2 \rangle = \Phi_2(\mathbf{x})$;
(3) *if* $Q^2 \cdot \mathbf{x} = 0$, $\langle \mathbf{x}, Q^2, Q \rangle = \Phi_3(\mathbf{x})$;
(4) *if* $\langle \mathbf{x}, Q, Q^2 \rangle = 0$, $\langle \mathbf{x}, Q, Q^2, Q \rangle = \Phi_4(\mathbf{x})$.

In fact, while for simplicity we have limited our exposition to Massey products involving only Q, Q^2 and V, the result naturally generalizes to the analogous Massey products involving QV^i, Q^2V^j and V^{k+1}. Let us for example point out how to compute $\langle Q, \mathbf{x}, V^{k+1} \rangle$, where of course we assume $Q \cdot \mathbf{x} = V^{k+1} \cdot \mathbf{x} = 0$. As $Q\mathbf{x} = 0$ implies that $\mathbf{x} = \pi(\mathbf{y})$, and because $V^{k+1} \cdot \mathbf{x} = 0$, we have that $\pi_*(U^{2k+2} \cdot \mathbf{y}) = 0$, so that $U^{2k+2} \cdot \mathbf{y} = \iota(\mathbf{z})$. We have then $\langle Q, \mathbf{x}, V^{k+1} \rangle = \mathbf{z}$. Furthermore, it will be clear from the proof that the statement holds also for Massey products for the left $\mathcal{A}_\infty$-structure; for example, $\Phi_3(\mathbf{x}) = \langle Q, Q^2, \mathbf{x} \rangle$.

Remark 2.3. Looking at the Gysin sequence of S^3, we obtain a direct proof (i.e. without relying on an argument involving the Eilenberg–Moore spectral sequence as in [23]) of the fact that $\langle Q, Q^2, Q, Q^2 \rangle = V$ (here we apply the theorem above to the left $\mathcal{A}_\infty$-structure).

Remark 2.4. While our main result involves specific Massey products, one can in general exploit the natural $\mathcal{A}_\infty$-structure in homology provided by Kadeishvili's homotopy transfer theorem (see [16] and [37]) to obtain more information. Let us for example consider the (classical) Massey product $\langle \mathbf{x}|Q^2|Q|Q^2 \rangle$, where we assume $\mathbf{x} \cdot Q = \langle \mathbf{x}|Q^2|Q \rangle = 0$. Recalling the vanishing of the triple products in $\mathcal{R}$ and the relation $\langle Q|Q^2|Q|Q^2 \rangle = V$, we obtain after substituting the latter in the $\mathcal{A}_\infty$-relations, the relation

$$\mathbf{x} \cdot V = \langle \mathbf{x}|Q^2|Q|Q^2 \rangle \cdot Q.$$

In several cases, this is enough to determine $\langle \mathbf{x}|Q^2|Q|Q^2 \rangle$.

The proof of this result occupies the rest of the section. Recall first from [25] that the Gysin exact sequence arises as the long exact sequence in homology associated to the short exact sequence of chain complexes

$$0 \longrightarrow \widehat{C}^J_\bullet(Y, \mathfrak{s}) \hookrightarrow \widehat{C}_\bullet(Y, \mathfrak{s}) \xrightarrow{1+J} (1+J)\widehat{C}_\bullet(Y, \mathfrak{s}) \longrightarrow w0 \qquad (6)$$

where $\widehat{C}_\bullet(Y, \mathfrak{s})$ is the Floer chain complex underlying $\widehat{HM}_\bullet(Y, \mathfrak{s})$ and the chain complexes $\widehat{C}^J_\bullet(Y, \mathfrak{s})$ and $(1 + J)\widehat{C}_\bullet(Y, \mathfrak{s})$ both have homology $\widehat{HS}_\bullet(Y, \mathfrak{s})$. We first review how the connecting map in the induced long exact sequence is identified with multiplication by Q. Consider a representative x of a class $\mathbf{x} \in \widehat{HS}_\bullet(Y, \mathfrak{s})$. Consider its image under the map induced by the cobordism $(I \times Y) \setminus \text{int}(B^4)$ with cylindrical ends attached, where we look at the solutions converging to the first negative critical submanifold C_{-1} on the additional incoming S^3 end, or, equivalently the element $\hat{m}_2(x|C_{-1})$ obtained from the product map

$$\hat{m}_2 \colon \widehat{C}_\bullet(Y, \mathfrak{s}) \otimes \widehat{C}_\bullet(S^3) \longrightarrow \widehat{C}_\bullet(Y, \mathfrak{s})$$

by considering the chain C_{-1} on the second factor. As this map induces the identity in homology, this element is also a representative of $\mathbf{x}$. Recall that C_{-1} is a copy of S^2 on which J acts as the antipodal map. Denote by D^2 the upper hemisphere, and by $S^1 = \partial D^2$ the equator (notice that the latter is J-invariant). We have then $S^2 = D^2 \cup J(D^2) = (1 + J)D^2$, so that

$$\hat{m}_2(x|C_{-1}) = \hat{m}_2(x|(1 + J)D^2) = (1 + J)\hat{m}_2(x|D^2) = (1 + J)(y)$$

where $y = \hat{m}_2(x|D^2) \in \widehat{C}_\bullet(Y, \mathfrak{s})$. Now, as $\partial x = 0$, we have

$$\partial y = \partial(\hat{m}_2(x|D^2)) = \hat{m}_2(x|\partial D^2) = \hat{m}_2(x|S^1)$$

which is a J-invariant cycle, hence in the image of the inclusion $\widehat{C}^J_\bullet(Y, \mathfrak{s}) \hookrightarrow \widehat{C}_\bullet(Y, \mathfrak{s})$. By definition, its class in $\widehat{HS}_\bullet$ represents the image of $\mathbf{x}$ under the boundary map in the induced long exact sequence. On the other hand, as S^1 is a representative of Q in $\widehat{HS}_\bullet(S^3) = \mathcal{R}$, y also represents $Q \cdot \mathbf{x}$.

In a similar spirit, we now provide the proof of Lemma 2.1.

Proof of Lemma 2.1. Let p be a point in C_{-1}. Then the point p is a cycle representing $U \in \widehat{HM}_\bullet(S^3)$, while $p \cup Jp$ is an invariant cycle representing $Q^2 \in \widehat{HS}_\bullet(S^3)$. If $y \in \widehat{C}_\bullet(Y, \mathfrak{s})$ represents $\mathbf{y} \in \widehat{HM}_\bullet(Y, \mathfrak{s})$, then $U \cdot \mathbf{y}$ is represented by $\hat{m}_2(y|p)$, while if $x \in \widehat{C}^J_\bullet(Y, \mathfrak{s})$ represents $\mathbf{x} \in \widehat{HS}(Y, \mathfrak{s})$, then $Q^2 \cdot \mathbf{y}$ is represented by $\hat{m}_2(x|p \cup Jp)$. On the other hand we have for $x \in \widehat{C}^J_\bullet(Y, \mathfrak{s})$ that

$$\hat{m}_2(x|p \cup Jp) = \hat{m}_2(x|(1 + J)p) = (1 + J)\hat{m}_2(x|p),$$

and the result follows. $\square$

With these simple computations in mind, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Throughout the proof, let us fix a representative $x \in \hat{C}_\bullet^J(Y, \mathfrak{s})$ of $\mathbf{x}$. We will prove the various statements separately (with the warning that the proof of (4) builds on the proof of (2)).

Proof of (1). Recall that the action of V on $\widetilde{\mathrm{HS}}_\bullet$ and of U^2 on $\widetilde{\mathrm{HM}}_\bullet$ are both obtained by multiplication by the second negative critical submanifold C_{-2} on the additional incoming S^3 end. Let

$$y = \hat{m}_2(x|D^2) \in \hat{C}_\bullet(Y, \mathfrak{s})$$

as above, so that its image under $1 + J$ is a representative of $\mathbf{x}$, and $\partial y = \hat{m}_2(x|S^1)$ is a J-invariant chain cycle representing $Q \cdot \mathbf{x} = 0$. Hence we have $\hat{m}_2(x|S^1) = \partial t$ for $t \in \hat{C}_\bullet^J(Y, \mathfrak{s})$. Consider also $\hat{m}_2(x|C_{-2})$, which represents $V \cdot \mathbf{x} = 0$ and hence is ∂s for $s \in \hat{C}_\bullet^J(Y, \mathfrak{s})$. Then by definition the triple Massey product $\langle Q, \mathbf{x}, V \rangle$ is represented by

$$\hat{m}_2(t|C_{-2}) + \hat{m}_2(S^1|s) + \hat{m}_{2,2}(S^1|x|C_{-2}) \in \hat{C}_\bullet^J(Y, \mathfrak{s}).$$

Consider the image of this cycle in $\hat{C}_\bullet(Y, \mathfrak{s})$. By adding to it the (non J-invariant) boundaries

$$\partial \hat{m}_{2,2}(D^2|x|C_{-2})$$
$$= \hat{m}_{2,2}(S^1|x|C_{-2}) + \hat{m}_2(\hat{m}_2(D^2|x)|C_{-2}) + \hat{m}_2(D^2|\hat{m}_2(x|C_{-2}))$$

and

$$\partial \hat{m}_2(D^2|s) = \hat{m}_2(S^1|s) + \hat{m}_2(D^2|\hat{m}_2(x|C_{-2}))$$

we see that $\iota_*(\langle Q, \mathbf{x}, V \rangle) \in \widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$ is represented by

$$\hat{m}_2(t + \hat{m}_2(D^2|x)|C_{-2}) \in \hat{C}_\bullet(Y, \mathfrak{s}). \tag{7}$$

As t is J-invariant, $t + \breve{m}_2(D^2|x)$ is a cycle in $\hat{C}_\bullet(Y, \mathfrak{s})$ whose image under π_* is a representative of $\mathbf{x}$; furthermore, the chain (7) represents its image under the action of $V = U^2$, so that the result follows. $\triangle$

Proof of (2). Recall from Example 1.8 that $Q^2 \cdot Q$ is zero at the chain level. Consider as above the cycle $\hat{m}_2(x|S^2)$, and let t be a J-invariant chain such that $\partial t = \hat{m}_2(x|S^1)$, where again we use $Q \cdot \mathbf{x} = 0$. Then the Massey product $\langle \mathbf{x}, Q, Q^2 \rangle$ is by definition represented by

$$\hat{m}_2(t|p \cup Jp) + \hat{m}_3(t|S^1|p \cup Jp)$$
$$= (1 + J)(\hat{m}_2(t|p) + \hat{m}_3(x|S^1|p)) \in (1 + J)\hat{C}_\bullet(Y, \mathfrak{s}). \tag{8}$$

Of the two natural disks D^2 and $_J D^2$ whose boundary is S^1, we can assume without loss of generality that $\hat{m}_2(D^2|p) = 0$. Adding then to the expression above

$$\partial[(1 + _J)(\hat{m}_3(x|D^2|p))] = (1 + _J)(\hat{m}_3(x|S^1|p) + \hat{m}_2(\hat{m}_2(x|D^2)|p))$$

we see that the Massey product is represented by

$$(1 + _J)[\hat{m}_2(t + \hat{m}_2(x|D^2)|p)].$$

Now $t + \hat{m}_2(x|D^2) \in \widehat{C}_\bullet(Y, \mathfrak{s})$ is again a cycle mapping to a representative of **x**, and the result follows.   △

Proof of (3). As $Q^2 \cdot \mathbf{x} = 0$, by Lemma 2.1 we can consider $z \in \widehat{C}_\bullet^J(Y, \mathfrak{s})$ such that $\partial z = \hat{m}_2(x|p \cup _J p)$. Then by definition the Massey product $\langle \mathbf{x}, Q^2, Q \rangle$ is represented by

$$\hat{m}_2(z|S^1) + \hat{m}_3(x|p \cup _J p|S^1) \in \widehat{C}_\bullet^J(Y, \mathfrak{s}).$$

Consider its image under the inclusion in $\widehat{C}_\bullet(Y, \mathfrak{s})$. Assume $\hat{m}_2(p|D^2) = 0$, and set $p' = \hat{m}_2(_J p|D^2)$. We can add the (non $_J$-invariant) boundary

$$\partial \hat{m}_3(x|p \cup _J p|D^2) = \hat{m}_3(x|p \cup _J p|S^1) + \hat{m}_2(\hat{m}_2(x|p \cup _J p)|D^2) + \hat{m}_2(x|p').$$

We see that the image in $\widehat{C}_\bullet(Y, \mathfrak{s})$ of the triple Massey product is also represented by

$$\hat{m}_2(z|S^1) + \hat{m}_2(\hat{m}_2(x|p \cup _J p)|D^2) + \hat{m}_2(x|p') = \hat{m}_2(x|p') + \partial \hat{m}_2(z|D^2),$$

and the result follows.   △

Proof of (4). Suppose $\langle \mathbf{x}, Q, Q^2 \rangle = 0$. Then, in the notation of the proof of the second bullet, we have that

$$(1 + _J)[\hat{m}_2(t + \hat{m}_2(x|D^2)|p)] = \partial w. \tag{9}$$

for some $w \in (1 + _J)\widehat{C}_\bullet(Y, \mathfrak{s})$ so that for the chain representative in equation (8) we have

$$(1 + _J)(\hat{m}_2(t|p) + \hat{m}_3(x|S^1|p)) = \partial(w + (1 + _J)\beta) \tag{10}$$

with $\beta = \hat{m}_3(x|D^2|p)$. As the products $Q \cdot Q^2$, $Q^2 \cdot Q$ and $\langle Q, Q^2, Q \rangle$ are zero

at the chain level (see Example 1.8), the formula (3) for the 4-fold Massey product greatly simplifies. In particular, we have that $\langle \mathbf{x}, Q, Q^2, Q \rangle$ is represented by

$$\hat{m}_2(w + (1 + J)\beta|S^1) + \hat{m}_4(x|S^1|p \cup Jp|S^1) + \hat{m}_3(t|p \cup Jp|S^1).$$

Let us point out that while in the proof of (2) we worked up to boundaries, it is important here that we work with the actual chain representative in equation (8) of the Massey products. We claim that under the inclusion into $\hat{C}_\bullet(Y, \mathfrak{s})$ this maps to the same class as $\hat{m}_2(t + \hat{m}_2(x|D^2)|p)$, so that the result follows. Let us choose again disks D^2 such that $\hat{m}_2(p|D^2) = 0 = \hat{m}_2(D^2|p) = 0$. Here the two disks are not the same, but we will not incorporate that in our already heavy notation as it should not create confusions. We have

$$\begin{aligned}\hat{m}_4(x|S^1|p|S^1) = {} & \partial \hat{m}_4(x|S^1|p|D^2) + \hat{m}_2(\hat{m}_3(x|S^1|p)|D^2) \\ & + \hat{m}_3(\hat{m}_2(x|S^1)|p|D^2) + \hat{m}_2(x|\hat{m}_3(S^1|p|D^2))\end{aligned}$$

and

$$\hat{m}_3(t|p|S^1) = \partial \hat{m}_3(t|p|D^2) + \hat{m}_2(\hat{m}_2(t|p)|D^2) + \hat{m}_3(\hat{m}_2(x|S^1)|p|D^2).$$

where we used in both cases $\hat{m}_2(p|D^2) = 0$. Notice that, regarding the last term in the first sum, $\hat{m}_3(S^1|p|D^2)$ is a collection of points, so that $1 + J$ of the last term represents a multiple of $Q^2 \cdot \mathbf{x} = 0$; we can ignore the last term. Hence the image $\iota_*(\langle \mathbf{x}, Q, Q^2, Q \rangle) \in \widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$ is represented by the cycle

$$\begin{aligned}& \hat{m}_2(w + (1 + J)\beta|S^1) + (1 + J)[\hat{m}_2(\hat{m}_3(x|S^1|p)|D^2) + \hat{m}_2(\hat{m}_2(t|p)|D^2)] \\ & = \hat{m}_2(w|S^1) + (1 + J)[\hat{m}_2(\beta|S^1) + \hat{m}_2(\hat{m}_3(x|S^1|p)|D^2) + \hat{m}_2(\hat{m}_2(t|p)|D^2)].\end{aligned}$$

Notice that

$$\begin{aligned}\hat{m}_2(w|S^1) & = \hat{m}_2(w|\partial D^2) \\ & = \partial \hat{m}_2(w|D^2) + \hat{m}_2([(1 + J)(\hat{m}_2(t + \hat{m}_2(x|D^2)|p))]|D^2),\end{aligned}$$

and, using now $\hat{m}_2(D^2|p) = 0$,

$$\begin{aligned}\hat{m}_2(\hat{m}_3(x|S^1|p)|D^2) & = \hat{m}_2(\hat{m}_3(x|\partial D^2|p)|D^2) \\ & = \hat{m}_2(\partial \hat{m}_3(x|D^2|p) + \hat{m}_2(\hat{m}_2(x|D^2)|p)|D^2).\end{aligned}$$

Hence ι of our cycle has the form (up to boundaries)

$$\hat{m}_2((1+\jmath)\alpha|D^2) + (1+\jmath)[\hat{m}_2(\alpha+\partial\beta|D^2) + \hat{m}_2(\beta|S^1)]$$

where $\alpha = \hat{m}_2(t + \hat{m}_2(x|D^2)|p)$, or equivalently

$$\hat{m}_2(\jmath\alpha|D^2 \cup \jmath D^2) + \partial[(1+\jmath)\hat{m}_2(\beta|D^2)].$$

Here $\hat{m}_2(\jmath\alpha|D^2 \cup \jmath D^2)$ represents the same class as $\jmath\alpha$. Finally, relation (9) implies that this is also the class of α itself, and the result follows. △

This concludes the proofs of Theorem 2.2. □

Let us point out some immediate consequences regarding correction terms. There is a plethora of numerical invariants of homology cobordism of rational homology spheres that one can in principle extract from the $\mathcal{R}$-module structure in Pin(2)-monopole Floer homology (these might go under the name *generalized correction terms*). On the other hand, the simplest correction term, namely the Frøyshøv invariant $\delta(Y, \mathfrak{s})$ arising from usual monopole Floer homology cannot be recovered from the $\mathcal{R}$-module structure (see [35]). We briefly recall the definition of the latter (which is $-h(Y, \mathfrak{s})$ in the notation of [17]). We have that, given a rational homology sphere Y,

$$\overline{\mathrm{HM}}_\bullet(Y, \mathfrak{s}) \cong \mathbb{F}[U^{-1}, U]]$$

and that the map $i_*: \overline{\mathrm{HM}}_\bullet(Y, \mathfrak{s}) \to \widetilde{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$ is an isomorphism in degrees high enough and vanishes in degrees low enough. We then define

$$\delta(Y, \mathfrak{s}) = \frac{1}{2} \min\{\deg(\mathbf{x}) \mid \mathbf{x} \in \mathrm{Im}(i_*)\}.$$

We have the following characterization of $\delta(Y, \mathfrak{s})$ purely in terms of the Massey products in Pin(2)-monopole Floer homology. From this, Theorem 3 in the Introduction directly follows.

