

HOMFLYPT homology for links in handlebodies via type A Soergel bimodules

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Abstract. We define a triply-graded invariant of links in a genus g handlebody, generalizing the colored HOMFLYPT (co)homology of links in the 3-sphere. Our main tools are the description of these links in terms of a subgroup of the classical braid group, and a family of categorical actions built from complexes of (singular) Soergel bimodules.

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1. Introduction

The HOMFLYPT polynomial is a classical invariant of links $\ell \subset \mathcal{S}^3$ in the 3-sphere \mathcal{S}^3 with interesting and deep connections to representation theory. As pioneered by Jones [Jon87], the HOMFLYPT polynomial may be defined using representations of the classical n strand braid group $\mathcal{B}r(n)$ on the type A Hecke algebra. Indeed, we may use Alexander’s theorem to present a link as a braid closure, and the

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HOMFLYPT polynomial then results by mapping the braid to the Hecke algebra and applying the so-called Jones–Ocneanu trace. (For the duration of the introduction, if not explicitly stated otherwise, “Hecke algebra” and related notions are always of type A.)

This approach to the HOMFLYPT polynomial was categorified in work of Khovanov [Kho07]. In this work, Khovanov shows that the triply-graded Khovanov–Rozansky homology $\mathcal{H}\mathcal{H}\mathcal{H}^\bullet(\ell)$ of $\ell \subset \mathcal{S}^3$, originally defined in [KR08b], admits a construction paralleling Jones’s approach at the categorical level. This approach proceeds by replacing the Hecke algebra by the corresponding Hecke category, i.e. the category of Soergel bimodules. The latter admits a categorical action of $\mathcal{B}r(n)$ via so-called Rouquier complexes [Rou06], and the link homology results by taking Hochschild (co)homology, which provides a categorical analogue of the Jones–Ocneanu trace.

In addition to their triply-graded invariant, for each $m \geq 2$ Khovanov and Rozansky define a doubly-graded homology theory for links $\ell \subset \mathcal{S}^3$ [KR08a] that categorifies the \mathfrak{sl}_m specialization of the HOMFLYPT polynomial. In the $m = 2$ case, which coincides with Khovanov’s categorification of the Jones polynomial [Kho00], Asaeda–Przytycki–Sikora have extended this link homology to links in 3-manifolds $\mathcal{M} \neq \mathcal{S}^3$, see [APS04], namely to links in thickened surfaces. Of particular interest is the case of the thickened annulus, where the so-called annular Khovanov homology has deep connections to both Floer theory and representation theory, see e.g. [Rob13], [GW10], and [GLW18]. In [QR18], an analogue of doubly-graded Khovanov–Rozansky homology was constructed for annular links, extending annular Khovanov homology, and its connection to representation theory, to general m . Unfortunately, the above approaches to link homology in 3-manifolds $\mathcal{M} \neq \mathcal{S}^3$ do not extend to the triply-graded setting.

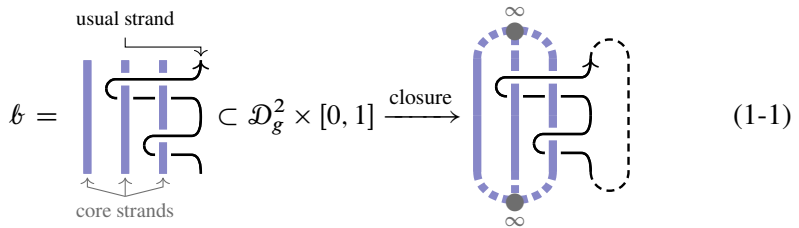
In this paper, we remedy this by constructing generalizations of the triply-graded link homology for links in 3-manifolds distinct from the 3-sphere, namely in genus g handlebodies. Specifically, we fix $g \geq 0$ and a corresponding genus g handlebody, and construct HOMFLYPT homology for links in this fixed handlebody that is invariant under ambient isotopy (but not under orientation-preserving diffeomorphisms of the handlebody; these notions disagree for $g > 0$, contrasting the $g = 0$ case). For $g = 1$, this is the case of links in the thickened annulus. Our key insights are: (1) to consider various generalizations of the classical braid group that are related to links in handlebodies, and (2) that certain structures in categorical representation theory model the topology of the handlebody. We now detail our approach.

1A. An overview of our construction. Throughout, we let $g, n \in \mathbb{N}_0$. Recall that Khovanov’s construction of $\mathcal{H}\mathcal{H}\mathcal{H}^\bullet(\ell)$ for $\ell \subset \mathcal{S}^3$ requires the following:

- Alexander’s Theorem, which states that, up to isotopy, any link $\ell \subset \mathcal{S}^3$ can be presented as the closure of a braid b in the classical n -strand braid group $\mathcal{B}r(n)$;
- Markov’s Theorem, which gives necessary and sufficient conditions for two distinct braids to have isotopic closures;
- a categorical action of $\mathcal{B}r(n)$ on the Hecke category via Rouquier complexes, which allows for the assignment of a chain complex of Soergel bimodules to each $b \in \mathcal{B}r(n)$;
- Hochschild (co)homology, which produces a Markov invariant triply-graded vector space from this complex of Soergel bimodules.

In [HOL02] (see also [Lam93]), it is shown that analogues of Alexander’s and Markov’s Theorems hold for links in the genus g handlebody \mathcal{H}_g . Playing the role of the classical braid group is the n -strand braid group $\mathcal{B}r(g, n)$ of the g -times punctured disk \mathcal{D}_g^2 . The classical story here is the $g = 0$ case, where $\mathcal{B}r(n) = \mathcal{B}r(0, n)$.

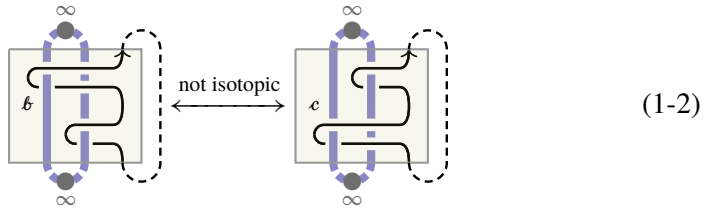
As we more fully detail in Section 2B, braids in $\mathcal{B}r(g, n)$ can be pictured as classical braids in the presence of non-intersecting “core strands”. We then obtain a link in \mathcal{H}_g by allowing the tops and bottoms of the core strands to meet at ∞ , and by taking a closure of the “usual strands”. The latter then form a link in the complement of the (glued) core strands, which is a handlebody \mathcal{H}_g :



The analogue of the Alexander Theorem here shows that, up to isotopy, every link in \mathcal{H}_g arises in this way, and the corresponding Markov Theorem characterizes when distinct braids give rise to isotopic closures.

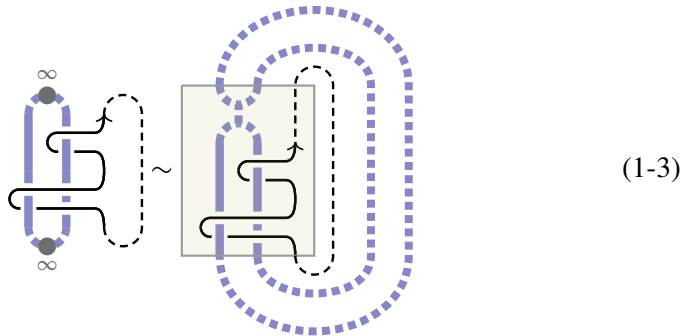
Issues arise, however, when attempting to carry out the last two steps in the construction of triply-graded link homology in this setting. Indeed, for general g , the groups $\mathcal{B}r(g, n)$ are not known to be Artin–Tits groups (see Section 1B for further discussion), so, to our knowledge, there are no associated Soergel bimodules

and/or Rouquier complexes. Further, the Markov Theorem has a weaker notion of conjugation than in the classical case, e.g. we have



even though the indicated (boxed) braids $b, c \in \mathcal{B}r(2, 1)$ are conjugate. Hence, even with a categorical representation of $\mathcal{B}r(g, n)$ in hand, one cannot simply apply Hochschild cohomology to obtain an invariant of links that is sensitive to the topology of \mathcal{H}_g .

We simultaneously resolve both these problems as follows. We expand the point at infinity to a small segment, which we move close to the top of the core strands. As a result, we can view the closure of the “usual strands” as a link in the handlebody given by the complement of the graph determined by the core strands and the segment at infinity, e.g.



In this modified presentation we are able to assign an invariant to the link $l \subset \mathcal{H}_g$ using known structures in categorical representation theory. Indeed, for any labeling of the core strands, the boxed diagram in (1-3) determines a complex of singular Soergel bimodules. The latter determine a (2-)category that contains the Hecke category [Wil11], and categorifies the Schur algebra, a certain idempotent completion of the Hecke algebra. Further, the closure procedure now does not involve the point at infinity, and hence can be carried out algebraically as usual, using Hochschild cohomology. In this way, we obtain a triply-graded homology for links $l \subset \mathcal{H}_g$. We show that this indeed produces a well-defined invariant of handlebody links, and that it is sensitive to the topology of the handlebody, e.g. it distinguishes the links in (1-2).

1B. A digression on Artin–Tits groups. Our motivation for this project, from which we have now somewhat strayed, was to further our understanding of the connection between low-dimensional topology and Artin–Tits groups. Recall that a Coxeter diagram $\Gamma = (V, E)$ consists of a simple, complete graph with finitely many vertices V whose edges $e = (i, j) \in E$ carry a label $m_{ij} = m_{ji} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. To any such diagram, we may associate the Artin–Tits group:

$$AT(\Gamma) := \langle \beta_i, i \in V \mid \underbrace{\dots \beta_i \beta_j \beta_i}_{m_{ij} \text{ factors}} = \underbrace{\dots \beta_j \beta_i \beta_j}_{m_{ij} \text{ factors}} \rangle. \tag{1-4}$$

This group is an extension of the corresponding Coxeter group:

$$W(\Gamma) := \langle \sigma_i, i \in V \mid \sigma_i^2 = 1, \underbrace{\dots \sigma_i \sigma_j \sigma_i}_{m_{ij} \text{ factors}} = \underbrace{\dots \sigma_j \sigma_i \sigma_j}_{m_{ij} \text{ factors}} \rangle. \tag{1-5}$$

The jumping-off point is the classical observation that $\mathcal{B}r(n) = \mathcal{B}r(0, n)$ is isomorphic to the Artin–Tits braid group of type A, while $\mathcal{B}r(1, n)$ is isomorphic to the Artin–Tits group of type C = B and extended affine type A. More surprising is the lesser-known fact that $\mathcal{B}r(2, n)$ is isomorphic to the Artin–Tits group of affine type C [All02]. The following table summarizes these known connections, details of which can be found in e.g. [All02, Section 4] and [Bri73].

Genus	type A	type C
$g = 0$	$\mathcal{B}r(n) \cong AT(A_{n-1})$?
$g = 1$	$\mathcal{B}r(1, n) \cong \mathbb{Z} \ltimes AT(\tilde{A}_{n-1}) \cong AT(\hat{A}_{n-1})$	$\mathcal{B}r(1, n) \cong AT(C_n)$
$g = 2$?	$\mathcal{B}r(2, n) \cong AT(\tilde{C}_n)$
$g \geq 3$?	?

(1-6)

Herein, A_{n-1} denotes the type A Coxeter diagram with $n - 1$ nodes, while \tilde{A}_{n-1} denotes the affine type A Coxeter diagram with n nodes and \hat{A}_{n-1} is the corresponding extended affine type. Similarly, C_n and \tilde{C}_n denote the type C = B and affine type C (but not affine type B) Coxeter diagrams with n and $n + 1$ nodes, respectively.

As mentioned above, the first row of the type A column in (1-6) underpins Jones’s construction of the HOMFLYPT polynomial, and the second row has similarly been exploited in topological considerations, see e.g. [OR07] and [EI18].

The type C column, however, has received less attention, especially in the affine, $g = 2$ case, where not much appears to be known about connections to link invariants. (However, this case has been explored from a representation-theoretic point of view, see e.g. [DR18].) A notable example is work of Geck–Lambropoulou [GL97] in the $g = 1$ case, where a HOMFLYPT polynomial for links in \mathcal{H}_1 is constructed via the analogue of Jones’s construction in type C. The results in [Rou17] and [WW11] should pair to give a categorification of this invariant. In a companion paper, we plan to study this invariant, and develop its genus two analogue, using type C and affine type C Hecke algebras and Soergel bimodules.

