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A graph TQFT for hat Heegaard Floer homology

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Abstract. We construct maps on hat Heegaard Floer homology for cobordisms decorated with graphs. The graph TQFT allows for cobordisms with disconnected ends. Our construction uses Juhász's sutured Floer TQFT. We compute the maps for several elementary graph cobordisms. As an application, we compute the action of the fundamental group on hat Heegaard Floer homology.

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1. Introduction

Ozsváth and Szabó constructed a powerful set of invariants for closed 3-manifolds, and cobordisms between them [12] [13]. To a closed, oriented 3-manifold Y, they constructed a finitely generated abelian group

$$\widehat{HF}(Y),$$

as well as $\mathbb{Z}[U]$-modules $HF^-(Y)$, $HF^+(Y)$ and $HF^\infty(Y)$. We focus mostly on $\widehat{HF}$ in our present paper. Also, we work over the field $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$.

To a compact, connected, and oriented cobordism W between two connected 3-manifolds, Y_0 and Y_1, they constructed a linear map

$$\hat{F}_W : \widehat{HF}(Y_1) \longrightarrow \widehat{HF}(Y_2).$$

If $W = W_2 \cup_Y W_1$, where Y is a closed, connected 3-manifold, then

$$\hat{F}_W = \hat{F}_{W_2} \circ \hat{F}_{W_1}.$$

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An important subtlety is that the construction of $\widehat{HF}(Y)$ requires a choice of basepoint in Y. Similarly, the construction of $\widehat{F}_W$ implicitly relies on choosing an arc in W, connecting the two basepoints in ∂W. To make the dependence explicit, we will write $\widehat{HF}(Y, p)$ and $\widehat{F}_{W,\gamma}$ for the groups and maps defined with an auxiliary choice of basepoint p or arc γ.

1.1. Maps for graph cobordisms. The main construction of this paper is an extension of Ozsváth and Szabó's cobordism maps to the following category:

Definition 1.1. The *graph cobordism category* $\mathsf{Cob}^{\Gamma}_{3+1}$ has the following objects and morphisms:

- the objects are pairs $(Y, \mathbf{p})$, where Y is a closed and oriented 3-manifold (possibly disconnected or empty), and $\mathbf{p}$ is a finite collection of basepoints in Y, such that each component of Y has at least one basepoint;
- a morphism from $(Y_0, \mathbf{p}_0)$ to $(Y_1, \mathbf{p}_1)$ is a pair (W, Γ) such that
  (1) W is an oriented, compact cobordism from Y_0 to Y_1, and
  (2) $\Gamma \subseteq W$ is an embedded graph, such that $\Gamma \cap Y_i = \mathbf{p}_i$, Γ has no valence 0 vertices, and $\mathbf{p}_i \subseteq \Gamma$ are all valence 1.

Generalizing their construction of Heegaard Floer homology for singly pointed 3-manifolds [12], Ozsváth and Szabó also defined a group $\widehat{HF}(Y, \mathbf{p})$, whenever $(Y, \mathbf{p})$ is a closed, oriented 3-manifold with a finite collection of basepoints [15]. The construction extends via a tensor product to disconnected 3-manifolds, as long as each component of Y contains at least one basepoint.

In this paper, we construct cobordism maps for graph cobordisms, and prove the following:

Theorem 1.2. *If* (W, Γ) *is a graph cobordism from* $(Y_0, \mathbf{p}_0)$ *to* $(Y_1, \mathbf{p}_1)$*, then the construction of this paper gives a well-defined map*

$$\widehat{F}_{W,\Gamma} \colon \widehat{HF}(Y_0, \mathbf{p}_0) \longrightarrow \widehat{HF}(Y_1, \mathbf{p}_1),$$

satisfying the following:

(1) $\widehat{F}_{[0,1]\times Y, [0,1]\times \mathbf{p}} = \mathrm{id}_{\widehat{HF}(Y,\mathbf{p})}$;

(2) *if* $(W, \Gamma) = (W_2, \Gamma_2) \cup (W_1, \Gamma_1)$*, then*

$$\widehat{F}_{W,\Gamma} = \widehat{F}_{W_2,\Gamma_2} \circ \widehat{F}_{W_1,\Gamma_1};$$

(3) *if* $(W,\Gamma)\colon (Y_0,p_0)\to (Y_1,p_1)$ *is a cobordism such that* Γ *is a path connecting* p_0 *and* p_1*, then* $\widehat{F}_{W,\Gamma}$ *coincides with the map of Ozsváth and Szabó* [13].

Theorem 1.2 implies that Heegaard Floer homology extends to a functor from $\mathsf{Cob}^{\Gamma}_{3+1}$ to the category of vector spaces over $\mathbb{F}$. Our construction of $\widehat{F}_{W,\Gamma}$ uses Juhász's sutured Floer homology TQFT [7] [8].

1.2. Elementary graph cobordisms. In Sections 4 and 5, we compute the maps for the following elementary graph cobordisms, whose underlying 4-manifolds are $[0,1]\times Y$.

(Γ-1) *Free-stabilization cobordisms*: the graph Γ consists of $[0,1]\times \mathbf{p}$, for a non-empty collection of basepoints $\mathbf{p}\subseteq Y$, together with one half-arc of the form $\left[0,\tfrac{1}{2}\right]\times\{p\}$ or $\left[\tfrac{1}{2},1\right]\times\{p\}$, for some $p\notin\mathbf{p}$.

(Γ-2) *Merging and splitting cobordisms*: Γ consists of $[0,1]\times\mathbf{p}$, for a (possibly empty) collection of basepoints $\mathbf{p}\subseteq Y$, as well as one wye-shaped component which merges or splits two basepoints.

(Γ-3) *Spliced loop cobordisms*: Γ consists of $[0,1]\times\mathbf{p}$, for a non-empty collection of basepoints $\mathbf{p}\subseteq Y$, as well as one loop γ in $\left\{\tfrac{1}{2}\right\}\times Y$, which intersects $[0,1]\times\{p\}$ for exactly one $p\in\mathbf{p}$.

(Γ-4) *Broken path cobordisms*: Γ consists of $[0,1]\times\mathbf{p}$, for a (possibly empty) collection of basepoints $\mathbf{p}\subseteq Y$, together with one broken arc $\left(\left[0,\tfrac{1}{3}\right]\cup\left[\tfrac{2}{3},1\right]\right)\times\{p\}$, for some $p\notin\mathbf{p}$.

The elementary graph cobordisms (Γ-1)–(Γ-4) are depicted in Figure 1.1.

1.3. The action of the fundamental group. Since a basepoint is implicitly used in the construction of the Heegaard Floer groups, the naturality theorem of [9] implies only that elements of the *based* mapping class group $\mathrm{MCG}(Y,p)$ induce well defined endomorphisms of Heegaard Floer homology.

There is a fibration

$$\mathrm{Diff}(Y,p)\longrightarrow \mathrm{Diff}(Y)\xrightarrow{\mathrm{ev}_p} Y,$$

where ev_p denotes evaluation at p. The long exact sequence of homotopy groups for this fibration gives a homomorphism

$$\pi_1(Y,p)\longrightarrow \mathrm{MCG}(Y,p).$$

By exactness, the image of $\pi_1(Y,p)$ in $\mathrm{MCG}(Y,p)$ is the kernel of the forgetful map $\mathrm{MCG}(Y,p)\to\mathrm{MCG}(Y)$.