Proposition 2.5. *Let* $\mathbf{x}$ *be the bottom element of the γ-tower which is not in the image of Q. Then*

$$\delta(Y, \mathfrak{s}) = \begin{cases} \dfrac{1}{2}(\deg(\mathbf{x}) - 2) & \textit{if either } Q^2 \cdot \mathbf{x} \neq 0 \textit{ or } \langle Q, Q^2, \mathbf{x} \rangle \neq 0, \\ \dfrac{1}{2}\deg(\mathbf{x}) & \textit{otherwise.} \end{cases}$$

Proof. Let $\mathbf{x}$ be the bottom of the element in the γ-tower which is not in the image of Q. By exactness of the Gysin sequence, $\iota_*(\mathbf{x}) \neq 0$. Furthermore, by comparing with the Gysin sequence in degrees high enough, we see that $\iota_*(\mathbf{x})$ is in fact an element of the U-tower of $\widetilde{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$. We claim that $U^2 \iota_*(\mathbf{x}) = 0$. In fact, we would have otherwise

$$0 \neq U^2 \cdot \iota_*(\mathbf{x}) = V \cdot \iota_*(\mathbf{x}) = \iota_*(V\mathbf{x}),$$

so that $V\mathbf{x}$, which is an element in the γ-tower, is not in the image of Q, which contradicts our choice of $\mathbf{x}$. The bottom of the U-tower is therefore given by either $\iota_*(\mathbf{x})$ or $U \cdot \iota_*(\mathbf{x})$. In the latter case, by exactness we have that either $U \cdot \iota_*(\mathbf{x})$ is in the image of ι_*, or its image under π_* is non-vanishing. These two cases correspond, thanks to part (3) of Theorem 2.2 and Lemma 2.1, to respectively $\langle Q, Q^2, \mathbf{x} \rangle \neq 0$ or $Q^2 \cdot \mathbf{x} \neq 0$, and the result follows. $\square$

In fact, a more natural correction term to study in Pin(2)-monopole Floer homology (introduced in [35]) is

$$\delta'(Y, \mathfrak{s}) = \frac{1}{2}(\min\{\deg(\mathbf{x}) \mid \mathbf{x} \in \gamma\text{-tower } \mathbf{x} \notin \mathrm{Im}\, Q\} - 2)$$

which, by the discussion above, coincides with either $\delta(Y, \mathfrak{s})$ or $\delta(Y, \mathfrak{s}) + 1$. Furthermore, $\delta'(Y, \mathfrak{s})$ reduces modulo 2 to $-\mu(Y, \mathfrak{s})$. While we have

$$\delta(-Y, \mathfrak{s}) = -\delta(Y, \mathfrak{s}),$$

the effect of orientation reversal on $\delta'(Y, \mathfrak{s})$ cannot be described purely in term of the module structure (see also Remark 1.6). On the other hand, it can be described in terms of Massey products as follows.

Proposition 2.6. *Let* $\mathbf{x}$ *be the bottom element of the* γ-*tower such that either* $Q^2 \cdot \mathbf{x} \neq 0$ *or* $\langle \mathbf{x}, Q^2, Q \rangle \neq 0$*, and set*

$$\delta''(Y, \mathfrak{s}) = \frac{1}{2}(\deg(\mathbf{x}) - 2).$$

Then we have $\delta'(-Y, \mathfrak{s}) = -\delta''(Y, \mathfrak{s})$.

The key observation here is that one can use Poincaré duality and the long exact sequence (2) to obtain relations between $\widetilde{\mathrm{HS}}_\bullet(Y)$ and $\widetilde{\mathrm{HS}}_\bullet(-Y)$. As this will be used repeatedly in the final part of the paper, let us discuss here in detail the simplest manifestation of these ideas, which is at the core of the proof of Proposition 2.6.

Lemma 2.7. *There is an element* $\mathbf{z} \neq 0$ *in degree* k *in the* γ*-tower of* $\widetilde{\mathrm{HS}}_\bullet(Y)$ *which is in the image of* Q *if and only if there is an element* $\mathbf{x} \neq 0$ *in degree* $-k$ *in the* γ*-tower of* $\widetilde{\mathrm{HS}}_\bullet(-Y)$ *for which* $Q^2 \cdot \mathbf{x} = 0$ *and* $\langle \mathbf{x}, Q^2, Q \rangle \neq 0$.

Proof. Let us show in detail the forward implication. Given $\mathbf{y} \in \widetilde{\mathrm{HS}}_{k+1}(Y)$ such that $Q \cdot \mathbf{y}$ is in the γ-tower, notice that its image $j_*(\mathbf{y})$ in $\widehat{\mathrm{HS}}_{k+1}(Y)$ is annihilated by Q. We claim that $\langle j_*(\mathbf{y}), Q, Q^2 \rangle$ is $Q^2 \cdot \mathcal{V}$-based. To see this, we notice that the exactness of the Gysin exact sequence implies that $\mathbf{y}$ is not in the image of π_*, while $j_*(\mathbf{y})$ is. Comparing this with the exact triangle relating the three flavors of usual monopole Floer homology, we see by a simple diagram chasing that $j_*(\mathbf{y})$ is in the image of a non U-torsion element $\mathbf{w} \in \widehat{\mathrm{HM}}_\bullet(Y, \mathfrak{s})$. For such a $\mathbf{w}$, we have that $\pi_*(U \cdot \mathbf{w}) \in \widehat{\mathrm{HS}}_{k-1}(Y)$ is $Q^2 \cdot \mathcal{V}$-based (again by a simple diagram chasing), and the claim follows by bullet (2) of Theorem 2.2.

Now, the Poincaré duality isomorphisms

$$\widetilde{\mathrm{HM}}^*(-Y) \cong \widehat{\mathrm{HM}}_{-1-*}(Y), \quad \widetilde{\mathrm{HS}}^*(-Y) \cong \widehat{\mathrm{HS}}_{-1-*}(Y)$$

provide a natural identification of the two diagrams

$$
\begin{array}{ccc}
\widehat{\mathrm{HM}}_{k+1}(Y) & \xrightarrow{\pi_*} & \widehat{\mathrm{HS}}_{k+1}(Y) \\
{\scriptstyle U\cdot}\downarrow & & \\
\widehat{\mathrm{HM}}_{k-1}(Y) & \xrightarrow{\pi_*} & \widehat{\mathrm{HS}}_{k-1}(Y)
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\mathrm{HS}}^{-k}(-Y) & \xleftarrow{\iota^*} & \widetilde{\mathrm{HM}}^{-k}(-Y) \\
& & \uparrow{\scriptstyle U\cdot} \\
\widetilde{\mathrm{HS}}^{-k-2}(-Y) & \xleftarrow{\iota^*} & \widetilde{\mathrm{HM}}^{-k-2}(-Y)
\end{array}
$$

exchanging the role of π_* and ι^*. Under this identification we see therefore (using the natural cohomological version of Theorem 2.2) that there is an element $\iota^*(\mathbf{y}') \in \widetilde{\mathrm{HS}}^{k-2}(-Y)$ for which $\langle \iota^*(\mathbf{y}'), Q, Q^2 \rangle \in \widetilde{\mathrm{HS}}^k(-Y)$ is $Q^2 \cdot \mathcal{V}$-based. Now, the cohomological diagram on the right is dual to the homological diagram below:

$$
\begin{array}{ccc}
\widetilde{\mathrm{HS}}_{-k}(-Y) & \xrightarrow{\iota_*} & \widetilde{\mathrm{HM}}_{-k}(-Y) \\
& & \downarrow{\scriptstyle U\cdot} \\
\widetilde{\mathrm{HS}}_{-k-2}(-Y) & \xrightarrow{\iota_*} & \widetilde{\mathrm{HM}}_{-k-2}(-Y)
\end{array}
$$

We can fix an identification

$$\widetilde{\mathrm{HM}}^{-k-2}(-Y) = \mathrm{im}(\iota^*) \oplus \ker(\iota^*),$$

and observe that

$$\iota^*(U \cdot \ker(\iota^*)) = \iota^*(U \cdot \mathrm{im}(\pi^*)) = Q^2 \cdot \mathrm{im}(\pi^*)$$

does not contain $Q^2 \cdot \mathcal{V}$-based elements (as there are no $\mathcal{V}$-based elements in degree $k - 2$). We can therefore fix a splitting

$$\widetilde{\mathrm{HM}}^{-k-2}(-Y) = \mathbb{F}\langle \mathbf{y}' \rangle \oplus W \oplus \ker(\iota^*)$$

with the properties that

- $\iota^*(U \cdot (W \oplus \ker(\iota^*)))$ does not contain $Q^2 \cdot \mathcal{V}$-based elements,
- $\iota^*(U \cdot \mathbf{y}')$ is $Q^2 \cdot \mathcal{V}$-based,
- ι^* restricted to $\mathbb{F}\langle \mathbf{y}' \rangle \oplus W$ is injective.

On the dual homological diagram, this implies that there exists an element $\mathbf{x} \in \widetilde{\mathrm{HS}}_{-k}(-Y)$ belonging to the γ tower such that $U \cdot \iota_*(\mathbf{x})$ is in the image of ι_*, and the result follows by Theorem 2.2. The proof of the reverse implication is analogous. $\square$

Proof of Proposition 2.6. Consider the element $\mathbf{z}$ in degree $2\delta'(Y) + 2$ in the γ-tower of $\widetilde{\mathrm{HS}}_\bullet(Y)$, so that it realizes the minimum in the definition of δ'. Then we have that $V\mathbf{z}$ is either zero, or is in the image of Q. In either case, we get a non-zero element $\mathbf{x}$ in the γ-tower of $\widetilde{\mathrm{HS}}_\bullet(-Y)$ in degree $-2\delta'(Y) + 2$ which satisfies $Q^2 \cdot \mathbf{x} \neq 0$ in the former case, or $Q^2 \cdot \mathbf{x} = 0$ and $\langle \mathbf{x}, Q^2, Q \rangle \neq 0$ in the latter. Finally, the same arguments above show that $\mathbf{x}$ indeed realize $\delta''(-Y)$. $\square$

3. Examples

In this section we discuss several classes of manifolds for which the description of the Massey products in terms of the Gysin exact triangle from the previous section is very explicit.

Manifolds of simple type M_n. We introduce a special class of homology spheres which play a central role in Pin(2)-monopole Floer homology. Let us first discuss the relevant algebraic definitions; these are slightly different according to whether n is even or odd.

In the case $n = 2k$, we define

$$M_n = \mathbb{F}[[V]] \oplus \mathbb{F}[[V]]\langle 4k - 1 \rangle \oplus \mathbb{F}[[V]]\langle 4k - 2 \rangle \oplus \mathbb{F}[[V]]/(V^{k-1})\langle 4k - 3 \rangle,$$

where the action of V respects the direct sum decomposition and the action of Q maps one column to the one on the right and has maximal possible rank. This

module can be depicted graphically as
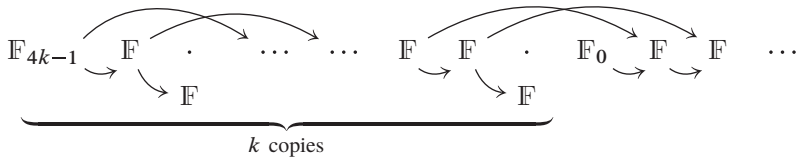
where the arrows in the upper and lower rows represent respectively the actions of V and Q, and the bottom row corresponds to the $\mathbb{F}[[V]]/(V^{k-1})\langle 4k-3 \rangle$ summand. Let 1 be the generator of the first summand and qv^{-k} be the generator of the second summand. They lie in degrees respectively 0 and $4k-1$, and are $\mathcal{V}$ and $Q \cdot \mathcal{V}$-based.

In the case $n = 2k + 1$ for $k \geq 0$, we define

$$M_n = \mathbb{F}[[V]]\langle -2 \rangle \oplus \mathbb{F}[[V]]\langle 4k + 1 \rangle \oplus \mathbb{F}[[V]]\langle 4k \rangle \oplus \mathbb{F}[[V]]/(V^{k-1})\langle 4k - 1 \rangle.$$

where again the action of V respects the direct sum decomposition and the action of Q maps one column to the one on the right and has maximal possible rank. More visually,
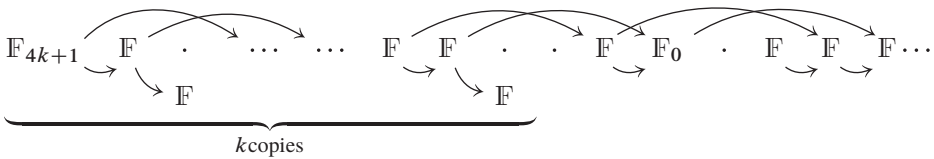
We denote the generator of the first summand by v and the generator of the second summand by qv^{-k}. They lie in degrees -2 and $4k + 1$ respectively, and are $\mathcal{V}$ and $Q \cdot \mathcal{V}$-based.

The following is the key definition of this section (an analogous concept of manifolds of projective type was introduced in [36]).

Definition 3.1. A homology sphere Y has Pin(2)-*simple type* M_n if there is a direct sum decomposition as $\mathcal{R}$-modules

$$\widehat{\mathrm{HS}}_\bullet(Y) = M_n\langle -1 \rangle \oplus J$$

with $p_*(J) = 0$ and no non-trivial Massey products between the two summands.

From the Gysin exact sequence it readily follows that the part of $\widehat{\mathrm{HM}}_\bullet(Y)$ interacting with $M_n\langle -1 \rangle$ has the form

$$\mathbb{F}[[U]]\langle -1 \rangle \oplus \mathbb{F}[[U]]/U^n\langle 2n - 2 \rangle.$$

This implies that if Y has simple type M_n, $\delta(Y) = 0$ and $\alpha(Y) = \beta(Y) = n$,

$$\gamma(Y) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{otherwise,} \end{cases}$$

and that the Rokhlin invariant of Y is simply the parity of n.

Surgery on L-space knots.   There are several examples of manifolds with simple type M_n obtained by surgery on a knot in S^3 (this should be compared, in the Heegaard Floer setting, to the results in [13]). In what follows, we prefer to work for notational reasons with the *to* homologies $\widetilde{\mathrm{HS}}_\bullet$. Let us introduce a notation for the standard U and V-towers

$$\mathcal{U}^+ = \mathbb{F}[[U^{-1}, U]]/\mathbb{F}[[U]],$$
$$\mathcal{V}^+ = \mathbb{F}[[V^{-1}, V]]/\mathbb{F}[[V]],$$

where the bottom element lies in degree zero. Recall that a knot $K \subset S^3$ is called an *L-space knot* if for large enough $r > 0$, the manifold $S_r^3(K)$ is an L-space (i.e. it has vanishing reduced monopole Floer homology $\mathrm{HM}_\bullet$). Given an L-space knot, we have

$$\widetilde{\mathrm{HM}}_\bullet(S_0^3(K), \mathfrak{s}_0) = \mathcal{U}^+\langle -1 \rangle \oplus \mathcal{U}^+\langle -2n \rangle \tag{11}$$

for some $n \geq 0$. Here $\mathfrak{s}_0$ denotes the unique torsion spin^c structure. We say in this case that K has *type n*. The analogous fact in Heegaard Floer homology is well known [32], and implies our claim via the isomorphism with monopole Floer homology (see [19], [2], and subsequent papers). Our key source of examples is the following.