By contrast, our construction in the present paper exploits the relation between $\mathcal{B}r(g, n)$ and $\mathcal{B}r(g + n)$, and the fact that the latter is an Artin–Tits group. Indeed, our construction can be recast as follows. By viewing the distinguished strands as usual strands, we obtain an injective group homomorphism $\mathcal{B}r(g, n) \hookrightarrow \mathcal{B}r(g + n)$. Since the latter is an Artin–Tits group, we can assign a complex of Soergel bimodules to any braid $b \in \mathcal{B}r(g, n)$. Now, before taking Hochschild cohomology (doing so immediately would give an invariant less-sensitive to the topology of the handlebody), we glue on an additional Soergel bimodule that allows invariance under the Markov Theorem for $\mathcal{B}r(g, n)$, but not for $\mathcal{B}r(g + n)$. In fact, our procedure is slightly more general in that it uses an embedding of $\mathcal{B}r(g, n)$ into the colored braid group, and singular Soergel bimodules.

1C. Future outlook. In addition to our planned investigation in type C, we believe there are a number of interesting future directions.

- **The relation between $\mathcal{B}r(g, n)$ and Hecke algebras.** There exists a Hecke-like algebra associated to $\mathcal{B}r(g, n)$ for general g , see e.g. [Lam00]. In the $g = 0, 1$ cases, this algebra matches the Hecke algebras associated to the Artin–Tits groups in the type A column of (1-6). These algebras have not been widely studied, e.g. to our knowledge it is not known whether they admit Markov traces or categorifications.

In another direction, it is an interesting problem to extend the type C column of (1-6) to higher genus. The presentation of $\mathcal{B}r(g, n)$ given below in Definition 2.4 hints to a connection to the Artin–Tits group associated to the Coxeter diagram that is obtained from the type A_n diagram by adjoining g additional vertices. These vertices are attached to each other with ∞ -labeled edges, and to the first type A vertex with 4-labeled edges. (Something very similar was also observed in [Lam00, Remark 4].) For example, the $g = 3$

Convention 1.2. We will find it convenient to depict morphisms in certain categories (and 1-morphisms in certain 2-categories) diagrammatically. We will read such diagrams from bottom-to-top (and in the presence of a monoidal or 2-categorical structure, also right-to-left). These reading conventions are summarized by

$$\begin{array}{c} \text{C} \\ \text{b} \quad \text{d} \\ \text{B} \\ \text{a} \quad \text{c} \\ \text{A} \end{array} \iff (bd) \circ (ac): A \longrightarrow B \longrightarrow C. \quad (1-8)$$

Moreover, all such diagrams are invariant under (distant) height exchange isotopy (up to isomorphism, in the 2-categorical context). Finally, we will occasionally omit certain data (e.g. labelings) from such diagrams when they may be recovered from the given data, or are not important for the argument in question.

Convention 1.3. We will work with \mathbb{Z}^k -graded categories throughout, for $k = 1, 2, 3$. The three gradings of importance are the internal degree \mathbf{q} , the homological degree \mathbf{t} (both appearing from Section 3 onward), and the Hochschild degree \mathbf{a} (making its appearance in Section 4).

There are competing notions of what is meant by a graded category, so we now detail our conventions, focusing on the \mathbf{q} -degree. Let \mathcal{C} be a category enriched in \mathbb{Z} -graded abelian groups, i.e. for objects X and Y , $\text{Hom}_{\mathcal{C}}(X, Y)$ is a \mathbb{Z} -graded abelian group:

$$\text{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(X, Y)_d \quad (1-9)$$

Given such a category, we can introduce a formal grading-shift functor \mathbf{q} and consider the category $\mathcal{C}^{\mathbf{q}}$ in which objects are given by formal shifts $\mathbf{q}^s X$ of objects in \mathcal{C} , and

$$\text{Hom}_{\widetilde{\mathcal{C}}^{\mathbf{q}}}(\mathbf{q}^s X, \mathbf{q}^t Y) = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(X, Y)_{d+t-s}. \quad (1-10)$$

i.e. $\widetilde{\mathcal{C}}^{\mathbf{q}}$ is again enriched in \mathbb{Z} -graded abelian groups. Finally, we let $\mathcal{C}^{\mathbf{q}}$ be the category with the same objects as $\widetilde{\mathcal{C}}^{\mathbf{q}}$, but where we restrict to \mathbf{q} -degree zero morphisms, i.e.

$$\text{Hom}_{\mathcal{C}^{\mathbf{q}}}(\mathbf{q}^s X, \mathbf{q}^t Y) = \text{Hom}_{\mathcal{C}}(X, Y)_{s-t}. \quad (1-11)$$

Note that $\mathcal{C}^{\mathfrak{q}}$ is not enriched in \mathbb{Z} -graded abelian groups, but is equipped with an autoequivalence shift functor \mathfrak{q} . It is categories of this form that will be of primary interest in this work.

We note, however, that it is possible to recover the \mathbb{Z} -graded abelian group $\text{Hom}_{\mathcal{C}}(X, Y)$ from the category $\mathcal{C}^{\mathfrak{q}}$. Indeed, we can consider the \mathbb{Z} -graded abelian group

$$\text{HOM}_{\mathcal{C}^{\mathfrak{q}}}(X, Y) := \bigoplus_d \text{Hom}_{\mathcal{C}^{\mathfrak{q}}}(\mathfrak{q}^d X, Y) \tag{1-12}$$

and we note that

$$\text{HOM}_{\mathcal{C}^{\mathfrak{q}}}(\mathfrak{q}^s X, \mathfrak{q}^t Y) = \mathfrak{q}^{t-s} \text{HOM}_{\mathcal{C}^{\mathfrak{q}}}(X, Y), \tag{1-13}$$

where on the right-hand side the power \mathfrak{q} denotes a shift of the indicated \mathbb{Z} -graded abelian group.

Our consideration of \mathbb{Z}^2 - and \mathbb{Z}^3 -graded categories is analogous – in these cases we have additional shift functors \mathfrak{t} and \mathfrak{a} , and we restrict to \mathfrak{t} - and \mathfrak{a} -degree zero maps, unless otherwise indicated. However, we will reserve the capitalization notation HOM when considering “graded Homs” with respect to the \mathfrak{q} -degree only.

Lastly, we note that these considerations carry over to 2-categories as well, where the above applies to the Hom -categories in our 2-category, i.e. to the 1- and 2-morphisms.

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2. Links and braids in handlebodies

In this section we collect result concerning links and braids in handlebodies.

2A. Topological recollections. Recall that a handlebody \mathcal{H}_g of genus g is the compact, orientable 3-manifold with boundary obtained by attaching g 1-handles to the closed 3-ball \mathcal{H}_0 . An explicit model for \mathcal{H}_g , that we call the standard presentation, is given by the “inside” of a standardly embedded genus g surface $\Sigma_g \subset \mathcal{S}^3$ in the 3-sphere \mathcal{S}^3 , i.e. the 3-manifold given by the union of Σ_g with the component of $\mathcal{S}^3 \setminus \Sigma_g$ that does not contain the point at infinity.

We will typically work with another presentation for \mathcal{H}_g , given by the closure in \mathcal{S}^3 of the complement of an auxiliary handlebody \mathcal{H}_g^c . We view \mathcal{H}_g^c as consisting of g parallel 1-handles that are attached to the closure of a small neighborhood of $\infty \in \mathcal{S}^3$. See the gray portion of the first figure in Example 2.2 for the $g = 3$ case. In order to connect with the categorical representation theory used to produce our link invariant, we note that \mathcal{H}_g^c is isotopic to a closed neighborhood of the embedded graph obtained by taking g parallel edges, called the “core strands”, each meeting a $(g + 1)$ -valent vertex, together with an additional “edge at infinity” joining the two vertices. We will typically view the edge at infinity as being near the top of the core strands, hereby viewing \mathcal{H}_g^c as being obtained from the core strands by first gluing on a graph with g 1-valent vertices at its top and bottom (and two $(g + 1)$ -valent vertices) and then taking their closure. See the gray portion of the second figure in Example 2.2.

We will refer to this presentation of $\mathcal{H}_g = \overline{\mathcal{S}^3 \setminus \mathcal{H}_g^c}$ as the costandard presentation, and note that it contains a copy of the standard presentation, to which it is isotopic, given by intersecting with a closed 3-ball that meets each core strand in a segment.

We consider oriented links $\ell \subset \mathcal{H}_g$, which, in the costandard presentation, are given by links in $\mathcal{S}^3 \setminus \mathcal{H}_g^c$. Equivalently, using the isotopy with the standard presentation, such links are given by links in the closed 3-ball that avoid its intersection with the core strands. Finally, two links in $\ell, \ell' \subset \mathcal{H}_g$ are isotopic, denoted by $\ell \sim \ell'$, if and only if the corresponding links in \mathcal{S}^3 (in the costandard presentation) are isotopic through an isotopy that keeps \mathcal{H}_g^c fixed pointwise.

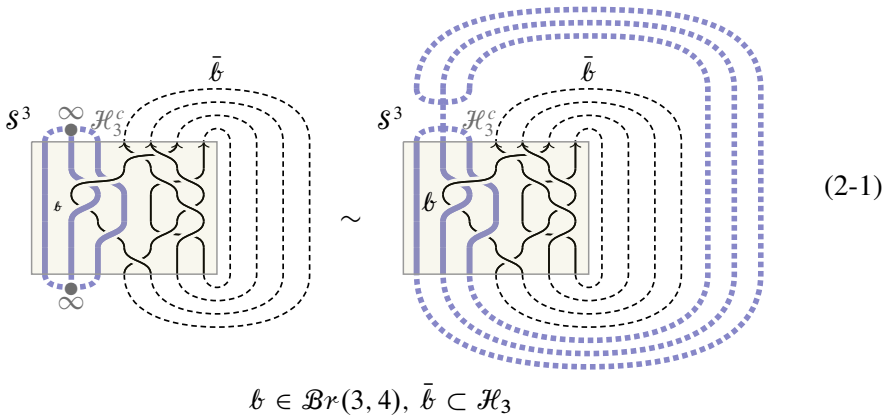
2B. Alexander’s theorem. There is a corresponding notion of n strand braids in a genus g handlebody. Strictly speaking, we define $\mathcal{B}r(g, n)$ to be the braid group of the surface \mathcal{D}_g^2 given as the complement of g disjoint open disks in the closed disk \mathcal{D}^2 . The standard presentation of \mathcal{H}_g may be identified with the product $\mathcal{D}_g^2 \times [0, 1] \subset \mathcal{S}^3$, so braids in $\mathcal{B}r(g, n)$ “live in” \mathcal{H}_g . The group $\mathcal{B}r(g, n)$, which

we call the handlebody braid group, can be equivalently described as follows. There is a subgroup of the classical braid group $\mathcal{B}r(g+n)$ on $g+n$ strands consisting of braids that are pure on the first g strands, and a homomorphism from this subgroup to the classical braid group $\mathcal{B}r(g)$ given by forgetting the final n strands. The kernel of this homomorphism is precisely $\mathcal{B}r(g,n)$. (Informally, $\mathcal{B}r(g,n)$ consists of braids on $g+n$ strands in which the first g strands do not braid among themselves.) Slightly abusing notation, we will again refer to the first g strands of a braid in $\mathcal{B}r(g,n)$ as core strands; they correspond to the core strands above as we now describe.

The handlebody braid group $\mathcal{B}r(g,n)$ is related to links in \mathcal{H}_g in a manner paralleling the relation between the classical braid group $\mathcal{B}r(n)$ and links in \mathcal{S}^3 . That is, given a braid $b \in \mathcal{B}r(g,n)$, one obtains a link $\bar{b} \subset \mathcal{H}_g$ via a closure procedure as follows: the first g strands in b are joined at each of their ends to the point at infinity, and the remainder of the braid is closed as in the classical case. In this way, we obtain a link $\bar{b} \subset \mathcal{H}_g$ where the closure of the last n strands constitutes \bar{b} , and the first g strands in b become the core strands in \mathcal{H}_g^c . As in our discussion of \mathcal{H}_g^c above, we will typically work with an equivalent closure procedure, which again corresponds to expanding the point at infinity to an edge, and moving it near the top of the core strands. Specifically, the closure procedure consists of merging the g core strands to meet the strand at infinity, then splitting the strand at infinity into g strands, and finally taking the standard closure of all strands.

Example 2.1. We have $\mathcal{B}r(0,n) \cong \mathcal{B}r(n)$, which corresponds to links in the closed 3-ball \mathcal{H}_0 ; we call this the classical case. In genus one, $\mathcal{B}r(1,n)$ consists of all braids in $\mathcal{B}r(1+n)$ that are pure on the first strand, and \mathcal{H}_1 is a solid torus.

Example 2.2. Here we illustrate the closure procedure for $b \in \mathcal{B}r(3,4)$:



The braid itself is depicted as the solid strands in the indicated rectangle, while the dashed edges correspond to the closure procedure described above. The thin, black components (both solid and dashed) give the link \bar{b} , while the thick, gray graph (again, both solid and dashed) depicts \mathcal{H}_g^c .

The next result shows that, up to isotopy, all links in \mathcal{H}_g arise from the closure procedure for handlebody braids described above. The proof is analogous to the classical case.

