Heegaard Floer invariants of contact structures on links of surface singularities

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Abstract. Let a contact 3-manifold (Y, ξ_0) be the link of a normal surface singularity equipped with its canonical contact structure ξ_0. We prove a special property of such contact 3-manifolds of "algebraic" origin: the Heegaard Floer invariant $c^+(\xi_0) \in HF^+(-Y)$ cannot lie in the image of the U-action on $HF^+(-Y)$. It follows that Karakurt's "height of U-tower" invariants are always 0 for canonical contact structures on singularity links, which contrasts the fact that the height of U-tower can be arbitrary for general fillable contact structures. Our proof uses the interplay between the Heegaard Floer homology and Némethi's lattice cohomology.

1. Introduction and background

Consider a complex surface $\Sigma \subset \mathbb{C}^N$ with an isolated critical point at the origin. For a sufficiently small $\varepsilon > 0$, the intersection $Y = \Sigma \cap S_\varepsilon^{2N-1}$ with the sphere $S_\varepsilon^{2N-1} = \{|z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 = \varepsilon\}$ is a smooth 3-manifold called the *link of the singularity*. The complex structure on Σ induces the *canonical* contact structure ξ_0 on Y given by the distribution of complex tangencies; in this setting, (Y, ξ_0) is also called *Milnor fillable*. The contact manifold (Y, ξ_0) is independent of the choice of ε, up to contactomorphism. Moreover, it is shown in [2] that the Milnor fillable contact structure on Y is unique (note that in general, a link of singularity may support a number of tight or Stein fillable contact structures). The Milnor fillable structure can be thought of as the contact structure closely related

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to the algebraic origin of the manifold Y as link of singularity (and potentially carry information about the singularity). We would like to address

Question 1.1. *Are there any special features that distinguish the canonical contact structure from other contact structures on the link of singularity?*

It is known, for example, that ξ_0 is always Stein fillable [1] and universally tight [9]. In this paper, we work with Ozsváth–Szabó's Heegaard Floer homology [15] and Némethi's lattice cohomology [11, 12] to establish special properties of the Heegaard Floer contact invariant c^+ (introduced in [17]) of canonical contact structures. Recall that for a 3-manifold Y, the Heegaard Floer homology $\mathrm{HF}^+(Y)$ is an $\mathbb{F}[U]$-module (coefficients are assumed to be $\mathbb{F} = \mathbb{Z}/2$, see Remark 5.8 for $\mathbb{Z}$ coefficients). We review the context and background after stating our main result in terms of the U-action.

Theorem 1.2. *Let (Y, ξ_0) be a rational homology sphere link of a normal surface singularity with its canonical contact structure, and $c^+(\xi_0) \in \mathrm{HF}^+(-Y)$ its contact invariant. Assume that the singularity is **not** rational. Then $c^+(\xi) \notin \mathrm{Im}\, U$.*

A singular point p is *normal* when bounded holomorphic functions defined in its punctured neighborhood can be extended over p. More importantly to us, normality together with the homological assumption on Y is equivalent to saying that Y is the boundary of a negative definite 4-manifold which is a plumbing of disk bundles over 2-spheres such that the plumbing graph is a tree (see section 2).

Given a 3-manifold Y, recall that its Heegaard Floer homology, developed in [15] and sequels, is an $\mathbb{F}[U]$-module $\mathrm{HF}^+(Y)$ that decomposes as a direct sum of components corresponding to Spin^c-structures on Y. When Y is a rational homology sphere, $\mathrm{HF}^+(Y, \mathfrak{s}) = \mathcal{T} \oplus \mathrm{Torsion}$, where $\mathcal{T}$ is isomorphic to $\mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$ (with the appropriate grading) and Torsion is annihilated by U^d for some large d. A rational homology sphere Y is called an L-space when its Heegaard Floer homology is the simplest possible, i.e. $\mathrm{HF}^+(Y, \mathfrak{s}) = \mathcal{T}$ for every $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. In the case where Y is the link of a normal surface singularity, it is known that Y is an L-space if and only if the singularity is *rational*, see [16, 13]. (We will not discuss algebro-geometric definition of rational singularities here; in fact the reader can take the L-space criterion above as a definition.)

Given a contact 3-manifold (Y, ξ), its invariant $c^+(\xi)$ is defined as a distingushed element of the Heegaard Floer group $\mathrm{HF}^+(-Y)$, see [17]. More precisely, $c^+(\xi) \in \mathrm{HF}^+(-Y, \mathfrak{t}_\xi)$, where $\mathfrak{t}_\xi$ is the Spin^c-structure induced by ξ. For Stein fillable contact structures, the invariant is non-zero, in particular, $c^+(\xi_0) \neq 0$ for the

canonical contact structure ξ_0 on a link of any surface singularity. For an arbitrary contact 3-manifold (Y, ξ), the contact invariant is annihilated by the U-action, i.e. $c^+(\xi) \in \operatorname{Ker} U$.

The $\mathbb{F}[U]$-module structure was used by Karakurt in [7] to define a related numerical invariant of contact structures. More precisely, Karakurt considers the height of U-tower over $c^+(\xi)$ to define

$$ht(\xi) = \max\{n : c^+(\xi) \in U^n \cdot \operatorname{HF}^+(-Y)\}.$$

We have taken the liberty of changing the sign in Karakurt's original definition; in [7], the invariant is defined as $\sigma(\xi) = -ht(\xi)$. Karakurt computes ht for a number of contact structures obtained by Legendrian surgery, and shows that ht can take arbitrary integer values from 0 to $+\infty$. In [8], Karakurt and Öztürk show that the height of tower is 0 for canonical contact structures on links of "almost rational" (AR) singularities, using the fact that Heegaard Floer homology is isomorphic to Némethi's lattice cohomology [11, 12] for 3-manifolds of this type. For rational singularities, it is easy to see that $ht = +\infty$ for every contact structure on the link: this follows from the fact that the link Y of a rational singularity is an L-space, i.e. $\operatorname{HF}^+(-Y, \mathfrak{s}) = \mathcal{T}$ for every Spin^c-structure s on Y, see [16, 11]. Karakurt and Öztürk ask whether height of tower can take arbitrary integer values for canonical contact structures on links of general normal surface singularities [8, Question 6.2]. It follows immediately from Theorem 1.2 that the answer is manifestly no:

Corollary 1.3. *Consider a normal surface singularity which is not rational and its link is a rational homology sphere. Let ξ_0 be the canonical contact structure on the link. Then $ht(\xi_0) = 0$.*

When starting this work, our initial goal was to use the height of tower invariants (together with their monotonicity under Stein cobordisms [7]) to obstruct certain deformations of surface singularities. The above corollary means, however, that the ht invariant contains very little information about the given singularity! (One could use ht to show that rational singularities cannot be deformed into non-rational, but this is a well-known fact and a special case of the semicontinuity of the geometric genus, see [3]. The algebro-geometric proof of this fact is non-trivial, so the Heegaard Floer argument may still be of interest.)

Similarly to [8], our proof also uses the interplay between Heegaard Floer homology and lattice cohomology of [11, 12]. Lattice cohomology is defined in a combinatorial way, using the intersection lattice of the plumbing graph (the dual resolution graph of the singularity). Under certain rather restrictive conditions (for

example, for links of AR singularities), the Heegaard Floer homology and lattice cohomology are known to be isomorphic [16, 12]. For arbitrary 3-manifolds, a spectral sequence from lattice homology to Heegaard Floer homology was found in [19]; this spectral sequence collapses in certain special cases, but in general, isomorphism between Heegaard Floer and lattice (co)homologies has not been established. The isomorphism between the Heegaard Floer and lattice theories in the case of AR singularities is the key tool in Karakurt's and Karakurt-Öztürk's proofs in [7, 8]. Our approach is different in that we only use an $\mathbb{F}[U]$-equivariant map from the Heegaard Floer homology to the lattice cohomology and do not require an isomorphism, thus our argument works in general. The homomorphism we use comes from [16] and maps $\mathrm{HF}^+(-Y)$ to the 0-dimensional part $\mathbb{H}_0^+(\Gamma)$ of lattice cohomology. (The latter is much simpler that the full lattice cohomology $\mathbb{H}^*(\Gamma)$; note that for AR-singularities, lattice cohomology vanishes in dimensions $n > 0$, [14].) Another difference between our work and [8] is that we use general properties of graded roots without resorting to Laufer sequences specific to the AR case. Finally, we use a somewhat different approach than [8] to address the case where the minimal resolution of the surface singularity does not have a normal crossings exceptional divisor (see discussion in Section 2 and Section 5).

It is intriguing that the proof of Theorem 1.2 works with lattice cohomology to establish a statement about Heegaard Floer invariants, even in the absence of isomorphism between the two theories. It would be very interesting to find further similar applications of lattice cohomology.

The paper is organized as follows. In Sections 2 and 3, we collect some background facts on resolutions and describe different constructions of lattice cohomology. In Section 4, we prove Theorem 1.2 under the additional assumption that there is a *good* resolution that carries a Stein structure; in the general case we indicate how the proof of Theorem 1.2 follows from some technical lemmas that are relegated to Section 5. Section 5 examines lattice homology in presence of blowups and completes the proof of Theorem 1.2.

2. Resolutions and plumbing graphs

Consider a resolution $\pi\colon \widetilde{\Sigma} \to \Sigma$ of a normal surface singularity $(\Sigma, 0)$. The inverse image in $\widetilde{\Sigma}$ of a small neighborhood of 0 in Σ is a 4-manifold X with $\partial X = Y$, where Y is the link of the singularity. If the resolution is minimal (i.e. X has no smooth rational curves of self-intersection -1), by [1] the manifold $X_{\min}$ carries a Stein structure J so that (X, J) is a Stein filling for the canonical contact structure ξ_0 on Y.

