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Web calculus and tilting modules in type C_2

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Abstract. Using Kuperberg's web calculus (1996), and following Elias and Libedinsky, we describe a "light leaves" algorithm to construct a basis of morphisms between arbitrary tensor products of fundamental representations for $\mathfrak{sp}_4$ (and the associated quantum group). Our argument has very little dependence on the base field. As a result, we prove that when $[2]_q \neq 0$, the Karoubi envelope of the C_2 web category is equivalent to the category of tilting modules for the divided powers quantum group $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$.

Contents

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathbf{Rep}(\mathfrak{g})$ denote the category of finite-dimensional modules for $\mathfrak{g}$. By Weyl's theorem on complete reducibility $\mathbf{Rep}(\mathfrak{g})$ is a semisimple category, so as an abelian category $\mathbf{Rep}(\mathfrak{g})$ is determined by the number of its simple objects. Since isomorphism classes of finite-dimensional irreducible $\mathfrak{g}$-modules are in bijection with the countably infinite set of dominant integral weights X_+, $\mathbf{Rep}(\mathfrak{g}) \cong \mathbf{Rep}(\mathfrak{g}')$ as abelian categories, for any two semisimple Lie algebras.

A Lie algebra acts on the tensor product of two representations, so $\mathbf{Rep}(\mathfrak{g})$ is a monoidal category. Viewing $\mathbf{Rep}(\mathfrak{g})$ as a monoidal semisimple category, we capture much more information about $\mathfrak{g}$ (the amount of information can be made precise

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through Tannaka–Krein duality). One then may ask for a presentation by generators and relations of the monoidal category ($\mathbf{Rep}(\mathfrak{g}), \otimes$). A modern point of view on this problem is to find a combinatorial replacement for $\mathbf{Rep}(\mathfrak{g})$ and then use planar diagrammatics to describe the combinatorial replacement by generators and relations.

By combinatorial replacement, we mean a full subcategory of $\mathbf{Rep}(\mathfrak{g})$ monoidally generated by finitely many objects, such that all objects in $\mathbf{Rep}(\mathfrak{g})$ are direct sums of summands of objects in the subcategory. We will focus on the combinatorial replacement $\mathbf{Fund}(\mathfrak{g})$, which is the full subcategory of $\mathbf{Rep}(\mathfrak{g})$ monoidally generated by the irreducible modules $V(\varpi)$ of highest weight ϖ for all fundamental weights ϖ. Note that $\mathbf{Fund}(\mathfrak{g})$ is not an additive category.

We use the terminology $\mathfrak{g}$-webs to refer to a diagrammatic category equivalent to $\mathbf{Fund}(\mathfrak{g})$. The history of $\mathfrak{g}$-webs begins with the Temperley–Lieb algebra [28, 34] for $\mathfrak{sl}_2$ and Kuperberg's "rank two spiders" [19] for $\mathfrak{sl}_3$, $\mathfrak{sp}_4 \cong \mathfrak{so}_5$, and $\mathfrak{g}_2$. D. Kim gave a conjectural presentation for $\mathfrak{sl}_4$-webs [18], and then Morrison gave a conjectural description of $\mathfrak{sl}_n$-webs [22]. Proving that the diagrammatic category was equivalent to $\mathbf{Fund}(\mathfrak{sl}_n)$ turned out to be difficult, but was eventually carried out by Cautis, Kamnitzer, and Morrison using skew Howe duality [6]. Recently, a conjectural description of $\mathfrak{sp}_6$-webs has appeared in a preprint by Rose and Tatham [26].

The Lie algebras $\mathfrak{g}$ for which there are $\mathfrak{g}$-web categories which are known to be equivalent to $\mathbf{Fund}(\mathfrak{g})$ are

$$\mathfrak{g} \in \{\mathfrak{sl}_n, \mathfrak{gl}_n, \mathfrak{sp}_4 \cong \mathfrak{so}_5, \mathfrak{g}_2\}.$$

Each of these $\mathfrak{g}$-web categories has a q-deformed integral form, which we denote by $\mathcal{D}_{\mathfrak{g}}$, over $\mathbb{Z}[q, q^{-1}]$ (or some localization). On the representation theory side, we have Lusztig's divided powers form of the quantum group, denoted $U_q^{\mathbb{Z}}(\mathfrak{g})$. This algebra has modules $V^{\mathbb{Z}}(\varpi)$, which are lattices inside $V(\varpi)$, for each fundamental weight. One should keep in mind that these lattices may not be irreducible after scalar extension to a field. The full subcategory monoidally generated by the modules $V^{\mathbb{Z}}(\varpi)$ will be denoted $\mathbf{Fund}(U_q^{\mathbb{Z}}(\mathfrak{g}))$.

Let $\Bbbk$ be a field and let $q \in \Bbbk^{\times}$. We can specialize the integral versions of both the diagrammatic category and the combinatorial replacement category to $\Bbbk$. It is natural to ask if these two categories are equivalent [3, Section 5A.4]. Taking all sums of summands of objects in $\mathbf{Fund}(\Bbbk \otimes U_q^{\mathbb{Z}}(\mathfrak{g}))$, one obtains the category of tilting modules $\mathbf{Tilt}(\Bbbk \otimes U_q^{\mathbb{Z}}(\mathfrak{g}))$. So, a positive answer to this question means we have found generators and relations for the monoidal category of tilting modules.

For $\mathfrak{g} = \mathfrak{gl}_n$, an answer to this question appears in a paper of Elias [8]. Using ideas from Libedinsky's work [20] on constructing bases for maps between Soergel bimodules, Elias constructs a set of diagrams, denoted $\mathbb{LL}$ and referred to as double ladders, in the $\mathbb{Z}[q, q^{-1}$-linear category $\mathcal{D}_{\mathfrak{gl}_n}$. There are two main arguments in [8].

First, a diagrammatic argument shows that $\mathbb{LL}$ spans the category over $\mathbb{Z}[q, q^{-1}]$. Second, Elias describes a functor $\Gamma\colon \mathcal{D}_{\mathfrak{gl}_n} \to \mathbf{Fund}(U_q^{\mathbb{Z}}(\mathfrak{gl}_n))$ and proves that $\Gamma(\mathbb{LL})$ is linearly independent. After observing that the ranks of homomorphism spaces in $\mathbf{Fund}(\mathbb{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{gl}_n))$ are equal to $\#\mathbb{LL}$ (see [7]), it follows that the diagrams $\mathbb{k} \otimes \mathbb{LL}$ are a basis for $\mathbb{k} \otimes \mathcal{D}_{\mathfrak{gl}_n}$ and the functor $\mathbb{k} \otimes \Gamma$ is an equivalence.

Kuperberg proved [19] there is a monoidal equivalence

$$\mathbb{k} \otimes \mathcal{D}_{\mathfrak{sp}_4} \to \mathbf{Fund}(\mathbb{k} \otimes U_q(\mathfrak{sp}_4)),$$

when $\mathbb{k} = \mathbb{C}(q)$and when $\mathbb{k} = \mathbb{C}$ and $q = 1$. Our goal is to prove this equivalence with as few restrictions on $\mathbb{k}$ and q as possible.

The present work is completely indebted to Elias's approach, and the basis we construct for Kuperberg's $\mathcal{D}_{\mathfrak{sp}_4}$ webs is the analogue of Elias's light ladder basis for $\mathfrak{sl}_n$-webs in [8]. However, our arguments take less effort, since we can use Kuperberg's result [19] that non-elliptic webs span $\mathcal{D}_{\mathfrak{sp}_4}$ over $\mathbb{Z}[q, q^{-1}]$, and are a basis for $\mathcal{D}_{\mathfrak{sp}_4}$ over $\mathbb{C}$, when $q = 1$. Most of our work is to carefully construct an explicit functor $\Xi\colon \mathcal{D}_{\mathfrak{sp}_4} \to U_q^{\mathbb{Z}}(\mathfrak{sp}_4)$-mod.

The following theorem is the main result of the paper.

Theorem 1.1. *If $\mathbb{k}$ is a field and $q \in \mathbb{k}^\times$ is such that $q + q^{-1} \neq 0$, then the functor*

$$\Xi\colon \mathbb{k} \otimes \mathcal{D}_{\mathfrak{sp}_4} \to \mathbf{Fund}(\mathbb{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4)).$$

is a monoidal equivalence, and therefore induces a monoidal equivalence between the Karoubi envelope of $\mathbb{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ and the category $\mathbf{Tilt}(\mathbb{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4))$.

Remark 1.2. The reader who is already well acquainted with [19] may wonder why we are talking about type C_2 and $\mathfrak{sp}_4$, instead of type B_2 and $\mathfrak{so}_5$. This certainly makes no difference classically, since $\mathfrak{sp}_4(\mathbb{C}) \cong \mathfrak{so}_5(\mathbb{C})$. For the purposes of this paper there is no difference over other fields either. Under our hypothesis that $q + q^{-1} \neq 0$ (note that this includes the possibility that $q = 1$ and $\mathbb{k}$ is not characteristic two), there is an isomorphism $\mathbb{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4) \cong \mathbb{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{so}_5)$, as well as an equivalence between $\mathbb{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ and the base change from $\mathbb{Z}[q, q^{-1}]$ to $\mathbb{k}$ of Kuperberg's B_2 spider category.

We chose C_2 over B_2 hoping it would prevent confusion, since the defining relations in $\mathcal{D}_{\mathfrak{sp}_4}$ are slightly different than the relations in Kuperberg's B_2 spider.

Remark 1.3. If we take $\mathbb{k}$ to be an algebraically closed field of characteristic p and let $q = 1$, then $\mathbf{Tilt}(\mathbb{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4))$ is equivalent to the category of tilting modules for the reductive algebraic group $\mathrm{Sp}_4(\mathbb{k})$ [15, Section H.6]. Very little is known about tilting modules for reductive groups in characteristic $p > 0$, and our results apply in this setting as well for all $p > 2$.

Remark 1.4. If $q + q^{-1} = 0$, then the fundamental representation $\Bbbk \otimes V^{\mathbb{Z}}(\varpi_2)$ is not tilting. So, if one is interested in tilting objects the category $\mathbf{Fund}(\mathfrak{g})$ is not the correct category to study. Also, the category $\mathcal{D}_{\mathfrak{sp}_4}$ is not defined when $q + q^{-1} = 0$, because some relations have coefficients with $q + q^{-1}$ in the denominator. One could clear denominators in the relations and obtain a category which is defined when $q+q^{-1}=0$. However, we do not know what this diagrammatic category would describe.

The following result is a consequence of our main theorem, and is new even if $\Bbbk = \mathbb{C}$ and $q = 1$ or if $\Bbbk = \mathbb{C}(q)$.

Theorem 1.5. *Let $\Bbbk$ be a field and let $q \in \Bbbk^{\times}$ so that $q + q^{-1} \neq 0$. The double ladder diagrams defined in Section 2.5 form a basis for the morphism spaces in $\Bbbk \otimes \mathcal{D}_{\mathfrak{sp}_4}$.*

Remark 1.6. As already mentioned, Kuperberg's B_2 web category is spanned by the same non-elliptic diagrams over $\mathbb{Z}[q, q^{-1}]$. The work of Sikora and Westbury [30] proves that these diagrams are linearly independent whenever $q + q^{-1} \neq 0$. Although their techniques are quite different than ours and certainly are worth studying, their result is a consequence of ours.

Suppose that one could show that either double ladder diagrams span or are linearly independent. Since the number of double ladders is equal to the number of non-elliptic webs, the result from [30] would imply that the double ladder diagrams are a basis.

However, it is not possible to obtain our main theorem with just their result. Even though their paper and some basic representation theory implies the dimensions of homomorphism spaces in $\Bbbk \otimes \mathcal{D}_{\mathfrak{sp}_4}$ and $\mathbf{Fund}(\Bbbk \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4))$ are equal, it is not enough to deduce that $\Bbbk \otimes \Xi$ is an equivalence. The difficulty is best illustrated via analogy: the lattice $\mathbb{Z}$ becomes a one-dimensional vector space after base change to any field, but the map $\mathbb{Z} \xrightarrow{x \mapsto 2x} \mathbb{Z}$ is not an isomorphism after tensoring with a field of characteristic two. We really need to know that the map $\Bbbk \otimes \Xi$ is an isomorphism and to do this we must explicitly construct and analyze the functor Ξ.

Remark 1.7. It remains an open problem to adapt the arguments in [8] to prove that double ladder diagrams span $\mathcal{D}_{\mathfrak{sp}_4}$ without using Kuperberg's results about non-elliptic webs. The first steps in this adaptation would be to rewrite every composition of elementary light ladder diagrams of the form $L_\mu \circ (\mathrm{id} \otimes L_\nu)$ as a linear combination of double ladder diagrams. This is an easy exercise which may convince the reader that such an adaptation is possible. The second step is to prove that any diagram of the form $(\mathrm{id} \otimes L_\mu \otimes \mathrm{id}) \circ N \circ (\mathrm{id} \otimes \mathbb{D}(L_\nu) \otimes \mathrm{id})$, where N is an arbitrary neutral diagram, is a linear combination of double ladder diagrams. The case when N is the identity is another easy exercise, and considering the case of arbitrary N may help convince the reader that writing a complete adaptation of [8] would be non-trivial.

Remark 1.8. It is work in progress of Victor Ostrik and Noah Snyder to find the precise relationship between Kuperberg's G_2 webs and tilting modules.

1.1. Potential applications

Let $\mathbb{k} = \mathbb{C}$ and let $q = e^{\pi i/\ell}$. Soergel conjectured [31] and then proved [32] a formula for the character of a tilting module for $\mathbb{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{g})$ when $\ell > h$, where h is the Coxeter number of $\mathfrak{g}$.

In (5.4), we will prove that the category $\mathcal{D}_{\mathfrak{sp}_4}$ is a strictly object adapted cellular category [10]. Thus, the discussion in [12, Section 11.5] allows one to adapt the algorithm in [16] from the context of Soergel bimodules to $\mathfrak{sp}_4$-webs. Using this algorithm, which we outline in Section 5.2, one can compute tilting characters for the quantum group at a root of unity as long as $\ell \geq 3$ (the $\ell = 2$ case is ruled out by the assumption in our theorem that $q + q^{-1} \neq 0$). The Coxeter number of $\mathfrak{sp}_4$ is $h = 4$. This means that when $\ell = 3$, Soergel's conjecture for tilting characters does not apply but the diagrammatic category $\mathbb{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ does still describe tilting modules.

There may be a conjecture for the characters of tilting modules of quantum groups that includes $\ell \leq h$, along the lines of [11, Section 8.1] and [25, Theorem 1.6]. Ideally, the conjecture would relate tilting characters for the quantum group at a root of unity to singular, antispherical Kazhdan–Lusztig polynomials. One could use $\mathfrak{sp}_4$ webs to check such a conjecture for small weights.

