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A unification of the ADO and colored Jones polynomials of a knot

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Abstract. In this paper we prove that the family of colored Jones polynomials of a knot in S^3 determines the family of ADO polynomials of this knot. More precisely, we construct a two variables knot invariant unifying both the ADO and the colored Jones polynomials. On the one hand, the first variable q can be evaluated at $2r$ roots of unity with $r \in \mathbb{N}^*$ and we obtain the ADO polynomial over the Alexander polynomial. On the other hand, the second variable A evaluated at $A = q^n$ gives the colored Jones polynomials. From this, we exhibit a map sending, for any knot, the family of colored Jones polynomials to the family of ADO polynomials. As a direct application of this fact, we will prove that every ADO polynomial is holonomic and is annihilated by the same polynomial as of the colored Jones function. The construction of the unified invariant will use completions of rings and algebra. We will also show how to recover our invariant from Habiro's quantum $\mathfrak{sl}_2$ completion studied by Habiro in [J. Pure Appl. Algebra 211 (2007), 265–292], showing that it corresponds in fact to the two-variable colored Jones invariant defined by Habiro in [Invent. Math. 171 (2008), 1–81].

1. Introduction

Main results. In [1], Akutsu, Deguchi, and Ohtsuki gave a generalisation of the Alexander polynomial, building a colored link invariant at each root of unity. These *ADO invariants,* also known as *colored Alexander's polynomials,* can be obtained as the action on 1-1 tangles of the usual ribbon functor on some representation category of a version of quantum $\mathfrak{sl}_2$ at roots of unity (see [4, 8]). On the other hand, we have the colored Jones polynomials, a family of invariants obtained by taking the usual ribbon functor of quantum $\mathfrak{sl}_2$ on finite-dimensional representations. It is known ([5]), that, given the ADO polynomials of a knot, one can recover the colored Jones polynomials of this knot. One of the results of the present paper is to show the other way around: given the Jones polynomials of a knot, one can recover the ADO polynomials of this knot.

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We denote by $\mathrm{ADO}_r(A, \mathcal{K})$ the ADO invariant at $2r$ root of unity seen as a polynomial in the variable A. Also, we denote by $J_n(q, \mathcal{K})$ the n-th colored Jones polynomial in the variable q and by $A_{\mathcal{K}}(A)$ the Alexander polynomial in the variable A.

Result 1. *There is a well-defined map such that, for any knot $\mathcal{K}$ in S^3,*

$$\{J_n(q, \mathcal{K})\}_{n\in\mathbb{N}^*} \mapsto \{\mathrm{ADO}_r(A, \mathcal{K})\}_{r\in\mathbb{N}^*}.$$

(Detailed version: Theorem 69.)

The above result is a consequence of the construction of a unified knot invariant containing both the ADO polynomials and the colored Jones polynomials of the knot. This unified invariant is in fact equal to the two-variable colored Jones invariant defined by Habiro in [13] and answer positively to the conjectures of its behaviour at roots of unity. Briefly put, we obtain it by looking at the action of the universal invariant (see [14, 15, 17]) on some Verma module with coefficients in some ring completion. For the sake of simplicity, let us state the result for 0-framed knots.

Result 2. *In some ring completion of $\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$ equipped with suitable evaluation maps, for any 0-framed knot $\mathcal{K}$ in S^3, there exists a well-defined knot invariant $F_\infty(q, A, \mathcal{K})$ such that*

$$F_\infty(\zeta_{2r}, A, \mathcal{K}) = \frac{\mathrm{ADO}_r(A, \mathcal{K})}{A_{\mathcal{K}}(A^{2r})}, \quad F_\infty(q, q^n, \mathcal{K}) = J_n(q^2, \mathcal{K}).$$

(Detailed version: Theorem 63 and Corollary 59.)

A visual representation of the relationship between all these invariants is given in Figure 1.

Let us set $J_\bullet(q^2, \mathcal{K}) = \{J_n(q, \mathcal{K})\}_{n\in\mathbb{N}^*}$ and call it *colored Jones function of $\mathcal{K}$*.

The holonomy of the unified invariant and of the ADO polynomials will follow as a simple application of the two previous results and of the q-holonomy of the colored Jones function as shown in [6]. Mainly, there are two operators Q and E on the set of discrete function over $\mathbb{Z}[q^{\pm 1}]$ that forms a quantum plane and for any knot $\mathcal{K}$ and there is a two-variable polynomial $\alpha_{\mathcal{K}}$ such that $\alpha_{\mathcal{K}}(Q, E)J_\bullet(q^2, \mathcal{K}) = 0$. We say that the colored Jones function is *q-holonomic*.

This paper gives a proof that the same polynomial $\alpha_{\mathcal{K}}$, in some similar operators as Q and E, annihilates the unified invariant $F_\infty(q, A, \mathcal{K})$ and, at roots of unity, annihilates $\mathrm{ADO}_r(A, \mathcal{K})$.

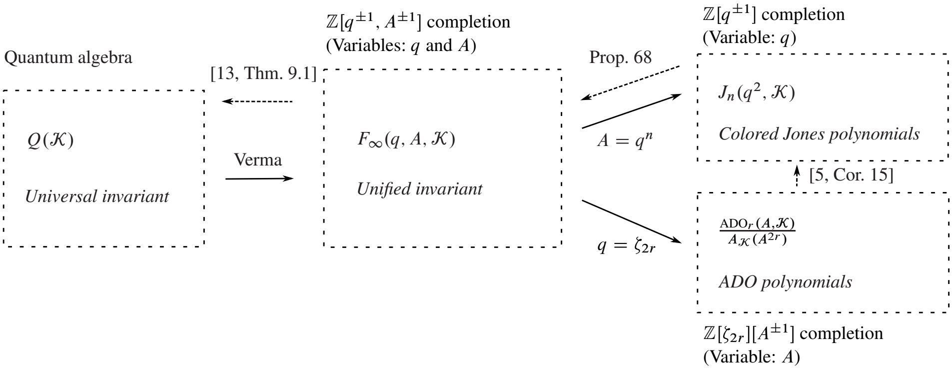


Figure 1. Visual representation of the unified knot invariant.

Result 3. *For any* 0*-framed knot* $\mathcal{K}$ *and any* $r \in \mathbb{N}^*$*:*

- *the unified invariant* $F_\infty(q, A, \mathcal{K})$ *is* q*-holonomic;*
- *the ADO invariant* $\mathrm{ADO}_r(A, \mathcal{K})$ *is* ζ_{2r}*-holonomic.*

Moreover, they are annihilated by the same polynomial as of the colored Jones function. (Detailed version: Theorems 71 and 73.)

Remark 1. Keep in mind that these results cover only the case of a knot $\mathcal{K}$ in S^3.

Summary of the paper. A way to build a unified element for ADO invariants is to do it by hand. First, one can explicit a formula for the ADO invariant at a $2r$ root of unity by decomposing it as a sum of what we will call *state diagrams*. This explicit formula will allow us to see what are the obstructions to unify the invariants: first, it will depend on the root of unity ζ_{2r}; secondly the range of the sums coming from the action of the truncated R-matrix will depend on the order $2r$ of the root of unity. The first obstruction is easy to overcome, since taking a formal variable q instead of each occurrence of ζ_{2r} will do the trick. But, for the second one, one could ask that the ranges go to infinity, and this will bring some convergence issues. A way to make these sums convergent is to use a completion of the ring $\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$ denoted by $\widehat{R}^{\widehat{I}}$. This will allows us to define a good candidate for the unification.

But then, we will have to check that this element contains the ADO invariants. We will show that, at each root of unity of order $2r$ with $r \in \mathbb{N}^*$, one can define an evaluation map that evaluates q in ζ_{2r}, and that the result can be factorized into a product of an invertible element of the complete ring and the ADO invariant.

So, we will get an element containing ADO invariants, but the way we built this element depends on the chosen diagram of the knot. A way to prove that this element is really a knot invariant itself is to recover it with a more advance machinery: the universal invariant of a knot. The universal invariant of a knot was introduced in [14, 15]; the construction can also be found in [17]. It is a knot invariant and an element of the h-adic version of quantum $\mathfrak{sl}_2$, we will use this fact to construct an integral subalgebra in which the universal invariant of a 0-framed knot will lie. The integrality of the subalgebra will allow us to build a Verma module of it whose coefficients will lie in $\widehat{R}^{\widehat{I}}$, and on which the scalar action of the universal invariant gives our unified element. A corollary will be that the unified element is a knot invariant.

Completions were studied by Habiro in [12]. For the sole purpose of the factorization at roots of unity, we had to use a different completion than the ones mentioned in [12]. But, as we will see, we can also recover our unified invariant from his algebraic setup. Moreover, the unified invariant corresponds to its two-variable colored Jones invariant defined in [13].

Once we have this connection between quantum $\mathfrak{sl}_2$, the two-variable colored Jones invariant and this unified invariant, we can henceforth relate it also to the colored Jones polynomials. This will allow us to use the Melvin–Morton–Rozansky conjecture proved by Bar-Natan and Garoufalidis in [2] in order to get some information on the factorization at roots of unity: briefly put, the unified invariant factorize at root of unity as ADO polynomial over the Alexander polynomial. This theorem answers positively to [13, Conjecture 7.5 and subsequent paragraph] about the two-variable colored Jones invariant at roots of unity.

Now, we have a unified invariant for both the ADO polynomials and the colored Jones polynomials; the maps recovering them are also well understood. This will allow us to prove that, given the colored Jones polynomials, one may recover the ADO polynomials.

From the fact that the colored Jones polynomials recovers the unified invariant and from the factorisation at roots of unity, we will prove that the unified invariant and ADO polynomials follow the same holonomic rule as of the colored Jones function (see [6]). In the same time this paper was made, Brown, Dimofte Garoufalidis, and Geer got a more general result covering the case of links in [3, Theorem 4.3].

We will also see that the unified invariant is an integral version of the h-adic loop expansion of the colored Jones function and remark that, even if it is not clear in general if it's a power series, it has similar properties as the power series invariant conjectured by Gukov and Manolescu in [9, Conjectures 1.5 and 1.6] .

Finally, we will give some computations of the unified invariant and its factorization at roots of unity, showing how the inverse of the Alexander polynomial appears.

Nota bene. In this article, any knots and links are in S^3 and supposed oriented and framed. We will use the term link or knot invariant to refer to framed oriented link or knot invariants.

2. The ADO invariant for knots

2.1. Definition of the ADO invariants from quantum algebra

We will expose in this section how to obtain ADO invariants for links [1], also called *colored Alexander's polynomials,* from a non-semi-simple category of module over an unrolled version of $U_q(\mathfrak{sl}_2)$. A more detailed and thorough construction can be found in [4, 8].

For any variable q, we set

$$\{n\} = q^n - q^{-n}, \quad [n] = \frac{\{n\}}{\{1\}}, \{n\}! = \prod_{i=1}^{n}\{i\}, \quad [n]! = \prod_{i=1}^{n}[i],$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[n-k]![k]!}, \quad \text{with convention} \begin{bmatrix} n \\ k \end{bmatrix}_q = 0 \text{ if } n < 0.$$

In order to define ADO invariants for knots, and for the sake of simplicity, in this section q will be an even root of unity.

Definition 2. Let $q = e^{\frac{i\pi}{r}} = \zeta_{2r}$ root of unity. We work with an "unrolled" version of $U_{\zeta_{2r}}(\mathfrak{sl}_2)$ denoted by $U^H_{\zeta_{2r}}(\mathfrak{sl}_2)$ and defined as follow:

- GENERATORS: E, F, K, K^{-1}, H;
- RELATIONS:

$$KK^{-1} = K^{-1}K = 1, \quad KE = \zeta_{2r}^2 EK, \quad KF = \zeta_{2r}^{-2} FK, \quad [E, F] = \frac{K - K^{-1}}{\zeta_{2r} - \zeta_{2r}^{-1}},$$
$$KH = HK, \qquad [H, E] = 2E, \quad [H, F] = -2F, \qquad E^r = F^r = 0.$$

This algebra has a Hopf algebra structure:

$$\begin{aligned}
&\Delta(E) = 1 \otimes E + E \otimes K, && \varepsilon(E) = 0, && S(E) = -EK^{-1},\\
&\Delta(F) = K^{-1} \otimes F + F \otimes 1, && \varepsilon(F) = 0, && S(F) = -KF,\\
&\Delta(H) = 1 \otimes H + H \otimes 1, && \varepsilon(H) = 0, && S(H) = -H,\\
&\Delta(K) = K \otimes K, && \varepsilon(K) = 1, && S(K) = K^{-1},\\
&\Delta(K^{-1}) = K^{-1} \otimes K^{-1}, && \varepsilon(K^{-1}) = 1, && S(K^{-1}) = K.
\end{aligned}$$

Now, we can look at some category of finite-dimensional representation of this algebra and endow it with a ribbon structure.

Definition 3. Let *Rep* be the category of finite-dimensional $U^H_{\zeta_{2r}}(\mathfrak{sl}_2)$-modules such that

(1) the action of H is diagonalizable;

(2) the action of K and of ζ_{2r}^H are the same.

Here, $\zeta_{2r}^{H}\colon V \to V$ is defined by $\zeta_{2r}^{H}.v = \zeta_{2r}^{\lambda} v$ if v is a eigenvector of H with eigenvalue λ.

Proposition 4. *The irreducible representations of* $U_{\zeta_{2r}}^{H}(\mathfrak{sl}_2)$ *are*

- V_α *for* $\alpha \in (\mathbb{C} - \mathbb{Z}) \cup r\mathbb{Z}$ *and*
- S_i *for* $i \in \{0, \ldots, r-2\}$,

where S_i *is the highest weight module of weight* i *and dimension* $i+1$, *and* V_α *is the highest weight module of weight* $\alpha + r - 1$ *and dimension* r.

Definition 5. In V_α, we say that v *has weight level* n if $H.v = (\alpha + r - 1 - 2n)v$.

We can endow *Rep* with a ribbon structure by giving the action of a R-matrix and a ribbon element. For $V, W \in$ *Rep*, we define

$$\zeta_{2r}^{\frac{H \otimes H}{2}} : V \otimes W \to V \otimes W$$

by

$$\zeta_{2r}^{\frac{H \otimes H}{2}}.v \otimes w = \zeta_{2r}^{\frac{\lambda\beta}{2}} v \otimes w$$

if $H.v = \lambda v$ and $H.w = \beta w$. We set

$$F^{(n)} = \frac{\{1\}^n F^n}{[n]!} \quad \text{for } 0 \leq n < r-1.$$

Proposition 6. *The element*

$$R = \zeta_{2r}^{\frac{H \otimes H}{2}} \sum_{n=0}^{r-1} \zeta_{2r}^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}$$

is an R*-matrix whose action is well defined on Rep and it's inverse is*

$$R^{-1} = \Big(\sum_{n=0}^{r-1} (-1)^n \zeta_{2r}^{-\frac{n(n-1)}{2}} E^n \otimes F^{(n)}\Big) \zeta_{2r}^{-\frac{H \otimes H}{2}}.$$

Proposition 7. K^{1-r} *is a pivotal element for* $U_{\zeta_{2r}}^{H}(\mathfrak{sl}_2)$ *compatible with the braiding.*

We can now take the usual ribbon functor RT in order to obtain a link invariant, but on the V_α it will be 0 (since the quantum trace is 0). Hence, we need to be more subtle in order to retrieve some information.

On irreducible representations, a 1-1 tangle can be seen as a scalar. If L is a link obtained by closure of a 1-1 tangle T, we set $\mathrm{RT}(T)v_0 = \mathrm{ADO}(T)v_0$ where v_0 is a highest weight vector. Notice that it depends on the 1-1 tangle T chosen. In order to have a link invariant, we must multiply it by a "modified trace."