Theorem 2.3. (Alexander’s Theorem in a handlebody; [HOL02, Theorem 2].) *Given a link $\ell \subset \mathcal{H}_g$ there exists a braid $b \in \mathcal{B}r(g, n)$ such that $\bar{b} \sim \ell \subset \mathcal{H}_g$.*

2C. Generators and relations for braids in handlebodies. We now recall the algebraic presentation of $\mathcal{B}r(g, n)$.

Definition 2.4. The group $\text{Br}(g, n)$ is the group generated by b_1, \dots, b_{n-1} and t_1, \dots, t_g , called braid and twist generators, respectively, subject to the relations

$$b_j b_i b_j = b_i b_j b_i \text{ if } |i - j| = 1, \quad b_j b_i = b_i b_j \text{ if } |i - j| > 2, \quad (2-2)$$

$$b_1 t_i b_1 t_i = t_i b_1 t_i b_1, \quad b_i t_j = t_j b_i \text{ if } i > 2, \quad (2-3)$$

$$(b_1 t_i b_1^{-1}) t_j = t_j (b_1 t_i b_1^{-1}) \text{ for } i < j. \quad (2-4)$$

By convention, $\text{Br}(g, 0) = \{1\}$, and we omit the twist generators when $g = 0$ and the braid generators when $n = 1$.

The following theorem identifies $\mathcal{B}r(g, n)$ and $\text{Br}(g, n)$, and we likewise do for the duration. In particular, we identify $\text{Br}(n) = \text{Br}(0, n)$ and $\mathcal{B}r(n)$.

Proposition 2.5. ([Ver98, Theorem 1] & [Lam00, Section 5].) *There is an isomorphism of groups*

$$\text{Br}(g, n) \cong \mathcal{B}r(g, n). \quad (2-5)$$

□

An explicit isomorphism realizing Proposition 2.5 is given on the braid and the twist generators as follows:

$$b_i \mapsto \left[\dots \left| \begin{array}{c} \uparrow \dots \uparrow \\ \text{---} \\ \downarrow \dots \downarrow \end{array} \right. \dots \right] \quad \& \quad t_i \mapsto \left[\dots \left| \begin{array}{c} \dots \dots \dots \\ \dots \dots \dots \\ \dots \dots \dots \\ \dots \dots \dots \end{array} \right. \dots \right] \quad (2-6)$$

The inverse of b_i is given, as usual, by the corresponding opposite crossing, and the inverse of t_i is given as above, but with the braid strand wrapping the i^{th} core strand oppositely (however, it still crosses over the other core strands).

Example 2.6. Under this map, the first relation in (2-2) corresponds to the braid-like Reidemeister III relation, and the second relations in (2-2) and (2-3) correspond to planar isotopy. The four term relation in (2-3) and the relation in (2-4) become e.g. the relations (2-7) and (2-8), respectively:

$$b_1 t_i b_1 t_i = t_i b_1 t_i b_1 \iff \text{diagram} = \text{diagram} \tag{2-7}$$

$$(b_1 t_i b_1^{-1}) t_j = t_j (b_1 t_i b_1^{-1}) \iff \text{diagram} = \text{diagram} \tag{2-8}$$

The non-trivial statement in Proposition 2.5 is that these relations are sufficient.

2D. Markov's theorem. Let $\mathcal{B}r(g, \infty) := \bigsqcup_{n \in \mathbb{N}} \mathcal{B}r(g, n)$, the set of all braids in \mathcal{H}_g .

Definition 2.7. Let $\mathcal{B}r(g, \infty)^{Ma} := \mathcal{B}r(g, \infty) / \sim$ be the quotient given by conjugation (2-9) in (each) $\mathcal{B}r(g, n)$ by elements $s \in \langle b_1, \dots, b_{n-1} \rangle$ and stabilization (2-10), i.e.

$$b \sim s b s^{-1} \tag{2-9}$$

for $b \in \mathcal{B}r(g, n)$, $s \in \langle b_1, \dots, b_{n-1} \rangle$

$$(c\uparrow)b_n(b\uparrow) \sim cb \sim (c\uparrow)b_n^{-1}(b\uparrow)$$

for $b, c \in \mathcal{B}r(g, n)$,

(2-10)

where $b\uparrow \in \mathcal{B}r(g, n + 1)$ is the braid obtained from $b \in \mathcal{B}r(g, n)$ by adding a strand to the right.

Remark 2.8. The conjugation (2-9) is weaker than in the classical case – there one can conjugate by any element, instead of just by certain elements. This will play an important role in our construction of the HOMFLYPT invariant, see Proposition 4.8. On the other hand, the stabilization (2-10) is stronger than the classical case when considered on its own, but together with the classical conjugation relation is equivalent to the classical stabilization.

Theorem 2.9. (Markov’s Theorem in a handlebody; [HOL02, Theorem 5].) Let $b, c \in \mathcal{B}r(g, \infty)$, then $\bar{b} \sim \bar{c} \subset \mathcal{H}_g$ if and only if $b = c \in \mathcal{B}r(g, \infty)^{\text{Ma}}$.

Remark 2.10. Although it may appear that Definition 2.7 omits conjugation by certain elements that clearly give isotopic closures e.g. by the “maximal loop” $\omega = t_g \dots t_1$ or its inverse, [HOL02, Section 5] shows how conjugation by such elements can be described in terms of the above Markov moves.

2E. From handlebody braids to classical braids. Recall from Section 1B that one of our main ingredients in constructing homological invariants of links in \mathcal{H}_g for all $g \geq 0$ is the relation between $\mathcal{B}r(g, n)$ and (a colored variant of) the type A braid group $\mathcal{B}r(g, n)$: We have a group homomorphism $\mathcal{B}r(g, n) \rightarrow \mathcal{B}r(0, g + n)$ given by viewing the core strands as “usual” strands, e.g.

(2-11)

As discussed above, this map is clearly injective, hence we have:

Proposition 2.11. The map induced by (2-11) gives rise to an embedding of groups

$$\mathcal{B}r(g, n) \hookrightarrow \mathcal{B}r(g + n). \tag{2-12}$$

□

However, Proposition 2.11 is only one ingredient in our construction, since invariance under the procedure

(2-13)

(i.e. under “conjugation in $\mathcal{B}r(g + n)$ ”) is not desirable for an invariant of

$$l \sim \bar{b} \subset \mathcal{H}_g,$$

cf. Remark 2.8. As such, we will use the theory of singular Soergel bimodules to mimic the merging and splitting of the core strands in the closure procedure for $\mathcal{B}r(g, n)$, which will lead to invariants of \bar{b} that are not invariant under (2-13).

3. Braids in handlebodies and singular type A Soergel bimodules

In the present section, we construct a map from $\mathcal{B}r(g, n)$ to the 2-category of singular Soergel bimodules.

3A. Parabolic subgroups and Frobenius extensions. Fix $N \in \mathbb{N}_{\geq 1}$ and let $R := R_N := \mathbb{K}[x_1, \dots, x_N]$ be the \mathfrak{q} -graded polynomial ring with $\mathfrak{q}\text{deg}(x_i) = 2$ for all i (by convention, $R_0 := \mathbb{K}$). The symmetric group $S(N) = W(A_{N-1})$ acts on R via

$$\sigma_i \cdot x_j = \begin{cases} x_{i+1} & \text{if } i = j, \\ x_i & \text{if } i = j + 1, \\ x_j & \text{else.} \end{cases} \tag{3-1}$$

Remark 3.1. Recall that Tits defined a faithful representation of any Coxeter group $W(\Gamma)$ on a real vector space of dimension $|V|$, commonly called the reflection representation of $W(\Gamma)$. (Recall our notation from Section 1B.) This representation is a crucial ingredient in the original definition of the associated category of Soergel bimodules, see [Soe92, Section 1.4]. In our case, this is the standard (irreducible) representation of $S(N)$ of dimension $N - 1$. By contrast, the representation given by (3-1) is built from the N -dimensional permutation representation, which decomposes as a direct sum of the standard representation and the trivial representation. By e.g. [EK10, Section 4.6] and [EL17, Theorem 2.7

and Proposition 2.10], the difference (akin to the difference between considering \mathfrak{gl}_N rather than \mathfrak{sl}_N) will not play a role in the present work, in the sense that all results from the cited literature hold in this case as well.

Fix any tuple $\mathbf{I} = (k_1, \dots, k_r) \in \mathbb{N}_{\geq 1}^r$ with $k_1 + \dots + k_r = N$. (Note further that choosing \mathbf{I} also determines N since $N = k_1 + \dots + k_r$. We will tacitly use this throughout.) By definition, the corresponding parabolic subgroup is

$$S_{\mathbf{I}}(N) := S(k_1) \times \dots \times S(k_r) \subset S(N). \tag{3-2}$$

Since there is a bijection between tuples and parabolic subgroups, we will implicitly identify them, e.g. $\mathbf{I} \subset \mathbf{J}$ denotes an inclusion of parabolic subgroups.

Given a parabolic subgroup \mathbf{I} , we let $R^{\mathbf{I}} := R^{S_{\mathbf{I}}(N)}$ be the ring of invariants. This ring is \mathfrak{q} -graded, since the action in (3-1) is \mathfrak{q} -homogeneous.

Example 3.2. The parabolic subgroups in (3-2) can alternatively be defined by choosing corresponding subsets of the vertices $V = \{1, \dots, N-1\}$ of the type A_{N-1} Coxeter diagram (with the left-right order of the vertices). For type A_3 one gets

$$\begin{aligned} (1, 1, 1, 1) &\leftrightarrow \emptyset, & (1, 2, 1) &\leftrightarrow \{2\}, \\ (2, 1, 1) &\leftrightarrow \{1\}, & (1, 1, 2) &\leftrightarrow \{3\}, \\ (3, 1) &\leftrightarrow \{1, 2\}, & (1, 3) &\leftrightarrow \{2, 3\}, \\ (2, 2) &\leftrightarrow \{1, 3\}, & (4) &\leftrightarrow \{1, 2, 3\}. \end{aligned} \tag{3-3}$$

Above we have listed all choices of tuples and the associated parabolic subgroups. Thus, $R^{(1,1,1,1)} = R^{\emptyset}$ is R itself, while $R^{(4)} = R^{\{1,2,3\}}$ is the \mathbb{K} -algebra of symmetric polynomials in four variables.

For the duration, we will use the following ordering convention for parabolic subgroups $\mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{L}$ and their rings of invariants:

$$\tag{3-4}$$

The \mathfrak{q} -degree 0 inclusion

$$\iota_{\mathbf{I}}^{\mathbf{J}}: R^{\mathbf{J}} \hookrightarrow R^{\mathbf{I}} \tag{3-5}$$

of \mathbb{K} -algebras is a \mathbf{q} -graded Frobenius extension (see [ESW17]), meaning that $\mathbb{R}^{\mathbf{I}}$ is a \mathbf{q} -graded, free $\mathbb{R}^{\mathbf{J}}$ -module of finite rank, possessing a non-degenerate, $\mathbb{R}^{\mathbf{J}}$ -linear trace map $\mathbb{R}^{\mathbf{I}} \rightarrow \mathbb{R}^{\mathbf{J}}$. In the present case, the latter is built using the Demazure operators $\partial_{\sigma_i}: \mathbb{R} \rightarrow \mathbb{R}^{\{\mathbf{i}\}} \subset \mathbb{R}$, given by $\partial_{\sigma_i}(f) := (f - \sigma_i \cdot f)/\alpha_i$ for the roots $\alpha_i = \mathbf{x}_i - \mathbf{x}_{i+1}$. The collection $\{\partial_{\sigma_i}\}$ satisfies the classical braid relations, and thus gives a well-defined map ∂_w associated to any $w \in S(N)$ using a reduced expression for w . Using these, the aforementioned trace map is given by

$$\partial_{\mathbf{I}}^{\mathbf{J}}: \mathbb{R}^{\mathbf{I}} \longrightarrow \mathbb{R}^{\mathbf{J}}, \quad f \longmapsto \partial_{w_{\mathbf{I}}w_{\mathbf{J}}^{-1}}(f) \tag{3-6}$$

and is of \mathbf{q} -degree $2\ell(\mathbf{I}) - 2\ell(\mathbf{J})$. Here $w_{\mathbf{I}}$ is the longest element in $S_{\mathbf{I}}(N)$, and $\ell(\mathbf{I})$ denotes its length.