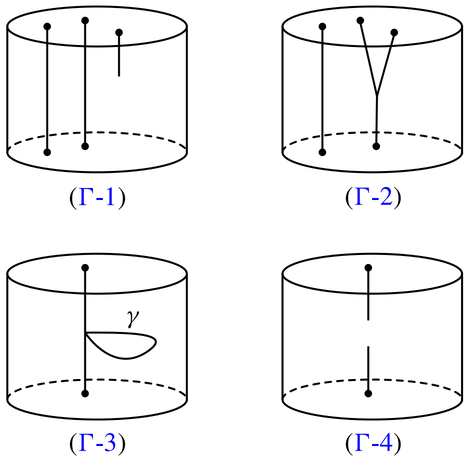

Figure 1.1. The four elementary graph cobordisms in $[0,1] \times Y$.

Suppose $p \in \mathbf{p}$ and $\gamma \in \pi_1(Y, p)$. We write

$$\gamma_*: \widehat{HF}(Y, \mathbf{p}) \longrightarrow \widehat{HF}(Y, \mathbf{p})$$

for the induced endomorphism.

Using the graph TQFT, we prove the following:

Theorem 1.3. *Suppose that $(Y, \mathbf{p})$ is a multi-pointed 3-manifold and that $p \in \mathbf{p}$. If $\gamma \in \pi_1(Y, p)$, then the induced endomorphism γ_* of $\widehat{HF}(Y, \mathbf{p})$ satisfies*

$$\gamma_* = \mathrm{id} + \Phi_p \circ A_{[\gamma]},$$

where $A_{[\gamma]}$ denotes the standard action of $[\gamma] \in H_1(Y; \mathbb{Z})/\operatorname{Tors}$, and

$$\Phi_p: \widehat{HF}(Y, \mathbf{p}) \longrightarrow \widehat{HF}(Y, \mathbf{p})$$

is the broken path graph cobordism labeled (Γ-4) in Figure 1.1.

In Section 5, we identify the broken path graph cobordism Φ_p with the *basepoint action* for the point p, which counts holomorphic disks on a Heegaard diagram with multiplicity 1 at p. See Proposition 5.1.

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2. Background

2.1. Heegaard Floer homology. Suppose $(Y, \mathbf{p})$ is a multi-pointed 3-manifold, $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, and $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p})$ is a Heegaard diagram for $(Y, \mathbf{p})$. Ozsváth and Szabó [15] construct chain complexes $\widehat{CF}(\mathcal{H}, \mathfrak{s})$, $CF^-(\mathcal{H}, \mathfrak{s})$, $CF^+(\mathcal{H}, \mathfrak{s})$ and $CF^\infty(\mathcal{H}, \mathfrak{s})$, as follows. We focus on the case that Y is connected. If Y is disconnected, then $\widehat{CF}(\mathcal{H}, \mathfrak{s})$ is defined by tensoring over $\mathbb{F}$ the complexes for each connected component.

The chain complex $\widehat{CF}(\mathcal{H}, \mathfrak{s})$ is generated by intersection points $\mathbf{x}$ of the two half dimensional tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g(\Sigma)+n-1} \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g(\Sigma)+n-1},$$

in $\mathrm{Sym}^{g(\Sigma)+n-1}(\Sigma)$ (where $n = |\mathbf{p}|$), which satisfy $\mathfrak{s}_{\mathbf{p}}(\mathbf{x}) = \mathfrak{s}$. The differential is given by the formula

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1 \\ n_{\mathbf{p}}(\phi)=0}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot \mathbf{y},$$

where $\mathcal{M}(\phi)$ denotes the moduli space of holomorphic disks in $\mathrm{Sym}^{g(\Sigma)+n-1}(\Sigma)$, representing a given homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

We define

$$\widehat{CF}(\mathcal{H}) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{CF}(\mathcal{H}, \mathfrak{s}). \tag{2.1}$$

Although we mostly focus on $\widehat{CF}$ in this paper, in Section 5, we consider CF^-, which we review presently. Write $\mathbf{p} = \{p_1, \dots, p_n\}$, and $R_n := \mathbb{F}[U_1, \dots, U_n]$. The module $CF^-(\mathcal{H}, \mathfrak{s})$ is the free R_n-module with generators $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_{\mathbf{p}}(\mathbf{x}) = \mathfrak{s}$. The differential on $CF^-(\mathcal{H}, \mathfrak{s})$ is

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U_1^{n_{p_1}(\phi)} \cdots U_n^{n_{p_n}(\phi)} \cdot \mathbf{y}.$$

Unlike for $\widehat{CF}$, it is usually not possible to define a total complex $CF^-(\mathcal{H})$ as a direct sum over all Spin^c structures, analogous to equation (2.1), since CF^- requires a stronger version of admissibility than $\widehat{CF}$, which cannot normally be simultaneously realized for all Spin^c structures on a single diagram [12, Section 4.2.2]. Hence, on CF^-, we usually work with one Spin^c structure at a time.

2.2. Sutured Floer homology. *Sutured manifolds* were defined by Gabai [3] to study foliations on 3-manifolds. Juhász [7] defined an extension of Heegaard Floer homology for sutured manifolds, called *sutured Floer homology*. Juhász [8] also described a $(3+1)$-dimensional TQFT for sutured Floer homology. In this section, we recall some background about sutured manifolds and the sutured Floer homology TQFT.

The following is a slight restriction of Gabai's original definition, but is sufficient for our purposes:

Definition 2.1. A *sutured manifold* (M,γ) is a compact, oriented 3-manifold M with boundary, together with a set of pairwise disjoint, simple closed curves $\gamma \subseteq \partial M$ (the sutures) which are oriented. The surface $\partial M \setminus \gamma$ is partitioned into two sets of components, $R_+(\gamma)$ and $R_-(\gamma)$, which are oriented so that the normal of $R_+(\gamma)$ points out of M, while the normal of $R_-(\gamma)$ points into M. Finally, we require γ to be consistently oriented with respect to the boundary orientation of $R_+(\gamma)$ and $R_-(\gamma)$.

The main difference between Definition 2.1 and Gabai's original definition is that we exclude toroidal sutures. We say that a sutured manifold (M,γ) is *balanced* if $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$. To a balanced sutured manifold (M,γ) with no closed components, Juhász constructs an $\mathbb{F}$-vector space

$$SFH(M,\gamma).$$

If Y is a closed, oriented 3-manifold, and $\mathbf{p}$ is a collection of basepoints, then we write $Y(\mathbf{p})$ for the sutured manifold (M,γ) where

$$M := Y \setminus \operatorname{int} N(\mathbf{p})$$

and γ consists of one simple closed curve per component of ∂M. We note that

$$SFH(Y(\mathbf{p})) = \widehat{HF}(Y,\mathbf{p}),$$

since a Heegaard diagram for $(Y,\mathbf{p})$ may be obtained from a diagram for $Y(\mathbf{p})$ by collapsing each suture to a basepoint.