We set

$$\{\alpha\}_{\zeta_{2r}} = \zeta_{2r}^{\alpha} - \zeta_{2r}^{\alpha}, \quad \{\alpha + k\}_{\zeta_{2r}} = \zeta_{2r}^{\alpha+k} - \zeta_{2r}^{-\alpha-k},$$

$$\{\alpha; n\}_{\zeta_{2r}} = \prod_{i=0}^{n-1} \{\alpha - i\}_{\zeta_{2r}}.$$

Proposition 8. *If* L *a link and* T *is any* 1-1 *tangle whose closure is* L *such that the open component is colored with* V_α*, set*

$$d(\alpha) = \frac{\{\alpha\}}{\{r\alpha\}};$$

then

$$\mathrm{ADO}'_r(L) := d(\alpha)\,\mathrm{ADO}_r(T)$$

is a framed oriented link invariant.

Although we do not have to specify α and obtain a polynomial in q^α, we cannot do the same for q. The root of unity q must be fixed in order to define the invariant, hence it is a natural question for one to ask how such invariants behave when the root of unity changes.

2.2. Useful form of the ADO invariant

From now on, we will only work with oriented framed knots. To see how the ADO polynomials behave when the root of unity changes, we will explicit a formula for the invariant using the ribbon functor on a diagram D of a knot.

Let $\mathcal{K}$ be a knot colored by $V_{\alpha-r+1}$ and T a 1-1 tangle whose closure is $\mathcal{K}$, since we are working with knots $\mathrm{ADO}_r(A, \mathcal{K}) := \mathrm{ADO}_r(T)$ is well defined, where A is the free variable ζ_{2r}^{α}. Let us study this element, by choosing a basis of $V_{\alpha-r+1}$ and computing the invariant with state diagrams.

Remark 9. $V_{\alpha-r+1}$ is generated by $v_0, v_1, \ldots, v_{r-1}$ where v_0 is a highest weight vector, and $v_i = \frac{F^{(i)}.v_0}{\{\alpha;i\}_{\zeta_{2r}}}$.

Proposition 10. *We have*

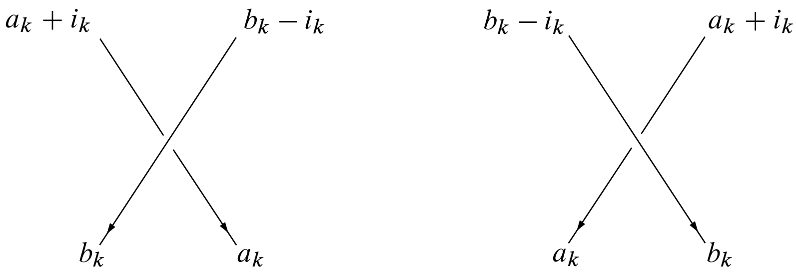
$$E.v_0 = 0, \qquad E.v_i = v_{i-1},$$

$$F.v_i = [i+1][\alpha - i]v_{i+1}, \qquad F^{(k)}.v_i = \begin{bmatrix} k+i \\ k \end{bmatrix}_{\zeta_{2r}} \{\alpha - i; k\}_{\zeta_{2r}} v_{k+i},$$

$$K.v_i = \zeta_{2r}^{\alpha-2i} v_i, \qquad \zeta_{2r}^{\frac{H\otimes H}{2}}.v_i \otimes v_j = \zeta_{2r}^{\frac{\alpha^2}{2}} \zeta_{2r}^{-(i+j)\alpha}, \zeta_{2r}^{2ij} v_i \otimes v_j.$$

Corollary 11. *We have* $\mathrm{ADO}_r(A,\mathcal{K}) \in \zeta_{2r}^{\frac{f\alpha^2}{2}}\mathbb{Z}[\zeta_{2r},\zeta_{2r}^{\pm\alpha}]$, *where* f *is the framing of the knot.*

More precisely, in order to calculate a useful form of this invariant one may look at *state diagram* of a knot. For any knot seen as a $(1,1)$ tangle, take a diagram D, label the top and bottom strands 0 and starting from the bottom strand, and label the strand after the k-th crossing encountered with the rule described in Figure 2. The resulting diagram is called a *state diagram* of D.



(a) Positive crossing. **(b)** Negative crossing.

Figure 2. The two possibilities for the k-th crossing in D.

Let $\mathcal{K}$ be a knot and D a diagram of the knot seen as a $(1,1)$ tangle. Suppose the diagram has N crossings. Now, for any state diagram of D we can associate an element

$$D_r(i_1,\dots,i_N) = \left(\prod_{j=1}^{S}\zeta_{2r}^{\pm(r-1)(\alpha-2\varepsilon_j)}\right)\prod_{k\in\mathrm{pos}}\zeta_{2r}^{\frac{i_k(i_k-1)}{2}}\begin{bmatrix}a_k+i_k\\ i_k\end{bmatrix}_{\zeta_{2r}}\{\alpha-a_k;i_k\}_{\zeta_{2r}}$$
$$\times\,\zeta_{2r}^{-(a_k+b_k)\alpha}\zeta_{2r}^{2(a_k+i_k)(b_k-i_k)}\prod_{k\in\mathrm{neg}}(-1)^{i_k}\zeta_{2r}^{-\frac{i_k(i_k-1)}{2}}$$
$$\times\begin{bmatrix}a_k+i_k\\ i_k\end{bmatrix}_{\zeta_{2r}}\{\alpha-a_k;i_k\}_{\zeta_{2r}}\zeta_{2r}^{(a_k+b_k)\alpha}\zeta_{2r}^{-2a_kb_k},$$

where $\mathrm{neg}\cup\mathrm{pos} = [|1,N|]$ and $k\in\mathrm{pos}$ if the k-th crossing of D is positive, else $k\in\mathrm{neg}$. a_k, b_k are the strands labels at the k-th crossing of the state diagram (see Figure 2), S is the number of + appearing in the diagram, and ε_j the strand label at the j-th or , the $\pm$ sign is positive if and negative if .

Remark 12. Note that the a_k and b_k appearing are defined in terms of i_j. You can find some examples of state diagrams in Section 5, Figures 7 and 8.

Proposition 13. *If D is a diagram of $\mathcal{K}$ seen as a* 1-1 *tangle, we have*

$$
\begin{aligned}
\mathrm{ADO}_r(A,\mathcal{K}) &= \zeta_{2r}^{\frac{f\alpha^2}{2}} \sum_{\bar{\imath}=0}^{r-1} D_r(i_1,\dots,i_N)\\
&= \zeta_{2r}^{\frac{f\alpha^2}{2}} \sum_{\bar{\imath}=0}^{r-1} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\pm(r-1)(\alpha-2\varepsilon_j)}\Big) \prod_{k\in\mathrm{pos}} \zeta_{2r}^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_{\zeta_{2r}}\\
&\quad\times \{\alpha-a_k;i_k\}_{\zeta_{2r}} \zeta_{2r}^{-(a_k+b_k)\alpha} \zeta_{2r}^{2(a_k+i_k)(b_k-i_k)} \prod_{k\in\mathrm{neg}} (-1)^{i_k}\\
&\quad\times \zeta_{2r}^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_{\zeta_{2r}} \{\alpha-a_k;i_k\}_{\zeta_{2r}} \zeta_{2r}^{(a_k+b_k)\alpha} \zeta_{2r}^{-2a_k b_k},
\end{aligned}
$$

where $\bar{\imath}=(i_1,\dots,i_N)$, N is the number of crossings, S the number of  $+$  *and f is the framing of the knot.*

Proof. Notice that $\zeta_{2r}^{\frac{f\alpha^2}{2}} D_r(i_1,\dots,i_N)$ is the element obtained by adding to the k-th crossing a coupon labeled with

$$
q^{\frac{H\otimes H}{2}} q^{\frac{i_k(i_k-1)}{2}} E^{i_k} \otimes F^{(i_k)}
$$

if positive and

$$
q^{\frac{-H\otimes H}{2}} q^{\frac{-i_k(i_k-1)}{2}} E^{i_k} \otimes F^{(i_k)}
$$

if negative. Then add a coupon to  labeled K^{r-1} and to  labeled K^{1-r}. We get an element of $U_{\zeta_{2r}}^{H}(\mathfrak{sl}_2)$, whose action on $v_0\in V_{\alpha+r-1}$, the highest weight vector, gives the element

$$
\zeta_{2r}^{\frac{f\alpha^2}{2}} D_r(i_1,\dots,i_N).
$$

Summing them over i_k for all k gives the ADO polynomial. ■

Now that we have an explicit formula, can we construct from it a suitable element that can be evaluated at roots of unity and recover the ADO invariants?

We have two main issues here. The first is that ζ_{2r} appears in the formula, so we will have to replace each occurrence with some variable q, in order to see it as a polynomial or a formal series. The second one is more difficult to solve: the action of the R-matrices makes appear sums that range to $r-1$, which depends on the order of the root of unity. A solution to this problem, as we will explicit it, is to let the sum range to infinity and define a ring in which such sums converge. Then we will see how to factorize the ADO invariant from this new unified form.

3. Unified form for ADO invariants of knots

The approach here will be to unify the invariants: using completions of rings and algebras, we will explicit an integral invariant in some variable q that can be evaluated at any root of unity, recovering ADO invariants defined previously.

In Section 3.1, will create the right setup to define a unified form inspired by the useful form of the ADO invariant in Proposition 13. Using a completion of the ring of integral Laurent polynomials in two variables q, A, we define a unified form $F_\infty(q, A, D)$ by taking the previous form of ADO, replacing the root of unity ζ_{2r} by q, replacing ζ_{2r}^α by A, and letting the truncated sums coming from the R-matrices action go to infinity. Note that at this point, the defined form is not a knot invariant, as it *a priori* depends on the diagram D of the knot.

In Section 3.2, will make the bridge between this new element and the ADO polynomials. By evaluating the unified form at roots of unity ζ_{2r} with $r \in \mathbb{N}^*$, we factor out the ADO invariant. We will then explicit a map sending the unified form of a knot to the corresponding ADO invariants. This will show that the ADO invariants are contained in the unified form and that we can recover them from it.

3.1. Ring completion for the unified form

Let us lay the groundwork for an unified form to exist. It must be a ring in which infinite sums previously mentioned converge.

Let $R = \mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$. We will construct a completion of that ring. For the sake of simplicity, we will use the notation $q^\alpha := A$ and use previous notation for quantum numbers. Keep in mind that, here, α is just a notation, not a complex number. We set

$$\{\alpha\}_q = q^\alpha - q^{-\alpha}, \quad \{\alpha + k\}_q = q^{\alpha+k} - q^{-\alpha-k},$$
$$\{\alpha; n\}_q = \prod_{i=0}^{n-1} \{\alpha - i\}_q.$$

Definition 14. Let I_n be the ideal of R generated by the set $\{\{\alpha + l; n\}_q, l \in \mathbb{Z}\}$.

Lemma 15. *I_n is generated by elements of the form $\{n; i\}\{\alpha; n - i\}$, $i \in \{0, \dots n\}$.*

Proof. The proof can be found in Habiro's article [12]. Replacing K (resp. K^{-1}) by q^α (resp. $q^{-\alpha}$) in [12, Proposition 5.1], one gets the proof of this lemma. ∎

We then have a projective system

$$\hat{I} : I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots .$$

From which we can define the completion of R, taking the projective limit.

Definition 16. Let

$$\widehat{R}^{\widehat{I}} = \varprojlim_n \frac{R}{I_n} = \Big\{(a_n)_{n\in\mathbb{N}^*} \in \prod_{i=1}^{\infty} \frac{R}{I_n} \;\Big|\; \rho_n(a_{n+1}) = a_n\Big\},$$

where

$$\rho_n\colon \frac{R}{I_{n+1}} \to \frac{R}{I_n}$$

is the projection map.

This completion is a bigger ring containing R:

Proposition 17. *The canonical projection maps induce an injective map $R \hookrightarrow \widehat{R}^{\widehat{I}}$.*

Proof. It is sufficient to prove that

$$\bigcap_{n\in\mathbb{N}^*} I_n = \{0\}.$$

Since $R = \mathbb{Z}[q^{\pm 1}][A^{\pm 1}]$, it is a Laurent polynomial ring. Let us denote by $\deg_q(x)$ and $\mathrm{val}_q(x)$ the degree and valuation of x in the variable q, respectively.

Let $f_k\colon \mathbb{Z}[q^{\pm 1}, A^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}]$, $A \mapsto q^k$. We have $f_k(I_n) \subset \{n\}!\mathbb{Z}[q^{\pm 1}]$ because I_n is generated by elements of the form $\{n;i\}\{\alpha;n-i\}$ that maps to $\{n;i\}\{k;n-i\}$, which is divisible by $\{n\}!$. Hence, if $x \in \bigcap_{n\in\mathbb{N}^*} I_n$, $f_k(x) \in \{n\}!\mathbb{Z}[q^{\pm 1}]$ for all n, since $\mathbb{Z}[q^{\pm 1}]$ is factorial, $f_k(x) = 0$ for all k.

Take $x \in \bigcap_{n\in\mathbb{N}^*} I_n$, written $x = \sum a_n A^n$ with $a_n \in \mathbb{Z}[q^{\pm 1}]$. Take N such that $\deg_q(x) < N$ and $\mathrm{val}_q(x) > -N$. This implies that $\deg_q(a_n) < N$ and $\mathrm{val}_q(a_n) > -N$ (since it is the case for x and any higher or lower terms could not compensate since the power of A is different before each a_n).

Thus, since $f_{2N}(x) = 0$, $\sum a_n q^{2Nn} = 0$, we have $\deg_q(a_n q^{2Nn}) < N(1+2n)$ and $\mathrm{val}_q(a_n q^{2Nn}) > N(2n-1)$, and then each terms $a_n q^{2Nn}$ must be 0. Hence, $a_n = 0$ for all n, meaning that $x = 0$. ∎

Remark 18. If $b_0 \in R$ and $b_n \in I_{n-1}$ for $n \geq 1$, the partial sums $\sum_{i=0}^{N} b_n$ converges in $\widehat{R}^{\widehat{I}}$ as N goes to infinity. We use the notation

$$\sum_{i=0}^{+\infty} b_n := \overline{\Big(\sum_{i=0}^{N} b_n\Big)}_{N\in\mathbb{N}^*}.$$

Conversely, if $a = (\overline{a_n})_{n\in\mathbb{N}^*} \in \widehat{R}^{\widehat{I}}$, let $a_n \in R$ be any representative of $\overline{a_n}$ in R, then $a = \sum_{i=0}^{+\infty} b_n$, where $b_0 = a_1$ and $b_n = a_{n+1} - a_n$ for $n \in \mathbb{N}^*$.

We proceed similarly as in the paragraph preceding Proposition 13. Let $\mathcal{K}$ be a knot seen as a $(1,1)$ tangle and D a diagram of it. For a state diagram of D we define

$$
\begin{aligned}
D(i_1,\dots,i_N) = &\Big(\prod_{j=1}^{S} q^{\mp(\alpha-2\varepsilon_j)}\Big) \prod_{k\in \mathrm{pos}} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_q \{\alpha-a_k;i_k\}_q \\
&\times q^{-(a_k+b_k)\alpha} q^{2(a_k+i_k)(b_k-i_k)} \prod_{k\in \mathrm{neg}} (-1)^{i_k} q^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_q \\
&\times \{\alpha-a_k;i_k\}_q q^{(a_k+b_k)\alpha} q^{-2a_kb_k}
\end{aligned}
$$

where $\mathrm{neg}\cup\mathrm{pos} = [|1,N|]$ and $k\in\mathrm{pos}$ if the k-th crossing of D is positive, else $k\in\mathrm{neg}$. Here, a_k, b_k are the strands labels at the k-th crossing of the state diagram (see Figure 2), S is the number of  +  appearing in the diagram, and ε_j the strand label at the j-th  or , the $\mp$ sign is negative if  and positive if .

Remark 19. Note that the a_k and b_k appearing are defined in terms of i_j. As mentioned previously, you can find some examples of state diagrams in Section 5, Figures 7 and 8.

