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Factorization homology and 4D TQFT

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Abstract. In 2010, B. Balsam and A. Kirillov, Jr. showed that the Turaev–Viro invariants defined for a spherical fusion category $\mathcal{A}$ extends to invariants of 3-manifolds with corners. In 2021, A. Kirillov, Jr. described an equivalent formulation for the 2-1 part of the theory (2-manifolds with boundary) using the space of "stringnets with boundary conditions" as the vector spaces associated to 2-manifolds with boundary. Here we construct a similar theory for the 3-2 part of the 4-3-2 theory by L. Crane and D. Yetter (1993).

1. Introduction

The notion of factorization homology for topological manifolds was introduced by Ayala and Francis (see [2, 3]) following the earlier work of Beilinson and Drinfeld and many others. The main idea of this construction is quite natural: it allows one to construct invariants of n-dimensional manifolds by "gluing" local data associated to balls embedded in M. A simple example of such a construction is the usual homology $H_*(M, A)$, where A is an abelian group. A more general form of factorization homology uses as input the following algebraic data:

- an object $\mathcal{A}$ in a symmetric monoidal ∞-category $\mathcal{V}$ (this is the object assigned to the ball);
- a structure of an algebra over the operad of (framed) n-disks on $\mathcal{A}$; this is used to define the gluing of local data.

As an output, the factorization homology of an n-dimensional manifold M with coefficients in $\mathcal{A}$ (denoted $\int_M \mathcal{A}$) gives again an object of $\mathcal{V}$.

Unfortunately, the precise definition of factorization homology has some drawbacks. First, it is given in the language of ∞-categories, so it is rather technical. More importantly, the factorization homology is defined by suitable universality properties, so this definition is not very explicit; in fact, even the existence of such an object is non-trivial.

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The main goal of our note is to give an explicit construction of the factorization homology in two special cases:

(1) $n = 1$, $\mathcal{A}$ is a spherical fusion category;

(2) $n = 2$, $\mathcal{A}$ is a premodular category.

In both cases,the target category $\mathcal{V}$ will be the $(2,1)$ category $\mathcal{R}ex$ of essentially small finitely co-complete **k**-linear categories and right exact functors, as defined in [7]; the symmetric monoidal structure on $\mathcal{R}ex$ is given by Kelly product. In particular, it contains the $(2,1)$ category of small **k**-linear abelian categories as a full subcategory. We will discuss category $\mathcal{R}ex$ in more detail in Section 2.

The case $n = 1$ is rather simple: the only non-trivial 1-manifold is S^1, and it is easy to show that, for a spherical fusion category $\mathcal{A}$, $\int_{S^1} \mathcal{A} = Z(\mathcal{A})$ is the Drinfeld center of $\mathcal{A}$ (as an abelian category; the monoidal structure on the Drinfeld center requires additional construction). Yet, we include this case as it is necessary to understand the $n = 2$ case.

The case $n = 2$ has been studied in the papers by Ben-Zvi, Brochier, and Jordan [7, 8].

In both cases, we show that one can give an explicit definition of factorization homology $\int_M \mathcal{A}$ using suitable colored graphs in dimension $n + 1$ modulo an equivalence relation generated by local moves; this follows the general ideas first suggested by Walker in [29]. For $n = 1$, such local relations were first explicitly written by physicists Levin and Wen in [21], who dubbed such graphs on surfaces "stringnets". A rewriting of this notion in a more mathematical language can be found in [20], where it is shown that the stringnets and their boundary conditions coincide with the 2–1 part of Turaev–Viro $(2 + 1)$-dimensional TQFT.

For $n = 2$, the corresponding colored graphs are ribbon graphs in 3 dimensions; the space of such graphs modulo local relations is commonly called the *skein module*, see e.g., [19]. This space is part of a $(3 + 1)$-dimensional TQFT, which is usually called Crane–Yetter TQFT, introduced in [10].

The main results of this note are Theorem 7.5, which shows that the space of colored graphs satisfies the excision property and thus coincides with factorization homology (Corollary 7.6), and Theorem 9.7, which shows that in the case when $n = 2$ and $\mathcal{A}$ is modular, the category $Z_{\mathrm{CY}}(\Sigma)$ assigned to a surface Σ in Crane–Yetter theory based on category $\mathcal{A}$, only depends on the number of boundary components of Σ. In particular, when Σ is closed, then $Z_{\mathrm{CY}}(\Sigma)$ is trivial and does not depend on the genus of Σ. This result has been widely expected before (see, e.g., the unpublished notes by Freed and Teleman [16, 17]), but, to the best of our knowledge, no formal proof has been published yet.

Notation and conventions

Throughout the paper, the word "manifold" stands for a smooth manifold which admits a finite open cover such that any finite intersection of the open subsets forming the cover is diffeomorphic to an affine space. It is known that this is equivalent to requiring that M be diffeomorphic to the interior of a compact manifold with boundary.

Throughout the paper, we fix an algebraically closed field $\mathbf{k}$ of characteristic 0. All abelian categories will be locally finite $\mathbf{k}$-linear categories (see [13, Section 1.8]). In particular, we denote by $\mathcal{V}ec$ the category of finite-dimensional vector spaces over $\mathbf{k}$.

We will denote by $\boxtimes$ the Deligne tensor product of abelian categories, see [13, Section 1.11]; it is well defined for locally finite $\mathbf{k}$-linear categories and it coincides with the Kelly tensor product in $\mathcal{R}ex$ (see [7, Section 3.2] for references).

We will heavily use the notions of fusion category, pivotal category, spherical fusion category, and premodular category. We refer the reader to [13] for definitions and basic properties of such categories. In most of our formulas and computations, we will suppress both the associativity morphism and the unit morphisms as well as the pivotal morphism $V \simeq V^{**}$.

We will also use graphical presentations of morphisms. We will adopt the convention that morphisms go from top to bottom, and the braiding of a braided category is represented by the " \ " strand going over the " / " strand; see Appendix A for more details.

Note. While this paper was in preparation, we have received a preprint of Juliet Cooke [9], which contains results that are very similar to ours. Yet, both the proof and the exposition are different, so we hope that many readers will find our work useful.

2. Factorization homology overview

In this section, we give a brief summary of the theory of factorization homology. We only try to cover as much as is necessary for our purposes, referring the reader to the review [2] and original papers cited there for details.

To keep things simple, we will only consider the theory for oriented manifolds, ignoring other possible choices of framing structures. All manifolds considered here will be mooth and finite dimensional, i.e., those that are interiors of compact manifolds with (possibly empty) boundary.

We define the symmetric monoidal category $\mathcal{D}isk_n^{\mathrm{or}}$ as the category whose objects are finite disjoint unions of copies of $\mathbb{R}^n$ (or, equivalently, the open unit ball B^n) and morphisms are orientation-preserving embeddings. The set of embeddings is considered as a topological space, with compact-open C^∞ topology, so $\mathcal{D}isk_n^{\text{or}}$ becomes a topological category, and thus, an ∞-category. The monoidal structure is given by the disjoint union.

Given a symmetric monoidal ∞-category $\mathcal{V}$, an *n-disk algebra* in $\mathcal{V}$ is a symmetric monoidal functor of ∞-categories

$$\mathcal{D}isk_n^{\text{or}} \to \mathcal{V}.$$

In particular, this defines an object $A \in \text{Obj}(\mathcal{V})$ (namely, the image of the standard unit n-disk). By abuse of language, we will also call A a "disk algebra."

Given such a disk algebra A, one defines for any oriented n-manifold M the factorization homology

$$\int_M A \in \text{Obj}\,\mathcal{V}$$

as a certain colimit, over all embeddings of collections of oriented disks into M. Note that the existence of such a factorization homology is not guaranteed: in order for it to be defined, we need $\mathcal{V}$ to have sufficiently many colimits. We do not reproduce the definition here; instead, we state some of the properties of this construction, and give a list of properties (Theorem 2.4) which characterizes factorization homology ([1,4]). We refer the reader to the original papers for details and proofs.

Theorem 2.1. *So defined, the factorization homology satisfies the following properties:*

(1) *For an open n-ball B^n, we have*

$$\int_{B^n} A = A;$$

(2) *it is functorial with respect to open embeddings: for any open embedding of oriented n-manifolds $i\colon M \hookrightarrow N$, we have a functor*

$$i_*\colon \int_M A \to \int_N A;$$

(3) *it sends disjoint unions to tensor products in $\mathcal{V}$:*

$$\int_{M \sqcup N} A = \Big(\int_M A\Big) \boxtimes \Big(\int_N A\Big).$$

Let us now restrict our attention to the special case when the target category $\mathcal{V}$ is the $(2,1)$ category $\mathcal{R}ex$ of finitely co-complete $\mathbf{k}$-linear categories with right exact functors, as defined in [7]. We will not repeat the definition, listing instead the properties of this category; we refer the reader to [7, Section 3] for proofs.

(1) The category $\mathcal{R}ex$ is equivalent (as a $(2,1)$ category) to the category $\mathcal{P}r_c$ of compactly generated presentable **k**-linear categories with compact and cocontinuous functors [7, Section 3.1].

(2) The category $\mathcal{R}ex$ is closed under small colimits [7, Proposition 3.5].

(3) The category $\mathcal{P}r_c$ (and thus $\mathcal{R}ex$) includes as a full subcategory the 2-category of small **k**-linear abelian categories, with right exact functors.

(4) The category $\mathcal{R}ex$ has a symmetric monoidal structure, given by the so-called *Kelly product*. If $\mathcal{A}$, $\mathcal{B}$ are abelian category such that Deligne tensor product $\mathcal{A} \boxtimes \mathcal{B}$ is defined (see [13, Section 1.11]), then the Kelly tensor product coincides with the Deligne tensor product [7, Section 3.2]. From now on, we use notation $\boxtimes$ for the Kelly tensor product.

(5) Let $\mathcal{A}$ be an E_1 algebra in $\mathcal{R}ex$. Then we can define the notion of a left (respectively, right) $\mathcal{A}$-module in $\mathcal{V}$ and define the relative tensor product $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ as an object of $\mathcal{V}$. In the special case when $\mathcal{A}$ is a multitensor category (i.e., a locally finite **k**-linear abelian rigid monoidal category) and $\mathcal{M}$, $\mathcal{N}$ are abelian categories, the relative Kelly product $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ is also an abelian category and coincides with the balanced tensor product (see Section 3 below) whenever the latter is defined [7, Remark 3.15 and Corollary 3.18].

(6) Balanced braided tensor categories are 2-disk algebras in $\mathcal{R}ex$ (see [7, Section 3.3]).

Using these properties, it was shown in [7] that the factorization homology with values in $\mathcal{V}$ is well defined (they were working in the case $n = 2$; however, it is trivial to see that the same also applies in the case $n = 1$).

We will now formulate the final property of the factorization homology, called the *excision property*. For simplicity, we will only do so in the case when the target category $\mathcal{V}$ is the $(2,1)$ category $\mathcal{R}ex$, even though it also holds for any factorization homology.

Before formulating the excision property, we need the following lemma about state some simple corollaries of the properties above.

Lemma 2.2. *Let $n = 1$ or 2 and let A be an n-disk algebra in $\mathcal{V} = \mathcal{R}ex$.*

(1) *For any $(n-1)$-dimensional oriented manifold N, let $A(N) = \int_{N \times I} A$, where $I = (0,1)$ is the open interval. Then $A(N)$ has a canonical structure of an E_1 algebra in $\mathcal{R}ex$, with the multiplication coming from embedding $(N \times I) \sqcup (N \times I) \to N \times I$ ("stacking").*

(2) *Let M be an n-dimensional manifold with boundary; we denote by M^o the interior of M. Let N be a boundary component of M. Assume that we are given a diffeomorphism of a neighborhood of N in M^o with $N \times (0,1)$; we will call such an isomorphism a* collaring *or a* collared structure *at N. Then this gives on $\int_{M^o} A$ a natural structure of a module over $A(N)$.*

One of the goals of this paper is to show that, in some special cases, this algebra and the corresponding module structure coincide with the so-called *skein algebra* (respectively, the *skein module*), see [19, 29].

We can now state the final property of the factorization homology.

Theorem 2.3. *With the assumptions of Lemma 2.2, the factorization homology satisfies the following* excision property. *Let M_1, M_2 be n-manifolds with boundary, and M_i^o the interior of M_i. Let N_1, N_2 be connected components of the boundary of M_1, M_2 respectively, together with a diffeomorphism $N_1 \simeq \overline{N_2}$ (where bar stands for opposite orientation). Moreover, assume we are given a collared structure at N_1, N_2 as in Lemma 2.2.*

Let M be the manifold obtained by gluing together M_1 with M_2 using φ; the choice of collared structures gives a smooth structure on M.

Then, one has an equivalence of categories

$$\int_M A = \Big(\int_{M_1^o} A\Big) \boxtimes_{A(N)} \Big(\int_{M_2^o} A\Big),$$

where $\boxtimes_A$ is the relative tensor product in $\mathcal{R}ex$.

Finally, the following properties uniquely characterize the factorization homology (this is a recasting of [7, Theorem 2.5], in turn based on [1, 4]):

Theorem 2.4. *Let A be a 2-disk algebra in $\mathcal{R}ex$. Then the functor $\int_{\_} A$ satisfies, and is characterized by, the following properties.*

(1) *If U is contractible, then there is an equivalence in $\mathcal{R}ex$,*

$$\int_U A \simeq A.$$

(2) *The E_1 "stacking" structure on $A(N)$ in Lemma 2.2 is unique (any two diffeomorphisms $Y \simeq N \times I$ respecting the fibre structure $N \times I \xrightarrow{\text{proj}} I$ induce an equivalent E_1-algebra structures on $\int_Y A$).*

(3) *The excision property of Theorem 2.3 holds.*

3. Module categories, balanced tensor product, and center

In this section, we review the results about balanced tensor product of module categories. Our main goal is to give two constructions of the center of an $\mathcal{C}$-bimodule category $\mathcal{M}$ - $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ (Definition 3.1) and $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$ (Definition 3.7), and show that when $\mathcal{C}$ is pivotal multifusion, they are equivalent (Theorem 3.10).

Recall our convention that all categories considered in this paper are locally finite $\mathbf{k}$-linear. Most of the time, they will be abelian; however, in some cases, we will need to use $\mathbf{k}$-linear additive (but not necessarily abelian) categories. For such a category $\mathcal{A}$, we will denote by $\mathrm{Kar}(\mathcal{A})$ the *Karoubi envelope* (also known as *idempotent completion*) of $\mathcal{A}$. By definition, an object of $\mathrm{Kar}(\mathcal{A})$ is a pair (A, p), where A is an object of $\mathcal{A}$ and $p \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ is an idempotent: $p^2 = p$. Morphisms in $\mathrm{Kar}(\mathcal{A})$ are defined by

$$\mathrm{Hom}_{\mathrm{Kar}(\mathcal{A})}((A_1, p_1), (A_2, p_2)) = \{f \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \mid p_2 f p_1 = f\}.$$

Throughout this section, $\mathcal{C}$ is a pivotal category, though in the definitions $\mathcal{C}$ is only required to be monoidal. When $\mathcal{C}$ is multifusion, we use the conventions and notation laid out in Appendix A. In particular, $\mathrm{Irr}(\mathcal{C})$ is the set of isomorphism classes, $\mathrm{Irr}_0(\mathcal{C})$ are those simple objects appearing as direct summands of the unit $\mathbf{1}$, $\{X_i\}$ will be a fixed set of representatives of $\mathrm{Irr}(\mathcal{C})$, d_i^R is the (right) dimension of X_i, and we will use graphical presentation of morphisms.

We assume that the reader is familiar with the notions of module categories; for a left module category $\mathcal{M}$ over $\mathcal{C}$, we will denote the action of $A \in \mathcal{C}$ on $M \in \mathcal{M}$ by $A \triangleright M$. Similarly, we use $M \triangleleft A$ for right action. In this paper, all module categories are assumed to be semisimple (as abelian categories).

This section is organized as follows: Section 3.1 provides the definition and some properties of $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$; Section 3.2 does so for $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$; and Section 3.3 shows that when $\mathcal{C}$ is pivotal multifusion, these definitions are essentially the same.

3.1. $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$

The following definition is essentially given in [18, Definition 2.1] (there $\mathcal{C}$ is assumed to be fusion).

Definition 3.1. Let $\mathcal{C}$ be a finite multitensor category, and let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category. The center of $\mathcal{M}$, denoted $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$, is the category with the following objects and morphisms.

Objects: pairs (M, γ), where $M \in \mathcal{M}$ and γ is an isomorphism of functors $\gamma_A : A \triangleright M \to M \triangleleft A$, $A \in \mathcal{C}$ (half-braiding) satisfying natural compatibility conditions.

Morphisms: $\mathrm{Hom}((M,\gamma),(M',\gamma')) = \{f \in \mathrm{Hom}_{\mathcal{M}}(M,M') \mid f\gamma = \gamma' f\}$.

In particular, in the special case $\mathcal{M} = \mathcal{C}$, this construction gives the Drinfeld center $\mathcal{Z}(\mathcal{C})$.

Remark 3.2. Equivalently, the center $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ can be described as the category of $\mathcal{C}$-bimodule functors $\mathcal{C} \to \mathcal{M}$; see [18] for details.

Theorem 3.3. *Let $\mathcal{C}$ be pivotal multifusion, and $\mathcal{M}$ a $\mathcal{C}$-bimodule category.*

Let $F\colon \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \to \mathcal{M}$ be the natural forgetful functor $F\colon (M,\gamma) \mapsto M$. Then it has a two-sided adjoint functor $I\colon \mathcal{M} \to \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$, given by

$$I(M) = \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} X_i \triangleright M \triangleleft X_i^* \tag{3.1}$$

with the half-braiding shown in Figure 1.

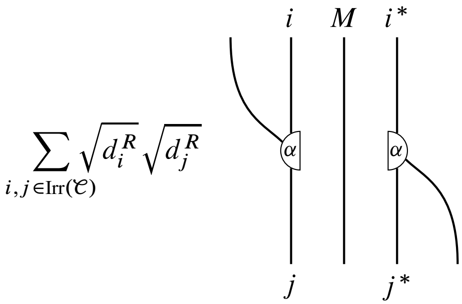
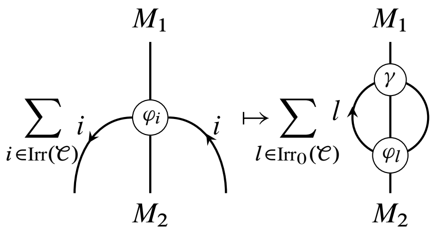


Figure 1. Half-braiding on $I(M)$. See Notation A.3 for the definition of α.

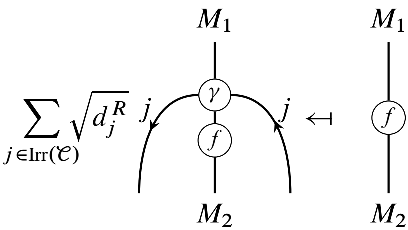
The adjunction isomorphism for $F\colon \mathcal{Z}(\mathcal{M}) \rightleftharpoons \mathcal{M}\colon I$,

$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{M})}((M_1,\gamma), I(M_2)) \simeq \mathrm{Hom}_{\mathcal{M}}(M_1,M_2),$$

is given by



(3.2)



(3.3)

(Note the sum on the right in (3.2) *is over* $\mathrm{Irr}_0(\mathcal{C})$ *and not* $\mathrm{Irr}(\mathcal{C})$*.) The other adjunction isomorphism for* $I: \mathcal{M} \rightleftharpoons \mathcal{Z}(\mathcal{M}) : F$*,*

$$\mathrm{Hom}_{\mathcal{M}}(M_1, M_2) \simeq \mathrm{Hom}_{\mathcal{Z}(\mathcal{M})}(I(M_1), (M_2, \gamma)),$$

is given by a similar formula, essentially obtained by rotating all the diagrams above.