The Frobenius extension data allows for the definition of maps between certain $\mathbb{R}^{\mathbf{I}}$ -bimodules, that will serve as important morphisms between singular Soergel bimodules (we recall the definition of the latter below). To wit, given a basis $\{a_i\}$ for $\mathbb{R}^{\mathbf{I}}$ over $\mathbb{R}^{\mathbf{J}}$, we can find a dual basis $\{a_i^*\}$ satisfying $\partial_{\mathbf{I}}^{\mathbf{J}}(a_i a_j^*) = \delta_{ij}$. Given this, we obtain the Frobenius element $\mathbf{a} := \sum_i a_i \otimes a_i^*$, which is of \mathbf{q} -degree $2\ell(\mathbf{J}) - 2\ell(\mathbf{I})$ and independent of the choice of $\{a_i\}$. This gives multiplication and comultiplication maps

$$\begin{aligned} \mu_{\mathbf{I}}^{\mathbf{J}}: \mathbb{R}^{\mathbf{I}} \otimes_{\mathbb{R}^{\mathbf{J}}} \mathbb{R}^{\mathbf{I}} &\longrightarrow \mathbb{R}^{\mathbf{I}}, & f \otimes g &\longmapsto fg, & \mathbf{q}\text{-degree } 0, \\ \Delta_{\mathbf{I}}^{\mathbf{J}}: \mathbb{R}^{\mathbf{I}} &\longrightarrow \mathbb{R}^{\mathbf{I}} \otimes_{\mathbb{R}^{\mathbf{J}}} \mathbb{R}^{\mathbf{I}}, & f &\longmapsto f\mathbf{a}, & \mathbf{q}\text{-degree } 2(\ell(\mathbf{J}) - \ell(\mathbf{I})). \end{aligned} \tag{3-7}$$

These morphisms of bimodules are unital and counital with respect to $\iota_{\mathbf{I}}^{\mathbf{J}}$ and $\partial_{\mathbf{I}}^{\mathbf{J}}$, respectively.

Example 3.3. For $\mathbf{I} = \emptyset$ and $\mathbf{J} = \{\mathbf{i}\}$, we have $w_{\mathbf{I}} = 1$ and $w_{\mathbf{J}} = \sigma_i$. It follows that $\{1, \frac{1}{2}\alpha_i\}$ and $\{1^* = \frac{1}{2}\alpha_i, (\frac{1}{2}\alpha_i)^* = 1\}$ are dual bases for \mathbb{R} as an $\mathbb{R}^{\{\mathbf{i}\}}$ -module, and $\mathbf{a} = \frac{1}{2}(1 \otimes \alpha_i + \alpha_i \otimes 1)$.

Finally, let us explicitly identify the rings $\mathbb{R}^{\mathbf{I}}$ for all $\mathbf{I} = (k_1, \dots, k_r)$. To this end, we consider r alphabets \mathbb{X}_i (we tend to omit the alphabets if no confusion can arise) with k_i variables, and write $\otimes_{\mathbb{K}} = \otimes$. A classical result about symmetric functions gives that

$$\mathbb{R}^{\mathbf{I}} \cong \mathbb{K}[\mathbf{e}_1(\mathbb{X}_1), \dots, \mathbf{e}_{k_1}(\mathbb{X}_1)] \otimes \dots \otimes \mathbb{K}[\mathbf{e}_1(\mathbb{X}_r), \dots, \mathbf{e}_{k_r}(\mathbb{X}_r)], \tag{3-8}$$

where $\mathbf{e}_j(\mathbb{X}_i)$ denotes the j^{th} elementary symmetric function in the variables \mathbb{X}_i . Note that $\mathbf{qdeg}(\mathbf{e}_j) = 2j$. In particular,

$$\mathbf{qdim}_{\mathbb{K}}(\mathbb{R}^{\mathbf{I}}) = \prod_{j=1}^r \prod_{i=1}^{k_j} \frac{1}{1-\mathbf{q}^{2i}}. \tag{3-9}$$

3B. A reminder on type A singular Soergel bimodules. We now briefly recall the category of singular Soergel bimodules $\mathbf{SS}^{\mathfrak{q}}(N) = \mathbf{SS}^{\mathfrak{q}}(A_{N-1})$ of type A_{N-1} , which categorifies the Hecke/Schur algebra of type A [Wil11, Theorem 1.2] in characteristic 0. Details (in more generality) can be found e.g. in [Wil11], or [EL17] and [ESW17] for the underlying diagrammatic calculus.

Define the merge (“restriction”) and split (“induction”) bimodules as follows:

$${}_J M_I := \mathfrak{q}^{\ell(I)-\ell(J)} R^J \otimes_{R^J} R^I, \quad {}_I S_J := R^I \otimes_{R^J} R^J, \quad (3-10)$$

where we follow the conventions from (3-4). Here, we have indicated the left/right actions using left/right subscripts, a convention that we will use throughout. There is a (horizontal) composition of such bimodules given by tensoring over the common (“middle”) ring, which we denote e.g. by ${}_L M_J M_I = {}_L M_J \otimes_{R^J} {}_J M_I$. In particular, we have the following \mathfrak{q} -degree 0 bimodule isomorphisms that we implicitly use below:

$${}_L M_J M_I \cong {}_L M_I \cong {}_L M_K M_I, \quad {}_I S_J S_L \cong {}_I S_L \cong {}_I S_K S_L. \quad (3-11)$$

All of the isomorphism in (3-11) are essentially identities, as the careful reader is invited to check. (Note e.g. that $f \otimes g \otimes h = 1 \otimes 1 \otimes fgh \in {}_L M_J M_I$.)

Definition 3.4. Let $\mathbf{SS}^{\mathfrak{q}}(N)$ be the \mathbb{K} -linear, \mathfrak{q} -graded 2-category given as the additive Karoubi 2-closure (meaning taking direct sums and summands) of the 2-category where objects are parabolic subgroups $I \subset S(N)$, 1-morphisms are generated by \mathfrak{q} -shifts of

$$R^I: I \longrightarrow I, \quad {}_J M_I: I \longrightarrow J, \quad {}_I S_J: J \longrightarrow I \quad (3-12)$$

for $I \subset J$, and 2-morphisms are (all) bimodule maps of \mathfrak{q} -degree 0.

Example 3.5. We have $\mathfrak{q}^{-1} R \otimes_{R^{\{i\}}} R \cong {}_{1,1} S_2 M_{1,1}$, which the reader familiar with (usual) Soergel bimodules of type A (see e.g. [EW14]) might recognize as being so-called Bott–Samelson bimodules. In particular, Soergel bimodules of type A_{N-1} can be identified with the 2-category $\mathbf{End}_{\mathbf{SS}^{\mathfrak{q}}(N)}(\emptyset)$, which has just one object (hence is a monoidal category).

3C. Web diagrammatics. Following e.g. ideas in [MSV11, Section 3], the generating 1-morphisms from (3-12), and compositions thereof, admit a description in terms of an MOY-type calculus, which we now sketch.

The basic building blocks are the identity, merge, and split bimodules, which are depicted using the following (local) graphical notation:

$$\begin{array}{c} k_i \\ | \\ k_i \end{array} \longleftrightarrow R^{k_i}, \quad \begin{array}{c} k+l \\ \swarrow \quad \searrow \\ k \quad l \end{array} \longleftrightarrow {}_{k+l} M_{k,l}, \quad \begin{array}{c} k \quad l \\ \swarrow \quad \searrow \\ k+l \end{array} \longleftrightarrow {}_{k,l} S_{k+l}. \quad (3-13)$$

Here, and for the duration, we use the abbreviation \mathbb{R}^k for the ring associated to $\mathbb{I} = (k)$.

Recall that, by Convention 1.2, vertical concatenation of such pictures corresponds to composition of 1-morphisms, and e.g. composing on the left corresponds to stacking on the top. Moreover, we can place such diagrams side-by-side, which corresponds to taking the tensor product over \mathbb{K} . Hence, we can associate a singular Soergel bimodule to each trivalent graph that we can build from these diagrams via these operations. (Note that webs corresponding to singular Soergel bimodules never have edges of negative label, but we will allow them in formulas for convenience of notation, with the understanding that the corresponding bimodules are zero.)

Example 3.6. For $N = 2$, the standard way to depict the Soergel bimodule from Example 3.5 (see e.g. [Kho07, Figure 2]) is built into our conventions:

$$\begin{array}{c} 1 & 1 \\ \cup & \\ \cap & \\ 1 & 1 \end{array} := \begin{array}{c} 1 & 1 \\ \cup & \\ 2 & \end{array} \otimes_{\mathbb{R}^2} \begin{array}{c} 2 \\ \cap \\ 1 & 1 \end{array} \iff \mathbf{q}^{-1}\mathbb{R} \otimes_{\mathbb{R}\{i\}} \mathbb{R} \cong {}_{1,1}\mathbb{S}_2\mathbb{M}_{1,1}. \tag{3-14}$$

Also of importance will be the ladder-rung bimodules:

$$\begin{array}{c} | & a & | \\ \diagdown & & / \\ k & & l \end{array} := \begin{array}{c} \cap \\ | & a & | \\ \cup \\ k & & l \end{array} \quad \& \quad \begin{array}{c} | & a & | \\ / & & \diagdown \\ k & & l \end{array} := \begin{array}{c} \cup \\ | & a & | \\ \cap \\ k & & l \end{array} \tag{3-15}$$

that will be used to build the square bimodules appearing in the complexes in (3-18) below.

Example 3.7. There exist \mathbf{q} -degree 0 bimodule isomorphisms

$$\begin{array}{c} k+l+m \\ \cap \\ k & l & m \end{array} \cong \begin{array}{c} k+l+m \\ \cup \\ k & l & m \end{array} \quad \& \quad \begin{array}{c} k & l & m \\ \cup \\ k+l+m \end{array} \cong \begin{array}{c} k & l & m \\ \cap \\ k+l+m \end{array} \tag{3-16}$$

that follow from the isomorphisms in (3-11). Hence, we can unambiguously write

$$\begin{array}{c} k_1 + \dots + k_r \\ \cap \\ k_1 & \dots & k_r \end{array} \quad \& \quad \begin{array}{c} k_1 & k_r \\ \cup \\ k_1 + \dots + k_r \end{array} \tag{3-17}$$

3D. Rickard–Rouquier complexes

Definition 3.8. Given an additive category \mathcal{C} , we denote its bounded homotopy category by $\mathcal{K}^b(\mathcal{C})$. This is the category whose objects are bounded chain complexes, and whose morphisms are homotopy classes of chain maps. We will use \simeq to denote isomorphisms in $\mathcal{K}^b(\mathcal{C})$, i.e. homotopy equivalence.

Recalling Section 1D, we can view the objects in $\mathcal{K}^b(\mathcal{C})$ as finite direct sums $\bigoplus_i \mathbf{t}^{k_i} \mathbf{X}_i$, equipped with a differential d with $\mathbf{tdeg}(d) = -1$. There is a \mathbf{t} -degree zero inclusion of categories $\mathcal{C} \hookrightarrow \mathcal{K}^b(\mathcal{C})$ given by considering objects of \mathcal{C} as one-term complexes concentrated in \mathbf{t} -degree 0. We also remark that we can consider $\mathcal{K}^b(\mathcal{C})$ for a 2-category \mathcal{C} , by passing to the homotopy category in each Hom-category. In particular, if \mathcal{C} is monoidal, then so is $\mathcal{K}^b(\mathcal{C})$.

We now recall Rickard–Rouquier complexes, i.e. complexes of singular Soergel bimodules that determine maps from the (colored) braid group(oid) into certain Hom-categories in $\mathcal{K}^b(\mathbf{SS}^q(N))$. Our terminology here arises as these complexes correspond to the Rickard complexes (originally defined for symmetric groups) in categorified quantum groups, but also agree with the type A Rouquier complexes in the “uncolored” $k = l = 1$ case.

They are given as follows:

$$\begin{array}{c}
 \begin{array}{c} l & k \\ \diagdown & / \\ k & l \end{array} := \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \xrightarrow{d_0^+} \mathbf{t} \mathbf{q}^{-1} \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \xrightarrow{d_1^+} \dots \xrightarrow{d_{m-1}^+} \mathbf{t}^m \mathbf{q}^{-m} \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \\
 \\
 \begin{array}{c} l & k \\ \diagdown & / \\ k & l \end{array} := \mathbf{t}^{-m} \mathbf{q}^m \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \xrightarrow{d_{m-1}^-} \dots \xrightarrow{d_1^-} \mathbf{t}^{-1} \mathbf{q} \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \xrightarrow{d_0^-} \begin{array}{c} l & k \\ | & | \\ k & l \end{array}
 \end{array} \tag{3-18}$$

where $m = \min(k, l)$. Our notation denotes e.g. that, as a \mathbf{tq} -graded bimodule, $\llbracket \beta_i \rrbracket_{k,l}$ is the direct sum of the indicated terms, and the arrows depict the non-zero components of the differentials. Recalling the bimodule maps from (3-11), (3-5), (3-6), (3-7), and omitting the \mathbf{tq} -shifts, these are given by

$$\begin{array}{c}
 d_i^+ : \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \begin{array}{c} \xrightarrow{\partial} \\ \xleftarrow{\iota} \end{array} \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \begin{array}{c} \xrightarrow{(3-11)} \\ \xleftarrow{(3-11)} \end{array} \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \begin{array}{c} \xrightarrow{\mu} \\ \xleftarrow{\Delta} \end{array} \begin{array}{c} l & k \\ | & | \\ k & l \end{array} \begin{array}{c} \xrightarrow{\iota} \\ \xleftarrow{\partial} \end{array} \begin{array}{c} l & k \\ | & | \\ k & l \end{array} : d_i^-
 \end{array} \tag{3-19}$$

Here the corresponding parabolic subsets, which determine the bimodule maps, can be read from the indicated sequence of webs, and we use e.g. $\iota_{\mathbb{I}}^{\mathbb{J}}$ as

$$\iota_{\mathbb{I}}^{\mathbb{J}}: \mathbb{R}^{\mathbb{J}} \cong \mathbb{R}^{\mathbb{J}} \otimes_{\mathbb{R}^{\mathbb{J}}} \mathbb{R}^{\mathbb{J}} \otimes_{\mathbb{R}^{\mathbb{J}}} \mathbb{R}^{\mathbb{J}} \hookrightarrow \mathbb{R}^{\mathbb{J}} \otimes_{\mathbb{R}^{\mathbb{J}}} \mathbb{R}^{\mathbb{I}} \otimes_{\mathbb{R}^{\mathbb{J}}} \mathbb{R}^{\mathbb{J}} = {}_{\mathbb{J}}M_{\mathbb{I}}S_{\mathbb{J}}. \quad (3-20)$$

Remark 3.9. We note that the differential in the Rickard–Rouquier complexes can be described diagrammatically using type A singular Soergel calculus, see e.g. [EL17, Section 2]. Alternatively, we could work with the $n \rightarrow \infty$ limit of the 2-category of \mathfrak{gl}_n foams to describe these 2-morphisms in $\mathbf{SS}^{\mathfrak{q}}(N)$ (here, n is a parameter independent of N). In fact, these two descriptions are equivalent, as the type A singular Soergel calculus corresponds to the “calculus of seams” in the foam framework. (See e.g. [QRS18, Section 5.2] for a precise statement.)