Our main tool in this paper is lattice homology, an invariant constructed from a *good* resolution of $(\Sigma, 0)$. A resolution is good if the irreducible components of the exceptional divisor $\pi^{-1}(0)$ are smooth complex curves that intersect transversely at double points only. (In other words, $\pi^{-1}(0)$ is required to be a normal crossing divisor.) The dual resolution graph Γ is the graph whose vertices correspond to irreducible components of the exceptional divisor and the edges record intersections of these components. Each vertex is decorated with an integer weight equal to the self-intersection of the corresponding curve. The resolution yields a 4-manifold $X(\Gamma)$ such that $\partial X(\Gamma) = Y$, where Y is the link of singularity. For normal singularities, $X(\Gamma)$ is negative definite, and Y is a rational homology sphere if and only if Γ is a tree and each vertex corresponds to a 2-sphere. (See for example [12, Sections 2.1–2.2] for details.) The manifold $X(\Gamma)$ can be obtained by plumbing disk bundles over 2-spheres (with Euler numbers given by weights of vertices) as dictated by the graph Γ, so Γ is often called a plumbing graph. If the resolution of a normal singularity $(\Sigma, 0)$ is not good, it can still be encoded by a graph Γ, but one needs to record the singularities of the components of the exceptional divisor and specify multiple intersections.

It is important to note that the minimal resolution of $(\Sigma, 0)$ does not have to be good, but one can always obtain a good resolution from a minimal resolution by some blowups. (This is a standard fact, see for example [6, Theorem V.3.9] for a closely related result. See also [10, Section 1] for discussion and examples; bad minimal resolutions can appear even in very simple situations.) Accordingly, a good resolution does not have to be minimal, so $X(\Gamma)$ is not Stein in general. However, $X(\Gamma)$ always carries a symplectic form ω_0 such that $(X(\Gamma), \omega_0)$ is a strong symplectic filling for (Y, ξ_0). Indeed, $\widetilde{\Sigma}$ is Kähler since it lives in a blowup of $\mathbb{C}^N$, in particular, $\widetilde{\Sigma}$ has a symplectic form ω_0 compatible with the complex structure. For a good resolution, the irreducible components of the exceptional divisor $\pi^{-1}(0)$ are smooth complex curves, thus they are symplectic surfaces in $X(\Gamma) \subset \widetilde{\Sigma}$ with respect to ω_0. The contact manifold (Y, ξ_0) is the convex boundary of the plumbing $X(\Gamma)$ of these symplectic surfaces.

We work with bad minimal resolutions and their good blowups in Section 5. We need a minimal resolution to use the Stein structure, but if $\pi^{-1}(0)$ is not a normal crossing divisor, $X_{\min}$ cannot be used to define the lattice homology of the link. We will need to perform some additional blowups on $X_{\min}$ to obtain a complex surface X which is a *good* resolution with plumbing graph Γ, and examine special features of the lattice homology construction in presence of (-1) vertices of Γ.

3. Lattice Cohomology

In this section, we discuss the necessary background on lattice cohomology, [16, 11, 12]. Lattice cohomology $\mathbb{H}^*(\Gamma)$ was defined by Némethi in [12] as a combinatorial theory conjecturally parallel to Heegaard Floer homology. Starting with a plumbing graph Γ that defines a 4-manifold with boundary $X(\Gamma)$, Némethi's construction uses cellular cohomology of certain CW-complexes associated to the lattice $L = H_2(X(\Gamma), \mathbb{Z})$ equipped with a weight function. We do not give the general definition of $\mathbb{H}^*(\Gamma)$ here as we will only work with its 0-dimensional part $\mathbb{H}_0^+(\Gamma)$. (The reader will get a glimpse of the CW-complexes in the graded roots discussion below.) However, we will use several equivalent definitions of the 0-dimensional cohomology, those from [12] and its precursors [16, 11]. We also use specific isomorphisms between these constructions, so we review this material in some detail. (Everything we need is contained in [11] but some of the statements are implicit and somewhat difficult to extract from [11].)

As before, let Y be a rational homology sphere which is a link of normal surface singularity. Let Γ be a negative definite connected plumbing graph as above, defining a 4-manifold $X = X(\Gamma)$ with boundary $\partial X = Y = Y(\Gamma)$.

Consider the lattice $L = H_2(X, \mathbb{Z})$; the intersection form on L can be read off the graph Γ. Indeed, the vertices of Γ give a basis for L; v will usually denote both a vertex and its corresponding homology class. Then, the self-intersection $v \cdot v$ equals the weight decoration of the vertex v, and for two different vertices v, w we have $v \cdot w = 1$ if v, w are connected by an edge in Γ, and 0 otherwise.

Set $L' = H^2(X, \mathbb{Z})$ and $H = H_1(Y, \mathbb{Z})$. Since X has no 1-handles, from the Poincaré duality, the universal coefficient theorem and the homology exact sequence of the pair (X, Y) we have

$$L' = H^2(X, \mathbb{Z}) \simeq H_2(X, Y, \mathbb{Z}) \simeq \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}),$$

and our assumption that Y is a rational homology sphere gives a short exact sequence $0 \to L \to L' \to H \to 0$. We will use the map PD: $L \to L'$

defined by composing the Poincaré duality $H_2(X, \mathbb{Z}) \to H^2(X, Y, \mathbb{Z})$ with the cohomological inclusion $H^2(X, Y, \mathbb{Z}) \to H^2(X)$.

Let $\mathrm{Char}(\Gamma) \subset H^2(X, \mathbb{Z})$ be the set of characteristic vectors, that is,

$$\mathrm{Char} = \mathrm{Char}(\Gamma) = \{K \in L' : \langle K, x \rangle \equiv x \cdot x \pmod 2 \text{ for all } x \in L\},$$

where $\langle K, x \rangle$ is the evaluation of $K \in L' = H^2(X, \mathbb{Z}) \simeq \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$ on $x \in H_2(X, \mathbb{Z})$ and $x \cdot x$ is the self-intersection of x by the intersection form on $L = H_2(X, \mathbb{Z})$.

We have $\mathrm{Char} = K + 2L'$ for any fixed $K \in \mathrm{Char}$. The natural action $K \mapsto K + 2\,\mathrm{PD}(x)$ (for any $x \in L$) of L on Char has orbits of form $K + 2\,\mathrm{PD}(L)$. We will denote an orbit of this form by $[K] \subset \mathrm{Char}$.

Since $X(\Gamma)$ is simply connected, we know that $\mathrm{Char}(\Gamma)$ is isomorphic to the set of Spin^c-structures on X, and the identification is given by the first Chern class of the determinant line bundle associated with a given Spin^c-structure (see e.g. [5, Proposition 2.4.16]). If $\mathfrak{s}$ is any fixed Spin^c-structure on X and $\mathfrak{t} = \mathfrak{s}|_Y$ is its restriction to Y, the Spin^c-structures on X which restrict to $\mathfrak{t}$ are exactly those whose first Chern classes form an orbit of form $[K] = K + 2\,\mathrm{PD}(L)$, where $K = c_1(\mathfrak{s}) \in \mathrm{Char}(\Gamma)$. Thus, Spin^c-structures on Y can be identified with orbits of the L-action on $\mathrm{Char}(X)$, and we will sometimes use the notation $\mathfrak{t} = [K] \in \mathrm{Spin}^c(Y)$.

For any $K \in \mathrm{Char}$, we will also consider the (in general, rational) number K^2 defined by using the intersection pairing on $H^2(X; \mathbb{Q}) \simeq H_2(X, Y; \mathbb{Q}) \simeq H_2(X, \mathbb{Q})$ (the latter isomorphism holds because Y is a rational homology sphere). Rational coefficients are needed since $H^2(X; \mathbb{Z}) \simeq H_2(X, Y; \mathbb{Z})$ doesn't have a well-defined intersection pairing.

All the lattice cohomologies discussed below are taken with coefficients in $\mathbb{F} = \mathbb{Z}/2$ and have the structure of $\mathbb{F}[U]$-modules (these modules are graded but we omit the gradings since they will not be important to us). See Remark 5.8 for coefficients in $\mathbb{Z}$.

Let $\mathcal{T}_0^+$ denote the module $\mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]$. We will use the notation $1 = U^0 \in \mathcal{T}_0^+$ for the corresponding generator.

3.1. Lattice cohomology via functions on Char. This is a review of the construction due to Ozsváth and Szabó [16, Section 1].

Define a weight function w on Char by setting $w(K) = -(K^2 + |\Gamma|)/8$, where $|\Gamma|$ stands for the number of vertices in the plumbing graph, i.e. the number of basis elements of $H_2(X, \mathbb{Z})$ provided by the exceptional divisors.

Definition 3.1. The lattice cohomology $\mathbb{H}_0^+(\Gamma) \subset \mathrm{Hom}(\mathrm{Char}(\Gamma), \mathcal{T}_0^+)$ in dimension 0 is the set of functions satisfying the following adjunction relations for characteristic vectors $K \in \mathrm{Char}(\Gamma)$ and vertices v of Γ. If n is an integer such that $2n = \langle K, v \rangle + v \cdot v$ (or, equivalently, $w(K) - w(K + 2\,\mathrm{PD}(v)) = n$), we require for every $\phi \in \mathbb{H}_0^+(\Gamma) \subset \mathrm{Hom}(\mathrm{Char}(\Gamma), \mathcal{T}_0^+)$ that

$$
\begin{aligned}
U^n \cdot \phi(K + 2\,\mathrm{PD}[v]) &= \phi(K) && \text{if } n \geq 0, \\
\phi(K + 2\,\mathrm{PD}[v]) &= U^{-n} \cdot \phi(K) && \text{if } n \leq 0.
\end{aligned}
\tag{1}
$$

We introduce U-action on $\mathbb{H}_0^+(\Gamma)$ by setting $(U\phi)(K) = U(\phi(K))$ for every characteristic vector $K \in \mathrm{Char}(\Gamma)$, thus $\mathbb{H}_0^+(\Gamma)$ becomes an $\mathbb{F}[U]$-module.