Proposition 3.2. *Let K be an L-space knot of type n. Then the manifold $-S_{-1}^3(K)$ has simple type M_n, where $-Y$ denotes Y with the orientation reversed.*

Example 3.3. Recall that all positive torus knots have a positive lens space, hence L-space, surgery. More concretely, the torus knot $T(2, 4n - 1)$ is an L-space knot of type n, and furthermore $S_{-1}^3(T(2, 4n - 1))$ is the Seifert space $\Sigma(2, 4n - 1, 8n - 1)$. Therefore, $-\Sigma(2, 4n - 1, 8n - 1)$ has simple type M_n.

Remark 3.4. In fact, in the setting of the statement above, the same proof will show that $-S_{-1/2k+1}^3(K)$ has simple type M_n for $k \geq 0$.

Notice that the Arf invariant of an L-space knot K is the same as the parity of its type n. The proof of Proposition 3.2 is simpler in the case of Arf $= 1$, and essentially follows from the computations involving the surgery exact triangle in [24]. The proof of the Arf $= 0$ case is more subtle, and follows from the content of the unpublished note [22]. We here discuss the main ideas involved for both cases. We denote by $\breve{S}_{1/q}$ the group $\widetilde{\mathrm{HS}}_\bullet(S^3_{1/q}(K))$. For each q, the main result of [24] implies that there is an exact triangle

$$
\begin{array}{ccc}
\breve{S}_{1/q} & \xrightarrow{\quad \breve{A}_q \quad} & \breve{S}_0 \\
& & \\
& \searrow \quad \nwarrow & \swarrow \; \breve{B}_{q+1} \\
& \breve{S}_{1/(q+1)} &
\end{array}
$$

where the maps $\breve{A}_q^s$ and $\breve{B}_q^s$ are those induced by the (spin) cobordisms given by handle attachments. The key observation is the following.

Lemma 3.5. *The composite*

$$
\breve{B}_q \circ \breve{A}_q \colon \breve{S}_{1/q} \longrightarrow \breve{S}_{1/q} \tag{12}
$$

is given by multiplication by Q.

By contrast, the analogous map in the usual setting of monopole Floer homology vanishes, see [18].

Proof. The composition of the two cobordisms is described by the Kirby diagram in Figure 1. The cobordism from $Y_{1/q}(K)$ to $Y_0(K)$ is given by a two handle attachment along the knot K', while the following one from $Y_0(K)$ to $Y_{1/q}(K)$ is given by attaching another two handle along a zero framed meridian of K'. If we trade this second handle for a 1-handle (i.e. adding a dot in the notation of [11]), we obtain a pair of canceling 1 and 2-handles. Hence the composite cobordism is obtained from the product cobordism $[0,1] \times Y_{1/q}(K)$ by removing a neighborhood $S^1 \times D^3$ of a loop and replacing it by $S^2 \times D^2$. The result then follows from the fact that the map induced by $S^2 \times S^2$ with two ball removed induces multiplication by Q (see the proof of Theorem 5 in [24]). $\qquad\square$

When K has Arf invariant zero, we have from [24], or more generally [26], that

$$
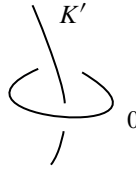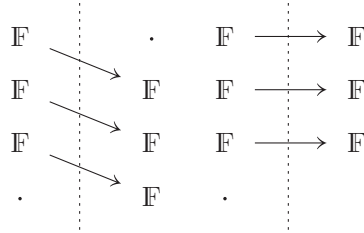\bar{S}_0 = \widetilde{\mathcal{R}}\langle -1 \rangle \oplus \widetilde{\mathcal{R}}, \tag{13}
$$

Figure 1. A handlebody description of the composite of the cobordisms defining the map $\breve{B}_q \circ \breve{A}_q$. This link is inside $Y_{1/q}$.

where we fix the identification so that so that $\overline{A}_0^s$ is an isomorphism onto the first summand, so that the triangle looks schematically like

repeated in both directions four-periodically. Here the three columns denote $\overline{S}_\infty^s$, $\overline{S}_0^s$ and $\overline{S}_1^s$ respectively. Lemma 3.5 implies that for $q = -1$ (or, in general, q odd) the triangle looks like
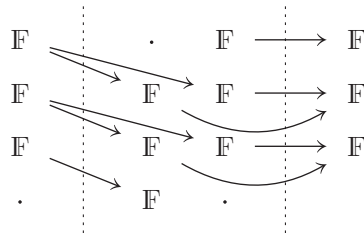
repeated in both directions four-periodically.

The final observation is that the isomorphism (11) implies via the Gysin exact sequence that, setting

$$\mathcal{R}^+ = \widetilde{\mathrm{HS}}_\bullet(S^3) = \mathcal{V}^+ \oplus Q \cdot \mathcal{V}^+ \oplus Q^2 \cdot \mathcal{V}^+,$$

we have

$$\widetilde{\mathrm{HS}}_\bullet(S_0^3(K)) = \mathcal{R}^+\langle -1 \rangle \oplus \mathcal{R}^+\langle -2n \rangle.$$

With this in hand, the proof of Proposition 3.2 in the Arf zero case follows as in the next example.

Example 3.6. Consider the case of the torus knot $K = T(2, 7)$, which is an L-space knot of type 2. We have then the identification of the triangle with $q = -1$ with
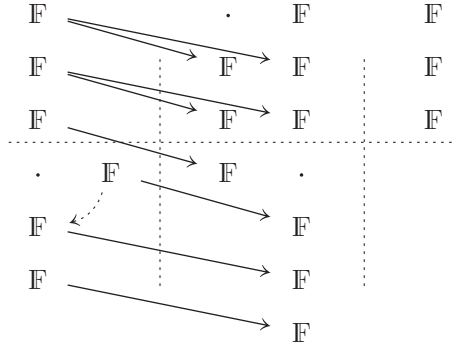

where for simplicity we have only depicted $\breve{A}_0$, and we have omitted summands in $\widetilde{HS}_\bullet(S^3_{-1}(K))$ arising from the non-torsion spin^c structures on $S^3_0(K)$, as they do not have interesting Massey products due to naturality. Here the dotted arrow denotes a Q action, and the horizontal dotted line represents grading zero. The result then follows from Poincaré duality.

The case in which K has Arf invariant one is significantly simpler. Indeed, in this case we have

$$\overline{HS}_\bullet(S^3_0(K)) = (\mathcal{V} \otimes \mathbb{F}[Q]/Q^2) \oplus (\mathcal{V} \otimes \mathbb{F}[Q]/Q^2)\langle -2\rangle.$$

so that the maps on $\overline{HS}_\bullet$ in the exact triangle are uniquely determined, see [24] for the details.

Remark 3.7. In fact, we see that $\overline{HS}_\bullet(S^3_0(K))$, where K has Arf invariant 1, has many interesting Massey products itself. The Gysin exact triangle looks in this case like
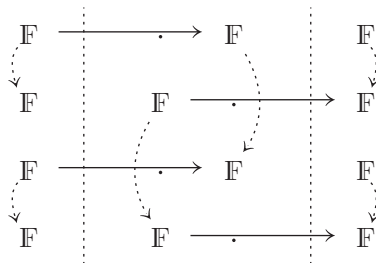
where the solid arrows depict the maps ι_* and π_*, while the dotted arrows denote the actions of Q and U. In light of Theorem 2.2, we see that there are many non-trivial Massey products of the form $\langle \cdot, Q, Q^2 \rangle$ and $\langle \cdot, Q^2, Q \rangle$.

Manifolds of simple type $-M_n$. Of course, from the view point of the Massey products discussed in Section 2, manifolds of simple type M_n are not particularly interesting. On the other hand, the manifolds obtained by orientation reversal, to which we refer as manifold with simple type $-M_n$, have a richer structure. For simplicity we will focus on the case in which n is even. Let us start from $n = 2$. The $\mathcal{R}$-module structure for a manifold of type $-M_2$ is given by

$$\mathbb{F}_0 \quad \cdot \quad \mathbb{F}_{-2} \quad \cdot \quad \mathbb{F} \rightsquigarrow \mathbb{F} \rightsquigarrow \mathbb{F} \quad \cdots$$

As a notation, we will denote the generator in degree zero by $\mathbf{z}$, while the generators in degree -2 and -4 by q^2 and v respectively. For a general manifold Y of simple type $-M_2$, we will have a decomposition $\overline{\mathrm{HS}}_\bullet(Y) = -M_2 \oplus C$ as $\mathcal{R}$-modules, where furthermore there are no non-trivial Massey-products between the summands. The key observation is the following.

Lemma 3.8. *We have the triple Massey products*

$$\langle \mathbf{z}, Q, Q^2 \rangle = q^2 \quad \text{and} \quad \langle V, \mathbf{z}, Q \rangle = v.$$

Proof. We use the Gysin sequence characterization of the triple Massey product, Theorem 2.2. The corresponding component in $\widehat{\mathrm{HM}}_\bullet$ is given by $\mathbb{F}[[U]] \oplus \mathbb{F}[[U]]/U^2$. In degrees ≥ -6, the Gysin sequence looks like
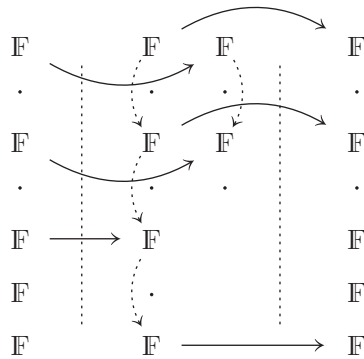

where the dotted arrows represent the U action. Consider the element $1 \in \mathbb{F}[[U]]$. Then comparing with the Gysin triangle in $\overline{\mathrm{HS}}_\bullet$, we see that $\pi_*(1) = \mathbf{z}$. Furthermore $V\iota_*(q^2) = \iota_*(q^2 v) = 0$, hence U is not in the image of ι_* and $\pi_*(U) = q^2$. The second statement follows in the same way. $\square$

In the case of general even $n = 2k$, we have that the module structure is given by the $\mathcal{R}$-module

$$(\mathbb{F}[V]\langle -4k \rangle \oplus \mathbb{F}[V]\langle -4k - 1 \rangle) \oplus \mathbb{F}[V]\langle -2 \rangle \oplus \mathbb{F}_0[V]/V^n$$

where the action of Q is injective from the first tower to the second, and from the second to the third. We will denote the direct sum of the first three terms by N_{2k}. Graphically, in the case $n = 4$, M_n is given by
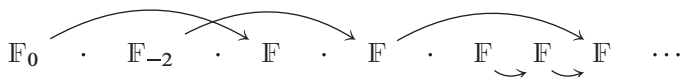
Denote the generators of the summands by q^2, v^n, qv^n and $\mathbf{z}$. We then have for example the relations

$$\langle \mathbf{z}, Q, Q^2 \rangle = q^2, \quad \langle V^n, \mathbf{z}, Q \rangle = v^n.$$

This follows in the same way as Lemma 3.8, using the fact that the corresponding component in $\widehat{\mathrm{HM}}_\bullet$ is given by $\mathbb{F}[U] \oplus \mathbb{F}[U]/U^{2k}$.

Seifert spaces.　Another large class of manifolds for which the Massey products described in Theorem 2.2 can be understood explicitly is given by Seifert spaces. The main observation here is that we can assume that all irreducible solutions have odd degree for a suitable choice of orientation (see [31], and also [3] for a discussion of the more general case of plumbed manifolds).
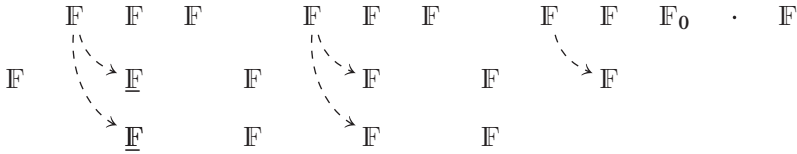
Example 3.9. Rather than describing a general theory (which would be analogous to parts of the content of [6]), let us focus on an interesting example (due to Duncan McCoy) that involves three or more $\mathbb{F}[[U]]$-summands. Consider the Seifert space $Y = \Sigma(13, 21, 34)$. Then, up to grading shifts, we have that

$$\widehat{\mathrm{HM}}_\bullet(Y) = \mathbb{F}[[U]]_0 \oplus (\mathbb{F}[[U]]/U^6)_{11} \oplus (\mathbb{F}[[U]]/U^5)_{11} \oplus (\mathbb{F}[[U]]/U^4)_9 \oplus J^{\oplus 2}$$

where the involution action exchanges the two copies of $J^{\oplus 2}$. As for this orientation there are only irreducible critical points of odd degree, it is straightforward to reconstruct the underlying chain complex $\widehat{C}_\bullet(Y)$ (where we forget about the $J^{\oplus 2}$ summand as it is irrelevant for our purposes):
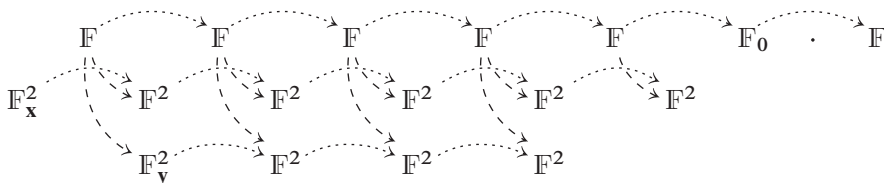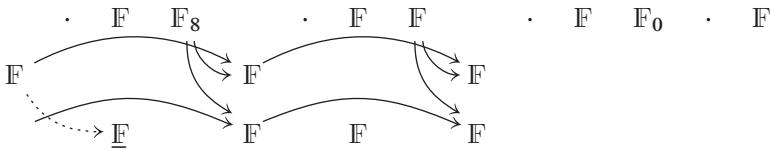
Here the first row represents the tower corresponding to the reducible solution, while the second and third rows correspond to the irreducible solutions; the dotted arrows depict the action of U, while the dashed ones represent the differential (where each dashed arrow sends the generator of $\mathbb{F}$ to the sum of the generators of $\mathbb{F}^2$). The natural involution $\jmath$ fixes the first row and exchanges the summands in each copy of $\mathbb{F}^2$. Also, we labeled the irreducible generators as $\mathbb{F}[[U]]$-modules by $\mathbf{x}$, $\mathbf{y}$ and their conjugates via $\jmath$. The invariant chain complex $\hat{C}^{\jmath}_{\bullet}(Y)$ is therefore

$$
\begin{array}{ccccccccccccc}
\mathbb{F} & \mathbb{F} & \mathbb{F} & & \mathbb{F} & \mathbb{F} & \mathbb{F} & & \mathbb{F} & \mathbb{F} & \mathbb{F}_0 & \cdot & \mathbb{F} \\
& & & & & & & & & & & & \\
\mathbb{F} & & \underline{\mathbb{F}} & & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & & \\
& & \underline{\mathbb{F}} & & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & & & &
\end{array}
$$

where the two underlined summands are generated by respectively $(1 + \jmath)U\mathbf{x}$ and $(1 + \jmath)\mathbf{y}$.

The Pin(2)-monopole Floer homology $\widehat{\mathrm{HS}}_{\bullet}(Y)$ is then

$$
\begin{array}{ccccccccccccc}
\cdot & \mathbb{F} & \mathbb{F}_8 & & \cdot & \mathbb{F} & \mathbb{F} & & \cdot & \mathbb{F} & \mathbb{F}_0 & \cdot & \mathbb{F} \\
\mathbb{F} & & & & & & \mathbb{F} & & & & \mathbb{F} & & \\
& & \underline{\mathbb{F}} & & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & &
\end{array}
$$

where the solid arrows denote the Q and V actions. From this description, and the fact that the Gysin exact sequence is the long exact sequence induced by the short exact sequence of chain complexes (6), one can determine the non-trivial Massey products. Let us spell out a specific example. Denote the underlined class in $\widehat{\mathrm{HS}}_{\bullet}(Y)$ by $\mathbf{z}$. It is represented by either $(1 + \jmath)U\mathbf{x}$ or $(1 + \jmath)\mathbf{y}$. The Massey product $\langle \mathbf{z}, Q^2, Q \rangle$ is by Theorem 2.2 a class mapping via ι_* to the class in $\widehat{\mathrm{HM}}_{\bullet}(Y)$ of either $(1 + \jmath)U^2\mathbf{x}$ or $(1 + \jmath)U\mathbf{y}$. Each of these generates one of the $\mathbb{F}$ summands in degree 7. Of course, once we quotient by the image of Q, they are identified, as their sum is the image of the class in degree 8. Even though involving a different definition, the Massey product $\langle \mathbf{z}, Q, Q^2 \rangle$ also consists of the same two classes, which are again identified under the image of Q^2. Finally, we leave to the reader to identify the dotted arrow with a Massey product of the form $\langle \cdot, Q, Q^2 \rangle$.