Note that the isomorphisms here differ slightly from that of [20]. The proof is given in Appendix A.

An important special case is when $\mathcal{M} = \mathcal{M}_1 \boxtimes \mathcal{M}_2$, where $\mathcal{M}_1$ is a right module category over a pivotal multifusion category $\mathcal{C}$, and $\mathcal{M}_2$ is a left module category over $\mathcal{C}$. In this case, by [15, Proposition 3.8], one has that $\mathcal{Z}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2)$ is naturally equivalent to the balanced tensor product of categories:

$$\mathcal{Z}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2) \simeq \mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2, \tag{3.4}$$

where the balanced tensor product is defined by the following universal property: for any abelian category $\mathcal{A}$, we have a natural equivalence

$$\mathcal{F}un_{\mathrm{bal}}(\mathcal{M}_1 \times \mathcal{M}_2, \mathcal{A}) = \mathcal{F}un(\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2, \mathcal{A})$$

where $\mathcal{F}un$ (resp., $\mathcal{F}un_{\mathrm{bal}}$) stand for the category of $\mathbf{k}$-linear additive functors (resp., the category of $\mathbf{k}$-linear additive $\mathcal{C}$-balanced functors); see details in [15, Definition 3.3].

Under the equivalence (3.4), the natural functor $\mathcal{M}_1 \boxtimes \mathcal{M}_2 \to \mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2$ is identified with the functor $I: \mathcal{M}_1 \boxtimes \mathcal{M}_2 \to \mathcal{Z}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2)$ constructed in Theorem 3.3.

Recall that a functor $F: \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is abelian and $\mathcal{A}$ additive (not necessarily abelian), is called *dominant* if any object of $\mathcal{B}$ appears as a subquotient of $F(X)$ for some $X \in \mathrm{Obj}\,\mathcal{A}$. Similarly, we say that a full subcategory $\mathcal{A} \subset \mathcal{B}$ is *dominant* if any object of $\mathcal{B}$ appears as a subquotient of some $X \in \mathrm{Obj}\,\mathcal{A}$. In the case when $\mathcal{A}$ is a full additive subcategory in a semisimple abelian category $\mathcal{B}$, this immediately implies that the Karoubi envelope of $\mathcal{A}$ is equivalent to $\mathcal{B}$ (in particular, this implies that $\mathrm{Kar}(\mathcal{A})$ is abelian).

Proposition 3.4. *Under the hypotheses of Theorem* 3.3*, the functor* $I: \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ *is dominant. Moreover, any object* (M, γ) *is a direct summand of* $I(M)$*.*

Proof. The adjunction isomorphism applied to $\mathrm{id}_M \in \mathrm{Hom}_{\mathcal{M}}(M, M)$ provides the inclusion $(M, \gamma) \subseteq I(M)$, and the other adjunction isomorphism gives the projection $I(M) \to (M, \gamma)$; see Lemma A.8 for details. ■

Proposition 3.5. *Under the hypotheses of Theorem 3.3, if $\mathcal{M}$ is finite semisimple, then so is $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$.*

Proof. Using the exactness in $\mathcal{M}$ of the left and right actions, the abelianness of $\mathcal{M}$ transfers to $\mathcal{Z}(\mathcal{M})$. For example, the kernel K of a morphism $f : M_1 \to M_2$ such that $f \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{M})}((M_1, \gamma^1), (M_2, \gamma^2))$ would inherit a half-braiding $\gamma^1|_K$. Semisimplicity follows easily. Finiteness follows from Proposition 3.4; I ensures there cannot be too many simple objects in $\mathcal{Z}(\mathcal{M})$. See [27] for a similar proof for $\mathcal{M} = \mathcal{C}$. ■

For applications, we will need to consider centers over a full, dominant, monoidal subcategory $\mathcal{C}' \subseteq \mathcal{C}$. Equivalently, $\mathcal{C}'$ is a pivotal category whose Karoubi envelope is multifusion.

Lemma 3.6. *Let $\mathcal{C}'$ be a pivotal locally finite* **k***-linear additive category whose Karoubi envelope $\mathcal{C} = \mathrm{Kar}(\mathcal{C}')$ is multifusion. Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category, and hence naturally a $\mathcal{C}'$-bimodule category (as before, we assume that $\mathcal{M}$ is a semisimple abelian category). Then there is a natural equivalence*

$$\mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \simeq \mathcal{Z}_{\mathcal{C}'}(\mathcal{M}).$$

In particular, for a right $\mathcal{C}$-module category $\mathcal{M}_1$ and a left $\mathcal{C}$-module category $\mathcal{M}_2$, there is a natural equivalence

$$\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2 \simeq \mathcal{M}_1 \boxtimes_{\mathcal{C}'} \mathcal{M}_2.$$

Proof. The equivalence is given as follows: objects (M, γ) in $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ are naturally objects in $\mathcal{Z}_{\mathcal{C}'}(\mathcal{M})$ by forgetting some of the half-braiding, i.e., $(M, \gamma|_{\mathcal{C}'})$; morphisms $f : (M, \gamma) \to (M', \gamma')$ are naturally morphisms $f : (M, \gamma|_{\mathcal{C}'}) \to (M', \gamma|_{\mathcal{C}'})$. We need to check that this is an equivalence.

The functor is essentially surjective: any half-braiding over $\mathcal{C}'$ can be completed to a half-braiding over $\mathcal{C}$. To see this, let γ be a half-braiding over $\mathcal{C}'$. Let $X \in \mathrm{Obj}\, \mathcal{C} \setminus \mathrm{Obj}\, \mathcal{C}'$, and let it be a direct summand of some $Y \in \mathrm{Obj}\, \mathcal{C}'$, $X \underset{p}{\overset{\iota}{\rightleftharpoons}} Y$. We define the extension of γ to X by $\gamma_X = (\mathrm{id}_{M_2} \lhd p) \circ \gamma_Y \circ (\iota \rhd \mathrm{id}_{M_1})$. It is easy to check, using the semisimplicity of $\mathcal{C}$, that γ_X is independent on the choice of Y and p, ι. It is also easy to check that the resulting extension is indeed natural in X.

For morphisms, it is clear that this functor is faithful. To show fullness, consider $f \in \mathrm{Hom}_{\mathcal{Z}_{\mathcal{C}'}(\mathcal{M})}((M_1, \gamma^1), (M_2, \gamma^2))$. We need to check that it also intertwines half-braiding with $X \in \mathcal{C}$, but this follows easily from the definition of the extension of half-braidings given above.

Note that, since γ has a unique extension to all of $\mathcal{C}$, this proof actually shows that the equivalence is an isomorphism. ■

Note also that, in the proof above, we do not use the rigidity of $\mathcal{C}$, but we need it to conclude the second statement concerning balanced tensor products.

3.2. $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$

Next we define the other notion of center.

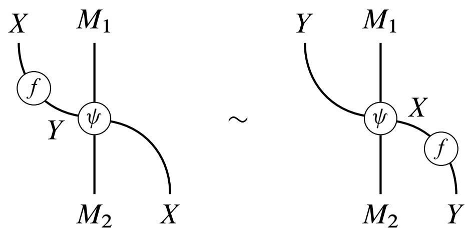
Definition 3.7. Let $\mathcal{C}$ be monoidal, and $\mathcal{M}$ a $\mathcal{C}$-bimodule category. Define the *horizontal trace* $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$ as the category with the following objects and morphisms.

Objects: the same as in $\mathcal{M}$.

Morphisms: we set

$$\mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1, M_2) = \bigoplus_X \mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2)/\sim,$$

where $\mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2) := \mathrm{Hom}_{\mathcal{M}}(X \triangleright M_1, M_2 \triangleleft X)$, the sum is over all (not necessarily simple) objects $X \in \mathcal{C}$, and $\sim$ is the equivalence relation generated by the following. For any $\psi \in \mathrm{Hom}^{Y,X}_{\mathcal{M}}(M_1, M_2) := \mathrm{Hom}_{\mathcal{M}}(Y \triangleright M_1, M_2 \triangleleft X)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have



In other words,

$$\mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1, M_2) = \int^X \mathrm{Hom}^{X,X}_{\mathcal{M}}(M_1, M_2)$$

is the coend of the functor $\mathrm{Hom}^{-,-}_{\mathcal{M}}(M_1, M_2)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}ec$ (see e.g., [22]).

Composition: given by

$$\mathrm{Hom}^Y_{\mathcal{M}}(M_2, M_3)) \otimes \mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2) \to \mathrm{Hom}^{Y\otimes X}_{\mathcal{M}}(M_1, M_3)$$

which sends $\psi \otimes \varphi$ to

$$Y \triangleright (X \triangleright M_1) \xrightarrow{\mathrm{id}_Y \triangleright \psi} Y \triangleright (M_2 \triangleleft X) \simeq (Y \triangleright M_2) \triangleleft X$$
$$\xrightarrow{\varphi \triangleleft \mathrm{id}_X} (M_3 \triangleleft Y) \triangleleft X.$$

For a right $\mathcal{C}$-module category $\mathcal{M}_1$ and a left $\mathcal{C}$-module category $\mathcal{M}_2$, we denote

$$\mathcal{M}_1 \hat{\boxtimes}_{\mathcal{C}} \mathcal{M}_2 = \mathrm{hTr}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2).$$

When the context is clear, we will drop the subscript $\mathrm{hTr} = \mathrm{hTr}_{\mathcal{C}}$. We will write $[\varphi] \in \mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1, M_2)$ for the morphism represented by $\varphi \in \mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2)$ for some X.

It can be shown that, in a certain sense, this definition is dual to the definition of center given above and is closely related to the notion of co-center as described in [11, Section 3.2.2]. However, we will not be discussing the exact relation here.

It is easy to see that the category $\mathrm{hTr}(\mathcal{M})$ is additive but not necessarily abelian.

There is a natural inclusion functor $\mathrm{hTr}\colon \mathcal{M} \to \mathrm{hTr}(\mathcal{M})$ which is the identity on objects, and on morphisms is the natural map

$$\mathrm{Hom}_{\mathcal{M}}(M_1, M_2) = \mathrm{Hom}^{\mathbf{1}}_{\mathcal{M}}(M_1, M_2) \to \mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1, M_2).$$

The horizontal trace construction is functorial, and in particular, we have:

Lemma 3.8. *Given a functor of $\mathcal{C}$-bimodule categories $F\colon \mathcal{M} \to \mathcal{M}'$, there is a natural functor $\mathrm{hTr}(F)\colon \mathrm{hTr}(\mathcal{M}) \to \mathrm{hTr}(\mathcal{M}')$ that is the same as F on objects.*

Proof. Straightforward exercise left to the reader. ∎

We also consider $\mathcal{C}' \subseteq \mathcal{C}$ as in Lemma 3.6, but here we need neeither the rigidity nor the semisimplicity of $\mathcal{C}$:

Lemma 3.9. *Let $\mathcal{C}'$ be monoidal, and let $\mathcal{C} = \mathrm{Kar}(\mathcal{C}')$ be its Karoubi envelope. Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category, and hence naturally a $\mathcal{C}'$-bimodule category. Then there is a natural equivalence*

$$\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M}) \simeq \mathrm{hTr}_{\mathcal{C}}(\mathcal{M}).$$

In particular, for a right $\mathcal{C}$-module category $\mathcal{M}_1$ and a left $\mathcal{C}$-module category $\mathcal{M}_2$, there is a natural equivalence

$$\mathcal{M}_1 \hat{\boxtimes}_{\mathcal{C}} \mathcal{M}_2 \simeq \mathcal{M}_1 \hat{\boxtimes}_{\mathcal{C}'} \mathcal{M}_2.$$

Proof. The equivalence is given by the identity map on objects, and, for two objects $M_1, M_2 \in \mathrm{Obj}\,\mathcal{M}$, the map on morphisms is given by completing the bottom arrow:

$$\begin{array}{ccc} \bigoplus_{X \in \mathcal{C}'} \mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2) & \longrightarrow & \bigoplus_{X \in \mathcal{C}} \mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})}(M_1, M_2) & \longrightarrow & \mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1, M_2) \end{array}$$

It remains to prove that the bottom arrow is an isomorphism.

Let us first observe the following. Let $X, Y \in \operatorname{Obj}\mathcal{C}'$, and suppose X is a direct summand of Y, with $X \underset{p}{\overset{\iota}{\rightleftharpoons}} Y$. Let $\varphi \in \operatorname{Hom}^X_{\mathcal{M}}(M_1, M_2)$. Then,

$$\varphi = \varphi \circ p \circ \iota \sim \iota \circ \varphi \circ p \in \operatorname{Hom}^Y_{\mathcal{M}}(M_1, M_2),$$

where we write p, ι instead of $p \rhd \mathrm{id}_{M_1}, \mathrm{id}_{M_2} \lhd \iota$ for simplicity. This works for $\mathcal{C}$ too. Thus, one can identify $\operatorname{Hom}^X_{\mathcal{M}}(M_1, M_2)$ with a subspace of $\operatorname{Hom}^Y_{\mathcal{M}}(M_1, M_2)$.

Surjectivity. Essentially, we need to show that any morphism in $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$ can be "absorbed" into $\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})$. Let $[\varphi] \in \operatorname{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1, M_2)$ be represented by some $\varphi \in \operatorname{Hom}^X_{\mathcal{M}}(M_1, M_2)$. By the above observation, we can choose $Y \in \operatorname{Obj}\mathcal{C}'$ with X a direct summand of Y, then $\varphi \in \operatorname{Hom}^X_{\mathcal{M}}(M_1, M_2)$ is identified with some morphism in $\operatorname{Hom}^Y_{\mathcal{M}}(M_1, M_2)$, so $[\varphi]$ is in the image.

Injectivity. Essentially, we need to show that relations can also be "absorbed" into $\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})$. Let $[\varphi] \in \operatorname{Hom}_{\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})}(M_1, M_2)$ that is sent to 0. By the observation above, we may represent it by some $\varphi \in \operatorname{Hom}^Y_{\mathcal{M}}(M_1, M_2)$ for some $Y \in \operatorname{Obj}\mathcal{C}'$. Since it is 0 in $\operatorname{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1, M_2)$, there exist

- a finite collection of objects $J = \{A_j\} \subset \operatorname{Obj}\mathcal{C}$ so that $A_0 = Y$,
- $\Phi_i \in \operatorname{Hom}^{A_{m_i}, A_{n_i}}_{\mathcal{M}}(M_1, M_2)$,
- $f_i \colon A_{n_i} \to A_{m_i}$,

such that $\varphi = \sum_i f_i \circ \Phi_i - \Phi_i \circ f_i \in \bigoplus_{A_j \in J} \operatorname{Hom}^{A_j}_{\mathcal{M}}(M_1, M_2)$.

We want to be able to replace the A_j's with objects in $\mathcal{C}'$. For each $j \neq 0$, choose some $B_j \in \operatorname{Obj}\mathcal{C}'$ such that A_j is a direct summand of B_j:

$$A_j \underset{p_j}{\overset{\iota_j}{\rightleftharpoons}} B_j.$$

For $j = 0$, we take $B_0 = A_0 = Y$ and $\iota_0 = p_0 = \mathrm{id}_Y$. This gives us maps $\Theta_j : \psi \mapsto \iota_j \circ \psi \circ p_j : \operatorname{Hom}^{A_j}_{\mathcal{M}}(M_1, M_2) \to \operatorname{Hom}^{B_j}_{\mathcal{M}}(M_1, M_2)$. Denote $\Theta = \sum \Theta_j$.

Now, consider

- $L = \{B_j\} \subset \operatorname{Obj}\mathcal{C}'$,
- $\Psi_i = \iota_{n_i} \circ \Phi_i \circ p_{m_i} \in \operatorname{Hom}^{B_{m_i}, B_{n_i}}_{\mathcal{M}}(M_1, M_2)$,
- $g_i = \iota_{m_i} \circ f_i \circ p_{n_i} \colon B_{n_i} \to B_{m_i}$.

It is a simple matter to verify that

$$g_i \circ \Psi_i - \Psi \circ g_i = \Theta(f_i \circ \Phi_i - \Phi_i \circ f_i).$$

Hence,

$$\varphi = \Theta(\varphi) = \Theta\Big(\sum f_i \circ \Phi_i - \Phi_i \circ f_i\Big) = \sum g_i \circ \Psi_i - \Psi \circ g_i$$

is 0 in $\operatorname{Hom}_{\mathrm{hTr}_{\mathcal{C}'}}(M_1, M_2)$. ■

3.3. Equivalence

Theorem 3.10. *Let $\mathcal{C}$ be pivotal multifusion and $\mathcal{M}$ be a $\mathcal{C}$-bimodule category. One has a natural equivalence*

$$\mathrm{Kar}(\mathrm{hTr}(\mathcal{M})) \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{M}).$$

Under this equivalence, the inclusion functor $\mathrm{hTr}\colon \mathcal{M} \to \mathrm{hTr}(\mathcal{M})$ *is identified with the functor* $I\colon \mathcal{M} \to \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$.

In particular, for a right $\mathcal{C}$-module category $\mathcal{M}_1$ and a left $\mathcal{C}$-module category $\mathcal{M}_2$, we have

$$\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2 \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2) \simeq \mathrm{Kar}(\mathcal{M}_1 \hat{\boxtimes}_{\mathcal{C}} \mathcal{M}_2).$$

Before proving the theorem, we will need the following lemma.

Lemma 3.11. *The natural linear map*

$$\bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{M}}^{X_i}(M_1, M_2) \to \mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1, M_2) \tag{3.5}$$

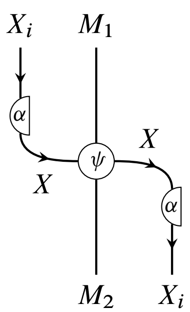
is an isomorphism.

Proof. To prove the statement, we define a linear map

$$\mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1, M_2) \to \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{M}}^{X_i}(M_1, M_2)$$

by

$$\psi \mapsto \sum_{i \in \mathrm{Irr}(\mathcal{C})} d_i^R \tag{3.6}$$



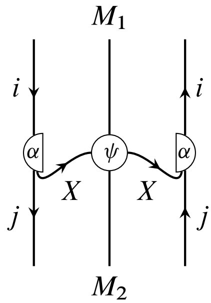
for $\psi \in \mathrm{Hom}_{\mathcal{M}}^{X}(M_1, M_2)$; α is a sum over dual bases – see Notation A.3. The map (3.6) is well defined by Lemma A.1. Using Lemma A.4, it is easy to see that (3.5) and (3.6) are mutually inverse. ■

Proof of Theorem 3.10. Define the functor $G\colon \mathrm{hTr}(\mathcal{M}) \to \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ on objects by

$$G(M) = I(M),$$

and on morphisms by

$$G(\psi) = \sum_{i,j \ \mathrm{Irr}(\mathcal{C})} \sqrt{d_i^R}\sqrt{d_j^R} \qquad (3.7)$$



for $\psi \in \mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2)$; once again, see Notation A.3 for the definition of α.

It is easy to check the following properties.

(1) G is well defined on morphisms (i.e., it preserves the equivalence relation): this follows from Lemma A.7.

(2) G is dominant: any $Y \in \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ appears as a direct summand of $G(M)$ for some $M \in \mathcal{M}$. Namely, if $Y = (M, \gamma)$, then it appears as a direct summand of $G(M)$; the projection to Y is, up to a factor, $G(\sum d_i^R \gamma_{X_i})$ (see Lemma A.8 for the proof; compare Proposition 3.4).