There are other open questions related to tilting modules when ℓ is large enough for the diagrammatic category to be equivalent to the category of tilting modules, but ℓ is still less than the Coxeter number. For example, what is the semisimplification of the category of tilting modules for such ℓ? The solution to this problem when $\ell > h$ is very well known, and provides a wealth of examples of finite tensor categories. When $\ell > h$ satisfies certain congruence conditions based on the root system of $\mathfrak{g}$ (for $\mathfrak{sp}_4$ the condition is ℓ is even) the semisimplification of the category of tilting modules is a modular category [27] which gives rise to a Reshetikhin–Turaev 3-manifold invariant [35]. Theorem (1.1) implies that $\mathcal{D}_{\mathfrak{sp}_4}$ can be used to aid in the calculation of these three manifold invariants.

By interpreting $\mathfrak{gl}_n$-webs in terms of Schur algebras, Brundan, Entova-Aizenbud, Etingof, and Ostrik [5] use results of Donkin to reprove that

$$\mathbb{k} \otimes \mathcal{D}_{\mathfrak{gl}_n} \cong \mathbb{k} \otimes \mathbf{Fund}(U_q^{\mathbb{Z}}(\mathfrak{gl}_n))$$

for any field when $q = 1$. The main result of [5] is that when $q = 1$ and char $\mathbb{k} < h$, the semisimplification of $\mathbf{Tilt}(\mathbb{k} \otimes U_q(\mathfrak{gl}_n))$ is a semisimple monoidal category, which may have infinitely many objects, and is related to Kazhdan–Lusztig cells the affine Hecke algebra.

When Soergel's results on tilting characters of the quantum group are known to hold, Ostrik proved [23] that there is a bijection between cells in the antispherical module for the Langlands dual affine Hecke algebra and thick monoidal ideals in the category of tilting modules $\mathbf{Tilt}(\Bbbk \otimes U_q^{\mathbb{Z}}(\mathfrak{g}))$. On the other hand, a deep theorem of Lusztig [21] is that there is also a bijection between cells in the antispherical module for affine Weyl group and orbits in the nilpotent cone of G. Note that the bijection between nilpotent orbits and thick monoidal ideals no longer appears to involve Langlands duality.

The maximal thick monoidal ideal in the category of tilting modules corresponds to the "highest" cell in the antispherical module which in turn corresponds to the regular nilpotent orbit. This maximal ideal coincides with the ideal of negligible morphisms, denoted by $\mathcal{N}$, and therefore the quotient is what is referred to as the semisimplification of the category of tilting modules. Since Soergel's methods of proof do not apply when $\ell < h$, it follows that Ostrik's results also do not apply. There may still be a non-trivial negligible ideal, but it might be that the objects in it now correspond to a different cell in the antispherical module and correspondingly a different nilpotent orbit.

When $\mathfrak{g} = \mathfrak{sp}_4$ and $\ell = 3$, we still have a non-trivial semisimplification (this is not the case when $\ell = 2$) and now the "highest" cell is replaced by the unique reduced expression cell. The unique reduced expression cell corresponds via Lusztig's bijection to the sub regular nilpotent orbit $\mathcal{O}_{\text{subreg}}$. The group $\mathrm{Sp}_4(\mathbb{C})$ acts on this orbit by conjugation. Now, fix a point $u \in \mathcal{O}_{\text{subreg}}$ in the orbit. The stabilizer of u is an algebraic group with maximal reductive quotient, denoted G_u, a two component disconnected group with a one-dimensional torus for the identity component. As an abstract group G_u is an extension of $\mathbb{Z}/2$ by $\mathbb{C}^\times$. We conjecture that G_u is a split but non-trivial extension.

Motivated by these observations, we expect the following. Let $\Bbbk = \mathbb{C}$ and let $q \in \mathbb{C}$ be a primitive 2ℓ-th root of unity for $\ell = 3$ or 4. There is an equivalence of monoidal categories $\mathbf{Tilt}(\Bbbk \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4))/\mathcal{N} \to \mathbf{Rep}(\mathbb{C}^\times \rtimes \mathbb{Z}/2)$. In order to prove this we will certainly need to use the results of this paper, as well as develop something like webs for the group $\mathbb{C}^\times \rtimes \mathbb{Z}/2$. Other work in progress of the author which stems from the results in this paper is adapting Elias's clasp conjectures [8] to $\mathfrak{sp}_4$ webs. Work in progress of Ben Elias and Geordie Williamson uses $\mathcal{D}_{\mathfrak{sp}_4}$ to extend the quantum algebraic Satake equivalence [9] to type B_2/C_2.

1.2. Structure of the paper

In Section 2 we discuss how to decompose tensor products of representations for $\mathfrak{sp}_4$. Then use the plethysm patterns to describe an algorithm for light ladder diagrams. Finally, we define the double ladder diagrams. In Section 3 we define an evaluation functor from the diagrammatic category to the representation theoretic category. After reviewing some of the theory of tilting modules for quantum groups/reductive algebraic groups, we interpret the image of the evaluation functor as an integral form of the category of tilting modules. Then we argue that the main theorem follows from linear independence of the image of the double ladder diagrams. In Section 4 we argue that the double ladder diagrams are linearly independent. Then we deduce that $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}$ is an object adapted cellular category, and describe an algorithm to compute tilting characters.

2. Light ladders in type C_2

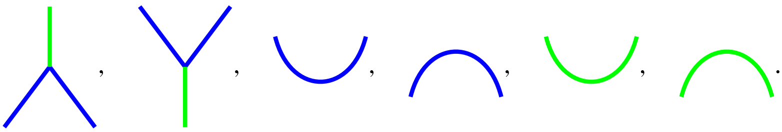
2.1. C_2-webs

We use the convention that the quantum integers in $\mathbb{Z}[q, q^{-1}]$ are defined as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{for } n \in \mathbb{Z}. \tag{2.1}$$

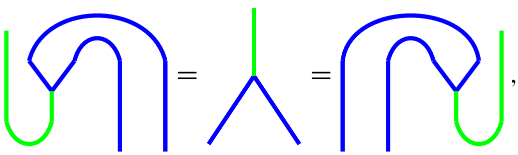
Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}, [2]_q^{-1}]$, the ring $\mathbb{Z}[q, q^{-1}]$ localized at $[2]_q$.

Definition 2.1. Let $\mathcal{D}$ be the $\mathcal{A}$-linear monoidal category defined by generators and relations. The generating objects are 1 and 2, the generating morphisms are the following diagrams:

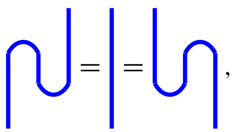


(2.2)

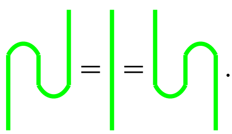
The relations are the following local relations on diagrams:



(2.3)



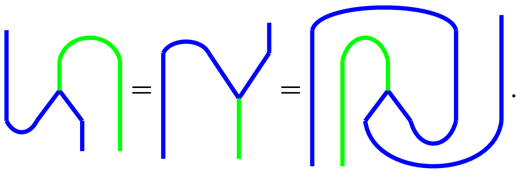
(2.4)



(2.5)

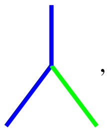
Remark 2.2. Our convention is that diagrams are read as morphisms from the bottom boundary to the top boundary. Composition of morphisms is vertical stacking. The monoidal structure on objects is concatenation of words and the monoidal unit is the empty word. The monoidal product on morphisms is horizontal concatenation of diagrams, and the identity morphism of the empty word is the empty diagram.

Notation 2.3. The defining relations in $\mathcal{D}$ imply the following equalities of morphisms in $\mathrm{Hom}_{\mathcal{D}}(12, 1)$:



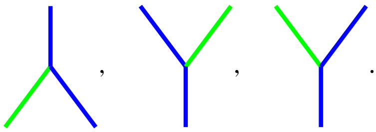
(2.6)

We will denote any one of these morphisms by the following trivalent vertex diagram in $\mathrm{Hom}_{\mathcal{D}}(12, 1)$:



(2.7)

There are similar equalities for every possible vertical and horizontal reflection, and we will write the corresponding trivalent morphisms as follows:



(2.8)

Thanks to this notation, we may now view morphisms in $\mathcal{D}$ as $\mathcal{A}$-linear combinations of isotopy classes trivalent graphs; see, e.g., Figure 1.

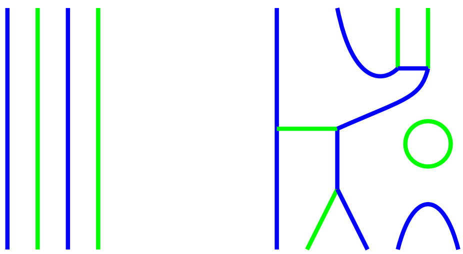
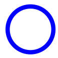
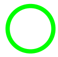
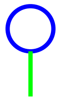


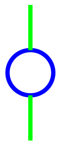
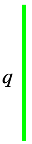
Figure 1. The identity morphism of 1211 and a morphism from 12111 to 1122.

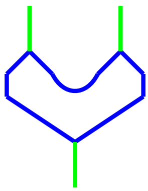
Definition 2.4. The $\mathcal{A}$-linear monoidal category $\mathcal{D}_{\mathfrak{sp}_4}$ is the quotient of $\mathcal{D}$ by the following local relations:

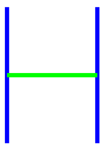
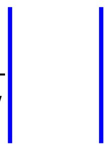
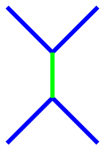
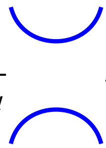
 $= -\dfrac{[6]_q[2]_q}{[3]_q}$, (2.9)

 $= \dfrac{[6]_q[5]_q}{[3]_q[2]_q}$, (2.10)

 $= 0$, (2.11)

 $= -[2]_q$  , (2.12)

 $= 0$, (2.13)

 $= \dfrac{1}{[2]_q}$  $+$  $- \dfrac{1}{[2]_q}$  . (2.14)

Notation 2.5. When $\Bbbk$ is an $\mathcal{A}$-algebra, we can base change the category $\mathcal{D}_{\mathfrak{sp}_4}$ to $\Bbbk$, denoted $\Bbbk \otimes \mathcal{D}_{\mathfrak{sp}_4}$. The category $\Bbbk \otimes \mathcal{D}_{\mathfrak{sp}_4}$ has the same objects as $\mathcal{D}_{\mathfrak{sp}_4}$ and we apply $\Bbbk \otimes_{\mathcal{A}} (-)$ to homomorphism spaces. We may also write $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4} := \Bbbk \otimes \mathcal{D}_{\mathfrak{sp}_4}$ for short.

Remark 2.6. The coefficients in the circle relations are written as fractions but are actually elements of $\mathcal{A}$, as can be observed in the following quantum number calculations:

$$\begin{aligned} [5]_q - [1]_q &= \frac{([5]_q - [1]_q)[3]_q}{[3]_q} \\ &= \frac{[7]_q + [5]_q + [3]_q - [3]_q}{[3]_q} \\ &= \frac{[6]_q[2]_q}{[3]_q}. \end{aligned} \tag{2.15}$$

and

$$
\begin{aligned}
[7]_q - [5]_q + [3]_q &= \frac{[8]_q + [2]_q}{[2]_q} \\
&= \frac{[10]_q + [8]_q + [6]_q + [4]_q + [2]_q}{[3]_q[2]_q} \\
&= \frac{[6]_q[5]_q}{[3]_q[2]_q}. \qquad (2.16)
\end{aligned}
$$

Remark 2.7. The category $\mathcal{D}_{\mathfrak{sp}_4}$ is almost the B_2 spider category in [19]. But we replaced q with q^2 and rescaled the trivalent vertex by $[2]_q^{-1/2}$. The trivalent vertex in $\mathcal{D}_{\mathfrak{sp}_4}$ may seem less natural since the relations now require us to insist $[2]_q$ is invertible, but when we connect the diagrammatic category to representation theory the rescaled trivalent vertex in $\mathcal{D}_{\mathfrak{sp}_4}$ will be more natural.

2.2. Decomposing tensor products in Rep($\mathfrak{sp}_4(\mathbb{C})$)

We now recall some basic facts about $\mathfrak{sp}_4(\mathbb{C})$ and its representation theory. Some of this is worked out in detail in [13, Lecture 16]. Then we will record some formula's describing the decomposition of certain tensor products in **Rep**($\mathfrak{sp}_4$).

Let $X = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ be the weight lattice for $\mathfrak{sp}_4(\mathbb{C})$. The weights $\varpi_1 = \varepsilon_1$ and $\varpi_2 = \varepsilon_1 + \varepsilon_2$ are called the fundamental weights, and $X_+ = \mathbb{Z}_{\geq 0}\varpi_1 \oplus \mathbb{Z}_{\geq 0}\varpi_2$ is the set of dominant weights.

Let **Fund**($\mathfrak{sp}_4(\mathbb{C})$) be the full monoidal subcategory of **Rep**($\mathfrak{sp}_4(\mathbb{C})$) generated by $V(\varpi_1)$ and $V(\varpi_2)$. The decomposition

$$
V(\varpi_1) \otimes V(\varpi_1) \cong V(2\varpi_1) \oplus V(\varpi_2) \oplus V(0). \qquad (2.17)
$$

implies there is a one-dimensional space of maps between $V(\varpi_1) \otimes V(\varpi_1)$ and $V(\varpi_2)$. We will later prove that there is a choice for this map so that sending the trivalent vertex to the chosen map gives a well-defined monoidal functor from $\mathbb{C} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ to **Fund**($\mathfrak{sp}_4(\mathbb{C})$). We will then show that this functor is full and faithful.

For now we will take the equivalence on faith, and use it to guide our intuition for constructing a basis for Hom spaces in $\mathcal{D}_{\mathfrak{sp}_4}$. Let λ and μ be dominant integral weights. There is a direct sum decomposition

$$
V(\lambda) \otimes V(\mu) \cong \bigoplus_{\nu \in X(\lambda,\mu) \subset \mathrm{wt}(V(\mu))} V(\lambda + \nu), \qquad (2.18)
$$

where wt($V(\mu)$) is the multiset of weights in $V(\mu)$ and $X(\lambda, \mu)$ is a submultiset. Our goal is to determine the set $X(\lambda, \mu)$.