Finally, the fact that these indeed are complexes follows e.g. by comparing (3-18) to the Rickard complex in the categorified quantum group, as in Remark 3.9.

Example 3.10. In the uncolored case $k = l = 1$ the complexes are

$$\begin{array}{c} 1 & 1 \\ \diagdown & / \\ & \\ / & \diagdown \\ 1 & 1 \end{array} = \begin{array}{c} 1 & 1 \\ | & | \\ \diagdown & / \\ 1 & 1 \end{array} \xrightarrow{\mu_{1,1}^2} \mathbf{t} \mathbf{q}^{-1} \begin{array}{c} 1 & 1 \\ | & | \\ 1 & 1 \end{array} \quad \& \quad \begin{array}{c} 1 & 1 \\ \diagdown & / \\ & \\ / & \diagdown \\ 1 & 1 \end{array} = \mathbf{t}^{-1} \mathbf{q} \begin{array}{c} 1 & 1 \\ | & | \\ 1 & 1 \end{array} \xrightarrow{\Delta_{1,1}^2} \begin{array}{c} 1 & 1 \\ | & | \\ \diagdown & / \\ 1 & 1 \end{array} \quad (3-21)$$

Remark 3.11. The conventions in Example 3.10 are the same as in [Rou06], except that in that work, there is no shift on the bimodule ${}_{1,1}S_2M_{1,1}$.

Example 3.12. There exist \mathfrak{q} -degree 0 isomorphisms in $\mathcal{K}^b(\mathbf{SS}^{\mathfrak{q}}(N))$

$$\begin{array}{c} l+m & k \\ | & | \\ \diagdown & / \\ k & l & m \end{array} \cong \begin{array}{c} l+m & k \\ | & | \\ \diagdown & / \\ k & l & m \end{array} \quad \& \quad \begin{array}{c} k+l \\ | & | \\ \diagdown & / \\ k & l \end{array} \cong \mathbf{q}^{kl} \begin{array}{c} k+l \\ | & | \\ \diagdown & / \\ k & l \end{array} \quad \& \quad \begin{array}{c} k+l \\ | & | \\ \diagdown & / \\ k & l \end{array} \cong \mathbf{q}^{-kl} \begin{array}{c} k+l \\ | & | \\ \diagdown & / \\ k & l \end{array} \quad (3-22)$$

as well as variants with analogous \mathfrak{q} -shifts involving split bimodules.

Let \mathbb{I} be a parabolic subgroup, and let $\#\mathbb{I}$ denote the number of entries in the corresponding tuple (i.e. for an r -tuple \mathbb{I} , $\#\mathbb{I} = r$). Given a braid generator $b_i \in \mathcal{B}^r(\#\mathbb{I})$, we let $[[b_i^{\pm 1}]]_{\mathbb{I}}$ denote the complex given by placing appropriately labeled vertical strands next to the corresponding complex in (3-18), i.e. by taking tensor product over \mathbb{K} with the rings $\mathbb{R}^{(k_1, \dots, k_{i-1})}$ and $\mathbb{R}^{(k_i+2, \dots, k_{\#\mathbb{I}})}$.

Definition 3.13. For I and $b \in \mathcal{B}r(\#I)$, fix an expression $b = b_{i_1}^{\pm 1} \dots b_{i_r}^{\pm 1}$. Define

$$[[b]]_I := [[b_{i_1}^{\pm 1}]]_{I'} \dots [[b_{i_r}^{\pm 1}]]_I \tag{3-23}$$

where, on the right-hand side, we use composition in $\mathcal{K}^b(\mathbf{SS}^q(N))$, i.e. tensor product of the complexes of singular Soergel bimodules.

By e.g. the results in [QRS18, Section 5.2], the complex $[[b]]_I$ does not depend, up to isomorphism, on the choice of expression for b . Thus, the assignment $b \mapsto [[b]]_I$ gives an action of $\mathcal{B}r(\#I)$ on $\mathcal{K}^b(\mathbf{SS}^q(N))$. We get:

Proposition 3.14. *There is an action of $\mathcal{B}r(g, n)$ on $\mathcal{K}^b(\mathbf{SS}^q(N))$ determined by the assignment $b \mapsto [[b]]_I$.*

Proof. By the discussion above, we have an action of the classical braid group. Composing this action with the map from Proposition 2.11 gives the desired action of the handlebody braid group. □

4. Colored HOMFLYPT homology for links in handlebodies

In this section, we proceed to construct our triply-graded invariant of links in \mathcal{H}_g , with Theorem 4.7 and Corollary 4.13 being the main statements. We keep the notation from the previous sections and begin with some preliminaries.

4A. A reminder on Hochschild cohomology. Let A be a \mathfrak{q} -graded \mathbb{K} -algebra, and recall that we may regard any \mathfrak{q} -graded A -bimodule B as a \mathfrak{q} -graded left module over the enveloping algebra $A \otimes A^{\text{op}}$. The Hochschild cohomology of A with coefficients in B is the $\mathfrak{a}\mathfrak{q}$ -graded \mathbb{K} -vector space

$$\mathcal{H}\mathcal{H}^\bullet(A, B) := \bigoplus_{a \in \mathbb{Z}} \mathcal{H}\mathcal{H}^a(A, B) \tag{4-1}$$

with a -degree component defined by

$$\mathcal{H}\mathcal{H}^a(A, B) := \text{EXT}_{A \otimes A^{\text{op}}}^a(A, B) = \bigoplus_{s \in \mathbb{Z}} \text{Ext}_{A \otimes A^{\text{op}}}^a(\mathfrak{q}^s A, B). \tag{4-2}$$

(Compare our notation here to Convention 1.3.)

The relevant case for our considerations is when $A = \mathbf{R}^I = (\mathbf{R}^I)^{\text{op}}$. Here, for $I = \emptyset$, Khovanov showed that the triply-graded link homology from [KR08b] can

be constructed using the Hochschild homology (defined using Tor instead of Ext) of Soergel bimodules; see [Kho07, Section 1.1]. Recall from (3-8) that $\mathbb{R}^{\mathbb{I}}$ is a polynomial ring, so Hochschild homology and cohomology are isomorphic (up to a shift). We work with the latter since e.g. in this framework the invariant of the (colored) unknot inherits a natural algebra structure [Hog18], which is important for various considerations.

Example 4.1. Let $\mathbb{I} = (k_1, \dots, k_r)$. Recall that $\mathbb{R}^{\mathbb{I}}$ is a polynomial ring (and, in particular, is Koszul). Hence, we can compute Hochschild cohomology using the Koszul resolution of $\mathbb{R}^{\mathbb{I}}$, which is the free resolution of $\mathbb{R}^{\mathbb{I}}$ as an $\mathbb{R}^{\mathbb{I}}$ -bimodule given by

$$\bigotimes_{j=1}^r \left(\bigotimes_{i=1}^{k_j} (\mathbf{h}\mathbf{q}^{2i} \mathbb{R}^{\mathbb{I}} \otimes \mathbb{R}^{\mathbb{I}} \xrightarrow{e_i \otimes 1 - 1 \otimes e_i} \mathbb{R}^{\mathbb{I}} \otimes \mathbb{R}^{\mathbb{I}}) \right). \tag{4-3}$$

Here \mathbf{h} denotes a shift up in an auxiliary homological degree, and the outer tensor products are taken over $\mathbb{R}^{\mathbb{I}} \otimes \mathbb{R}^{\mathbb{I}}$. Given a $\mathbb{R}^{\mathbb{I}}$ -bimodule \mathbf{B} , taking the “internal” \mathbf{q} -graded Hom of complexes $\mathcal{H}\mathcal{O}\mathcal{M}_{\mathbb{R}^{\mathbb{I}} \otimes \mathbb{R}^{\mathbb{I}}}^{\bullet}(-, \mathbf{B})$ (i.e. applying $\text{HOM}_{\mathbb{R}^{\mathbb{I}} \otimes \mathbb{R}^{\mathbb{I}}}(-, \mathbf{B})$ to the terms and differentials of a chain complex to obtain a cochain complex) gives a complex concentrated in non-negative cohomological degree \mathbf{a} , which is the negative of the \mathbf{h} -degree. The a^{th} cohomology of this complex is $\mathcal{H}\mathcal{H}^a(\mathbb{R}^{\mathbb{I}}, \mathbf{B})$.

Computing for $\mathbf{B} = \mathbb{R}^{\mathbb{I}}$ gives the following. For each j , fix a set of variables $\{\theta_i \mid 1 \leq i \leq k_j\}$ with $\mathbf{aqdeg}(\theta_i) = (1, -2i)$, and recall that $\mathbf{aqdeg}(e_i) = (0, 2i)$. We then have an isomorphism of \mathbf{aq} -graded \mathbb{K} -vector spaces

$$\mathcal{H}\mathcal{H}^{\bullet}(\mathbb{R}^{\mathbb{I}}, \mathbb{R}^{\mathbb{I}}) \cong \bigotimes_{j=1}^r (\mathbb{K}[\mathbf{e}_1, \dots, \mathbf{e}_{k_j}] \otimes \bigwedge^{\bullet} \{\theta_i \mid 1 \leq i \leq k_j\}), \tag{4-4}$$

where $\bigwedge^{\bullet} \{\theta_i \mid 1 \leq i \leq k_j\}$ denotes the exterior algebra.

Since Hochschild cohomology is functorial with respect to bimodule morphisms, we can apply $\mathcal{H}\mathcal{H}^{\bullet}$ to a complex of $\mathbb{R}^{\mathbb{I}}$ -bimodules term-wise to obtain a complex of \mathbf{aq} -graded \mathbb{K} -vector spaces. (In fact, since our ring is commutative, these \mathbb{K} -vectors spaces inherit an action of $\mathbb{R}^{\mathbb{I}}$, so can be thought of as trivial $\mathbb{R}^{\mathbb{I}}$ -bimodules.)

In particular, let $\mathbb{R}^{\mathbb{I}}\text{Bim}^{\mathbf{q}}$ denote the category of \mathbf{q} -graded, finitely-generated $\mathbb{R}^{\mathbb{I}}$ -bimodules, and let $\mathcal{K}^b(\mathbb{R}^{\mathbb{I}}\text{Bim}^{\mathbf{q}})$ be its homotopy category. We get a functor

$$\mathcal{H}\mathcal{H}_{\mathbb{I}}^{\bullet}(-) := \bigoplus_{a \in \mathbb{Z}} \mathcal{H}\mathcal{H}_{\mathbb{I}}^a(-): \mathcal{K}^b(\mathbb{R}^{\mathbb{I}}\text{Bim}^{\mathbf{q}}) \longrightarrow \mathcal{K}^b(\mathbb{K}\text{Vec}^{\mathbf{aq}}) \tag{4-5}$$

whose a -degree component is the functor

$$\mathcal{H}\mathcal{H}_{\mathbb{I}}^a(-) := \mathcal{H}\mathcal{H}^a(\mathbb{R}^{\mathbb{I}}, -): \mathcal{K}^b(\mathbb{R}^{\mathbb{I}}\text{Bim}^{\mathbf{q}}) \longrightarrow \mathcal{K}^b(\mathbb{K}\text{Vec}^{\mathbf{q}}). \tag{4-6}$$

4B. Towards handlebody HOMFLYPT homology. Fix integers $M \in \mathbb{N}_{\geq 1}$ and $l_1, \dots, l_n \in \mathbb{N}_{\geq 1}$, called the core and link colors, respectively. For any $g \geq 0$, these choices determine a parabolic subset $\mathbb{M} := (M, \dots, M, l_1, \dots, l_n)$ with $\#\mathbb{M} = g + n$. We view \mathbb{M} as providing a coloring for braids $b \in \mathcal{B}r(g, n)$ as in Section 3D, where strands are colored at the bottom by the entries of \mathbb{M} . We will call a colored braid (b, \mathbb{M}) balanced if the colors at the top and bottom of the i^{th} position agree for all i . For the duration, we only consider balanced colorings and any braid or link will be colored by default.