(3) G is bijective on morphisms: by the adjointness property (Theorem 3.3), we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Z}_{\mathcal{C}}(\mathcal{M})}(I(M), I(M')) &\cong \mathrm{Hom}_{\mathcal{M}}(I(M), M') \\ &= \bigoplus_i \mathrm{Hom}_{\mathcal{M}}(X_i \triangleright M \triangleleft X_i^*, M') \end{aligned}$$

and, by Lemma 3.11, the right-hand side coincides with $\mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M, M')$.

This immediately implies the statement of the theorem by the universal properties of Karoubi envelopes. ■

By Lemma 3.6 and Lemma 3.9, we extend the above theorem to $\mathcal{C}' \subseteq \mathcal{C}$:

Corollary 3.12. *Let $\mathcal{C}'$ be a pivotal category whose Karoubi envelope $\mathcal{C} = \mathrm{Kar}(\mathcal{C}')$ is multifusion. Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category, and hence naturally a $\mathcal{C}'$-bimodule category. Then we have*

$$\mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})) \simeq \mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})) \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \simeq \mathcal{Z}_{\mathcal{C}'}(\mathcal{M}).$$

Note $\mathrm{Kar}(\mathcal{M})$ inherits a $\mathcal{C}'$-bimodule structure from $\mathcal{M}$. For example,

$$A \triangleright (M, p) = (A \triangleright M, \mathrm{id}_A \triangleright p).$$

We compare these constructions for $\mathcal{M}$ and its Karoubi envelope:

Lemma 3.13. *Under the same hypotheses as Corollary 3.12,*

$$\mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})) \simeq \mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathrm{Kar}(\mathcal{M}))).$$

In particular, if $\mathcal{M}'$ is a dominant submodule category of $\mathcal{M}$, then

$$\mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M}')) \simeq \mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})).$$

Proof. The natural inclusion $\mathcal{M} \to \mathrm{Kar}(\mathcal{M})$ is a full, dominant functor of $\mathcal{C}'$-bimodules, and it is easy to see that the corresponding functor $\mathrm{hTr}(\mathcal{M}) \to \mathrm{hTr}(\mathrm{Kar}(\mathcal{M}))$ is also full and dominant. It follows that the induced functor on their Karoubi envelopes is an equivalence.

The second statement follows because $\mathrm{Kar}(\mathcal{M}') \simeq \mathrm{Kar}(\mathcal{M})$. ∎

4. Colored graphs in Turaev–Viro theory

In this section, we recall the definition of colored graphs (called *stringnets* in [20]) in Turaev–Viro theory. This is intended to serve as a reminder only; proofs are omitted. Details and proofs can be found in [20]. For a description of the Turaev–Viro theory as an extended theory in terms of cell decompositions, see [6].

Throughout this section, all surfaces are assumed to be oriented. We denote by $\mathcal{A}$ a spherical fusion category. We will heavily use the graphical presentation of morphisms in $\mathcal{A}$; we give a summary of our notations and conventions in Appendix A.

For a finite graph Γ embedded in surface Σ, we denote by $E(\Gamma)$ the set of its edges. Note that the edges are not oriented. Let E^{or} be the set of oriented edges, i.e., pairs $\mathbf{e} = (e, \text{orientation of } e)$; for such an oriented edge $\mathbf{e}$, we denote by $\bar{\mathbf{e}}$ the edge with opposite orientation.

If Σ has a boundary, the graph is allowed to have uncolored one-valent vertices on $\partial\Sigma$, but no other common points with $\partial\Sigma$; all other vertices will be called interior. We will call the edges of Γ terminating at these one-valent vertices *legs*.

Definition 4.1. Let Σ an oriented surface (possibly with boundary) and $\Gamma \subset \Sigma$ – an embedded graph as defined above. A *coloring* of Γ is given as follows.

- Choose an object $V(\mathbf{e}) \in \mathrm{Obj}\,\mathcal{A}$ for every oriented edge $\mathbf{e} \in E^{\mathrm{or}}(\Gamma)$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.
- Choose a vector $\varphi(v) \in \langle V(\mathbf{e}_1), \dots, V(\mathbf{e}_n)\rangle$ (see (A.2)) for every interior vertex v, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are edges incident to v, taken in counterclockwise order and with outward orientation (see Figure 4 in Appendix A).

We will denote the set of all colored graphs on a surface Σ by $\mathrm{Graph}(\Sigma)$.

Note that, if Σ has a boundary, then every colored graph Γ defines a collection of points $B = \{b_1, \ldots, b_n\} \subset \partial\Sigma$ (the endpoints of the legs of Γ) and a collection of objects $V_b \in \text{Obj } \mathcal{A}$ for every $b \in B$: the colors of the legs of Γ taken with outgoing orientation. We will denote the pair $(B, \{V_b\})$ by $\mathbf{V} = \Gamma \cap \partial\Sigma$ and call it *boundary value*. We define

$$\text{Graph}(\Sigma, \mathbf{V}) = \text{set of all colored graphs in } \Sigma \text{ with boundary value } \mathbf{V}.$$

We can also consider formal linear combinations of colored graphs. Namely, a for fixed boundary value $\mathbf{V}$ as above, we will denote

$$\text{VGraph}(\Sigma, \mathbf{V}) = \{\text{formal linear combinations of graphs } \Gamma \in \text{Graph}(\Sigma, \mathbf{V})\}. \quad (4.1)$$

In particular, if $\partial\Sigma = \varnothing$, then the only possible boundary condition is trivial ($B = \varnothing$); in this case, we will just write VGraph(Σ).

It follows from the result of Reshetikhin and Turaev that, for every colored graph Γ in a disk $D \subset \mathbb{R}^2$, one can define its "evaluation"

$$\langle \Gamma \rangle_D \in \langle V(\mathbf{e}_1), \ldots, V(\mathbf{e}_n) \rangle \quad (4.2)$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the edges of Γ meeting the boundary of D (legs), taken in counterclockwise order and with outgoing orientation; in particular, in the case when Γ is a star graph, with one vertex colored by $\varphi \in \langle V(\mathbf{e}_1), \ldots, V(\mathbf{e}_n) \rangle$, then $\langle \Gamma \rangle = \varphi$.

We call a formal linear combination of colored graphs

$$\Gamma = \sum c_i \Gamma_i \in \text{VGraph}(\Sigma, \mathbf{V})$$

a *null graph* if there exists an embedded disk $D \hookrightarrow \Sigma$ such that all graphs Γ_i meet the boundary of D transversally, all Γ_i coincide outside of D (as colored graphs) and

$$\langle \Gamma \rangle_D = \sum c_i \langle \Gamma_i \cap D \rangle_D = 0.$$

We will say Γ is *null with respect to* D. We can now give the main definition of this section.

Definition 4.2. For an oriented surface Σ and boundary condition $\mathbf{V} = (B, \{V_b\})$ on $\partial\Sigma$, we define the stringnet space by

$$Z_{\text{TV}}(\Sigma, \mathbf{V}) = \text{VGraph}(\Sigma, \mathbf{V})/N \quad (4.3)$$

where N is the subspace spanned by all null graphs (for all possible embedded disks).

As an example, it was shown in [20] that

$$Z_{\text{TV}}(S^2) = Z_{\text{TV}}(\mathbb{R}^2) = \mathbf{k}.$$

We can now define the category of boundary conditions.

Definition 4.3. Let N be an oriented 1-dimensional manifold, possibly non-compact. Suppose first N has no boundary. Define $\hat{Z}_{\text{TV}}(N)$ as the category whose objects are finite subsets $B \subset N$ together with a choice of object $V_b \in \text{Obj}\,\mathcal{A}$ for every point $b \in B$; we will use the notation $\mathbf{V} = (B, \{V_b\})$ for such an object, and B is called the *set of marked points of* $\mathbf{V}$. Define the morphisms in $\hat{Z}_{\text{TV}}(N)$ by

$$\text{Hom}_{\hat{Z}_{\text{TV}}(N)}(\mathbf{V}, \mathbf{V}') = Z_{\text{TV}}(N \times [0,1]; \mathbf{V}^*, \mathbf{V}'), \quad \mathbf{V} = (B, \{V_b\}), \quad \mathbf{V}' = (B', \{V_{b'}\}),$$

where $\mathbf{V}^*, \mathbf{V}'$ means the boundary condition obtained by putting points $b \in B$ on the "top" $N \times \{1\}$, colored by objects V_b^* for outgoing legs (and thus colored by V_b for incoming legs), and putting points $b' \in B'$ on the "bottom" $N \times \{0\}$, colored by objects $V_{b'}$ for outgoing legs (see Figure 2 below).

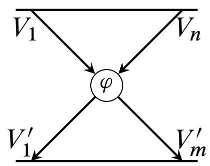


Figure 2. Morphisms in $\hat{Z}_{\text{TV}}(N)$

This category is additive and $\mathbf{k}$-linear. We denote by

$$Z_{\text{TV}}(N) = \text{Kar}(\hat{Z}_{\text{TV}}(N)) \tag{4.4}$$

its Karoubi envelope.

For N with boundary, we define

$$\hat{Z}_{\text{TV}}(N) = \hat{Z}_{\text{TV}}(N \backslash \partial N), \quad Z_{\text{TV}}(N) = Z_{\text{TV}}(N \backslash \partial N).$$

It is immediate from the definition that

$$Z_{\text{TV}}(I) \simeq \mathcal{A}.$$

where I is an open/closed interval.

It has been shown in [20] that $Z_{\text{TV}}(S^1) = \mathcal{Z}(\mathcal{A})$ is the Drinfeld center of $\mathcal{A}$. We will reprove it (in a slightly different way) as a special case of a more general result later.

5. Skeins in Crane–Yetter theory

In this section, we give a definition of colored graphs/skeins in Crane–Yetter theory, mirroring closely the previous section, and we will reuse many definitions. This definition essentially coincides with those given in [9, 19]; we use framed graphs instead of ribbons and coupons.

Throughout this section, all 3-manifolds are assumed to be oriented, and may be non-compact and/or with boundary. $\mathcal{A}$ will be a skeletal premodular category; see Appendix A for a summary of notation and conventions.

We will consider finite framed graphs Γ in a 3-manifold M, that is, Γ is a smoothly embedded graph in M with finitely many edges, and each edge comes with a transversal ray field along it (that is, to each point p on an edge is assigned a ray ρ_p in T_pM emanating from the origin, varying smoothly with p); the transversal ray field ρ_p is the *framing* of the edge. We also impose the condition that the edges are not tangent to each other at a vertex (this is necessary for the "infinitesimal spheres" discussion below). From here on, we will simply refer to finite framed graphs as *graphs.*

Graphs are allowed to intersect the boundary ∂M transversally; each point of intersection of Γ with ∂M should be a vertex of Γ, and they are the *boundary vertices* of Γ. Other vertices of Γ are the *interior vertices*. Furthermore, the framing on Γ induces at each boundary vertex b a ray in $T_b(\partial M)$, a *framing* on b. This makes the boundary ∂M an *extended surface*, a surface together with a configuration of finitely many framed points.

For each interior vertex v, the "infinitesimal sphere" at v also acquires an extended surface structure as follows. The space of rays emanating from the origin in T_vM is a sphere S^2_v, which we call the *infinitesimal sphere.* An edge e leaving v has a tangent vector ν_e at v, which gives us a point $\overline{\nu_e} \in S^2_v$. The framing on e at v is a ray ρ_v in T_vM; the quarter plane spanned by ν_e and ρ_v in T_vM defines a ray in $T_{\overline{\nu_e}}S^2_v$, i.e., a framing of $\overline{\nu_e}$. The collection of such framed points $\overline{\nu_e}$ is the extended surface structure that S^2_v inherits from the graph ($\overline{\nu_e}$ are distinct by the extra condition of non-tangency of edges at vertices).

Given an input premodular category $\mathcal{A}$, and given an extended sphere S where each marked point p_i is colored with an object $V_i \in \mathcal{A}$, the Reshetikhin–Turaev construction functorially yields a vector space $Z_{\mathrm{RT}}(S; V_1, \dots, V_k)$, see [24]. In particular, this vector space is (non-canonically) isomorphic to $\langle V_1, \dots, V_k \rangle$.

Definition 5.1. A coloring of a graph $\Gamma \subset M$ is the following data:

- choice of an object $V(\mathbf{e}) \in \operatorname{Obj} \mathcal{A}$ for each oriented edge $\mathbf{e} \in E^{\mathrm{or}}(\Gamma)$, so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$;
- choice of a vector $\varphi(v) \in Z_{\mathrm{RT}}(S^2_v; V(\mathbf{e}_1), \dots, V(\mathbf{e}_n))$, for each interior vertex v, where $\mathbf{e}_i$ are the edges incident to v, taken with outward orientation (pointing away from v).

If M has boundary, then we can color each boundary vertex of Γ with the color of the incident edge (taken with outgoing orientation). The pair $(B, \{V_b\})$ of the set of boundary vertices with a coloring is the *boundary value* of Γ. We will denote

$$\mathrm{Graph}(M, \mathbf{V}) = \text{set of all colored graphs in } M \text{ with boundary value } \mathbf{V}$$

and similarly consider formal linear combinations

$$\mathrm{VGraph}(M, \mathbf{V}) = \{\text{formal linear combinations of graphs } \Gamma \in \mathrm{Graph}(M, \mathbf{V})\}.$$

It follows from the result of Reshetikhin and Turaev that for every colored graph Γ in a ball $D \subset \mathbb{R}^3$, one can define its "evaluation"

$$\langle \Gamma \rangle_D \in Z_{\mathrm{RT}}(\partial D; V(\mathbf{e}_1), \ldots, V(\mathbf{e}_n)) \cong \langle V(\mathbf{e}_1), \ldots, V(\mathbf{e}_n) \rangle$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the edges of Γ meeting the boundary of D (legs), taken with outgoing orientation; in particular, in the case when Γ is a star graph in the unit ball in $\mathbb{R}^3$, with one vertex at the center colored by $\varphi \in Z_{\mathrm{RT}}(S^2_v; V(\mathbf{e}_1), \ldots, V(\mathbf{e}_n))$, then[1] $\langle \Gamma \rangle = \varphi$.

We call a formal linear combination of colored graphs

$$\Gamma = \sum c_i \Gamma_i \in \mathrm{VGraph}(M, \mathbf{V})$$

a *null graph* if there exists an embedded closed ball $D \hookrightarrow M$ such that all Γ_i meet ∂D transversally, all Γ_i coincide outside of D as colored graphs, and

$$\langle \Gamma \rangle_D = \sum c_i \langle \Gamma_i \rangle_D = 0.$$

(Note D is allowed to touch the boundary ∂M.) We will say Γ is *null* with respect to D.

We can now give the main definition of this section:

Definition 5.2. For an oriented 3-manifold M and boundary condition $\mathbf{V} = (B, \{V_b\})$ on ∂M, we define the space of skeins by

$$Z_{\mathrm{CY}}(M, \mathbf{V}) = \mathrm{VGraph}(M, \mathbf{V})/N$$

where N is the subspace spanned by all null graphs (for all possible embedded disks).

We can now define the category of boundary conditions.

Definition 5.3. Let Σ be an oriented surface, possibly non-compact. Suppose first Σ has no boundary. Define $\hat{Z}_{\mathrm{CY}}(\Sigma)$ as the category whose objects are finite subsets $B \subset \Sigma$, together with a framing and coloring $V_b \in \mathrm{Obj}\, \mathcal{A}$ for each point $b \in B$; we

[1] Here the identification $Z_{\mathrm{RT}}(S^2_v; V(\mathbf{e}_i)) \cong Z_{\mathrm{RT}}(\partial D; V(\mathbf{e}_i))$ is made using the natural maps $\partial D \hookrightarrow \mathbb{R}^3 \backslash 0 \simeq T_0\mathbb{R}^3 \backslash 0 \to S^2_v$.

will use the notation $\mathbf{V} = (B, \{V_b\})$ for such an object (suppressing the framing), and we call B the *set of marked points of* $\mathbf{V}$. Define the morphisms in $\hat{Z}_{\mathrm{CY}}(\Sigma)$ by

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\Sigma)}(\mathbf{V}, \mathbf{V}') = Z_{\mathrm{CY}}(\Sigma \times [0,1]; \mathbf{V}^*, \mathbf{V}'), \quad \mathbf{V} = (B, \{V_b\}), \quad \mathbf{V}' = (B', \{V_{b'}\}),$$

where $\mathbf{V}^*, \mathbf{V}'$ means the boundary condition obtained by putting points $b \in B$ on the "top" $\Sigma \times \{1\}$, colored by objects V_b^* for outgoing legs (and thus colored by V_b for incoming legs), and putting points $b' \in B'$ on the "bottom" $N \times \{0\}$, colored by objects $V_{b'}$ for outgoing legs.

$\hat{Z}_{\mathrm{CY}}(\Sigma)$ is additive and $\mathbf{k}$-linear. We denote by

$$Z_{\mathrm{CY}}(\Sigma) = \mathrm{Kar}(\hat{Z}_{\mathrm{CY}}(\Sigma)) \tag{5.1}$$

its Karoubi envelope.

For N with boundary, we define

$$\hat{Z}_{\mathrm{CY}}(N) = \hat{Z}_{\mathrm{CY}}(N \backslash \partial N), \quad Z_{\mathrm{CY}}(N) = Z_{\mathrm{CY}}(N \backslash \partial N).$$

It is immediate from the definition that for a 2-disk D^2, $Z_{\mathrm{CY}}(D^2) \simeq \mathcal{A}$.

6. Generalities of skein modules and categories of boundary values

We consider properties of skein modules and categories of boundary values that are common for both the Turaev–Viro theory and the Crane–Yetter theory. Section 6.1 is focused on the space of relations (i.e., the null graphs $N \subset \mathrm{VGraph}(Y, \mathbf{V})$); in particular, on how they are generated. In Section 6.2, we exhibit a "stacking" monoidal structure on the category of boundary values of manifolds of the form $P \times (0,1)$, and show it to be pivotal.

Throughout this section, $n = 1$ or 2. We will use $Z, \hat{Z}$ to denote either $Z_{\mathrm{TV}}, \hat{Z}_{\mathrm{TV}}$ (when $n = 1$) or $Z_{\mathrm{CY}}, \hat{Z}_{\mathrm{CY}}$ (when $n = 2$), so that $Z(n\text{-manifold})$ is a category, and $Z((n+1)\text{-manifold}; \mathbf{V})$ is a vector space. $\mathcal{A}$ is spherical fusion for $n = 1$, and is premodular for $n = 2$. Denote by $I = (0,1)$, the *open* interval.

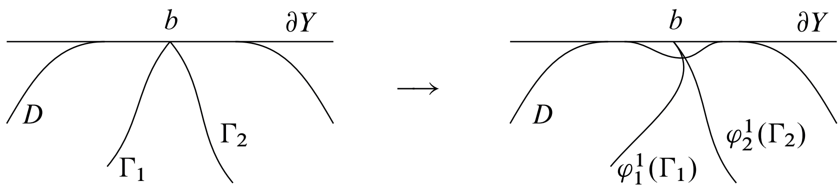
6.1. Skein modules

Recall that a null graph in Y is null with respect to some $(n+1)$ ball D, and D is allowed to touch the boundary ∂Y. In future applications, it will be convenient to only consider balls D that do not meet ∂Y, such balls can be displaced by ambient isotopy but balls meeting ∂Y may not. Boundary vertices are univalent, so graphs have simple behavior near the boundary. If we exclude balls D that meet ∂Y, the resulting space of null graphs N' will be strictly smaller than N, but not by much; the following lemma says we just need to include equivalence of graphs under ambient isotopy rel boundary:

Lemma 6.1. *Let Y be an $(n+1)$-manifold, possibly with boundary or non-compact, and let $\mathbf{V} \in \mathrm{Obj}\, \widehat{Z}(\partial Y)$ be a fixed boundary value. Define $N' \subset N \subset \mathrm{VGraph}(Y, \mathbf{V})$ to be the subspace generated by graphs that are null with respect to a ball that does not meet the boundary ∂Y. Define $N'' \subset \mathrm{VGraph}(Y, \mathbf{V})$ to be relations obtained by ambient isotopy, i.e., generated by graphs $\Gamma^1 - \Gamma^0$, where $\Gamma^t = \varphi^t(\Gamma)$, φ^t is a compactly-supported ambient isotopy fixing ∂Y. Then $N = N' + N''$.*

Proof. It suffices to show that $N \subset N' + N''$. Let $\Gamma = \sum c_i \Gamma_i$ be a null graph with some boundary value $\mathbf{V}$, null with respect to a ball $D \subset Y$, and suppose D meets the boundary ∂Y. We would like to shrink D to not meet ∂Y while maintaining that Γ be null with respect to it. Clearly, if D does not meet any point in $\mathbf{V}$, then we can do this, and then $\Gamma \in N'$.