Juhász also defines cobordism maps for sutured Floer homology [8]. He uses the following notion of cobordism between sutured manifolds:

Definition 2.2. A *cobordism of sutured manifolds*

$$\mathcal{W} = (W, Z, [\xi])\colon (M_0,\gamma_0) \longrightarrow (M_1,\gamma_1)$$

is a triple such that

(1) W is a compact, oriented 4-manifold with boundary,

(2) Z is a compact, codimension-0 submanifold of ∂W, and $\partial W \setminus \operatorname{int}(Z) = -M_0 \sqcup M_1$,

(3) $[\xi]$ is an equivalence class of positive contact structures on Z, such that ∂Z is a convex surface with dividing set γ_i on ∂M_i, for $i \in \{0, 1\}$.

The notion of equivalence between contact structures used in Definition 2.2 can be found in [8, Definition 2.3].

If $\mathcal{W}\colon (M_0, \gamma_0) \to (M_1, \gamma_1)$ is a cobordism between balanced sutured manifolds, Juhász [8] constructs a well-defined map

$$F_{\mathcal{W}}\colon SFH(M_0, \gamma_0) \longrightarrow SFH(M_1, \gamma_1),$$

which is functorial in the following sense. If ξ is a $[0, 1]$-invariant contact structure on $[0, 1] \times \partial M$, such that $\{0, 1\} \times \partial M$ is convex, with dividing set γ, then $\mathcal{W} = ([0, 1] \times M, [0, 1] \times \partial M, [\xi])$ is a sutured manifold cobordism from (M, γ) to itself. The induced cobordism map

$$F_{\mathcal{W}}\colon SFH(M, \gamma) \longrightarrow SFH(M, \gamma)$$

is the identity. Furthermore, if $\mathcal{W}$ decomposes as the composition of two sutured manifold cobordisms $\mathcal{W}_2 \circ \mathcal{W}_1$, then

$$F_{\mathcal{W}} = F_{\mathcal{W}_2} \circ F_{\mathcal{W}_1}.$$

See [8, Theorem 11.3].

We outline the construction of the sutured cobordism maps in Section 2.4, after we outline one of its constituents, the *contact gluing map*.

2.3. The contact gluing map. We now recall Honda, Kazez, and Matić's *contact gluing map* for sutured Floer homology [6], as well as the contact-handle formulation given by Juhász and the author [10].

Definition 2.3. Suppose that (M, γ) and (M', γ') are sutured manifolds. We say that (M, γ) is a *sutured submanifold* of (M', γ') if $M \subseteq \operatorname{int}(M')$.

If (M, γ) is a sutured submanifold of (M', γ'), and ξ is a positive contact structure on $Z := M' \setminus \operatorname{int}(M)$ which induces the dividing set $\gamma \cup \gamma'$, then Honda, Kazez and Matić [6] define a *contact gluing map*

$$\Phi_{Z,\xi}\colon SFH(-M, \gamma) \longrightarrow SFH(-M', \gamma').$$

In [10], Juhász and the author gave a reformulation of the contact gluing map in terms of *contact handles*, which facilitates computations. Contact handles were defined by Giroux [4]. See Ozbagci [11] for an exposition. We take the following as the definition of a contact handle:

Definition 2.4. Suppose (M, γ) is a sutured submanifold of (M', γ'), and ξ is a positive contact structure on $Z := M' \setminus \operatorname{int}(M)$, with dividing set $\gamma \cup \gamma'$. We say that (Z, ξ) is a *contact handle of index k* if there is a contact vector field v on Z that points into Z on ∂M, and out of Z on $\partial M'$, as well as a decomposition $Z = Z_0 \cup h$ such that

(1) Z_0 is diffeomorphic to $[0, 1] \times \partial M$,

(2) v is non-vanishing on Z_0, points into Z_0 on $\{0\} \times \partial M$ and out of Z_0 on $\{1\} \times \partial M$, and each flowline of v is an arc from $\{0\} \times \partial M$ to $\{1\} \times \partial M$,

(3) h is smooth 3-ball with corners, and ξ is tight on h.

We have the following additional requirements, depending on k.

- $k = 0$: $h = D^3$ (with no corners) and $h \cap Z_0 = \emptyset$. The dividing set on ∂h is a single circle.
- $k = 1$: $h = [0, 1] \times D^2$ and $h \cap Z_0 = \{0, 1\} \times D^2$. The dividing set on ∂h is a single closed curve, consisting of an arc on $\{0\} \times D^2$ and $\{1\} \times D^2$, and two longitudinal arcs on $[0, 1] \times \partial D^2$.
- $k = 2$: $h = [0, 1] \times D^2$ and $h \cap Z_0 = [0, 1] \times \partial D^2$. The dividing set is as in $k = 1$ case.
- $k = 3$: $h = D^3$ (with no corners), and $h \cap Z_0 = \partial h$. The dividing set on ∂h is a single circle.

We now state the description from [10] of the contact gluing maps of Honda, Kazez and Matić when $M' \setminus M$ is a contact handle.

If Z is a contact 0-handle, we extend the Heegaard surface into Z_0 using the flow of v, and then add a disk to the Heegaard surface which lies in h and intersects ∂D^3 along the sutures. We add no new alpha or beta curves. The map on sutured Floer homology is the tautological one.

If Z is a contact 1-handle, we extend Σ into Z_0 using the flow of v, and then attach a band to the boundary of the Heegaard surface, which is contained in h and intersects the boundary along the dividing set. We add no alpha or beta curves. Similar to the contact 0-handle map, the map on sutured Floer homology is the tautological one.

If Z is a contact 2-handle, we extend the Heegaard surface into Z_0 using the flow of v, as before. Now ∂h intersects ∂Z in an annulus. Let c denote the core of the attaching annulus. The curve c may be taken to intersect the dividing set in two points. Let λ_+ denote the subarc of c which intersects R_+, and let λ_- denote the subarc which intersects R_-. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a diagram for (M, γ), we may obtain a diagram for (M', γ') by adjoining a band to $\partial\Sigma$, and adding a new alpha curve α, and a new beta curve β. The curves α and β have a single intersection point in the band region, as in Figure 2.1. Outside of the band region, β consists of λ_+, projected onto $\Sigma \setminus \boldsymbol{\beta}$, and α consists of λ_-, projected on $\Sigma \setminus \boldsymbol{\alpha}$. The map

$$\Phi_{Z,\xi}\colon CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}) \longrightarrow CF(\Sigma \cup B, \boldsymbol{\beta} \cup \{\beta\}, \boldsymbol{\alpha} \cup \{\alpha\})$$

is given by $\mathbf{x} \mapsto \mathbf{x} \times c$. According to [10, Lemma 3.13], the map $\Phi_{Z,\xi}$ is a chain map, and hence induces a homomorphism on homology groups.

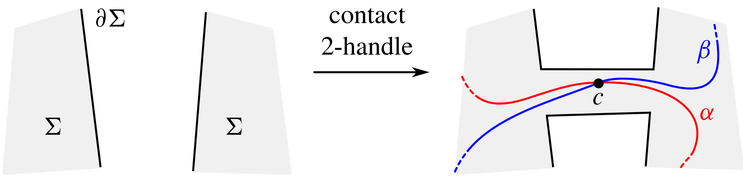

Figure 2.1. A contact 2-handle on Heegaard diagrams.