Example 4.2. The prototypical example of a balanced coloring is the case where the link is uncolored, i.e. where $l_1 = \dots = l_n = 1$ and M is arbitrary. In general, M should be viewed as being “very large,” i.e. $M \gg l_i$ for all i ; compare e.g. to [ILZ18], where the core of the solid torus is colored by a Verma module.

Remark 4.3. It is possible to work with any balanced coloring of $b \in \mathcal{B}r(g, n)$, using e.g. colors M_1 to M_g for the core strands (recall, we have fixed a handlebody, so these core strands are distinguishable). However, if one wishes e.g. to further incorporate the action of the mapping class group of \mathcal{H}_g into our picture, then the core strands should be indistinguishable, hence uniformly colored. Given this, and to avoid overly cumbersome exposition in the following, we have chosen to not work in full generality.

Consider $\text{SS}^{\mathfrak{q}}(\mathbb{I}) := \mathbf{End}_{\text{SS}^{\mathfrak{q}}(N)}(\mathbb{I})$ which is a \mathfrak{q} -graded, full, monoidal subcategory of $\mathbb{R}^{\mathbb{I}}\text{Bim}^{\mathfrak{q}}$. The monoidal structure is inherited from the horizontal composition in $\text{SS}^{\mathfrak{q}}(N)$, i.e. it is given by tensor product over $\mathbb{R}^{\mathbb{I}}$. We will occasionally denote this by $\otimes_{\mathbb{R}^{\mathbb{I}}}$, in addition to our previous notation for this operation, which was simply concatenation.

Recalling Example 3.7 and Proposition 3.14, and motivated by Remark 2.8, we define:

Definition 4.4. For $b \in \mathcal{B}r(g, n)$ and (b, \mathbb{M}) a balanced coloring, we let

$$[[b]]_{\mathcal{H}_g} := \left(\begin{array}{cccc} M & M & l_1 & l_n \\ \vdots & \vdots & | & | \\ \text{⋈} & & \vdots & \vdots \\ \vdots & \vdots & | & | \\ M & M & l_1 & l_n \end{array} \right) \otimes_{\mathbb{R}^{\mathbb{M}}} [[b]]_{\mathbb{M}} \in \mathcal{K}^b(\text{SS}^{\mathfrak{q}}(\mathbb{M})). \tag{4-7}$$

Example 4.5. In the cases $g = 0, 1$ we have $[[b]]_{\mathcal{H}_g} = [[b]]_{\mathbb{M}}$. For $g = 2$, we have

$$b = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \curvearrowright \quad \& \quad [[b]]_{\mathbb{M}} = \begin{array}{c} M \quad M \quad l \\ | \quad | \quad | \\ \text{---} \\ | \\ \text{---} \\ | \\ M \quad M \quad l \end{array} \quad \& \quad [[b]]_{\mathcal{H}_2} = \begin{array}{c} M \quad M \quad l \\ | \quad | \quad | \\ \text{---} \\ | \\ \text{---} \\ | \\ M \quad M \quad l \end{array} \quad (4-8)$$

For $g \geq 2$, we generally have $[[b]]_{\mathcal{H}_g} \neq [[b]]_{\mathbb{M}}$, cf. Proposition 4.8 below.

Definition 4.6. For $b \in \mathcal{B}r(g, n)$ and (b, \mathbb{M}) a balanced coloring we let

$$\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b) := \bigoplus_{a \in \mathbb{Z}} \mathcal{H}\mathcal{H}^a([[b]]_{\mathcal{H}_g}). \quad (4-9)$$

By Proposition 3.14, $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b)$ is an invariant of the colored braid $b \in \mathcal{B}r(g, n)$ taking values in $\mathcal{K}^b(\mathbb{K}\text{Vec}^{\text{aq}})$.

4C. Colored handlebody HOMFLYPT homology. In (4-46) below we use $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b)$ to define $\mathcal{H}\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b)$, an invariant of the colored link $\bar{b} \subset \mathcal{H}_g$, valued in $\mathbb{K}\text{Vec}^{\text{atq}}$. To do so, we establish the following.

Theorem 4.7. *The assignment*

$$\mathcal{B}r(g, n) \longrightarrow \mathcal{K}^b(\mathbb{K}\text{Vec}^{\text{aq}}), \quad b \longmapsto \mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b) \quad (4-10)$$

is invariant under the conjugation (2-9) and stabilization (2-10) relations for $\mathcal{B}r(g, n)$, up to homotopy equivalence and grading normalization. Moreover, it is not generally invariant under the classical conjugation relation (2-13) for $\mathcal{B}r(g + n)$.

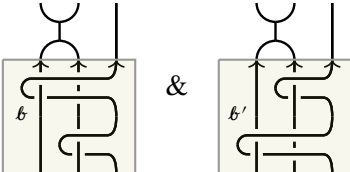
The remainder of this section constitutes a proof of this theorem. Namely, invariance under conjugation holds as a special case of the corresponding result for colored, triply-graded link homology in S^3 , and Lemma 4.12 establishes invariance (up to a grading shift) under stabilization. Proposition 4.8 shows the failure of invariance under the classical conjugation relation. We stress the importance of this latter fact: any invariant of the classical braid group $\mathcal{B}r(g + n)$, that additionally is invariant under classical conjugation and stabilization, gives rise to invariants of links in \mathcal{H}_g using the inclusion $\mathcal{B}r(g, n) \hookrightarrow \mathcal{B}r(g + n)$. However, such invariants are less-sensitive to the topology of \mathcal{H}_g , as our results show.

Proposition 4.8. *For $g > 1$ and $n > 0$, there exists handlebody braids $b, b' \in \mathcal{B}r(g, n)$ that are conjugate in $\mathcal{B}r(g + n)$, but satisfy $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b) \neq \mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b')$.*

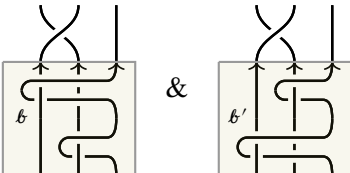
This shows that our handlebody homology, which is defined below to be the cohomology of a renormalization of $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(-)$, distinguishes these handlebody links, while the invariant obtained by including into $\mathcal{B}r(g+n)$ and using the classical (colored) triply-graded homology does not.

Proof. It suffices to give an example, and we provide one in the $g = 2$ and $n = 1$ case that immediately generalizes to any $g \geq 2$ and $n \geq 1$. Let $b = t_2 t_1$ and $b' = t_1 t_2$, which are conjugate braids in $\mathcal{B}r(g+n)$. We claim that $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b) \not\cong \mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b')$, and exhibit this explicitly in the case that $M = 1$.

Indeed, if they were homotopy equivalent, then the Euler characteristics (i.e. alternating sums of $\mathbf{a}\mathbf{q}$ -graded dimensions) of these complexes would agree. However, since the category of (usual) type A Soergel bimodules categorifies the type A Hecke algebra, and Hochschild cohomology categorifies the Jones–Ocneanu trace, this would imply that the Jones–Ocneanu traces of the following braided, trivalent graphs agree:


(4-11)

Using the decategorification of the first equation in Example 3.10, this in turn would imply that the HOMFLYPT polynomials of the links given as the closures of


(4-12)

agree. However, a computation shows that the difference between their (reduced) HOMFLYPT polynomials is $(a - a^{-1})^2 - (q - q^{-1})^2$, where a, q are variables (at the decategorified level) corresponding to \mathbf{a}, \mathbf{q} . \square

We now turn our attention to the behavior of $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b)$ under stabilization (2-10). Our main technical tool will be the partial Hochschild trace from [Hog18, Section 3], which we now adapt to the colored setting. The construction of this functor is motivated as follows. Since Hochschild cohomology satisfies the classical conjugation relation (i.e. the relation (2-13)), we informally view this operation as a mean to take the closure of (the singular Soergel bimodule associated

to) a web diagram. In order to study the stabilization relation, we would like to be able to take this closure “one strand at a time” in a manner that is compatible with taking Hochschild cohomology.

Recall that the Hochschild cohomology of an R^I -bimodule M is defined as

$$\mathcal{H}\mathcal{H}^a(R^I, M) = \text{EXT}_{R^I \otimes R^I}^a(R^I, M) \cong \text{HOM}_{\mathcal{D}^b(R^I \text{Bim}^{\mathfrak{q}})}(R^I, \mathbf{h}^a M), \quad (4-13)$$

where here we follow Convention 1.3 for the \mathfrak{q} -graded Hom. Here $\mathcal{D}^b(R^I \text{Bim}^{\mathfrak{q}})$ is the bounded derived category, and we emphasize that the homological degree therein is not the \mathbf{t} -degree from Section 3D, but rather the \mathbf{h} -degree from Example 4.1 (which should be viewed as “perpendicular” to the homological degree of the Rickard–Rouquier complexes). Our discussion above suggests that we should consider functors between the categories $\mathcal{D}^b(R^I \text{Bim}^{\mathfrak{q}})$ for various I that are compatible with the functors $\text{HOM}_{\mathcal{D}^b(R^I \text{Bim}^{\mathfrak{q}})}(R^I, -)$.

To this end, given $I = (k_1, \dots, k_r)$ we let $I^- := (k_1, \dots, k_{r-1})$, i.e. I^- is obtained from I by removing the last entry. Let

$$Q_I^{k_r} := R^I \otimes R^I / (e_i(X_r) \otimes 1 - 1 \otimes e_i(X_r))_{i=1}^{k_r}.$$

Using the notation in (3-8), we have

$$R^I \cong Q_I^{k_r} \otimes_{R^{I^-} \otimes R^{I^-}} R^{I^-}, \quad (4-14)$$

which suggests that we consider $\mathcal{J}_I: \mathcal{D}^b(R^{I^-} \text{Bim}^{\mathfrak{q}}) \rightarrow \mathcal{D}^b(R^I \text{Bim}^{\mathfrak{q}})$ given by derived tensor product \otimes^L with $Q_I^{k_r}$ over $R^{I^-} \otimes R^{I^-}$. We then obtain

$$\mathcal{J}_I: \mathcal{D}^b(R^I \text{Bim}^{\mathfrak{q}}) \longrightarrow \mathcal{D}^b(R^{I^-} \text{Bim}^{\mathfrak{q}}),$$

which we define to be the right adjoint to \mathcal{J}_I , using derived tensor-hom adjunction.

The functors \mathcal{J}_I and \mathcal{J}_I admit the following explicit descriptions. We have an isomorphism

$$Q_I^{k_r} \cong \bigotimes_{i=1}^{k_r} (\mathbf{h}\mathfrak{q}^{2i} R^I \otimes R^I \xrightarrow{e_i \otimes 1 - 1 \otimes e_i} R^I \otimes R^I) =: K_I^{k_r} \quad (4-15)$$

in $\mathcal{D}^b(R^I \text{Bim}^{\mathfrak{q}})$, where the (outer) tensor product is taken over $R^I \otimes R^I$. Since $K_I^{k_r}$ is a complex of free $R^I \otimes R^I$ -modules, given any complex $M \in \mathcal{D}^b(R^I \text{Bim}^{\mathfrak{q}})$, $\mathcal{J}_I(M)$ is the complex

$$\mathcal{H}\text{OM}_{R^I \otimes R^I}^\bullet(K_I^{k_r}, M) \cong \bigotimes_{i=1}^{k_r} (M \xrightarrow{e_i \otimes 1 - 1 \otimes e_i} \mathbf{a}\mathfrak{q}^{-2i} M). \quad (4-16)$$

(In the case that M is a complex, we interpret the latter as a shift of the cone of the indicated chain map.) Similarly, for $N \in \mathcal{D}^b(\mathbb{R}^{I^-} \text{Bim}^{\mathfrak{q}})$, we have that

$$\mathcal{J}_I(N) = K_I^{k_r} \otimes_{\mathbb{R}^{I^-} \otimes \mathbb{R}^{I^-}} N, \tag{4-17}$$

where we again interpret the latter as the total complex of this double complex. Here, since $K_I^{k_r} \cong Q_I^{k_r}$ and the latter is a free \mathbb{R}^{I^-} -bimodule, we also have

$$\mathcal{J}_I(N) = Q_I^{k_r} \otimes_{\mathbb{R}^{I^-} \otimes \mathbb{R}^{I^-}} N. \tag{4-18}$$

Our next result collects the salient features of \mathcal{J}_I and \mathcal{T}_I needed for our considerations, both of which follow immediately from the definition of \mathcal{J}_I and \mathcal{T}_I .