Suppose D does contain some boundary vertex $b \in \mathbf{V}$. For each i, apply a small ambient isotopy φ_i^t supported in a small neighborhood of b so that the resulting graphs $\varphi_i^1(\Gamma_i)$ agree in a (possibly smaller) neighborhood of b.



Then we can push D slightly inwards away from the boundary at b, and note that this new graph $\Gamma' = \sum c_i \varphi_i^1(\Gamma_i)$ will be null with respect to the deformed D. This reduces the number of points in $\mathbf{V}$ that D contains, so after performing this finitely many times, we are back to the case considered above where D does not contain any boundary vertices. Thus we see that repeated applications of isotopies (i.e., relations in N'') takes Γ to another graph $\Gamma' \in N'$; in other words, $\Gamma \in N'' + N'$. ■

The following lemma says that isotopies can be broken into a sequence of "smaller" ones:

Lemma 6.2. *Let φ^t be an isotopy of diffeomorphisms $\varphi^t : Y \to Y$ that is supported on some compact set K. Let $\{U_i\}$ be a finite open cover. Then there exists a sequence of isotopies φ_j^t such that each φ_j^t is supported on some $U_{a_j} \cap K$, and the isotopies concatenate to give a piecewise-smooth isotopy from φ^0 to φ^1.*

The proof can be found in [12, Corollary 1.3].

In other words, given two diffeomorphisms φ^0 and φ^1 that are isotopic, there is another sequence of isotopies that takes φ^0 to φ^1 such that each is supported on a subset of Y. One can make the new isotopies as close to the original isotopy as needed.

Finally, we show that the subspace of null graphs are spanned by those that are null with respect to "small" balls. More precisely,

Proposition 6.3. *Let Y be an $(n+1)$-manifold, possibly with boundary or non-compact. Let $\{U_i\}$ be a finite open cover of Y. Let $\mathbf{V} \in \mathrm{Obj}\,\widehat{Z}(\partial Y)$ be a fixed boundary value. Define $N_i \subset N \subset \mathrm{VGraph}(Y, \mathbf{V})$ to be the subspace of null graphs in Y with boundary value $\mathbf{V}$ that are null with respect to some closed ball D contained in U_i. Then the space of null graphs is generated by N_i's, i.e.,*

$$N = \sum N_i.$$

Proof. Let $\Gamma = \sum c_j \Gamma_j \in N$ be a null graph. By Lemma 6.1, Γ can be written as a sum of null graphs $\Gamma' + \Gamma''$, where $\Gamma' = \sum c'_j \Gamma'_j$ is a sum of graphs, each Γ'_j is null with respect to some ball not meeting ∂Y, and $\Gamma'' = \sum c''_j \Gamma''_j$ is a sum of graphs, each Γ''_j is of the form $(\Gamma''_j)^1 - (\Gamma''_j)^0$ for some smooth isotopy $(\Gamma''_j)^t$.

Consider one such Γ''_j, and suppose that $\varphi^t : Y \to Y$ is an ambient isotopy supported on a compact subset $K \subset Y$, such that $(\Gamma''_j)^t = \varphi^t(\Theta)$ for some graph Θ. By Lemma 6.2, there is a sequence of isotopies φ^t_k, such that each φ^t_k is supported on some $U_{a_k} \cap K$, and the isotopies concatenate to give a piecewise-smooth isotopy from φ^0 to φ^1. Then $\Gamma''_j = \varphi^1(\Theta) - \varphi^0(\Theta) = \sum_k \varphi^1_k(\Theta) - \varphi^0_k(\Theta) \in \sum N_i$. Thus, in the sum $\Gamma = \Gamma' + \Gamma''$, we have $\Gamma'' \in \sum N_i$.

Now, consider a term Γ'_j in Γ', and suppose it is null with respect to some ball D not meeting ∂Y. There exists an ambient isotopy of identity $\varphi^t : Y \to Y$ that moves D into some open set U_a. Then $\varphi^1(\Gamma'_j) \in N_a$. But by the same argument as above, $\varphi^1(\Gamma'_j) - \Gamma'_j \in \sum N_i$. Hence, we conclude that $\Gamma' = \sum c_j \Gamma'_j \in \sum N_i$, and we are done. ∎

6.2. Categories of boundary values

Lemma 6.4. *Let X_1, X_2 be n-manifolds without boundary, possibly non-compact. Let $\varphi : X_1 \to X_2$ be an orientation-preserving embedding. Then φ induces an obvious inclusion functor*

$$\varphi_* : \widehat{Z}(X_1) \to \widehat{Z}(X_2)$$

that sends objects to their image under φ, and sends morphisms to their image under $\varphi \times \mathrm{id}_I$. This descends to the Karoubi envelopes

$$\varphi_* : Z(X_1) \to Z(X_2).$$

Furthermore, an isotopy $\varphi^t : X_1 \to X_2$ induces a natural isomorphism from φ^0_* to φ^1_*, and isotopic isotopies induce the same natural isomorphisms.

Proof. Clear. ∎

Lemma 6.5. *Under the same hypothesis above,*

$$\hat{Z}(X_1 \sqcup X_2) \simeq \hat{Z}(X_1) \boxtimes \hat{Z}(X_2),$$
$$Z(X_1 \sqcup X_2) \simeq Z(X_1) \boxtimes Z(X_2).$$

Proof. The proof for $\hat{Z}$ is clear: the inclusions of X_1 and X_2 into $X_1 \sqcup X_2$ together induce $\hat{Z}(X_1) \boxtimes \hat{Z}(X_2) \to \hat{Z}(X_1 \sqcup X_2)$, and this is easily seen to be an isomorphism of categories. The equivalence for Z then follows by universal property, and the fact that the Deligne–Kelly tensor product of two finite semisimple abelian categories is also a finite semisimple abelian category. ∎

Finally, we discuss the "stacking" monoidal structure of some special n-manifolds. Let P be a $(n-1)$-manifold without boundary, possibly disconnected (with finitely many components) or non-compact. For $n = 1$, P is just a collection of points. For $n = 2$, P is a collection of open intervals and circles.

Let $I = (0, 1)$, and let $m\colon I \sqcup I \to I$ be $x/2$ on the first I and $(x+1)/2$ on the second I. This is part of an A_∞-space structure, as defined in [26]: m is not associative, but there is a "straight line" isotopy $m_3^t\colon I \sqcup I \sqcup I \to I$ from $m_3^0 = m \circ (m \sqcup \mathrm{id}_I)$ to $m_3^1 = m \circ (\mathrm{id}_I \sqcup m)$, relating two extreme ways of including three intervals into one.

Let

$$\tilde{m}\colon P \times I \sqcup P \times I = P \times (I \sqcup I) \to P \times I,$$
$$\tilde{m}_3^t\colon P \times I \sqcup P \times I \sqcup P \times I = P \times (I \sqcup I \sqcup I) \to P \times I.$$

Proposition 6.6. *There is a monoidal structure on $\hat{Z}(P \times I)$ given as follows.*

- *The tensor product is*

$$\otimes := \tilde{m}_*\colon \hat{Z}(P \times I) \boxtimes \hat{Z}(P \times I) \to \hat{Z}(P \times I).$$

- *The unit* **1** *is the empty configuration. (Left, right unit constraints are given in proof.)*
- *The associativity constraint α is the natural isomorphism that is induced by $\tilde{m}_3^t$.*

Similarly, there is a monoidal structure on $Z(P \times I)$.

Proof. The left unit constraint $l_A\colon A \otimes \mathbf{1} \to A$ is given by a "straight line" graph, likewise for the right unit constraint. That α satisfies the pentagon relations follows from the fact that any two inclusions $I^{\sqcup 4} \hookrightarrow I$ are isotopic, and any two isotopies are themselves isotopic. The result for $Z(P \times I)$ follows from the universal property. ∎

Proposition 6.7. *The monoidal structure on $\hat{Z}(P \times I)$ and $Z(P \times I)$ given in Proposition 6.6 is pivotal.*

Remark 6.8. The input category $\mathcal{A}$ has to be spherical, but the resulting categories $Z(P \times I)$ may be not; in a future work, we will show that $Z(S^1 \times I)$ is pivotal, but not spherical.

Proof. It suffices to prove this for $\hat{Z}(P \times I)$, since its Karoubi envelope will inherit the pivotal structure.

The rigid and pivotal structures come from topological constructions. Denote by $\theta: P \times I \to P \times I$ the orientation-reversing diffeomorphism which flips I, i.e., $(p, x) \mapsto (p, 1 - x)$. Denote by $\Theta: P \times I \times [0,1] \to P \times I \times [0,1]$ the orientation-preserving diffeomorphism that rotates the $I \times [0,1]$ rectangle by 180°, i.e.,

$$(p, x, t) \mapsto (p, 1 - x, 1 - t).$$

Denote by υ the map that takes $P \times I \times [0,1]$, squeezes it in half along the direction I, bends it like an accordion so that the left side collapses, and puts it back in $P \times I \times [0,1]$ so that the top and bottom are now attached to the top (see Figure 3). υ', η, η' are defined similarly.

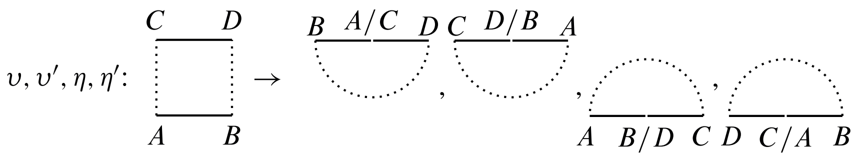


Figure 3. The maps $\upsilon, \upsilon', \eta, \eta'$ for $P = \{*\}$

Let $\mathbf{V} = (B, \{V_b\}) \in \operatorname{Obj} \hat{Z}(P \times I)$. Its left dual $\mathbf{V}^*$ is given by $(\theta(B), \{V_b^*\})$, that is, apply the flipping diffeomorphism θ defined above to the marked points, and label them by the left duals of the original labeling. Similarly, the right dual is ${}^*\mathbf{V} = (\theta(B), \{{}^*V_b\})$. (It is not too important to distinguish V_b^* from ${}^*V_b$ since $\mathcal{A}$ itself is pivotal.)

The left evaluation and coevaluation morphisms for $\mathbf{V}$ are obtained by applying υ and η to $\mathrm{id}_{\mathbf{V}}$, respectively. Similarly, the right evaluation and coevaluation morphisms for $\mathbf{V}$ are obtained by applying υ' and η' to $\mathrm{id}_{\mathbf{V}}$, respectively. It is easy to see that these morphisms have the required properties.

Given a morphism $f \in \operatorname{Hom}_{\hat{Z}(P \times I)}(\mathbf{V}, \mathbf{V}')$ represented by a graph Γ, it is easy to check that its left and right duals are given by applying the rotation Θ to Γ, and keeping all orientations and labels of the edges of Γ.

The pivotal structure is essentially the identity morphism, but with one vertex on each vertical line labeled by δ, the pivotal structure of $\mathcal{A}$. ∎

Example 6.9. We pointed out at the end of Section 4 that $Z_{\mathrm{TV}}(I) \simeq \mathcal{A}$. Giving $Z_{\mathrm{TV}}(I)$ the stacking monoidal structure above, we see that this equivalence is a tensor equivalence respecting the pivotal structure.

Example 6.10. Similarly, we had $Z_{\mathrm{CY}}(I \times I) \simeq \mathcal{A}$. $I \times I$ can stack in two ways, along the first copy of I (horizontal stacking) or the second (vertical stacking). They both give monoidal structures equivalent to that of $\mathcal{A}$.

Proposition 6.11. *The E_1-algebra structure of $Z(P \times I)$ is unique in the sense of Theorem 2.4; that is, any automorphism of $P \times I$ induces an E_1-algebra self-equivalence on $Z(P \times I)$.*

Proof. It is not hard to see that it suffices to consider P connected. For $P = *, I, S^1$, the space of self-diffeomorphisms of P is connected, so any fiber-preserving automorphism of $P \times I$ is isotopic to the identity, so the induced functor of the automorphism is an equivalence of E_1-algebras. ■

Next, we consider (left) module categories over $Z(P \times I)$. First, I is a left module over the A_∞ space I as follows. Let $f: I \to (1/2, 1) \subset I$ be some inclusion that is identity near 1. The embedding $n = (\cdot/2) \sqcup f: I \sqcup I \to I$ gives left multiplication, and it is associative up to some isotopy, that is, the two inclusions $n_3^0 := n \circ (\mathrm{id}_I \sqcup n)$ and $n_3^1 := n \circ (m \sqcup \mathrm{id}_I)$ are isotopic via some isotopy n_3^t. It is not hard to see that any two such left module structures are equivalent.

Now, let X be a collared n-manifold, i.e., we have an embedding $P \times I \hookrightarrow X$, where the 0 end in I escapes to infinity in X. Crossing with P, we can upgrade the above left module structure on I to X, obtaining a left multiplication $\tilde{n}: P \times I \sqcup X \to X$ and an isotopy $\tilde{n}_3^t$ from $\tilde{n} \circ (\mathrm{id}_{P \times I} \sqcup \tilde{n})$ to $\tilde{n} \circ (\tilde{m} \sqcup \mathrm{id}_X)$. (See Lemma 2.2.)

Proposition 6.12. *Given a collared n-manifold X, there is a left $\widehat{Z}(P \times I)$-module category structure on $\widehat{Z}(X)$ given by*

$$\triangleright := \tilde{n}_*: \widehat{Z}(P \times I) \boxtimes \widehat{Z}(X) \to \widehat{Z}(X)$$

and the associativity constraint is given by the natural isomorphism induced by the isotopy $\tilde{n}_3^t$. Such a structure is unique up to equivalence.

Similarly, there is a left $Z(P \times I)$-module category structure on $Z(X)$.

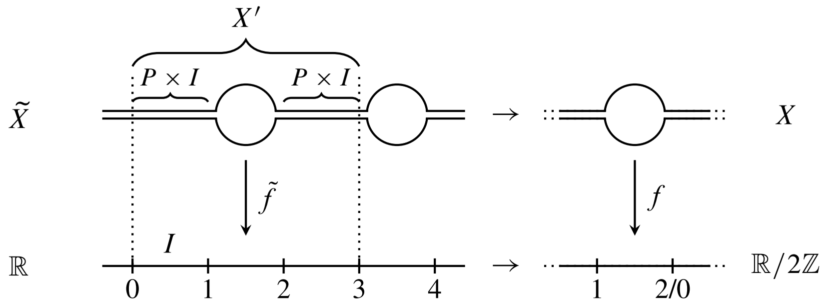
Proof. Similar to Proposition 6.6. ■

There is a similar story for the right module structure, where X is a collared n-manifold so that 1 escapes to infinity.

7. Excision for Z_{TV}, Z_{CY}

In this section, we prove the main result of the paper, that both Z_{TV} and Z_{CY} satisfy excision. As in the previous section, essentially the same proof works for both the Turaev–Viro and Crane–Yetter theory, so we adopt the same notation as before, namely $Z, \widehat{Z}$ stands for either of the theories.

Let X be an n-manifold without boundary, with finitely many components, possibly non-compact. To present X as the quotient of some n-manifold X' by some gluing, consider a smooth function $f\colon X \to S^1 = \mathbb{R}/2\mathbb{Z}$, together with a trivialization of P-bundles $P \times I \simeq f^{-1}(I)$ for some $(n-1)$-manifold P. Take X' to be the "preimage of $(0,3)$ under f"; more precisely, pullback f along the universal covering map $\mathbb{R} \to \mathbb{R}/2\mathbb{Z}$ to get $\tilde{f}\colon \widetilde{X} \to \mathbb{R}$, and take $X' = \tilde{f}^{-1}((0,3))$ (see figure below). So, X is obtained from X' by gluing the parts over $(0,1)$ and $(2,3)$.



Remark 7.1. Excision is usually phrased in terms of gluing two collared manifolds. In the above language, that will correspond to the case when $X' = X_1 \sqcup X_2$, where $X_1 = \tilde{f}^{-1}((0,1.5))$, $X_2 = \tilde{f}^{-1}((1.5,3))$, so that the pullback map $X' \to X$ is the gluing/overlapping of X_1 and X_2 over $(0,1)$, the collared neighborhoods.

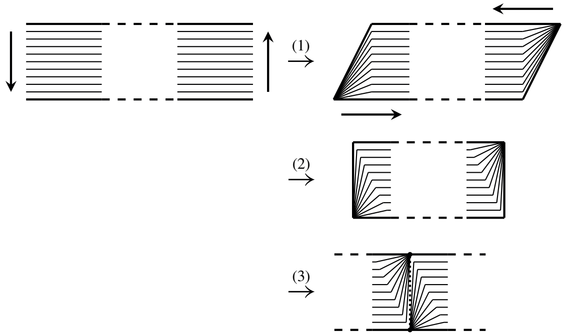
Since $\tilde{f}^{-1}((0,1)) \simeq \tilde{f}^{-1}((2,3)) \simeq f^{-1}(I)$ naturally, the trivialization $P \times I \simeq f^{-1}(I)$ gives a left and right $P \times I$-module structure on X', and makes $\widehat{Z}(X')$ a $\widehat{Z}(P \times I)$-bimodule category (likewise for Z).

The natural gluing map $X' \to X$ is the composition $X' \subset \widetilde{X} \to X$. We can also embed X' in X as follows: consider a sequence of maps $X' \to P \times I \sqcup X' \sqcup P \times I \to X' \to X$; the first map is just the obvious inclusion, the second one is the left and right module map "squeezing" X' into itself, and the third map is the natural quotient map. It is easy to see that the composition is an embedding, in fact a diffeomorphism onto $X \backslash f^{-1}(0.5)$. We denote this composition by i.

Since $i\colon X' \to X$ is an embedding, it induces a functor $i_*\colon \widehat{Z}(X') \to \widehat{Z}(X)$. Recall that there is a natural functor $\mathrm{hTr}\colon \widehat{Z}(X') \to \mathrm{hTr}(\widehat{Z}(X'))$ that is the identity on objects.