4. Some homological algebra over $\mathcal{R}$

In this section, we discuss some homological algebra relevant in the description of the E^2-page of our spectral sequence

$$
\mathrm{Tor}^{\mathcal{R}}_{*,*}(M, N)
$$

for a pair of graded $\mathcal{R}$-modules M and N. Recall from Section 1 that $\text{Tor}^{\mathcal{R}}_{*,*}$ arises in our setting naturally as the homology of the tensor product of M with the bar resolution of N. While the latter object has nice formal properties, it is quite unmanageable for actual explicit computations. As the computation of $\text{Tor}^{\mathcal{R}}_{*,*}$ is independent of the choice of projective resolution, we first discuss how to compute a particularly nice projective resolution of N, called *minimal free resolution*. As the name suggests, this will be very efficient in terms of size. On the other hand, when discussing higher differentials we will need to represent classes in E^2 as elements of the bar complex, and the second part of the section will be devoted to translating back in this language the construction using minimal resolutions.

While there is in general no satisfactory classification of finitely generated modules over $\mathcal{R}$, the theory of their resolutions is quite well understood, see for example [8] and [33] for a more general and detailed treatment. The ring $\mathcal{R}$ is local with maximal ideal $\mathfrak{m}$ generated by Q and V. Given a graded module L, we will denote by $\bar{L}$ the $\mathbb{F}$-vector space $L/\mathfrak{m}L$. We say that a graded $\mathcal{R}$-module homomorphism $u: L \to L'$ is *minimal* if

- u is surjective;

- $\ker(u) \subset \mathfrak{m}L$.

This is equivalent (by Nakayama's lemma) to requiring that the induced map $\bar{u}: \bar{L} \to \bar{L}'$ is an isomorphism. A *graded minimal free resolution* of a graded $\mathcal{R}$-module N is a graded resolution of the form

$$N \xleftarrow{d_0} \mathcal{R}^{n_1} \xleftarrow{d_1} \mathcal{R}^{n_2} \xleftarrow{d_2} \mathcal{R}^{n_3} \longleftarrow \cdots$$

where $d_i: \mathcal{R}^{n_i} \to \ker(d_{i-i}) \subset \mathcal{R}^{n_{i-1}}$ is minimal for each i. Here we omit from the notation the grading shift of each $\mathcal{R}$ component. The main result from [8] is the following.

Theorem 4.1. *Every finitely generated $\mathcal{R}$-module N admits a graded minimal free resolution. Any two minimal free resolutions are (non-canonically) isomorphic. Furthermore, minimal free resolutions are two-periodic for $i \geq 3$ (up to grading shift), and we have therefore we have the isomorphism*

$$\text{Tor}^{\mathcal{R}}_{i+2,k-3}(M, N) \cong \text{Tor}^{\mathcal{R}}_{i,k}(M, N)$$

for $i \geq 2$.

For our purposes, the existence statement is the most important, and we now quickly review the explicit construction. Notice first that for any module finitely generated module M, there is a minimal map $u: \mathcal{R}^n \to M$. This can be constructed by choosing a basis $\{\bar{e}_i\}$, $i = 1, \ldots, n$ of $\bar{M}$ over $\mathbb{F}$. Lifting these elements to $e_i \in M$ provides a minimal

$$u: \mathcal{R}^n \longrightarrow M.$$

Given now an $\mathcal{R}$-module N, choose a minimal map $d_0: \mathcal{R}^{n_0} \to N$. Inductively, we can choose a minimal map $d_i: \mathcal{R}^{n_i} \to \ker(d_{i-1})$, and these form a minimal resolution.

Let us comment about the rest of the statement. The two-periodicity of the minimal free resolution is a general consequence of the fact that we are considering the coordinate ring of a hypersurface, namely the zero set of the polynomial $(Q^3) \subset \mathbb{F}[[Q, V]]$, see [8] (notice that while several results in the paper do not hold for finite fields, the results about two-periodic resolutions in Section 5 and 6 hold). Furthermore, the dimension n_i is independent of $i \geq 3$. Therefore, to each module N we can associate a matrix factorization

$$\mathbb{F}[[Q, V]]^n \underset{B}{\overset{A}{\rightleftarrows}} \mathbb{F}[[Q, V]]^n$$

where A and B are the $n \times n$ matrices corresponding to d_{2i} and d_{2i+1} for $i \gg 0$, respectively. These have the property that $AB = BA = Q^3 \cdot I$.

Given this general discussion, let us provide some concrete examples in which we describe the minimal free resolution.

Example 4.2. The trivial $\mathcal{R}$-module $\mathbb{F}$ (thought of in degree zero) has a projective resolution
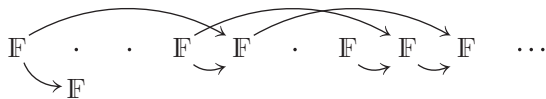
$$\mathbb{F} \xleftarrow{d_0} \mathcal{R} \xleftarrow{d_1} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_2} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_3} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_4} \cdots$$

where, in matrix notation, $d_1 = [V \quad Q]$, and for $i \geq 0$ we have

$$d_i = \begin{cases} \begin{bmatrix} Q & 0 \\ V & Q^2 \end{bmatrix} & \text{if } i \text{ is even,} \\ \begin{bmatrix} Q^2 & 0 \\ V & Q \end{bmatrix} & \text{if } i \text{ is odd.} \end{cases}$$

This is clearly two-periodic for $i \geq 2$.

Example 4.3. Consider the graded $\mathcal{R}$-module

This has the minimal free resolution

$$N \xleftarrow{d_0} \mathcal{R}^3 \xleftarrow{d_1} \mathcal{R}^4 \xleftarrow{d_2} \mathcal{R}^4 \xleftarrow{d_3} \mathcal{R}^4 \longleftarrow \cdots$$
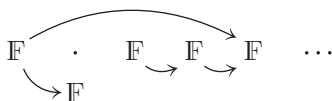
where

$$d_1 = \begin{bmatrix} Q & 0 & 0 & 0 \\ V & Q^2 & Q & 0 \\ 0 & 0 & V & Q^2 \end{bmatrix}$$

and for $i \geq 1$ we have

$$d_{2i} = \begin{bmatrix} Q^2 & 0 & 0 & 0 \\ V & Q & 0 & 0 \\ 0 & 0 & Q^2 & 0 \\ 0 & 0 & V & Q \end{bmatrix}, \quad d_{2i+1} = \begin{bmatrix} Q & 0 & 0 & 0 \\ V & Q^2 & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & V & Q^2 \end{bmatrix}.$$

Example 4.4. Consider the graded $\mathcal{R}$-module

This has the minimal free resolution

$$N \xleftarrow{d_0} \mathcal{R}^2 \xleftarrow{d_1} \mathcal{R}^2 \xleftarrow{d_2} \mathcal{R}^2 \xleftarrow{d_3} \mathcal{R}^2 \longleftarrow \cdots$$

where

$$d_1 = \begin{bmatrix} Q^2 & 0 \\ V & Q^2 \end{bmatrix}$$

and for $i \geq 1$

$$d_{2i} = \begin{bmatrix} Q^2 & 0 \\ V & Q \end{bmatrix}, \quad d_{2i+1} = \begin{bmatrix} Q & 0 \\ V & Q^2 \end{bmatrix}.$$

Example 4.5. Consider the module

This has the minimal free resolution

$$N \xleftarrow{d_0} \mathcal{R} \xleftarrow{d_1} \mathcal{R}^3 \xleftarrow{d_2} \mathcal{R}^3 \xleftarrow{d_3} \mathcal{R}^3 \longleftarrow \cdots$$

where

$$d_1 = \begin{bmatrix} V^2 & QV & Q^2 \end{bmatrix}$$

and for $i \geq 1$ we have

$$d_{2i} = \begin{bmatrix} Q & 0 & 0 \\ V & Q & 0 \\ 0 & V & Q \end{bmatrix}, \quad d_{2i+1} = \begin{bmatrix} Q^2 & 0 & 0 \\ QV & Q^2 & 0 \\ V^2 & QV & Q^2 \end{bmatrix}.$$

We now discuss how to represent classes in $\mathrm{Tor}_{*,*}^{\mathcal{R}}(M, N)$ in terms of the bar resolution. Let us start with the simplest case of $\mathrm{Tor}_{*,*}^{\mathcal{R}}(\mathbb{F}, \mathbb{F})$, which is well studied in light of the classical Eilenberg–Moore spectral sequence in algebraic topology (see for example [30]). In this case, recall that for graded algebras A and B over $\mathbb{F}$ there is an isomorphism of graded bimodules

$$\mathrm{Tor}_{*,*}^{A \otimes B}(\mathbb{F}, \mathbb{F}) = \mathrm{Tor}_{*,*}^{A}(\mathbb{F}, \mathbb{F}) \otimes \mathrm{Tor}_{*,*}^{B}(\mathbb{F}, \mathbb{F})$$

which is indeed also an isomorphism of coalgebras. Denoting the bar resolution by $\widehat{B}$, this is induced by the map

$$\widehat{B}(A) \otimes \widehat{B}(B) \xrightarrow{\mathrm{EZ}} \widehat{B}(A \otimes B).$$

The map EZ is the shuffle map appearing in the proof of the Eilenberg–Zilber theorem, namely

$$\mathrm{EZ}((a_1|\cdots|a_p) \otimes (b_1|\cdots|b_q)) = \sum_{(p,q)-\text{shuffle } \sigma} c_{\sigma(1)}|\cdots|c_{\sigma(p+q)}$$

where

$$c_{\sigma(i)} = \begin{cases} a_{\sigma(i)} \otimes 1 & \text{if } 1 \leq \sigma(i) \leq p, \\ 1 \otimes b_{\sigma(i)-p} & \text{if } p+1 \leq \sigma(i) \leq p+q, \end{cases}$$

and a (p, q)-shuffle is a permutation σ of $\{1, \ldots, p+q\}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p-1) < \sigma(p)$$
$$\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q-1) < \sigma(p+q).$$

The computation in our case is quite simple:

$$\mathrm{Tor}^{\mathcal{R}}_{*,*}(\mathbb{F},\mathbb{F}) = \mathrm{Tor}^{\mathbb{F}[[V]]}_{*,*}(\mathbb{F},\mathbb{F}) \otimes \mathrm{Tor}^{\mathbb{F}[Q]/Q^3}_{*,*}(\mathbb{F},\mathbb{F}).$$

The group $\mathrm{Tor}^{\mathbb{F}[[V]]}(\mathbb{F},\mathbb{F})$ is non zero only in bidegrees $(0,0)$ and $(1,-4)$ (generated by the empty product $[]$ and V respectively), while $\mathrm{Tor}^{\mathbb{F}[Q]/Q^3}(\mathbb{F},\mathbb{F})$ is non zero in degrees $(2i,-3i)$ and $(2i+1,-3i-1)$ for $i \geq 0$. If we define the n-tuple

$$Q_n = \begin{cases} (Q, Q^2, Q, \cdots, Q, Q^2, Q) & \text{if } n \text{ is odd,} \\ (Q^2, Q, Q^2, \cdots, Q, Q^2, Q) & \text{if } n \text{ is even,} \end{cases}$$

then the generator of $\mathrm{Tor}^{\mathbb{F}[Q]/Q^3}_{*,*}(\mathbb{F},\mathbb{F})$ in homological degree n is represented by Q_n. Putting these pieces together, we see that $\mathrm{Tor}^{\mathcal{R}}_{*,*}(\mathbb{F},\mathbb{F})$ is a copy of $\mathbb{F}$ in the cases in which (i,j) is

- $(0,0)$;

- $(2n,-3n)$ and $(2n,-3n-2)$ for $n \geq 0$;

- $(2n+1,-3n-1)$ and $(2n+1,-3n-4)$ for $n \geq 0$.

The generators of $\mathrm{Tor}^{\mathcal{R}}_{*,*}(\mathbb{F},\mathbb{F})$ can then be described explicitly in terms of the shuffle map. Given an ordered n-tuple $(a_1,\ldots,a_n)$ and an element b in $\mathcal{R}$, we define their shuffle as

$$\mathrm{sh}((a_1,\ldots,a_n),b) = \sum_{i=0}^{n} a_1|\cdots|a_i|b|a_{i+1}|\cdots|a_n \in \mathcal{R}^{\otimes(n+1)}.$$

The representatives of the classes described above are given respectively by

- $[\,]$;

- Q_{2n} and $\mathrm{sh}(Q_{2n-1}, V)$;

- Q_{2n+1} and $\mathrm{sh}(Q_{2n}, V)$.

The following picture represents the groups for $i \leq 7$, where the top left element has bigrading $(0,0)$. It should make apparent the two-periodicity of the $\mathrm{Tor}^{\mathcal{R}}_{*,*}(\mathbb{F},\mathbb{F})$:

$$
\begin{matrix}
\mathbb{F} & \cdot & & & & & & & & \\
\cdot & \mathbb{F} & & & & & & & & \\
 & \cdot & & & & & & & & \\
 & \cdot & \mathbb{F} & & & & & & & \\
\mathbb{F} & \cdot & \mathbb{F} & & & & & & & \\
 & \mathbb{F} & \cdot & & & & & & & \\
 & & \cdot & \mathbb{F} & & & & & & \\
 & & \mathbb{F} & \cdot & \mathbb{F} & & & & & \\
 & & & \mathbb{F} & \cdot & & & & & \\
 & & & & \cdot & \mathbb{F} & & & & \\
 & & & & \mathbb{F} & \cdot & & & & \\
 & & & & & \mathbb{F} & & & &
\end{matrix}
$$

For general $\mathrm{Tor}^{\mathcal{R}}_{*,*}(M, N)$, we can adapt this approach involving shuffle maps by taking into account a minimal resolution of one of the two modules. Consider the minimal free resolution

$$
N \xleftarrow{d_0} \mathcal{R}^{n_1} \xleftarrow{d_1} \mathcal{R}^{n_2} \xleftarrow{d_2} \mathcal{R}^{n_3} \xleftarrow{d_3} \mathcal{R}^{n_4} \longleftarrow \cdots ,
$$

so that $\mathrm{Tor}^{\mathcal{R}}_{*,*}(M, N)$ is the homology of the chain complex

$$
M^{n_1} \xleftarrow{1_M \otimes d_1} M^{n_2} \xleftarrow{1_M \otimes d_2} M^{n_3} \xleftarrow{1_M \otimes d_3} M^{n_4} \longleftarrow \cdots
$$

obtained by tensoring over $\mathcal{R}$ with M. Our goal is to define a canonical quasi-isomorphism $\{\varphi_i\}$ between the minimal free resolution

$$
\mathcal{R}^{n_1} \xleftarrow{d_1} \mathcal{R}^{n_2} \xleftarrow{d_2} \mathcal{R}^{n_3} \xleftarrow{d_3} \mathcal{R}^{n_4} \longleftarrow \cdots
$$

and the bar resolution

$$
\mathcal{R} \otimes N \xleftarrow{\delta_1} \mathcal{R} \otimes \mathcal{R} \otimes N \xleftarrow{\delta_2} \mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R} \otimes N \xleftarrow{\delta_3} \cdots
$$

Of course, the map

$$
\varphi_1 \colon \mathcal{R}^{n_1} \longrightarrow \mathcal{R} \otimes N
$$

is given by

$$
\mathbf{x} \longmapsto 1 \otimes d_0(\mathbf{x}).
$$

Suppose now inductively that we are given

$$\varphi_i \colon \mathcal{R}^{n_i} \longrightarrow \mathcal{R}^i \otimes N.$$

We have $d_{i+1}(e_j) = \sum r_{jk} e_k'$ where e_j and e_k' are the standard bases of $\mathcal{R}^{n_{i+1}}$ and $\mathcal{R}^{n_i}$ respectively. We then define

$$\varphi_{i+1} \colon \mathcal{R}^{n_{i+1}} \longrightarrow \mathcal{R}^{i+1} \otimes N,$$

$$e_j \longmapsto \sum 1 \otimes (r_{jk} \cdot \varphi_i(e_k')).$$

This is readily checked to be a chain map, and as the complexes are acyclic in degrees ≥ 1 it is a quasi-isomorphism. Using this quasi-isomorphism, one can describe elements in $\mathrm{Tor}^{\mathcal{R}}_{*,*}$ in terms of a minimal free resolution of N. Indeed, if $\underline{m} = (m_j) \in M^{n_k}$ is in the kernel of $1_M \otimes d_{k-1}$, then it corresponds to the cycle

$$\sum m_j \otimes \varphi_k(e_j) \in M \otimes_{\mathcal{R}} \mathcal{R}^k \otimes N = M \otimes \mathcal{R}^{k-1} \otimes N.$$

Let us discuss this rather abstract construction in a very concrete example.