Lemma 4.9. *For all $M \in \mathcal{D}^b(\mathbb{R}^I \text{Bim}^{\mathfrak{q}})$ and all $a \in \mathbb{Z}$, there is a functorial isomorphism*

$$\mathcal{H}\mathcal{H}^a(\mathbb{R}^I, M) \cong \mathcal{H}\mathcal{H}^a(\mathbb{R}^{I^-}, \mathcal{J}_I(M)). \tag{4-19}$$

Additionally, given $N, P \in \mathcal{D}^b(\mathbb{R}^{I^-} \text{Bim}^{\mathfrak{q}})$, we have

$$\mathcal{J}_I(\mathcal{J}_I(N) \otimes_{\mathbb{R}^I}^L M \otimes_{\mathbb{R}^I}^L \mathcal{J}_I(P)) \cong N \otimes_{\mathbb{R}^{I^-}}^L \mathcal{J}_I(M) \otimes_{\mathbb{R}^{I^-}}^L P \tag{4-20}$$

Finally, setting $I^1 := \{k_1, \dots, k_s\}$ and $I^2 := \{k_{s+1}, \dots, k_r\}$, we have

$$\mathcal{T}_I(M_1 \otimes_{\mathbb{K}} M_2) \cong M_1 \otimes_{\mathbb{K}} \mathcal{T}_I(M_2). \tag{4-21}$$

for $M_1 \in \mathcal{D}^b(\mathbb{R}^{I^1} \text{Bim}^{\mathfrak{q}})$ and $M_2 \in \mathcal{D}^b(\mathbb{R}^{I^2} \text{Bim}^{\mathfrak{q}})$.

We will use the functors \mathcal{J}_I and \mathcal{T}_I to give a “local” proof of invariance under stabilization. Note that there is a \mathfrak{q} -degree 0, fully faithful inclusion functor $\text{SS}^{\mathfrak{q}}(I) \hookrightarrow \mathcal{D}^b(\mathbb{R}^I \text{Bim}^{\mathfrak{q}})$ given by viewing a singular Soergel bimodule as a complex concentrated in \mathfrak{h} -degree zero. Further, this functor is monoidal (with respect to $\otimes_{\mathbb{R}^I}$ on the former and $\otimes_{\mathbb{R}^I}^L$ on the latter) since singular Soergel bimodules are free as either left or right \mathbb{R}^I -modules. (The latter can be deduced from the fact that \mathbb{R}^I is free over \mathbb{R}^J for $I \subseteq J$, cf. Section 3A.)

Given this, we now develop a graphical interpretation for the action of the functors \mathcal{J}_I and \mathcal{T}_I on singular Bott–Samelson bimodules, again adapting [Hog18, Section 3.3] to the singular setting. Since our eventual aim is to apply these results to $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(-)$, we will focus on the (bimodules appearing in the) complex $[[b]]_{\mathcal{H}_g}$. Let $M := (M, \dots, M, l_1, \dots, l_n)$, then, for $B \in \text{SS}^{\mathfrak{q}}(M^-)$ and $C \in \text{SS}^{\mathfrak{q}}(M)$, we depict \mathcal{J}_M and \mathcal{T}_M as follows:

$$\mathcal{J}_M \left(\begin{array}{c} \vdots \\ \vdots \\ \text{---} B \text{---} \\ \vdots \\ \vdots \end{array} \right) = \begin{array}{c} \vdots \\ \vdots \\ \text{---} B \text{---} \\ \vdots \\ \vdots \end{array} \quad \& \quad \mathcal{T}_M \left(\begin{array}{c} \vdots \\ \vdots \\ \text{---} C \text{---} \\ \vdots \\ \vdots \end{array} \right) = \begin{array}{c} \vdots \\ \vdots \\ \text{---} C \text{---} \\ \vdots \\ \vdots \end{array} \tag{4-22}$$

Similarly, taking Hochschild cohomology will be depicted by closing all (non-core and core) strands. In this language, the first statement in Lemma 4.9 says that we obtain the same result whether we close all strands at once or one at a time, while the second and third are

(4-23)

Next, we compute the value of the colored partial trace on the “merge-split” bimodule. (Strictly speaking, we will only use the $k = l = 1$ case of Lemma 4.10, which is given e.g. in [Hog18, equation (3.1b)]. However, as we are developing the skein calculus for the colored partial trace, and since we anticipate applications of this formula to explicit computations of our invariant, we take the opportunity to extend loc. cit. to the colored setting.)

Lemma 4.10. *For $k, l \geq 0$, there is an \mathbf{atq} -degree 0 isomorphism*

$$\begin{array}{c} k \\ \text{---} \\ \text{---} \\ \text{---} \\ k \end{array} \Big|_l \cong \prod_{i=1}^l \frac{\mathbf{q}^k + \mathbf{a}\mathbf{q}^{k-2i}}{1 - \mathbf{q}^{2i}} \Big|_k \tag{4-24}$$

Proof. In the $k = 0$ case, the result simply claims that the l -colored circle is a \mathbb{K} -vector space of \mathbf{aq} -graded dimension $\prod_{i=1}^l \frac{1 + \mathbf{a}\mathbf{q}^{-2i}}{1 - \mathbf{q}^{2i}}$. This follows directly from Example 4.1.

We thus assume that $k \geq 1$, and proceed as in the proof of [Hog18, Proposition 3.10]. Namely, we explicitly write down the value of $\mathcal{T}_{\mathbb{I}}$ on the bimodule in the left-hand side of (4-24), apply a change of variables, and use this to explicitly identify the result in the derived category.

To this end, we assign alphabets of \mathbf{q} -degree 2 variables to the boundary points of the corresponding web as follows:

(4-25)

where $\#\mathbb{X}_1 = k = \#\mathbb{X}'_1$ and $\#\mathbb{X}_2 = l = \#\mathbb{X}'_2$. Precisely, by this assignment we identify the (singular) Bott–Samelson bimodule ${}_{k,l}\mathbf{S}_{k+l}\mathbf{M}_{k,l}$ with the following quotient of the (shifted) polynomial ring generated by the elementary symmetric functions in these alphabets:

$$\mathbf{q}^{-kl}\mathbb{K}[\mathbf{e}_r(\mathbb{X}_1), \mathbf{e}_r(\mathbb{X}'_1), \mathbf{e}_s(\mathbb{X}_2), \mathbf{e}_s(\mathbb{X}'_2)]/(\mathbf{e}_t(\mathbb{X}_1 \cup \mathbb{X}_2) - \mathbf{e}_t(\mathbb{X}'_1 \cup \mathbb{X}'_2)). \quad (4-26)$$

Here r, s, t are indices ranging $1 \leq r \leq k, 1 \leq s \leq l$ and $1 \leq t \leq k + l$ (i.e. we slightly abuse notation and let $\mathbf{e}_r(\mathbb{X}_1)$ denote $\mathbf{e}_1(\mathbb{X}_1), \dots, \mathbf{e}_k(\mathbb{X}_1)$, etc.). The latter is quasi-isomorphic to the object in $\mathcal{D}^b(\mathbf{R}^{k,l}\mathbf{Bim}^{\mathbf{q}})$ given by the dg algebra

$$\mathbf{K} := \mathbf{q}^{-kl}\mathbb{K}[\mathbf{e}_r(\mathbb{X}_1), \mathbf{e}_r(\mathbb{X}'_1), \mathbf{e}_s(\mathbb{X}_2), \mathbf{e}_s(\mathbb{X}'_2)] \otimes_{\mathbb{K}} \bigwedge^{\bullet}\{\theta_t\}, \quad (4-27)$$

where $\mathbf{aqdeg}(\theta_t) = (-1, 2t)$ and $d(\theta_t) = \mathbf{e}_t(\mathbb{X}_1 \cup \mathbb{X}_2) - \mathbf{e}_t(\mathbb{X}'_1 \cup \mathbb{X}'_2)$. Computing partial trace then gives that

$$\mathcal{T}_{k,l}(\mathbf{K}) \cong \mathbf{a}^l \mathbf{q}^{-kl} \mathbf{q}^{-l(l+1)} \mathbb{K}[\mathbf{e}_r(\mathbb{X}_1), \mathbf{e}_r(\mathbb{X}'_1), \mathbf{e}_s(\mathbb{X}_2), \mathbf{e}_s(\mathbb{X}'_2)] \otimes_{\mathbb{K}} \bigwedge^{\bullet}\{\theta_t, \xi_s\} \quad (4-28)$$

where $\mathbf{aqdeg}(\xi_s) = (-1, 2s)$ and $d(\xi_s) = \mathbf{e}_s(\mathbb{X}_2) - \mathbf{e}_s(\mathbb{X}'_2)$.

Since the right-hand side of (4-24) is quasi-isomorphic to a direct sum of copies of the Koszul complex associated to the elements $\mathbf{e}_r(\mathbb{X}_1) - \mathbf{e}_r(\mathbb{X}'_1)$, we now aim to change variables in $\mathcal{T}_{k,l}(\mathbf{K})$, with the hope of identifying it as such. Note that

$$\begin{aligned} d(\theta_t) &= \mathbf{e}_t(\mathbb{X}_1 \cup \mathbb{X}_2) - \mathbf{e}_t(\mathbb{X}'_1 \cup \mathbb{X}'_2) \\ &= \sum_{j=0}^t \mathbf{e}_{t-j}(\mathbb{X}_1)\mathbf{e}_j(\mathbb{X}_2) - \sum_{j=0}^t \mathbf{e}_{t-j}(\mathbb{X}'_1)\mathbf{e}_j(\mathbb{X}'_2) \\ &= \mathbf{e}_t(\mathbb{X}_1) - \mathbf{e}_t(\mathbb{X}'_1) + \sum_{j=1}^t \mathbf{e}_{t-j}(\mathbb{X}_1)(\mathbf{e}_j(\mathbb{X}_2) - \mathbf{e}_j(\mathbb{X}'_2)) \\ &\quad + \sum_{j=1}^t (\mathbf{e}_{t-j}(\mathbb{X}_1) - \mathbf{e}_{t-j}(\mathbb{X}'_1))\mathbf{e}_j(\mathbb{X}'_2) \\ &= \mathbf{e}_t(\mathbb{X}_1) - \mathbf{e}_t(\mathbb{X}'_1) + \sum_{j=1}^t \mathbf{e}_{t-j}(\mathbb{X}_1)d(\xi_j) \\ &\quad + \sum_{j=1}^t (\mathbf{e}_{t-j}(\mathbb{X}_1) - \mathbf{e}_{t-j}(\mathbb{X}'_1))\mathbf{e}_j(\mathbb{X}'_2). \end{aligned} \quad (4-29)$$

This suggests that we recursively define

$$\Theta_t := \theta_t - \sum_{j=0}^t \mathbf{e}_{t-j}(\mathbb{X}_1)\xi_j - \sum_{j=0}^t \Theta_{t-j}\mathbf{e}_j(\mathbb{X}'_2). \quad (4-30)$$

By (4-29), this gives

$$d(\Theta_t) = e_t(\mathbb{X}_1) - e_t(\mathbb{X}'_1). \tag{4-31}$$

and, in particular, $d(\Theta_t) = 0$ for $t > k$.

It then follows that we have quasi-isomorphisms

$$\begin{aligned} \mathcal{T}_{k,l}(\mathbb{K}) &\cong \mathbf{a}^l \mathbf{q}^{-l(k+l+1)} \mathbb{K}[e_r(\mathbb{X}_1), e_r(\mathbb{X}'_1), e_s(\mathbb{X}_2), e_s(\mathbb{X}'_2)] \otimes_{\mathbb{K}} \bigwedge^{\bullet} \{\Theta_t, \xi_s\} \\ &\cong \mathbf{a}^l \mathbf{q}^{-l(k+l+1)} \mathbb{K}[e_r(\mathbb{X}_1), e_r(\mathbb{X}'_1), e_s(\mathbb{X}_2)] \otimes_{\mathbb{K}} \bigwedge^{\bullet} \{\Theta_t\} \\ &\cong \mathbf{a}^l \mathbf{q}^{-l(k+l+1)} \mathbb{R}^k \otimes_{\mathbb{K}} \mathbb{K}[e_s(\mathbb{X}_2)] \otimes_{\mathbb{K}} \bigwedge^{\bullet} \{\Theta_b\}, \end{aligned} \tag{4-32}$$

where, in this last equation, the index b ranges from $k + 1, \dots, k + l$.