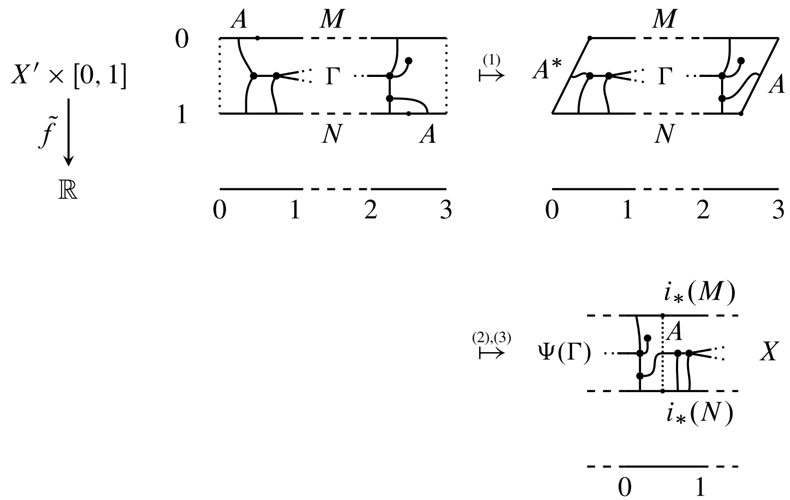
Lemma 7.2. *The inclusion functor* $i_*\colon \widehat{Z}(X') \to \widehat{Z}(X)$ *extends along* hTr *to a functor* $i_*\colon \mathrm{hTr}(\widehat{Z}(X')) \to \widehat{Z}(X)$.

Proof. Consider a map $\Psi\colon X' \times [0,1] \to X \times [0,1]$ described as a composition of operations given by the figures below (with further explanations later):



The first figure depicts $X' \times [0,1]$, with the $[0,1]$ factor going in the vertical direction. The foliation depicted consists of the obvious horizontal leaves $X' \times \{r\}$; we depict the foliation only to better explain the operations we perform below. The left and right parts are the $P \times I$ portions that would glue to give X. Operation (1) "pinches" the left vertical side down and the right vertical side up. Operation (2) "squeezes" the bottom left and top right portions. Operation (3) glues the two vertical sides.

In the diagram below, we depict a graph Γ in $X' \times [0,1]$ with incoming boundary value $A \rhd M$ and outgoing boundary value $N \lhd A$, representing an element of $\mathrm{Hom}_{\mathrm{hTr}(\widehat{Z}(X'))}(M, N)$. It is sent to a graph $\Psi(\Gamma)$ in $X \times [0,1]$ with incoming boundary M and outgoing boundary N, representing an element of $\mathrm{Hom}_{\widehat{Z}(X)}(M, N)$.



Note that operation (1) creates corners in the top left and bottom right, so Ψ is not exactly a smooth map; however, it is an embedding when restricted to $X' \times (0, 1)$, and can easily be slightly perturbed to be a smooth embedding. As we ultimately care about the images of graphs $\Psi(\Gamma)$ up to isotopy, we will not bother with the details of this perturbation nor the non-smoothness of Ψ at the corner.

The only points in $X \times [0, 1]$ that are hit more than once are in $f^{-1}(0.5)$; we call this the *seam*. In the figure above, the seam is depicted as the vertical dotted line in the right most figure. The seam is also the image of the top left and bottom right boundary pieces (the parts labeled A). The image of $\Psi|_{X' \times (0,1)}$ is exactly $X \times (0, 1) \backslash \text{seam}$.

We claim that the following map is well defined:

$$\mathrm{Hom}_{\mathrm{hTr}(\widehat{Z}(X'))}(M, N) \to \mathrm{Hom}_{\widehat{Z}(X)}(i_*(M), i_*(N)),$$
$$\Gamma \mapsto \Psi(\Gamma).$$

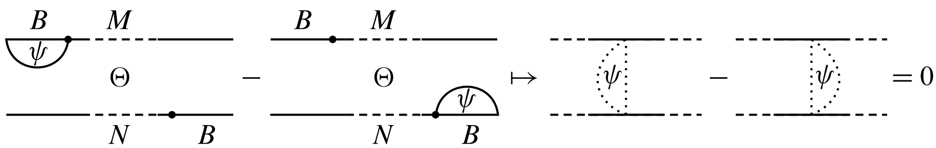
It is not hard to see that the assignment $\Gamma \mapsto \Psi(\Gamma)$ yields a well-defined map

$$\mathrm{Hom}^{A}_{\widehat{Z}(X')}(M, N) \to \mathrm{Hom}_{\widehat{Z}(X)}(i_*(M), i_*(N));$$

a graph $\Gamma = \sum c_i \Gamma_i$ that is null with respect to some ball D would have image $\Psi(\Gamma)$ null with respect to $\Psi(D)$. We need to check that the relations $\sim$ in

$$\mathrm{Hom}_{\mathrm{hTr}(\widehat{Z}(X'))}(M, N) = \bigoplus \mathrm{Hom}^{A}_{\widehat{Z}(X')}(M, N)/\sim$$

are satisfied. Recall that relations are generated by $\Theta \circ (\psi \triangleright \mathrm{id}_M) - (\mathrm{id}_N \triangleleft \psi) \circ \Theta$, where $\Theta \in \mathrm{Hom}^{A,B}_{\widehat{Z}(X')}(M, N)$ and $\psi \in \mathrm{Hom}_{\widehat{Z}(P \times I)}(B, A)$. We see that



The proof that the composition is respected is a simple exercise. ∎

We want to show that i_* is an equivalence, and will consider Ψ^{-1} applied to graphs. It is not clear that this is well defined, e.g., moving parts of a graph in $X \times [0, 1]$ across the seam could result in different graphs with different boundary conditions in $X' \times [0, 1]$. However, the relation $\Theta \circ (\psi \triangleright \mathrm{id}_M) - (\mathrm{id}_N \triangleleft \psi) \circ \Theta$ essentially takes care of this ambiguity.

Let us make this precise. Consider a small neighborhood $P \times (0.5 - \varepsilon, 0.5 + \varepsilon) \times [0, 1]$ of the seam in $X \times [0, 1]$. Consider the following vector field ν: at $(p, x, t) \in P \times (0.5 - \varepsilon, 0.5 + \varepsilon) \times [0, 1]$, the vector field has value $\sigma(x) \sin(\pi t) \frac{\partial}{\partial x}$, where $\sigma(x)$ is a smooth non-negative cut-off function on $(0, 1)$ that has support exactly $(0.5 - \varepsilon,$

$0.5+\varepsilon$). This vector field ν has the following displacing property: for any compact subset K in $P \times (0,5-\varepsilon, 0.5+\varepsilon) \times (0,1)$ (i.e., near the seam and not touching the boundary), the flow eventually pushes K off of the seam, i.e., there is some α such that the flow under ν after time α does not intersect the seam.

Let ζ^α be the isotopy generated by ν. Denoting by L_0 the seam, we define $L_\alpha = \zeta^\alpha(L_0)$. Let Ψ_α be the composition $\zeta^\alpha \circ \Psi$. Then L_α is the "seam" for Ψ_α.

Suppose a graph Γ in $X \times [0,1]$ intersects the seam L_0 transversally, in that the edges meet L_0 transversally and no vertices are on L_0. Then Γ defines a boundary value at the seam: the marked points are the points of intersection, and coloring is the color associated to the edge taken with right-ward orientation (that is, in direction of ν). In particular, the boundary value of $i_*(\Gamma)$ in the figure above is A. If Γ intersects L_α transversally, then we can also define its boundary value at L_α similarly; to be precise, it is the boundary value of $\zeta^{-\alpha}(\Gamma)$ at the seam.

Lemma 7.3. *Let Γ be a graph in $X \times [0,1]$ that represents a morphism in $\mathrm{Hom}_{\widehat{Z}(X)}(i_*(M), i_*(N))$ for some $M, N \in \mathrm{Obj}\,\mathrm{hTr}(\widehat{Z}(X'))$. Choose some α such that Γ is transverse to L_α, and suppose it defines the boundary value A_α. We see that $\Psi_\alpha^{-1}(\Gamma)$ is a graph in $X' \times [0,1]$ representing a morphism in $\mathrm{Hom}_{\widehat{Z}(X')}(A_\alpha \triangleright M, N \triangleleft A_\alpha)$. Then as a morphism in $\mathrm{Hom}_{\mathrm{hTr}(\widehat{Z}(X'))}(M,N)$, $\Psi_\alpha^{-1}(\Gamma)$ is independent of such a choice of α.*

Proof. Clear from the picture. ∎

We come to the main "topological" result of the paper:

Theorem 7.4. *The extension $i_*\colon \mathrm{hTr}(\widehat{Z}(X')) \to \widehat{Z}(X)$ is an equivalence.*

Proof. It was already evident from the object map that $\mathrm{hTr}(\widehat{Z}(X')) \to \widehat{Z}(X)$ is essentially surjective – it only misses objects that have marked points on $f^{-1}(0.5)$, but such an object is isomorphic to an object with points moved slightly off of $f^{-1}(0.5)$.

To show that i_* is fully faithful, fix objects $M, N \in \mathrm{hTr}(\widehat{Z}(X'))$. By Lemma 7.3, the family Ψ_α^{-1} of maps defines a map $\Phi\colon \mathrm{VGraph}(X \times [0,1]; i_*(M)^*, i_*(N)) \to \mathrm{Hom}_{\mathrm{hTr}(\widehat{Z}(X'))}(M,N)$.

Let us show that Φ factors through the projection

$$\mathrm{VGraph}(X \times [0,1]; i_*(M)^*, i_*(N)) \to \mathrm{Hom}_{\widehat{Z}(X)}(i_*(M), i_*(N)).$$

We make the following observation. If $\Gamma = \sum c_i \Gamma_i$ is null with respect to some closed ball $D \subset X \times [0,1]$, and there is some L_α that does not meet D and is transversal to Γ, then $\Phi(\Gamma) = \Psi_\alpha^{-1}(\Gamma)$ is null with respect to $\Psi_\alpha^{-1}(D)$.

Let $0 < \beta < 0.5$ be such that $i_*(M)$ and $i_*(N)$ do not have any marked points in $f^{-1}((0.5-\beta, 0.5+\beta)) \subset X$; denote $J = (0.5-\beta, 0.5+\beta)$. Consider the open cover $\{U_1, U_2\}$ of $X \times [0,1]$, where $U_1 = f^{-1}(J)$ and $U_2 = X \times [0,1] \backslash L_0$. By Proposition 6.3, the space of null graphs is generated by graphs that are null with respect to balls D contained in either U_1 or U_2, thus it suffices to show that Φ sends such graphs to 0. By the previous observation, it suffices to check that there exists an α that does not intersect such D.

For $D \subset U_2$, such L_α exists by Sard's theorem – for small enough α, L_α does not intersect D, so it suffices to consider the transversality with Γ, which is a generic condition.

Now, suppose $D \subset U_1$. Since there are no marked points on the boundary in U_1, by Lemma 6.1, we may assume that D does not meet the boundary. As we pointed out, the vector field ν defining the isotopy ζ^α has the property that it will displace D off of L_0. So, if $\zeta^\alpha(D)$ does not intersect L_0, we can take $L_{-\alpha+\varepsilon}$, where small ε is chosen to get transversality with Γ, and we are done. ∎

Combining the topological result above with the algebraic results of Section 3, we have the main result of the paper:

Theorem 7.5. *There is an equivalence*

$$\mathcal{Z}_{Z(P\times I)}(Z(X')) \simeq Z(X).$$

In particular, when $X = X_1 \cup X_2$ as in Remark 7.1,

$$Z(X_1) \boxtimes_{Z(P\times I)} Z(X_2) \simeq \mathcal{Z}_{Z(P\times I)}(Z(X_1 \sqcup X_2)) \simeq Z(X).$$

Proof. We claim that $Z(P \times I)$ is multifusion; we justify this claim later. By Proposition 6.7, $\hat{Z}(P \times I)$ is pivotal. In reference to the notation in Section 3, take $\mathcal{C}' = \hat{Z}(P \times I), \mathcal{C} = Z(P \times I), \mathcal{M}' = \hat{Z}(X'), \mathcal{M} = Z(X')$. Then,

$$\begin{aligned}
\mathcal{Z}_{Z(P\times I)}(Z(X')) &\simeq \mathrm{Kar}(\mathrm{hTr}_{\hat{Z}(P\times I)}(Z(X'))) && \text{(by Corollary 3.12)}\\
&\simeq \mathrm{Kar}(\mathrm{hTr}_{\hat{Z}(P\times I)}(\hat{Z}(X'))) && \text{(by Lemma 3.13)}\\
&\simeq \mathrm{Kar}(\hat{Z}(X)) && \text{(by extending } i_* \text{ from Theorem 7.4 to Kar)}\\
&= Z(X).
\end{aligned}$$

The second statement follows from the first by applying Lemma 6.5 and (3.4).

Now, we need to justify $Z(P \times I)$ being multifusion. This is true for $P = \{*\}$ and for $P = I$. By Example 8.2, which uses the argument above for $P = I$, we have $Z(S^1 \times I) \simeq \mathcal{Z}(\mathcal{A})$ as a $\mathbf{k}$-linear abelian category (but not as a monoidal category, see Remark 8.3); in particular, this implies that it is semisimple with finitely many simple objects. Since, by Proposition 6.7, the stacking monoidal structure on $Z(P \times I)$ is rigid and pivotal, this shows that $Z(P \times I)$ is a pivotal multifusion category.

So, $Z(P \times I)$ is pivotal multifusion for any connected P; the claim follows for a disjoint union of finitely many such P's. ■

Corollary 7.6. *In each of the two cases below:*

- $n = 1$, *$\mathcal{A}$ a spherical fusion category,*
- $n = 2$, *$\mathcal{A}$ a premodular category,*

for an n-manifold X the category $Z(X)$ of boundary values for colored graphs constructed above coincides with the factorization homology $\int_X \mathcal{A}$.

Proof. We verify that $Z(-)$ satisfies the three characterizing properties laid out in Theorem 2.4.

(1) For both cases of n, $\mathcal{A}$ defines an n-disk algebra in $\mathcal{V} = \mathcal{R}ex$, thus defining factorization homologies $\int_{\_} \mathcal{A}$ which coincide with $Z(-)$ on the n-disk.

(2) Proposition 6.11 proves the uniqueness of the E_1-algebra structure on $Z(P \times I)$.

(3) Theorem 7.5 proves the excision property (as in Theorem 2.3).

Thus, $Z(X) \simeq \int_X \mathcal{A}$. ■

Corollary 7.7. *In the assumptions of Corollary 7.6, $Z(X)$ is a finite semisimple category.*

Proof. Any connected X can be built from I^n by a sequence of gluings of collared manifolds. For example, for $n = 2$, gluing opposite edges of a square gives an annulus, and gluing boundaries of the annulus together gives the torus.

By Theorem 7.5, the corresponding category $Z(X)$ thus can be constructed from the Deligne product of several copies of $Z(I^n) \simeq \mathcal{A}$ by repeatedly applying the center construction, replacing a category $\mathcal{M}$ by $\mathcal{Z}_{Z(P \times I)}(\mathcal{M})$. Since it was shown in the proof of Theorem 7.5 that $Z(P \times I)$ is pivotal multifusion, it now follows from Proposition 3.5 that applying the center construction always gives a finite semisimple category. Thus, $Z(X)$ is a finite semisimple category. ■

8. Examples and computations

In this section, we present some examples and computations using the results obtained so far.

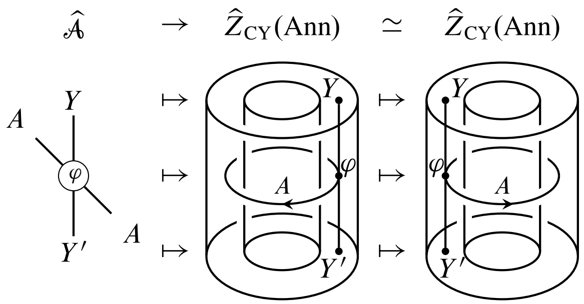
Example 8.1. $Z_{\mathrm{TV}}(S^1) \simeq \mathcal{Z}(\mathcal{A})$. This follows applying Theorem 7.5 to $X' = (0, 3)$, $X = S^1 = \mathbb{R}/2\mathbb{Z}$ (see Example 6.9).

This example, is, of course, well known: see, e.g., [11, 20].

Example 8.2. $Z_{\mathrm{CY}}(\mathrm{Ann}) \simeq \mathcal{Z}(\mathcal{A})$ as abelian categories, where $\mathrm{Ann} = I \times S^1$ is the annulus. Here we get Ann by gluing $I \times I$ to itself in the vertical direction (see Example 6.10). The result follows from applying Theorem 7.5 to $X' = I \times (0,3)$, $X = \mathrm{Ann} = I \times \mathbb{R}/2\mathbb{Z}$, with $P = I$.

Again, this result is not new: see, e.g., [7].

Let us flesh out some details. Define $\widehat{\mathcal{A}} = \mathrm{hTr}(\mathcal{A})$, where $\mathcal{A}$ is an $\mathcal{A}$-bimodule by left, right multiplication. Theorem 7.4 gives an equivalence $\widehat{\mathcal{A}} \simeq \widehat{Z}_{\mathrm{CY}}(\mathrm{Ann})$, pictorially given by the following figure on the left:



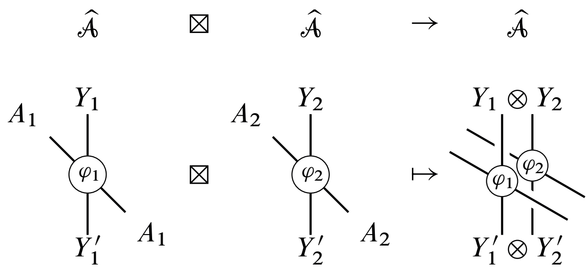
(8.1)

Here the loop A is given a trivial (e.g., always horizontal) framing. It is clear from this picture that $\mathrm{End}_{\widehat{\mathcal{A}}}(\mathbf{1})$ is commutative.

By Proposition 6.6, $\mathrm{Ann} = S^1 \times I$ has a horizontal stacking operation that, under the equivalence $\widehat{\mathcal{A}} \simeq \widehat{Z}_{\mathrm{CY}}(\mathrm{Ann})$ above, is given by a map

$$\mathrm{Hom}_{\mathcal{A}}^{A_1}(Y_1, Y_1') \otimes \mathrm{Hom}_{\mathcal{A}}^{A_2}(Y_2, Y_2') \to \mathrm{Hom}_{\mathcal{A}}^{A_1 \otimes A_2}(Y_1 \otimes Y_2, Y_1' \otimes Y_2)$$

described as follows:



(8.2)

This stacking operation gives rise to the monoidal structure that is defined in Proposition 6.6, where we take $P = S^1$.

Remark 8.3. Note that the stacking operation in Example 8.2 does *not* result in the usual tensor product on the Drinfeld center $\mathcal{Z}(\mathcal{A})$ (the latter can be defined as the functor assigned to pair of pants in Turaev–Viro theory). This is explored in more detail in [28], where the tensor product is defined purely in terms of $\mathcal{A}$ (i.e., without recourse to topology); it is shown that the stacking tensor product is typically not spherical (but pivotal) and not fusion (but multifusion).

Next, we will be concerned with relating Z_{CY} of a surface Σ with that of a punctured one Σ_0, that is, $\Sigma_0 = \Sigma \backslash \{p\}$. We will think of Σ as obtained from Σ_0 by gluing with an open disk, "sealing" the puncture: $\Sigma = \Sigma_0 \cup \mathbb{D}^2$, implicitly choosing some collared structure on Σ_0 and $\mathbb{D}^2$.