To simplify notation, we may write $V(a,b)$ in place of $V(a\varpi_1 + b\varpi_2)$. The following formulas are easy to work out using classical theory. For example, one can use [24, Theorem 2.1]:

$$V(a,b) \otimes V(1,0) \cong \begin{cases} V(1,0) & \text{if } a=b=0, \\ V(a+1,0) \oplus V(a-1,1) \oplus V(a-1,0) & \text{if } a \geq 1, b=0, \\ V(1,b) \oplus V(1,b-1) & \text{if } a=0, b \geq 1, \\ V(a+1,b) \oplus V(a-1,b+1) & \\ \quad \oplus V(a-1,b) \oplus V(a+1,b-1), & \text{if } a \geq 1, b \geq 1; \end{cases} \tag{2.19a}$$

$$V(a,b) \otimes V(0,1) \cong \begin{cases} V(0,1) & \text{if } a=b=0, \\ V(0,b+1) \oplus V(2,b-1) \oplus V(0,b-1) & \text{if } a=0, b \geq 1, \\ V(1,1) \oplus V(1,0) & \text{if } a=1, b=0, \\ V(1,b+1) \oplus V(1,b) \oplus V(3,b-1) \oplus V(1,b-1) & \text{if } a=1, b \geq 1, \\ V(a,1) \oplus V(a,0) \oplus V(a-2,1) & \text{if } a \geq 2, b=0, \\ V(a,b+1) \oplus V(a+2,b-1) & \\ \quad \oplus V(a,b-1) \oplus V(a,b) \oplus V(a-2,b+1) & \text{if } a \geq 2, b \geq 1. \end{cases} \tag{2.19b}$$

Notation 2.8. We will write $V(1) = V(\varpi_1) = V(1,0)$ and $V(2) = V(\varpi_2) = V(0,1)$ as well as $\operatorname{wt} 1 = \varpi_1$ and $\operatorname{wt} 2 = \varpi_2$. Also, for a sequence $\underline{w} = (w_1, \dots, w_n)$, $w_i \in \{1, 2\}$ we will write $V(\underline{w}) = V(w_1) \otimes \cdots \otimes V(w_n)$, $\operatorname{wt} \underline{w} = \operatorname{wt} w_1 + \operatorname{wt} w_2 + \cdots + \operatorname{wt} w_n$, and $\underline{w}_{\leq k} = (w_1, w_2, \dots, w_k)$.

Definition 2.9. Let $\underline{w} = (w_1, \dots, w_n)$ with $w_i \in \{1, 2\}$. A sequence $(\mu_1, \dots, \mu_n)$ where $\mu_i \in \operatorname{wt}(V(w_i))$ is a *dominant weight subsequence* of $\underline{w}$ if

(1) μ_1 is dominant;

(2) $V(\mu_1 + \cdots + \mu_{i-1} + \mu_i)$ is a summand of $V(\mu_1 + \cdots + \mu_{i-1}) \otimes V(w_i)$.

We write $E(\underline{w})$ for the set of all dominant weight subsequences of $\underline{w}$ and

$$E(\underline{w}, \lambda) := \{(\mu_1, \dots, \mu_n) \in E(\underline{w}) \colon \mu_1 + \cdots + \mu_n = \lambda\} \tag{2.20}$$

for all $\lambda \in X_+$.

Lemma 2.10. *Let $\underline{w} = (w_1, \dots, w_n)$, $w_i \in \{1, 2\}$, then*

$$V(\underline{w}) \cong \bigoplus_{(\mu_1,\dots,\mu_n) \in E(\underline{w})} V(\mu_1 + \cdots + \mu_n). \tag{2.21}$$

If we denote the multiplicity of $V(\lambda)$ as a summand of $V(\underline{w})$ by $[V(\underline{w}) : V(\lambda)]$, then

$$[V(\underline{w}) : V(\lambda)] = \#E(\underline{w}, \lambda). \tag{2.22}$$

Proof. If we begin with $V(\emptyset) = \mathbb{C}$ and tensor with $V(w_1)$, then there is only one irreducible summand. This summand corresponds to the dominant weight in $\operatorname{wt} V(w_1)$, which we record as μ_1. Then we tensor $V(w_1)$ by $V(w_2)$ and note that $V(w_1) \otimes V(w_2)$ contains $V(\mu_1) \otimes V(w_2)$ as a summand. Choose a summand of $V(\mu_1) \otimes V(w_2)$. The chosen summand is isomorphic to $V(\mu_1 + \mu_2)$ for some weight $\mu_2 \in \operatorname{wt} V(w_2)$, and we record this choice of summand by the weight $\mu_2 \in \operatorname{wt} V(w_2)$. Next, we tensor $V(w_1) \otimes V(w_2)$ by $V(w_3)$, observe that $V(w_1) \otimes V(w_2) \otimes V(w_3)$ contains a summand isomorphic to $V(\mu_1 + \mu_2) \otimes V(w_3)$, and choose a summand of this summand. The chosen summand is isomorphic to $V(\mu_1 + \mu_2 + \mu_3)$ and we record the choice by the weight $\mu_3 \in \operatorname{wt} V(w_3)$. Iterating this procedure, we end up with a sequence of weights $(\mu_1, \ldots, \mu_n)$, which is a dominant weight subsequence of $\underline{w}$, and a summand in $V(\underline{w})$ isomorphic to $V(\mu_1 + \cdots + \mu_n)$. Furthermore, all summands of $V(\underline{w})$ can be realized uniquely as the end result of the process we just described. ∎

Lemma 2.11. *Let $\underline{u} = (u_1, \ldots, u_n)$ be a sequence with $u_i \in \{1, 2\}$, then*

$$\dim \operatorname{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})) = \sum_{\lambda \in X_+} [V(\underline{w}) : V(\lambda)][V(\underline{u}) : V(\lambda)]. \tag{2.23}$$

Proof. Thanks to Lemma (2.10), this is consequence of Schur's lemma. ∎

2.3. Motivating the light ladder algorithm

We outline a well-known construction of a basis of homomorphism spaces in the category $\mathbf{Fund}(\mathfrak{sp}_4(\mathbb{C}))$.

Suppose that $(\mu_1, \ldots, \mu_m) \in E(\underline{w}, \lambda)$. For $i = 1, \ldots, m$ there is a projection map

$$P_{(\mu_1, \ldots, \mu_i)} \colon V(w_1) \otimes \cdots \otimes V(w_i) \to V(\mu_1 + \cdots + \mu_i).$$

The map $P_{(\mu_1, \ldots, \mu_i)}$ is the projection

$$P_{(\mu_1, \ldots, \mu_{i-1})} \colon V(w_1) \otimes \cdots \otimes V(w_{i-1}) \to V(\mu_1 + \cdots + \mu_{i-1})$$

postcomposed with the projection

$$p_{\mu_i} \colon V(\mu_1 + \cdots + \mu_{i-1}) \otimes V(w_i) \to V(\mu_1 + \cdots + \mu_i).$$

Let $(\nu_1, \ldots, \nu_n) \in E(\underline{v}, \lambda)$. Now, for $i = 1, \ldots, n$ there are inclusion maps

$$I^{(\nu_1, \ldots, \nu_i)} \colon V(\nu_1 + \cdots + \nu_i) \to V(u_1) \otimes \cdots \otimes V(u_i).$$

Composing the projection with the inclusion we get a map

$$I^{(\nu_1,\dots,\nu_n)} \circ P_{(\mu_1,\dots,\mu_m)} \colon V(\underline{w}) \to V(\underline{u}),$$

factoring through $V(\lambda)$.

Since $[V(\lambda) : V(\underline{w})] = E(\underline{w}, \lambda)$ and $[V(\lambda) : V(\underline{u})] = E(\underline{u}, \lambda)$, the maps

$$\bigcup_{\substack{\lambda \in X_+ \\ (\mu_1,\dots,\mu_m) \in E(\underline{w},\lambda) \\ (\nu_1,\dots,\nu_n) \in E(\underline{u},\lambda)}} \{I^{(\nu_1,\dots,\nu_n)} \circ P_{(\mu_1,\dots,\mu_m)}\} \tag{2.24}$$

form a basis in $\mathrm{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u}))$.

The maps $P_{(\mu_1,\dots,\mu_n)}$ are built inductively out of the p_{μ_i}'s in a way that is analogous to how we will define light ladder diagrams in terms of elementary light ladder diagrams. The inclusion map $I^{(\nu_1,\dots,\nu_n)} \colon V(\lambda) \to V(\underline{u})$ is analogous to what we will call upside down light ladder diagrams. We will define double ladder diagrams as the composition of a light ladder diagram and an upside down light ladder diagram, in analogy with the $I \circ P$'s. Then our work will be to argue that double ladder diagrams are a basis.

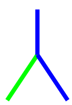
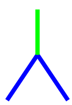
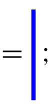
Remark 2.12. The projection and inclusion maps we discuss here are not the image of the light ladder diagrams under a functor $\mathcal{D}_{\mathfrak{sp}_4} \to \mathbf{Fund}(\mathfrak{sp}_4(\mathbb{C}))$. There are at least two reasons for this. The first being that the object $V(\lambda)$ is not in the category $\mathbf{Fund}(\mathfrak{sp}_4(\mathbb{C}))$, so we have to construct light ladder maps not from $V(\underline{w})$ to $V(\lambda)$, but from $V(\underline{w})$ to $V(\underline{x})$ where $\mathrm{wt}\,\underline{x} = \lambda$.

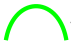
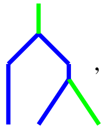
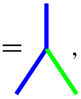
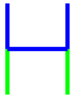
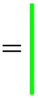
The second reason is that we want to construct a basis for the diagrammatic category which descends to a basis in **Fund** for fields other than $\mathbb{C}$. Over other fields the representation theory is no longer semisimple so $V(\lambda)$ may not be a summand of $V(\underline{w})$. There will still be the same number of maps from $V(\underline{w})$ to a suitable version of $V(\lambda)$ but they may not be inclusions and projections.

2.4. Light ladder algorithm

Now, we define some morphisms in the diagrammatic category.

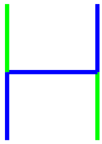
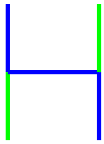
Definition 2.13. An *elementary light ladder diagram* is one of the following diagrams in $\mathcal{D}_{\mathfrak{sp}_4}$. We will say that L_μ is the elementary light ladder diagram of weight μ:

$L_{(-1,0)} =$ , $L_{(1,-1)} =$ , $L_{(-1,1)} =$ , $L_{(1,0)} =$ ; (2.25)

$L_{(0,-1)} =$ , $L_{(-2,1)} =$ , $L_{(0,0)} =$ , $L_{(2,-1)} =$ , $L_{(0,1)} =$ . (2.26)

Remark 2.14. If $L_\mu : \underline{u}* \to \underline{w}$, for $* \in \{1, 2\}$, then $\mu \in \operatorname{wt} V(*)$ and $\operatorname{wt} \underline{w} = \operatorname{wt} \underline{u} + \mu$.

Definition 2.15. A *neutral diagram* is any diagram which is the horizontal and/or vertical composition of identity maps and the following *basic neutral diagrams*:

$N_{12}^{21} =$ , $N_{21}^{12} =$ . (2.27)

An example of a neutral diagram appears in Figure 2.

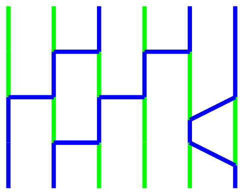


Figure 2. A neutral diagram from 112221 to 221211.

Definition 2.16. Fix an object $\underline{w} = (w_1, \ldots, w_n)$ in $\mathcal{D}_{\mathfrak{sp}_4}$, a dominant weight subsequence $\vec{\mu} = (\mu_1, \ldots, \mu_n) \in E(\underline{w})$, and an object $\underline{v} = (v_1, \ldots, v_m)$ in $\mathcal{D}_{\mathfrak{sp}_4}$ such that $\operatorname{wt} \underline{v} = \mu_1 + \cdots + \mu_n$. We will describe an algorithm, which we will refer to as the *light ladder algorithm*, to construct a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ with source $\underline{w}$ and target $\underline{v}$. This diagram will be denoted $LL_{\underline{w},\vec{\mu}}^{\underline{v}}$ and we will call it a *light ladder diagram*.

We define the diagrams inductively, starting by defining $LL_{\emptyset,(\emptyset)}^{\emptyset}$ to be the empty diagram. Suppose we have constructed $LL_{\underline{w}_{\leq n-1},(\mu_1,\ldots,\mu_{n-1})}^{\underline{u}}$, where

$$\operatorname{wt}(\underline{u}) = \mu_1 + \cdots + \mu_{n-1}.$$

Then we define

$$LL_{\underline{w},(\mu_1,\ldots,\mu_n)}^{\underline{v}} = N_?^{\underline{v}} \circ (\operatorname{id} \otimes L_{\mu_n}) \circ (N_{\underline{u}}^? \otimes \operatorname{id}) \circ (LL_{\underline{w}_{\leq n-1},(\mu_1,\ldots,\mu_{n-1})}^{\underline{u}} \otimes \operatorname{id}_{w_n}), \tag{2.28}$$

where $N_?^?$ is a neutral diagram with appropriate source (subscript) and target (superscript). The graphical interpretation of this algorithm is summarized in Figure 3.

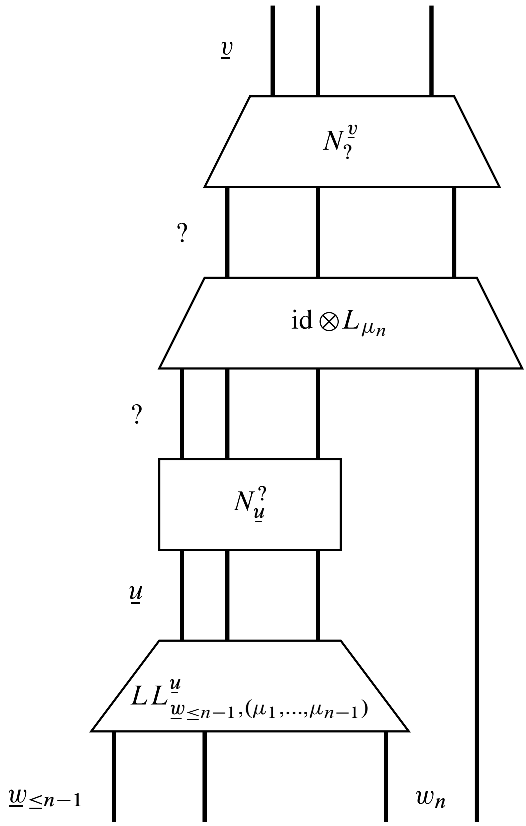
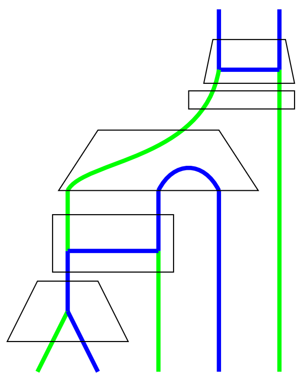


Figure 3. A schematic for the inductive definition of a light ladder diagram $LL_{\underline{w},(\mu_1,\ldots,\mu_n)}$.

To further aid the readers understanding of the light ladder construction we give an example and some clarifying comments.

Example 2.17. A light ladder diagram $LL^{11}_{21212,((0,1),(1,-1),(0,1),(-1,0),(2,-1))}$:



$$. \tag{2.29}$$

Our convention of rectangles and trapezoids is to indicate whether a diagram is a neutral diagram or a diagram of the form id $\otimes$ elementary light ladder diagram. We omitted the first and third steps corresponding to $\mu = (0, 1)$.