As the compatibility condition (1) above involves relations between elements of $\mathrm{Char}(\Gamma)$ that differ by an element in $2\,\mathrm{PD}(L)$, the $\mathbb{F}[U]$-module $\mathbb{H}_0^+(\Gamma)$ decomposes as a direct sum according to the Spin^c-structures on Y:

$$
\mathbb{H}_0^+(\Gamma) = \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} \mathbb{H}_0^+(\Gamma, \mathfrak{t}).
$$

We will use the notation $\mathbb{H}_0^+(\Gamma, [K])$ to denote the direct summand of the above decomposition which corresponds to the Spin^c-structure $\mathfrak{t}$. Here, $[K]$ is the L-orbit formed by the first Chern classes of Spin^s structures on X restricting to $\mathfrak{t}$ on Y. One can think of elements of $\mathbb{H}_0^+(\Gamma, [K])$ as functions in $\mathrm{Hom}([K], \mathcal{T}_0^+)$ satisfying the compatibility conditions (1).

3.2. Lattice cohomology via functions on homology lattice.

We now describe a slightly different construction by Némethi, introduced in [11, Proposition 4.7].

Given any characteristic element $K \in \mathrm{Char}(\Gamma)$, define the weight function χ_K on L such that for any $x \in L$ we set

$$
\chi_K(x) = -\frac{1}{2}(\langle K, x \rangle + x \cdot x).
\tag{2}
$$

Definition 3.2. For a fixed characteristic vector $K \in \mathrm{Char}(\Gamma)$, the lattice cohomology $\mathbb{HL}_0^+(\Gamma, K) \subset \mathrm{Hom}(L, \mathcal{T}_0^+)$ is the set of functions $\varphi \colon L \to \mathcal{T}_0^+$ satisfying the following relations for elements $x \in L$ and vertices v of Γ. If n is an integer such that $2n = \langle K, v \rangle + v \cdot v + 2x \cdot v$, or, equivalently, if $\chi_K(x) - \chi_K(x + v) = n$ we require that

$$
\begin{aligned}
U^n \cdot \varphi(x + v) &= \varphi(x) && \text{if } n \geq 0, \\
\varphi(x + v) &= U^{-n} \cdot \varphi(x) && \text{if } n \leq 0.
\end{aligned}
\tag{3}
$$

This is also naturally an $\mathbb{F}[U]$ module by setting $(U\varphi)(x) = U(\varphi(x))$.

Lemma 3.3 ([11, Proposition 4.7]). *We have*

$$\mathbb{HL}_0^+(\Gamma, K) \cong \mathbb{H}_0^+(\Gamma, [K]).$$

Proof. The isomorphism is constructed as follows. Let $\iota_K \colon L \to [K] = K + 2\,\mathrm{PD}(L) \subset \mathrm{Char}$ be the mapping defined by $\iota_K(x) = K + 2\,\mathrm{PD}(x)$. Let $\iota_K^* \colon \mathbb{H}_0^+(\Gamma, [K]) \to \mathbb{HL}_0^+(\Gamma, K)$ be the induced dual map, that is, $\mathrm{Hom}(L, \mathcal{T}_0^+) \ni \varphi = \iota_K^*(\phi)$ for $\phi \in \mathrm{Hom}([K], \mathcal{T}_0^+)$ if $\varphi = \phi \circ \iota_K$. This map is well defined as the two compatibility conditions (1) and (3) correspond to each other: setting $K' = K + 2\,\mathrm{PD}(x)$, we see that for a basis element v of L corresponding to a vertex of the plumbing graph,

$$\langle K', v \rangle + v \cdot v = \langle K, v \rangle + v \cdot v + 2x \cdot v.$$

In the language of the weight functions, this is exactly the fact $w(\iota_K(x + v)) - w(\iota_K(x)) = \chi_K(x + v) - \chi_K(x)$. $\qquad\square$

3.3. Lattice cohomology via graded roots.

Here we review Némethi's main construction from [11, Section 4].

Fix $K \in \mathrm{Char}(\Gamma)$ and again consider the weight function $\chi_K \colon L \to \mathbb{Z}$ defined by equation (2) above. We consider sublevel sets of the function χ_K in the lattice L. For each $n \in \mathbb{Z}$, let $\bar{L}_{K,\leq n}$ be a finite 1-dimensional cell complex whose 0-skeleton is the set

$$L_{K,\leq n} = \{x \in L \colon \chi_K(x) \leq n\},$$

and the 1-cells are constructed as follows. If $x \in L$ and v is the basis element of $L = H_2(X, \mathbb{Z})$ corresponding to a vertex of Γ, then we connect x and $x + v$ by a unique 1-cell in $\bar{L}_{K,\leq n}$ whenever x and $x + v$ are both in $L_{K,\leq n}$. Clearly, such cell complexes can be built as subsets of $L \otimes \mathbb{R}$, taking the 1-cells to be straight line segments connecting their endpoints. Then we have $\bar{L}_{K,\leq n} \subset \bar{L}_{K,\leq m}$ for $n < m$.

Consider the set $\pi_0(\bar{L}_{K,\leq n})$ of the connected components of $\bar{L}_{K,\leq n}$, and let C_w denote the component corresponding to $w \in \pi_0(\bar{L}_{K,\leq n})$. If $m > n$, each C_w is contained in a component $C_{w'}$ for some $w' \in \pi_0(\bar{L}_{K,\leq m})$, and $C_{w'}$ may contain several distinct components of $\bar{L}_{K,\leq n}$. These inclusion relations are codified by the *graded root* (R_K, χ_K), which is a graph with an integer-valued grading function. The grading on the graph is closely related to the U-action on cohomology.

The vertices $\mathcal{V}(R_K)$ of (R_K, χ_K) are given by the set $\cup_{n \in \mathbb{Z}} \pi_0(\bar{L}_{K,\leq n})$. The grading, $\mathcal{V}(R_K) \to \mathbb{Z}$, still denoted by χ_K, is defined by $\chi_K|_{\pi_0(\bar{L}_{K,\leq n})} = n$. Finally, all edges are obtained by connecting vertices of the form $w_n \in \pi_0(\bar{L}_{K,\leq n})$ and $w_{n+1} \in \pi_0(\bar{L}_{K,\leq n+1})$ such that $C_{w_n} \subset C_{w_{n+1}}$, where the inclusion is understood in the sense described above.

Remark 3.4. As we mentioned, the elements of $\mathrm{Char}(\Gamma)$ fall into equivalence classes of form $[K]$ corresponding to Spin^c-structures on Y. It turns out that the graded roots corresponding to two characteristic elements K, K' belonging to the same orbit (that is, $K - K' \in 2\,\mathrm{PD}(L)$) are the same up to a grading shift, so one can associate a well-defined graded root (R_t, χ_t) to a Spin_c structure $t \in \mathrm{Spin}^c(Y)$ if one fixes the grading so that $\min \chi_t|_{R_t} = 0$, see [11, Section 4] for details. As we do not work with absolute gradings on cohomology modules, we will not make the grading shift and will simply use the grading given by χ_K.

Definition 3.5. Fix a characteristic element $K \in \mathrm{Char}$, let χ_K be a weight function as in (2), and consider the graded root (R_K, χ_K) as above with the vertex set $\mathcal{V} = \mathcal{V}(R_K)$. The associated $\mathbb{F}[U]$ module $\mathbb{H}(R, \chi)$ is defined as the set of functions $\psi \colon \mathcal{V} \to \mathcal{T}_0^+$ satisfying the condition

$$U \cdot \psi(v) = \psi(w) \text{ if } v, w \text{ are connected by an edge of } R \text{ and } \chi(v) < \chi(w). \quad (4)$$

Note that by the construction of the graded root, for v, w as above we have in fact $\chi(v) + 1 = \chi(w)$. As before, there is obvious U-action on $\mathbb{H}(R, \chi)$, so that $(U\psi)(v) = U(\psi(v))$. See [11, Definition 3.5] and discussion therein for details.

Lemma 3.6 ([11, Proposition 4.7]). $\mathbb{HL}_0^+(\Gamma, K) \cong \mathbb{H}(R_K, \chi_K)$.

Proof. The isomorphism of [11, Proposition 4.7] is constructed as follows. For an element $x \in L$ with $\chi_K(x) = n$, the map $\theta \colon L \to \mathcal{V}(R_K)$ associates to x the component of $\pi_0(\bar{L}_{K, \leq n})$ containing x.

This induces a map $\theta^* \colon \mathrm{Hom}(\mathcal{V}(R_K), \mathcal{T}_0^+) \to \mathrm{Hom}(L, \mathcal{T}_0^+)$ given by $\theta^*(\psi) = \varphi$ if $\varphi = \psi \circ \theta$. One can check that this is indeed a well-defined mapping from $\mathbb{H}(R_K, \chi_K)$ to $\mathbb{HL}_0^+(\Gamma, K)$, as the compatibility conditions (3) and (4) are matching. By some more work, it is also easy to see that it is an isomorphism (for the details, see the proof of [11, Proposition 4.7]). $\square$

We will also need a special property of the graded root corresponding to the canonical class. Some care with notation and terminology is needed here: in [11], Némethi defines the canonical class K_{can} as the first Chern class of the canonical line bundle, also uniquely determined by the relations

$$\langle K_{\mathrm{can}}, v \rangle = -v \cdot v - 2$$

for every basis element of L corresponding to a vertex of Γ. We work instead with the anticanonical class $K_0 = c_1(TX, J)$. If $\mathfrak{s}_{\mathrm{can}}$ is the Spin^c-structure on X induced by J, we have $K_0 = c_1(\mathfrak{s}_{\mathrm{can}})$. Note that $\mathfrak{s}_{\mathrm{can}}$ is usually referred to as the

canonical Spin^c-structure; its restriction to the 3-manifold Y is the Spin^c-structure on Y induced by the canonical contact structure ξ_0. We have

$$\langle c_1(TX, J), v \rangle = v \cdot v + 2$$

by adjunction, and the relation between our class K_0 and Némethi's canonical class K_{can} is $K_0 = -K_{\mathrm{can}}$. We will use [11, Theorem 6.1(c, d)], which Némethi proves for K_{can}. However, $\chi_{K_0}(x) = \chi_{K_{\mathrm{can}}}(-x)$ for any $x \in L$, and this symmetry implies that any statements about the connected components of level sets with respect to these two weight functions will be the same (cf. [11, Section 5.1]). This enables us to state the lemma below for the class K_0.