Recall $\widehat{\mathcal{A}} := \text{hTr}(\mathcal{A})$ from Example 8.2. There is a right action of $\text{Hom}_{\widehat{\mathcal{A}}}(\mathbf{1},\mathbf{1})$ on the morphisms of $\hat{Z}_{\text{CY}}(\Sigma_0)$, by "pushing in" from the puncture, i.e.,

$$\text{Hom}_{\hat{Z}_{\text{CY}}(\Sigma_0)}(Y, Y') \otimes \text{Hom}_{\widehat{\mathcal{A}}}(\mathbf{1},\mathbf{1}) \to \text{Hom}_{\hat{Z}_{\text{CY}}(\Sigma_0)}(Y \lhd \mathbf{1}, Y' \lhd \mathbf{1}) \cong \text{Hom}_{\hat{Z}_{\text{CY}}(\Sigma_0)}(Y, Y').$$

It is easy to see that for $\Gamma \in \text{Hom}_{\hat{Z}_{\text{CY}}(\Sigma_0)}(Y, Y')$ and $f, g \in \text{Hom}_{\widehat{\mathcal{A}}}(\mathbf{1},\mathbf{1})$,

$$\Gamma \lhd (f \circ g) = (\Gamma \lhd f) \lhd g.$$

Moreover, for $\Gamma' \in \text{Hom}_{\hat{Z}_{\text{CY}}(\Sigma_0)}(Y', Y'')$,

$$(\Gamma' \circ \Gamma) \lhd (f \circ g) = (\Gamma' \lhd f) \circ (\Gamma \lhd g).$$

Let $\pi = \sum d_i/\mathcal{D} \cdot \text{id}_{X_i} \in \bigoplus \text{Hom}_{\mathcal{A}}^{X_i}(\mathbf{1},\mathbf{1}) = \text{Hom}_{\widehat{\mathcal{A}}}(\mathbf{1},\mathbf{1})$. (Note: $\mathcal{D}$ and simple objects X_i are of $\mathcal{A}$, and not of $\mathcal{Z}(\mathcal{A})$.) π is an idempotent in $\text{Hom}_{\widehat{\mathcal{A}}}(\mathbf{1},\mathbf{1})$, and hence also acts as an idempotent on $\text{Hom}_{\hat{Z}_{\text{CY}}(\Sigma_0)}(Y, Y')$.

Proposition 8.4. *Let $\Sigma_0 = \Sigma \backslash \{p\}$ as above. Consider the category $\widehat{\mathcal{B}}$ consisting of the same objects as $\hat{Z}_{\text{CY}}(\Sigma_0)$, but morphisms given by*

$$\text{Hom}_{\widehat{\mathcal{B}}}(Y, Y') = \text{im}(\text{Hom}_{\hat{Z}_{\text{CY}}(\Sigma_0)}(Y, Y') \circlearrowleft \pi).$$

Then the restriction to $\widehat{\mathcal{B}}$ of the inclusion functor corresponding to $i \colon \Sigma_0 \hookrightarrow \Sigma$ is an equivalence:

$$i_*|_{\widehat{\mathcal{B}}} \colon \widehat{\mathcal{B}} \simeq \hat{Z}_{\text{CY}}(\Sigma).$$

Proof. First note that $\widehat{\mathcal{B}}$ is indeed closed under composition of morphisms because π is idempotent. It is clear that $i_*|_{\widehat{\mathcal{B}}}$ is essentially surjective. To prove fully faithfulness, consider two objects $Y, Y' \in \hat{Z}_{\text{CY}}(\Sigma_0)$. By abuse of notation, we also denote $i_*(Y), i_*(Y') \in \text{Obj}\, \hat{Z}_{\text{CY}}(\Sigma_0)$ by Y, Y'. We call the vertical segment $p \times [0,1] \subset \Sigma \times [0,1]$ the *pole*, so that $\Sigma_0 \times [0,1] = \Sigma \times [0,1] \backslash \text{pole}$.

We construct an inverse map to i_*. Let U be a small open neighborhood of p in Σ, and let $\mathcal{N} = U \times [0,1] \subset \Sigma \times [0,1]$ be a small open neighborhood of the pole. Choose U small enough so that it does not contain any marked points of Y, Y'. Consider a graph $\Gamma \in \mathrm{Graph}(\Sigma \times [0,1]; Y^*, Y')$. Define $j(\Gamma)$ as follows: if Γ intersects the pole, then use an isotopy supported in $\mathcal{N}$ to push Γ off of it, resulting in a new graph Γ'. Now, Γ' can be considered a graph in $\mathrm{Graph}(\Sigma_0 \times [0,1]; Y^*, Y')$. Then we define $j(\Gamma) = \Gamma' \triangleleft \pi$.

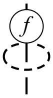
We need to check that j is well defined. Firstly, the (linear combination of) graphs $\Gamma' \triangleleft \pi$ is independent of the choice of isotopy – this follows from the sliding lemma (Lemma A.5). More generally, it means that for any isotopy φ of $\Sigma \times [0,1]$ supported on $\mathcal{N}$, $j(\Gamma) = j(\varphi(\Gamma))$.

Now, we check that j sends null graphs to 0. Take the two set open cover $\{\mathcal{N}, \Sigma_0 \times [0,1]\}$ of $\Sigma \times [0,1]$, and apply Proposition 6.3. Let $\Gamma = \sum c_i \Gamma_i$ be null with respect to some ball D. If $D \subset \Sigma_0 \times [0,1]$, clearly $j(\Gamma)$ is null with respect to D. If $D \subset \mathcal{N}$, we may assume D does not touch the boundary (by choice of U), so we can isotope it with some isotopy φ supported on $\mathcal{N}$ so that $\varphi(D)$ does not meet the pole. Then clearly $j(\varphi(\Gamma))$ is null with respect to $\varphi(D)$.

Finally, it is easy to see that j is inverse to i_*. For example, $i_* \circ j$ amounts to adding a trivial dashed circle, which is equivalent to 1 by Lemma A.6. ■

Corollary 8.5. *$Z_{\mathrm{CY}}(S^2) \simeq \mathcal{Z}^{\mathrm{Mü}}(\mathcal{A})$, the Müger center of $\mathcal{A}$, and in particular, when $\mathcal{A}$ is modular, $Z_{\mathrm{CY}}(S^2) \simeq \mathcal{V}ec$.*

Proof. Think of the disk $\mathbb{D}^2$ as a punctured sphere; by Proposition 8.4, we have that $\widehat{Z}_{\mathrm{CY}}(S^2) \simeq \widehat{\mathcal{B}}$, where $\widehat{\mathcal{B}}$ is the category with the same objects as $\widehat{Z}_{\mathrm{CY}}(\mathbb{D}^2) \simeq \mathcal{A}$, but morphisms are, for $A, A' \in \mathcal{A}$,

$$\mathrm{Hom}_{\widehat{\mathcal{B}}}(A, A') = \left\{ \frac{1}{\mathcal{D}} \right.$$  $$\left. \;\middle|\; f \in \mathrm{Hom}_{\mathcal{A}}(A, A') \right\}.$$

In particular, when $A = A' = X_i$ a simple object, it follows from [23, Corollary 2.14] that simple objects that are not transparent, i.e., not in the Müger center, are killed:

$$\mathrm{End}_{\widehat{\mathcal{B}}}(X_i) = \begin{cases} \mathrm{End}_{\mathcal{A}}(X_i) & \text{if } X_i \in \mathcal{Z}^{\mathrm{Mü}}(\mathcal{A}), \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\widehat{\mathcal{B}}$ coincides with the Müger center, which is already abelian, and so $Z_{\mathrm{CY}}(S^2) \simeq \mathrm{Kar}(\widehat{\mathcal{B}}) = \mathcal{Z}^{\mathrm{Mü}}(\mathcal{A})$. ■

9. Crane–Yetter and the elliptic Drinfeld center

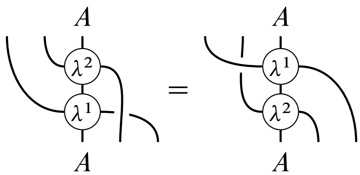
In [27], the second author constructed a category similar to the Drinfeld center, but instead the objects have two half-braidings that satisfy some compatibility conditions.

In this section, we show that this category is the category of boundary values on the once-punctured torus.

We note that all morphisms depicted using graphical calculus are over $\mathcal{A}$, but they may represent morphisms in a different category. In particular, dashed lines do not need an orientation and in makes sense to use the circular α instead of the semicircular one.

For the reader's convenience, we recall the definition and some properties of the elliptic Drinfeld center:

Definition 9.1. Let $\mathcal{A}$ be a premodular category. The category $\mathcal{Z}^{\mathrm{el}}(\mathcal{A})$ consists of objects of the form $(A, \lambda^1, \lambda^2)$, where λ^1, λ^2 are half-braidings on A that satisfy



(9.1)

We call the relation (9.1) "COMM". The morphisms

$$\mathrm{Hom}_{\mathcal{Z}^{\mathrm{el}}(\mathcal{A})}((A, \lambda^1, \lambda^2), (A', \mu^1, \mu^2)$$

are morphisms of $\mathcal{A}$ that intertwine both half-braidings, i.e.,

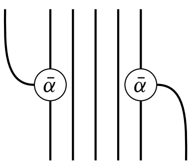
$$\begin{aligned}&\mathrm{Hom}_{\mathcal{Z}^{\mathrm{el}}(\mathcal{A})}((A, \lambda^1, \lambda^2), (A', \mu^1, \mu^2))\\ &\quad := \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}((A, \lambda^1), (A', \mu^1)) \cap \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}((A, \lambda^2), (A', \mu^2)).\end{aligned}$$

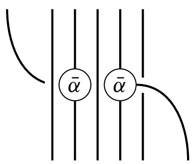
Proposition 9.2 ([27, Proposition 3.4]). *$\mathcal{Z}^{\mathrm{el}}(\mathcal{A})$ is a finite semisimple category.*

Proposition 9.3 ([27, Propositions 3.5 and 3.8]). *The forgetful functor $\mathcal{F}^{\mathrm{el}}\colon \mathcal{Z}^{\mathrm{el}}(\mathcal{A}) \to \mathcal{A}$ has a two-sided adjoint $\mathcal{I}^{\mathrm{el}}\colon \mathcal{A} \to \mathcal{Z}^{\mathrm{el}}(\mathcal{A})$, where on objects, $\mathcal{I}^{\mathrm{el}}$ sends*

$$A \mapsto \Big(\bigoplus_{i,j} X_i X_j A X_j^* X_i^*, \Gamma^1, \Gamma^2\Big)$$

where

$\Gamma^1 =$ 

, $\Gamma^2 =$ 

where $\bar{\alpha}$ is defined in Lemma A.7.

On morphisms, $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$,

$$\mathcal{I}^{\mathrm{el}}(f) = \bigoplus_{i,j} \mathrm{id}_{X_i X_j} \otimes f \otimes \mathrm{id}_{X_j^* X_i^*}$$

We refer to [27] *for the functorial isomorphisms giving the adjunction. Furthermore, $\mathcal{I}^{\mathrm{el}}$ is dominant.*

Theorem 9.4 ([27, Theorem 4.3]). *When $\mathcal{A}$ is modular, there is an equivalence*

$$\begin{aligned} \mathcal{A} &\simeq \mathcal{Z}^{\mathrm{el}}(\mathcal{A}), \\ A &\mapsto \Big(\bigoplus_i X_i A X_i^*, \Gamma, \Omega\Big), \end{aligned}$$

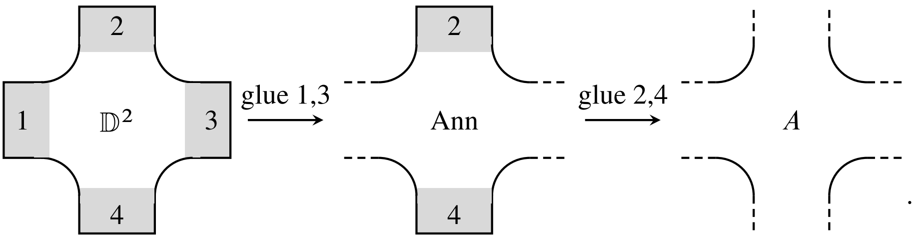
where Γ is the half-braiding on $I(X)$ in Theorem 3.3, and $\Omega = c^{-1}_{X_i^*,-} \circ c_{-,A} \circ c_{-,X_i}$, where $c_{-,-}$ is the braiding on $\mathcal{A}$.

Proposition 9.5. *Let $\mathbf{T}^2_0$ be the once-punctured torus. There is an equivalence*

$$Z_{\mathrm{CY}}(\mathbf{T}^2_0) \cong \mathcal{Z}^{\mathrm{el}}(\mathcal{A}).$$

Under this equivalence, the inclusion functor $\mathcal{A} \simeq Z_{\mathrm{CY}}(\mathbb{D}^2) \to Z_{\mathrm{CY}}(\mathbf{T}^2_0)$ is identified with $\mathcal{I}^{\mathrm{el}}\colon \mathcal{A} \to \mathcal{Z}^{\mathrm{el}}(\mathcal{A})$.

Proof. Think of the once-punctured torus as an open disk, drawn like a '+' sign, with opposite sides identified (Ann $= S^1 \times I$):



The left most figure shows how $Z_{\mathrm{CY}}(\mathbb{D}^2) \simeq \mathcal{A}$ is a module category over $Z_{\mathrm{CY}}(I \times I) \simeq \mathcal{A}$ in four ways; we think of the 1,2 edges as acting on the left, 3,4 edges as acting on the right. The actions are just usual left and right multiplication.

By Theorem 7.4, the first "glue 1,3" arrow induces an equivalence

$$\widehat{Z}_{\mathrm{CY}}(\mathrm{Ann}) \simeq \mathrm{hTr}_{\widehat{Z}_{\mathrm{CY}}(I\times I)}(\widehat{Z}_{\mathrm{CY}}(\mathbb{D}^2)) \simeq \mathrm{hTr}_{\mathcal{A}}(\mathcal{A})$$

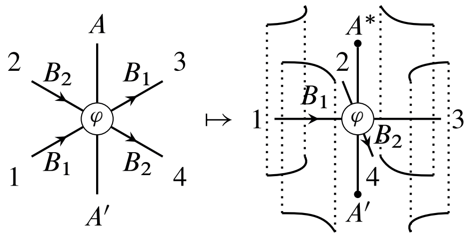
(see also Example 8.2). Again, by Theorem 7.4, the second "glue 2,4" arrow induces an equivalence

$$\widehat{Z}_{\mathrm{CY}}(\mathbf{T}^2_0) \simeq \mathrm{hTr}_{\widehat{Z}_{\mathrm{CY}}(I\times I)}(\widehat{Z}_{\mathrm{CY}}(\mathrm{Ann})) \simeq \mathrm{hTr}_{\mathcal{A}}(\mathrm{hTr}_{\mathcal{A}}(\mathcal{A})).$$

Let us give a more explicit description of the last equivalence. For $A, A' \in \operatorname{Obj} \mathcal{A}$,

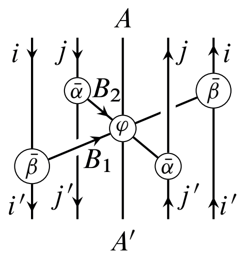
$$\begin{aligned} \operatorname{Hom}_{\mathrm{hTr}(\mathrm{hTr}(\mathcal{A}))}(A, A') &\cong \int^{B_2} \operatorname{Hom}_{\mathrm{hTr}(\mathcal{A})}(B_2 \otimes A, A' \otimes B_2) \\ &\cong \int^{B_2} \int^{B_1} \operatorname{Hom}_{\mathcal{A}}(B_1 \otimes B_2 \otimes A, A' \otimes B_2 \otimes B_1). \end{aligned}$$

Under the equivalence, a morphism $\varphi \in \operatorname{Hom}_{\mathcal{A}}(B_1 \otimes B_2 \otimes A, A' \otimes B_2 \otimes B_1)$, shown on the left in the figure below, is sent to the graph in $\mathbf{T}_0^2 \times [0, 1]$ shown on the right:



(9.2)

Now, we define a functor $\mathrm{hTr}_{\mathcal{A}}(\mathrm{hTr}_{\mathcal{A}}(\mathcal{A})) \to \mathcal{Z}^{\mathrm{el}}(\mathcal{A})$. On objects, it sends $A \mapsto \mathcal{I}^{\mathrm{el}}(A)$. On morphisms, the morphism in (9.2) is sent to



(9.3)

It is clear that this assignment respects the composition of morphisms. The following sequence of isomorphisms shows that this functor is fully faithful:

$$\begin{aligned} &\operatorname{Hom}_{\mathrm{hTr}(\mathrm{hTr}(\mathcal{A}))}(A, A') \\ &\cong \bigoplus_{i_1, i_2 \in \operatorname{Irr}(\mathcal{A})} \operatorname{Hom}_{\mathcal{A}}(X_{i_1} \otimes X_{i_2} \otimes A, A' \otimes X_{i_2} \otimes X_{i_1}) && \text{(by Lemma 3.11)} \\ &\cong \operatorname{Hom}_{\mathcal{A}}(X_{i_1} \otimes X_{i_2} \otimes A \otimes X_{i_2}^* \otimes X_{i_1}^*, A') \\ &\cong \operatorname{Hom}_{\mathcal{Z}^{\mathrm{el}}(\mathcal{A})}(\mathcal{I}^{\mathrm{el}}(A), \mathcal{I}^{\mathrm{el}}(A')). && \text{(by Proposition 9.3)} \end{aligned}$$

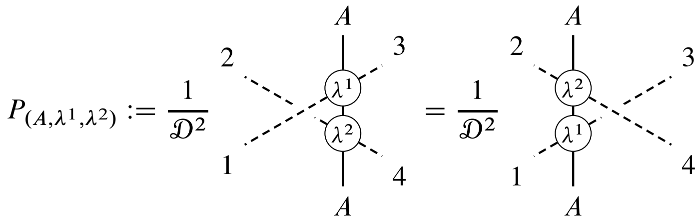
Since $\mathcal{Z}^{\mathrm{el}}(\mathcal{A})$ is abelian, we have that the extension to the Karoubi envelope is an equivalence:

$$Z_{\mathrm{CY}}(\mathbf{T}_0^2) \simeq \operatorname{Kar}(\mathrm{hTr}_{\mathcal{A}}(\mathrm{hTr}_{\mathcal{A}}(\mathcal{A}))) \simeq \mathcal{Z}^{\mathrm{el}}(\mathcal{A})$$

and we are done. But before we end the proof, we provide an explicit inverse functor that will be useful later: on objects,

$$(A, \lambda^1, \lambda^2) \mapsto \operatorname{im}(P_{(A,\lambda^1,\lambda^2)}) \tag{9.4}$$

where



the equality of diagrams follows from the COMM requirement (9.1), and the dashed line represents a weighted sum over simple objects (see Appendix A). On morphisms,

$$\operatorname{Hom}_{\mathcal{Z}^{\mathrm{el}}(\mathcal{A})}((A, \lambda^1, \lambda^2), (A', \mu^1, \mu^2)) \ni f \mapsto P_{(A',\mu^1,\mu^2)} \circ f \circ P_{(A,\lambda^1,\lambda^2)}$$

Thus we have a 2-commutative diagram

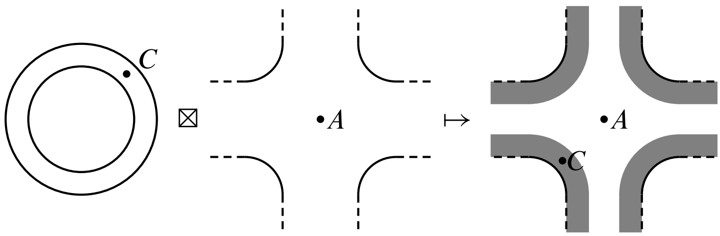
$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{I}^{\mathrm{el}}} & \mathcal{Z}^{\mathrm{el}}(\mathcal{A}) \\ \downarrow \simeq & & \downarrow \simeq \\ Z_{\mathrm{CY}}(\mathbb{D}^2) & \xrightarrow{\mathrm{incl.}_*} & Z_{\mathrm{CY}}(\mathbf{T}^2_0) \end{array} \tag{9.5}$$

This concludes the proof of Proposition 9.5. ∎

In particular, when $\mathcal{A}$ is modular, we have an equivalence

$$\mathrm{incl.}_*\colon Z_{\mathrm{CY}}(\mathbb{D}^2) \xrightarrow{\simeq} Z_{\mathrm{CY}}(\mathbf{T}^2_0).$$

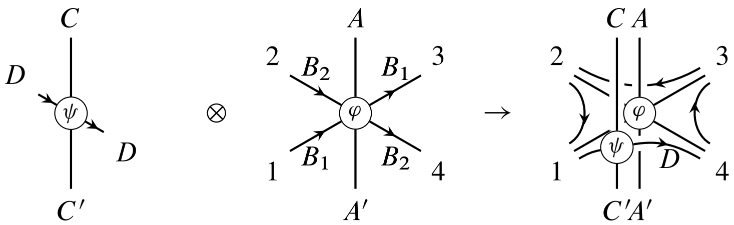
Our next task is to upgrade this equivalence to an equivalence of left $Z_{\mathrm{CY}}(\mathrm{Ann})$-modules. In the rightmost figure in (9.6), the gray area is a collar neighborhood of the puncture of $\mathbf{T}^2_0$. By Proposition 6.12, there is a (left) $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$-module structure on $\hat{Z}_{\mathrm{CY}}(\mathbf{T}^2_0)$: on objects,



(9.6)

while on morphisms, the module structure, employing the equivalences of (9.2) and (8.1), is given as follows:

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})}(C, C') \otimes \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbf{T}_0^2)}(A, A') \to \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbf{T}_0^2)}(A, A')$$

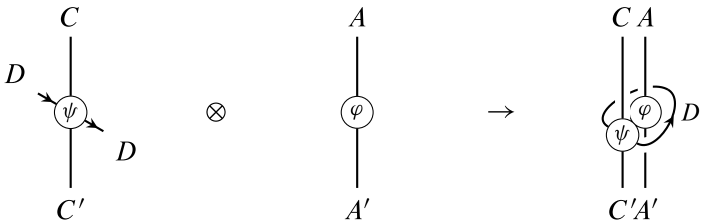


(9.7)

(The D-labeled strand originally goes around the annulus in Ann $\times[0,1]$; after inserting into $\mathbf{T}_0^2 \times [0,1]$, it wraps around like the gray area in (9.6)). This extends to a left $Z_{\mathrm{CY}}(\mathrm{Ann})$-module structure on $Z_{\mathrm{CY}}(\mathbf{T}_0^2)$.