Definition 20. Let $\mathcal{K}$ be a knot and T 1-1 tangle whose closure is $\mathcal{K}$. Let D be a diagram of T. We define

$$
\begin{aligned}
F_\infty(q,A,D) &:= q^{\frac{f\alpha^2}{2}} \sum_{\bar{i}=0}^{+\infty} D(i_1,\dots,i_N) \\
&= q^{\frac{f\alpha^2}{2}} \sum_{\bar{i}=0}^{+\infty} \Big(\prod_{j=1}^{S} q^{\mp(\alpha-2\varepsilon_j)}\Big) \prod_{k\in \mathrm{pos}} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_q \{\alpha-a_k;i_k\}_q \\
&\quad \times q^{-(a_k+b_k)\alpha} q^{2(a_k+i_k)(b_k-i_k)} \prod_{k\in \mathrm{neg}} (-1)^{i_k} q^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k+i_k \\ i_k \end{bmatrix}_q \\
&\quad \times \{\alpha-a_k;i_k\}_q q^{(a_k+b_k)\alpha} q^{-2a_kb_k},
\end{aligned}
$$

where $\bar{i} = (i_1,\dots,i_N)$, N is the number of crossings, S the number of  +  and f is the framing of the knot. We have that $F_\infty(q,A,D)$ is a well-defined element of $q^{\frac{f\alpha^2}{2}}\widehat{R}^{\widehat{I}}$.

Note that it is not clear that this element is a knot invariant, it could depend *a priori* on the diagram D and we will have to prove later that it does not.

3.2. Recovering the ADO invariant

In this section, we will see how to evaluate at a root of unity an element of $\widehat{R}^{\widehat{I}}$. We will first need some useful lemma.

Let r any integer, $R_r = \mathbb{Z}[\zeta_{2r}, A^{\pm 1}]$, we use the same previous notations and

$$\zeta_{2r}^{\alpha} := A.$$

Lemma 21. *For any k,*

$$\{\alpha - k; r\}_{\zeta_{2r}} = (-1)^k \zeta_{2r}^{\frac{-r(r-1)}{2}} \{r\alpha\}_{\zeta_{2r}}.$$

Proof. We have

$$\begin{aligned}\{\alpha - k; r\} &= \{\alpha - k\} \dots \{\alpha - k - r + 1\}\\ &= (-1)\{\alpha - k + 1\} \dots \{\alpha - k - r + 2\}\\ &= (-1)^k \{\alpha\} \dots \{\alpha - r + 1\}\\ &= (-1)^k \{\alpha; r\}\end{aligned}$$

and

$$\begin{aligned}\{\alpha; r\} &= \prod_{j=0}^{r-1} (\zeta_{2r}^{\alpha - j} - \zeta_{2r}^{-\alpha + j})\\ &= \zeta_{2r}^{\frac{-r(r-1)}{2}} \zeta_{2r}^{-r\alpha} \prod_{j=0}^{r-1} (\zeta_{2r}^{2\alpha} - \zeta_{2r}^{2j})\\ &= \zeta_{2r}^{\frac{-r(r-1)}{2}} \zeta_{2r}^{-r\alpha} (\zeta_{2r}^{2r\alpha} - 1)\\ &= \zeta_{2r}^{\frac{-r(r-1)}{2}} \{r\alpha\},\end{aligned}$$

where the fourth equality is obtained by developing the factorized form of $X^r - 1$ at $X = \zeta_{2r}^{2\alpha}$. ■

Let $I = \{r\alpha\}_{\zeta_{2r}} R_r$. We build the I-adic completion of R_r:

Definition 22. Let

$$\hat{R}_r^I = \varprojlim_n \frac{R_r}{I^n} = \left\{ (a_n)_{n \in \mathbb{N}^*} \in \prod_{i=1}^{\infty} \frac{R_r}{I^n} \;\middle|\; \rho_n'(a_{n+1}) = a_n \right\},$$

where

$$\rho_n' \colon \frac{R_r}{I^{n+1}} \to \frac{R_r}{I^n}$$

is the projection map.

This completion is a bigger ring containing R_r:

Proposition 23. *The canonical projection maps induce an injective map $R_r \hookrightarrow \hat{R}_r^I$.*

Proof. It is sufficient to prove that $\bigcap_{n\in\mathbb{N}^*} I^n = \{0\}$. Since $R_r = \mathbb{Z}[\zeta_{2r}][A^{\pm 1}]$, it is a Laurent polynomial ring. Hence, any non-zero element x can be uniquely written $x = \sum_{i=l}^{n} a_n A^n$ where $a_k \in \mathbb{Z}[\zeta_{2r}]$, for all $k \in \{l, l+1, \ldots, n-1, n\}$ and $a_n, a_l \neq 0$. Let us define $\mathrm{len}(x) = n - l$. We have that $\mathrm{len}(xy) = \mathrm{len}(x) + \mathrm{len}(y)$. Thus, if $x \in \bigcap_{n\in\mathbb{N}^*} I^n$ is non-zero, of length n, there exists $y \in R_r$ such that $x = \{r\alpha\}^n y$, hence $\mathrm{len}(x) = 2rn + \mathrm{len}(y)$, contradiction. ■

Let us now define the evaluation map from $\widehat{R}^{\widehat{I}}$ to $\widehat{R}_r^I$. At the level of R and R_r we have a well-defined evaluation map,

$$\mathrm{ev}_{\zeta_{2r}} \colon R \to R_r, \quad q \mapsto \zeta_{2r}.$$

We will extend this map to the completions.

Proposition 24. $\mathrm{ev}_{\zeta_{2r}}(I_{rn}) = I^n$

Proof. Direct application of Lemma 21. ■

Hence, ev_r factorize into maps $\psi_n \colon R/I_{rn} \to R_r/I^n$, we can then define the map extension:

Proposition 25. *We have a well-defined map*

$$\mathrm{ev}_r \colon \widehat{R}^{\widehat{I}} \to \widehat{R}_r^I$$

such that, if $(a_n)_{n\in\mathbb{N}^*} \in \widehat{R}^{\widehat{I}}$, *then* $\mathrm{ev}_r((a_n)_{n\in\mathbb{N}^*}) = (\psi_n(a_{rn}))_{n\in\mathbb{N}^*}$.

Proof. If we denote by $\lambda_n \colon R/I_{r(n+1)} \to R/I_{rn}$ the projective maps, the proof lies on the fact that the following diagram is commutative:

$$\begin{array}{ccc} R/I_{r(n+1)} & \xrightarrow{\psi_{n+1}} & R_r/I^{n+1} \\ \Big\downarrow \lambda_n & & \Big\downarrow \rho'_n \\ R/I_{rn} & \xrightarrow{\psi_n} & R_r/I^n \end{array}$$

■

It is now time to study the element

$$F_\infty(\zeta_{2r}, A, D) := \mathrm{ev}_r(F_\infty(q, A, D)),$$

we will see that the ADO invariant $\mathrm{ADO}_r(A, \mathcal{K})$ can be factorized from it. In order to do so, we will need some useful computations.

Lemma 26. *We have the following factorizations:*

$$\zeta_{2r}^{\frac{(i+rl)(i+rl-1)}{2}} = (-1)^{il}\zeta_{2r}^{\frac{rl(rl-1)}{2}}\zeta_{2r}^{\frac{i(i-1)}{2}}, \tag{1}$$

$$\begin{aligned}&\{\alpha-a-ru; i+rl\}_{\zeta_{2r}}\\ &\quad = (-1)^{al+rul+ui+li}\zeta_{2r}^{\frac{-rl(r-1)}{2}}\zeta_{2r}^{\frac{-rl(l-1)}{2}}\{r\alpha\}_{\zeta_{2r}}^{l}\{\alpha-a;i\}_{\zeta_{2r}},\end{aligned} \tag{2}$$

$$\begin{bmatrix} a+i+r(u+l) \\ i+rl \end{bmatrix}_{\zeta_{2r}} = (-1)^{al+rul+ui}\binom{u+l}{l}\begin{bmatrix} a+i \\ i \end{bmatrix}_{\zeta_{2r}}, \tag{3}$$

$$\zeta_{2r}^{\frac{-rl(r-1)}{2}}\zeta_{2r}^{\frac{-rl(l-1)}{2}} = \zeta_{2r}^{\frac{-rl(rl-1)}{2}}. \tag{4}$$

Proof. (1) The first part is obtained by developing the product.

(2) The second part is an application of Lemma 21. First,

$$\{\alpha-a-ru; i+rl\} = \zeta_{2r}^{(i+rl)ru}\{\alpha-a;i+rl\} = (-1)^{iu}(-1)^{rul}\{\alpha-a;i+rl\}.$$

Then,

$$\begin{aligned}\{\alpha-a;i+rl\} &= \{\alpha-a;rl\}\{\alpha-a-rl;i\}\\ &= (-1)^{al}\zeta_{2r}^{\frac{-rl(l-1)}{2}}\{\alpha;rl\}\{\alpha-a-rl;i\}\\ &= (-1)^{al}\zeta_{2r}^{\frac{-rl(l-1)}{2}}\zeta_{2r}^{\frac{-rl(r-1)}{2}}\{r\alpha\}^{l}\{\alpha-a-rl;i\}.\end{aligned}$$

Finally,

$$\{\alpha-a-rl;i\} = (-1)^{li}\{\alpha-a;i\}.$$

Putting together, we get

$$\begin{aligned}&\{\alpha-a-ru; i+rl\}_{\zeta_{2r}}\\ &\quad = (-1)^{al+rul+ui+li}\zeta_{2r}^{\frac{-rl(r-1)}{2}}\zeta_{2r}^{\frac{-rl(l-1)}{2}}\{r\alpha\}_{\zeta_{2r}}^{l}\{\alpha-a;i\}_{\zeta_{2r}}.\end{aligned}$$

(3) The third part follows from the fact that

$$\mathrm{ev}_{\zeta_{2r}}\left(\frac{\{rk\}_q}{\{r\}_q}\right) = (-1)^{1-k}k.$$

In $\begin{bmatrix} a+i+r(u+l) \\ i+rl \end{bmatrix}_{\zeta_{2r}}$, seen as $\frac{\{a+i+r(u+l)\}!}{\{a+ru\}!\{i+rl\}!}$, taking only the terms $\{rk\}$, we extract $(-1)^{ul}\binom{u+l}{l}$. Now, we only have to deal with non-multiples of quantum r. We use the equality $\{t+r\} = (-1)\{t\}$ in order to have consecutive terms in the denominators (excepted from multiple of r), indeed

$$\{a+ru\}! = \{ru\}!\{a+ru;a\}$$

and

$$\{i+rl\})(-1)^{u(i+rl)}\{i+rl+ru; i+rl\};$$

hence

$$\begin{aligned}\frac{\{a+i+r(u+l)\}!}{\{a+ru\}!\{i+rl\}!} &= (-1)^{u(i+rl)}\frac{\{a+i+r(u+l);a\}}{\{a+ru;a\}}\\ &= (-1)^{u(i+rl)}(-1)^{au}(-1)^{a(u+l)}\frac{\{a+i;a\}}{\{a;a\}}\\ &= (-1)^{ui}(-1)^{rul}(-1)^{al}\begin{bmatrix}a+i\\ i\end{bmatrix}_{\zeta_{2r}}.\end{aligned}$$

Putting things together with the quantum r multiple part, we get the desired result.

(4) The last part is obtained as follow:

$$\begin{aligned}\zeta_{2r}^{\frac{-rl(rl-1)}{2}} &= \prod_{k=0}^{rl-1}\zeta_{2r}^{-k} = \prod_{j=0}^{l}\prod_{k=0}^{r-1}\zeta_{2r}^{-k-rj}\\ &= \prod_{j=0}^{l}\zeta_{2r}^{-rj}\prod_{k=0}^{r-1}\zeta_{2r}^{-k}\\ &= \zeta_{2r}^{\frac{-rl(r-1)}{2}}\zeta_{2r}^{\frac{-rl(l-1)}{2}}.\end{aligned}$$

■

We proceed similarly as in the paragraph preceding Definition 20 and define an element to each state diagram of D that will be used to factorise $F_\infty(q, A, D)$. Let $\mathcal{K}$ be a knot seen as a $(1,1)$ tangle and D a diagram of it. For a state diagram of D, we define

$$\begin{aligned}D_{C,r}(l_1,\dots,l_N) = \left(\prod_{j=1}^{S}\zeta_{2r}^{\mp r\alpha}\right)\prod_{k\in\text{pos}}\binom{u_k+l_k}{l_k}\{r\alpha\}_{\zeta_{2r}}^{l_k}\zeta_{2r}^{-(u_k+v_k)r\alpha}\\ \times\prod_{k\in\text{neg}}(-1)^{l_k}\binom{u_k+l_k}{l_k}\{r\alpha\}_{\zeta_{2r}}^{l_k}\zeta_{2r}^{(u_k+v_k)r\alpha},\end{aligned}$$

where $\text{neg}\cup\text{pos} = [|1, N|]$ and $k\in\text{pos}$ if the k-th crossing of D is positive, else $k\in\text{neg}$, $a_k, b_k \in [|0,\dots,r-1|]$, and $a_k + ru_k, b_k + rv_k$ are the strands labels at the k-th crossing of the state diagram (see Figure 3), S is the number of  +  appearing in the diagram, and ε_j the strand label at the j-th  or , the $\mp$ sign is negative if  and positive if .

Proposition 27. *For a knot $\mathcal{K}$ and a diagram of the knot D, $r\in\mathbb{N}^*$, we have the following factorization in $\widehat{R}_r^I$:*

$$F_\infty(\zeta_{2r}, A, D) = C_\infty(r, A, D)\times \text{ADO}_r(A,\mathcal{K})$$

where

$$
\begin{aligned}
C_\infty(r, A, D) &= \sum_{\overline{l}=0}^{+\infty} D_{C,r}(l_1, \dots, l_N)\\
&= \sum_{\overline{l}=0}^{+\infty} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\mp r\alpha}\Big) \prod_{k\in \mathrm{pos}} \binom{u_k + l_k}{l_k} \{r\alpha\}_{\zeta_{2r}}^{l_k} \zeta_{2r}^{-(u_k+v_k)r\alpha}\\
&\quad \times \prod_{k\in \mathrm{neg}} (-1)^{l_k} \binom{u_k + l_k}{l_k} \{r\alpha\}_{\zeta_{2r}}^{l_k} \zeta_{2r}^{(u_k+v_k)r\alpha}
\end{aligned}
$$

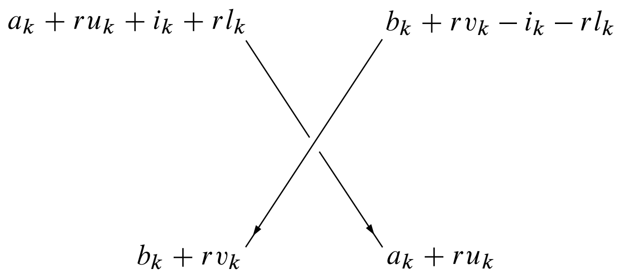
where $\overline{l} = (l_1, \dots, l_N)$*,* N *is the number of crossings,* S *the number of*  +  *and* f *is the framing of the knot.*

Proof. For the sake of simplicity, we will only consider positive crossings in the following proof. We factorize as follows:

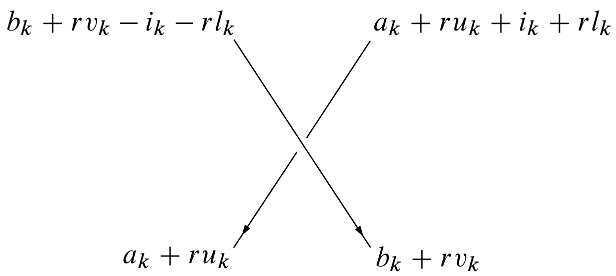
$$
\begin{aligned}
F_\infty(\zeta_{2r}, A, D) &= \zeta_{2r}^{\frac{f\alpha^2}{2}} \sum_{\overline{s}=0}^{+\infty} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\mp(\alpha-2\varepsilon_j)}\Big) \prod_{k=1}^{N} \zeta_{2r}^{\frac{s_k(s_k-1)}{2}} \begin{bmatrix} z_k + s_k \\ s_k \end{bmatrix}_{\zeta_{2r}}\\
&\quad \times \{\alpha - z_k; s_k\}_{\zeta_{2r}} \zeta_{2r}^{(-z_k-y_k)\alpha} \zeta_{2r}^{2(z_k+s_k)(y_k-s_k)}\\
&= \zeta_{2r}^{\frac{f\alpha^2}{2}} \sum_{\overline{\imath+rl}=0}^{+\infty} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\mp(\alpha-2\varepsilon_j)}\Big) \prod_{k=1}^{N} \zeta_{2r}^{\frac{(i_k+rl_k)(i_k+rl_k-1)}{2}}\\
&\quad \times \begin{bmatrix} a_k + i_k + r(u_k + l_k) \\ i_k + rl_k \end{bmatrix}_{\zeta_{2r}} \{\alpha - (a_k + ru_k); i_k + rl_k\}_{\zeta_{2r}}\\
&\quad \times \zeta_{2r}^{(-(a_k+ru_k)-(b_k+rv_k))\alpha} \zeta_{2r}^{2((a_k+ru_k)+(i_k+rl_k))(b_k+rv_k-(i_k+rl_k))}\\
&= \zeta_{2r}^{\frac{f\alpha^2}{2}} \sum_{\overline{\imath}=0}^{r-1} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\pm(r-1)(\alpha-2\varepsilon_j)}\Big) \prod_{k=1}^{N} \zeta_{2r}^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_{\zeta_{2r}}\\
&\quad \times \{\alpha - a_k; i_k\}_{\zeta_{2r}} \zeta_{2r}^{(-a_k-b_k)\alpha} \zeta_{2r}^{2(a_k+i_k)(b_k-i_k)}\\
&\quad \times \sum_{\overline{l}=0}^{+\infty} \Big(\prod_{j=1}^{S} \zeta_{2r}^{\mp r\alpha}\Big) \prod_{k=1}^{N} \binom{u_k + l_k}{l_k} \{r\alpha\}_{\zeta_{2r}}^{l_k} \zeta_{2r}^{(-u_k-v_k)r\alpha}.
\end{aligned}
$$

The second equality is obtained by changing variables $s_k = i_k + rl_k$ $0 \le i_k \le r-1$ and writing the strands labels at crossings z_k as $z_k = a_k + ru_k$ $0 \le a_k \le r-1$ and y_k as $y_k = b_k + rv_k$ $0 \le b_k \le r-1$. Note that a_k, b_k solely depends on i_k and u_k, b_k on l_k. This relies on the fact that $\begin{bmatrix} n+m \\ n \end{bmatrix}_q = 0$ at $q = \zeta_{2r}$ if $n, m \le r-1$ and $n + m \ge r$.

The third equality is obtained by replacing each term with its factorization given by Lemma 26, the crossed terms between i_k and l_k are just signs, that eventually compensate. Hence, we have the factorization. ■



(a) Positive crossing.



(b) Negative crossing.

Figure 3. The two possibilities for the k-th crossing in D when factorizing.

In order to get back $\mathrm{ADO}_r(A, \mathcal{K})$ from $F_\infty(\zeta_{2r}, A, D)$, we need to prove that $C_\infty(r, A, D)$ is a unit in $\widehat{R}_r^I$.

Proposition 28. *If $a = (a_n)_{n\in\mathbb{N}^*} \in \widehat{R}_r^I$ and $a_1 \in R_r/I$ is a unit, then a is a unit in $\widehat{R}_r^I$.*

Proof. Let $a = (a_n)_{n\in\mathbb{N}^*} \in \widehat{R}_r^I$ such that a_1 is a unit of R_r/I. Let us prove that a_n is also a unit in R_r/I^n. If y is an element of R_r/I^n such that $a_n y = a_1 y = 1 \mod I$, then there exists $z \in I.R_r/I^n$ such that $a_n y = 1 + z$, $z = a_n y - 1$, thus $0 = z^n = (a_n y - 1)^n$, which proves that a_n is invertible. Hence, $a^{-1} = (a_n^{-1})_{n\in\mathbb{N}^*}$ is the inverse of a in $\widehat{R}_r^I$. ■

Since $C_\infty(r, A, D) = (\prod_{j=1}^{S} \zeta_{2r}^{\mp r\alpha}) \mod\{r\alpha\}_{\zeta_{2r}}$ is an invertible element of R_r/I, then $C_\infty(r, A, D)$ is a unit of $\widehat{R}_r^I$.

Corollary 29. $\mathrm{ADO}_r(A, \mathcal{K}) = F_\infty(\zeta_{2r}, A, D) C_\infty(r, A, D)^{-1}$.

Finally, one can recover $C_\infty(r, A, D)$ with $F_\infty(q, A, D)$, this will prove that not only that ADO is contained in $F_\infty(q, A, D)$ but that it's possible to extract them with the sole datum of $F_\infty(q, A, D)$.

For $r = 1$, one gets

$$\begin{aligned} \mathrm{ev}_1(F_\infty(q, A, D)) &= F_\infty(\zeta_2, A, D) \\ &= C_\infty(1, A, D) \times \mathrm{ADO}_1(A, \mathcal{K}) \\ &= q^{\frac{f\alpha^2}{2}} C_\infty(1, A, D). \end{aligned}$$

Remark 30. Note that $\mathrm{ADO}_1(A, \mathcal{K})$ is only defined as the case $r = 1$ in Proposition 13, which is well defined. Nevertheless, the algebraic setup at Section 2 fails at $r = 1$ since $[E, F]$ is not well defined.

But then

$$C_\infty(1, A, D) \in \widehat{\mathbb{Z}[A^{\pm 1}]^{\{\alpha\}}} := \varprojlim_n \frac{\mathbb{Z}[A^{\pm 1}]}{\{\alpha\}^n},$$

for each r we have a well-defined map

$$g_r \colon \widehat{\mathbb{Z}[A^{\pm 1}]^{\{\alpha\}}} \to \widehat{\mathbb{Z}[A^{\pm 1}]^{\{r\alpha\}}}, \quad q^\alpha \mapsto q^{r\alpha},$$

such that $g_r(C_\infty(1, A, D)) = C_\infty(r, A, D)$.

This proves the following proposition:

Proposition 31. *For all r, we have a well-defined map*

$$FC_r = g_r \circ \mathrm{ev}_1 \colon \widehat{R}^{\hat{I}} \to \widehat{\mathbb{Z}[A^{\pm 1}]^{\{r\alpha\}}}$$

and, for any knot $\mathcal{K}$ and any diagram D of the knot,

$$F_\infty(q, A, D) \mapsto C_\infty(r, A, D).$$

Corollary 32. *For all r, we have a well-defined map*

$$\mathrm{ev}_r \times \frac{1}{FC_r} \colon (\widehat{R}^{\hat{I}})^\times \to (\widehat{R}_r^I)^\times$$

and, for any knot $\mathcal{K}$ and any diagram D of the knot,

$$F_\infty(q, A, D) \mapsto \mathrm{ADO}_r(A, \mathcal{K}).$$

Proof. Let $x \in (\widehat{R}^{\widehat{I}})^{\times}$ an invertible element, since FC_r is a ring morphism, $FC_r(x)$ is invertible. Then,

$$\mathrm{Id} \times \frac{1}{FC_r}(F_\infty(q, A, D)) = F_\infty(\zeta_{2r}, A, D) \times C_\infty(r, A, D)^{-1} = \mathrm{ADO}_r(A, \mathcal{K}). \quad \blacksquare$$

4. Universal invariant and Verma module

We have built by hand an element $F_\infty(q, A, D)$ in some completion of a ring, from which we have evaluation maps that recovers the ADO invariants. This element is built from the diagram of a knot, thus it depends *a priori* on it. In order to prove that this element is indeed a knot invariant, we will see how to obtain it using Hopf algebra machinery.

Section 4.1 will be dedicated to create an integral subalgebra of the h-adic version of quantum $\mathfrak{sl}_2$ containing the universal invariant of a 0-framed knot.

This will allow us to define, in Section 4.2, a Verma module on it. Since the algebra previously defined is integral, this will also be the case for the Verma module, whose coefficients will lie in $\widehat{R}^{\widehat{I}}$. The unified form $F_\infty(q, A, D)$ will be seen as the scalar action of the universal invariant on this Verma module. Since the universal invariant is a knot invariant, so will be $F_\infty(q, A, D)$.

This algebraic setup is made to get back the unified form and prove its invariance, and it is a completion which is very close to that of Habiro's in [12]. But they are not the same, and we will see in Section 4.3 how to connect this work to Habiro's setup in the article. We will interpret our ring completion $\widehat{R}^{\widehat{I}}$ as some subalgebra completion found in [12], allowing to prove some nice properties on the ring structure (integral domain, subring of some h-adic ring). Moreover, we will show that the unified invariant can also be recovered from Habiro's algebraic setup, using the same process as in Section 4.1, but with his completions. This will allow us to show that the unified invariant is equal to the two-variable colored Jones invariant defined in [13, Section 7.1] .

Using this fact, we will see that we can also recover the colored Jones polynomials from the unified invariant. First this will allow us to study the factorisation in Proposition 27, and find that $C_\infty(r, A, D)$ is just the inverse of the Alexander polynomial. Lastly, using the unified invariant as a bridge between the family of colored Jones polynomials and the family of ADO polynomials, we will show that they are equivalent, meaning that we can recover one family with the other.

As a direct application of this facts, we will show that the unified invariant and every ADO polynomials follow the same holonomic rule as the colored Jones function (see [6]).

Remark 33. The variable q in this paper corresponds to the variable v in [12, 13].

4.1. The universal invariant

In order to build $F_\infty(q, A, D)$ from Hopf algebra, we will need some "big enough" integral version quantum $\mathfrak{sl}_2$, but not too big in order to have a $\widehat{R}^{\widehat{I}}$ Verma module on it.

First let us define the biggest integral quantum $\mathfrak{sl}_2$, U_h.

Definition 34. We set

$$U_h := U_h(\mathfrak{sl}_2),$$

the $\mathbb{Q}[[h]]$ algebra topologically generated by H, E, F and relations

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

where $q = e^h$ and $K = q^H = e^{hH}$.

It is endowed with an Hopf algebra structure

$$\begin{aligned}
\Delta(E) &= 1 \otimes E + E \otimes K, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1},\\
\Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF,\\
\Delta(H) &= 1 \otimes H + H \otimes 1, & \varepsilon(H) &= 0, & S(H) &= -H,
\end{aligned}$$

and an R-matrix

$$R = q^{\frac{H \otimes H}{2}} \sum_{i=0}^{\infty} \frac{\{1\}^n q^{\frac{n(n-1)}{2}}}{[n]!} E^n \otimes F^n,$$

$$R^{-1} = \sum_{i=0}^{\infty} \frac{(-1)^n \{1\}^n q^{\frac{-n(n-1)}{2}}}{[n]!} E^n \otimes F^n q^{-\frac{H \otimes H}{2}}.$$

Altogether with a ribbon element: $K^{-1}u$ where $u = \sum S(\beta)\alpha$ if $R = \sum \alpha \otimes \beta$.

Hence, if $\mathcal{K}$ is a knot and T a 1-1 tangle whose closure is $\mathcal{K}$. We set $Q^{U_h}(\mathcal{K}) \in U_h$ the universal invariant associated to T in U_h. The definition of this element is given in Ohtsuki's book [17, Section 4.2]. It is a knot invariant.

Let us now build a suitable subalgebra of U_h and a $\widehat{R}^{\widehat{I}}$ Verma module on it. We will then see that the universal invariant is in some extent inside the subalgebra and its scalar action on the Verma module will give us $F_\infty(q, A, D)$. The subalgebra considered is an integral version of $U_q(\mathfrak{sl}_2)$ defined as follows.

Definition 35. We set

$$\mathcal{U} := U_q^D(\mathfrak{sl}_2),$$

the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of U_h generated by E, $F^{(n)}$, K, where $F^{(n)} = \frac{\{1\}^n F^n}{[n]!}$.

$\mathcal{U}$ inherits the Hopf algebra structure from U_h.

Remark 36. The R-matrix is not an element of $\mathcal{U}$ but we have

$$R = q^{\frac{H\otimes H}{2}} \sum_{i=0}^{\infty} \frac{\{1\}^n q^{\frac{n(n-1)}{2}}}{[n]!} E^n \otimes F^n = q^{\frac{H\otimes H}{2}} \sum_{i=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}.$$

Hence, aside from $q^{\frac{H\otimes H}{2}}$ (that we can control in the universal invariant as we will see further on), we need the convergence of $\sum_{i=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}$ in some tensor product of the algebra with itself. Thus we need to complete the algebra $\mathcal{U}$.

We set

$$\{H+m\}_q = Kq^m - K^{-m}q^{-m},$$

$$\{H+m;n\}_q = \prod_{i=0}^{n-1} \{H+m-i\}_q.$$

Definition 37. Let L_n be the $\mathbb{Z}[q^{\pm 1}]$ ideal generated by $\{n\}!$. Let J_n be the $\mathcal{U}$ two sided ideal generated by the following elements:

$$F^{(i+k)}\{H+m;n-i\}_q,$$

where $m \in \mathbb{Z}$, $i \in \{0,\dots,n\}$ and $k \in \mathbb{N}$.

Lemma 38. *J_n is generated by elements of the form $F^{(i+k)}\{n-i;j\}\{H;n-i-j\}$, $j \in \{0,\dots,n-i\}$, $i \in \{0,\dots,n\}$, and $k \in \mathbb{N}$.*

Proof. The proof can be found in Habiro's article [12, Proposition 5.1]. ∎

Following the completion described by Habiro in is article [12, Section 4]. We have

(1) $J_{n+1} \subset J_n$,

(2) $L_n \subset J_n$ (see [12, Proposition 5.1]),

(3) $\Delta(J_n) \subset \sum_{i+j=n} J_i \otimes J_j$,

(4) $\varepsilon(J_n) \subset L_n$,

(5) $S(J_n) \subset J_n$.

Thus, we can define the completion

$$\widehat{\mathcal{U}} := \varprojlim_{n} \frac{\mathcal{U}}{J_n}$$

as a $\widehat{\mathbb{Z}[q^{\pm 1}]} := \varprojlim_n \frac{\mathbb{Z}[q^{\pm 1}]}{L_n}$-algebra ($\widehat{\mathbb{Z}[q^{\pm 1}]}$ is Habiro's ring). And it is endowed with a complete Hopf algebra structure:

$$\widehat{\Delta}\colon \widehat{\mathcal{U}} \to \widehat{\mathcal{U}}\widehat{\otimes}\widehat{\mathcal{U}}, \quad \widehat{\varepsilon}\colon \widehat{\mathcal{U}} \to \widehat{\mathbb{Z}[q^{\pm 1}]}, \quad \widehat{S}\colon \widehat{\mathcal{U}} \to \widehat{\mathcal{U}}$$

where

$$\widehat{\mathcal{U}}\widehat{\otimes}\widehat{\mathcal{U}} = \varprojlim_{k,l} \frac{\widehat{\mathcal{U}} \otimes_{\widehat{\mathbb{Z}[q^{\pm 1}]}} \widehat{\mathcal{U}}}{\widehat{\mathcal{U}} \otimes_{\widehat{\mathbb{Z}[q^{\pm 1}]}} \overline{J_k} + \overline{J_l} \otimes_{\widehat{\mathbb{Z}[q^{\pm 1}]}} \widehat{\mathcal{U}}}$$

and $\overline{J_n}$ is the closure of J_n in $\widehat{\mathcal{U}}$.