Lemma 3.7 ([11, Theorem 6.1(c, d)]). *Let $K_0 = c_1(TX, J) \in \mathrm{Char}(\Gamma)$.*

(1) *Consider the sublevel set $\bar{L}_{K_0, \leq 0}$, and let C_0 be its connected component containing $0 \in L = H_2(X, \mathbb{Z})$. Then C_0 contains no points x with $\chi_{K_0}(x) < 0$, i.e. $\chi_{K_0}|_{C_0}$ is identically zero.*

(2) *The sublevel set $\bar{L}_{K_0, \leq n}$ is connected for $n \geq 1$.*

(3) *The graded root (R_{K_0}, χ_{K_0}) has a distinguished vertex w_0 of valency one, which is the end vertex of an infinite (sub)chain consisting of vertices $w_0, w_1, w_2, \ldots$ such that $\chi_{K_0}(w_i) = i$ and there is an edge between w_i and w_{i+1} for every $i \in \mathbb{Z}_0^+$. Moreover, for every $i > 0$, the only vertex v of the graded root with $\chi_{K_0}(v) = i$ is $v = w_i$.*

The third part of the above lemma directly follows (using the construction of the graded root) from the first two parts which are explicitly stated in [11, Theorem 6.1(c, d)]. The distinguished vertex w_0 is the connected component C_0 containing $0 \in L$ in $\pi_0(\bar{L}_{K_0, \leq 0})$, and the vertex w_i for $i > 0$ is the single connected component of the connected sublevel set $\bar{L}_{K_0, \leq i}$.

We will call the infinite (sub)chain $w_0, w_1, w_2, \ldots$ the *main trunk* of the canonical graded root (R_{K_0}, χ_{K_0}). Note that the canonical graded root in general can have many complicated branches outside the main trunk (if the singularity is not rational, see the proof of Lemma 4.4 later), but those other branches, if present, connect to the main trunk at the level-one vertex w_1, see Figure 1.

4. The contact invariants: the Stein filling case

In this section, we prove Theorem 1.2 in the case where the minimal resolution $X_{\min}$ has a good graph Γ. We build on ideas from [7, 8]: in these papers,
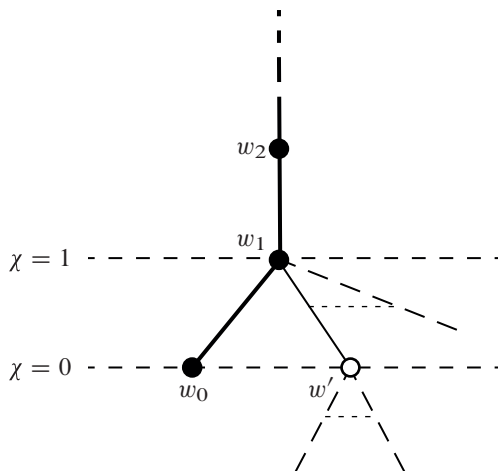
Figure 1. A sketch of a graded root with its main trunk. At least one vertex w' not on the main trunk is present on the 0-level, if the singularity is not rational.

Karakurt [7] and Karakurt and Öztürk [8] consider the case where the Heegaard Floer homology is isomorphic to $\mathbb{H}_0^+(\Gamma)$ (namely, they work with AR-graphs, for the definition see [11, Section 8]), and study the image of the contact invariant $c^+(\xi) \in HF^+(-Y)$ in the lattice homology under this isomorphism. Even when the isomorphism no longer holds, we are able to apply a similar strategy.

First, we state a lemma for an arbitrary negative definite plumbing graph Γ (the minimality assumption is not needed yet). Let $W(\Gamma)$ be the cobordism from S^3 to $Y = Y(\Gamma)$ given by the plumbing graph; $W(\Gamma)$ is obtained by cutting a small ball out of $X(\Gamma)$; Spin^c-structures on $W(\Gamma)$ are naturally identified with those on $X(\Gamma)$, and in turn with Char(Γ). We can think of $W(\Gamma)$ as cobordism from $-Y$ to S^3. Let $F_{W(\Gamma),\mathfrak{s}}^+ : HF^+(-Y) \to HF^+(S^3)$ be the map on Heegaard Floer homology induced by the Spin^c cobordism $(W(\Gamma), \mathfrak{s})$ (see [18]). Following [16], define the map

$$T^+ \colon HF^+(-Y) \longrightarrow \mathbb{H}_0^+(\Gamma)$$

as follows: for $x \in HF^+(-Y)$, let $T^+(x) \colon \mathrm{Char}(\Gamma) \to \mathcal{T}_0^+$ be given by

$$T^+(x)(K) = F_{W(\Gamma),\mathfrak{s}}^+(x) \in HF^+(S^3) = \mathcal{T}_0^+,$$

where K is the element of Char associated with the Spin^c-structure $\mathfrak{s}$.

Lemma 4.1 ([16, Proposition 2.4]). *The map T^+ induces an $\mathbb{F}[U]$-equivariant map from $HF^+(-Y(\Gamma), \mathfrak{t})$ to $\mathrm{Hom}(\mathrm{Char}_\mathfrak{t}(\Gamma), \mathcal{T}_0^+)$, whose image lies in $\mathbb{H}_0^+(\Gamma, \mathfrak{t})$.*

Again, it is important to note that the above lemma only uses basic properties of Heegaard Floer cobordism maps and requires no additional assumptions on the negative definite graph Γ. (This map is an isomorphism for AR-graphs, see [11, Theorem 8.3].)

We would like to find an explicit lattice-homological description of the element $c = T^+(c^+(\xi_0)) \in \mathbb{H}_0^+(\Gamma)$. In the next lemma, we will do so under the additional assumption that the graph Γ contains no (-1) vertices, i.e. $X(\Gamma)$ is minimal and thus gives a Stein filling of $Y(\Gamma)$ (see Section 2). In the next section, we will expand the argument to the general case.

Lemma 4.2. *Assume that the graph Γ contains no (-1) vertices, i.e. $X(\Gamma)$ is Stein. Consider the element $c = T^+(c^+(\xi_0)) \in \mathbb{H}_0^+(\Gamma)$. Let $K_0 = c_1(TX, J)$.*

(1) *The element c is given by a function $\phi_0 \in \mathbb{H}_0^+(\Gamma)$ such that $\phi_0(K_0) = 1$ in the degree 0 part of $\mathcal{T}_0^+$ and $\phi_0(K) = 0$ for any other characteristic class $K \neq K_0$. In particular, $c \in \mathbb{H}_0^+(\Gamma, [K_0])$.*

(2) *Under the isomorphism i_{K_0} of Lemma 3.3, the function ϕ_0 corresponds to the function $\varphi_0 \in \mathbb{HL}_0^+(\Gamma, K_0)$ such that $\varphi_0(0) = 1 \in \mathcal{T}_0^+$ in degree 0 and $\varphi_0(x) = 0$ for any $x \neq 0$, $x \in L$.*

(3) *Under the isomorphism θ^* of Lemma 3.6, the function φ_0 corresponds to the function $\psi_0 \colon \mathcal{V}(R_{K_0}) \to \mathcal{T}_0^+$ such that $\psi_0(w_0)$ is the generator of $\mathcal{T}_0^+$ in degree 0, and $\psi_0 = 0$ for all other vertices of $R_{\mathrm{t_{can}}}$. The vertex w_0 corresponds to the connected component of 0 in the homology lattice.*

Proof. The argument is essentially the same as Karakurt's observation in [7], based on the main theorem of [20]. Indeed, the homomorphism $c = T^+(c^+(\xi_0))$ is defined by

$$c(K) = F_{W(\Gamma), K}^+(c^+(\xi_0)) \in \mathrm{HF}^+(S^3) = \mathcal{T}_0^+.$$

By [17], $c^+(\xi_0) \in \mathrm{HF}^+(Y, \mathfrak{s}_{\mathrm{can}}|_Y)$, so it follows immediately that $c \in \mathbb{H}_0^+(\Gamma, [K_0])$ since the map T^+ respects Spin^c-structures.

By our assumption, $X(\Gamma)$ carries a Stein structure J so that (X, J) is a Stein filling for the canonical contact structure ξ_0 on Y. In this case [20, Theorem 4] asserts that $F_{W(\Gamma), \mathfrak{s}_{\mathrm{can}}}^+(c^+(\xi_0))$ is the generator of $\mathcal{T}_0^+$ in degree 0, and $F_{W(\Gamma), \mathfrak{s}}^+(c^+(\xi_0)) = 0$ for any other Spin^c-structure $\mathfrak{s}$ on $W(\Gamma)$. Since $c_1(\mathfrak{s}_{\mathrm{can}}) = c_1(TX, J) = K_0$ by our definition of the class K_0, this means that $c = \phi_0$.