Similarly, there is a left $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$-module structure on $\hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)$ (which extends to Z_{CY}):

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})}(C, C') \otimes \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)}(A, A') \to \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)}(A, A')$$



(9.8)

In light of (9.5), the following theorem is an upgrade of Theorem 9.4:

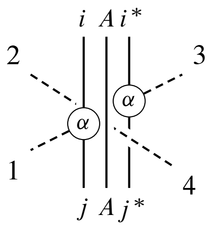
Theorem 9.6. *Let $\mathcal{A}$ be modular. There is an equivalence of left* $Z_{\mathrm{CY}}(\mathrm{Ann})$*-modules*

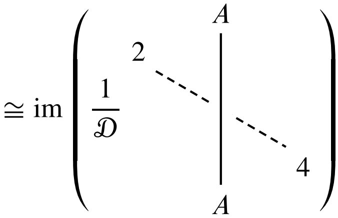
$$Z_{\mathrm{CY}}(\mathbb{D}^2) \simeq Z_{\mathrm{CY}}(\mathbf{T}_0^2).$$

Proof. Under the equivalence $\mathcal{Z}^{\mathrm{el}}(\mathcal{A}) \simeq Z_{\mathrm{CY}}(\mathbf{T}_0^2)$, it is easy to see that the equivalence of Theorem 9.4 can be rewritten as

$$Z_{\mathrm{CY}}(\mathbb{D}^2) \simeq \mathcal{A} \simeq Z_{\mathrm{CY}}(\mathbf{T}_0^2),$$

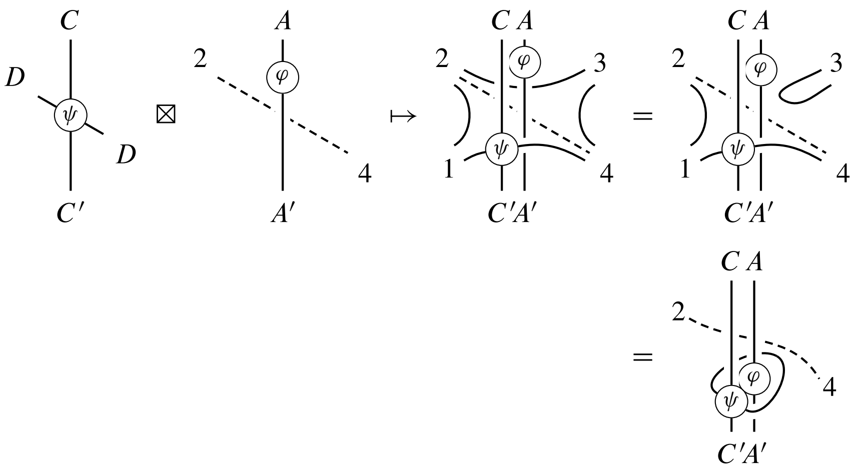
$$A \mapsto \mathrm{im}\left(\sum_{i,j} \frac{\sqrt{d_i}\sqrt{d_j}}{\mathcal{D}^2} \right)$$





where the isomorphism is essentially given by $\widehat{P}'_A$ and $\check{P}'_A$ from Lemma A.9 (with $\mathcal{M} = \mathcal{C}$), which is easily seen to be natural in A.

Then we see that



where we use the sliding lemma (Lemma A.5) for both equalities, and isotopies to move the strands around. The final diagram is what one obtains if one applies ψ to $\varphi \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ first, and then send it to $Z_{\mathrm{CY}}(\mathbf{T}^2_0)$. Hence, the equivalence does respect the module structure and we are done. ∎

Finally, we state the main result of this section:

Theorem 9.7. *Let $\mathcal{A}$ be modular. Let Σ be a connected compact oriented surface with b boundary components and genus g, and let $S_{0,b} = S^2 \backslash (\mathbb{D}^2)^{\sqcup b}$ be a genus 0 surface with b boundary components. Then*

$$Z_{\mathrm{CY}}(\Sigma) \cong Z_{\mathrm{CY}}(S_{0,b}).$$

In particular,

$$Z_{\mathrm{CY}}(\textit{closed surface}) \cong Z_{\mathrm{CY}}(S^2) \cong \mathcal{V}ec$$

and

$$Z_{\mathrm{CY}}(\textit{once-punctured surface}) \cong Z_{\mathrm{CY}}(\mathbb{D}^2) \cong \mathcal{A}.$$

Proof. Suppose $g > 0$, so that we can present Σ as a connect sum $\Sigma' \# \mathbf{T}^2$, where Σ' is a connected compact oriented surface with b boundary components and genus $g-1$. We think of the connect sum as $\Sigma = \Sigma'_0 \cup_{\mathrm{Ann}} (\mathbf{T}^2_0)$, where $\Sigma'_0 = \Sigma' \backslash \{\mathrm{pt}\}$ is a punctured surface. Then by Theorem 7.5 and Theorem 9.6, $Z_{\mathrm{CY}}(\Sigma) \simeq Z_{\mathrm{CY}}(\Sigma'_0) \boxtimes_{Z_{\mathrm{CY}}(\mathrm{Ann})} Z_{\mathrm{CY}}(\mathbf{T}^2_0) \simeq Z_{\mathrm{CY}}(\Sigma'_0) \boxtimes_{Z_{\mathrm{CY}}(\mathrm{Ann})} Z_{\mathrm{CY}}(\mathbb{D}^2) \simeq Z_{\mathrm{CY}}(\Sigma'_0 \cup_{\mathrm{Ann}} \mathbb{D}^2) = Z_{\mathrm{CY}}(\Sigma')$. Thus, by induction on the genus, we have $Z_{\mathrm{CY}}(\Sigma) \simeq Z_{\mathrm{CY}}(S_{0,b})$.

The final statements follow from the cases $b = 0, 1$ and Corollary 8.5. ■

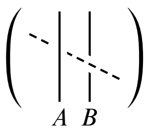
Remark 9.8. Here is an alternative proof to Theorem 9.7 and Theorem 9.6, pointed out by Jin-Cheng Guu, which avoids passage to the elliptic Drinfeld center and may be of independent interest. Recall the well-known equivalence

$$\mathcal{A} \boxtimes \mathcal{A} \simeq \mathcal{Z}(\mathcal{A}), \tag{9.9}$$
$$A \boxtimes B \mapsto (A \otimes B, c^{-1} \otimes c), \tag{9.10}$$

when $\mathcal{A}$ is modular [23]. This can be interpreted as an equivalence

$$Z_{\mathrm{CY}}(\mathbb{D}^2 \sqcup \mathbb{D}^2) \simeq Z_{\mathrm{CY}}(\mathrm{Ann})$$

which, using the equivalences established in Example 8.2, is given by

$A \boxtimes B \mapsto \mathrm{im}$  .

By similar reasoning as in the proof of Theorem 9.6, this can be shown to be a $Z_{\mathrm{CY}}(\mathrm{Ann})$-bimodule equivalence. Thus, performing a surgery (replacing an annulus with two disks or vice versa) does not affect the Z_{CY} of a surface. In particular, this yields

$$Z_{\mathrm{CY}}(\Sigma) \simeq \mathcal{A}^{\boxtimes b}$$

where b is the number of boundary components in Σ.

A. Pivotal multifusion categories conventions

This appendix is dedicated to the notation and the basic results about pivotal multifusion categories. It is adapted from [20], modified to accommodate for the non-spherical non-fusion case. We note that Ingo Runkel [25] also has similar results from developing a theory of string-net models for non-spherical pivotal fusion categories. We also point the reader to [13, Chapter 4] and [14] for further references.

Let $\mathcal{C}$ be a $\mathbf{k}$-linear pivotal multifusion category, where $\mathbf{k}$ is an algebraically closed field of characteristic 0. In all our formulas and computations, we will suppress the associativity and unit morphisms; we also suppress the pivotal morphism $V \simeq V^{**}$ when there is little cause for confusion.

We denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects in $\mathcal{C}$, and by $\mathrm{Irr}_0(\mathcal{C}) \subseteq \mathrm{Irr}(\mathcal{C})$ the subset of simple objects appearing in the direct sum decomposition of the unit object $\mathbf{1}$; it is known that $\mathbf{1}$ decomposes into a direct sum of distinct simple objects, so $\mathrm{End}(\mathbf{1}) \cong \bigoplus_{l \in \mathrm{Irr}_0(\mathcal{C})} \mathrm{End}(\mathbf{1}_l)$. We fix a representative X_i for each isomorphism class $i \in \mathrm{Irr}(\mathcal{C})$; by abuse of language, we will frequently use the same letter i for both a simple object and its isomorphism class. Rigidity gives us an involution $-^*$ on $\mathrm{Irr}(\mathcal{C})$; it is known that $l^* = l$ for $l \in \mathrm{Irr}_0(\mathcal{C})$. For $l \in \mathrm{Irr}_0(\mathcal{C})$, we may use the notation $\mathbf{1}_l := X_l$ to emphasize that it is part of the unit.

For $k, l \in \mathrm{Irr}_0(\mathcal{C})$, let $\mathcal{C}_{kl} := \mathbf{1}_k \otimes \mathcal{C} \otimes \mathbf{1}_l$, so that $\mathcal{C} = \bigoplus_{k,l \in \mathrm{Irr}_0(\mathcal{C})} \mathcal{C}_{kl}$. Any simple X_i is contained in exactly one of these $\mathcal{C}_{kl}$'s, or in other words, there are unique $k_i, l_i \in \mathrm{Irr}_0(\mathcal{C})$ such that $\mathbf{1}_{k_i} \otimes X_i \otimes \mathbf{1}_{l_i} \neq 0$. Since $\mathcal{C}_{kl}^* = \mathcal{C}_{lk}$, we have that $k_{i^*} = l_i$.

When $\mathcal{C}$ is spherical fusion, the categorical dimension is a scalar, defined as a trace, but here the non-simplicity of $\mathbf{1}$ and non-sphericality complicates things. To avoid confusion, denote by $\delta\colon V \to V^{**}$ the pivotal morphism. The *left dimension* of an object $V \in \mathrm{Obj}\,\mathcal{C}$ is the morphism

$$d_V^L := (\mathbf{1} \xrightarrow{\mathrm{coev}} V \otimes V^* \xrightarrow{\delta \otimes \mathrm{id}} V^{**} \otimes V^* \xrightarrow{\mathrm{ev}} \mathbf{1}) \in \mathrm{End}(\mathbf{1}).$$

Similarly, the *right dimension* of V is the morphism

$$d_V^R := (\mathbf{1} \xrightarrow{\mathrm{coev}} {}^*V \otimes V \xrightarrow{\mathrm{id} \otimes \delta^{-1}} {}^*V \otimes {}^{**}V \xrightarrow{\mathrm{ev}} \mathbf{1}) \in \mathrm{End}(\mathbf{1}).$$

Note that these are *vectors* and not scalars, since $\mathbf{1}$ may not be simple. It is easy to see that $d_V^R = d_{V^*}^L = d_{{}^*V}^L$. When $\mathcal{C}$ is spherical, we will drop the superscripts.

When $V = X_i$ is simple, we can interpret its left and right dimensions as scalars as follows. We have $X_i \in \mathcal{C}_{k_i l_i}$, so $\mathrm{Hom}(\mathbf{1}, X_i \otimes X_i^*) = \mathrm{Hom}(\mathbf{1}, \mathbf{1}_{k_i} \otimes X_i \otimes X_i^*) \simeq \mathrm{Hom}(\mathbf{1}_{k_i}, X_i \otimes X_i^*)$, and likewise $\mathrm{Hom}(X_i \otimes X_i^*, \mathbf{1}) \simeq \mathrm{Hom}(X_i \otimes X_i^*, \mathbf{1}_{k_i})$, so $d_{X_i}^L$ factors through $\mathbf{1}_{k_i}$, and hence we may interpret $d_{X_i}^L$ as an element of $\mathrm{End}(\mathbf{1}_{k_i}) \cong \mathbf{k}$. Similarly, $d_{X_i}^R$ may be interpreted as an element of $\mathrm{End}(\mathbf{1}_{l_i}) \cong \mathbf{k}$. We denote these scalar dimensions by d_i^L, d_i^R, and fix square roots such that $\sqrt{d_i^L} = \sqrt{d_{i^*}^R}$. The dimensions of simple objects are nonzero.

The *dimension* of $\mathcal{C}_{kl}$ is the sum

$$\mathcal{D} := \sum_{i \in \mathrm{Irr}(\mathcal{C}_{kl})} d_i^R d_i^L. \tag{A.1}$$

By [14, Proposition 2.17], this is the same for all pairs $k, l \in \mathrm{Irr}_0(\mathcal{C})$, and by [14, Theorem 2.3], they are nonzero.

We define functors $\mathcal{C}^{\boxtimes n} \to \mathcal{V}ec$ by

$$\langle V_1, \dots, V_n \rangle = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \cdots \otimes V_n), \tag{A.2}$$

$$\begin{aligned} \langle V_1, \dots, V_n \rangle_l &= \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_l, V_1 \otimes \cdots \otimes V_n) \\ &\simeq \langle \mathbf{1}_l, V_1, \dots, V_n \rangle \quad \text{for } l \in \mathrm{Irr}_0(\mathcal{C}), \end{aligned} \tag{A.3}$$

for any collection $V_1, \dots, V_n$ of objects of $\mathcal{C}$. Clearly,

$$\langle V_1, \dots, V_n \rangle = \textstyle\bigoplus_l \langle V_1, \dots, V_n \rangle_l .$$

Note that the pivotal structure gives functorial isomorphisms

$$z\colon \langle V_1, \dots, V_n \rangle \simeq \langle V_n, V_1, \dots, V_{n-1} \rangle \tag{A.4}$$

such that $z^n = \mathrm{id}$ (see [5, Section 5.3]); thus, up to a canonical isomorphism, the space $\langle V_1, \dots, V_n \rangle$ only depends on the cyclic order of $V_1, \dots, V_n$. In general, z does not preserve the direct sum decomposition of $\langle V_1, \dots, V_n \rangle$ above. For example, for a simple $X_i \in \mathcal{C}_{k_i l_i}$, we have $z\colon \langle X_i, X_i^* \rangle_{k_i} \simeq \langle X_i^*, X_i \rangle_{l_i}$.

We will commonly use graphic presentation of morphisms in a category, representing a morphism $W_1 \otimes \cdots \otimes W_m \to V_1 \otimes \cdots \otimes V_n$ by a diagram with m strands at the top, labeled by $W_1, \dots, W_m$, and n strands at the bottom, labeled $V_1, \dots, V_n$. (Note: this differs from the convention in many other papers!) We will allow diagrams with oriented strands, using the convention that a strand labeled by V is the same as the strands labeled by V^* with opposite orientation (suppressing isomorphisms $V \simeq V^{**}$).

We will show a morphism $\varphi \in \langle V_1, \dots, V_n \rangle$ by a round circle labeled by φ with outgoing edges labeled $V_1, \dots, V_n$ in counter-clockwise order, as shown in Figure 4. By (A.4) and the fact that $z^n = \mathrm{id}$, this is unambiguous. We will draw a morphism $\varphi \in \langle V_1, \dots, V_n \rangle_l$ by a semicircle labeled by φ and l as shown in Figure 4; in contrast with a circular node, a semicircle imposes a strict ordering on the outgoing legs, not just a cyclic ordering.

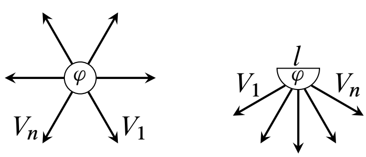


Figure 4. Labeling of colored graphs

We have a natural composition map

$$\langle V_1, \dots, V_n, X\rangle \otimes \langle X^*, W_1, \dots, W_m\rangle \to \langle V_1, \dots, V_n, W_1, \dots, W_m\rangle,$$
$$\varphi \otimes \psi \mapsto \varphi \underset{X}{\circ} \psi = \mathrm{ev}_{X^*} \circ (\varphi \otimes \psi), \tag{A.5}$$

where $\mathrm{ev}_{X^*}\colon X \otimes X^* \to \mathbf{1}$ is the evaluation morphism (the pivotal structure is suppressed).

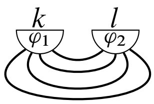
Repeated applications of the composition map above gives us a non-degenerate pairing

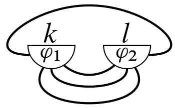
$$\langle V_1, \dots, V_n\rangle \otimes \langle V_n^*, \dots, V_1^*\rangle \to \mathrm{End}(\mathbf{1}). \tag{A.6}$$

More precisely, when restricted to the subspaces,

$$\langle V_1, \dots, V_n\rangle_k \otimes \langle V_n^*, \dots, V_1^*\rangle_l \to \mathrm{End}(\mathbf{1}), \tag{A.7}$$

the pairing is 0 if $k \neq l$, and is non-degenerate if $k = l$. The pairing is illustrated below for $\varphi_1 \in \langle V_1, \dots, V_n\rangle_k, \varphi_2 \in \langle V_n^*, \dots, V_1^*\rangle_l$:

$(\varphi_1, \varphi_2) =$ 

if $\mathcal{C}$ not spherical $\not\equiv$ 

$= (z^{-1} \cdot \varphi_1, z \cdot \varphi_2)$.