This completion is a bigger algebra than $\mathcal{U}$:

Proposition 39. *The canonical projection maps induce an injective map $\mathcal{U} \hookrightarrow \widehat{\mathcal{U}}$.*

Proof. Take the projective maps $j_n\colon \mathcal{U} \to \mathcal{U}/J_n$, they induce a map $j\colon \mathcal{U} \to \widehat{\mathcal{U}}$. This map in injective because if $j(x) = 0$ then $x \in \bigcap_{n\in\mathbb{N}^*} J_n$. But since $J_n \subset h^n U_h$ then $\bigcap_{n\in\mathbb{N}^*} J_n \subset \bigcap_{n\in\mathbb{N}^*} h^n U_h$. It is a well-known fact that $\bigcap_{n\in\mathbb{N}^*} h^n U_h = \{0\}$. ■

Moreover, since $J_n \subset h^n U_h$ we have a map $i\,\widehat{\mathcal{U}} \to U_h$. Since we do not know if this map is injective, we consider $\widetilde{\mathcal{U}} := i(\widehat{\mathcal{U}})$ the image in U_h. It is also an Hopf algebra.

Remark 40. $\sum_{i=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)} \in \widetilde{\mathcal{U}}\widehat{\otimes}\widetilde{\mathcal{U}}$.

We will need a lemma to compute some commutation rules.

Lemma 41. *We have*

$$(E \otimes 1) \times q^{\frac{H\otimes H}{2}} = q^{\frac{H\otimes H}{2}} \times (E \otimes 1) \times (1 \otimes K)$$

and

$$(F^{(n)} \otimes 1) \times q^{\frac{H\otimes H}{2}} = q^{\frac{H\otimes H}{2}} \times (F^{(n)} \otimes 1) \times (1 \otimes K^{-n}).$$

Proof. Notice that since

$$EH^n = (H+2)^n E \quad \text{and} \quad q^{\frac{H\otimes H}{2}} = \sum \Big(\frac{h^n}{2^n n!}\Big) H^n \otimes H^n,$$

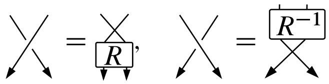
then

$$(E \otimes 1) \times q^{\frac{H\otimes H}{2}} = q^{\frac{(H+2)\otimes H}{2}} \times (E \otimes 1) = q^{\frac{H\otimes H}{2}} \times (1 \otimes K) \times (E \otimes 1).$$

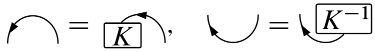
The same can be done for $F^{(n)}$. ■

Let us now construct the universal invariant $Q^{U_h}(\mathcal{K})$ by hand, seeing it as the $(1,1)$-tangle with coupons.

We can picture it as a 1-1 tangle with $(2,2)$ coupons for R-matrix, i.e.,



and also with $(1,1)$ coupons for the pivotal element:



The rest of the tangle remains unchanged.

By replacing the R-matrix with its formula, we get sums of diagrams with $(1,1)$-coupons E^n, $F^{(n)}$, K and $(2,2)$-coupons $q^{\frac{H\otimes H}{2}}$, $q^{-\frac{H\otimes H}{2}}$ (that we can decompose into sum of $(1,1)$-coupons if seen as exponentials). To compute the universal invariant, start from the top of the tangle and multiply (to the right) every coupons encountered.

Now, let us see that we can separate the universal invariant into two pieces. The example of the trefoil knot will illustrate the process all along. The first step is to represent the knot with R-matrices (2,2)-coupons and $K^{\pm 1}$ (1,1)-coupons for the pivotal elements, as illustrated in Figure 4b. Now, we write R as a sum, so the $(2,2)$-coupons labeled by R become the composition of (2,2)-coupons labeled by $q^{\frac{H\otimes H}{2}}$ and $(1,1)$-coupons labeled by E^n or $F^{(n)}$ (see Figure 4c).

We now slide down – following the orientation – the $(1,1)$-coupons E^n, $F^{(n)}$ and K^n, taking first (at any step) the closest to the bottom (see Figure 5a). During this process, the only non-trivial commutations that appear are between $E^n\otimes 1$ or $1\otimes E^n$ or $F^{(n)}\otimes 1$ or $1\otimes F^{(n)}$ and $q^{\frac{H\otimes H}{2}}$ or $q^{-\frac{H\otimes H}{2}}$. By Lemma 41, this only add some $(1,1)$ coupons labeled by $K^{\pm n}$ (see Figure 5b).

We are only left with coupons labeled by $q^{\frac{H\otimes H}{2}}$ and $q^{-\frac{H\otimes H}{2}}$, you can see Figure 6a for the example of the trefoil knot. These are exponentials and we can see them as the sum

$$q^{\pm\frac{H\otimes H}{2}}=\sum\Big(\frac{(\pm h)^n}{2^n n!}\Big)H^n\otimes H^n,$$

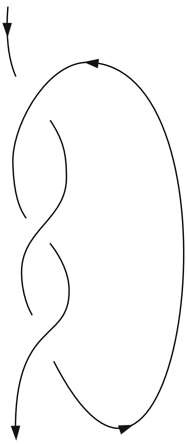
the $(2,2)$-coupons now become a sum of $(1,1)$-coupons as shown for the trefoil knot in Figure 6b. Now, take one of the $(1,1)$-coupon labeled by H^n and slide it towards the second $(1,1)$-coupon labeled by H^n (see Figure 6b). Since one is only left with coupons labeled by powers of H, everything commutes and we get coupons H^{2n} (as shown in Figure 6c), summing them over n gives us $(1,1)$-coupons labeled by $q^{\pm\frac{H^2}{2}}$ (see Figure 6d).

Now, since we control the quadratic part $q^{\frac{H\otimes H}{2}}$ and since the sum in the R-matrix converge in $\widetilde{U}$ by construction, we thus have the following proposition:

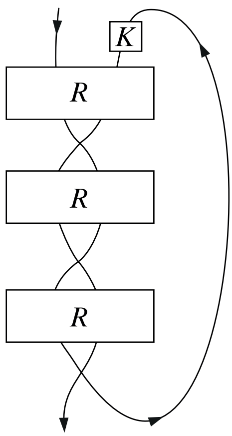
Proposition 42. *If $\mathcal{K}$ a knot and D a diagram of a* 1-1 *tangle T whose closure is $\mathcal{K}$, then*

$$Q^{U_h}(\mathcal{K}) = q^{f\frac{H^2}{2}} Q^{\widetilde{\mathcal{U}}}(D),$$

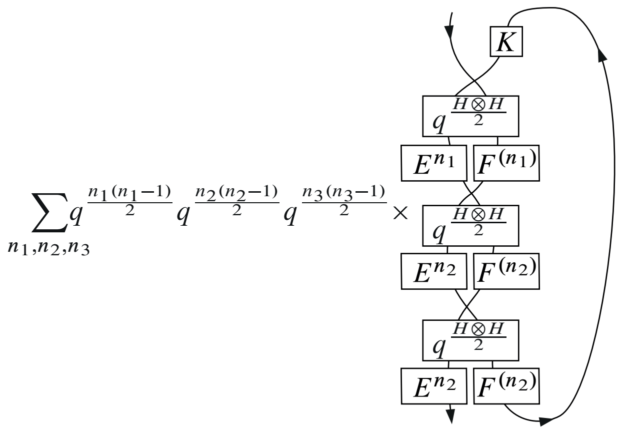
where $Q^{\widetilde{\mathcal{U}}}(D) \in \widetilde{\mathcal{U}} \subset U_h$ and f is the writhe of the diagram.



(a) The trefoil knot.

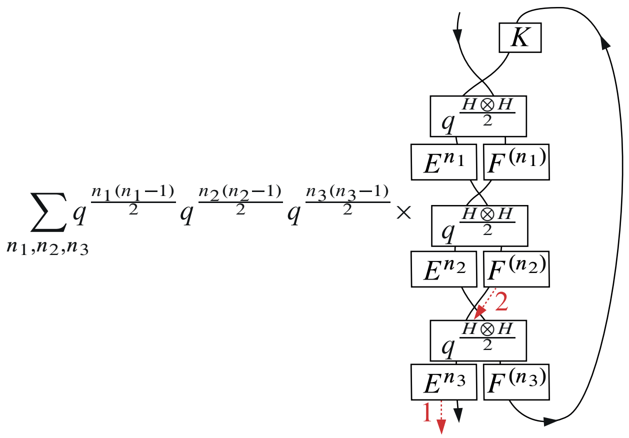


(b) The universal invariant.

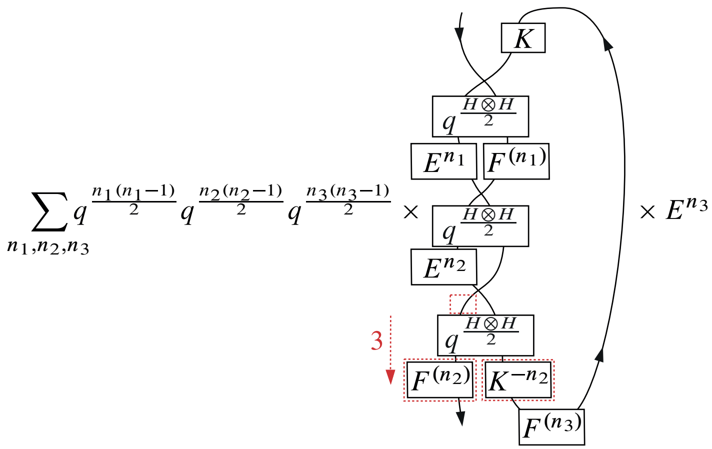


(c) R-matrices seen has sums.

Figure 4. The example of the trefoil knot.



(a) The first two steps, sliding coupons.



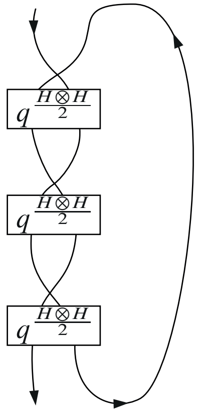
(b) The third step, passing through the (2,2) coupons.

Figure 5. Sliding coupons example with the trefoil knot.

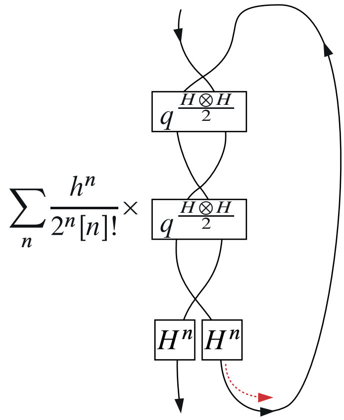
4.2. The completed Verma module

Now, it is time to construct the Verma module on which the universal invariant will act as $F_\infty(q, A, D)$. Let V^α be a R-module freely generated by vectors $\{v_0, v_1, \dots\}$, and we endow it with a $\mathcal{U}$-module structure

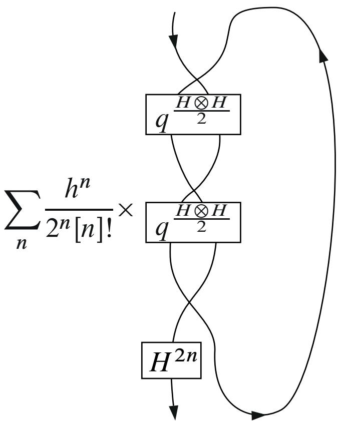
$$E.v_0 = 0, \quad E.v_{i+1} = v_i, \quad K.v_i = q^{\alpha-2i} v_i,$$
$$F^{(n)}.v_i = \begin{bmatrix} n+i \\ i \end{bmatrix}_q \{\alpha - i; n\}_q v_{n+i}.$$



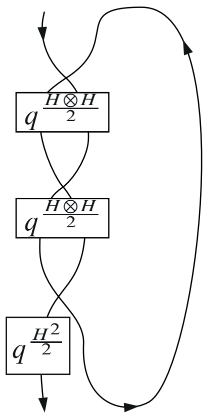
(a) Quadratic part of trefoil universal invariant.



(b) Illustration of quadratic simplification: first step.



(c) Illustration of quadratic simplification: second step.



(d) Illustration of quadratic simplification: third step.

Figure 6. Quadratic factorization and simplification for the trefoil knot.

We define the *completed Verma module* as the $\widehat{R}^{\widehat{I}}$-module

$$\widehat{V^\alpha} = \lim_{\overleftarrow{n}} \frac{V^\alpha}{I_n V^\alpha}$$

where $I_n V^\alpha$ is the ideal generated by elements of the form $\lambda \times v$, $\lambda \in I_n$, $v \in V^\alpha$.

Since $J_n . V^\alpha \subset I_n V^\alpha$, we can naturally endow it with a $\widetilde{\mathcal{U}}$ module structure. Moreover, since $\bigcap I_n = 0$ then $\bigcap I_n V^\alpha = 0$ and thus $V^\alpha \subset \widehat{V^\alpha}$ as a R-module.

We denote this representation by $\rho\colon \widetilde{\mathcal{U}} \to \operatorname{End}(\widehat{V^\alpha})$; if $A \in \widetilde{\mathcal{U}}$ and $v \in \widehat{V^\alpha}$ we will write

$$Av := \rho(A)(v).$$

Proposition 43. *If $\mathcal{K}$ is a knot and D is a diagram of a* 1-1 *tangle T whose closure is $\mathcal{K}$, then*

$$Q^{U_h}(\mathcal{K}).v_0 = q^{f\frac{H^2}{2}} Q^{\widetilde{\mathcal{U}}}(D).v_0 = F_\infty(q, A, D)v_0.$$

Hence, in particular, $F_\infty(q, A, D)$ is independent of the choice of the diagram and we denote it by $F_\infty(q, A, \mathcal{K})$.

Proof. The identity comes from the definition of the state diagrams contribution $D(i_1, \dots, i_N)$ and the definition of the action of the R-matrices, where i_j is the index of the sum corresponding to the R-matrix at the j-th crossing.

Indeed, recall that

$$\begin{aligned} F_\infty(q, A, D) :=\,& q^{\frac{f\alpha^2}{2}} \sum_{\bar{\imath}=0}^{+\infty} D(i_1, \dots, i_N) \\ =\,& q^{\frac{f\alpha^2}{2}} \sum_{\bar{\imath}=0}^{+\infty} \Big(\prod_{j=1}^{S} q^{\mp(\alpha - 2\varepsilon_j)} \Big) \prod_{k \in \text{pos}} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \{\alpha - a_k; i_k\}_q \\ & \times q^{-(a_k+b_k)\alpha} q^{2(a_k+i_k)(b_k-i_k)} \prod_{k \in \text{neg}} (-1)^{i_k} q^{-\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \\ & \times \{\alpha - a_k; i_k\}_q q^{(a_k+b_k)\alpha} q^{-2a_k b_k}, \end{aligned}$$

where $\bar{\imath} = (i_1, \dots, i_N)$, N is the number of crossings, S the number of  and f is the framing of the knot, and

$$\begin{aligned} R.(v_{b_k} \otimes v_{a_k}) = q^{\frac{\alpha^2}{2}} \sum_{i_k} q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \{\alpha - a_k; i_k\}_q \\ \times q^{-(a_k+b_k)\alpha} q^{2(a_k+i_k)(b_k-i_k)} v_{b_k-i_k} \otimes v_{a_k+i_k}. \end{aligned}$$

Hence,

$$Q^{U_h}(\mathcal{K}).v_0 = q^{f\frac{H^2}{2}} Q^{\widetilde{\mathcal{U}}}(D).v_0 = F_\infty(q, A, D)v_0. \qquad \blacksquare$$

We have a bit better, the action of $Q^{U_h}(\mathcal{K})$ is in fact scalar.