This implies that $\mathcal{T}_{k,l}(\mathbb{K})$ is quasi-isomorphic to a direct sum of

$$\mathbf{a}^l \mathbf{q}^{-l(k+l+1)} \prod_{i=1}^l \frac{1 + \mathbf{a}^{-1} \mathbf{q}^{2(k+i)}}{1 - \mathbf{q}^{2i}} = \prod_{i=1}^l \frac{\mathbf{q}^k + \mathbf{a} \mathbf{q}^{-k-2i}}{1 - \mathbf{q}^{2i}} \tag{4-33}$$

copies of \mathbb{R}^k , as desired. □

Lemma 4.11. *Let $\mathbb{I} = (k_1, \dots, k_r)$ and $\mathbb{J} = (k_1, \dots, k_{r-1} + k_r)$. We also let $\mathbb{B} \in \mathcal{D}^b(\mathbb{R}^{\mathbb{J}}\text{-}\mathbb{R}^{\mathbb{I}}\text{Bim}^{\mathfrak{q}})$ and $\mathbb{C} \in \mathcal{D}^b(\mathbb{R}^{\mathbb{I}}\text{-}\mathbb{R}^{\mathbb{J}}\text{Bim}^{\mathfrak{q}})$, then we have*

$$\begin{aligned} \text{B} \otimes_{\mathbb{R}^{\mathbb{J}}} \text{B} &\cong \mathbf{q}^{2k_{r-1}k_r} \text{B} \otimes_{\mathbb{R}^{\mathbb{J}'}} \text{B} \\ \text{C} \otimes_{\mathbb{R}^{\mathbb{I}}} \text{C} &\cong \mathbf{q}^{2k_{r-1}k_r} \text{C} \otimes_{\mathbb{R}^{\mathbb{I}'}} \text{C} \end{aligned} \tag{4-34}$$

Proof. We show the first quasi-isomorphism in (4-34), as the proof of the second is similar. The idea for the proof is easy: simply pass to Koszul resolutions at the places where the tensor products take place. Formally, let us consider the case when $r = 2$ now, as the general proof differs only in requiring more cumbersome notation. The left-hand side of the first isomorphism in (4-34) is

$$\mathcal{T}_{\mathbb{I}}(\mathcal{T}_{\mathbb{I}}(\mathbb{I}\mathbb{S}_{\mathbb{J}} \otimes_{\mathbb{R}^{\mathbb{J}}} \mathbb{B})) \cong \mathbf{a}^{k_1+k_2} \mathbf{q}^{-k_1^2-k_1-k_2^2-k_2} (\mathbb{I}\mathbb{S}_{\mathbb{J}} \otimes_{\mathbb{R}^{\mathbb{J}}} \mathbb{J}\mathbb{B}_{\mathbb{I}'}) \otimes_{\mathbb{K}} \bigwedge^{\bullet} \{\xi_r, \zeta_s\}, \tag{4-35}$$

where $1 \leq r \leq k_1$, $1 \leq s \leq k_2$ and with differential given by $d(\xi_r) = e_r(\mathbb{X}_1) - e_r(\mathbb{X}'_1)$, $d(\zeta_s) = e_s(\mathbb{X}_2) - e_s(\mathbb{X}'_2)$ for alphabets of size $|\mathbb{X}_1| = k_1 = |\mathbb{X}'_1|$ and $|\mathbb{X}_2| = k_2 = |\mathbb{X}'_2|$, respectively. Here, polynomials in the relevant alphabets act as indicated by the subscripts on the bimodules. Passing to a Koszul resolution of the diagonal \mathbb{R}^J -bimodule, we see this is quasi-isomorphic to

$$\mathbf{a}^{k_1+k_2} \mathbf{q}^{-k_1^2-k_2^2-k_2} ({}_I\mathbb{S}_J \otimes_{\mathbb{K}} {}_J\mathbb{B}_{I'}) \otimes_{\mathbb{K}} \bigwedge^\bullet \{\xi_r, \zeta_s, \theta_t\}, \tag{4-36}$$

where here (additionally) $1 \leq t \leq k_1 + k_2$, $d(\theta_t) = e_t(\mathbb{X}) - e_t(\mathbb{X}')$, and $|\mathbb{X}| = k_1 + k_2 = |\mathbb{X}'|$.

Similarly, the right-hand side is

$$\begin{aligned} & \mathbf{q}^{2k_1k_2} {}_J\mathbb{B}_{I'}(\mathbb{B} \otimes_{\mathbb{R}^I} {}_I\mathbb{S}_J) \\ & \cong \mathbf{a}^{k_1+k_2} \mathbf{q}^{2k_1k_2} \mathbf{q}^{-(k_1+k_2)(k_1+k_2+1)} ({}_J\mathbb{B}_I \otimes_{\mathbb{R}^I} {}_I\mathbb{S}_J) \otimes_{\mathbb{K}} \bigwedge^\bullet \{\Theta_t\}, \end{aligned} \tag{4-37}$$

where $1 \leq t \leq k_1 + k_2$, $d(\Theta_t) = e_t(\mathbb{Y}') - e_t(\mathbb{Y})$, and $|\mathbb{Y}| = k_1 + k_2 = |\mathbb{Y}'|$. Passing to a Koszul resolution of \mathbb{R}^I gives that this is quasi-isomorphic to

$$\mathbf{a}^{k_1+k_2} \mathbf{q}^{-(k_1+k_2)(k_1+k_2+1)+2k_1k_2} ({}_J\mathbb{B}_{I'} \otimes_{\mathbb{K}} {}_I\mathbb{S}_J) \otimes_{\mathbb{K}} \bigwedge^\bullet \{\Theta_t, \Xi_r, Z_s\}, \tag{4-38}$$

with $1 \leq r \leq k_1$, $1 \leq s \leq k_2$ and differential given by $d(\Xi_r) = e_r(\mathbb{Y}'_1) - e_r(\mathbb{Y}_1)$, $d(Z_s) = e_s(\mathbb{Y}'_2) - e_s(\mathbb{Y}_2)$ for alphabets of size $|\mathbb{Y}_1| = k_1 = |\mathbb{Y}'_1|$ and $|\mathbb{Y}_2| = k_2 = |\mathbb{Y}'_2|$. The result now follows by comparing (4-36) with (4-38). \square

Lemma 4.12. *For $k \geq 0$, there are \mathbf{atq} -degree 0 isomorphisms*

$$\begin{array}{c} k \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ k \end{array} \simeq \mathbf{t}^k \mathbf{q}^{-k} \left| \begin{array}{c} k \\ \text{---} \\ k \end{array} \right. \quad \& \quad \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ k \end{array} \simeq \mathbf{a}^k \mathbf{q}^{-2k^2-k} \left| \begin{array}{c} k \\ \text{---} \\ k \end{array} \right. \tag{4-39}$$

This result, which implies the invariance of the usual colored triply-graded link homology under stabilization, is well-known, and follows from the equivalence of the definition in terms of singular Soergel bimodules with the constructions in [WW17] and [Cau17]. We give the (well-known) argument for the sake of completion, and to determine the exact degree shifts (given our grading conventions for the Rickard–Rouquier complexes) so that we may be precise in (4-46) below.

Proof. We induct on k , starting with $k = 1$. By Example 3.10 and (4-24), we have

$$\begin{array}{c} 1 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ 1 \end{array} \simeq \frac{\mathbf{q} + \mathbf{aq}^{-3}}{1 - \mathbf{q}^2} \left| \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} \right. \longrightarrow \frac{\mathbf{tq}^{-1} + \mathbf{atq}^{-3}}{1 - \mathbf{q}^2} \left| \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} \right. \quad \& \quad \begin{array}{c} 1 \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array} \simeq \frac{\mathbf{t}^{-1}\mathbf{q} + \mathbf{at}^{-1}\mathbf{q}^{-1}}{1 - \mathbf{q}^2} \left| \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} \right. \longrightarrow \frac{\mathbf{q} + \mathbf{aq}^{-3}}{1 - \mathbf{q}^2} \left| \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} \right. \tag{4-40}$$

The proof of [Hog18, Proposition 3.10] identifies the differentials in these complexes, giving homotopy equivalences

$$\begin{array}{c} 1 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array} \simeq \mathbf{tq}^{-1} \begin{array}{c} 1 \\ | \\ 1 \end{array} \quad \& \quad \begin{array}{c} 1 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array} \simeq \mathbf{aq}^{-3} \begin{array}{c} 1 \\ | \\ 1 \end{array} \quad (4-41)$$

that follow from ‘‘Gaussian elimination’’ of all terms for which the \mathbf{aq} -degrees coincide. For the inductive step, we compute, using Lemma 4.11 and (3-22), that

$$\begin{aligned} \frac{\mathbf{q}^k - \mathbf{q}^{-k}}{\mathbf{q} - \mathbf{q}^{-1}} \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ k \end{array} &\simeq \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 1 \\ | \\ k \end{array} \\ &\simeq \mathbf{q}^{-2(k-1)} \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ k \end{array} \begin{array}{c} k-1 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array} \simeq \mathbf{a}^{k-1} \mathbf{q}^{-2k^2+k+1} \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ k \end{array} \begin{array}{c} 1 \\ | \\ 1 \end{array} \\ &\simeq \mathbf{a}^{k-1} \mathbf{q}^{-2k^2+k+1} \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ k \end{array} \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ k \end{array} \simeq \mathbf{a}^k \mathbf{q}^{-2k^2+k-2} \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 1 \\ | \\ k \end{array} \\ &\simeq \frac{\mathbf{q}^k - \mathbf{q}^{-k}}{\mathbf{q} - \mathbf{q}^{-1}} \mathbf{a}^k \mathbf{q}^{-2k^2-k} \begin{array}{c} k \\ | \\ k \end{array} \end{aligned} \quad (4-42)$$

and the result follows for the negative crossing using the Krull–Schmidt property of the derived category, see e.g. [Wu14, Lemma 4.20]. The case of the positive crossing follows from an analogous computation. \square

Together with (4-23), Lemma 4.12 proves stabilization invariance of $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b)$ (up to grading shift), and consequently completes the proof of Theorem 4.7.

Hence, given a balanced coloring (b, \mathbb{M}) of a handlebody braid, we define

$$w_{(b, \mathbb{M})} := \sum_{k=1}^{\infty} k \cdot \left(\# \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right)_k - \# \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right)_k \right), \tag{4-43}$$

i.e. it is a weighted sum of the difference between the number of purely k -colored positive and negative crossings. Similarly, define

$$W_{(b, \mathbb{M})} := \sum_{k=1}^{\infty} k^2 \cdot \left(\# \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right)_k - \# \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right)_k \right). \tag{4-44}$$

Passing to half-integral values of the **at**-gradings, we set

$$\mathbf{x}(b, \mathbb{M}) := \mathbf{a}^{\frac{1}{2}(w_{(b, \mathbb{M})} - \sum_{i=1}^n l_i)} \mathbf{t}^{\frac{1}{2}(-w_{(b, \mathbb{M})} - \sum_{i=1}^n l_i)} \mathbf{q}^{-W_{(b, \mathbb{M})} + \sum_{i=1}^n l_i^2 + l_i} \tag{4-45}$$

and define

$$\mathcal{H}\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b, \mathbb{M}) := H^\bullet(\mathbf{x}(b, \mathbb{M})\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b)), \tag{4-46}$$

where $H^\bullet(-)$ denotes taking homology.

Corollary 4.13. *For balanced, colored $(b, \mathbb{M}) \in \mathcal{B}\mathcal{r}(g, n)$, the triply-graded vector space $\mathcal{H}\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b, \mathbb{M}) \in \mathbb{K}\text{Vec}^{\text{atq}}$ is an invariant of the handlebody link $\bar{b} \subset \mathcal{H}_g$. In general, $\mathcal{H}\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b, \mathbb{M})$ is not an invariant of the link corresponding to the closure in \mathcal{S}^3 of the non-core strands in b .*

Proof. First observe that our normalization factor (4-45) is invariant under the relations in the handlebody braid group, i.e. if $b, b' \in \mathcal{B}\mathcal{r}(g, n)$ are colored handlebody braids related by (2-2)–(2-4), then $\mathbf{x}(b, \mathbb{M}) = \mathbf{x}(b', \mathbb{M})$. Thus, since $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(-)$ is an invariant of handlebody braids, the same is true for $\mathcal{H}\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b, \mathbb{M})$. Conjugation invariance follows from the conjugation invariance of $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet(b)$, up to homotopy, given in Theorem 4.7, together with the observation that $\mathbf{x}(b, \mathbb{M}) = \mathbf{x}(\mathfrak{s}b\mathfrak{s}^{-1}, \mathfrak{s} \cdot \mathbb{M})$ (here $\mathfrak{s} \cdot \mathbb{M}$ is obtained from \mathbb{M} by applying the permutation corresponding to \mathfrak{s}).

Invariance under stabilization follows from (4-23) and (4-39), together with a careful inspection of (4-45).

Finally, the second statement follows from (the proof of) Proposition 4.8, since this shows that the homology of $\mathcal{H}\mathcal{H}_{\mathcal{H}_g}^\bullet$ for the braids therein are not isomorphic up to a degree shift. □

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