The elementary light ladder diagrams have fixed source and target. As a result one can construct $LL^{\underline{u}}_{\underline{w}_{\leq n-1},(\mu_1,\ldots,\mu_{n-1})}$, then see that $V(\mu_n)$ is a summand of $V(\mu_1 + \cdots + \mu_{n-1}) \otimes V(w_n)$, but still not guarantee there is an object $\underline{y}$ in $\mathcal{D}_{\mathfrak{sp}_4}$ such that $\underline{y}w_n$ is the source of $\mathrm{id} \otimes L_{\mu_n}$. An example of this problem appears in Figure 4.

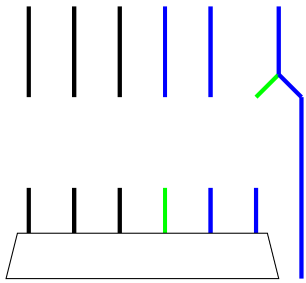


Figure 4. An example of what can go wrong without neutral diagrams.

Basic neutral diagrams encode isomorphisms $12 \to 21$ and $21 \to 12$, while arbitrary neutral diagrams encode isomorphisms $\underline{w} \to \underline{w}'$.

Remark 2.18. The reason we use basic neutral diagrams instead of the braiding is the latter is a non-trivial linear combination of diagrams in $\mathcal{D}_{\mathfrak{sp}_4}$, while the former is a single diagram in $\mathcal{D}_{\mathfrak{sp}_4}$.

Lemma 2.19. *Given two sequences $\underline{w}$ and $\underline{w}'$ such that $\mathrm{wt}\,\underline{w} = \mathrm{wt}\,\underline{w}'$, there is a neutral diagram connecting $\underline{w}$ to $\underline{w}'$.*

Proof. Suppose that $\mathrm{wt}\,\underline{w} = a\varpi_1 + b\varpi_2 = \mathrm{wt}\,\underline{w}'$. Connect both $\underline{w}$ and $\underline{w}'$ via colored neutral diagrams to the standard sequence $1^{\otimes a} \otimes 2^{\otimes b}$ and then compose the neutral diagram from $\underline{w}$ to the standard sequence with the vertical flip of the neutral diagram from the standard diagram to $\underline{w}'$. ∎

The following lemma uses this observation to fix the problem, in the light ladder algorithm, of elementary diagrams having fixed source and target. We illustrate the solution provided by this lemma in Figure 5.

Lemma 2.20. *Let $(\mu_1, \ldots, \mu_n) \in E(\underline{w})$ (in particular, $V(\mu_n)$ is a summand of $V(\mu_1 + \cdots + \mu_{n-1}) \otimes V(w_n)$). Suppose we have constructed $LL^{\underline{u}}_{\underline{w}_{\leq n-1},(\mu_1,\ldots,\mu_{n-1})}$. There is an object $\underline{y}$ in $\mathcal{D}_{\mathfrak{sp}_4}$ and a neutral map $N^{\underline{y}}_{\underline{w}_{\leq n-1}}$ such that $\underline{y} \otimes w_n$ is the source of* $\mathrm{id} \otimes L_{\mu_n}$.

Proof. We will argue this for the elementary diagram $L_{(1,-1)}$, so $\mu_n = (1, -1)$ and $w_n = 1$. The arguments for the rest of the cases follow the same pattern. From the tensor product decomposition formulas (2.19) we see that $V(1, -1)$ being a summand of $V(\mu_1 + \cdots + \mu_{n-1})$ implies that, if $\mu_1 + \cdots + \mu_{n-1} = a\varpi_1 + b\varpi_2$, then $b \geq 1$. Thus, in the sequence $\underline{u} = (u_1, \ldots, u_k)$ there is some k such that $u_k = 2$. By Lemma (2.19) there is a neutral diagram from the sequence $\underline{u}$ to a sequence which ends in 2. The target of this neutral diagram will be an object $\underline{y}$ such that $\underline{y} \otimes 1$ is the source of $\mathrm{id} \otimes L_{(1,-1)}$. ∎

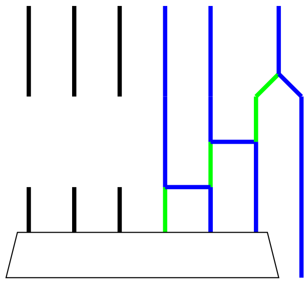


Figure 5. Using a neutral map to fix the problem.

Comparing the tensor product decompositions in (2.19) with the elementary light ladder diagrams, it is evident that dominant weight subsequences always produce a light ladder diagram. Neutral diagrams from one word to another are not unique. The choice of neutral diagram could result in several different light ladder diagrams for a given dominant weight subsequence.

Remark 2.21. For any $\underline{w}$ and $\underline{u}$ such that $\mathrm{wt}\,\underline{w} = \mathrm{wt}\,\underline{u}$, there is a distinguished choice of neutral diagram corresponding to the minimal coset representative in the symmetric group realizing the shuffle from one sequence to the other. However, we do not require that we choose particular elements as our neutral diagrams in the light ladder algorithm.

2.5. Double ladders

We define a contravariant endofunctor $\mathbb{D}$ on the category $\mathcal{D}_{\mathfrak{sp}_4}$ by requiring that $\mathbb{D}$ fixes objects and turns diagrams upside down. Note that $\mathbb{D}^2 = \mathrm{id}_{\mathcal{D}_{\mathfrak{sp}_4}}$, so $\mathbb{D}$ is a duality on the category.

Definition 2.22. Let $LL^{\underline{v}}_{\underline{w},\vec{\mu}}$ be a light ladder diagram. The associated *upside down light ladder diagram* is defined to be

$$\mathbb{D}(LL^{\underline{v}}_{\underline{w},\vec{\mu}}). \tag{2.30}$$

In terms of graphical calculus, this is reflecting a light ladder diagram through the horizontal axis, see for example Figure 6.

$$\mathbb{D}(LL^{121}_{112121,((1,0),(1,0),(-2,1),(1,0),(2,-1),(-1,1))}) =$$

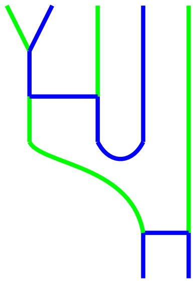


Figure 6. An upside down light ladder diagram.

For each dominant weight λ fix a word $\underline{x}_\lambda$ in the alphabet $\{1, 2\}$ corresponding to a sequence of fundamental weights which sum to λ. For all words $\underline{w}$ and for each dominant weight subsequence $\vec{\mu} \in E(\underline{w}, \lambda)$, we choose one light ladder diagram from $\underline{w}$ to $\underline{x}_\lambda$. If $\underline{w} = \underline{x}_\lambda$ and each μ_i is dominant, then we choose the identity diagram. From now on, we denote this chosen light ladder diagram by $L_{\underline{w},\vec{\mu}}$.

Remark 2.23. The choice of $LL_{\underline{x}_\lambda,\vec{\lambda}} = \mathrm{id}_{\underline{x}_\lambda}$ when the λ_i are all dominant is not essential for our arguments, but does ensure our construction is aligned with other conventions. For example, this is required in the definition of an object adapted cellular category in [10].

Definition 2.24. If $\underline{w}$ and $\underline{u}$ are fixed words in $\{1, 2\}$ and λ is a dominant weight, then for $\vec{\mu} \in E(\underline{w}, \lambda)$ and $\vec{\nu} \in E(\underline{u}, \lambda)$ we obtain a *double ladder diagram* (associated to our choices of $\underline{x}_\lambda$'s and our choices of light ladder diagrams)

$$\mathbb{LL}^{\underline{u},\vec{\nu}}_{\underline{w},\vec{\mu}} = \mathbb{D}(LL_{\underline{u},\vec{\nu}}) \circ LL_{\underline{w},\vec{\mu}}. \tag{2.31}$$

Remark 2.25. One reason for fixing an $\underline{x}_\lambda$ for all λ is so the composition on the right-hand side of (2.31) is well defined. To emphasize that a double ladder diagram factors through the object in the middle, we find it is useful to think of double ladder diagrams as a composition of two trapezoids, as appears in Figure 7.

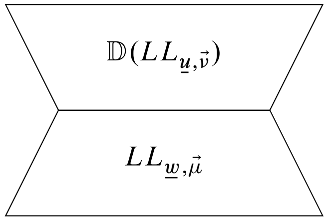
$\mathbb{LL}^{\underline{u},\vec{\nu}}_{\underline{w},\vec{\mu}} =$ 

Figure 7. A schematic for a double ladder diagram.

Remark 2.26. Note that light ladder diagrams ending in $\underline{x}_\lambda$ are double ladder diagrams, where the upside-down light ladder happens to be the identity diagram.

Definition 2.27. We define the set of all double ladder diagrams from $\underline{w}$ to $\underline{u}$ factoring through λ (associated to our choice of $\underline{x}_\lambda$'s and light ladder diagrams) to be

$$\mathbb{LL}^{\underline{u}}_{\underline{w}}(\lambda) = \{\mathbb{LL}^{\underline{u},\vec{\nu}}_{\underline{w},\vec{\mu}} : \vec{\mu} \in E(\underline{w}, \lambda),\ \vec{\nu} \in E(\underline{u}, \lambda)\}, \tag{2.32}$$

and define the set of all double ladder diagrams from $\underline{w}$ to $\underline{u}$ (associated to our choice of $\underline{x}_\lambda$'s and light ladder diagrams) to be

$$\mathbb{LL}^{\underline{u}}_{\underline{w}} = \bigcup_{\lambda \in X_+} \mathbb{LL}^{\underline{u}}_{\underline{w}}(\lambda). \tag{2.33}$$

Remark 2.28. Anytime we write $\mathbb{LL}^{\underline{u},\vec{\nu}}_{\underline{w},\vec{\mu}}$ or $\mathbb{LL}^{\underline{u}}_{\underline{w}}$, we have already fixed choices of $\underline{x}_\lambda$'s and choices of light ladder diagrams. The notation does not account for these choices, but we will not be comparing double ladders for different choices so the notation should not lead to confusion.

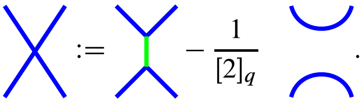
2.6. Relating non-elliptic webs to double ladders

Our next goal is to define an evaluation functor from $\mathcal{D}_{\mathfrak{sp}_4}$ to the category $\mathbf{Fund}(\mathfrak{sp}_4)$, and then to prove that the functor is an equivalence. That the functor is an equivalence will follow from showing that double ladder diagrams span the category $\mathcal{D}_{\mathfrak{sp}_4}$, and map to a set of linearly independent morphisms in $\mathbf{Fund}(\mathfrak{sp}_4)$. This approach is modeled on the work on type A webs in [8], where most of the work goes into showing that double ladder diagrams span the diagrammatic category. Checking linear independence is comparatively easy once you know the functor explicitly. But for $\mathcal{D}_{\mathfrak{sp}_4}$, the extra work to show double ladders span can be circumvented by bootstrapping known results about B_2 webs which we recall below.

Kuperberg's paper [19, pp. 14–15] introduces a tetravalent vertex in the B_2 web category which can be used to remove all internal double edges. Let $\mathbf{B}$ be the set of B_2 diagrams with no internal double edges and with no faces having one, two, or three adjacent edges. These diagrams are called non-elliptic in [19]. There are local relations in the B_2 category (now including the tetravalent vertex) which can be used to reduce triangular faces, bigons, monogons, and circles to sums of diagrams with fewer crossings (i.e., $\mathbf{B}$ is the set of irreducible webs with respect to the relations). It follows that the set $\mathbf{B}$ spans the B_2 category over $\mathbb{Z}[q, q^{-1}]$. Let $\mathbf{B}_{\underline{w}}$ be the set of diagrams in $\mathbf{B}$ with $\underline{w}$ on the boundary. One of the main results of [19] is that

$$\#\mathbf{B}_{\underline{w}} = \dim V(\underline{w})^{\mathfrak{sp}_4(\mathbb{C})}. \tag{2.34}$$

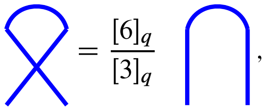
Notation 2.29. If we work in the $\mathcal{A}$-linear category $\mathcal{D}_{\mathfrak{sp}_4}$, there is an analogous 90 degree rotation invariant morphism, which we will call the tetravalent vertex, in $\mathrm{End}_{\mathcal{D}_{\mathfrak{sp}_4}}(11)$.



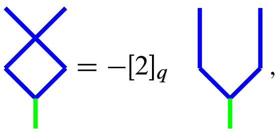
(2.35)

There is an augmented graphical calculus in which the generating diagrams are the cups, caps, and trivalent vertices in the definition of $\mathcal{D}_{\mathfrak{sp}_4}$ along with the tetravalent vertex (2.35). For the remainder of this section when we say a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$, we mean a diagram in the augmented graphical calculus.

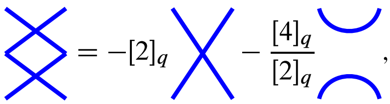
Since $[2]_q$ is invertible in our ground ring, we can use this tetravalent vertex to remove all internal green labeled edges in any diagram in $\mathcal{D}_{\mathfrak{sp}_4}$. The tetravalent vertex satisfies the following relations in $\mathcal{D}_{\mathfrak{sp}_4}$:



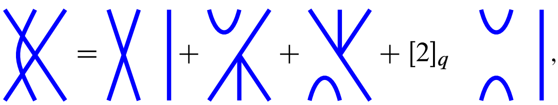
(2.36)



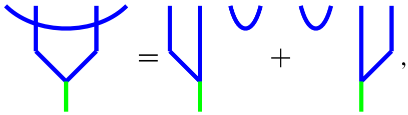
(2.37)



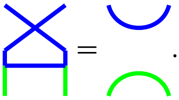
(2.38)



(2.39)



(2.40)



(2.41)

Remark 2.30. Due to the identity $[2n]_q/[n]_q = [n+1]_q - [n-1]_q$, the coefficients in these relations all lie in the ring $\mathbb{Z}[q, q^{-1}]$.

Definition 2.31. A *face* of a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ is a simply connected component of the complement of the diagram, which does not touch the boundary.

Definition 2.32. A *non-elliptic diagram* in $\mathcal{D}_{\mathfrak{sp}_4}$ is a diagram such that all faces have more than three sides (i.e., a diagram with no triangular faces, bigons, monogons, or circles).

Definition 2.33. An *internal* 2 *edge* of a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ is a 2 edge in the diagram which does not connect to the boundary. Examples of how a having a face or being non-elliptic interacts with internal 2 edges appear in Figures 8–10.

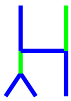


Figure 8. An example of a non-elliptic web with internal 2 edges in $\mathcal{D}_{\mathfrak{sp}_4}$. This diagram is a light ladder for the dominant weight sequence $((1, 0), (-1, 1), (1, 0))$.

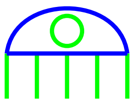


Figure 9. An example of an elliptic web with internal 2 edges in $\mathcal{D}_{\mathfrak{sp}_4}$. The only face is the interior of the 2 circle.