Proposition 44. *The $\widetilde{\mathcal{U}}$-endomorphism of $\widehat{V^\alpha}$ are scalars, i.e.,*

$$\operatorname{End}_{\widetilde{\mathcal{U}}}(\widehat{V^\alpha}) = \hat{R}^{\hat{I}} Id_{\widehat{V^\alpha}}.$$

Proof. Let $f \in \mathrm{End}_{\widetilde{\mathcal{U}}}(\widehat{V^\alpha})$. We get $K.f(v_i) = f(K.v_i) = q^{\alpha-2i} f(v_i)$, thus there exists $\lambda_i \in \widehat{R}^{\hat{I}}$ such that $f(v_i) = \lambda_i v_i$. Now, since $E.f(v_{i+1}) = f(E.v_{i+1}) = f(v_i)$, then $\lambda_{i+1} v_i = \lambda_i v_i$; hence we define $\lambda := \lambda_i$ and we have $f = \lambda Id_{\widehat{V^\alpha}}$. ∎

Remark 45. Since $\rho(Z(\widetilde{\mathcal{U}})) \subset \mathrm{End}_{\widetilde{\mathcal{U}}}(\widehat{V^\alpha})$, there is a well-defined map

$$f\colon Z(\widetilde{\mathcal{U}}) \to \widehat{R}^{\hat{I}}, \quad x \mapsto \lambda_x,$$

where $\rho(x) = \lambda_x Id_{\widehat{V^\alpha}}$

We set

$$q^{\pm\frac{H^2}{2}} v_i = q^{\pm\frac{(\alpha-2i)^2}{2}} v_i \in q^{\pm\frac{\alpha^2}{2}} \widehat{V^\alpha}.$$

Proposition 46. *$Q^{U_h}(\mathcal{K})$ is in the center of U_h.*

Proof. See [11, Proposition 8.2]. ∎

Corollary 47. $f(Q^{U_h}(\mathcal{K})) = F_\infty(q, A, \mathcal{K})$.

4.3. Connection to Habiro's work

This section will be dedicated to connect our setup (the ring setup $\widehat{R}^{\hat{I}}$ and the quantum algebra setup $\widetilde{\mathcal{U}}$) to Habiro's algebra setup in [12]. As we will see, we will get that our ring $\widehat{R}^{\hat{I}}$ is contained in some h-adic ring, and hence that it is an integral domain. Moreover, we will see how to get back our unified invariant $F_\infty(q, A, \mathcal{K})$ from Habiro's quantum algebra completions.

Once this is done, we will show that $F_\infty(q, A, \mathcal{K})$ is in fact the two-variable Jones polynomial $J_{\mathcal{K}}(q^\alpha, q)$ defined by Habiro in [13, Section 7] .

Remark 48. Recall that the variable q in this paper corresponds to the variable v in [12, 13]

4.3.1. Unified invariant from Habiro's quantum algebra. We define the subalgebras generated by H.

- In U_h, we set U_h^0 the subalgebra topologically generated by H.
- In $\mathcal{U}$, we set $\mathcal{U}^0$ the subalgebra generated by H.

We can complete the algebra into

$$\widehat{\mathcal{U}}^0 = \varprojlim_n \frac{\mathcal{U}^0}{\langle \{H+m; n\},\ m \in \mathbb{Z} \rangle}.$$

We then have the following proposition.

Proposition 49. *$U_h^0 \cong \mathbb{Q}[\alpha][[h]]$, the h-adic completion of the polynomial ring $\mathbb{Q}[\alpha]$, and $\widehat{\mathcal{U}}^0 \cong \hat{R}^{\hat{I}}$.*

Proof. The first statement is the definition of U_h^0 replacing H with formal variable α. The second statement comes from the fact that, replacing K by A, $\mathcal{U}^0 \cong \mathbb{Z}[q^{\pm 1}, A^{\pm 1}]$ and $\{H + m; n\} \cong I_n$. ∎

Now, one can use Proposition 6.8 and 6.9 in Habiro's article [12] and we have:

Proposition 50. *We have that $\hat{R}^{\hat{I}} \subset \mathbb{Q}[\alpha][[h]]$, and thus $\hat{R}^{\hat{I}}$ is an integral domain.*

Moreover, elements in $\hat{R}^{\hat{I}}$ can be uniquely expressed. This fact comes from [12, Corollary 5.5]. Recall that $q^\alpha := A$ and let $\{\alpha; n\}' = \prod_{i=0}^{n-1}(q^{2\alpha} - q^i)$ we have the following proposition:

Proposition 51. *We have the following isomorphism:*

$$\hat{R}^{\hat{I}} \cong \varprojlim_n \frac{\widehat{\mathbb{Z}[q]}[A]}{(\{\alpha; n\}')}.$$

Moreover, any element $t \in \hat{R}^{\hat{I}}$ can be uniquely written $\sum_{n=0}^{\infty} t_n \{\alpha; n\}'$ where $t_n \in \widehat{\mathbb{Z}[q]} + \widehat{\mathbb{Z}[q]}A$.

Proof. See [12, Corollary 5.5]. ∎

Remark 52. This means that the unified invariant $F_\infty(q, A, \mathcal{K})$ can be uniquely written as a series $\sum_{n=0}^{\infty} t_n \{\alpha; n\}'$ where $t_n \in \widehat{\mathbb{Z}[q]} + \widehat{\mathbb{Z}[q]}A$.

Now, let us present the quantum algebra setup used by Habiro. Our algebra and completion was done for the sole purpose of getting a nice form for our unified invariant, allowing us to factorize it at each roots of unity. Habiro's quantum algebra setup has been studied more in details, and thus have more proprieties.

Let $\mathcal{U}_{\mathrm{Hab}}$ be the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of U_h generated by elements $K^{\pm 1}, e, F^{[n]}$, where

$$e = \{1\}E \quad \text{and} \quad F^{[n]} = \frac{F^n}{\{n\}!}.$$

Let $\widetilde{J}_n$ be the ideal generated by elements $e^i\{H + m; n - i\}$ for all $m \in \mathbb{Z}$ We set

$$\widehat{\mathcal{U}}_{\mathrm{Hab}} := \varprojlim_n \frac{\mathcal{U}_{\mathrm{Hab}}}{\widetilde{J}_n}.$$

Habiro proved that there is an injective map

$$i \colon \widehat{\mathcal{U}}_{\mathrm{Hab}} \to U_h$$

(see [13, Propositions 6.8 and 6.9]).

Now, that his algebra setup is stated, let us make the connection with our unified invariant $F_\infty(q, A, \mathcal{K})$. To do so, we will build a unified invariant with Habiro's setup and prove that it is in fact $F_\infty(q, A, \mathcal{K})$.

First remark that, since the sums in the R-matrix also converge in $\widehat{\mathcal{U}}_{\mathrm{Hab}}$:

Remark 53. If $\mathcal{K}$ a knot and D a diagram of a 1-1 tangle T whose closure is $\mathcal{K}$, then

$$Q^{U_h}(\mathcal{K}) = q^{f\frac{H^2}{2}} Q^{\widetilde{\mathcal{U}}}(D),$$

where $Q^{\widetilde{\mathcal{U}}}(D) \in i(\widehat{\mathcal{U}}_{\mathrm{Hab}}) \subset U_h$ and f is the writhe of the diagram.

We can then define the corresponding Verma module on $\hat{R}^{\hat{I}}$. We can endow V^α with a $\mathcal{U}_{\mathrm{Hab}}$-module structure and we denote it by V^α_{Hab}:

$$e.v_0 = 0, \quad e.v_{i+1} = \{\alpha - i\}_q v_i, \quad K.v_i = q^{\alpha - 2i} v_i, \quad F^{[n]}.v_i = \begin{bmatrix} n+i \\ i \end{bmatrix}_q v_{n+i}.$$

Moreover, since $\widetilde{J}_n.V^\alpha_{\mathrm{Hab}} \subset I_n V^\alpha_{\mathrm{Hab}}$, we can naturally endow $\widehat{V^\alpha}$ with a $\widehat{\mathcal{U}}_{\mathrm{Hab}}$-module structure and we denote it by $\widehat{V^\alpha}_{\mathrm{Hab}}$.

We denote this representation by

$$\rho_{\mathrm{Hab}} \colon \widetilde{\mathcal{U}} \to \mathrm{End}(\widehat{V^\alpha}_{\mathrm{Hab}});$$

if $A \in \widehat{\mathcal{U}}_{\mathrm{Hab}}$ and $v \in \widehat{V^\alpha}_{\mathrm{Hab}}$ we write

$$A.v := \rho_{\mathrm{Hab}}(A)(v).$$

Then, in a similar fashion, we have that:

Remark 54. If $\mathcal{K}$ is a knot and D is a diagram of a 1-1 tangle T whose closure is $\mathcal{K}$, then there exists an element $F^{\mathrm{Hab}}_\infty(q, A, \mathcal{K}) \in q^{f\frac{\alpha^2}{2}} \times \hat{R}^{\hat{I}}$ such that

$$Q^{U_h}(\mathcal{K}).v_0 = q^{f\frac{H^2}{2}} Q^{\widetilde{\mathcal{U}}}(D).v_0 = F^{\mathrm{Hab}}_\infty(q, A, \mathcal{K}) v_0.$$

Let us $V^\alpha_h = V^\alpha \otimes_R \mathbb{Q}[\alpha][[h]]$. We define the *h-adic completed Verma module*

$$\widehat{V^\alpha_h} = \varprojlim_n \frac{V^\alpha_h}{h^n V^\alpha_h}.$$

$\widehat{V^\alpha_h}$ is the $\mathbb{Q}[\alpha][[h]]$-module topologically generated by vectors $\{v_0, v_1, \dots\}$, and we endow it with a U_h-module structure:

$$E.v_0 = 0, \quad E.v_{i+1} = v_i, \quad H.v_i = (\alpha - 2i) v_i, \quad F.v_i = [\alpha - i]_q v_{i+1}.$$

We can also use another useful topological basis $\{w_0, w_1, \dots\}$ such that $w_i = [\alpha; i]_q v_i$ and get hence

$$E.w_0 = 0, \quad E.w_{i+1} = [\alpha - i]_q w_i, \quad H.w_i = (\alpha - 2i) w_i, \quad F.w_i = w_{i+1}.$$

Using $\widehat{R}^{\widehat{I}} \subset \mathbb{Q}[\alpha][[h]]$, we have that $\widehat{V^\alpha} \subset \widehat{V_h^\alpha}$ as $\widetilde{\mathcal{U}}$-modules and that $\widehat{V^\alpha}_{\mathrm{Hab}} \subset \widehat{V_h^\alpha}$ as $\widehat{\mathcal{U}}_{\mathrm{Hab}}$-modules.

Proposition 55. *$\widehat{V^\alpha} \subset \widehat{V_h^\alpha}$ as $\widetilde{\mathcal{U}}$-modules and $\widehat{V^\alpha}_{\mathrm{Hab}} \subset \widehat{V_h^\alpha}$ as $\widehat{\mathcal{U}}_{\mathrm{Hab}}$-modules.*

Proof. We have a well-defined $\widetilde{\mathcal{U}}$-modules map

$$i \colon \widehat{V^\alpha} \to \widehat{V_h^\alpha}, \quad v_i \mapsto v_i,$$

since $I_n V^\alpha \subset h^n V_h^\alpha$ and the action of $\widetilde{\mathcal{U}}$ coincide on the topological basis. Moreover,

$$\begin{aligned}\widehat{V^\alpha} &= \left\{ (\overline{u_k})_{k \in \mathbb{N}^*} \in \frac{V^\alpha}{I_k V^\alpha} \,\middle|\, \overline{u_{k+1}} = \overline{u_k} \mod I_k V^\alpha \right\} \\ &= \Bigg\{ \Big(\sum_n \overline{a_{k,n}} v_n \Big)_{k \in \mathbb{N}^*} \in \frac{V^\alpha}{I_k V^\alpha} \,\Bigg|\, \overline{a_{k,n}} \in \frac{R}{I_k}, (\overline{a_{k,n}})_{n \in \mathbb{N}} \text{ have finite support,} \\ &\qquad\qquad \overline{a_{k+1,n}} = \overline{a_{k,n}} \mod I_k \Bigg\}.\end{aligned}$$

Note that $a_n := (a_{k,n})_{k \in \mathbb{N}^*} \in \widehat{R}^{\widehat{I}}$ by definition.

If $u = (\overline{u_k})_{k \in \mathbb{N}^*} = (\sum_n \overline{a_{k,n}} v_n)_{k \in \mathbb{N}^*} \in \widehat{V^\alpha}$ is such that $i(u) = 0$ then, using $j \colon \widehat{R}^{\widehat{I}} \hookrightarrow \mathbb{Q}[\alpha][[h]]$,

$$j(a_n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence, by injectivity,

$$a_n = 0 \quad \text{for all } n \in \mathbb{N}.$$

Thus, $u = 0$ and the map i is injective.

We proceed in the same fashion with $i \colon \widehat{V^\alpha}_{\mathrm{Hab}} \to \widehat{V_h^\alpha}$, $v_i \mapsto w_i$. ■

It follows that $F_\infty^{\mathrm{Hab}}(q, A, \mathcal{K}) = F_\infty(q, A, \mathcal{K})$ as elements of $\mathbb{Q}[\alpha][[h]]$. Since $\widehat{R}^{\widehat{I}} \subset \mathbb{Q}[\alpha][[h]]$, then $F_\infty^{\mathrm{Hab}}(q, A, \mathcal{K}) = F_\infty(q, A, \mathcal{K})$ as elements of $\widehat{R}^{\widehat{I}}$.

4.3.2. The unified invariant and the two-variable colored Jones invariant are the same. In [13, Section 7], Habiro define a two-variable colored Jones invariant unifying the colored Jones polynomials.

Remark 56. Recall that the notation v (resp. q) in [13] correspond to the notation q (resp. q^2) in this article.

This two-variables colored Jones invariant is defined as the universal invariant of a long knot seen in a completion ring

$$\varprojlim_{k} \mathbb{Z}[q,q^{-1},t,t^{-1}]/\Big(\prod_{-k<i<k}(t-q^{2i})\Big),$$

where t is defined, using the Casimir element $C=\{1\}^2FE+qK+q^{-1}K^{-1}$, by

$$C^2=t+t^{-1}+2.$$

The Casimir element is in the center of U_h and

$$Cv_0=(q^{\alpha+1}+q^{-\alpha-1})v_0.$$

We thus have the identification

$$t=q^{2(\alpha+1)}.$$

Let us come back to the unified invariant. For a 0-framed knot $\mathcal{K}$, we have

$$Q^{U_h}(\mathcal{K})\in Z(\hat{\mathcal{U}}_{\mathrm{Hab}}).$$

We also have, using [13, Theorem 9.1 and Section 9.2],

$$Z(\hat{\mathcal{U}}_{\mathrm{Hab}})\hookrightarrow \hat{\mathcal{U}}^0.$$

Since $\hat{\mathcal{U}}^0\cong \hat{R}^{\hat{I}}$, we have the following equality in $\hat{R}^{\hat{I}}$:

$$f(Q^{U_h}(\mathcal{K}))=F_\infty(q,q^\alpha,\mathcal{K}),$$

where $f\colon Z(\hat{\mathcal{U}}_{\mathrm{Hab}})\to \hat{R}^{\hat{I}}$ the scalar action of central elements on $\widehat{V^\alpha}_{\mathrm{Hab}}$.

In other words, the two-variables colored Jones invariant $J_{\mathcal{K}}(t,q^2)$ defined in [13, Section 7] verifies

$$J_{\mathcal{K}}(q^{2(\alpha+1)},q^2)=F_\infty(q,q^\alpha,\mathcal{K})$$

as elements in $\hat{R}^{\hat{I}}$.

4.4. About the colored Jones polynomials

The colored Jones polynomials and the study of $C_\infty(1,A,\mathcal{K})$. Knowing that the unified invariant comes from the universal invariant, we can use this fact to recover the colored Jones polynomials. When we evaluate $A=q^\alpha$ at q^n in $F_\infty(q,A,\mathcal{K})$, we obtain the n-colored Jones polynomial, denoted by $J_n\,(q,\mathcal{K})$ (with normalization $J_n(q,\text{unknot})=1$). Moreover, we can use this fact to compute $C_\infty(1,A,\mathcal{K})$ as the inverse of the Alexander polynomial, simplifying the factorization in Proposition 27.