Finally, suppose Z is a contact 3-handle, and let $S \subseteq \partial M$ denote the 2-sphere which is filled in by Z. Let S' denote a 2-sphere in $\operatorname{int}(M)$ obtained by pushing S into $\operatorname{int}(M)$. The contact 3-handle map is defined as the composition of the 4-dimensional 3-handle map for the 2-sphere S' (which leaves the disjoint union of (M', γ') and B^3), followed by the canonical isomorphism

$$SFH(M', \gamma') \otimes SFH(B^3) \cong SFH(M', \gamma').$$

2.4. Sutured cobordism maps. We now outline the construction of the sutured cobordism maps. Suppose

$$\mathcal{W} = (W, Z, [\xi])\colon (M_0, \gamma_0) \longrightarrow (M_1, \gamma_1),$$

is a cobordism of sutured manifolds, as in Definition 2.2. First, we remove some number of tight, contact 3-balls from Z, and add them to (M_0, γ_0) or (M_1, γ_1), so that each component of W intersects a component of M_0 and M_1 non-trivially. This does not affect the sutured Floer homology of (M_0, γ_0) or (M_1, γ_1), as there is a canonical isomorphism

$$SFH(M_0 \sqcup B^3, \gamma_0 \cup \gamma) \cong SFH(M_0, \gamma_0),$$

where $\gamma \subseteq B^3$ denotes a single closed curve.

Juhász calls a sutured cobordism $\mathcal{W}$ *special* if $Z = [0,1] \times \partial M_0$ and ξ is $[0,1]$-invariant. The cobordism map for a special cobordism is defined to be a composition of 1-handle, 2-handle and 3-handle maps, similar to the ones described by Ozsváth and Szabó [13].

If

$$\mathcal{W} = (W, Z, [\xi])\colon (M_0, \gamma_0) \longrightarrow (M_1, \gamma_1)$$

is a general sutured manifold cobordism, one may obtain a special cobordism

$$\mathcal{W}^s = (W, [0,1] \times \partial M_1, \xi_1)\colon (M_0 \cup Z, \gamma_1) \longrightarrow (M_1, \gamma_1),$$

where ξ_1 is a $[0,1]$-invariant contact structure on $\partial M_1 \times [0,1]$. The cobordism map $F_{\mathcal{W}}$ is defined as the composition

$$F_{\mathcal{W}} := F_{\mathcal{W}^s} \circ \Phi_{Z,\xi}. \tag{2.2}$$

3. Construction of the graph TQFT

Suppose (W, Γ) is a graph cobordism from $(Y_0, \mathbf{p}_0)$ to $(Y_1, \mathbf{p}_1)$. We define a sutured manifold cobordism

$$\mathcal{W}(W, \Gamma) = (W(\Gamma), Z(\Gamma), [\xi(\Gamma)])\colon Y_0(\mathbf{p}_0) \longrightarrow Y_1(\mathbf{p}_1),$$

as follows. We define the 4-manifold $W(\Gamma)$ to be $W \setminus \operatorname{int} N(\Gamma)$, and the 3-manifold $Z(\Gamma)$ to be $\partial W(\Gamma) \cap \partial N(\Gamma)$. We give $\partial Z(\Gamma)$ the same sutures as $Y(\mathbf{p}_0)$ and $Y(\mathbf{p}_1)$, for which we write γ_Z. We take $\xi(\Gamma)$ to be the unique tight contact structure with dividing set γ_Z, whose well definedness we sketch presently. The 3-manifold $Z(\Gamma)$ is homeomorphic to a disjoint union of connected sums of $S^1 \times S^2$, with some number of 3-balls removed. The sutures consist of a single closed curve on each copy of S^2 in $\partial Z(\Gamma)$. It is well known that up to isotopy, relative to $\partial Z(\Gamma)$, there is a unique tight contact structure on $Z(\Gamma)$ which has dividing set γ_Z. The case when $Z(\Gamma) = B^3$ follows from a result of Eliashberg [2]. The general case follows by decomposing $Z(\Gamma)$ along a collection of convex 2-spheres, until one obtains a disjoint union of tight, punctured 3-spheres, using convex surface theory, see [1] and [5].

Without further ado, we define

$$\widehat{F}_{W,\Gamma} := F_{\mathcal{W}(W,\Gamma)}.$$

4. Elementary graph cobordisms in $[0,1] \times Y$

In this section, we compute the maps induced by the elementary graph cobordisms shown in Figure 1.1, with the exception of the broken path cobordism, which is considered in Section 5.

4.1. Free-stabilization cobordisms. In this section we compute the maps for the free-stabilization cobordisms, which are labeled (Γ-1) in Figure 1.1. Let

$$\mathcal{W}_p^+ := ([0,1] \times Y, \Gamma_p^+) : (Y, \mathbf{p}) \longrightarrow (Y, \mathbf{p} \cup \{p\})$$

denote the free-stabilization graph cobordism which adds the basepoint p, and let $\mathcal{W}_p^-$ denote the free-stabilization graph cobordism which removes p.

We define

$$S_p^+ := \hat{F}_{\mathcal{W}_p^+} \quad \text{and} \quad S_p^- := \hat{F}_{\mathcal{W}_p^-}. \tag{4.1}$$

If $\mathcal{H}$ is a Heegaard diagram for $(Y, \mathbf{p})$, we may form a Heegaard diagram $\mathcal{H}_{(p)}$ for $(Y, \mathbf{p} \cup \{p\})$ by adding the basepoint p, encircled by a new pair of alpha and beta curves, α and β, as in Figure 4.1. After a sequence of handleslides, we may assume that α and β are immediately adjacent to another basepoint $p_0 \in \mathbf{p}$. The placement of basepoints makes it easy to verify that

$$\widehat{HF}(\mathcal{H}_{(p)}) \cong \widehat{HF}(\mathcal{H}) \otimes V, \tag{4.2}$$

where V is the 2-dimensional vector space $\mathbb{F}_{1/2} \oplus \mathbb{F}_{-1/2}$. We write θ^+ for the top degree generator of V, and θ^- for the bottom degree generator.

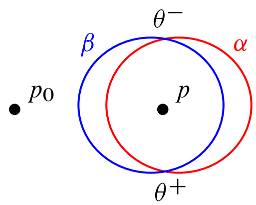

Figure 4.1. The diagram $\mathcal{H}_{(p)}$, obtained by adding a basepoint p to a diagram $\mathcal{H}$.

Lemma 4.1. *With respect to the isomorphism from equation (4.2), the maps S_p^+ and S_p^- satisfy*

$$S_p^+(\mathbf{x}) = \mathbf{x} \times \theta^+ \quad \textit{and} \quad S_p^-(\mathbf{x} \times \theta) = \begin{cases} \mathbf{x} & \textit{if } \theta = \theta^-, \\ 0 & \textit{if } \theta = \theta^+. \end{cases}$$

Proof. We first consider S_p^+. We may perform an index 0/1 handle cancellation to decompose the graph cobordism $\mathcal{W}_p^+$ as follows:

(1) a 0-handle B^4, containing an arc a, which intersects ∂B^4 in a single point;

(2) a 1-handle cobordism which merges (S^3, p) with $(Y, \mathbf{p})$, away from $\mathbf{p}$.