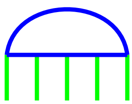


Figure 10. An example of a non-elliptic web with no internal 2 edges in $\mathcal{D}_{\mathfrak{sp}_4}$. There is only one face and it has five sides.

Definition 2.34. The set $\mathbf{D}$ is the collection of all non-elliptic diagrams in $\mathcal{D}_{\mathfrak{sp}_4}$ with no internal 2 edges, and the set $\mathbf{D}_{\underline{w}}^{\underline{u}}$ is the set of diagrams in $\mathbf{D} \cap \operatorname{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u})$.

Lemma 2.35. *The set* $\mathbf{D}$ *spans* $\mathcal{D}_{\mathfrak{sp}_4}$ *over* $\mathcal{A}$.

Proof. Let D be an arbitrary diagram in $\mathcal{D}_{\mathfrak{sp}_4}$. We will argue that D is a linear combination of non-elliptic webs with no internal 2 edges. If a 2 edge does not connect to a trivalent vertex, then you can use the bigon relation to introduce one. Thus, every 2 edge either connects to the boundary of D, or connects two trivalent vertices. Using the tetravalent vertex to remove all pairs of trivalent vertices, we can rewrite D as a linear combination of diagrams with no internal 2 edges. Thus, we may assume that D is a diagram with no internal 2 edges. Using the defining relations in $\mathcal{D}_{\mathfrak{sp}_4}$ along with the tetravalent relations, we can remove all faces with less than four edges. ■

Remark 2.36. In order to introduce a trivalent vertex, we used the bigon relation backwards, which required $[2]_q^{-1} \in \mathcal{A}$.

Lemma 2.37. *Let* $\Bbbk$ *be a field and let* $q \in \Bbbk^\times$ *be such that* $q + q^{-1} \neq 0$. *Then*

$$\dim \operatorname{Hom}_{\Bbbk \otimes \mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u}) \leq \dim \operatorname{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})). \tag{2.42}$$

Proof. There is an obvious bijection between the set $\mathbf{B}$ and the set $\mathbf{D}$. The result then follows from (2.34). ■

Remark 2.38. We sketch a more direct argument to deduce the inequality (2.42). The dimension of the $\mathfrak{sp}_4(\mathbb{C})$ invariants in $V(1)^{\otimes 2n}$ is known to be equal to the number of matchings of $2n$ points on the boundary of a disc such that there is no 6-point star in the matching [33] and [19, Section 8.4]. One can argue that the local condition of being non-elliptic implies the global condition of having no six point star. Then, noting that non-elliptic diagrams have a unique representative up to isotopy (there are no potential Reidemeister moves), it follows that there is a bijection between non-elliptic diagrams and matchings without a 6-point star. This proves that the inequality (2.42) holds when $\underline{w} = 1^{\otimes a}$ and $\underline{u} = 1^{\otimes b}$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Since 2 is a direct summand of 11 it follows that (2.42) holds for any words $\underline{w}$ and $\underline{u}$ in the alphabet $\{1, 2\}$.

We have defined a set $\mathbb{LL}_{\underline{w}}^{\underline{u}}$ of double ladders in $\mathcal{D}_{\mathfrak{sp}_4}$. It follows from the construction of $\mathbb{LL}_{\underline{w}}^{\underline{u}}$ and (2.23) that

$$\#\mathbb{LL}_{\underline{w}}^{\underline{u}} = \sum_{\lambda \in X_+} \#E(\underline{w}, \lambda) \#E(\underline{u}, \lambda) = \dim \operatorname{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})). \tag{2.43}$$

We want to show linear independence of the set of double ladders, or equivalently that the inequality of dimensions in (2.42) is in fact an equality, for a general choice of base ring $\Bbbk$. To this end we will define an evaluation functor from the diagrammatic category $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}$ to the representation theoretic category **Fund**$(\Bbbk \otimes U_q^{\mathcal{A}}(\mathfrak{sp}_4))$, and interpret the image of the evaluation functor in terms of tilting modules. If we can show that the image of the double ladder diagrams under the evaluation functor is a linearly independent set, then (2.37) will imply that the double ladder diagrams must be linearly independent in $\mathcal{D}_{\mathfrak{sp}_4}$. This implies that the inequality in (2.42) is an equality, and it follows that the evaluation functor maps bases to bases, so is fully faithful.

Remark 2.39. Since **D** spans $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}$ and is in a non-canonical bijection with the set of double ladder diagrams (for fixed choices of $\underline{x}_\lambda$ and fixed choices of light ladders), linear independence of the double ladder diagrams over $\Bbbk$ implies that both sets are bases.

Note that double ladders have many internal 2 label edges while the diagrams in **D** will have none. On the other hand, sometimes the double ladder diagrams will be non-elliptic webs with no internal 2 edges. A good exercise for the reader is to rewrite the diagram in Figure 10 as a double ladder diagram. A hint is that a double ladder diagram in $\mathrm{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(2^{\otimes 5}, \emptyset)$ will just be a light ladder diagram $LL^{\emptyset}_{2^{\otimes 5},?}$.

3. The evaluation functor and tilting modules

3.1. Defining the evaluation functor on objects

We are now going to be more precise about what category of representations associated to $\mathfrak{sp}_4$ we are considering. The discussion below is well known, but we reproduce it here to help the reader follow certain calculations which come later.

Our main reference for quantum groups is Jantzen's book [14]. Recall that $\mathfrak{sp}_4(\mathbb{C})$ gives rise to a root system Φ and a Weyl group W. We choose simple roots $\Delta = \{\alpha_s = \varepsilon_1 - \varepsilon_2, \alpha_t = 2\varepsilon_2\}$. There is a unique W invariant symmetric form $(-,-)$ on the root lattice $\mathbb{Z}\Phi$ such that the short roots pair with themselves to be 2. This is the form $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, restricted to the root lattice. For $\alpha \in \Phi$ we define the coroot $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, in particular $\alpha_s^\vee = \alpha_s$ and $\alpha_t^\vee = \alpha_t/2$ and the Cartan matrix $((\alpha_i^\vee, \alpha_j))$ is

$$\begin{pmatrix} \alpha_s^\vee(\alpha_s) & \alpha_s^\vee(\alpha_t) \\ \alpha_t^\vee(\alpha_s) & \alpha_t^\vee(\alpha_t) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

Define the algebra $U_q(\mathfrak{sp}_4)$ as the $\mathbb{Q}(q)$ algebra given by generators

$$F_s, F_t, K_s^{\pm 1}, K_t^{\pm 1} E_s, E_t$$

and relations

- $K_sK_s^{-1} = 1 = K_sK_s^{-1}, K_tK_t^{-1} = 1 = K_t^{-1}K_t, K_sK_t = K_tK_s,$
- $K_tE_t = q^4E_tK_t, K_tE_s = q^{-2}E_sK_t,$
- $K_sE_t = q^{-2}E_tK_s, K_sE_s = q^2E_sK_s,$
- $K_tF_t = q^{-4}F_tK_t, K_tF_s = q^2F_sK_t,$
- $K_sF_t = q^2F_tK_s, K_sF_s = q^{-2}F_sK_s,$
- $E_tF_s = F_sE_t, E_sF_t = F_tE_s,$
- $E_tF_t = F_tE_t + \dfrac{K_t - K_t^{-1}}{q^2 - q^{-2}},$
- $E_sF_s = F_sE_s + \dfrac{K_s - K_s^{-1}}{q - q^{-1}},$
- $E_t^2E_s - \dfrac{[4]_q}{[2]_q}E_tE_sE_t + E_sE_t^2 = 0,$
- $E_s^3E_t - [3]_qE_s^2E_tE_s + [3]_qE_sE_tE_s^2 - E_sE_t^3.$

Our convention is $[n]_q := \dfrac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q$.

Recall that $\mathcal{A} = \mathbb{Z}[q, q^{-1}, [2]_q^{-1}]$. Let $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ be the unital $\mathcal{A}$-subalgebra of $U_q(\mathfrak{sp}_4)$ spanned by $K_s^{\pm 1}, K_t^{\pm 1}$, and the divided powers

$$E_s^{(n)} = \frac{E_s^n}{[n]_q!}, F_s^{(n)} = \frac{F_s^n}{[n]_q!}, E_t^{(n)} = \frac{E_t}{[n]_{q^2}!}, F_t^{(n)} = \frac{F_t}{[n]_{q^2}!}$$

for all $n \in \mathbb{Z}_{\geq 1}$. So, $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ is Lusztig's divided powers quantum group [2].

Let $V^{\mathcal{A}}(\varpi_1)$ denote the free $\mathcal{A}$ module with basis

$$v_{(1,0)}, v_{(-1,1)}, v_{(1,-1)}, v_{(-1,0)}, \tag{3.1}$$

and action of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ given by

$$v_{(-1,0)} \underset{F_s=1}{\overset{E_s=1}{\rightleftarrows}} v_{(1,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(-1,1)} \underset{F_s=1}{\overset{E_s=1}{\rightleftarrows}} v_{(1,0)}. \tag{3.2}$$

Also, let $V^{\mathcal{A}}(\varpi_2)$ denote the free $\mathcal{A}$ module with basis

$$v_{(0,1)}, v_{(2,-1)}, v_{(0,0)}, v_{(-2,1)}, v_{(0,-1)}, \tag{3.3}$$

and action of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ given by

$$v_{(0,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(-2,1)} \underset{F_s=[2]_q}{\overset{E_s=1}{\rightleftarrows}} v_{(0,0)} \underset{F_s=1}{\overset{E_s=[2]_q}{\rightleftarrows}} v_{(2,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(0,1)}. \tag{3.4}$$

The elements K_α act on the basis vectors by

$$K_s \cdot v_{(i,j)} = q^i v_{(i,j)} \quad \text{and} \quad K_t \cdot v_{(i,j)} = q^{2j} v_{(i,j)}. \tag{3.5}$$

Our convention is that whenever we do not indicate the action of E_α or F_α they act by zero. The action of higher divided powers on these modules can be extrapolated from the given data. For example, $F_s^{(2)} v_{(-2,1)} = v_{(-2,1)}$.

Remark 3.1. Why are we using $\mathcal{A}$ instead of $\mathbb{Z}[q, q^{-1}]$? When $[2]_q = 0$, the Weyl module $\Bbbk \otimes V^{\mathcal{A}}(\varpi_2)$ is not irreducible and the correct choice of combinatorial category seems to be the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-linear monoidal category generated by V_q and $\Lambda^2(V_q)$. The module $\Lambda^2(V_q)$ has the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis

$$v_{(0,1)}, v_{(2,-1)}, X_0, Y_0, v_{(-2,1)}, v_{(0,-1)}, \tag{3.6}$$

and action of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ given by

$$v_{(0,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(-2,1)} \underset{F_s}{\overset{E_s}{\rightleftarrows}} X_0 \oplus Y_0 \underset{F_s}{\overset{E_s}{\rightleftarrows}} v_{(2,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(0,1)}, \tag{3.7}$$

where

$$\begin{aligned} E_s \cdot Y_0 = q^{-1/2} v_{(2,-1)}, \quad E_s \cdot X_0 = q^{1/2} v_{(2,-1)}, \\ E_s \cdot v_{(-2,1)} = q^{1/2} X_0 + q^{-1/2} Y_0; \\ F_s \cdot Y_0 = q^{-1/2} v_{(-2,1)}, \quad F_s \cdot X_0 = q^{1/2} v_{(-2,1)}, \\ F_s \cdot v_{(2,-1)} = q^{1/2} X_0 + q^{-1/2} Y_0. \end{aligned} \tag{3.8}$$

The module $V^{\mathcal{A}}(\varpi_2)$ can be defined over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. There is a map from $V^{\mathcal{A}}(\varpi_2)$ into $\Lambda^2(V_q)$, such that $v_{(0,0)} \mapsto q^{1/2} X_0 + q^{-1/2} Y_0$. Moreover, the cokernel of this inclusion map will be isomorphic to the trivial module. Thus, $\Lambda^2(V_q)$ is filtered by Weyl modules, and the filtration splits when $[2]_q \neq 0$. If $[2]_q = 0$, then $\Lambda^2(V_q)$ is indecomposable with socle and head isomorphic to the trivial module, and middle subquotient isomorphic to the irreducible module of highest weight ϖ_2.

The algebra $U_q(\mathfrak{sp}_4)$ is a Hopf algebra with structure maps (Δ, S, ε) defined on generators by

$$\begin{aligned} \Delta(E_\alpha) &= E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \Delta(F_\alpha) \\ &= F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, \Delta(K_\alpha) \\ &= K_\alpha \otimes K_\alpha, \\ S(E_\alpha) &= -K_\alpha^{-1} E_\alpha, S(F_\alpha) = -F_\alpha K_\alpha, S(K_\alpha) = K_\alpha^{-1}, \\ \varepsilon(E_\alpha) &= 0, \varepsilon(F_\alpha) = 0, \varepsilon(K_\alpha) = 1. \end{aligned}$$

Furthermore, the algebra $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ is a sub-Hopf-algebra of $U_q(\mathfrak{sp}_4)$, see [2]. Therefore, $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ will act on the tensor product of representations through the coproduct Δ.

Using the antipode S, we can define an action of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ on

$$V^{\mathcal{A}}(\varpi_1)^* = \mathrm{Hom}_{\mathcal{A}}(V^{\mathcal{A}}(\varpi_1), \mathcal{A}) \tag{3.9}$$

by

$$-q^4 v^*_{(1,0)} \underset{F_s=1}{\overset{E_s=1}{\rightleftarrows}} q^3 v^*_{(-1,1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} -q v^*_{(1,-1)} \underset{F_s=1}{\overset{E_s=1}{\rightleftarrows}} v^*_{(-1,0)}, \tag{3.10}$$

and on

$$V^{\mathcal{A}}(\varpi_2)^* = \mathrm{Hom}_{\mathcal{A}}(V^{\mathcal{A}}(\varpi_2), \mathcal{A}) \tag{3.11}$$

by

$$q^6 v^*_{(0,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} -q^4 v^*_{(-2,1)} \underset{F_s=[2]_q}{\overset{E_s=1}{\rightleftarrows}} q^2[2]_q v^*_{(0,0)} \underset{F_s=1}{\overset{E_s=[2]_q}{\rightleftarrows}} -q^2 v^*_{(2,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v^*_{(0,1)}. \tag{3.12}$$

Comparing (3.2) and (3.10) we see there is an isomorphism of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ modules

$$\varphi_1 \colon V^{\mathcal{A}}(\varpi_1) \to V^{\mathcal{A}}(\varpi_1)^* \tag{3.13}$$

such that basis elements in (3.2) are sent to the basis elements in (3.10). By comparing (3.4) and (3.12) we similarly obtain an isomorphism

$$\varphi_2 \colon V^{\mathcal{A}}(\varpi_2) \to V^{\mathcal{A}}(\varpi_2)^* \tag{3.14}$$

sending basis elements in (3.4) to the basis elements in (3.12).