The following lemma and proposition are a reformulation of the proof that the universal invariant contains the colored Jones polynomials as found in [13, Section 7.1].

Lemma 57. *If we denote by V_n the $(n+1)$-dimensional highest weight module of $\tilde{\mathcal{U}}$ and $\widehat{V^n}$ the Verma module of highest weight q^n and highest weight vector v_0, then $V_n \cong \tilde{\mathcal{U}}.v_0 \subset \widehat{V^n}$.*

Proof. It comes down to $F^{(n+1)}.v_0 = \{n; n+1\}v_{n+1} = 0 \times v_{n+1} = 0$. ■

Proposition 58. *If $\widehat{V^n}$ is the Verma of highest weight q^n,*

$$Q^{U_h}(\mathcal{K}).v_0 = q^{\frac{fn^2}{2}} q^{fn} J_n(q^2, \mathcal{K}) v_0.$$

Moreover, if we denote by $\widehat{\mathbb{Z}[q^{\pm 1}]}$ Habiro's ring completion of $\mathbb{Z}[q^{\pm 1}]$ by ideals $(\{n\}!)$, we have some well-defined evaluation maps

$$j_n \colon \hat{R}^{\hat{I}} \to \widehat{\mathbb{Z}[q^{\pm 1}]}, \quad q^\alpha \mapsto q^n.$$

This allows us to state the following corollary.

Corollary 59. $F_\infty(q, q^n, \mathcal{K}) = q^{\frac{fn^2}{2}} q^{fn} J_n(q^2, \mathcal{K})$.

Hence, $F_\infty(q, A, \mathcal{K})$ plays a double role in this dance: evaluating its first variable q at a root of unity ζ_{2r}, gives us the r-th ADO polynomial multiplied by this $C_\infty(r, A, \mathcal{K})$ element; but if one evaluates the second variable A at q^n, one gets the n-th colored Jones polynomial.

We will use this double role to study the factorization of ADO polynomials. Indeed, the Melvin–Morton–Rozansky conjecture (MMR) proved by Bar-Natan and Garoufalidis in [2] makes the junction between the inverse of the Alexander polynomial and the colored Jones polynomials. We will use the h-adic version that we state below in Theorem 60 (see [7, Theorem 2]).

We denote by $A_{\mathcal{K}}(t)$ the Alexander polynomial of the knot $\mathcal{K}$, with normalisation $A_{\text{unknot}}(t) = 1$ and $A_{\mathcal{K}}(1) = 1$.

Theorem 60 (Bar-Natan, Garoufalidis). *For $\mathcal{K}$ a knot, we have the following equality in $\mathbb{Q}[[h]]$:*

$$\lim_{n\to\infty} J_n(e^{h/n}) = \frac{1}{A_{\mathcal{K}}(e^h)}.$$

For the sake of simplicity, let us assume that the knot $\mathcal{K}$ is 0-framed so $f = 0$. Note that, since $F_\infty(q, q^n, \mathcal{K}) = J_n(q^2, \mathcal{K}) \in \mathbb{Z}[q^{\pm 1}]$ and $\mathbb{Z}[q^{\pm 1}] \subset \mathbb{Q}[[h]]$, we have a map $\mathbb{Q}[[h]] \to \mathbb{Q}[[h]]$, $h \mapsto \frac{h}{n}$ that sends $F_\infty(q, q^n, \mathcal{K}) \mapsto F_\infty(q^{1/n}, q, \mathcal{K})$ as elements of $\mathbb{Q}[[h]]$. But now $F_\infty(q^{1/n}, q, \mathcal{K})$ converges to $F_\infty(1, q, \mathcal{K})$ in the sense stated in [7, below Theorem 2], namely:

$$\lim_{n\to\infty} F_\infty(q^{1/n}, q, \mathcal{K}) = F_\infty(1, q, \mathcal{K})$$

$$\iff \lim_{n\to\infty} \operatorname{coeff}(F_\infty(q^{1/n}, q, \mathcal{K}), h^m) \\ = \operatorname{coeff}(F_\infty(1, q, \mathcal{K}), h^m) \quad \text{for all } m \in \mathbb{N},$$

where, for any analytic function f,

$$\operatorname{coeff}(f(h), h^m) = \frac{1}{m!}\frac{d^m}{dh^m} f(h)\Big|_{h=0}.$$

By Theorem 60,

$$F_\infty(1, q, \mathcal{K}) = \frac{1}{A_{\mathcal{K}}(q^2)} \quad \text{in } \mathbb{Q}[[h]].$$

On the other hand, if we denote by $\widehat{\mathbb{Z}[A^{\pm 1}]^{\{1\}_A}}$ the ring completion of $\mathbb{Z}[A^{\pm 1}]$ by ideals $((A - A^{-1})^n)$, then $F_\infty(1, A, \mathcal{K}) = C_\infty(1, A, \mathcal{K})$ in $\widehat{\mathbb{Z}[A^{\pm 1}]^{\{1\}_A}}$. Indeed, setting $q = 1$ in Definition 20 and looking at the definition of $C_\infty(1, A, \mathcal{K})$ in Proposition 27, one gets $F_\infty(1, A, \mathcal{K}) = C_\infty(1, A, \mathcal{K})$.

Since $\widehat{\mathbb{Z}[A^{\pm 1}]^{\{1\}_A}} \hookrightarrow \mathbb{Q}[[h]]$, $A \to e^h$ (see [12, Proposition 6.1] and [10, Corollary 4.1]), then we have the following proposition:

Proposition 61. *If $\mathcal{K}$ is 0-framed, $C_\infty(1, A, \mathcal{K}) = \frac{1}{A_{\mathcal{K}}(A^2)}$.*

By the discussion in the paragraph preceding Proposition 31, we have:

Corollary 62. *If $\mathcal{K}$ is 0-framed, then $C_\infty(r, A, \mathcal{K}) = \frac{1}{A_{\mathcal{K}}(A^{2r})}$.*

This allows us to state a factorization theorem:

Theorem 63 (factorization). *For a knot $\mathcal{K}$ and an integer $r \in \mathbb{N}^*$, we have the following factorization in $\widehat{R^I_r}$:*

$$F_\infty(\zeta_{2r}, A, \mathcal{K}) = \frac{A^{rf} \times \mathrm{ADO}_r(A, \mathcal{K})}{A_{\mathcal{K}}(A^{2r})},$$

where f is the framing of the knot.

Corollary 64. *As an immediate consequence,* [13, Conjecture 7.5 and subsequent paragraph] *are verified. In other word,*

$$J_{\mathcal{K}}(t, -1) = \frac{A_{\mathcal{K}}(-t)}{A_{\mathcal{K}}(t^2)}$$

where $t = q^{2(\alpha+1)}$; or, in terms of unified invariant,

$$F_\infty(i, q^\alpha, \mathcal{K}) = \frac{A_{\mathcal{K}}(q^{2\alpha})}{A_{\mathcal{K}}(q^{4\alpha})}.$$

Proof. Recall that $\mathrm{ADO}_2(q^\alpha, \mathcal{K}) = A_{\mathcal{K}}(q^{2\alpha})$. At $r = 2$, using the identification $t = q^{2(\alpha+1)}$, we have the following:

$$\begin{aligned} J_{\mathcal{K}}(t, -1) &= F_\infty(i, q^\alpha, \mathcal{K}) \\ &= \frac{\mathrm{ADO}_2(q^\alpha, \mathcal{K})}{A_{\mathcal{K}}(q^{4\alpha})} \\ &= \frac{A_{\mathcal{K}}(q^{2\alpha})}{A_{\mathcal{K}}(q^{4\alpha})} \\ &= \frac{A_{\mathcal{K}}(-t)}{A_{\mathcal{K}}(t^2)}. \end{aligned}$$

■

The colored Jones polynomials determine the ADO polynomials. One may ask what is the relationship between ADO invariants and the colored Jones polynomials. Does one family of polynomial determines completely the other? For the sake of simplicity the knot $\mathcal{K}$ is supposed 0-framed in this paragraph.

This is the case for $\{\mathrm{ADO}_r(A, \mathcal{K})\}_{r \in \mathbb{N}^*} \to \{J_n(q^2, \mathcal{K})\}_{n \in \mathbb{N}^*}$, knowing the ADO polynomials allows to find the colored Jones polynomials. This result was stated in [5, Corollary 15]. With our setup, we can get back this result as follows.

Remark 65. Notice that

$$F_\infty(\zeta_{2r}, \zeta_{2r}^N, \mathcal{K}) = J_N(\zeta_r, \mathcal{K}) = \frac{\mathrm{ADO}_r(\zeta_{2r}^N, \mathcal{K})}{A_{\mathcal{K}}(1)} = \mathrm{ADO}_r(\zeta_{2r}^N, \mathcal{K}).$$

Since J_N is a polynomial knowing an infinite number of value of it determines it. Given the family of polynomials $\{\mathrm{ADO}_r(A, \mathcal{K})\}_{r \in \mathbb{N}^*}$ we then know each value of J_N at any root of unity hence we know J_N entirely.

But we can also have the other way around:

$$\{J_n(q^2, \mathcal{K})\}_{n \in \mathbb{N}^*} \to \{\mathrm{ADO}_r(A, \mathcal{K})\}_{r \in \mathbb{N}^*}.$$

Knowing only the colored Jones polynomials recover the ADO polynomials. We will prove it by seeing that the colored Jones polynomials determines the unified invariant $F_\infty(q, A, \mathcal{K})$.

For all $k \in \mathbb{N}$, let

$$f_k \colon \mathbb{Q}[\alpha][[h]] \to \mathbb{Q}[[h]], \quad \alpha \mapsto k,$$

the evaluation map.

Proposition 66. $\cap_{k \in \mathbb{N}} \ker(f_k) = \{0\}$

Proof. Let $x \in \bigcap_{k \in \mathbb{N}} \ker(f_k)$, we write $x = \sum_n g_n(\alpha) h^n$ where $g_n(\alpha) \in \mathbb{Q}[\alpha]$. Then, for each $k \in \mathbb{N}$, we have that $g_n(k) = 0$ for all n. Since g_n are polynomials that vanish at an infinite number of point, they are 0. Hence, $\bigcap_{k \in \mathbb{N}} \ker(f_k) = \{0\}$. ■

Let

$$f\colon \mathbb{Q}[\alpha][[h]] \to \prod_{k\in\mathbb{N}} \mathbb{Q}[[h]], \quad x \mapsto (f_k(x))_{k\in\mathbb{N}}.$$

$\ker(f) = \bigcap_{k\in\mathbb{N}} \ker(f_k) = \{0\}$, hence f is injective.

Remark 67. For any knot $\mathcal{K}$, $f(F_\infty(q, A, \mathcal{K})) = \{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*}$.

Proposition 68. *For any knot $\mathcal{K}$, $F_\infty(q, A, \mathcal{K}) = f^{-1}(\{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*})$.*

Setting

$$g\colon \widehat{R}^{\widehat{I}} \to \prod_{r\in\mathbb{N}^*} \widehat{R}^I_r, \quad x \mapsto \left(\mathrm{ev}_r \times \frac{1}{FC_r}(x)\right)_{r\in\mathbb{N}^*},$$

we get the following theorem:

Theorem 69. *The map*

$$h = g \circ f^{-1}\colon \mathrm{Im}(f|_{\widehat{R}^{\widehat{I}}}) \to \prod_{r\in\mathbb{N}^*} \widehat{R}^I_r$$

is such that, for every knot $\mathcal{K}$,

$$\{\mathrm{ADO}_r(A, \mathcal{K})\}_{r\in\mathbb{N}^*} = h(\{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*}).$$

Application 1: the unified invariant and the ADO invariants are q-holonomic. The fact that the colored Jones polynomials are q-holonomic was proved in [6]. Let us state what it means and then let us prove that the unified invariant and the ADO polynomials verify the same holonomic rule. For the sake of simplicity we will work with 0-framed knot.

Let

$$Q\colon \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}^*} \to \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}^*} \quad \text{and} \quad E\colon \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}^*} \to \mathbb{Z}[q^{\pm 1}]^{\mathbb{N}^*}$$

be such that

$$(Qf)(n) = q^{2n} f(n), \quad (Ef)(n) = f(n+1).$$

Note that these operators can be extended to operators on $\mathbb{Q}[[h]]^{\mathbb{N}^*}$.

Let us denote by $J_\bullet(q^2, \mathcal{K}) = \{J_n(q^2, \mathcal{K})\}_{n\in\mathbb{N}^*}$ the colored Jones function. Now, from [6, Theorem 1], for any knot $\mathcal{K}$ there exists a polynomial $\alpha_{\mathcal{K}}(Q, E, q^2)$ such that $\alpha_{\mathcal{K}}(Q, E, q^2) J_\bullet(q^2, \mathcal{K}) = 0$. We say that $J_\bullet(q^2, \mathcal{K})$ is *q-holonomic.*

We define similar operators on $\mathbb{Q}[\alpha][[h]]$ and show that the same polynomial $\alpha_{\mathcal{K}}$, taken in terms of those new operators, annihilates $F_\infty(q, q^\alpha, \mathcal{K})$.

Let

$$\widetilde{Q}\colon \mathbb{Q}[\alpha][[h]] \to \mathbb{Q}[\alpha][[h]] \quad \text{and} \quad \widetilde{E}\colon \mathbb{Q}[\alpha][[h]] \to \mathbb{Q}[\alpha][[h]]$$

be such that if we take

$$x(\alpha) := \sum_{k=0}^{+\infty} x_k(\alpha)h^k \in \mathbb{Q}[\alpha][[h]]$$

with $x_k(\alpha) \in \mathbb{Q}[\alpha]$, then

$$\widetilde{Q}(x(\alpha)) = q^{2\alpha}x(\alpha), \quad \widetilde{E}(x(\alpha)) = x(\alpha+1),$$

where

$$x(\alpha+1) := \sum_{k=0}^{+\infty} x_k(\alpha+1)h^k.$$

Remark 70. Here, $\widetilde{Q}$ is just the multiplication of any element with $q^{2\alpha}$.

Notice that, if one take the injective map

$$f\colon \mathbb{Q}[\alpha][[h]] \to \mathbb{Q}[[h]]^{\mathbb{N}^*}, \quad x(\alpha) \mapsto (x(k))_{k\in\mathbb{N}^*}$$

previously defined, then

$$f \circ \widetilde{Q} = Q \circ f, \quad f \circ \widetilde{E} = E \circ f.$$

Hence, $f \circ \alpha_{\mathcal{K}}(\widetilde{Q}, \widetilde{E}, q^2) = \alpha_{\mathcal{K}}(Q, E, q^2) \circ f$. Since $\alpha_{\mathcal{K}}(Q, E, q^2)J_\bullet(q^2, \mathcal{K}) = 0$ and $f(F_\infty(q, q^\alpha, \mathcal{K})) = J_\bullet(q^2, \mathcal{K})$, we obtain $f \circ \alpha_{\mathcal{K}}(\widetilde{Q}, \widetilde{E}, q^2)(F_\infty(q, q^\alpha, \mathcal{K})) = 0$. The injectivity of f gives the following theorem.

Theorem 71. *For any* 0*-framed knot* $\mathcal{K}$*,* $\alpha_{\mathcal{K}}(\widetilde{Q}, \widetilde{E}, q^2)(F_\infty(q, q^\alpha, \mathcal{K})) = 0$.

Let us look at what happens at roots of unity. To do so we must restrict ourselves to a ring allowing evaluation at roots of unity such as $\widehat{R}^{\widehat{I}}$. Since $\widetilde{Q}(I_n) \subset I_n$ and $\widetilde{E}(I_n) \subset I_n$, we can restrict the operators $\widetilde{Q}$ and $\widetilde{E}$ to $\widehat{R}^{\widehat{I}}$, for the sake of simplicity we will still write them $\widetilde{Q}\colon \widehat{R}^{\widehat{I}} \to \widehat{R}^{\widehat{I}}$ and $\widetilde{E}\colon \widehat{R}^{\widehat{I}} \to \widehat{R}^{\widehat{I}}$.