We may similarly decompose $\mathcal{W}_p^-$ into a 3-handle cobordism followed by a 4-handle cobordism.

The graph cobordism map for $(B^4, a)\colon \emptyset \to (S^3, p)$ is easily seen to send the generator of $\widehat{HF}(\emptyset) \cong \mathbb{F}$ to the generator of $\widehat{HF}(S^3) \cong \mathbb{F}$, and similarly for the 4-handle cobordism in the opposite direction. The main claim now follows for S_p^+, since the stated formula coincides with the definition of the 1-handle map [8, Section 7]. The proof of S_p^- is similar. □

4.2. Merge and split cobordisms. We now compute the merge and split cobordism maps, which are labeled (Γ-2) in Figure 1.1. Suppose that p_1 and p_2 are two points in Y, λ is a path connecting p_1 to p_2, and p_0 is a point along λ. Suppose that $\mathbf{p}$ is a (possibly empty) collection of basepoints in $Y \setminus \{p_0, p_1, p_2\}$. Write

$$\mathcal{W}_\lambda^{\text{merge}}\colon (Y, \mathbf{p} \cup \{p_1, p_2\}) \longrightarrow (Y, \mathbf{p} \cup \{p_0\})$$

for the graph cobordism which merges p_1 and p_2 into p_0, along the path λ. Similarly write

$$\mathcal{W}_\lambda^{\text{split}}\colon (Y, \mathbf{p} \cup \{p_0\}) \longrightarrow (Y, \mathbf{p} \cup \{p_1, p_2\})$$

for the graph cobordism which splits p_0 into the pair p_1 and p_2. Write

$$Sp_\lambda := \widehat{F}_{\mathcal{W}_\lambda^{\text{split}}} \quad \text{and} \quad M_\lambda := \widehat{F}_{\mathcal{W}_\lambda^{\text{merge}}}.$$

Lemma 4.2. *Let $\mathcal{H}$ be a Heegaard diagram for $(Y, \mathbf{p} \cup \{p_0\})$, and let $\mathcal{H}_{p_1,(p_2)}$ be the Heegaard diagram for $(Y, \mathbf{p} \cup \{p_1, p_2\})$ obtained by relabeling p_0 as p_1, and adding new alpha and beta curves which bound small disks containing p_2, as in Figure 4.2. Furthermore, assume that λ is embedded in the Heegaard surface, as shown in Figure 4.2. With respect to the isomorphism in equation (4.2), we have*

$$Sp_\lambda(\mathbf{x}) = \mathbf{x} \times \theta^- \quad \textit{and} \quad M_\lambda(\mathbf{x} \times \theta) = \begin{cases} \mathbf{x} & \textit{if } \theta = \theta^+, \\ 0 & \textit{if } \theta = \theta^-. \end{cases}$$

Proof. We begin with the split cobordism $\mathcal{W}_\lambda^{\text{split}}$. Write (Z, ξ) for the contact portion of the boundary of the sutured manifold associated to $\mathcal{W}_\lambda^{\text{split}}$. The contact

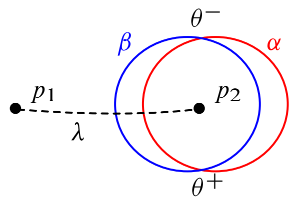

Figure 4.2. The diagram $\mathcal{H}_{p_1,(p_2)}$, considered in Lemma 4.2.

manifold (Z, ξ) is a thrice punctured, tight 3-ball. We glue Z to the boundary S^2 of $Y(\mathbf{p} \cup \{p_0\})$ associated to p_0. The special cobordism $(\mathcal{W}_\lambda^{\text{split}})^s$ is a product cobordism. Hence, by equation (2.2), $\widehat{F}_{\mathcal{W}_\lambda^{\text{split}}}$ coincides with the contact gluing map $\Phi_{Z,\xi}$. The contact manifold (Z, ξ) is a contact 2-handle, so the gluing map takes the form described in Section 2.3 (see specifically Figure 2.1). The description of the contact gluing map immediately gives the stated formula for Sp_λ. See Figure 4.3.

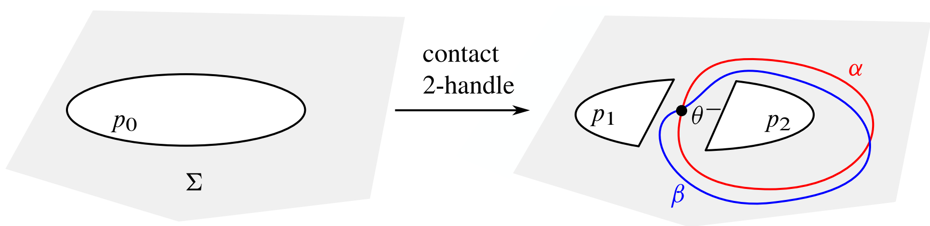

Figure 4.3. The map for a split cobordism coincides with a contact 2-handle map. The boundary circles represent the sutures of the manifolds $Y(\mathbf{p} \cup \{p_0\})$ and $Y(\mathbf{p} \cup \{p_1, p_2\})$.

We now compute the merge map. Note that the merge cobordism is obtained by turning around the split cobordism. A Morse theory argument (see [10, Lemma 6.7]) shows that sutured cobordism associated to $\mathcal{W}_\lambda^{\text{merge}}$ has the following description:

(1) a contact 1-handle which merges the two boundary components associated to p_1 and p_2. This turns the pair of boundary components into a single boundary component, and adds an $S^1 \times S^2$ summand;

(2) a 4-dimensional 2-handle which cancels the $S^1 \times S^2$ summand.

The stated formula for the merge map follows from an easy holomorphic triangle computation in the $S^1 \times S^2$ summand. See Figure 4.4. □