In (3.4) we will define a monoidal functor from $\mathcal{D}_{\mathfrak{sp}_4}$ to $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$-mod. The functor will send 1 to $V^{\mathcal{A}}(\varpi_1)$ and 2 to $V^{\mathcal{A}}(\varpi_2)$. The dual modules $V^{\mathcal{A}}(\varpi_1)^*$ and $V^{\mathcal{A}}(\varpi_2)^*$ will not be in the image of the functor Ξ. However, the maps φ_1 and φ_2 are fixed isomorphisms of these dual modules with modules which are in the image of the functor.

3.2. Caps and cups

Lemma 3.2. *If V is any finite rank $\mathcal{A}$ lattice with basis e_i, define maps*

$$\mathcal{A} \xrightarrow{u} V \otimes \mathrm{Hom}_{\mathcal{A}}(V, \mathcal{A}) \xrightarrow{c} \mathcal{A}, \tag{3.15}$$

$$\mathcal{A} \xrightarrow{u'} \mathrm{Hom}_{\mathcal{A}}(V, \mathcal{A}) \otimes V \xrightarrow{c'} \mathcal{A}, \tag{3.16}$$

where $u(1) = \sum e_i \otimes e_i^*$, $u'(1) = \sum e_i^* \otimes e_i$, $c(v \otimes f) = f(v)$, and $c'(f \otimes v) = f(v)$. Then

$$(\mathrm{id}_V \otimes c') \circ (u \otimes \mathrm{id}_V) = \mathrm{id}_V = (c \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes u') \tag{3.17}$$

and

$$(\mathrm{id}_{V^*} \otimes c) \circ (u' \otimes \mathrm{id}_{V^*}) = \mathrm{id}_{V^*} = (c' \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes u). \tag{3.18}$$

Proof. We will show that

$$(\mathrm{id}_V \otimes c') \circ (u \otimes \mathrm{id}_V) = \mathrm{id}_V$$

the arguments to establish the other three equalities in (3.17) and (3.18) are similar.

Let $v \in V$. Since e_i is a basis for V we can write $v = \sum v_i e_i$ for some $v_i \in \mathcal{A}$. Thus,

$$\begin{aligned}(\mathrm{id}_V \otimes c') \circ (u \otimes \mathrm{id}_V)(v) &= (\mathrm{id}_V \otimes c')\Big(\sum e_i \otimes e_i^* \otimes v\Big) \\ &= \sum e_i \cdot e_i^*(v) = \sum v_i e_i = v.\end{aligned}$$

■

Lemma 3.3. *Fix an isomorphism* $\varphi\colon V \to V^*$ *and write* $\mathbf{cap} = c' \circ (\varphi \otimes \mathrm{id})$ *and* $\mathbf{cup} = (\mathrm{id} \otimes \varphi^{-1}) \circ u$. *Then*

$$(\mathrm{id}_V \otimes \mathbf{cap}) \circ (\mathbf{cup} \otimes \mathrm{id}_V) = \mathrm{id}_V = (\mathbf{cap} \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \mathbf{cup}). \tag{3.19}$$

Proof. Using $\varphi \circ \varphi^{-1} = \mathrm{id} = \varphi^{-1} \circ \varphi$, (3.19) follows easily from (3.17) and (3.18). ■

The $\mathcal{A}$-linear maps

$$\mathcal{A} \xrightarrow{\mathbf{cup}_i := (\mathrm{id} \otimes \varphi_i^{-1}) \circ u_i} V^{\mathcal{A}}(\varpi_i) \otimes V^{\mathcal{A}}(\varpi_i) \xrightarrow{\mathbf{cap}_i := c_i' \circ (\varphi_i \otimes \mathrm{id})} \mathcal{A}, \quad \text{for } i = 1, 2, \tag{3.20}$$

are actually maps of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ modules, where $\mathcal{A}$ is the trivial module. The functor Ξ will send the cups and caps from the diagrammatic category to the maps $\mathbf{cup}_i$ and $\mathbf{cap}_i$

The module $V^{\mathcal{A}}(\varpi_1)$ has basis

$$\{v_{(1,0)}, v_{(-1,1)} = F_s v_{(1,0)}, v_{(1,-1)} = F_t F_s v_{(1,0)}, v_{(-1,0)} = F_s F_t F_s v_{(1,0)}\}, \tag{3.21}$$

and the module $V^{\mathcal{A}}(\varpi_2)$ has basis

$$\begin{aligned}\{v_{(0,1)}, v_{(2,-1)} = F_t v_{(0,1)}, v_{(0,0)} = F_s F_t v_{(0,1)}, v_{(-2,1)} \\ = F_s^{(2)} F_t v_{(0,1)}, v_{(0,-1)} = F_t F_s^{(2)} F_t v_{(0,1)}\}.\end{aligned} \tag{3.22}$$

With respect to these bases, we can write $\mathbf{cup}_1\colon \mathcal{A} \to V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1)$ as

$$\begin{aligned}1 \mapsto & -q^{-4} v_{(1,0)} \otimes v_{(-1,0)} + q^{-3} v_{(-1,1)} \otimes v_{(1,-1)} \\ & - q^{-1} v_{(1,-1)} \otimes v_{(-1,1)} + v_{(-1,0)} \otimes v_{(1,0)},\end{aligned} \tag{3.23}$$

and $\mathbf{cup}_2\colon \mathcal{A} \to V^{\mathcal{A}}(\varpi_2) \otimes V^{\mathcal{A}}(\varpi_2)$ as

$$\begin{aligned}1 \mapsto q^{-6} v_{(0,1)} \otimes v_{(0,-1)} - q^{-4} v_{(2,-1)} \otimes v_{(-2,1)} + \frac{q^{-2}}{[2]_q} v_{(0,0)} \otimes v_{(0,0)} \\ - q^{-2} v_{(-2,1)} \otimes v_{(2,-1)} + v_{(0,-1)} \otimes v_{(0,1)}.\end{aligned} \tag{3.24}$$

To record the maps $\mathbf{cap}_i$ in our basis we use the matrices

$$\mathbf{cap}_1(v_i \otimes v_j) = \begin{array}{c} \\ v_{(-1,0)} \\ v_{(1,-1)} \\ v_{(-1,1)} \\ v_{(1,0)} \end{array} \begin{array}{c} \begin{array}{cccc} v_{(-1,0)} & v_{(1,-1)} & v_{(-1,1)} & v_{(1,0)} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -q^4 \\ 0 & 0 & q^3 & 0 \\ 0 & -q & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{3.25}$$

and

$$\mathbf{cap}_2(v_i \otimes v_j) = \begin{array}{c} \\ v_{(0,-1)} \\ v_{(-2,1)} \\ v_{(0,0)} \\ v_{(2,-1)} \\ v_{(0,1)} \end{array} \begin{array}{c} \begin{array}{ccccc} v_{(0,-1)} & v_{(-2,1)} & v_{(0,0)} & v_{(2,-1)} & v_{(0,1)} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & q^6 \\ 0 & 0 & 0 & -q^4 & 0 \\ 0 & 0 & q^2[2]_q & 0 & 0 \\ 0 & -q^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{3.26}$$

.

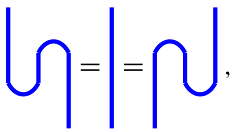
Example 3.4. We give two calculations to clarify how we arrived at these formulas:

$$\begin{aligned} \mathbf{cap}_2(v_{(0,0)} \otimes v_{(0,0)}) &= c_2' \circ (\varphi_2 \otimes \mathrm{id})(v_{(0,0)} \otimes v_{(0,0)}) \\ &= c_2'(q^2[2]_q v^*_{(0,0)} \otimes v_{(0,0)}) = q^2[2]_q, \end{aligned}$$

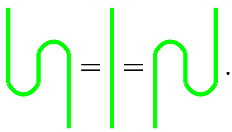
and

$$\begin{aligned} \mathbf{cup}_1(1) &= (\mathrm{id} \otimes \varphi_1^{-1}) \circ u_1(1) \\ &= v_{(-1,0)} \otimes \varphi_1^{-1}(v^*_{(-1,0)}) + v_{(1,-1)} \otimes \varphi_1^{-1}(v^*_{(1,-1)}) \\ &\quad + v_{(-1,1)} \otimes \varphi_1^{-1}(v^*_{(-1,1)}) + v_{(1,0)} \otimes \varphi_1^{-1}(v_{(1,0)}) \\ &= -q^{-4} v_{(1,0)} \otimes v_{(-1,0)} + q^{-3} v_{(-1,1)} \otimes v_{(1,-1)} \\ &\quad - q^{-1} v_{(1,-1)} \otimes v_{(-1,1)} + v_{(-1,0)} \otimes v_{(1,0)}. \end{aligned}$$

The maps $\mathbf{cup}_i$ and $\mathbf{cap}_i$ in $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$-mod are going to correspond to the colored cap and cup maps in $\mathcal{D}_{\mathfrak{sp}_4}$. In which case, the equation (3.19) corresponds to the isotopy relations:



(3.27)



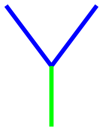
(3.28)

3.3. Trivalent vertices

Consider the module $V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1)$. We observe that the vector $q^{-1}v_{(1,0)} \otimes v_{(0,1)} - v_{(0,1)} \otimes v_{(1,0)}$ is annihilated by E_s and E_t. The action of K_s scales this vector by 1 and the action of K_t scales the vector by q^2. There is an $\mathcal{A}$-linear map

$$\begin{aligned}\mathbf{i}\colon V^{\mathcal{A}}(\varpi_2) &\to V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1),\\ v_{(0,1)} &\mapsto q^{-1}v_{(1,0)} \otimes v_{(-1,1)} - v_{(-1,1)} \otimes v_{(1,0)},\\ v_{(2,-1)} &\mapsto q^{-1}v_{(1,0)} \otimes v_{(1,-1)} - v_{(1,-1)} \otimes v_{(1,0)},\\ v_{(0,0)} &\mapsto q^{-1}v_{(1,0)} \otimes v_{(-1,0)} + q^{-2}v_{(-1,1)} \otimes v_{(1,-1)},\\ &\qquad - v_{(1,-1)} \otimes v_{(-1,1)} - q^{-1}v_{(-1,0)} \otimes v_{(1,0)},\\ v_{(-2,1)} &\mapsto q^{-1}v_{(-1,1)} \otimes v_{(-1,0)} - v_{(-1,0)} \otimes v_{(-1,1)},\\ v_{(0,-1)} &\mapsto q^{-1}v_{(1,-1)} \otimes v_{(-1,0)} - v_{(-1,0)} \otimes v_{(1,-1)}.\end{aligned} \tag{3.29}$$

Using the explicit description of $V^{\mathcal{A}}(\varpi_2)$ in (3.4), one checks that $\mathbf{i}$ is a map of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$-modules by computing the action of the generators of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ on the vectors appearing on the right-hand side of (3.29). The morphism $\mathbf{i}$ will correspond to the following diagram:



. (3.30)

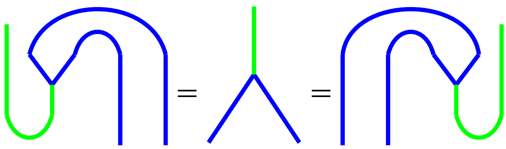
One can also easily check the equality of the following two elements of the space $\mathrm{Hom}_{U_q^{\mathcal{A}}(\mathfrak{sp}_4)}(V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1), V^{\mathcal{A}}(\varpi_2))$:

$$(\mathrm{id} \otimes \mathbf{cap}_1) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{cap}_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{i} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathbf{cup}_2 \otimes \mathrm{id} \otimes \mathrm{id}) \tag{3.31}$$

and

$$(\mathbf{cap}_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{cap}_1 \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{i} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{cup}_2). \tag{3.32}$$

Then we will unambiguously denote both maps by $\mathbf{p}$. In the graphical calculus this corresponds to the following:



. (3.33)

(3.31) $\mathbf{p}$ (3.32)

The equality of (3.31) and (3.32) follows from verifying that both maps act on a basis as follows:

$$
\begin{aligned}
&\mathbf{p}\colon V^{\mathcal{A}}(\varpi_1)\otimes V^{\mathcal{A}}(\varpi_1)\to V^{\mathcal{A}}(\varpi_2),\\
&v_{(1,0)}\otimes v_{(1,0)}\mapsto 0, && v_{(-1,1)}\otimes v_{(1,0)}\mapsto q v_{(0,1)},\\
&v_{(1,0)}\otimes v_{(-1,1)}\mapsto -v_{(0,1)}, && v_{(-1,1)}\otimes v_{(-1,1)}\mapsto 0,\\
&v_{(1,0)}\otimes v_{(1,-1)}\mapsto -v_{(2,-1)}, && v_{(-1,1)}\otimes v_{(1,-1)}\mapsto \frac{-1}{[2]_q}v_{(0,0)},\\
&v_{(1,0)}\otimes v_{(-1,0)}\mapsto \frac{-q}{[2]_q}v_{(0,0)}, && v_{(-1,1)}\otimes v_{(-1,0)}\mapsto -v_{(-2,1)},\\
&v_{(1,-1)}\otimes v_{(1,0)}\mapsto q v_{(2,-1)}, && v_{(-1,0)}\otimes v_{(1,0)}\mapsto \frac{q}{[2]_q}v_{(0,0)},\\
&v_{(1,-1)}\otimes v_{(-1,1)}\mapsto \frac{q^2}{[2]_q}v_{(0,0)}, && v_{(-1,0)}\otimes v_{(-1,1)}\mapsto q v_{(-2,1)},\\
&v_{(1,-1)}\otimes v_{(1,-1)}\mapsto 0, && v_{(-1,0)}\otimes v_{(1,-1)}\mapsto q v_{(0,-1)},\\
&v_{(1,-1)}\otimes v_{(-1,0)}\mapsto -v_{(0,-1)}, && v_{(-1,0)}\otimes v_{(-1,0)}\mapsto 0.
\end{aligned}
\tag{3.34}
$$

Remark 3.5. We sketch a method to compute (3.31) evaluated on $v_{(-1,1)}\otimes v_{(1,-1)}$, the other calculations follow the same pattern. The $\mathbf{cap}_1$'s in the definition of (3.31) are only non-zero on basis vectors of the form $v_\mu\otimes v_{-\mu}$. Also, in the formula for $\mathbf{i}$ (3.29) the only basis vector with a tensor of the form $v_{(-1,1)}\otimes v_{(1,-1)}$ is $v_{(0,0)}$. Therefore, (3.31) acts as

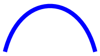
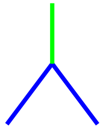
$$
\begin{aligned}
v_{(-1,1)}\otimes v_{(1,-1)}\mapsto{}& (\mathrm{id}\otimes\mathbf{cap}_1)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mathbf{cap}_1\otimes\mathrm{id})\\
&\times\Big(\frac{q^{-2}}{[2]_q}v_{(0,0)}\otimes\mathbf{i}(v_{(0,0)})\otimes v_{(-1,1)}\otimes v_{(1,-1)}\Big)\\
={}&q^{-2}\,\mathbf{cap}_1(v_{(1,-1)}\otimes v_{(-1,1)})\,\mathbf{cap}_1(v_{(-1,1)}\otimes v_{(1,-1)})\frac{q^{-2}}{[2]_q}v_{(0,0)}\\
={}&q^{-2}q^3(-q)\frac{q^{-2}}{[2]_q}v_{(0,0)}\\
={}&\frac{-1}{[2]_q}v_{(0,0)}.
\end{aligned}
\tag{3.35}
$$

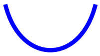
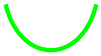
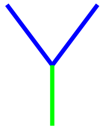
3.4. The definition of the evaluation functor

Theorem 3.6. *There is a monoidal functor*

$$\Xi\colon \mathcal{D}_{\mathfrak{sp}_4}\to U_q^{\mathcal{A}}(\mathfrak{sp}_4)\text{-mod}$$

defined on objects by defining $\Xi(1) = V^{\mathcal{A}}(\varpi_1)$ *and* $\Xi(2) = V^{\mathcal{A}}(\varpi_2)$ *and then extending monoidally. The functor* Ξ *is defined on morphisms by first defining*

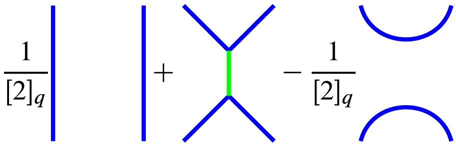
 $\mapsto \mathbf{cap}_1$  $\mapsto \mathbf{cap}_2$  $\mapsto \mathbf{p}$, (3.36)

 $\mapsto \mathbf{cup}_1$  $\mapsto \mathbf{cup}_2$  $\mapsto \mathbf{i}$, (3.37)

and then extending $\mathcal{A}$*-linearly so that horizontal concatenation of diagrams corresponds to tensor product of morphisms in* $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$-mod *and vertical composition of diagrams corresponds to composition of morphisms in* $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$-mod.