Example 4.6. Let us generalize the description of $\mathrm{Tor}^{\mathcal{R}}_{*,*}(\mathbb{F}, \mathbb{F})$ in terms of shuffles to $\mathrm{Tor}^{\mathcal{R}}_{*,*}(M, \mathbb{F})$, for any $\mathcal{R}$-module M. We computed above that the minimal free resolution of $\mathbb{F}$ is given by

$$\mathbb{F} \xleftarrow{d_0} \mathcal{R} \xleftarrow{d_1} \mathcal{R}^2 \xleftarrow{d_2} \mathcal{R}^2 \xleftarrow{d_3} \mathcal{R}^2 \xleftarrow{d_4} \cdots,$$

and using the description above one can write explicit representatives for all the cycles in $\mathrm{Tor}^{\mathcal{R}}_{*,*}(M, \mathbb{F})$ as follows. Let us denote by $\mathbf{z}$ the generator of $\mathbb{F}$. For $n \geq 1$, every element of $\mathrm{Tor}^{\mathcal{R}}_{n,*}(M, \mathbb{F})$ has a representative of the form

$$\mathbf{x}|\mathrm{sh}(Q_{n-1}, V)|\mathbf{z} + \mathbf{y}|Q_n|\mathbf{z}$$

where $d_{n-1}(\mathbf{x}, \mathbf{y}) = 0$. For example:

- every element in $\mathrm{Tor}^{\mathcal{R}}_{1,*}(M, \mathbb{F})$ has a representative of the form

$$\mathbf{x}|V|\mathbf{z} + \mathbf{y}|Q|\mathbf{z}$$

  with $V\mathbf{x} + Q\mathbf{y} = 0$, and such an element is zero if and only if $\mathbf{x} = Q\mathbf{a}$ and $\mathbf{y} = V\mathbf{a} + Q^2\mathbf{b}$ for some $\mathbf{a}, \mathbf{b}$;

- every element in $\mathrm{Tor}^{\mathcal{R}}_{2,*}(M, \mathbb{F})$ is represented by

$$\mathbf{a}|Q|V|\mathbf{z} + \mathbf{a}|V|Q|\mathbf{z} + \mathbf{a}|Q^2|Q|\mathbf{z}$$

  where $Q\mathbf{a} = 0$ and $V\mathbf{a} + Q^2\mathbf{b} = 0$, and such an element is zero if and only if $\mathbf{a} = Q^2\mathbf{x}$ and $\mathbf{b} = V\mathbf{x} + Q\mathbf{y}$ for some $\mathbf{x}, \mathbf{y}$;

- every element in $\text{Tor}^{\mathcal{R}}_{3,*}(M, \mathbb{F})$ is represented by

$$\mathbf{x}|Q^2|Q|V|\mathbf{z} + \mathbf{x}|Q^2|V|Q|\mathbf{z} + \mathbf{x}|V|Q^2|Q|\mathbf{z} + \mathbf{y}|Q|Q^2|Q|\mathbf{z},$$

  with $Q^2\mathbf{x} = V\mathbf{x} + Q\mathbf{y} = 0$, and such an element is zero if and only if $\mathbf{x} = Q\mathbf{a}$ and $\mathbf{y} = V\mathbf{a} + Q^2\mathbf{b}$ for some $\mathbf{a}, \mathbf{b}$.

The description then can be generalized to $n \geq 4$ in a two-periodic fashion.

5. Connected sums with manifolds of simple type

In this section we study the effect on Floer homology of the connected sum with a manifold of simple type M_n or its opposite $-M_n$, see Definition 3.1. Of course, if we are interested only in the information related to homology cobordism contained in $\widehat{\text{HS}}_\bullet$, we do not need to consider the additional summand J in Definition 3.1. We know from the previous section the general recipe to compute the E^2 page of the Eilenberg–Moore spectral sequence, and the goal of this section is to understand higher differentials and extensions. There are several different cases to discuss, and our treatment will combine general results with explicit examples. Before dwelling in our main cases of interest, let us discuss a warm up example.

Example 5.1. Suppose we have an $\mathcal{R}$-module decomposition $\widehat{\text{HS}}_\bullet(Y) = M \oplus \mathbb{F}$ without non-trivial Massey products among the two summands. We want to understand the contribution to $\widehat{\text{HS}}_\bullet(Y \# Y)$ of $\text{Tor}^{\mathcal{R}}_{*,*}(\mathbb{F}, \mathbb{F})$. The latter was described in detail in Section 4. As d_2 has bidegree $(-2, 1)$, we see that the only possible non-trivial d_2 differentials are from an $\mathbb{F}$ summand in bidegree $(2n, -3n)$ to a $\mathbb{F}$ summand in bidegree $(2n - 2, -3n + 1)$ for $n \geq 2$. On the other hand, the former is generated by
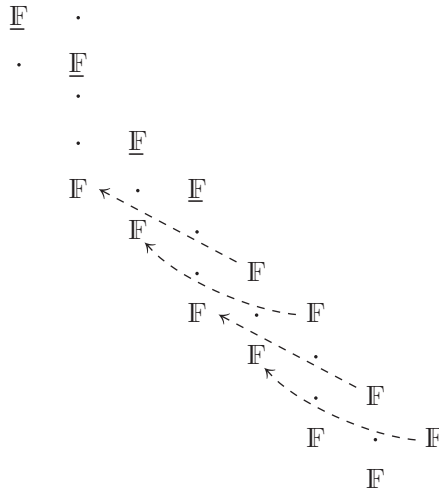
$$Q_{2n} = Q^2|Q|\cdots|Q^2|Q,$$

and we have that $d_2(Q_{2n}) = 0$. This is a direct consequence of the description of d_2 in Lemma 1.4 and the fact that

$$\langle Q, Q^2, Q \rangle = \langle Q^2, Q, Q^2 \rangle = 0,$$

which follows from Theorem 2.2 (see also Example 1.8). Regarding d_3, the natural generalization of Lemma 1.4 describes it in terms of four-fold Massey products; and the products $\langle Q, Q^2, Q, Q^2 \rangle = \langle Q^2, Q, Q^2, Q \rangle = V$ implies that we have the differentials

$$d_3(Q_i) = \text{sh}(Q_{i-4}, V)$$

for $i \geq 5$. Graphically, we see the differentials

$$
\begin{array}{ccccccc}
\mathbb{F} & & \cdot & & & & \\
& \cdot & & \mathbb{F} & & & \\
& & \cdot & & & & \\
& & & \cdot & & \mathbb{F} & \\
\mathbb{F} & \nwarrow & \cdot & & \underline{\mathbb{F}} & & \\
& & \mathbb{F} & & \cdot & & \\
& & \nwarrow & & & \cdot & \mathbb{F} \\
& & & & \mathbb{F} & \nwarrow & \cdot & \mathbb{F} \\
& & & & \mathbb{F} & & \cdot & \\
& & & & & & \mathbb{F} & \\
& & & & & & \mathbb{F}
\end{array}
$$

repeating in a two-periodic fashion. This implies that the spectral sequence collapses at the E^4 page, and (as there is no space for extensions), the final $\mathcal{R}$-module is a copy of $\mathbb{F}^2$ in degrees 0 and -1, corresponding to the underlined $\mathbb{F}$-summands.

Connected sum with M_{2k}. We observe that the $\mathcal{R}$-module M_{2k} has a very nice 2-step graded minimal free resolution

$$
M_{2k} \xleftarrow{d_0} \mathcal{R}\langle 4k-1\rangle \oplus \mathcal{R} \xleftarrow{d_1} \mathcal{R}\langle -1\rangle
$$

where d_0 sends, in the notation introduced in Section 3,

$$
(1, 0) \longmapsto qv^{-k},
$$
$$
(0, 1) \longmapsto 1,
$$

and d_1 is given in matrix notation by the matrix

$$
d_1 = \begin{bmatrix} V^k \\ Q \end{bmatrix}.
$$

In particular, the E^2-page of the Eilenberg–Moore spectral sequence for the connected sum $Y \# M_{2k}$ is supported on the first two columns; this implies that there are no higher differentials, so that $E^\infty \cong E^2$, and all we need to understand is the extension problem. Recall furthermore that

$$
\operatorname{Tor}^{\mathcal{R}}_{0,*}(\widehat{\mathrm{HS}}_\bullet(Y), M_{2k}) = \widehat{\mathrm{HS}}_\bullet(Y) \otimes_{\mathcal{R}} M_{2k},
$$

and, by the discussion in the previous section, $\text{Tor}_{1,*}^{\mathcal{R}}(\widehat{\text{HS}}_\bullet(Y), M_k)$ is in bijection with elements $\mathbf{x} \in \ker(d_1) = \ker V^k \cap \ker Q$ via the assignment

$$\mathbf{x} \to \mathbf{x}|V^k|qv^{-k} + \mathbf{x}|Q|1.$$

Hence, we need to understand the action of $\mathcal{R}$ on such an element. We have the following.

Proposition 5.2. *In the setup above, we have the identity*

$$Q \cdot (\mathbf{x}|V^k|qv^{-k} + \mathbf{x}|Q|1) = \langle Q, \mathbf{x}, V^k \rangle |qv^{-k} \in E_{0,*}^\infty.$$

If $V\mathbf{x} \neq 0$*, we have*

$$V \cdot (\mathbf{x}|V^k|qv^{-k} + \mathbf{x}|Q|1) = V\mathbf{x}|V^k|qv^{-k} + V\mathbf{x}|Q|1 \in E_{1,*}^\infty.$$

while if $V\mathbf{x} = 0$ *we have*

$$V \cdot (\mathbf{x}|V^k|qv^{-k} + \mathbf{x}|Q|1) = \langle V, \mathbf{x}, Q \rangle |1 \in E_{0,*}^\infty.$$

This implies that the $\mathcal{R}$-module structure of $Y\#M_{2k}$ is determined entirely by the triple Massey products of the form $\langle Q, \mathbf{x}, V^k \rangle$, which we have described in Theorem 2.2 in terms of the Gysin exact triangle. Let us discuss a simple example (see also Section 6 for more examples).
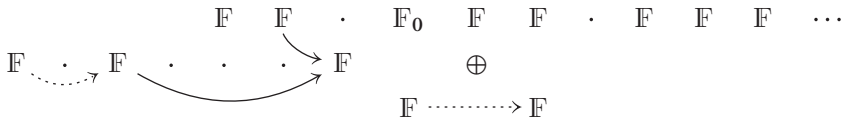
Example 5.3. Let us compute the homology of $-M_2\#M_4$, using the result above. The E^2 page is computed to be, graphically,
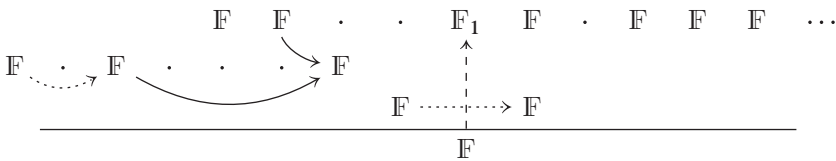
Above the line, we have depicted $\text{Tor}_{0,*}^{\mathcal{R}}(-M_2, M_4) = (-M_2) \otimes_{\mathcal{R}} M_4$. The first row, which is generated over $\mathcal{R}$ by $v|qv^{-2}$ and $v|1$, consists of based elements. The first two elements in the second row are $\mathbf{z}|qv^{-2}$ and $q^2|qv^{-2}$; we have depicted with a dotted arrow the Massey product relating them (whose existence follows from Lemma 3.8). The solid arrows represent the non obvious $\mathcal{R}$-actions. Under the line, we represented $\text{Tor}_{1,*}^{\mathcal{R}}(-M_2, M_4)$, which, by the lemma above, corresponds to $\ker V^k \cap \ker Q = \{\mathbf{z}\}$. It is represented by the element $\mathbf{z}|V^2|qv^{-2} + \mathbf{z}|Q|1$, and its image under the action of Q is

$$Q \cdot (\mathbf{z}|V^2|qv^{-2} + \mathbf{z}|Q|1) = \langle Q, \mathbf{z}, V^2 \rangle |qv^{-2} = v^2|qv^{-2} = v|qv^{-1},$$

as depicted by the dashed arrow. To sum up, the final result is

$$
\begin{array}{ccccccccccc}
\mathbb{F} & \mathbb{F} & \cdot & \mathbb{F}_0 & \mathbb{F} & \mathbb{F} & \cdot & \mathbb{F} & \mathbb{F} & \mathbb{F} & \cdots \\
\mathbb{F} \cdots \nearrow \mathbb{F} & \cdot & \cdot & \cdot \nearrow \mathbb{F} & & \oplus & & & & \\
& & & \mathbb{F} \cdots \dashrightarrow \mathbb{F} & & & & & & \\
\end{array}
$$

and in particular $\alpha = \beta = 2$, $\gamma = 0$.

In fact, we have the following more general observation.

Corollary 5.4. *The Manolescu correction terms of $Y \# M_k$ are determined the $\mathcal{R}$-module structure of $\widehat{\mathrm{HS}}_\bullet(-Y)$.*

In fact, the proof of the corollary implies that one can in principle write a (not particularly illuminating) formula for the correction terms of $Y \# M_k$ in terms of the $\mathcal{R}$-module structure of $\widehat{\mathrm{HS}}_\bullet(-Y)$. This is again a manifestation of the fact that one can use Poincaré duality and the long exact sequence (2) to relate $\widehat{\mathrm{HS}}_\bullet(Y)$ and $\widehat{\mathrm{HS}}_\bullet(-Y)$, see Lemma 2.7.

Proof. We need to interpret the statement:

(a) there exists $\mathbf{x} \in \widehat{\mathrm{HS}}_m(Y)$ for which $\mathbf{y} = \langle Q, \mathbf{x}, V^k \rangle \neq 0$ is based

purely in terms of the $\mathcal{R}$-module structure of $\widehat{\mathrm{HS}}_\bullet(-Y)$. Let us point out first that, as $\mathbf{y}$ is defined up to the image of Q and V^k, this implies that there are no based element such that its image under Q or V^k is in the same grading as $\mathbf{y}$. Now of course $p_*(\mathbf{x}) = 0$, so we have $\mathbf{x} = j_*(\mathbf{x}')$ for $\mathbf{x}' \in \widetilde{\mathrm{HS}}_\bullet(Y)$. If both $Q\mathbf{x}' = V^k\mathbf{x}' = 0$, then we would have by naturality of Massey products

$$
\mathbf{y} = \langle Q, \mathbf{x}, V^k \rangle = \langle Q, j_*(\mathbf{x}'), V^k \rangle = j_*(\langle Q, \mathbf{x}', V^k \rangle),
$$

so $\mathbf{y}$ would not be based. This implies that one of the two products is non-zero, and in fact by exactness of the Gysin sequence it a non-zero element in a tower. In fact, up to adding to $\mathbf{x}'$ the element in the tower in its same degree, we have shown that (a) implies the following:

(b) there exists $\mathbf{x}'$ in $\widetilde{\mathrm{HS}}_m(Y)$ such that $V^k\mathbf{x}' = 0$ and $Q\mathbf{x}'$ belongs to the tower.

In fact, it is easy to show that the reverse implication also holds. As this condition might seem slightly obscure, let us point out a simple instance of it. Suppose $\widehat{\mathrm{HS}}_\bullet(Y)$ contains $\mathbf{x}$ with $\mathbf{y} = \langle Q, \mathbf{x}, V \rangle \neq 0$ is $Q \cdot \mathcal{V}$-based. We represent this schematically as
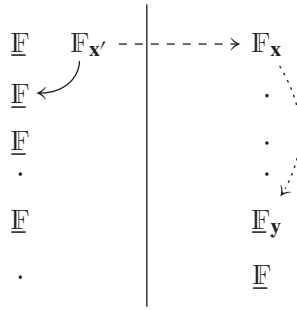
Here, the left part of the diagram represents $\widetilde{\mathrm{HS}}_\bullet(Y)$, the right part of the diagram represents $\widehat{\mathrm{HS}}_\bullet(Y)$, and the dashed arrow is the map j_*. The underlined elements represent respectively the tower and the based elements, and the dashed arrow represents the Massey product $\mathbf{y} = \langle Q, \mathbf{x}, V \rangle$. From the picture it should be clear that the fact that $V\mathbf{x}' = 0$ but $Q\mathbf{x}'$ is a non-zero element in the tower is a non-trivial constraint on the $\mathcal{R}$-module structure of $\widehat{\mathrm{HS}}_\bullet(Y)$; this is because element in the tower in the same degree is acted on non-trivially by V.