4.3. Spliced loop cobordisms. We now investigate the spliced loop cobordism, labeled (Γ-3) in Figure 1.1.

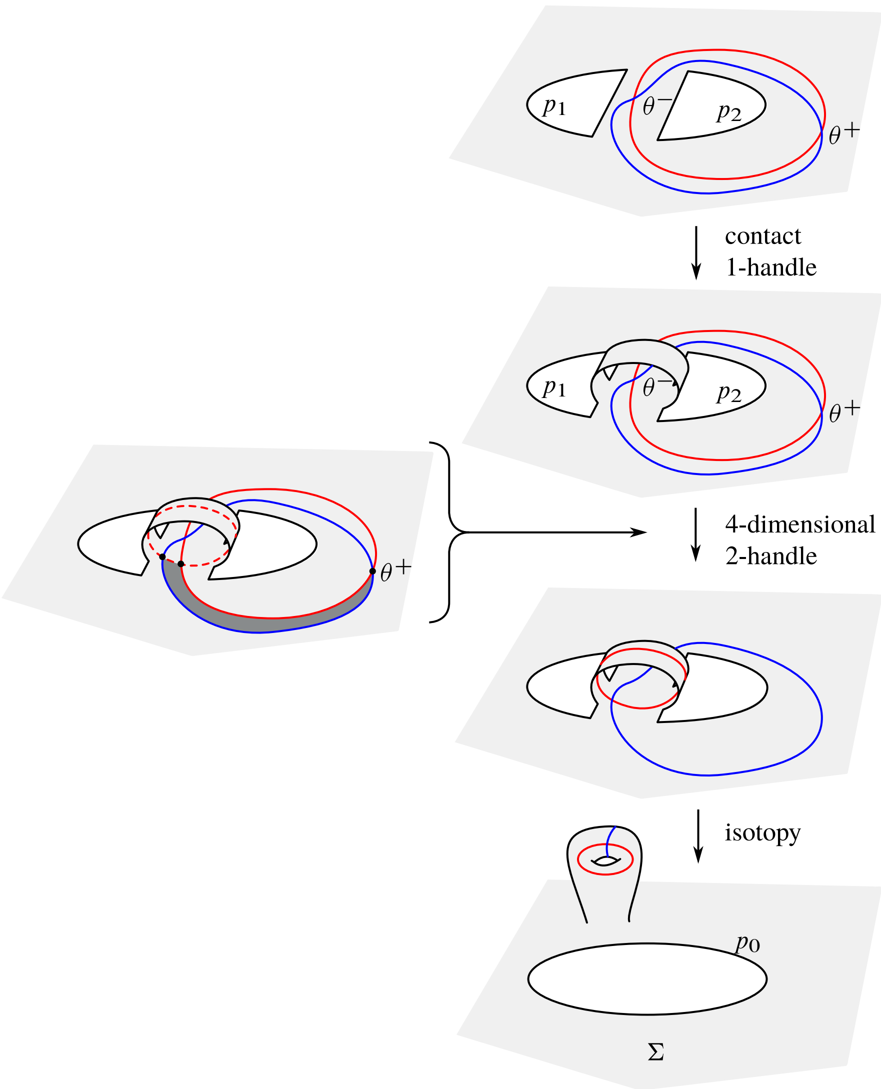

Figure 4.4. Computing the merge map. On the left side, an index 0 holomorphic triangle is shown.

Lemma 4.3. *If $\mathcal{W}_\gamma = ([0,1]\times Y, \Gamma_\gamma)\colon (Y, \mathbf{p}) \to (Y, \mathbf{p})$ is a spliced loop cobordism, then*

$$\hat{F}_{\mathcal{W}_\gamma} = A_{[\gamma]},$$

where $A_{[\gamma]}$ denotes the standard action of $H_1(Y;\mathbb{Z})/\operatorname{Tors}$ on $\widehat{HF}(Y, \mathbf{p})$.

Proof. Let $p \in \mathbf{p}$ denote the basepoint connected to the strand with the spliced loop. Write (Z, ξ) for the contact portion of the boundary of the sutured cobordism associated to $\mathcal{W}_\gamma$. The contact manifold (Z, ξ) has a component which consists of a twice punctured copy of $S^1 \times S^2$, one of whose boundary components is glued to the boundary S^2 in $Y(\mathbf{p})$ for p. The manifold (Z, ξ) may be decomposed into a contact 1-handle, which splits the suture for p into two circles (and adds no alpha or beta curves), as well as a contact 2-handle, which merges the two sutures together, and adds an alpha and beta curve. The resulting 3-manifold is $(Y \# S^1 \times S^2, \mathbf{p})$. Similar to Figure 4.3, the induced map is given by

$$\Phi_{Z,\xi}(\mathbf{x}) = \mathbf{x} \times \theta^-. \tag{4.3}$$

Let $\gamma_0 \subseteq Y \# S^1 \times S^2$ denote a curve which is supported in the $S^1 \times S^2$ summand, and represents a generator of $H_1(S^1 \times S^2)$. According to [14, Proposition 6.4], the map $A_{[\gamma_0]}$ is given by

$$A_{[\gamma_0]}(\mathbf{x} \times \theta^+) = \mathbf{x} \times \theta^- \quad \text{and} \quad A_{[\gamma_0]}(\mathbf{x} \times \theta^-) = 0. \tag{4.4}$$

There is also a 1-handle cobordism from $(Y, \mathbf{p})$ to $(Y \# (S^1 \times S^2), \mathbf{p})$, whose associated cobordism map is given by

$$F_1(\mathbf{x}) = \mathbf{x} \times \theta^+. \tag{4.5}$$

Combining equations (4.3), (4.4) and (4.5), we obtain

$$\Phi_{Z,\xi}(\mathbf{x}) = A_{[\gamma_0]}(F_1(\mathbf{x})).$$

The special cobordism associated to $\mathcal{W}_\gamma$ consists of a 2-handle, which cancels the new $S^1 \times S^2$ summand. The 2-handle is attached along a framed knot $\mathbb{K}$ whose underlying unframed knot is the splice $\gamma * \gamma_0$. The framing is irrelevant, since for any choice of integral framing on $\gamma * \gamma_0$, there is a canonical diffeomorphism between $(Y \# S^1 \times S^2)(\mathbb{K})$ and Y. Hence

$$\widehat{F}_{\mathcal{W}_\gamma} = F_{\mathbb{K}} \circ A_{[\gamma_0]} \circ F_1. \tag{4.6}$$

By definition, the right hand side of equation (4.6) represents Ozsváth and Szabó's map for the identity cobordism, twisted by the induced element $[\gamma_0]$ of $H_1([0,1] \times Y; \mathbb{Z}) / \operatorname{Tors}$. The class in $H_1(Y; \mathbb{Z})$ induced by the loop γ_0 coincides with $[\gamma]$, so the map induced by $\widehat{F}_{\mathcal{W}_\gamma}$ is exactly $A_{[\gamma]}$. $\square$

5. The broken path cobordism

We now investigate the broken path cobordism, labeled (Γ-4) in Figure 1.1. Let us write $\mathcal{B}_p$ for the induced cobordism map.

We first describe our candidate map. If $p \in \mathbf{p}$ and $\mathcal{H}$ is a Heegaard diagram for $(Y, \mathbf{p})$, then there is a map

$$\Phi_p \colon \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H}),$$

given by the formula

$$\Phi_p(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1 \\ n_p(\phi)=1 \\ n_{p'}(\phi)=0,\, p' \in \mathbf{p} \setminus \{p\}}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot \mathbf{y}.$$

By counting the ends of moduli spaces of index 2 holomorphic disks which cover p exactly once, we see that Φ_p is a chain map. By counting the ends of moduli spaces of index 2 holomorphic disks which cover p exactly twice, we obtain

$$\Phi_p^2 = \partial \circ H + H \circ \partial,$$

where H is the map which counts index 1 holomorphic disks representing classes ϕ with $n_p(\phi) = 2$ and $n_{p'}(\phi) = 0$ for all $p' \in \mathbf{p} \setminus \{p\}$.

In this section, we prove the following:

Proposition 5.1. *If $(Y, \mathbf{p})$ is a multi-pointed 3-manifold and $p \in \mathbf{p}$, then*

$$\mathcal{B}_p = \Phi_p,$$

as endomorphisms of $\widehat{HF}(Y, \mathbf{p})$.