Example 3.7. We illustrate how Ξ is defined on objects and on morphisms:

$$\Xi(122) = V^{\mathcal{A}}(122) = V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_2), \tag{3.38}$$



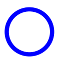
$$\stackrel{\Xi}{\mapsto} \frac{1}{[2]_q} \mathrm{id} \otimes \mathrm{id} - \mathbf{i} \circ \mathbf{p} + \frac{1}{[2]_q} \mathbf{cup}_1 \circ \mathbf{cap}_1 . \tag{3.39}$$

3.5. Checking relations

Since $\mathcal{D}_{\mathfrak{sp}_4}$ is defined by generators and relations, in order to verify the theorem we must check that the diagrammatic relations hold in $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$-mod.

Proof of Theorem 3.6. The isotopy relations follow from (3.19) and the equality of (3.31) and (3.32).

To verify the relation

 $= -\dfrac{[6]_q[2]_q}{[3]_q}$ (3.40)

it suffices to show that

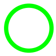
$$\mathbf{cap}_1 \circ \mathbf{cup}_1(1) = -\frac{[6]_q[2]_q}{[3]_q}. \tag{3.41}$$

Using (3.23) and (3.25) we find

$$\begin{aligned} \mathbf{cap}_1 \circ \mathbf{cup}_1(1) &= -q^{-4} \cdot 1 + q^{-3} \cdot (-q) - q^{-1} \cdot q^3 + 1 \cdot (-q^4) \\ &= -([5]_q - [1]_q). \end{aligned} \tag{3.42}$$

The desired equality (3.41) comes from the quantum number calculation in (2.15).

One can similarly argue that the relation



$$= \frac{[6]_q[5]_q}{[3]_q[2]_q} \tag{3.43}$$

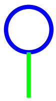
is satisfied. Use (3.24) and (3.26) to compute

$$\mathbf{cap}_2 \circ \mathbf{cup}_2(1) = q^{-6} + q^{-2} + 1 + q^2 + q^6 = [7]_q - [5]_q + [3]_q, \tag{3.44}$$

then use (2.16) to deduce

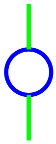
$$\mathbf{cap}_2 \circ \mathbf{cup}_2(1) = \frac{[6]_q[5]_q}{[3]_q[2]_q}. \tag{3.45}$$

To check the monogon relation



$$= 0 \tag{3.46}$$

and the bigon relation



$$= -[2]_q \tag{3.47}$$

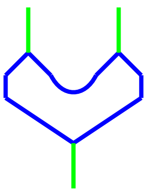
we need to show $\mathbf{cap}_1 \circ \mathbf{i} = 0$ and $\mathbf{p} \circ \mathbf{i} = -[2]_q$ id respectively. Since the module $V^{\mathcal{A}}(\varpi_2)$ is generated by the highest weight vector $v_{(0,1)}$ it suffices to show that $\mathbf{cap}_1 \circ \mathbf{i}(v_{(0,1)}) = 0$ and $\mathbf{p} \circ \mathbf{i}(v_{(0,1)}) = -[2]_q v_{(0,1)}$. The calculations go as follows:

$$\begin{aligned} \mathbf{cap}_1 \circ \mathbf{i}(v_{(0,1)}) &\overset{(3.29)}{=} \mathbf{cap}_1(q^{-1}v_{(1,0)} \otimes v_{(-1,1)} - v_{(-1,1)} \otimes v_{(1,0)}) \\ &\overset{(3.23)}{=} 0. \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} \mathbf{p} \circ \mathbf{i}(v_{(0,1)}) &\overset{(3.29)}{=} \mathbf{p}(q^{-1}v_{(1,0)} \otimes v_{(-1,1)} - v_{(-1,1)} \otimes v_{(1,0)}) \\ &\overset{(3.34)}{=} -q^{-1}v_{(0,1)} + qv_{(0,1)} \\ &= -[2]_q v_{(0,1)}. \end{aligned} \tag{3.49}$$

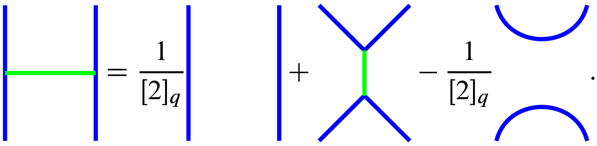
Verifying the trigon relation



$$= 0 \tag{3.50}$$

is left as an exercise (Hint: apply $(\mathbf{p} \otimes \mathbf{p}) \circ (\mathrm{id} \otimes \mathbf{cup}_1 \otimes \mathrm{id}) \circ \mathbf{i}$ to the vector $v_{(0,1)}$ and use (3.29) and (3.23) and (3.34)).

Now, we endeavor to check the $H = I$ relation:

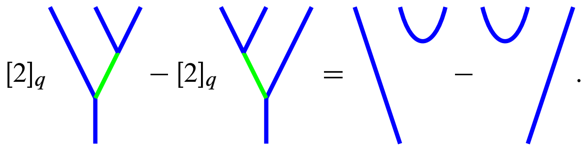


(3.51)

Precomposing with $\mathrm{id} \otimes \mathbf{cup}_1$ is an $\mathcal{A}$-linear map

$$\mathrm{Hom}_{U_q^{\mathcal{A}}(\mathfrak{sp}_4)}(V^{\mathcal{A}}(\varpi_1)^{\otimes 2}, V^{\mathcal{A}}(\varpi_1)^{\otimes 2}) \to \mathrm{Hom}_{U_q^{\mathcal{A}}(\mathfrak{sp}_4)}(V^{\mathcal{A}}(\varpi_1), V^{\mathcal{A}}(\varpi_1)^{\otimes 3}), \tag{3.52}$$

while postcomposing with $\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{cap}_1$ is an $\mathcal{A}$-linear map in the other direction. From (3.19) it follows that the two maps are mutually inverse isomorphisms of $\mathcal{A}$-modules, so we can instead check the following relation:



(3.53)

From the discussion in remark (2.3) it follows that we need to show

$$[2]_q(\mathrm{id} \otimes \mathbf{i}) \circ (\mathrm{id} \otimes \mathbf{p}) \circ (\mathbf{cup}_1 \otimes \mathrm{id}) - [2]_q(\mathbf{i} \otimes \mathrm{id})(\mathbf{p} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{cup}_1) \tag{3.54}$$

is equal to

$$\mathrm{id} \otimes \mathbf{cup}_1 - \mathbf{cup}_1 \otimes \mathrm{id}\,. \tag{3.55}$$

Since $V^{\mathcal{A}}(\varpi_1)$ is generated by the vector $v_{(1,0)}$ it suffices to check that (3.54) and (3.55) send $v_{(1,0)}$ to the same vector in $V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1)$.

From (3.23), (3.34), and (3.29) it follows that

$$\begin{aligned}&[2]_q(\mathrm{id} \otimes \mathbf{i}) \circ (\mathrm{id} \otimes \mathbf{p}) \circ (\mathbf{cup}_1 \otimes \mathrm{id})(v_{(1,0)})\\ &\quad - [2]_q(\mathbf{i} \otimes \mathrm{id})(\mathbf{p} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{cup}_1)(v_{(1,0)})\end{aligned} \tag{3.56}$$

is equal to

$$\begin{aligned}&-q^{-3}v_{(1,0)} \otimes \mathbf{i}(v_{(0,0)}) + q^{-2}[2]_q v_{(-1,1)} \otimes \mathbf{i}(v_{(2,-1)}) - [2]_q v_{(1,-1)} \otimes \mathbf{i}(v_{(0,1)})\\ &\quad + q^{-3}[2]_q \mathbf{i}(v_{(0,1)}) \otimes v_{(1,-1)} - q^{-1}[2]_q \mathbf{i}(v_{(2,-1)}) \otimes v_{(-1,1)} + q\,\mathbf{i}(v_{(0,0)}) \otimes v_{(1,0)}\end{aligned} \tag{3.57}$$

Using (3.23), we also find that

$$\mathrm{id} \otimes \mathbf{cup}_1(v_{(1,0)}) - \mathbf{cup}_1 \otimes \mathrm{id}(v_{(1,0)}) \tag{3.58}$$

is equal to

$$\begin{aligned} &v_{(1,0)} \otimes \big(- q^{-4} v_{(1,0)} \otimes v_{(-1,0)} + q^{-3} v_{(-1,1)} \otimes v_{(1,-1)} \\ &\qquad - q^{-1} v_{(1,-1)} \otimes v_{(-1,1)} + v_{(-1,0)} \otimes v_{(1,0)} \big) \\ &- \big(q^{-4} v_{(1,0)} \otimes v_{(-1,0)} - q^{-3} v_{(-1,1)} \otimes v_{(1,-1)} \\ &\qquad + q^{-1} v_{(1,-1)} \otimes v_{(-1,1)} - v_{(-1,0)} \otimes v_{(1,0)} \big) \otimes v_{(1,0)}. \end{aligned} \tag{3.59}$$

Using (3.29) to show that (3.57) = (3.59) is left as an exercise. ∎

3.6. Background on tilting modules

Let $\Bbbk$ be a field and let $q \in \Bbbk^\times$ be such that $q + q^{-1} \neq 0$. We will write $U_q^{\Bbbk}(\mathfrak{sp}_4) = \Bbbk \otimes U_q^{\mathcal{A}}(\mathfrak{sp}_4)$, and $U_q^{\Bbbk}(\mathfrak{sp}_4)$-mod for the category of finite-dimensional $U_q^{\Bbbk}(\mathfrak{sp}_4)$ modules which are direct sums of their weight spaces and such that K_α acts on the μ weight space as $q^{(\mu,\alpha^\vee)}$.

Everything we say in this section is well known to experts, but the results are essential for our arguments so we include some discussion for completeness. Two excellent references are Jantzen's book [15] (only the second edition contains the appendix on representations of quantum groups and the appendix on tilting modules) and the e-print [4]. To deal with specializations when q is an even root of unity we will also need some results from [29] and [17].

For each $\lambda \in X_+$ there is a *dual Weyl module* of highest weight λ, denoted $\nabla^{\Bbbk}(\lambda)$, which is defined as an induced module [15, Section H.11]. The dual Weyl modules are a direct sum of their weight spaces and therefore have formal characters. Recall that we wrote $V(\lambda)$ for the irreducible module $\mathfrak{sp}_4(\mathbb{C})$ module of highest weight λ. We will write $[V(\lambda)]$ for the formal character of $V(\lambda)$ in $\mathbb{Z}[X]$, the group algebra of the weight lattice. It is known that a q-analogue of Kempf's vanishing holds for any $\Bbbk$, see [29]. This implies that dual Weyl modules have formal character $[V(\lambda)]$, see [2, Theorem 5.12].

The dual Weyl module always has a unique simple submodule with highest weight λ. We will denote this module by $L^{\Bbbk}(\lambda)$. The module $L^{\Bbbk}(\lambda)$ should not be thought of as a base change of $V(\lambda)$. In fact, quite often the two modules will have distinct formal characters.

Since $U_q^{\Bbbk}(\mathfrak{sp}_4)$ is a Hopf-algebra, it acts on the dual vector space of any finite-dimensional representation. Then we define the *Weyl module* of highest weight λ by $V^{\Bbbk}(\lambda) = \nabla^{\Bbbk}(-w_0\lambda)^*$, see [15, Section H.15]. The dual Weyl module $V^{\Bbbk}(\lambda)$ has the same formal character as $\nabla^{\Bbbk}(\lambda)$, i.e., $[V(\lambda)]$, and $V^{\Bbbk}(\lambda)$ has a unique simple quotient isomorphic to $L^{\Bbbk}(\lambda)$.

Remark 3.8. In type C_2 the longest element w_0 acts on the weight lattice as -1. Therefore, $V^{\Bbbk}(\lambda) = \nabla^{\Bbbk}(\lambda)^*$.

Definition 3.9. A *tilting module* is a module which has a (finite) filtration by Weyl modules, and a (finite) filtration by dual Weyl modules. The category of tilting modules, denoted $Tilt(U_q^{\Bbbk}(\mathfrak{sp}_4))$, is the full subcategory of $U_q^{\Bbbk}(\mathfrak{sp}_4)$-mod where the objects are tilting modules.

Proposition 3.10. *The tensor product of two Weyl modules*

$$V^{\Bbbk}(\lambda_1) \otimes V^{\Bbbk}(\lambda_2)$$

has a filtration by Weyl modules.

Proof. That this holds over $\Bbbk$ follows from [17] where the result is shown to hold integrally using the theory of crystal bases. ∎

Corollary 3.11. *The tensor product of two tilting modules is a tilting module.*

Proof. Since $(-)^*$ is exact, it follows from proposition (3.10) that the tensor product of dual Weyl modules

$$V^{\Bbbk}(\lambda_1)^* \otimes V^{\Bbbk}(\lambda_2)^*$$

has a filtration by dual Weyl modules. Thus, the tensor product of two tilting modules will have a Weyl filtration and a dual Weyl filtration and is therefore a tilting module. ∎

Proposition 3.12. *Let* $\lambda, \mu \in X_+$. *Then* $\dim_{\Bbbk} \mathrm{Ext}^i(V^{\Bbbk}(\lambda), \nabla^{\Bbbk}(\mu)) = \delta_{i,0}\delta_{\lambda,\mu}$ *for all* $i \geq 0$.