Parts (2) and (3) of the lemma are immediate from the definitions of isomorphisms $\mathbb{H}_0^+(\Gamma, [K_0]) \simeq \mathbb{HL}_0^+(\Gamma, K_0) \simeq \mathbb{H}(R_{K_0}, \chi_{K_0})$ of Lemmas 3.3 and 3.6. $\square$

Remark 4.3. Under the hypothesis that Γ contains no (-1) vertices, the connected component C_0 of 0 in the level set $\overline{L}_{K_0,\leq 0}$ consists of a single point, ie $C_0 = \{0\}$. Indeed, due to the isomorphism of Lemma 3.6, the function φ_0 must be constant on the entire component C_0. It is also easy to check that $C_0 = \{0\}$ directly as follows. With the notation of Section 3.2, if v is a basis element, we use the identity $\langle K_0, v \rangle = v \cdot v + 2$ to see that

$$\chi_{K_0}(x) - \chi_{K_0}(x \pm v) = \pm x \cdot v + \begin{cases} v \cdot v + 1 \text{ or} \\ -1. \end{cases}$$

If Γ has no (-1) vertices, the inequality $v \cdot v \leq -2$ holds for any basis element v. Then for $x = 0$, we get $\chi_{K_0}(0) = 0$ but $\chi_{K_0}(\pm v) > 0$ for any basis element v, so $0 \in L$ is a single point in its connected component of the level set $\overline{L}_{K_0,\leq 0}$.

In the next section, we examine the general case where $X(\Gamma)$ may not be Stein. We will see that parts (1) and (2) of Lemma 4.2 no longer hold. However, it turns out that the support of the function φ_0 still matches the connected component C_0, so that part (3) of Lemma 4.2 holds in general. See Section 5.

We now return to graded roots to establish a useful property of the function ψ_0.

Lemma 4.4. *Let w_0 be the distinguished vertex of the canonical graded root (R_{K_0}, χ_{K_0}) in the sense of Lemma 3.7. Consider $\psi_0 \colon \mathcal{V}(R_{K_0}) \to \mathcal{T}_0^+$ such that $\psi_0(w_0) = 1 \in \mathcal{T}_0^+$ and $\psi_0(v) = 0$ for all other vertices v of R_{K_0}. Then $\psi_0 \in \mathbb{H}(R_{K_0}, \chi_{K_0})$, and $\psi_0 \in \operatorname{Ker} U$. Moreover, $\psi_0 \in \operatorname{Im} U$ if and only if the singularity is rational.*

Proof. We need to check that ψ_0 satisfies the compatibility conditions (4), which is immediate because the generator $1 \in \mathcal{T}_0^+$ is annihilated by U and by Lemma 3.7 part (1), there is no vertex v of the graded root connected to w_0 such that $\chi(v) < \chi(w_0)$ (w_0 is valency-one vertex of the graded root). Similarly, $\psi_0 \in \operatorname{Ker} U$ follows from the relations (4). (Alternatively, one can use the fact that by Lemma 4.2 and Lemma 5.7, ψ_0 is the image of $c^+(\xi_0)$ under the map from Heegaard Floer to lattice homology, and $c^+(\xi_0) \in \operatorname{Ker} U$ in Heegaard Floer homology by [17].)

As proved by Némethi [11, Theorem 6.3], the singularity is rational if and only if $\mathbb{H}(R_{K_0}, \chi_{K_0}) = \mathcal{T}_0^+$, and this happens exactly when the graded root is a single infinite chain with the end vertex w_0, that is, the graded root consists of nothing else but the main trunk.

Therefore, if the singularity is rational, it is easy to see that $\psi_0 \in \operatorname{Im} U$. If the singularity is not rational, the graded root R_{K_0} has non-trivial branches, i.e., at least one vertex $v \neq w_i$ ($i \in \mathbb{Z}_0^+$) outside its main trunk. Recall that w_1 is

the (unique) vertex connected to w_0 by an edge in R_{K_0} and $\chi_{K_0}(w_1) = 1$. By Lemma 3.7, all the vertices not on the main trunk must have non-positive χ_{K_0}-value, so there exists a vertex $w' \neq w_0$ such that $\chi(w') = 0$ and w' is connected to w_1, see Figure 1. Now, suppose that $\psi_0 = U\psi$ for some $\psi \in \mathbb{H}(R_{K_0})$. Then $\psi(w_1) = U\psi(w_0) = \psi_0(w_0) \neq 0$ and $\psi_0(w') = U\psi(w') = \psi(w_1) \neq 0$, which is a contradiction because we defined $\psi_0(v) = 0$ for all vertices $v \neq w_0$. $\square$

Remark 4.5. The above argument is similar to [8, Section 5.8], but Karakurt-Öztürk in [8] use Laufer sequences, an approach that only works in the special case of AR-singularities. Instead, we rely on the general graded root defined in [11] for any negative definite rational homology sphere plumbed manifold.

Proof of Theorem 1.2. The result follows immediately from Lemmas 4.4, 4.2, and 4.1 in the Stein case. Similarly, in the general case the proof follows from Lemmas 4.4, 4.1, and Lemma 5.7 established in the next section. Indeed, for the canonical Spin^c-structure $\mathfrak{t}$ there is an $\mathbb{F}[U]$-equivariant map $\mathrm{HF}^+(-Y(\Gamma), \mathfrak{t}) \to \mathbb{H}_0^+(\Gamma, [K_0]) \simeq \mathbb{H}\mathbb{L}_0^+(\Gamma, K_0) \simeq \mathbb{H}(R_{K_0}, \chi_{K_0})$ mapping $c^+(\xi_0)$ to ψ_0. Therefore, if $\psi_0 \notin \mathrm{Im}\, U$, then $c^+(\xi_0) \notin \mathrm{Im}\, U$. $\square$

5. The contact invariants in presence of blowups

In this section, we prove part (3) of Lemma 4.2 for the case where the minimal resolution $X_{\min} \to \Sigma$ of the singularity $(\Sigma, 0)$ does not have a normal crossings exceptional divisor. A similar statement is proved in [8] for AR-singularities, using the Ozsváth–Szabó algorithm [16, Proposition 3.2]. It was explained to us by Cagri Karakurt that similar ideas work in the general case. However, the proof is rather implicit in [8] and relies heavily on Laufer sequences specific to the AR-singularities, thus we take a different approach to give a self-contained argument. We work directly on the lattice homology side, explicitly describing the connected component of the canonical class in the relevant level set. An interested reader can compare our results in this section to Section 3.1 of [16] and to [8]. Example 5.3 below and the calculations given in Lemma 5.7 are similar in spirit to examples of [7, Theorem 7.1].

Consider the minimal resolution $X_{\min} \to \Sigma$ of the singularity $(\Sigma, 0)$. As discussed in Section 2, we can perform some additional blowups to obtain a normal crossing resolution $X = X(\Gamma)$, so that the resolution graph Γ is good. The blowups are performed in several steps; at every step we blow up one or several

points simultaneously. This gives a sequence of complex surfaces

$$X = X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \ldots \longrightarrow X_{\min} \longrightarrow \Sigma, \tag{5}$$

where the maps are the corresponding blowdowns. It will be convenient, even if not strictly necessary, to assume that blowdowns are performed simultaneously whenever the surface contains several rational curves with self-intersection -1. (Note that in a surface with negative definite intersection form, two distinct complex curves with self-intersection -1 must be disjoint.) The graph Γ encodes the exceptional divisor of the composite map $X \to \Sigma$. If $\widetilde{Z} \to Z$ is a blowup at a single point, we have $H_2(\widetilde{Z}) = H_2(Z) \oplus H_2(\overline{\mathbb{CP}^2})$, where the second summand corresponds to the exceptional divisor, and the map $H_2(Z) \hookrightarrow H_2(\widetilde{Z})$ is induced by the inclusion of the punctured copy of Z into $\widetilde{Z}$. Thus, we have homology inclusions $H_2(X_r) \hookrightarrow H_2(X)$ for each r.

We introduce notation for the components of the exceptional divisors at different stages of the blowup (5), as follows. Let $D_1^1, D_2^1, \ldots, D_{k(1)}^1$ be the collection of disjoint rational curves of self-intersection -1 in X_1 that are blown down to obtain X_2, and let $E_1^1, E_2^1, \ldots, E_{k(1)}^1$ denote the corresponding vertices of Γ. At the second step, there is a collection of disjoint exceptional curves $D_1^2, D_2^2, \ldots, D_{k(2)}^2$ in X_2; these are simultaneously blown down to obtain X_3. Under the blowup $X_1 \to X_2$, the strict transforms of $D_1^2, D_2^2, \ldots, D_{k(2)}^2$ become components of the exceptional divisor in X_1 encoded by Γ, so they correspond to certain vertices $E_1^2, E_2^2, \ldots, E_{k(2)}^2 \in \Gamma$. (Under our assumption, D_j^2 could not be blown down in X_1, so $E_j^2 \cdot E_j^2 \leq -2$, and the self-intersection increases in X_2 as a result of the blowdown.) Inductively, we blow down a collection $D_1^r, D_2^r, \ldots, D_{k(r)}^r$ of rational curves with self-intersection -1 in X_r to obtain the surface X_{r+1}, until we arrive to the surface $X_{R+1} = X_{\min}$ after R steps. For each surface, we have a map $X \to X_r$ given by (5); for the divisors $D_1^r, D_2^r, \ldots, D_{k(r)}^r \subset X_r$, their strict transforms in X correspond to some components of the exceptional divisor of the map $X \to \Sigma$. This exceptional divisor is encoded by Γ; let $E_1^r, E_2^r, \ldots, E_{k(r)}^r$ denote the vertices of Γ that correspond to (the strict transforms of) $D_1^r, D_2^r, \ldots, D_{k(r)}^r$.

Recall that the collection of divisors in X corresponding to all vertices of the graph Γ gives a basis of $H_2(X)$. Some, but not all, of these vertices appear in the sets $\{E_1^r, E_2^r, \ldots, E_{k(r)}^r\}$ above. The vertices that do not appear in these sets correspond to classes that come from $H_2(X_{\min})$.