To prove Proposition 5.1, it is helpful to consider the minus version of the Heegaard Floer chain complexes. Write $\mathbf{p} = \{p_1, \dots, p_n\}$. We now describe an algebraic interpretation of Φ_{p_i} in terms of the chain complex $CF^-(Y, \mathbf{p}, \mathfrak{s})$, which we recall is finitely generated and free over the ring

$$R_n := \mathbb{F}[U_1, \dots, U_n].$$

Given a Heegaard diagram $\mathcal{H}$ of $(Y, \mathbf{p})$, the intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_{\mathbf{p}}(\mathbf{x}) = \mathfrak{s}$ give a free basis of $CF^-(\mathcal{H}, \mathfrak{s})$ over R_n. The complex $\widehat{CF}(\mathcal{H}, \mathfrak{s})$ is obtained by setting $U_1 = U_2 = \cdots = U_n = 0$, or equivalently by taking a tensor product with the ring $\mathbb{F}$, with the trivial action of U_i.

We may write the differential of $CF^-(\mathcal{H}, \mathfrak{s})$ as a square matrix, using the basis of intersection points. The map Φ_{p_i} is given by taking this matrix, differentiating each entry with respect to U_i, and setting $U_1 = \cdots = U_n = 0$.

More generally, suppose (C^-, ∂^-) is a free, finitely generated chain complex over the ring R_n, with some chosen basis. Write $(\widehat{C}, \hat{\partial})$ for the chain complex obtained by setting $U_1 = \cdots = U_n = 0$. We may define a map

$$\Phi_{U_i}: \widehat{C} \longrightarrow \widehat{C},$$

by taking the matrix for ∂^-, and differentiating each entry with respect to U_i, and then setting all variables to be zero.

Lemma 5.2. (1) *Suppose (C_1^-, ∂_1^-) and (C_2^-, ∂_2^-) are free, finitely generated chain complexes over R_n, with fixed bases, and $F: C_1^- \to C_2^-$ is an R_n-equivariant chain map. Write $\widehat{F}: \widehat{C}_1 \to \widehat{C}_2$ for the induced map. Then*

$$\Phi_{U_i} \circ \widehat{F} + \widehat{F} \circ \Phi_{U_i} \simeq 0.$$

(2) *Suppose that $\mathbf{p} = \{p_1, \dots, p_n\}$ is a collection of basepoints on Y. The map $\Phi_p: \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$ is natural, in the sense that if $\mathcal{H}$ and $\mathcal{H}'$ are two diagrams for $(Y, \mathbf{p})$ then*

$$\Psi_{\mathcal{H}\to\mathcal{H}'} \circ \Phi_p \simeq \Phi_p \circ \Psi_{\mathcal{H}\to\mathcal{H}'},$$

where $\Psi_{\mathcal{H}\to\mathcal{H}'}$ denotes the naturality map from $\widehat{CF}(\mathcal{H})$ to $\widehat{CF}(\mathcal{H}')$.

Proof. The second statement follows from the first, since the naturality maps on $\widehat{CF}$ are restrictions of the naturality maps on CF^-.

To prove the first claim, we take the equation

$$0 = \partial^- \circ F + F \circ \partial^-,$$

and differentiate it with respect to U_i. Using the Leibniz rule for products of matrices, and then setting $U_1 = \cdots = U_n = 0$, we obtain

$$F \circ \Phi_{U_i} + \Phi_{U_i} \circ F = \hat{\partial} \circ \widehat{F'} + \widehat{F'} \circ \hat{\partial},$$

as maps from $\widehat{C}_1$ to $\widehat{C}_2$. Here, F' denotes the map obtained by taking the matrix for F, and differentiating each entry with respect to U_i, and $\widehat{F'}$ denotes the map resulting from setting $U_1 = \cdots = U_n = 0$. □

We prove an additional lemma, which concerns modifying the graph by adding an extra strand:

Lemma 5.3. *Suppose that (W, Γ) and (W, Γ') are two graph cobordisms, such that Γ' is obtained by adding an* ***interior leaf*** *to Γ, i.e. a new edge e, contained in $\operatorname{int}(W)$, such that $e \cap \Gamma$ consists of a single point. See Figure 5.1. Then*

$$F_{W,\Gamma} = F_{W,\Gamma'}.$$

Proof. The complements of Γ and Γ' are diffeomorphic, so the claim is immediate from the construction of the cobordism maps in Section 3. $\square$

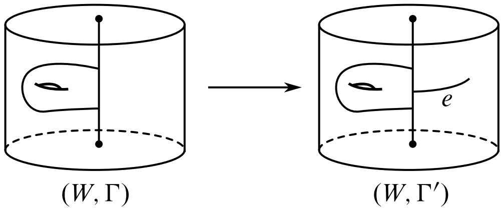

Figure 5.1. Adding an interior leaf to a graph cobordism.

Remark 5.4. The relation in Lemma 5.3 is called the *trivial strand relation* in [16].

Proof of Proposition 5.1. To disambiguate terms, let us write $\Phi_p^{(Y,\mathbf{p})}$ and $\mathcal{B}_p^{(Y,\mathbf{p})}$ for the endomorphisms Φ_p and $\mathcal{B}_p$ of $\widehat{HF}(Y, \mathbf{p})$.

As a first step, we show the claim when the component of Y which contains p also contains another basepoint p_0. In this case, the $\mathcal{B}_p^{(Y,\mathbf{p})}$ is equal to $S_p^+ \circ S_p^-$, where S_p^+ and S_p^- denote the free-stabilization and destabilization maps considered in Section 4.1. We may use the diagram $\mathcal{H}_{(p)}$ shown in Figure 4.1. Using Lemma 4.1, we obtain

$$\mathcal{B}_p^{(Y,\mathbf{p})}(\mathbf{x} \times \theta^+) = 0 \quad \text{and} \quad \mathcal{B}_p^{(Y,\mathbf{p})}(\mathbf{x} \times \theta^-) = \mathbf{x} \times \theta^+. \tag{5.1}$$

On the other hand, using the diagram in Figure 4.1, the only holomorphic curves of index 1 which have multiplicity 1 on p and multiplicity 0 on p_0 have domain consisting of the bigon going over p. Using this diagram, we see that $\Phi_p^{(Y,\mathbf{p})}$ coincides with equation (5.1). Hence, the claim follows if there is another basepoint in the component of Y which contains p.

We now consider the case when p is the only basepoint in its component of Y. In this case, we argue by adding an interior leaf to the graph, as shown in Figure 5.2. According to Lemma 5.3, this does not change the cobordism map. We decompose the broken path cobordism as follows:

(1) a free-stabilization cobordism, adding a new basepoint p_0;

(2) a broken path cobordism from $(Y, \{p, p_0\})$ to $(Y, \{p, p_0\})$ (which is broken over p);

(3) a basepoint merging cobordism, which merges p and p_0 along a path λ.

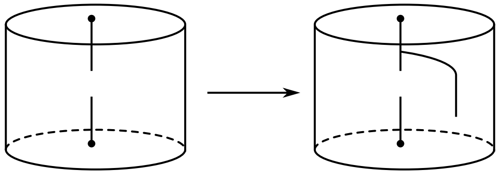

Figure 5.2. Computing the broken path cobordism map by adding an interior leaf.