Finally, using that $\widetilde{\mathrm{HS}}^\bullet(Y) = \widehat{\mathrm{HS}}_{-1-\bullet}(-Y)$ is the dual $\mathcal{R}$-module of $\widetilde{\mathrm{HS}}_\bullet(Y)$, condition (b) can be rephrased purely in terms of the $\mathcal{R}$-module structure of $\widehat{\mathrm{HS}}_\bullet(-Y)$. For example, in the concrete example above this corresponds to the following: there exists an Q^2V-based element in $\widehat{\mathrm{HS}}_{-1-m}(-Y)$ which is in the image of V but not in the image of Q. □

The key computations behind Proposition 5.2 are the following two general lemmas.

Lemma 5.5. *Suppose* $\mathbf{x}r = \mathbf{x}s = t\mathbf{x} = 0$ *and* $r\mathbf{y} + s\mathbf{z} = 0$, *and consider the element* $\mathbf{x}|r|\mathbf{y} + \mathbf{x}|s|\mathbf{z} \in \mathrm{Tor}^\mathcal{R}_{1,*}$. *If it survives in the* E^∞-*page, multiplication by* t *sends it to* $\langle t, \mathbf{x}, r \rangle|\mathbf{y} + \langle r, \mathbf{x}, s \rangle|\mathbf{z}$.

Proof. Suppose we have fixed cycles representing the homology classes $\mathbf{x}, \mathbf{y}, \mathbf{z}$, r, s, which we denote with the same letter. Choose chains a, b, c such that

$$\partial a = \mathbf{x}r, \quad \partial b = \mathbf{x}s, \quad \partial c = r\mathbf{y} + s\mathbf{z}, \quad \partial d = t\mathbf{x}.$$

Then

$$\mathbf{x}|r|\mathbf{y} + \mathbf{x}|s|\mathbf{z} + a|\mathbf{y} + b|\mathbf{z} + \mathbf{x}|c$$

is a cycle in the $\mathcal{A}_\infty$-tensor product whose image in the E^2-page is the class $\mathbf{x}|r|\mathbf{y} + \mathbf{x}|s|\mathbf{z}$. By definition, the action of t on it is given by the cycle

$$t\mathbf{x}|r|\mathbf{y} + t\mathbf{x}|s|\mathbf{z} + ta|\mathbf{y} + tb|\mathbf{z} + t\mathbf{x}|c + \hat{m}_3(t|\mathbf{x}|r)|\mathbf{y} + \hat{m}_3(t|\mathbf{x}|s)|\mathbf{z}.$$

Now, we have the identities

$$\partial(d\,|r\,|\mathbf{y} + d\,|s\,|\mathbf{z}) = t\mathbf{x}|r\,|\mathbf{y} + t\mathbf{x}|s|\mathbf{z} + dr\,|\mathbf{y} + d\,|r\mathbf{y} + ds|\mathbf{z} + d\,|s\mathbf{z},$$
$$\partial(d\,|c) = t\mathbf{x}|c + d\,|r\mathbf{y} + d\,|s\mathbf{z},$$

so that summing all the three equations we see that a representative of the action by t is

$$(ta + dr + \hat{m}_3(t\,|\mathbf{x}|r))|\mathbf{y} + (tb + ds + \hat{m}_3(t\,|\mathbf{x}|s))|\mathbf{z},$$

hence the result.   □

The second lemma is the following.

Lemma 5.6. *Suppose that $r\mathbf{x} = 0$. Then $\langle r, \mathbf{x}, r \rangle = 0$.*

The proof of this lemma is a special case of a more general result involving generalized Massey products that arise when studying higher differentials of elements which are not represented by simple tensors (see Lemma 1.4 and the following discussion). The simplest case is the following. Consider classes $\mathbf{x}$ and r, s with $\mathbf{x}r = \mathbf{x}s = 0$, where we do *not* assume $rs = 0$. Choose chains a, b, c such that

$$\partial a = \mathbf{x}r, \quad \partial b = \mathbf{x}s, \quad \partial c = rs + sr.$$

Then the following expression

$$as + br + \mathbf{x}c + \hat{m}_3(\mathbf{x}|s|r) + \hat{m}_3(\mathbf{x}|r|s) \tag{14}$$

is a cycle, and we define its homology class to be the *generalized Massey product* $\langle \mathbf{x}|r, s \rangle$. We then have the following.

Lemma 5.7. *Given classes $\mathbf{x}, r$ and s as above, we have $\langle \mathbf{x}|r, s \rangle = \langle r, \mathbf{x}, s \rangle$.*

For example, for $-M_2$ the identity

$$\langle \mathbf{z}|Q, V \rangle = \langle Q, \mathbf{z}, V \rangle = q^2$$

holds, see Lemma 3.8.

Proof. Let us first spell out some details implicit in the construction of the $\mathcal{A}_\infty$-bimodule structure in [23]. Suppose the family of metrics and perturbations has been chosen so that the $\mathcal{A}_\infty$-module structure $\{\hat{m}_n\}$ is defined (this takes as input an embedded ball in Y). The $\mathcal{A}_\infty$-bimodule structure takes as input a second embedded ball, disjoint from the first. Then, by suitably pulling back via a family of isotopies (as in the construction of [23]) the metric used to define $\{\hat{m}_n\}$ for our new

data (defined by the second embedded ball), we see that we can assume that our multiplication satisfies $\hat{m}_{2,1}(d\,|\mathbf{y}) = \hat{m}_{1,2}(\mathbf{y}|d)$ for all choices of $d \in \hat{C}^J_\bullet(S^3)$ and $\mathbf{y} \in \hat{C}^J_\bullet(Y)$. Notice that this does not imply that the multiplication μ_2 on $\hat{C}^J_\bullet(S^3)$ is commutative at the chain level (rather than just commutative up to homotopy), as the data on S^3 has been fixed a priori. The implication for our purposes is that, once this choice of data is made, we have that $\langle r, \mathbf{x}, s \rangle$ is represented by

$$as + rb + \hat{m}_3(r\,|\mathbf{x}|s),$$

where we are using the notation introduced above. To show that this cycle is cobordant to the one in equation (14), let us consider a family of metrics and perturbations on the manifold with cylindrical ends

$$(I \times Y \setminus (\operatorname{int} D^4 \amalg \operatorname{int} D^4))^*$$

parameterized by a hexagon as in Figure 2, and consider the chain obtained by taking fibered products on the incoming end with $\mathbf{x}, r$ and s. We provide a sketchy description of the metrics and perturbations involved - the details of the construction are very similar to those in [23]. The thick edges of the hexagon $\mathcal{H}$ correspond to stretching along the three pairs of hypersurfaces on the right of the figure, and taking fibered products one obtains the chains $\hat{m}_3(\mathbf{x}|r|s)$, $\hat{m}_3(\mathbf{x}|s|r)$ and $\hat{m}_{2,2}(r|\mathbf{x}|s)$ respectively. The top thin edge corresponds to a metric in which the top hypersurface is stretched to infinity, and we perform a chain homotopy realizing the commutativity of μ_2 on $\hat{C}^J_\bullet(S^3)$; the corresponding chain is c. The bottom thin lines correspond to a chain homotopy between $\hat{m}_{2,1}$ and $\hat{m}_{1,2}$ with one of the diagonal hypersurfaces stretched to infinity; as discussed above we can choose such a chain homotopy to be induced by an isotopy of metrics, so that the chains in consideration will just be $I \times (\mathbf{x}r)s$ and $I \times (\mathbf{x}s)r$, hence they are zero in our chain complex (as they are small, see Section 1). The boundary of the hexagon can be filled with a family of metrics and perturbations of the manifold with cylindrical ends, as the corresponding space is contractible; taking the fibered product with the moduli spaces parameterized by $\mathcal{H}$, we obtain a chain whose boundary (from the discussion above) is $\hat{m}_3(\mathbf{x}|r|s) + \hat{m}_3(\mathbf{x}|s|r) + \hat{m}_{2,2}(r|\mathbf{x}|s) + \mathbf{x}c$ and the result follows. $\qquad\square$

Proof of Proposition 5.2. Given the lemmas above, Proposition 5.2 follows immediately, with the additional observation for the last point that when $V\mathbf{x} = 0$, $\mathbf{x}|V^k|qv^{-k} + \mathbf{x}|Q|1$ is also represented by $\mathbf{x}|V|qv^{-1} + \mathbf{x}|Q|1$, as they differ by the boundary of $\mathbf{x}|V|V^{k-1}|qv^{-k}$. $\qquad\square$
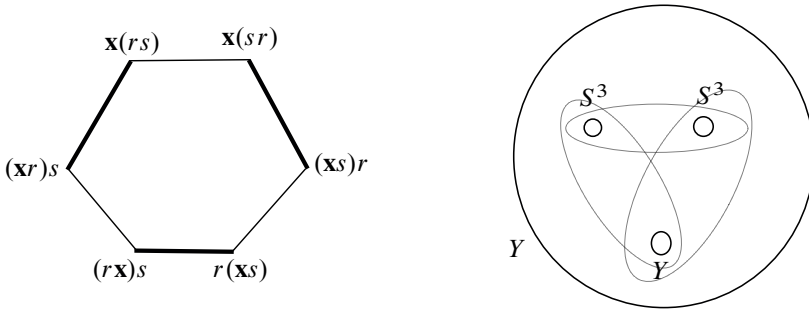
Figure 2. On the left, the hexagon $\mathcal{H}$ parameterizing the family of metrics and perturbations, where we have denoted the strata corresponding to each corner. On the right, we have depicted three hypersurfaces in a doubly punctured $I \times Y$; the top one is a copy of S^3 while the diagonal ones are copy of Y. Each pair determines a one parameter family of metrics and perturbations defining higher compositions $\hat{m}_3$ or $\hat{m}_{2,2}$.

Connected sums with $-M_{2k}$. We now discuss the effect of connected sums with manifolds of type $-M_{2k}$, i.e. manifolds obtained by manifolds of type M_{2k} by reversing the orientation (see Section 3). Of course, the correction terms of $Y \# - M_{2k}$ are determined by the correction terms of $-(Y \# - M_{2k}) = -Y \# M_{2k}$, so that Corollary 5.4 implies in turn that they are determined entirely by the module structure of $\widehat{\mathrm{HS}}_\bullet(Y)$. We will explain this fact by a direct inspection of the Eilenberg–Moore spectral sequence.

For simplicity, we will start with the case $k = 1$, in which case $-M_2 = \mathbb{F} \oplus N_2$ as $\mathcal{R}$-modules. A convenient projective resolution for the trivial module $\mathbb{F}$ was defined in Section 4. The module N also admits a simple two-periodic projective resolution

$$N_2 \xleftarrow{d_0} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_1} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_2} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_3} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_4} \cdots$$

where, in matrix notation, for $i \geq 0$ we have

$$d_i = \begin{cases} \begin{bmatrix} Q & 0 \\ V & Q^2 \end{bmatrix} & \text{if } i \text{ is even,} \\[2em] \begin{bmatrix} Q^2 & 0 \\ V & Q \end{bmatrix} & \text{if } i \text{ is odd.} \end{cases}$$

Here d_0 sends $(1, 0)$ to v and $(0, 1)$ to q^2.

As usual $\text{Tor}_{0,*}^{\mathcal{R}}(M, N_2) = M \otimes_{\mathcal{R}} N_2$. Then the general description of generators of $\text{Tor}_{i,*}^{\mathcal{R}}$, as given in Section 4, specializes to our case as follows. Define

$$\tilde{Q}_n = \begin{cases} (Q^2, Q, \cdots, Q, Q^2) & \text{if } n \text{ is odd,} \\ (Q, Q^2, \cdots, Q, Q^2) & \text{if } n \text{ is even.} \end{cases}$$

Lemma 5.8. *Let $n \geq 1$. Then every element of $\text{Tor}_{n,*}^{\mathcal{R}}(M, N_2)$ has a representative of the form*

$$\mathbf{x}|\tilde{Q}_n|v + \mathbf{x}|\text{sh}(Q_{n-1}, V)|q^2 + \mathbf{y}|Q_n|q^2$$

where $d_n(\mathbf{x}, \mathbf{y}) = 0$.

The elements in $\text{Tor}_{*,*}^{\mathcal{R}}(M, \mathbb{F})$ were described in detail in Example 4.6. Consider first an element in $\text{Tor}_{2,*}^{\mathcal{R}}$, which has the form

$$\alpha = \mathbf{a}|Q|V|\mathbf{z} + \mathbf{a}|V|Q|\mathbf{z} + \mathbf{b}|Q^2|Q|\mathbf{z}$$

where $Q\mathbf{a} = 0$ and $V\mathbf{a} + Q^2\mathbf{b} = 0$. Such an element is zero if and only if $\mathbf{a} = Q^2\mathbf{x}$ and $\mathbf{b} = V\mathbf{x} + Q\mathbf{y}$. Given elements $\mathbf{a}, \mathbf{b}$ satisfying $Q\mathbf{a} = 0$ and $V\mathbf{a} + Q^2\mathbf{b} = 0$, we can define their generalized triple Massey product $\phi(\mathbf{a}, \mathbf{b})$ in the same way as equation (14). More explicitly, suppose we have chosen chain representatives, and choose chains r, s such that $\partial r = Qa$ and $\partial s = Va + Q^2b$. Then $\phi(\mathbf{a}, \mathbf{b})$ is the class of

$$Vr + Qs + \hat{m}_3(Q|V|a) + \hat{m}_3(V|Q|a) + \hat{m}_3(Q|Q^2|b).$$

To see that this is a cycle, recall that the product of Q and Q^2 is zero at the chain level. Then the differential on the E^2-page is identified with

$$\begin{aligned} d_2(\alpha) &= \phi(\mathbf{a}, \mathbf{b})|\mathbf{z} + \mathbf{a}|(\langle Q, V|\mathbf{z}\rangle) + \mathbf{b}|\langle Q^2, Q, \mathbf{z}\rangle \\ &= \phi(\mathbf{a}, \mathbf{b})|\mathbf{z} + \mathbf{a}|v + \mathbf{b}|q^2. \end{aligned}$$

where in the last row we used Lemmas 3.8 and 5.7. This easily generalizes to the case of elements in $\text{Tor}_{n,*}^{\mathcal{R}}$ with n even. Specializing the discussion of Section 4, a general element is of the form

$$\alpha = \mathbf{a}|\text{sh}(Q_{n-1}, V)|\mathbf{z} + \mathbf{b}|Q_n|\mathbf{z}$$

where $Q\mathbf{a} = 0$ and $V\mathbf{a} + Q^2\mathbf{b} = 0$, and such an element is zero if and only if $\mathbf{a} = Q^2\mathbf{x}$ and $\mathbf{b} = V\mathbf{x} + Q\mathbf{y}$. Its differential is then given by

$$d_2(\alpha) = \langle \mathbf{a}, Q, Q^2 \rangle |\mathrm{sh}(Q_{n-3}, V)|\mathbf{z} + \phi(\mathbf{a}, \mathbf{b})|Q_{n-2}|\mathbf{z}$$
$$+ \mathbf{a}|\widetilde{Q}_{n-2}|v + \mathbf{a}|\mathrm{sh}(Q_{n-3}, V)|q^2 + \mathbf{b}|Q_{n-2}|q^2. \tag{15}$$

The key observation to have in mind is that this differential naturally decomposes in two parts: the one in the first row, which defines an element in $\mathrm{Tor}^{\mathcal{R}}_{n-2,*}(M, \mathbb{F})$, and the one in the second row in $\mathrm{Tor}^{\mathcal{R}}_{n-2,*}(M, N_2)$. In fact, the latter is the element corresponding to the pair $(\mathbf{a}, \mathbf{b})$ from Lemma 5.8.

An analogous description holds to the odd case too (provided of course that $n \geq 3$). For the sake of clarity, we first write down explicitly the case of $\mathrm{Tor}^{\mathcal{R}}_{3,*}(M, \mathbb{F})$. An element in this group has the form

$$\beta = \mathbf{x}|Q^2|Q|V|\mathbf{z} + \mathbf{x}|Q^2|V|Q|\mathbf{z} + \mathbf{x}|V|Q^2|Q|\mathbf{z} + \mathbf{y}|Q|Q^2|Q|\mathbf{z}$$

where $Q^2\mathbf{x} = 0$ and $V\mathbf{x} + Q\mathbf{y} = 0$. Such an element is zero if and only if $\mathbf{x} = Q\mathbf{a}$ and $\mathbf{y} = V\mathbf{a} + Q^2\mathbf{b}$. To a pair $\mathbf{x}, \mathbf{y}$ like this we can assign a generalized triple Massey product as in equation (14). We then have (using again the computation in Lemma 3.8)

$$d_2(\beta) = \langle \mathbf{x}, Q^2, Q \rangle |V|\mathbf{z} + \psi(\mathbf{x}, \mathbf{y})|Q|\mathbf{z} + \mathbf{x}|Q^2|v + \mathbf{x}|V|q^2 + \mathbf{y}|Q|q^2.$$

In the general case for n odd, we have that the general element of $\mathrm{Tor}^{\mathcal{R}}_{n,*}(M, \mathbb{F})$ is

$$\beta = \mathbf{x}|\mathrm{sh}(Q_{n-1}, V)|\mathbf{z} + \mathbf{y}|Q_n|\mathbf{z},$$

where $\mathbf{x}$ and $\mathbf{y}$ are as above. We then have

$$d_2(\beta) = \langle \mathbf{x}, Q^2, Q \rangle |\mathrm{sh}(Q_{n-3}, V)|\mathbf{z} + \psi(\mathbf{x}, \mathbf{y})|Q_{n-2}|\mathbf{z}$$
$$+ \mathbf{x}|\widetilde{Q}_{n-2}|v + \mathbf{x}|\mathrm{sh}(Q_{n-3}, V)|q^2 + \mathbf{y}|Q_{n-2}|q^2. \tag{16}$$

Again, the first row is an element in $\mathrm{Tor}^{\mathcal{R}}_{n-2,*}(M, \mathbb{F})$, and the second row is the element in $\mathrm{Tor}^{\mathcal{R}}_{n-2,*}(M, N_2)$ corresponding to the pair $(\mathbf{x}, \mathbf{y})$.