Remark 5.1. By construction, the smooth complex curve D_j^r lies in X_r. Its preimage under the map $X \to \cdots \to X_r$ is the total transform $\widetilde{D}_j^r \subset X$; generally, this divisor is reducible. The image of the homology class of D_j^r under

the inclusion $H_2(X_r) \hookrightarrow H_2(X)$ is the homology class of the total transform; we use the same notation D_j^r for this class in $H_2(X)$. Note that when the total transform consists of several components and is not smooth, this class cannot be realized by a smooth complex curve in X: if D were a smooth complex representative, then D would non-negatively intersect all components of the total transform $\widetilde{D}_i^r$, contradicting $D \cdot \widetilde{D}_i^r = D \cdot D = -1$. By contrast, each class D_j^r can be realized by a smooth *symplectic* sphere in X with self-intersection -1, so that all these symplectic spheres (for all r, j) are pairwise disjoint. (Recall that the complex surface X has a compatible symplectic structure ω_0, see section 2.) Indeed, we start with the first blowup in $X_{\min}$: for the map $X_R \to X_{\min} = X_{R+1}$, the exceptional divisors $D_1^R, D_2^R, \ldots, D_{k(R)}^R$ in X_R are disjoint smooth complex (and thus symplectic) curves. Next, we blow up X_R at one or more points to obtain X_{R-1}. If any of the blown-up points lie in the curves $D_1^R, D_2^R, \ldots, D_{k(R)}^R$ in X_R, we can push each such divisor D_j^R off these points by a C^∞-small isotopy in X_R. Since being a symplectic surface is an C^∞-open condition, the perturbed curves will be symplectic. These symplectic spheres are now unaffected by the blowups and thus remain smooth in X_{R-1}; they are obviously disjoint from the new exceptional divisors D_j^{R-1} in X_{R-1} (the curves D_j^{R-1} are the preimages of the blown-up points under $X_{R-1} \to X_R$). We continue this process: before blowing up X_{R-1} to get X_{R-2}, we perturb any of the affected spheres (D_j^R or D_j^{R-1}) off the blown-up points, and so on, until we arrive at $X = X_1$ and a collection of disjoint symplectic spheres representing homology classes $D_j^r \in H_2(X)$ for all r, j.

Let

$$\mathcal{D} = \{D_1^1, D_2^1, \ldots, D_{k(1)}^1, D_1^2, D_2^2, \ldots, D_{k(2)}^2, \ldots, D_1^R, D_2^R, \ldots, D_{k(R)}^R\} \subset H_2(X) \tag{6}$$

be the set of homology classes D_j^m in X constructed above. Let $\mathcal{S} \subset H_2(X)$ be the set of sums $D_{i_1} + D_{i_2} + \cdots + D_{i_r}$ of distinct elements from $\mathcal{D}$, where $\{D_{i_1}, D_{i_2}, \ldots D_{i_r}\}$ ranges over subsets of $\mathcal{D}$. Note that $\mathcal{S}$ contains 0 (which corresponds to the empty subset). In other words,

$$\mathcal{S} = \left\{ \sum_{n,i} \varepsilon_i^n D_i^n : D_i^n \in \mathcal{D}, \varepsilon_i^n = 0, 1 \right\} \subset H_2(X). \tag{7}$$

We will need to express the homology classes in $\mathcal{D}$ in terms of the basis elements $E_i^n \in H_2(X)$ corresponding to the vertices of Γ. It is convenient to use the notion of proximity (we tweak the usual definition of proximity of points to talk about proximity of divisors). Consider vertices $E_i^a, E_j^b \in \Gamma$, with $b \geq a$.

For the curve E_i^a in $X = X_1$, we can consider its images under projection to the blowdown surfaces $X_2, X_3, \ldots, X_a$. In X_a, the image is the curve D_i^a, it has self-intersection (-1) and gets blown down at the subsequent step. Similarly, E_j^b projects non-trivially to $X_1, X_2, \ldots, X_b$ and becomes a point after the blowdown to X_{b+1}. If D_i^a intersects the image of E_j^b in X_a, we say that E_i^a is *proximate* to E_j^b and use notation $E_i^a \rightsquigarrow E_j^b$. Equivalently, $E_i^a \rightsquigarrow E_j^b$ if, once E_i^a becomes a point in X_{a+1} after the blowdown $X_a \to X_{a+1}$, this point lies in the image of E_j^b in X_{a+1}. (Note that $E_i^a \rightsquigarrow E_j^b$ implies in fact that $b > a$: if $b = a$, then the projections of E_i^a and E_j^b to $X_a = X_b$ are the smooth complex curves D_i^a and D_j^b with self-intersection (-1), which must be disjoint in a negative definite manifold.)

Lemma 5.2. *The homology classes $D_j^r \in H_2(X)$ can be recursively expressed via the basis classes E_i^m as follows*:

$$D_j^m = E_j^m + \sum_{(i,n): E_i^n \rightsquigarrow E_j^m} D_i^n. \tag{8}$$

Proof. The lemma follows from the familiar relation between the homology classes of the strict transform and the total transform. Indeed, blowing up X_m to obtain X_{m-1}, we have

$$[\text{strict transform of } D_j^m \text{ in } X_{m-1}] = D_j^m - \sum_{i: E_i^{m-1} \rightsquigarrow E_j^m} D_i^{m-1},$$

because by definition, the proximate classes E_i^{m-1} correspond precisely to exceptional divisors $D_i^{m-1} \subset X_{m-1}$ that project to points in D_j^m under blowdown $X_{m-1} \to X_m$. Note that D_j^m is a smooth complex curve in X_m, so all these points are smooth, and all the divisors enter with multiplicity 1 in the above formula for the strict transform.

Further, observe that the strict transform of D_j^m in X_{m-1} (taken with respect to the map $X_{m-1} \to X_m$) is the same as the projection of E_j^m under the blowdown $X = X_1 \to \cdots \to X_{m-1}$. It follows that the vertices E_i^{m-2} proximate to E_j^m correspond exactly to those divisors D_i^{m-2} among $D_1^{m-2}, \ldots, D_{k-2}^{m-2}$ in X_{m-2} that project, under the blowdown $X_{m-2} \to X_{m-1}$, to points of this strict transform of D_j^m. All these points are smooth. Comparing the strict transform in X_{m-2} to the total transform, we get

$$[\text{strict transform of } D_j^m \text{ in } X_{m-2}] = D_j^m - \sum_{i: E_i^{m-1} \rightsquigarrow E_j^m} D_i^{m-1} - \sum_{i: E_i^{m-2} \rightsquigarrow E_j^m} D_i^{m-2}.$$

We continue inductively, obtaining similar expressions for strict transforms of D_j^m in X_{m-3}, etc, until we arrive at the formula for E_j^m, which is the strict transform of D_j^m in $X = X_1$:

$$E_j^m = D_j^m - \sum_{(i,n): E_i^n \rightsquigarrow E_j^m} D_i^n. \qquad \square$$

Explicit calculations (such as Example 5.3 below) can be done via handleslides on the plumbing graph, Kirby calculus-style. The graph Γ gives a Kirby diagram for X_1 (each vertex corresponds to an unknotted component of the link). At each step of the blowdown, we look for (-1)-framed unknots in the diagram, handleslide them away, and blow down; the classes in H_2 corresponding to these (-1) unknots in the diagram for X_r are $D_1^r, \ldots, D_{k(r)}^r$. Proximity relation $E_i^a \rightsquigarrow E_j^b$ means that in the diagram for X_a, the (-1)-framed unknot representing D_i^a is linked with the component that corresponds to E_j^b (this component survives in X_a because $b > a$). Accordingly, when we handleslide D_i^a away from a link component, the homology class of the handle corresponding to the latter changes by adding D_i^a. It is not hard to see that in this context, formula (8) is a consequence of handle addition operations and their effect on homology. Note that if we handleslide away the components corresponding to D_j^m but do not blow them down, the resulting diagram represents X at every step, and we explicitly see smooth representatives of the homology classes from $\mathcal{D}$ in X, given by all the (-1) framed unknots that we slide away.
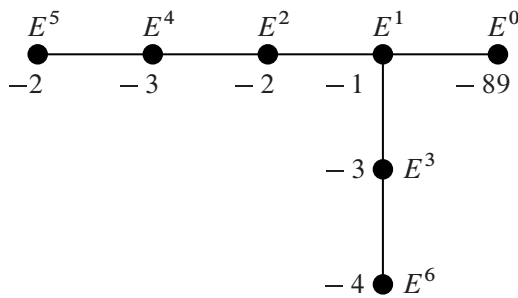

Figure 2. Plumbing graph of Example 5.3.

Example 5.3. Consider the negative definite plumbing graph of Figure 2. (This is the plumbing representation of the (-1)-surgery along the $(8, 11)$ algebraic torus knot: the part of the graph obtained by deleting E^0 is the plumbing of S^3 arising as the embedded good resolution of the plane curve singularity $x^8 = y^{11}$.)

In this example, at every blowdown step, we encounter one (-1)-divisor only, therefore, we omit the lower indices. The divisors $E^1, E^2, \ldots, E^6$ can be blown down in this order consecutively. Just before E^1 is blown down, it intersects E^3 and E^2, so E^1 is proximate to E^3 and also to E^2. Next, E^2 is blown down, and just before this happens in X_1, it intersects the strict transforms of E^4 and E^3, so E^2 is proximate to E^4 and E^3. Similarly, E^3 is proximate to E^4 and E^6, E^4 is proximate to E^5 and E^6, and E^5 is proximate to E^6 only.

This means that we start with $D^1 = E^1$. Then, since only E^1 is proximate to E^2, we have $D^2 = D^1 + E^2 = E^1 + E^2$. The divisors proximate to E^3 are E^2 and E^1, thus $D^3 = D^2 + D^1 + E^3 = 2E^1 + E^2 + E^3$. Divisors E^3 and E^2 are proximate to E^4, so $D^4 = D^3 + D^2 + E^4 = 3E^1 + 2E^2 + E^3 + E^4$. Only E^4 is proximate to E^5, so $D^5 = D^4 + E^5 = 3E^1 + 2E^2 + E^3 + E^4 + E^5$. Divisors E^5, E^4 and E^3 are all proximate to E^6: $D^6 = D^5 + D^4 + D^3 + E^6 = 8E^1 + 5E^2 + 3E^3 + 2E^4 + E^5 + E^6$. We will return to this example in the proof of Lemma 5.7.