Hence

$$\mathcal{B}_p^{(Y,p)} = M_\lambda \circ \mathcal{B}_p^{(Y,\{p,p_0\})} \circ S_{p_0}^+. \tag{5.2}$$

By the proof when there are at least 2 basepoints, equation (5.2) gives

$$\Phi_p^{(Y,p)} = M_\lambda \circ \Phi_p^{(Y,\{p,p_0\})} \circ S_{p_0}^+. \tag{5.3}$$

If we can show

$$\Phi_p^{(Y,\{p,p_0\})} \circ S_{p_0}^+ = S_{p_0}^+ \circ \Phi_p^{(Y,p)}, \tag{5.4}$$

then we can manipulate equation (5.3) to obtain

$$\begin{aligned} \mathcal{B}_p^{(Y,p)} &= M_\lambda \circ \Phi_p^{(Y,\{p,p_0\})} \circ S_{p_0}^+ \\ &= M_\lambda \circ S_{p_0}^+ \circ \Phi_p^{(Y,p)} \\ &= \Phi_p^{(Y,p)}, \end{aligned} \tag{5.5}$$

since $M_\lambda \circ S_{p_0}^+ = \mathrm{id}$, as the corresponding cobordism is the identity graph cobordism, with an extra interior leaf. Hence, it suffices to prove equation (5.4).

Consider the 2-variable polynomial ring $\mathbb{F}[U, U_0]$, where U is associated to p, and U_0 is associated to p_0. Note that $\widehat{CF}(Y, \{p, p_0\}, \mathfrak{s})$ is obtained from $CF^-(Y, \{p, p_0\}, \mathfrak{s})$ by setting $U = U_0 = 0$. Similarly, $\widehat{CF}(Y, p, \mathfrak{s})$ is also obtained from $CF^-(Y, p) \otimes_{\mathbb{F}} \mathbb{F}[U_0]$ by setting $U = U_0 = 0$. Hence, by Lemma 5.2(1), to show equation (5.4), it suffices to show that the map $S_{p_0}^+$ can be extended to an $\mathbb{F}[U, U_0]$-equivariant map from $CF^-(Y, p, \mathfrak{s}) \otimes_{\mathbb{F}} \mathbb{F}[U_0]$ to $CF^-(Y, \{p, p_0\}, \mathfrak{s})$.

Let $\mathcal{H}$ be a diagram for (Y, p), and consider a diagram $\mathcal{H}_{(p_0)}$ like the one shown in Figure 4.1, but with the basepoint p_0 encircled by the new alpha and beta circles. There is an obvious isomorphism of modules

$$CF^-(\mathcal{H}_{(p_0)}, \mathfrak{s}) \cong CF^-(\mathcal{H}, \mathfrak{s}) \otimes_{\mathbb{F}} \langle \theta^+, \theta^- \rangle \otimes_{\mathbb{F}} \mathbb{F}[U_0].$$

Furthermore, Ozsváth and Szabó [15, Equation 20] prove that there is an almost complex structure so that the differential on $CF^-(\mathcal{H}_{(p_0)}, \mathfrak{s})$ takes the form

$$\partial_{\mathcal{H}_{(p_0)}}(\mathbf{x} \times \theta^+) = \partial_{\mathcal{H}}(\mathbf{x}) \otimes \theta^+, \tag{5.6a}$$

$$\partial_{\mathcal{H}_{(p_0)}}(\mathbf{x} \times \theta^-) = \partial_{\mathcal{H}}(\mathbf{x}) \otimes \theta^- + (U_p + U_{p_0}) \cdot \mathbf{x} \times \theta^+. \tag{5.6b}$$

Equation (5.6) implies that the map $\mathbf{x} \mapsto \mathbf{x} \otimes \theta^+$, extended equivariantly over $\mathbb{F}[U, U_0]$, gives an $\mathbb{F}[U, U_0]$-equivariant chain map from $CF^-(\mathcal{H}, \mathfrak{s}) \otimes \mathbb{F}[U_0]$ to $CF^-(\mathcal{H}_{(p_0)}, \mathfrak{s})$, which restricts to $S_{p_0}^+$ when we set $U = U_0 = 0$. Lemma 5.2(1) implies equation (5.4), which allows us to perform the manipulation from equation (5.5), completing the proof. □

6. The action of the fundamental group

We are now ready to compute the action of the fundamental group on $\widehat{HF}(Y, \mathbf{p})$.

Theorem 6.1. *The action of $\gamma \in \pi_1(Y, p)$ on $\widehat{HF}(Y, \mathbf{p})$ is given by the formula*

$$\gamma_* = \mathrm{id} + A_{[\gamma]} \circ \Phi_p,$$

where $A_{[\gamma]}$ denotes the action of $H_1(Y; \mathbb{Z})/$ Tors.

As a helpful first step, we prove the relation shown in Figure 6.1.

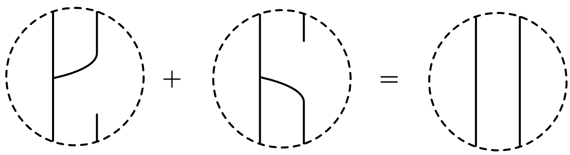

Figure 6.1. A local relation satisfied by the graph cobordisms.

Lemma 6.2. *The graph cobordism maps satisfy the local relation shown in Figure 6.1.*

Proof. We view the local relation as taking place in a cylinder $[0, 1] \times Y$. Let p_1 and p_2 be two basepoints of Y, corresponding to the two bottom points in the local relation, and let $\lambda \subseteq Y$ be the corresponding path connecting them. We may view the left cobordism of Figure 6.1 as a free-destabilization, followed by a basepoint splitting cobordism. The middle cobordism is a basepoint merge, followed by a free-stabilization. The right hand side is the identity. Hence, it is sufficient to check

$$Sp_\lambda \circ S_{p_2}^- + S_{p_2}^+ \circ M_\lambda = \mathrm{id}. \tag{6.1}$$

Equation (6.1) is easily verified from Lemmas 4.1 and 4.2. □

We now prove the formula for the π_1-action:

Proof of Theorem 6.1. We focus on the case when Y has a single basepoint, to simplify the notation. The diffeomorphism map γ_* coincides with the graph cobordism map for $([0,1] \times Y, \hat{\gamma})$, where

$$\hat{\gamma} := \{(t, \gamma(t)) : t \in [0,1]\} \subseteq [0,1] \times Y.$$

We apply the local relation from Figure 6.1 to the graph $\hat{\gamma}$, as shown in Figure 6.2. We obtain the sum of the two graph cobordisms shown on the right side of Figure 6.2. We may identify the right most term with the map $\Phi_p \circ A_{[\gamma]}$ using Lemma 4.3 and Proposition 5.1. The proof is complete. ☐

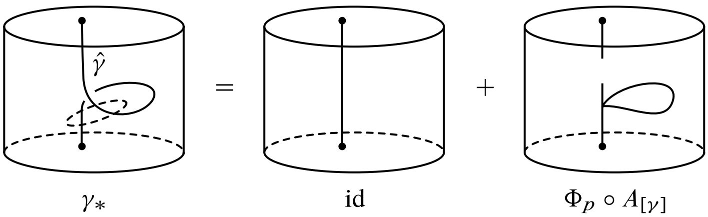

Figure 6.2. Obtaining the formula for the π_1-action by applying the local relation from Figure 6.1 to the graph $\hat{\gamma}$.

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