Now, let $r \in \mathbb{N}^*$ and let

$$\bar{Q}\colon \widehat{R}^I_r \to \widehat{R}^I_r \quad \text{and} \quad \bar{E}\colon \widehat{R}^I_r \to \widehat{R}^I_r$$

be such that if we take

$$x(\alpha) = \sum_{k=0}^{\infty} x_k(\alpha)\{r\alpha\}^k \in \widehat{R}^I_r$$

with $x_k(\alpha) \in \mathbb{Z}[\zeta_{2r}, A]$ (recall that $\zeta_{2r}^\alpha := A$), then

$$\bar{Q}(x(\alpha)) = \zeta_{2r}^{2\alpha}x(\alpha), \quad \bar{E}(x(\alpha)) = x(\alpha+1),$$

where

$$x(\alpha+1)=\sum_{k=0}^{\infty} x_k(\alpha+1)(-1)^k\{r\alpha\}^k.$$

Since $\mathrm{ev}_r \circ \widetilde{Q} = \bar{Q} \circ \mathrm{ev}_r$ and $\mathrm{ev}_r \circ \widetilde{E} = \bar{E} \circ \mathrm{ev}_r$, the same formula holds:

$$\alpha_{\mathcal{K}}(\bar{Q},\bar{E},\zeta_{2r}^2)(F_\infty(\zeta_{2r},\zeta_{2r}^\alpha,\mathcal{K}))=0.$$

By Theorem 63,

$$\alpha_{\mathcal{K}}(\bar{Q},\bar{E},\zeta_{2r}^2)\Big(\frac{\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K})}{A_{\mathcal{K}}(\zeta_{2r}^{2r\alpha})}\Big)=0.$$

Remark 72. We have the following identities:

$$\begin{aligned}\bar{Q}\Big(\frac{\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K})}{A_{\mathcal{K}}(\zeta_{2r}^{2r\alpha})}\Big)&=q^{2\alpha}\frac{\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K})}{A_{\mathcal{K}}(\zeta_{2r}^{2r\alpha})}\\&=\frac{\bar{Q}(\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K}))}{A_{\mathcal{K}}(\zeta_{2r}^{2r\alpha})}\end{aligned}$$

and

$$\begin{aligned}\bar{E}\Big(\frac{\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K})}{A_{\mathcal{K}}(\zeta_{2r}^{2r\alpha})}\Big)&=\frac{\bar{E}(\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K}))}{\bar{E}(A_{\mathcal{K}}(\zeta_{2r}^{2r\alpha}))}\\&=\frac{\bar{E}(\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K}))}{A_{\mathcal{K}}(\zeta_{2r}^{2r\alpha})}\end{aligned}$$

(because $\zeta_{2r}^{2r(\alpha+1)}=\zeta_{2r}^{2r\alpha}$).

Hence,

$$\frac{\alpha_{\mathcal{K}}(\bar{Q},\bar{E},\zeta_{2r}^2)(\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K}))}{A_{\mathcal{K}}(\zeta_{2r}^{2r\alpha})}=0,$$

which proves the following theorem:

Theorem 73. *For any* 0*-framed knot* $\mathcal{K}$,

$$\alpha_{\mathcal{K}}(\bar{Q},\bar{E},\zeta_{2r}^2)(\mathrm{ADO}_r(\zeta_{2r}^\alpha,\mathcal{K}))=0.$$

Remark 74. In the upcoming article [3], Brown, Dimofte, Garoufalidis, and Geer proved that the ADO invariant of links are q-holonomic (Theorem 4.3), which generalise Theorem 73. Their Theorem 4.4 actually gives a converse statement of Theorem 73: any polynomial annihilating the ADO family will also annihilate the colored Jones. This proves that the ADO and colored Jones family are annihilated by the same polynomials.

Application 2: the unified invariant is the loop expansion of the colored Jones function. Let us first introduce the loop expansion of the colored (see in [7, Section 2]). We can write the colored Jones polynomials as an expansion (see [19] for more details):

$$J_n(e^{2h}, \mathcal{K}) = \sum_{k=0}^{+\infty} \frac{P_k(e^{2nh})}{A_{\mathcal{K}}(e^{2nh})^{2k+1}} h^k$$

where $P_k(X) \in \mathbb{Q}[X, X^{-1}]$.

Hence, we get an element

$$J_\alpha(q^2, \mathcal{K}) = \sum_{k=0}^{+\infty} \frac{P_k(e^{2\alpha h})}{A_{\mathcal{K}}(e^{2\alpha h})^{2k+1}} h^k \in \mathbb{Q}[\alpha][[h]]$$

that is such that $f(J_\alpha(q^2, \mathcal{K})) = J_\bullet(q^2, \mathcal{K})$. This means that it evaluates into the colored Jones at $\alpha = n$, we call it *loop expansion of the colored Jones function*.

Proposition 75. *For any knot $\mathcal{K}$, we have the following identity in $\mathbb{Q}[\alpha][[h]]$:*

$$J_\alpha(q^2, \mathcal{K}) = F_\infty(q, q^\alpha, \mathcal{K}).$$

Proof. The fact that f is injective proves the proposition. ■

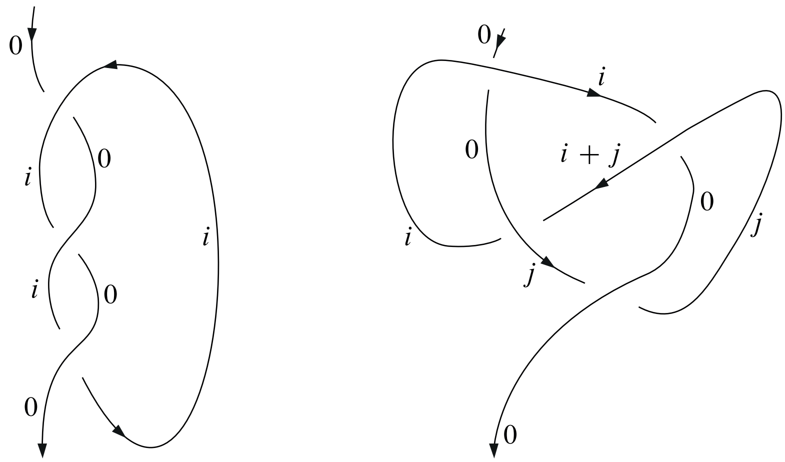
Remark 76. Putting everything together, the results of this section imply that the unified invariant $F_\infty(q, A, \mathcal{K})$ is an integral version of the colored Jones function, built in a ring allowing evaluations at roots of unity. The integrality and the existence of evaluation maps allow us to recover the ADO polynomials, the fact that the completion ring is a subring of an h-adic ring allows us to connect it to other notions of colored Jones function/invariants.

Another approach, described by Gukov and Manolescu in [9], would be to see the unified invariant as a power series in q, A (as opposed to a quantum factorial expansion as we have here). This would be another integral version of it. Indeed, because it verifies Proposition 75 and Theorem 71, the unified invariant $F_\infty(q, A, \mathcal{K})$ except being a power series, also verifies [9, Conjectures 1.5 and 1.6]. Thus, if $F_\infty(q, A, \mathcal{K})$ could be written as a power series, it would fully verify the conjectures. This is the case for positive braid knots, as show by Park in [18]. This means that for a positive braid knot, the unified invariant and the GM power series coincide.

5. Some computations

This section will be dedicated to compute the unified invariant $F_\infty(q, A, \mathcal{K})$ on some examples. We will also explicitly compute $C_\infty(1, A, \mathcal{K})$ and see that it is equal to the inverse of the Alexander polynomial.

To do so we will use state diagrams and compute the unified invariant from it. You can also use them to compute the ADO polynomials (see [16, Section 4]). Recall that $q^{\alpha} := A$.



(a) The trefoil knot. **(b)** The figure eight knot.

Figure 7. Examples of state diagrams to compute the invariants.

The trefoil knot. We denote by 3_1 the trefoil knot. We have

$$F_{\infty}(q, A, 3_1) = q^{\frac{3\alpha^2}{2}} \sum_i q^{\alpha - 2i} q^{\frac{i(i-1)}{2}} \{\alpha; i\}_q q^{-3i\alpha},$$

$$\begin{aligned} C_{\infty}(1, A, 3_1) &= q^{\alpha} \sum_i q^{-3i\alpha} \{\alpha\}_q^i \\ &= q^{\alpha} \frac{1}{1 - q^{-3\alpha}\{\alpha\}_q} \\ &= \frac{q^{3\alpha}}{A_{3_1}(q^{2\alpha})}. \end{aligned}$$

The figure eight knot. We denote by 4_1 the figure eight knot. We have

$$\begin{aligned} F_{\infty}(q, A, 4_1) &= \sum_{i,j} q^{2(i-j)} q^{i\alpha} q^{-j\alpha} (-1)^j q^{\frac{i(i-1)}{2}} \begin{bmatrix} i+j \\ j \end{bmatrix}_q \{\alpha - j; i\}_q q^{(i+j)\alpha} \\ &\quad \times q^{-2ij} q^{-\frac{j(j-1)}{2}} \{\alpha; j\}_q q^{-(i+j)\alpha} q^{2ij} \\ &= \sum_{i,j} q^{2(i-j)} q^{(i-j)\alpha} (-1)^j \begin{bmatrix} i+j \\ j \end{bmatrix}_q q^{\frac{i(i-1)}{2}} q^{-\frac{j(j-1)}{2}} \{\alpha; i+j\}_q, \end{aligned}$$

$$
\begin{aligned}
C_\infty(1, A, 4_1) &= \sum_{i,j} q^{(j-i)\alpha}(-1)^i \binom{i+j}{j} \{\alpha\}_q^{i+j} \\
&= \sum_N \sum_{i=0}^N q^{N\alpha} q^{-2i\alpha}(-1)^j \binom{i+j}{j} \{\alpha\}_q^N \\
&= \sum_N q^{N\alpha} \{\alpha\}_q^N \sum_{i=0}^N (-q^{-2\alpha})^i \binom{i+j}{j} \\
&= \sum_N q^{N\alpha} \{\alpha\}_q^N (1 - q^{-2\alpha})^N \\
&= \sum_N \{\alpha\}_q^{2N} \\
&= \frac{1}{1 - \{\alpha\}_q^2} \\
&= \frac{1}{A_{4_1}(q^{2\alpha})}.
\end{aligned}
$$

The cinquefoil knot. We denote by 5_1 the cinquefoil knot. We have

$$
\begin{aligned}
F_\infty(q, A, 5_1) = q^{\frac{5\alpha^2}{2}} \sum_{i,j,k} & q^{\alpha - 2(i-j+k)} q^{-5(i-j+k)\alpha} q^{2i(k-j)} q^{2k(i-j)} q^{\frac{i(i-1)}{2}} q^{\frac{j(j-1)}{2}} \\
& \times q^{\frac{k(k-1)}{2}} \{\alpha; i\}_q \{\alpha - k + j; j\}_q \{\alpha - i + j; k\}_q \\
& \times \begin{bmatrix} k \\ k-j \end{bmatrix}_q \begin{bmatrix} i-j+k \\ k \end{bmatrix}_q,
\end{aligned}
$$

$$
\begin{aligned}
C_\infty(1, A, 5_1) &= q^\alpha \sum_{i,j,k} q^{-5(i-j+k)\alpha} \{\alpha\}_q^{i+j+k} \binom{k}{k-j} \binom{i-j+k}{k} \\
&= q^\alpha \sum_{i,j,k} q^{-5(i-j+k)\alpha} \{\alpha\}_q^{i+j+k} \binom{i-j+k}{j, k-j, i-j} \\
&= q^\alpha \sum_N \sum_{j,k} q^{-5N\alpha} \{\alpha\}_q^{N+2j} \binom{N}{j, k-j, N-k} \\
&= q^\alpha \sum_N q^{-5N\alpha} \{\alpha\}_q^N \sum_{j,k} \{\alpha\}_q^{2j} \binom{N}{j, k-j, N-k} \\
&= q^\alpha \sum_N q^{-5N\alpha} \{\alpha\}_q^N (2 + \{\alpha\}_q^2)^N \\
&= q^\alpha \frac{1}{1 - q^{-5\alpha} \{\alpha\}_q (2 + \{\alpha\}_q^2)}
\end{aligned}
$$

$$= \frac{q^{5\alpha}}{A_{5_1}(q^{2\alpha})}.$$

The three twist knot. We denote by 5_2 the three twist knot. We have

$$\begin{aligned}
F_\infty(q, A, 5_2) = q^{\frac{-5\alpha^2}{2}} \sum_{i,j,k} & q^{2(i-k)-\alpha} q^{(5i+5j-3k)\alpha} q^{-2ij} q^{-2(j-k)(i+j)} q^{\frac{-i(i-1)}{2}} \\
& \times q^{\frac{-j(j-1)}{2}} q^{\frac{-k(k-1)}{2}} (-1)^{i+j+k} \{\alpha; i\}_q \{\alpha - i; j\}_q \{\alpha - j + k; k\}_q \\
& \times \begin{bmatrix} j \\ j-k \end{bmatrix}_q \begin{bmatrix} i+j \\ j \end{bmatrix}_q,
\end{aligned}$$

$$\begin{aligned}
C_\infty(1, A, 5_2) &= q^{-\alpha} \sum_{i,j,k} q^{(5i+5j-3k)\alpha} (-1)^{i+j+k} \{\alpha\}_q^{i+j+k} \binom{j}{j-k} \binom{i+j}{j} \\
&= q^{\alpha} \sum_{i,j,k} q^{(5i+5j-3k)\alpha} (-1)^{i+j+k} \{\alpha\}_q^{i+j+k} \binom{i+j}{k, j-k, i} \\
&= q^{-\alpha} \sum_N \sum_{j,k} q^{(5N-3k)\alpha} (-1)^{N+k} \{\alpha\}_q^{N+k} \binom{N}{k, j-k, N-i} \\
&= q^{-\alpha} \sum_N q^{5N\alpha} (-1)^N \{\alpha\}_q^N \sum_{j,k} q^{-3k\alpha} (-1)^k \{\alpha\}_q^k \\
&\quad \times \binom{N}{k, j-k, N-i} \\
&= q^{-\alpha} \sum_N q^{5N\alpha} (-1)^N \{\alpha\}_q^N (2 - q^{-3\alpha}\{\alpha\}_q)^N \\
&= q^{-\alpha} \frac{1}{1 + q^{5\alpha}\{\alpha\}_q (2 - q^{-3\alpha}\{\alpha\}_q)} \\
&= \frac{q^{-5\alpha}}{A_{5_2}(q^{2\alpha})}.
\end{aligned}$$

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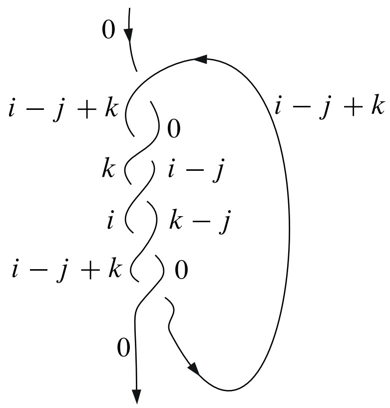
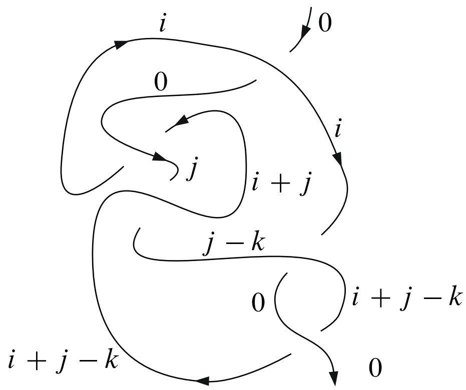


(a) The Cinquefoil Knot.



(b) The three twist Knot.

Figure 8. Examples of state diagrams to compute the invariants.

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