We will now use the blowup formula in Heegaard Floer homology to identify the Spin^c-structures on X with $F^+_{W(\Gamma),\mathfrak{s}}(c^+(\xi_0)) \neq 0$. By contrast with Lemma 4.2, when Γ has vertices of weight -1 it is no longer true that $F^+_{W(\Gamma),\mathfrak{s}}(c^+(\xi_0)) = 0$ for all Spin^c-structures $\mathfrak{s}$ other than the canonical one. Recall that $K_0 = c_1(TX, J)$, and let $\mathfrak{s}_{\min, \text{can}}$ denote the canonical Spin^c-structure on $X_{\min}$.

Lemma 5.4. *Let $\mathfrak{s}$ be a* Spin^c*-structure on X.*

(1) *$F^+_{W,\mathfrak{s}}(c^+(\xi_0))$ is the generator of $\mathcal{T}^+_0$ in degree 0 if $\langle c_1(\mathfrak{s}), D \rangle = \pm 1$ for all $D \in \mathcal{D}$, and $\mathfrak{s}|X_{\min} = \mathfrak{s}_{\min, \text{can}}$; otherwise, $F^+_{W,\mathfrak{s}}(c^+(\xi_0)) = 0$.*

(2) *Equivalently, $F^+_{W,\mathfrak{s}}(c^+(\xi_0)) = 1 \in \mathcal{T}^+_0$ if $c_1(\mathfrak{s}) = K_0 + 2\,\mathrm{PD}(D)$ for some $D \in \mathcal{S}$; otherwise, $F^+_{W,\mathfrak{s}}(c^+(\xi_0)) = 0$.*

Remark 5.5. In the sense of [16], the set of Spin^c-structures $\{K_0 + 2\,\mathrm{PD}(D)$: $D \in \mathcal{S}\}$ is a *full path* starting at the canonical class K_0, as described by [16, Proposition 3.2].

Corollary 5.6. (1) *The element $c = T^+(c^+(\xi_0)) \in \mathbb{H}^+_0(\Gamma)$ is given by the function $\phi_0 \in \mathbb{H}^+_0(\Gamma)$ such that*

$$\phi_0(\mathfrak{s}) = \begin{cases} 1 \in \mathcal{T}^+_0 & \text{if } \mathfrak{s} = K_0 + 2\,\mathrm{PD}(D), D \in \mathcal{S}, \\ 0 & \text{otherwise}. \end{cases}$$

(2) *Under the isomorphism $i_{K_0}^*$ of Lemma 3.3, the function ϕ_0 corresponds to the function $\varphi_0 \in \mathbb{HL}_0^+(\Gamma, K_0)$ such that for $x \in L$,*

$$\varphi_0(x) = \begin{cases} 1 \in \mathcal{T}_0^+ & \text{if } x \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Corollary 5.6. This follows from Lemma 5.4, by an argument completely analogous to the proof of Lemma 4.2 $\hfill\square$

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Proof of Lemma 5.4. (1) In the minimal case, the set $\mathcal{S}$ is empty, and the statement was already given in Lemma 4.2.

In the non-minimal case, we blow down X to $X_{\min}$, perhaps in several steps, and argue by induction on the number of exceptional divisors in this sequence of blowdowns.

Suppose the statement holds for the complex surface X' which is a blowup of $X_{\min}$, and X is obtained from X' by an additional blowup with exceptional divisor D_0. Let $\mathcal{D}'$ and $\mathcal{S}'$ be the sets of homology classes defined for X' as in formulas (6) and (7). Note that $H_2(X) = H_2(X') \oplus \mathbb{Z}$; identifying homology classes in X' with their images under inclusion in X, we have $\mathcal{D} = \mathcal{D}' \cup \{D_0\}$. Let W, W' denote the corresponding cobordisms from $-Y^3$ to S^3, and let $B = [0, 1] \times Y \# \overline{\mathbb{CP}^2}$ be the blowup cobordism. As a smooth manifold, W can be thought of as composition of cobordism B from $-Y$ to $-Y$ followed by cobordism W' from $-Y$ to S^3. Since $W = W' \# \overline{\mathbb{CP}^2}$, a Spin^c-structure $\mathfrak{s}$ on W is completely determined by its restrictions $\mathfrak{s}|_{W'}$ and $\mathfrak{s}|_B$, so by the composition law [18, Theorem 3.4], we have

$$F_{W',\mathfrak{s}|_{W'}}^+ \circ F_{B,\mathfrak{s}|_B}^+ = F_{W,\mathfrak{s}}^+.$$

We are interested in the canonical contact structure ξ_0 on $Y = \partial X' = \partial X$, so we focus on Spin^c-structures with $\mathfrak{s}|_Y = \mathfrak{s}_{\mathrm{can}}|_Y$, since $c^+(\xi_0) \in \mathrm{HF}^+(Y, \mathfrak{s}_{\mathrm{can}}|_Y)$.

The map $F_{B,\mathfrak{s}|_B}^+$ is given by the Ozsváth–Szabó blowup formula [18, Theorem 3.7]. Let D_0 be the exceptional divisor of the blowup, and

$$\langle c_1(\mathfrak{s}), D_0 \rangle = \langle c_1(\mathfrak{s}|B), D_0 \rangle = \pm(2l + 1), \quad l \geq 0.$$

Then

$$F_{B,\mathfrak{s}|_B}^+ \colon \mathrm{HF}^+(-Y, \mathfrak{s}|_Y) \to \mathrm{HF}^+(-Y, \mathfrak{s}|_Y)$$

is the multiplication by $U^{l(l+1)}$. By [17], the contact invariant lies in $\mathrm{Ker}\, U$, thus we get $F_{B,\mathfrak{s}|_B}^+(c^+(\xi_0)) = 0$ if $l \neq 0$. For $l = 0$, $F_{B,\mathfrak{s}|_B}^+(c^+(\xi_0)) = c^+(\xi_0)$ since $F_{B,\mathfrak{s}|_B}^+$ is the identity map.

By the induction hypothesis, $F^+_{W',\mathfrak{s}|_{W'}}(c^+(\xi_0))$ is the generator $1 \in \mathfrak{T}^+_0$ if

$$\mathfrak{s}|_{W_{\min}} = \mathfrak{s}_{\min,\,\text{can}}$$

and

$$\langle c_1(\mathfrak{s}), D' \rangle = \pm 1 \quad \text{for all } D' \in \mathcal{D}';$$

moreover, $F^+_{W',\mathfrak{s}|_{W'}}(c^+(\xi_0))$ is zero for all other Spin^c-structures. Then from the composition formula, $F^+_{W,\mathfrak{s}|_W}(c^+(\xi_0)) = 1 \in \mathfrak{T}^+_0$ if

$$\mathfrak{s}|_{W_{\min}} = \mathfrak{s}_{\min,\,\text{can}},$$

$$\langle c_1(\mathfrak{s}), D' \rangle = \pm 1 \quad \text{for all } D' \in \mathcal{D}',$$

and

$$\langle c_1(\mathfrak{s}), D_0 \rangle = \pm 1.$$

For all other Spin^c-structures on X, $F^+_{W,\mathfrak{s}}(c^+(\xi_0)) = 0$. Since $\mathcal{D} = \mathcal{D}' \cup \{D_0\}$, part (1) of the lemma holds for Spin^c-structures on X.

(2) Observe that for a Spin^c-structure s with $\mathfrak{s}|_{W_{\min}} = \mathfrak{s}_{\text{can}}$ and $\langle c_1(\mathfrak{s}), D \rangle = \pm 1$ for all $D \in \mathcal{D}$, we have

$$c_1(\mathfrak{s}) = c_1(\mathfrak{s}_{\text{can}, W_{\min}}) + \sum_{m,j} \mp \text{PD}(D^m_j).$$

The canonical Spin^c-structure $\mathfrak{s}_{\text{can}}$ on W is determined by the equalities $\mathfrak{s}|_{W_{\min}} = \mathfrak{s}_{\text{can}, W_{\min}}$ and $\langle c_1(\mathfrak{s}), D^m_j \rangle = +1$ for all $D^m_j \in \mathcal{D}$, so

$$K_0 = c_1(\mathfrak{s}_{\text{can}}) = c_1(\mathfrak{s}_{\text{can}, W_{\min}}) - \sum_{m,j} \text{PD}(D^m_j).$$

The statement now follows from the first part of the lemma. $\qquad \square$

We now show that part (3) of Lemma 4.2 holds in the general case even though parts (1) and (2) don't. Corollary 5.6 identifies the support of the function φ_0 as the set $\mathcal{S} \subset H_2(X; \mathbb{Z})$ given by sums of distinct homology classes from $\mathcal{D}$. To understand the function ψ_0 corresponding to φ_0 in the graded root description of lattice homology, we examine connected components of the level set $\overline{L}_{K_0, \leq 0}$ (compare with Remark 4.3).