With this discussion in hand, we are ready to prove the following.

Proposition 5.9. *The Eilenberg–Moore spectral sequence for the connected sum with $-M_2$ collapses at the E^3-page. Furthermore, $E^{\infty}_{i,*} = 0$ for $i \geq 2$, and all elements in $E^{\infty}_{1,*}$ have the form $\mathbf{x}|V|\mathbf{z} + \mathbf{y}|Q|\mathbf{z}$ for pairs $\mathbf{x}, \mathbf{y}$ of elements in M such that $V\mathbf{x} + Q\mathbf{y} = 0$. The action of V and Q on such an element are given respectively by $\mathbf{x}|v$ and $\mathbf{y}|q^2$.*

From this, it is clear as mentioned in the introduction that the correction terms of $Y \# - M_2$ are determined entirely by the module structure of Y.

Proof. Set $M = \widehat{\mathrm{HS}}_\bullet(Y)$. Let us consider first the part in degree $2n$ of the E^2-page, i.e.

$$\mathrm{Tor}^{\mathcal{R}}_{2n,*}(M, -M_2) = \mathrm{Tor}^{\mathcal{R}}_{2n,*}(M, N_2) \oplus \mathrm{Tor}^{\mathcal{R}}_{2n,*}(M, \mathbb{F}).$$

For $n \geq 1$, both summands can be identified with

$$W = \ker \begin{bmatrix} Q^2 & 0 \\ V & Q \end{bmatrix} / \mathrm{im} \begin{bmatrix} Q & 0 \\ V & Q^2 \end{bmatrix}$$

where we think the matrices as acting on M^2. Because of two periodicity, the part in even grading of the (E^2, d_2) page can be rewritten as the complex

$$\cdots \xrightarrow{d_2} W \oplus W \xrightarrow{d_2} W \oplus W \xrightarrow{d_2} W \oplus W \xrightarrow{d_2} \cdots \xrightarrow{d_2} W \oplus W \xrightarrow{d_2} W \oplus W.$$

When thought of as a 2-by-2 matrix, d_2 is upper triangular, and by (15) above we see that it has to be of the form

$$d_2 = \begin{bmatrix} A & \mathrm{Id}_W \\ 0 & B \end{bmatrix}.$$

Imposing $d_2^2 = 0$ we also obtain $A^2 = 0$ and $A = B$, so that

$$d_2 = \begin{bmatrix} A & \mathrm{Id}_W \\ 0 & A \end{bmatrix}.$$

We want to show that d_2 is exact. Suppose $d_2(\mathbf{x}, \mathbf{y}) = 0$. Hence, $A\mathbf{x} = \mathbf{y}$ and $A\mathbf{y} = 0$. Then

$$d_2(0, \mathbf{x}) = \begin{bmatrix} A & \mathrm{Id}_W \\ 0 & A \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

and the result follows.

The analogous argument holds for the odd part, as for each $n \geq 1$

$$\mathrm{Tor}^{\mathcal{R}}_{2n+1,*}(M, -M_2) = \mathrm{Tor}^{\mathcal{R}}_{2n+1,*}(M, N_2) \oplus \mathrm{Tor}^{\mathcal{R}}_{2n+1,*}(\mathbb{F}, \mathbb{F}).$$

each of the summands on the right can be identified with

$$W' = \ker \begin{bmatrix} Q & 0 \\ V & Q^2 \end{bmatrix} / \mathrm{im} \begin{bmatrix} Q^2 & 0 \\ V & Q \end{bmatrix}$$

and d_2 has an analogous shape, see equation (16). Finally, the $\mathcal{R}$-module structure can be easily computed using Lemma 3.8 and 5.5. $\square$

The general case of M_{2k} can be derived with few modifications, so that the analogue of Proposition 5.9 also holds. Let us discuss the key modifications involved. Recall that as $\mathcal{R}$-modules, $M_{2k} = N_{2k} \oplus \mathbb{F}[V]/V^k$. The module $\mathbb{F}[V]/V^k$ has minimal free resolution

$$\mathbb{F}[V]/V^k \xleftarrow{d_0} \mathcal{R} \xleftarrow{d_1} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_2} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_3} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_4} \cdots$$

where in matrix notation

$$d_1 = \begin{bmatrix} V^k & Q \end{bmatrix},$$

and for $i \geq 2$ we have

$$d_i = \begin{cases} \begin{bmatrix} Q & 0 \\ V^k & Q^2 \end{bmatrix} & \text{if } i \text{ is even,} \\[2ex] \begin{bmatrix} Q^2 & 0 \\ V^k & Q \end{bmatrix} & \text{if } i \text{ is odd,} \end{cases}$$

while N_{2k} has minimal projective resolution

$$N_{2k} \xleftarrow{d_0} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_1} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_2} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_3} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_4} \cdots$$

where for $i \geq 1$ we have

$$d_i = \begin{cases} \begin{bmatrix} Q & 0 \\ V^k & Q^2 \end{bmatrix} & \text{if } i \text{ is even,} \\[2ex] \begin{bmatrix} Q^2 & 0 \\ V^k & Q \end{bmatrix} & \text{if } i \text{ is odd.} \end{cases}$$

Here d_0 sends $(1, 0)$ to v^n and $(0, 1)$ to q^2. The result then follows from the identities from Section 3, $\langle V^n, \mathbf{z}, Q \rangle = v^n$ and $\langle \mathbf{z}, Q, Q^2 \rangle = q^2$, in the same way as for $-M_2$.

The odd case. We have seen in the previous section that connected sums with manifolds of type $\pm M_{2k}$ are rather simple to understand. We will discuss the case in which $n = 2k + 1$ is odd. Let us first discuss a suitable minimal projective resolution.

$$M_{2k+1} \xleftarrow{d_0} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_1} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_2} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_3} \mathcal{R} \oplus \mathcal{R} \xleftarrow{d_4} \cdots$$

where

$$
d_i = \begin{cases}
\begin{bmatrix} Q & 0 \\ V^{k+1} & Q^2 V^k \end{bmatrix} & \text{if } i = 1, \\[1.5em]
\begin{bmatrix} Q & 0 \\ V & Q^2 \end{bmatrix} & \text{if } i \text{ is odd } \geq 3, \\[1.5em]
\begin{bmatrix} Q^2 & 0 \\ V & Q \end{bmatrix} & \text{if } i \text{ is even,}
\end{cases}
$$

and d_0 maps $(1,0)$ to v and $(0,1)$ to qv^{-k}. The computation of the E^2 page, from the general description of Section 4, is the following.

Lemma 5.10. *Let m be odd. generators of* $\mathrm{Tor}^{\mathcal{R}}_{m,*}(M, M_1)$ *have representatives of the form*

$$
\mathbf{x}|Q_m|v + \mathbf{x}\,\mathrm{sh}(\tilde{Q}_{m-1}, V)|q + \mathbf{y}|\tilde{Q}_m|q.
$$

where $Q\mathbf{x} = V\mathbf{x} + Q^2\mathbf{y} = 0$. *Let m be even. Then the generators of* $\mathrm{Tor}_{m,*}(M, N_1)$ *have representatives of the form*

$$
\mathbf{x}|\tilde{Q}_m|v + \mathbf{x}|\,\mathrm{sh}(Q_{m-1}, V)|q + \mathbf{y}|Q_m|q
$$

where $Q^2\mathbf{x} = V\mathbf{x} + Q\mathbf{y} = 0$. *The description of* $\mathrm{Tor}^{\mathcal{R}}_{*,*}(M, M_{2k+1})$ *in the case* $k \geq 1$ *is analogous.*

The description of the E^∞ page is not as straightforward as in the even case. While the differential on E^2 can be described as in the even case in terms of certain generalized Massey products, the spectral sequence does not collapse at the E^3-page in general. On the other hand, because the relation

$$
\langle QV^i, Q^2V^j, QV^k, Q^2V^l \rangle = V^{1+i+j+k+l}
$$

holds, it can be shown that the spectral sequence collapses at the E^4 page (see also Example 5.1). Rather than discussing the quite involved general theory, let us work out in detail a specific example that enlightens the key aspects of the computation.

Example 5.11. Let us revisit the example of the connected sum $M_1 \# M_1$ discussed in [23] from our new perspective. The E^2 page of the Eilenberg–Moore spectral sequence is depicted below. Here, starting from the left the ith column represents

$\mathrm{Tor}^{\mathcal{R}}_{i,*}$, with the element on the top left having bidegree $(0, 2)$. The picture repeats two-periodically to the right as in Example 5.1:
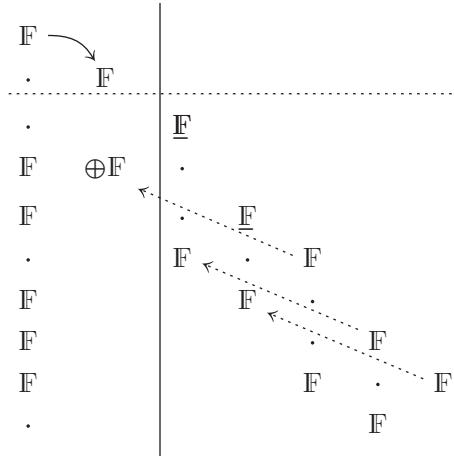
The first column in $\mathrm{Tor}^{\mathcal{R}}_{0,*}$ is generated by the based elements

$$q|q, \quad q|v, \quad v|v,$$

while the additional summand $\oplus \mathbb{F}$ is generated by $q|v + v|q$. The elements of the higher $\mathrm{Tor}^{\mathcal{R}}_{i,*}$ groups can be described thanks to Lemma 5.10. In particular, the generators of the top summand of each of the first three columns are given by

$$q|Q^2|q, \quad Qq|Q|Q^2|q, \quad q|Q^2|Q|Q^2|q.$$

The differential d_2 (described in Lemma 1.4) vanishes thanks to our description of the Massey products $\langle \cdot, Q^2, Q \rangle$ and $\langle \cdot, Q, Q^2 \rangle$ in Theorem 2.2. On the other hand, the differential d_3 is non-trivial. In the picture, the top dotted arrows represents

$$d_3(q|Q^2|Q|Q^2|q) = \langle q, Q^2, Q, Q^2 \rangle|q + q|\langle Q^2, Q, Q^2, q \rangle = v|q + q|v$$

where the Massey product $\langle q, Q^2, Q, Q^2 \rangle = v$ is computed as in Remark 2.4. Similarly, the other two dotted arrows represent

$$d_3(Qq|Q|Q^2|Q|Q^2|q) = v|Q^2|q + Qq|V|q + Qq|Q|v,$$
$$d_3(q|Q^2|Q|Q^2|Q|Q^2|q) = v|Q|Q^2|q + q|V|Q^2|q + q|Q^2|V|q + q|Q^2|Q|v,$$

and in general the whole two-periodic tail cancels out in this fashion as in Example 5.1. This implies that of the $\mathrm{Tor}^{\mathcal{R}}_{i,*}$ for $i \geq 1$ only the two underlined summands survive to the E^∞ page. There is only one non trivial extension, namely

$$Q \cdot (Qq|Q|Q^2|q) = v|q,$$

which can be again computed thanks to Theorem 2.2. The final result is therefore graphically depicted as
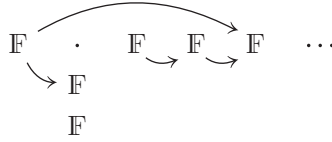
so that $\alpha = 2$ and $\beta = \gamma = 0$.

Example 5.12. Consider the connected sum of two manifolds of simple type M_3. The computation of the E^2 page is showed below. The group $\mathrm{Tor}^{\mathcal{R}}_{0,*}$ is as usual the tensor product $M_3 \otimes_{\mathcal{R}} M_3$; the first column consists of based elements and is generated over $\mathcal{R}$ by

$$qv^{-1}|qv^{-1}, \quad qv^{-1}|v, \quad v|v$$

lying in degrees respectively -10, -3 and 4, while the summands $\oplus \mathbb{F}$ are unbased and are represented by $qv^{-1}|v + v|qv^{-1}$ and its image under V respectively. Again there is a two-periodic infinite tail as in the case of $\mathrm{Tor}^{\mathcal{R}}_{*,*}(\mathbb{F}, \mathbb{F})$; its top generator in each of the first three columns is given respectively by

$$q|Q^2V|qv^{-1}, \quad Qq|Q|Q^2V|qv^{-1}, \quad q|Q^2|Q|Q^2V|qv^{-1}.$$

The main difference in this case is the presence of the extra summand $G \cong \mathbb{F}$, which is represented by $qv^{-1}|Q^2V|qv^{-1}$, and corresponds to

$$(0, qv^{-1}) \in \ker \begin{bmatrix} Q & 0 \\ V^2 & Q^2V \end{bmatrix}.$$

(see Figure 3.)

From this description, we readily recover the E^∞-page as in Example 5.11. In particular $E^\infty_{i,*}$ vanishes for $i \geq 3$, and of $\mathrm{Tor}^{\mathcal{R}}_{i,*}$, $i \geq 1$, only the underlined summands survive. The only non-trivial extension is given as in Example 5.11 by

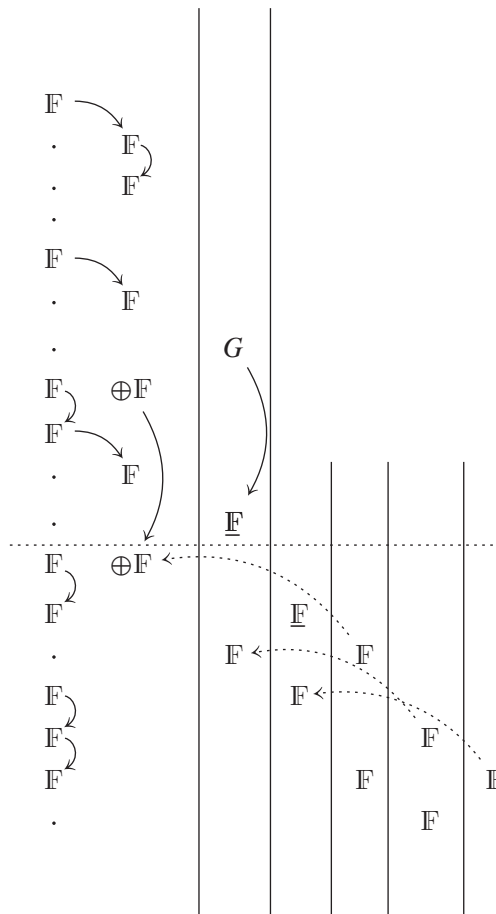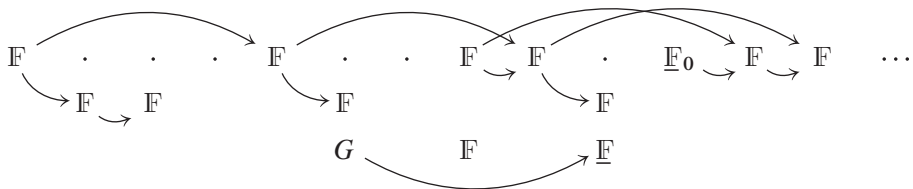$$Q \cdot (Qq|Q|Q^2V|qv^{-1}) = v^2|qv^{-1} = v|q.$$

Figure 3

The Floer homology of $M_3 \# M_3$ then looks like the following:

Therefore, $\alpha = 6$, $\beta = 2$ and $\gamma = 0$.