Thus, we have functorial isomorphisms

$$\langle V_1, \dots, V_n\rangle^* \simeq \langle V_n^*, \dots, V_1^*\rangle. \tag{A.8}$$

When $\mathcal{C}$ is spherical, this pairing is compatible with the cyclic permutations (A.4), in the sense that $(\varphi_1, \varphi_2) = (z \cdot \varphi_1, z^{-1} \cdot \varphi_2)$. Compatibility fails when $\mathcal{C}$ is not spherical; for example, it is easy to see that for $\varphi_1 = \varphi_2 = \mathrm{coev}_{X_i} \in \langle X_i, X_i^*\rangle$, one has $(\varphi_1, \varphi_2) = d_i^L$, while for $z \cdot \varphi_1 = z^{-1} \cdot \varphi_2 = \mathrm{coev}_{X_i^*} \in \langle X_i^*, X_i\rangle$, one has instead $(z \cdot \varphi_1, z^{-1}\varphi_2) = d_i^R$.

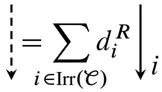
Lemma A.1. *For $\varphi \in \langle V_1, \dots, V_n\rangle_l, \varphi' \in \langle V_n^*, \dots, V_1^*\rangle_l, \psi \in \langle W_n^*, \dots, W_1^*\rangle_l$, and $f \in \mathrm{Hom}(V_1 \otimes \dots \otimes V_n, W_1 \otimes \dots \otimes W_n)$, we have*

$$(\varphi, \varphi') = (\varphi', \varphi), \tag{A.9}$$
$$(f \circ \varphi_1, \varphi_2) = (\varphi_1, f^* \circ \varphi_2). \tag{A.10}$$

Proof. Straightforward from definitions. ∎

We will make two additional conventions related to the graphic presentation of morphisms.

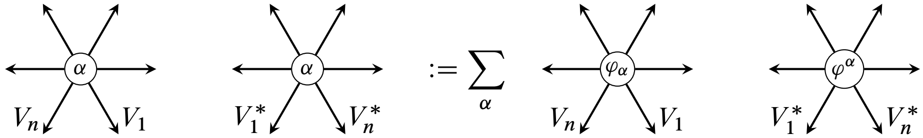
Notation A.2. A dashed line in the picture stands for the sum of all colorings of an edge by simple objects i, each taken with coefficient d_i^R:



(A.11)

When $\mathcal{C}$ is spherical, the orientation of such a dashed line is irrelevant.

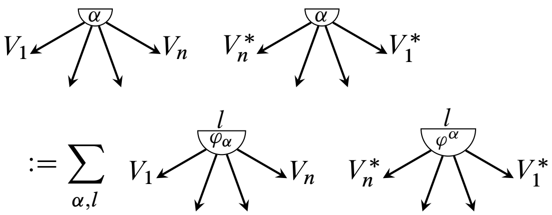
Notation A.3. Let $\mathcal{C}$ be spherical. If a figure contains a pair of circles, one with outgoing edges labeled $V_1, \dots, V_n$ and the other with edges labeled $V_n^*, \dots, V_1^*$, and the vertices are labeled by the same letter α (or β, or ...), then it will stand for summation over the dual bases:



(A.12)

where $\varphi_\alpha \in \langle V_1, \dots, V_n \rangle$, $\varphi^\alpha \in \langle V_n^*, \dots, V_1^* \rangle$ are dual bases with respect to pairing (A.6).

When $\mathcal{C}$ is not spherical, the pairing is no longer compatible with z from (A.4), so such notation can only make sense with semicircles:

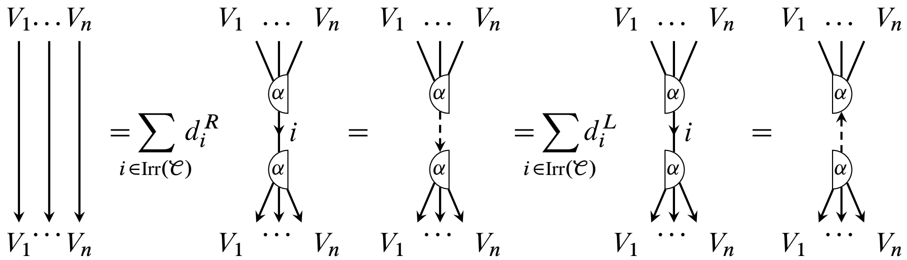


(A.13)

where $\varphi_\alpha \in \langle V_1, \dots, V_n \rangle_l$, $\varphi^\alpha \in \langle V_n^*, \dots, V_1^* \rangle_l$ are dual bases with respect to the pairing (A.6).

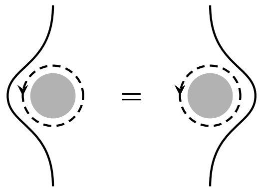
The following lemma illustrates the use of the notation above.

Lemma A.4. *For any $V_1, \dots, V_n \in \mathcal{C}$, we have*



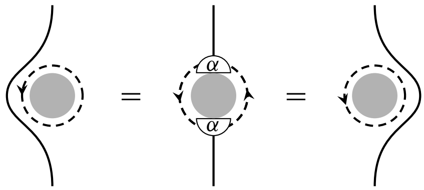
The proof of this lemma is straightforward: first show it for a simple X, then for direct sums; the interested reader can find a proof for a spherical $\mathcal{C}$ in [20].

Lemma A.5. *The following is a generalization of the "sliding lemma":*



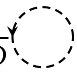
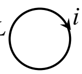
These relations hold regardless of the contents of the shaded region.

Proof.

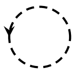
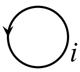


where we use Lemma A.4 in the equalities. See also [20, Corollary 3.5]. Note this trick does not work when the circle is oriented the other way (unless of course if $\mathcal{C}$ is spherical). ∎

Lemma A.6. *One has*

$$\frac{1}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}$$

$$= \mathrm{id}_{\mathbf{1}} = \frac{1}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}} \sum_{i\in\operatorname{Irr}(\mathcal{C})} d_i^L$$


Proof. Let $\operatorname{Irr}^{kl} = \operatorname{Irr}(\mathcal{C}_{kl})$, and let $\operatorname{Irr}^{k*} := \bigcup_l \operatorname{Irr}(\mathcal{C}_{kl})$, i.e., the set of simple objects X_i such that $\mathbf{1}_k \otimes X_i = X_i$. Then

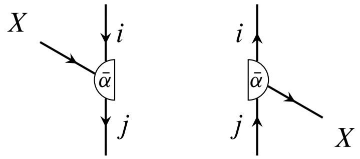

$$= \sum_{k\in\operatorname{Irr}_0(\mathcal{C})} \sum_{i\in\operatorname{Irr}^{k*}(\mathcal{C})} d_i^R$$


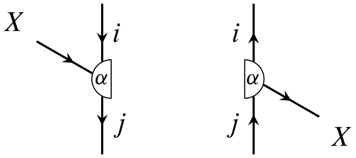
$$= \sum_{k\in\operatorname{Irr}_0(\mathcal{C})} \sum_{i\in\operatorname{Irr}^{k*}} d_i^R d_i^L \,\mathrm{id}_{\mathbf{1}_k}$$
$$= \sum_{k\in\operatorname{Irr}_0(\mathcal{C})} \sum_{l\in\operatorname{Irr}_0(\mathcal{C})} \sum_{i\in\operatorname{Irr}^{kl}} d_i^R d_i^L \,\mathrm{id}_{\mathbf{1}_k} = \sum_{k\in\operatorname{Irr}_0(\mathcal{C})} \sum_{l\in\operatorname{Irr}_0(\mathcal{C})} \mathcal{D}\,\mathrm{id}_{\mathbf{1}_k}$$
$$= |\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}\,\mathrm{id}_{\mathbf{1}}\,.$$

The second equality is proved in a similar manner. ∎

The following lemma is used to prove that Figure 1 is a half-braiding and the functor G in the proof of Theorem 3.10 respects the composition:

Lemma A.7. *For $X \in \operatorname{Obj} \mathcal{C}$, define $\Gamma_X \in \operatorname{Hom}(X \otimes X_i, X_j) \otimes \operatorname{Hom}(X_i^*, X_j^* \otimes X)$:*

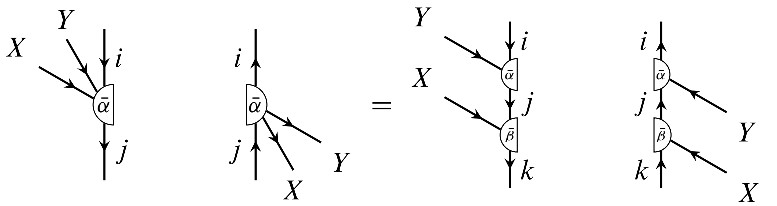
$\Gamma_X :=$ 

$:= \sum_{i,j \in \operatorname{Irr}(\mathcal{C})} \sqrt{d_i^R} \sqrt{d_j^R}$ 

(A.14)

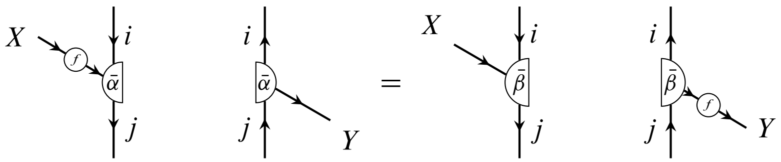
Γ_X satisfies the following properties:

(1) *Γ_{-} respects tensor products:*



(A.15)

(2) *Γ_X is natural in X: for $f : X \to Y$,*



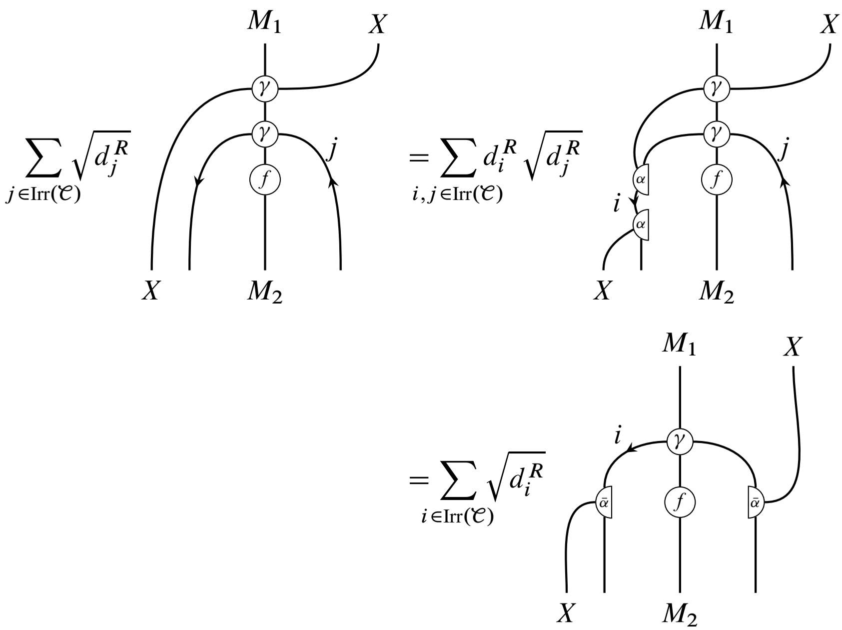
(A.16)

Proof. The second property follows from Lemma A.1. The first property follows from using Lemma A.1 to "pull" $\bar{\alpha}$ through $\bar{\beta}$, then use Lemma A.4 to contract the strand k. ∎

Finally, we give a proof of Theorem 3.3:

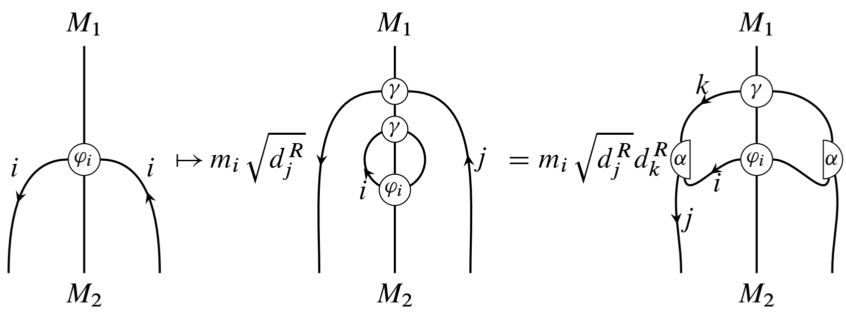
Proof of Theorem 3.3. This is essentially the same as when $\mathcal{C}$ is spherical, but we provide it to assuage any doubts that the non-sphericality, manifested in requiring semicircular morphisms α, does not lead to problems.

Let us check that the morphism on the left side of (3.3) intertwines half-braidings:

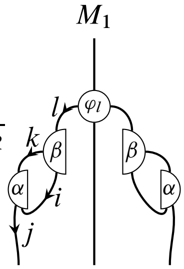


In the first equality, we use Lemma A.4; in the second equality, we use the naturality of γ to pull the top α to the right, and absorb the factor $\sqrt{d_i^R}\sqrt{d_j^R}$ into α to get $\bar{\alpha}$.

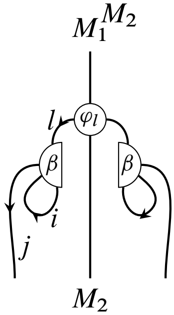
Next we check that if we apply (3.2), then (3.3) is the identity map. Let $m_i = 1$ if $i \in \mathrm{Irr}_0(\mathcal{C})$, 0 otherwise. In the following diagrams, we implicitly sum lowercase Latin letters over $\mathrm{Irr}(\mathcal{C})$. Then the composition is the map



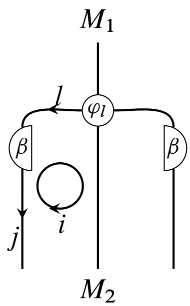
$$= m_i \sqrt{d_j^R} d_k^R \sqrt{d_i^R} \sqrt{d_l^R}$$



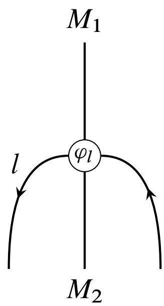
$$= m_i \sqrt{d_j^R} \sqrt{d_i^R} \sqrt{d_l^R}$$



$$= m_i \sqrt{d_j^R} \sqrt{d_i^R} \sqrt{d_l^R}$$



$=$



The first equality is the same as the previous computation. The second equality uses the fact that $\sum \varphi_i$ intertwines half-braidings, so that we "pull" the k strand through φ_i. The third equality comes from "pulling" α through β. The fourth equality comes from "pulling" the i loop through β. Finally, for the last equality, we observe that (1) only $j = k$ terms in the sum contribute, and so we have a d_j^R coefficient, and we may apply Lemma A.4; (2) since $d_i^R = 1$ for $i \in \mathrm{Irr}_0(\mathcal{C})$,

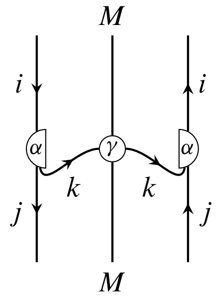
$$\sum_i m_i \sqrt{d_i^R} \bigcirc_i = \sum_{l \in \mathrm{Irr}_0(\mathcal{C})} \mathrm{id}_{\mathbf{1}_l} = \mathrm{id}_{\mathbf{1}} . \qquad \blacksquare$$

The following is a lemma used in Section 3:

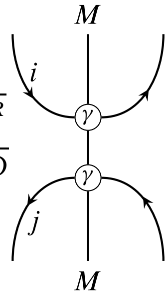
Lemma A.8. *Let* $(M,\gamma) \in \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$. *The morphism*

$$P_{(M,\gamma)} := \frac{1}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}} G\Big(\sum d_i^R \gamma_{X_i}\Big)$$

$$= \sum_{i,j,k \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_i^R}\sqrt{d_j^R} d_k^R}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}$$

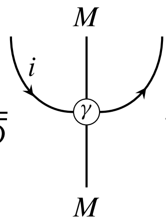


$$= \sum_{i,j \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_i^R}\sqrt{d_j^R}}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}} \tag{A.17}$$



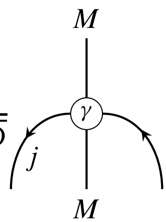
is a projection in $\operatorname{End}_{\mathcal{Z}_{\mathcal{C}}(\mathcal{M})}(I(M))$. *Furthermore, it can be written as a composition* $P_M = \widehat{P}_M \circ \check{P}_M$, *where*

$$\check{P}_M := \sum_{i \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_i^R}}{\sqrt{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}}$$



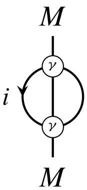
,

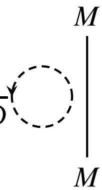
$$\widehat{P}_M := \sum_{j \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_j^R}}{\sqrt{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}} \tag{A.18}$$



such that $\check{P}_M \circ \widehat{P}_M = \mathrm{id}_{(M,\gamma)}$, *thus exhibiting* (M,γ) *as a direct summand of* $I(M)$.

Proof. The second equality in (A.17) follows from pulling α through γ and using Lemma A.4. $\widehat{P}_M$ was shown to be a morphism in $\operatorname{Hom}_{\mathcal{Z}_{\mathcal{C}}(\mathcal{M})}((M,\gamma), I(M))$ in the proof of Theorem 3.3, and one shows $\check{P}_M \in \mathrm{Hom}_{Z_{\mathcal{C}}(\mathcal{M})}(I(M), (M, \gamma))$ in a similar fashion. The following computation shows that $\check{P}_M \circ \hat{P}_M = \mathrm{id}_{(M,\gamma)}$:

$$\check{P}_M \circ \hat{P}_M = \sum_{i \in \mathrm{Irr}(\mathcal{C})} \frac{d_i^R}{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}$$ 

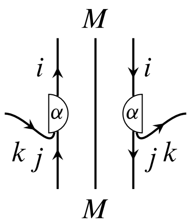
$$= \frac{1}{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}$$ 

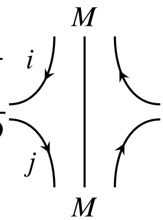
$$= \mathrm{id}_{(M,\gamma)}.$$

The second equality comes from "pulling" the loop j out to the left, and the last equality follows from Lemma A.6. ■

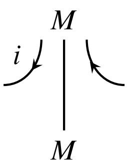
The following is a similar result, used in the proof of Theorem 9.6:

Lemma A.9. *Let $M \in \mathcal{M}$. The morphism*

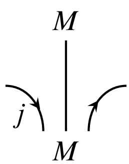
$$P'_M := \sum_{i,j,k \in \mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}\sqrt{d_j^L}\, d_k^R}{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}$$ 

$$= \sum_{i,j \in \mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}\sqrt{d_j^L}}{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}$$  (A.19)

is a projection in $\mathrm{End}_{\mathrm{hTr}(\mathcal{M})}(\bigoplus X_i \triangleright M \triangleleft X_i^*)$. *Furthermore, it can be written as a composition* $P'_M = \hat{P}'_M \circ \check{P}'_M$, *where*

$$\check{P}'_M := \sum_{i \in \mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}}{\sqrt{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}}$$  ,

(A.20)

$$\hat{P}'_M := \sum_{j \in \mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_j^L}}{\sqrt{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}}$$ 

such that $\check{P}'_M \circ \hat{P}'_M = \mathrm{id}_M$*, thus as objects in* $\mathrm{Kar}(\mathrm{hTr}(\mathcal{M}))$*, we have*

$$M \simeq \Big(\bigoplus X_i \triangleright M \triangleleft X_i^*, P'_M\Big).$$

Proof. Essentially the same as Lemma A.8. (Note the use of left dimensions d_i^L instead of right dimensions d_i^R.) ∎

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