Lemma 5.7. (1) *Let C_0 be the connected component of 0 in the level set $\overline{L}_{K_0, \leq 0}$. Then $\mathcal{S}$ is contained in C_0. (In fact, $\mathcal{S} = C_0$.)*

(2) *Under the isomorphism θ^* of Lemma 3.6, the function φ_0 corresponds to the function $\psi_0 \colon \mathcal{V}(R_{K_0}) \to \mathfrak{T}^+_0$ such that $\psi_0(w_0)$ is the generator of $\mathfrak{T}^+_0$ in degree 0, and $\psi_0 = 0$ for all other vertices of $R_{\text{t}_{\text{can}}}$. The vertex w_0 corresponds to the connected component of 0 in the homology lattice.*

Proof. (1) We saw that the homology classes D_j^m in the set $\mathcal{D}$ can be represented by pairwise disjoint symplectic spheres of self-intersection (-1) in $X = X_0$. It follows that these classes all lie in the zero level set of χ_{K_0}: by the adjunction formula, $\langle K_0, D_j^m \rangle = \langle c_1(TX, J), D_j^m \rangle = D_j^m \cdot D_j^m + 2 = 1$, therefore,

$$\chi_{K_0}(D_j^m) = 0.$$

Clearly, $D_j^m \cdot D_i^n = 0$ for any two distinct classes since they have disjoint representatives. The property of the weight function $\chi_{K_0}(x + y) = \chi_{K_0}(x) + \chi_{K_0}(y) - x \cdot y$ then implies that $\chi_{K_0} = 0$ for any of the sums forming $\mathcal{S}$, thus $\mathcal{S}$ lies in the zero level set $\bar{L}_{K_0, \leq 0}$. It will be convenient to refer to $D_1^m, D_2^m, \ldots, D_{k(m)}^m$ as elements of depth m, $1 \leq m \leq R$.

We want to show that $\mathcal{S}$ lies in the connected component C_0. Recall from section 3.2 that 1-cells in $\bar{L}_{K_0, \leq 0}$ correspond to basis vectors in the lattice $L = H_2(X, \mathbb{Z})$: we can walk from $x \in L$ to $x + v \in L$ along the edge corresponding to v if v is a basic vector. For each element of depth 1, the homology class $D_i^1 = E_i^1$ is in the basis, thus $D_i^1 = 0 + E_i^1$ lies in C_0 because it is connected to 0 by the egde corresponding to E_i^1. A typical lattice point $F \in \mathcal{S}$ is a sum of some elements D_j^m. We will show that $F \in C_0$ recursively, by reducing to sums of elements of lower depth. The key idea is formula (8), which recursively expresses classes D_j^m in terms of the basis elements $E_1^1, \ldots, E_{k(R)}^R$. Notice that in the above formula, each instance of n on the right hand side is strictly less than m, thus $D_j^m = E_j^m +$ sum of elements of lower depth.

First, we illustrate the recursive procedure for the lattice from Example 5.3. In fact, we construct the paths from 0 starting with elements of lower depth and building up to higher depth; the recursion will work backwards, reducing the depth. To begin, observe that $D^1 \in C_0$ as above. Next, $D^2 = D^1 + E^2$ is one step away from D^1 along the edge E^2, so $D^2 \in C_0$. Further, $D^1 + D^2 = D^2 + E^1$ is connected to D^2 by the edge E^1, so $D^1 + D^2 \in C_0$. Next, we see that $D^3 = D^1 + D^2 + E^3$ is connected to the vertex $D^1 + D^2 \in C_0$ by the edge E^3, so $D^3 \in C_0$. Then, we establish that $D^1 + D^3 = D^3 + E^1 \in C_0$, and then $D^2 + D^3 = D^3 + D^1 + E^1 \in C_0$, and then $D^1 + D^2 + D^3 = D^2 + D^3 + E^1 \in C_0$. Now we see that $D^4 = D^2 + D^3 + E^4 \in C_0$ because we already know that $D^2 + D^3 \in C_0$. We can write out the path from 0 to D^4:

$$D^4 = \underbrace{\underbrace{\underbrace{E^1}_{D^1} + E^2}_{D^2} + \underbrace{E^1}_{D^1} + E^3}_{D^3} + \underbrace{\underbrace{E^1}_{D^1} + E^2}_{D^2} + E^4.$$

In this form, $D^4 = \sum_{j=1}^{k} E^{i(j)}$ (with possibly repeating indices $i(j)$) is the sum of basis elements, showing steps along the corresponding edges. The initial partial sums $P^m = \sum_{j=1}^{m} E^{i(j)}$ we obtain after each step correspond to the vertices of the lattice L that our path goes through, starting from 0. The underbraces show that each such vertex is a sum of some *distinct* elements D^j. All of these sums lie in $\mathcal{S}$ which is contained the level set $\bar{L}_{K_0,\leq 0}$, thus we see that all the lattice points on the path lie in $\bar{L}_{K_0,\leq 0}$, showing that the entire path lies in the connected component of 0.

We continue in this fashion, consecutively building paths to elements $D^4 + D^1$, $D^4 + D^2$, $D^4 + D^2 + D^1$, $D^4 + D^3$, $D^4 + D^3 + D^1$ etc, until we reach all elements in $\mathcal{S}$. Here is the recursively constructed path showing that D^6 is in the connected component of 0:

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{E^1}_{D^1} + E^2}_{D^2} + \underbrace{E^1}_{D^1} + E^3}_{D^3} + \underbrace{\underbrace{E^1}_{D^1} + E^2 + E^4}_{D^2} + E^5}_{D^4}}_{D^5}}$$

$$+ \underbrace{\underbrace{\underbrace{\underbrace{\underbrace{E^1}_{D^1} + E^2}_{D^2} + \underbrace{E^1}_{D^1} + E^3}_{D^3} + \underbrace{\underbrace{E^1}_{D^1} + E^2 + E^4}_{D^2}}_{D^4}}$$

$$+ \underbrace{\underbrace{\underbrace{E^1}_{D^1} + E^2}_{D^2} + \underbrace{E^1}_{D^1} + E^3}_{D^3} + E^6 = D^6.$$

We now return to the proof of Lemma 5.7 and proceed with the formal recursion. To every lattice point $F \in \mathcal{S}$ we can assign an R-tuple of integers

$$t(F) := (s_R, \ldots, s_2, s_1),$$

where s_r is the number of summands in F that are elements of depth r. Namely, if $F = \sum_{a,i} \varepsilon_i^a D_i^a$, $\varepsilon_i^a \in \{0, 1\}$, $D_i^a \in \mathcal{D}$, we set $s_r := \sum_{i=1}^{k(r)} \varepsilon_i^r$, $1 \leq r \leq R$.

We define a partial ordering on $\mathcal{S}$ as follows. For $F_1, F_2 \in \mathcal{S}$ we say that $F_1 \prec F_2$ if $t(F_1) < t(F_2)$ with respect to the lexicographical ordering, looking

for the first differing number from the left (the largest index r where there is no equality). In other words, $F_1 \prec F_2$ if F_1 has fewer elements of depth r than F_2, and F_1 and F_2 have the same number of elements of depth $r + 1, r + 2, \dots, R$.

Take any element $F \in \mathcal{S}$. Let u be the smallest upper index among the summands of F, and write F in the form $F = \bar{F} + D_i^u$, where all summands in $\bar{F}$ have upper index at least u ($\bar{F}$ might be an empty sum). Looking at formula (8), we get $D_i^u = \bar{D} + E_i^u$, where all summands in $\bar{D}$ have upper index less than u. Again, $\bar{D}$ might be an empty sum.

It follows that $\bar{F}$ and $\bar{D}$ cannot share any summands, so the sum $\tilde{F} = \bar{F} + \bar{D}$ lies in $\mathcal{S}$. Moreover, due to our choice of the index u, we have $\tilde{F} \prec F$. Because $F = \tilde{F} + E_i^u$, this shows that any nontrivial element of $\mathcal{S}$ can be written as a sum of a lexicographically lower element and a basis element. To illustrate, we would decompose the element $F = D^4 + D^3$ in the above example as $F = \tilde{F} + E^3$, where $\tilde{F} = D^4 + D^2 + D^1$ is lexicographically lower than F; similarly, $D^4 + D^2 + D^1 = D^4 + D^2 + E^1$, with $D^4 + D^2 \prec D^4 + D^2 + D^1$.

The relation $F = \tilde{F} + E_i^u$ implies that F is connected to $\tilde{F}$ by the edge E_i^u. Therefore, if $\tilde{F}$ is in the connected component C_0, then so is F. Since the empty sum equals 0 and obviously lies in C_0, the above recursion shows that every element of $\mathcal{S}$ is in C_0.

It is worth noting that the above decomposition $F = \tilde{F} + E_i^u$ provides a recursive way to write any sum from $\mathcal{S}$ as a series of basis elements, indicating the path starting at zero and ending at the given lattice point, as shown in the example above.

We have shown that $\mathcal{S} \subset C_0$. Corollary 5.6 identifies $\mathcal{S}$ with the support of φ_0. The proof of Lemma 3.6 implies that the support of the function φ_0 is the union of several connected components of $\bar{L}_{K_0, \leq 0}$ (indeed, φ_0 must vanish on *entire* connected components). It follows that $\mathcal{S} = C_0$.

(2) The second statement of the lemma follows from the first statement, Corollary 5.6, and Lemma 3.6. (Compare with part (3) of Lemma 4.2.) $\square$

Remark 5.8. In this paper we worked with coefficients in $\mathbb{F} = \mathbb{Z}/2$ for simplicity, however our results hold for integer coefficients as well. When working with $\mathbb{Z}$ coefficients, the contact invariant $c^+(\xi)$ is only defined up to sign. The results of [20, 4] then assert that $F^+_{W(\Gamma), \mathfrak{s}_{can}}(c^+(\xi))$ is a generator $\pm 1 \in \mathcal{T}_0^+$, where $\mathcal{T}_0^+$ now stands for $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$. A further issue is that cobordism maps are only defined up to sign in [18], although [16, Section 2.1] explains how to define the map T^+ up to one overall sign (which can also be fixed). The isomorphisms between various constructions of lattice cohomology work with $\mathbb{Z}$ coefficients, and the distinguished elements ϕ_0, φ_0, and ψ_0 of Lemma 4.2

and its analogs in Section 5 correspond to one another. Thus, we see that the element $c^+(\xi_0) \in \mathrm{HF}^+(-Y(\Gamma), \mathfrak{t})/\pm 1$ is mapped to $(\pm\psi_0)$ under the map $\mathrm{HF}^+(-Y(\Gamma), \mathfrak{t})/\pm 1 \to \mathbb{H}(R_{K_0}, \chi_{K_0})/\pm 1$, and our proof goes through as before.

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