Remark 5.13. Recall that for the usual monopole Floer homology we have that

$$\widehat{\mathrm{HM}}_\bullet(Y_0 \# Y_1, \mathfrak{s}_0 \# \mathfrak{s}_1) = \mathrm{Tor}_{*,*}^{\mathbb{F}[U]}(\widehat{\mathrm{HM}}_\bullet(Y_0, \mathfrak{s}_0), \widehat{\mathrm{HM}}_\bullet(Y_1, \mathfrak{s}_1))\langle 1 \rangle,$$

see [23]. Hence the usual monopole Floer homology of $M_3 \# M_3$ is given by

$$\left(\mathbb{F}[U] \oplus \mathbb{F}[U]/U^3\langle 5\rangle \oplus \mathbb{F}[U]/U^3\langle 10\rangle\right) \oplus (\mathbb{F}[U]/U^3\langle 5\rangle)^{\oplus 2}.$$

The first summand is related via the Gysin exact triangle to the first two rows of our final result, while the second summand to the third row. This last computation implies the existence of some Massey products relating the three $\mathbb{F}$ summands in the third row of $\widehat{\mathrm{HS}}_\bullet$.

6. Connected sums with more summands

In this final section we discuss the proof of Theorems 1 and 2; the key point behind them is to understand connected sums involving multiple manifolds of simple type (possibly with both orientations). Let us begin by discussing the correction terms of connected sums with a given orientation.

Proposition 6.1. *Consider for $n_1 \geq n_2 \geq n_k \geq 1$ a connected sum of the form*

$$Y = Y_1 + Y_2 + \cdots + Y_k$$

where Y_i has simple type M_{n_i}. Then, if $\sum n_i$ is even, we have $\gamma(Y) = 0$ and

$$(\alpha(Y), \beta(Y)) = \begin{cases} (n_1 + n_2, n_1) & \text{if } n_1, n_2 \text{ are both even,} \\ (n_1 + n_2 - 1, n_1 - 1) & \text{if } n_1 \text{ is odd and } n_2 \text{ is even,} \\ (n_1 + n_2 - 1, n_1) & \text{if } n_1 \text{ is even and } n_2 \text{ is odd,} \\ (n_1 + n_2, n_1 - 1) & \text{if } n_1, n_2 \text{ are both odd.} \end{cases}$$

In other words, α and β are the largest even numbers smaller or equal than n_1 and $n_1 + n_2$ respectively. Similarly, if $\sum n_i$ is odd, we have $\gamma(Y) = 1$ and α and β are the largest odd numbers smaller than $n_1 + n_2$ and n_1 respectively.

Proof. This can be computed inductively in the number of summands, using the description of connected sums with manifolds of simple type discussed in Section 5. Let us start by considering the Floer homology of a connected sum $M_{2k_1} \# M_{2k_2}$ for $k_1 \geq k_2$. Recall that we have the identification

$$\mathrm{Tor}^{\mathcal{R}}_{0,*}(M^*_{2k_1}, M^*_{2k_2}) = M^*_{2k_1} \otimes_{\mathcal{R}} M^*_{2k_2}.$$

The latter can be written as a direct sum of two $\mathcal{R}$-modules, one of which is

$$\begin{aligned} \mathbb{F}[V] &\oplus \mathbb{F}[V]\langle 4k_1 - 1\rangle \oplus \mathbb{F}[[V]]\langle 4(k_1 + k_2) - 2\rangle \\ &\oplus \mathbb{F}[V]/V^{k_1+k_2}\langle 4(k_1 + k_2) - 3\rangle \oplus \mathbb{F}[V]/V^{k_2}\langle 4(k_1 + k_2) - 4\rangle, \end{aligned} \tag{17}$$

where the action of Q is not trivial from one summand to the one next to it on the right when there are two $\mathbb{F}$ summands that differ in degree by one, and the other is $\mathbb{F}[V]/V^{k_2}\langle 4k_2 - 1\rangle$. The group $\operatorname{Tor}^{\mathcal{R}}_{1,*}$ is isomorphic to $\mathbb{F}[V]/V^{k_2}\langle 4k_2 - 4\rangle$. In the case where $k_1 = 2$ and $k_2 = 1$, $\operatorname{Tor}^{\mathcal{R}}_{*,*}$ can be graphically described as follows:
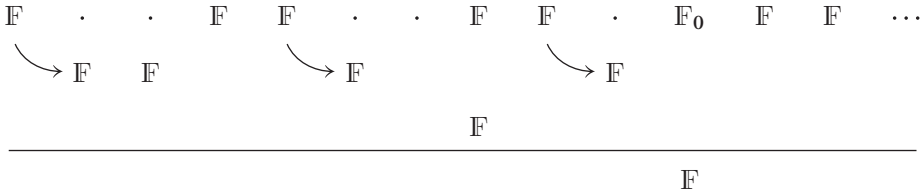
Here the first three rows represent $\operatorname{Tor}^{\mathcal{R}}_{0,*}$ (the first two rows being the summand in equation (17)) while the forth row represents $\operatorname{Tor}^{\mathcal{R}}_{1,*}$. For clarity, we have only depicted the $\mathcal{R}$-actions between elements of different rows. There are no trivial $\mathcal{R}$-extensions, so that

$$\alpha(Y_{k_1}\#Y_{k_2}) = 2k_1 + 2k_2, \quad \beta(Y_{k_1}\#Y_{k_2}) = 2k_1, \quad \gamma(Y_{k_1}\#Y_{k_2}) = 0,$$

The general case of a connected sum with all even summands now follows from this basic computation inductively by taking sums in decreasing order. Notice that there are no non-trivial Massey products of the form $\langle V^{k_3}, \mathbf{x}, Q\rangle$ in the Floer homology of the connected sum. This implies that when taking a connected sum with Y_{2k_3}, again only $\operatorname{Tor}^{\mathcal{R}}_{0,*}$ has to be taken account when computing correction terms. From here, a simple computation of the effect of tensoring with M_{2k_3} implies the claim; in particular, under the assumption $k_1 \geq k_2 \geq k_3$, the correction terms of the result are not affected. Finally, the general case in which also odd summands are involved, the strategy is the same and the only complication is that one should also keep track of some non-trivial extensions as in Example 5.12. □

With this computation in mind, we can now prove Theorem 1.

Proof of Theorem 1. As in the proof of the analogous result in [34], we need to show that a unique factorization property holds. Suppose we have a relation in the homology cobordism group of the form

$$[Y_{n_1}] + \cdots + [Y_{n_k}] = [Y_{m_1}] + \cdots + [Y_{m_l}]$$

where $n_1 \geq \cdots \geq n_k$ and $m_1 \geq \cdots \geq m_l$ where $[Y_{n_i}]$ and $[Y_{m_j}]$ have simple type M_{n_i} and M_{m_j} respectively. Then we also have a relation of the form

$$[Y_{n_1}] + \cdots + [Y_{n_k}] + [Y_1] = [Y_{m_1}] + \cdots + [Y_{m_l}] + [Y_1]$$

where Y_i has simple type M_1, and the sum of the indices has changed parity. Comparing this with the computation of the correction terms in Proposition 6.1, we conclude that $n_1 = m_1$, and the conclusion follows by induction. $\square$

Before proving Theorem 2, let us discuss some examples of connected sums where manifolds with both orientations appear.

Example 6.2. Consider $M = M_1 \# M_1$, whose relevant part of the homology was computed in Example 5.11 to be
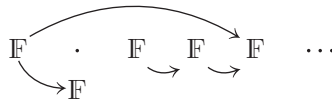
Let us consider the dual $-(M_1 \# M_1)$, which has Floer homology depicted as
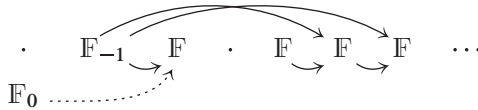
Here the dotted line represents the triple Massey product $\langle \cdot, Q, Q^2 \rangle$, which is determined by inspecting the Gysin exact sequence as in Proposition 2.5. Consider now $M' = -(M_1 \# M_1) \# M_2$. Using the description for connected sums with M_2 of Section 5, we see that the relevant part of its Floer homology is given by
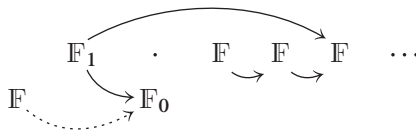
where the element in degree 2 is the tensor product of the element of degree zero in the homology of $-(M_1 \# M_1)$ and qv^{-1}. In particular, it comes with a non trivial Massey product onto the generator in degree 0, as depicted by the dotted arrow. M and M' can be distinguished up to homology cobordism just by looking at Massey products. Of course, in this case we already know that they cannot be homology cobordant because M_1 and M_2 are linearly independent.

Example 6.3. Let us discuss a slightly more involved example, namely $M_4 \# - M_3 \# - M_3$. To compute this we will regroup it as $M_4 \# - (M_3 \# M_3)$. The relevant part of $M_3 \# M_3$ was described in Example 5.12, so that by Poincaré duality (as in

the proof of Corollary 5.4) the relevant part of $-(M_3\#M_3)$ is
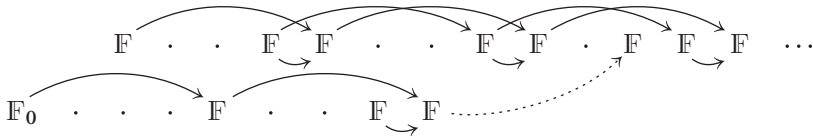
Here the elements in the top row are based, while the elements in the bottom row are not. Denote by $\mathbf{x}$ the element in degree zero in the bottom row. We have highlighted with a dotted line the Massey product $\langle Q, V^2\mathbf{x}, V \rangle = v^3$, which will be needed later. As in the proof of Corollary 5.4, this corresponds to the non-trivial Q-action into the tower in the *HS-to* Floer homology of $M_3\#M_3$. When connecting sum with M_4, $\mathrm{Tor}^{\mathcal{R}}_{1,*}$ corresponds to the elements in $\ker V^2 \cap \ker Q$, i.e. $V\mathbf{x}$ and $V^2\mathbf{x}$. Therefore, they give rise to the two elements

$$ V\mathbf{x}|V^2|qv^{-2} + V\mathbf{x}|Q|1, \qquad V^2\mathbf{x}|V^2|qv^{-2} + V^2\mathbf{x}|Q|1. \tag{18} $$

The action of V sends the first element to the second, and by Proposition 5.2 the second element is mapped via the action of Q to $v^3|qv^{-2} \in \mathrm{Tor}^{\mathcal{R}}_{0,*}$. The computation of the module structure of the connected sum is then readily obtained from that of $\mathrm{Tor}^{\mathcal{R}}_{0,*}$, i.e. the tensor product. The relevant part of the final result is
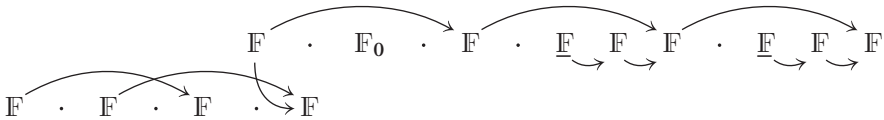
where the underlined $\mathbb{F}$ summands correspond to the classes (18) from $\mathrm{Tor}^{\mathcal{R}}_{1,*}$. We have in this case

$$ \alpha = 2, \quad \delta = 0, \quad \beta = \gamma = -2. $$

Again, the relevant Massey products can be inferred from both the tensor product formula or the Gysin exact triangle with

$$ \mathbb{F}[[U]] \oplus (\mathbb{F}[[U]]/U^4)_7 \oplus (\mathbb{F}[[U]]/U^3)_7 \oplus (\mathbb{F}[[U]]/U^3)_2 $$
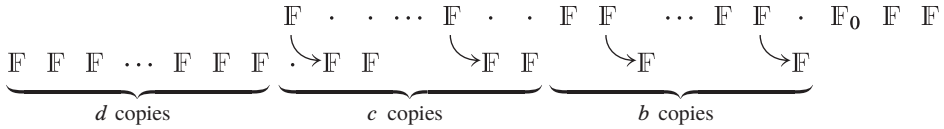
see also Example 3.9.

Going one step forward, we have the following computation.

Proposition 6.4. *Consider integers $a \geq b \geq c \geq d \geq 1$, and assume that the inequality $a \leq b+c+d$ holds. Then the manifold $Y = -M_{2b} - M_{2c} - M_{2d} + M_{2a}$ has*

$$\alpha(Y) = 2a - 2b,$$
$$\beta(Y) = 2a - 2b - 2c,$$
$$\gamma(Y) = 2a - 2b - 2c - 2d,$$

and of course $\delta(Y) = 0$.

Proof. Following the examples above, we start by computing the Floer homology of $M_{2b} + M_{2c} + M_{2d}$, whose relevant part is

Here the first row corresponds to the based elements, the second row is generated as an $\mathcal{R}$-module by the leftmost element, and we have only depicted the non-trivial Q-actions from the first row to the second row. As in the examples above, the relevant part of $-M_{2b} - M_{2c} - M_{2d}$ consists then of the based part
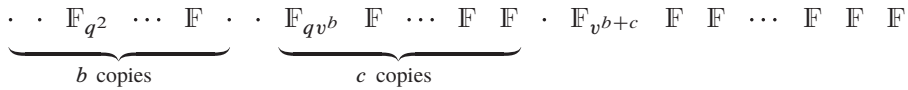
together with an $\mathcal{R}$-summand (which we will denote by L) of the form
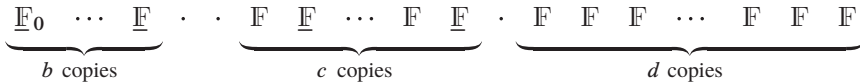
corresponding to the dual of the bottom row of the picture for $M_{2b} + M_{2c} + M_{2d}$. Furthermore, if $\mathbf{x}$ is one of the underlined elements, the Massey product $\langle Q, \mathbf{x}, V^k \rangle$ is a $\mathcal{V}$-based element provided k is large enough so that $V^k \mathbf{x} = 0$ (this again follows as in the proof of Corollary 5.4, and corresponds via the dualities to the arrows in the picture for $M_{2b} + M_{2c} + M_{2d}$). More precisely, if $\mathbf{y}$ is the rightmost underlined summand $\langle Q, \mathbf{y}, V^k \rangle = v^{b+c+k-1}$ provided $k \geq d + 1$.

We now need to compute the connected sum of this with M_{2a}. The based elements in $\text{Tor}^{\mathcal{R}}_{0,*}$ are given by $v^{b+c}|1$ (which is $\mathcal{V}$-based), $qv^b|1$ and $v^{b+c}|qv^{-a}$ (which are $Q \cdot \mathcal{V}$-based) and $q^2|1$ and $qv^b|qv^{-a}$ (which are $Q \cdot \mathcal{V}$-based). As in the previous examples, the extensions on the E^∞-page can only introduce new

$\mathcal{V}$-based elements, so we obtain right away the claimed computations of α and β. Recall that an element in $\mathrm{Tor}_{1,*}^{\mathcal{R}}$ has the form

$$\mathbf{x}|V^a|qv^{-a} + \mathbf{x}|Q|1$$

where $V^a\mathbf{x} = Q\mathbf{x} = 0$. Furthermore, V^a maps it to $\langle V^a|\mathbf{x}|Q\rangle|1$. Now, exactly the rightmost $a - d \geq 0$ underlined summands of L satisfy $V^a\mathbf{x} = Q\mathbf{x} = 0$. Denoting by $\mathbf{z}$ the one with highest degree (i.e. the $(a - d)$-th from the right), we have

$$V^a \cdot (\mathbf{z}|V^a|qv^{-a} + \mathbf{z}|Q|1) = v^{b+c+d}|1,$$

where we use that $a \leq b + c + d$. From this, the computation of γ follows. $\qquad\square$

Proof of Theorem 2. The construction in the previous proposition provides us with examples with α, β and γ even and $\delta = 0$ where $\alpha \geq 0 \geq \gamma$ and

$$\alpha - \beta \geq \beta - \gamma,$$

the last inequality corresponding to the assumption $c \geq d$. Given the formula provided there, it is straightforward to check that for any choice of α, β and γ satisfying these constraints, one can find a, b, c, d such $-M_{2b} - M_{2c} - M_{2d} + M_{2a}$ has the desired correction terms. The case in which the reverse inequality $\alpha - \beta \leq \beta - \gamma$ holds is obtained by considering the manifolds with opposite orientation $M_{2b} + M_{2c} + M_{2d} - M_{2a}$. The case in which $\delta = 0$ and α, β and γ are odd is treated in the same spirit by taking sums $-M_e - M_f - M_g + M_{2h}$ with some of the indices e, f, g odd; the details of the computation are analogous to the even case (and the various examples in this section) and are left to the reader. Finally, the case of general δ is obtained by taking further connected sums with the Poincaré homology sphere $\Sigma(2, 3, 5)$: as $\widehat{\mathrm{HS}}_\bullet(\Sigma(2, 3, 5)) \cong \mathcal{R}\langle -3\rangle$ (see [24]), we have

$$\widehat{\mathrm{HS}}_\bullet(\Sigma(2, 3, 5)\#Y) = \widehat{\mathrm{HS}}_\bullet(Y)\langle -2\rangle,$$

so that all four correction terms are shifted down by -1. $\qquad\square$

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