Proof. A standard argument [4, Proof of Claim 3.1] shows that the vanishing of higher extension groups follows from Kempf's vanishing [29]. ∎

Proposition 3.13. *The category* $\mathbf{Tilt}(U_q^{\Bbbk}(\mathfrak{sp}_4))$ *is closed under direct sums, direct summands, and tensor products. The isomorphism classes of indecomposable objects in the category are in bijection with* X_+. *We will write* $T^{\Bbbk}(\lambda)$ *for the indecomposable tilting module corresponding to the dominant integral weight* λ. *The module* $T^{\Bbbk}(\lambda)$ *is characterized as the unique indecomposable tilting module with a one-dimensional* λ *highest weight space.*

Proof. [15, Section E.3–E.6]. ∎

Lemma 3.14. *Weyl modules or dual Weyl modules give a basis for the Grothendieck group of the category* $U_q^{\Bbbk}(\mathfrak{sp}_4)$-mod.

Proof. Both $V^{\Bbbk}(\lambda)$ and $\nabla^{\Bbbk}(\lambda)$ have the same formal character: $[V(\lambda)]$. In particular, $V^{\Bbbk}(\lambda)$ and $\nabla^{\Bbbk}(\lambda)$ both have one-dimensional λ weight spaces. ∎

For a tilting module T, we will write $(T\colon V^{\Bbbk}(\lambda))$ to denote the filtration multiplicity. Formal character considerations also imply that $(T\colon V^{\Bbbk}(\lambda)) = (T\colon V^{\Bbbk}(\lambda)^*)$ [15, Section E.10].

Lemma 3.15. *The following are equivalent.*

1. *The Weyl module $V^{\Bbbk}(\lambda)$ is simple.*
2. $V^{\Bbbk}(\lambda) \cong \nabla^{\Bbbk}(\lambda)$
3. *The Weyl module $V^{\Bbbk}(\lambda)$ is a tilting module.*

Proof. It is not hard to see (1) implies (2) implies (3) [15, Section E.1]. That (3) implies (2) follows from Lemma (3.14), along with the equality of formal characters $[V^{\Bbbk}(\lambda)] = [\nabla^{\Bbbk}(\lambda)]$. To see that (2) implies (1), observe that the composition

$$L^{\Bbbk}(\lambda) \to \nabla^{\Bbbk}(\lambda) \xrightarrow{\sim} V^{\Bbbk}(\lambda) \to L^{\Bbbk}(\lambda)$$

is non-zero on the λ weight space. So, the composition is a non-zero endomorphism of a simple module and therefore is an isomorphism. Thus, $L^{\Bbbk}(\lambda)$ is a direct summand of $\nabla^{\Bbbk}(\lambda)$. Since $\nabla^{\Bbbk}(\lambda)$ has a simple socle, we may conclude that $\nabla^{\Bbbk}(\lambda) \cong L^{\Bbbk}(\lambda)$. ∎

Lemma 3.16. 1. *If X has a filtration by Weyl modules, then, for all $\lambda \in X_+$,*

$$dim\, \mathrm{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(X, \nabla^{\Bbbk}(\lambda)) = (X\colon V^{\Bbbk}(\lambda)).$$

2. *If Y has a filtration by dual Weyl modules, then, for all $\lambda \in X_+$,*

$$dim\, \mathrm{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(V^{\Bbbk}(\lambda), Y) = (Y\colon \nabla^{\Bbbk}(\lambda)).$$

Proof. Both claims follow from (3.12) and a long exact sequence argument. ∎

Proposition 3.17. *If T and T' are tilting modules, then*

$$\dim \mathrm{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(T, T') = \sum_{\lambda \in X_+} (T\colon V^{\Bbbk}(\lambda))(T'\colon V^{\Bbbk}(\lambda)). \tag{3.60}$$

Proof. Since T has both Weyl and dual Weyl filtrations, this follows from 3.16 and the fact that $(T'\colon \nabla^{\Bbbk}(\lambda)) = (T'\colon V^{\Bbbk}(\lambda))$. ∎

3.7. The image of the evaluation functor and tilting modules

We continue with our assumption that $\Bbbk$ is a field and $q \in \Bbbk^{\times}$ such that $q + q^{-1} \neq 0$.

Definition 3.18. The category $\mathbf{Fund}(U_q^{\Bbbk}(\mathfrak{sp}_4))$ is the full subcategory of the category $\mathrm{Rep}(U_q^{\Bbbk}(\mathfrak{sp}_4))$ with objects

$$V^{\Bbbk}(\underline{w}) = V^{\Bbbk}(w_1) \otimes V^{\Bbbk}(w_2) \otimes \cdots \otimes V^{\Bbbk}(w_n),$$

where $\underline{w} = w_1 w_2 \ldots w_n$ and $w_i \in \{1, 2\}$.

After changing coefficients to $\Bbbk$, the functor from Theorem (3.6) becomes

$$\Bbbk \otimes \Xi\colon \mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4} \to \mathbf{Fund}(U_q^{\Bbbk}(\mathfrak{sp}_4)) \tag{3.61}$$

We will abuse notation and write Ξ for $\Bbbk \otimes \Xi$.

Lemma 3.19. *The modules $V^{\Bbbk}(\underline{w})$ are tilting modules.*

Proof. From the description of the integral forms of the modules in (3.2) and (3.4), it is easy to see that $V^{\Bbbk}(\varpi_1)$ and $V^{\Bbbk}(\varpi_2)$ are irreducible with highest weight ϖ_1 and ϖ_2. They also have the same formal character as $[V(\varpi_1)]$ and $[V(\varpi_2)]$ respectively. So, (3.15) implies that $V^{\Bbbk}(\underline{w})$ is a tensor products of tilting modules and therefore is a tilting module. ∎

Remark 3.20. If $q + q^{-1} = 0$, then the Weyl module $V^{\Bbbk}(\varpi_1)$ is still simple and therefore tilting but the Weyl module $V^{\Bbbk}(\varpi_2)$ is not. In particular, $V^{\Bbbk}(\varpi_2)$ has two Jordan–Hölder factors, a simple socle isomorphic to $L^{\Bbbk}(0)$ and the simple quotient $L^{\Bbbk}(\varpi_2)$.

Lemma 3.21. *For all $\underline{w}$ and $\underline{u}$*

$$\dim_{\Bbbk} \mathrm{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(V^{\Bbbk}(\underline{w}), V^{\Bbbk}(\underline{u})) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})). \tag{3.62}$$

Proof. Suppose that

$$V(\underline{w}) \cong \bigoplus_{\lambda} V(\lambda)^{m_\lambda}, \tag{3.63}$$

so we have an equality of formal characters $[V(\underline{w})] = \sum m_\lambda [V(\lambda)]$. Since $[V^{\Bbbk}(\underline{w})] = [V(\underline{w})]$ and $[V^{\Bbbk}(\lambda)] = [V(\lambda)]$ it follows that $(V^{\Bbbk}(\underline{w}) : V^{\Bbbk}(\lambda)) = m_\lambda$. The claim then follows from proposition (3.17) and (2.23) ∎

Theorem 3.22. *The functor*

$$\Xi\colon \mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4} \to \mathbf{Fund}(U_q^{\Bbbk}(\mathfrak{sp}_4)).$$

is a monoidal equivalence.

Proof. The functor Ξ is monoidal and essentially surjective, so it suffices to prove Ξ is full and faithful.

Let $\underline{w}$ and $\underline{u}$ be objects in $\mathcal{D}_{\mathfrak{sp}_4}$. In the next section we will prove that $\Xi(\mathbb{LL}_{\underline{w}}^{\underline{u}})$ is a linearly independent set of homomorphisms in $\mathbf{Fund}(U_q^{\Bbbk}(\mathfrak{sp}_4))$.

Since

$$\#\mathbb{LL}_{\underline{w}}^{\underline{u}} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})) = \dim_{\Bbbk} \mathrm{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(V^{\Bbbk}(\underline{w}), V^{\Bbbk}(\underline{u})), \tag{3.64}$$

the linear independence of $\Xi(\mathbb{LL}_{\underline{w}}^{\underline{u}})$ implies that the functor Ξ maps $\mathbb{LL}_{\underline{w}}^{\underline{u}}$ to a basis in $\mathbf{Fund}(U_q^{\Bbbk}(\mathfrak{sp}_4))$. These observations imply that $\mathbb{LL}_{\underline{w}}^{\underline{u}}$ is a linearly independent set of homomorphisms in $\mathcal{D}_{\mathfrak{sp}_4}^{\Bbbk}$. From the inequality in Lemma (2.37) we deduce that $\mathbb{LL}_{\underline{w}}^{\underline{u}}$ is a basis. So, Ξ maps a basis to a basis and

$$\mathrm{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u}) \xrightarrow{\Xi} \mathrm{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(V^{\Bbbk}(\underline{w}), V^{\Bbbk}(\underline{u}))$$

is an isomorphism. ■

Corollary 3.23. *The functor Ξ induces a monoidal equivalence between the Karoubi envelope of $\mathcal{D}_{\mathfrak{sp}_4}^{\Bbbk}$ and the category* $\mathbf{Tilt}(U_q^{\Bbbk}(\mathfrak{sp}_4))$.

Proof. Tensor products and direct summands of tilting modules are tilting modules. Therefore, Lemma (3.19) implies that every direct summand of $V^{\Bbbk}(\underline{w})$ is a tilting module.

Let $\lambda \in X_+$, so $\lambda = a\varpi_1 + b\varpi_2$ for $a, b \in \mathbb{Z}_{\geq 0}$. The module $V^{\Bbbk}(1^{\otimes a} \otimes 2^{\otimes b})$ has a one-dimensional λ highest weight space and all other non-zero weight spaces in X_+ are less than λ. From (3.13) we deduce that $V^{\Bbbk}(1^{\otimes a} \otimes 2^{\otimes b})$ must contain $T^{\Bbbk}(\lambda)$ as a direct summand. Therefore, every indecomposable tilting module is a direct summand of some $V^{\Bbbk}(\underline{w})$. ■

4. Double ladders are linearly independent

4.1. Outline of the argument

In this section we will finish the proof of Theorem (3.22) by arguing that the set $\Xi(\mathbb{LL}_{\underline{w}}^{\underline{u}})$ is linearly independent for all words $\underline{w}$ and $\underline{u}$.

The idea of the proof is best illustrated as follows. Suppose we just wanted to prove that the image of light ladder diagrams from $\underline{w}$ to $\emptyset$ are linearly independent. Recall that $E(\underline{w}, 0)$ is the set of dominant weight subsequences $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, such that $\sum \mu_i = 0$. Assume that for each dominant weight subsequence in $E(\underline{w}, 0)$, we have fixed a choice of light ladder $LL_{\vec{\mu}}$ and a vector $v_{\vec{\mu}} \in V^{\Bbbk}(\underline{w})$. Since $\vec{\mu} \in E(\underline{w}, 0)$ is such that $\sum \mu_i = 0$, the light ladder $L_{\vec{\mu}}$ will map under Ξ to a homomorphism $V^{\Bbbk}(\underline{w}) \to V^{\Bbbk}(\emptyset) = \Bbbk$. If the following matrix of elements in $\Bbbk$

$$\left(\Xi(LL_{\vec{\mu}})(v_{\vec{\nu}})\right)_{\vec{\mu}, \vec{\nu} \in E(\underline{w}, 0)} \tag{4.1}$$

is upper triangular with invertible elements on the diagonal, then a non-trivial linear dependence among the maps $\Xi(LL_{\vec{\mu}})$ will give rise to a non-zero vector in the kernel of the matrix (4.1).

In Section 4.2 we will fix a choice of vectors associated to dominant weight subsequences. Then, since we want to argue double ladder diagrams are linearly independent, we must consider the image of the dominant weight subsequence vectors under both light ladders and upside down light ladders. The inductive construction of light ladders allows us to reduce these calculations to elementary light ladders, neutral ladders, and upside down elementary light ladders. In the end, we still deduce linear independence of double ladder diagrams from an upper triangularity argument.

4.2. Subsequence basis

Recall that the modules $V^{\Bbbk}(1)$ (3.21) and $V^{\Bbbk}(2)$ (3.22) both have a fixed basis of weight vectors v_ν for $\nu \in \operatorname{wt} V^{\Bbbk}(1) \cup \operatorname{wt} V^{\Bbbk}(2)$.

Definition 4.1. Fix $\underline{w} = (w_1, \dots, w_n)$, a word in the alphabet $\{1, 2\}$, and let

$$S(\underline{w}) := \{(\nu_1, \dots, \nu_n) : \nu_i \in \operatorname{wt} V^{\Bbbk}(w_i)\}. \tag{4.2}$$

We set

$$v_{\underline{w},+} := v_{w_1} \otimes v_{w_2} \otimes \cdots \otimes v_{w_n} \tag{4.3}$$

where $v_1 = v_{(1,0)}$ and $v_2 = v_{(0,1)}$. Also, for any sequence of weights $\vec{\nu} = (\nu_1, \dots, \nu_n) \in S(\underline{w})$, we define

$$v_{\underline{w},\vec{\nu}} := v_{\nu_1} \otimes \cdots \otimes v_{\nu_n} \in V^{\Bbbk}(\underline{w}). \tag{4.4}$$

The *subsequence basis* of $V^{\Bbbk}(\underline{w})$ is the set

$$\{v_{\vec{\nu}} : \vec{\nu} \in S(\underline{w})\}. \tag{4.5}$$

Lemma 4.2. *The subsequence basis of $V^{\Bbbk}(\underline{w})$ is a basis of $V^{\Bbbk}(\underline{w})$.*

Proof. This is clear. ∎

Definition 4.3. Let $\chi \in X_+$. The χ *weight space* of $V^{\Bbbk}(\underline{w})$, denoted $V^{\Bbbk}(\underline{w})[\chi]$, is the $\Bbbk$-span of the subsequence basis vectors $v_{\vec{\nu}}$ such that $\sum \nu_i = \chi$.

Note that $E(\underline{w}) \subset S(\underline{w})$. In particular, for each $\vec{\nu} \in E(\underline{w})$ we get a subsequence basis vector $v_{\underline{w},\vec{\nu}}$. In the special case that the dominant weight subsequence is such that $\nu_i = \operatorname{wt} w_i$ for all i, then $v_{\underline{w},\vec{\nu}} = v_{\underline{w},+}$. Also, there is a partition of the set of dominant weight subsequences of $\underline{w}$:

$$E(\underline{w}) = \bigcup_{\lambda \in X_+} E(\underline{w}, \lambda), \tag{4.6}$$

where $\vec{\nu} \in E(\underline{w})$ is in $E(\underline{w}, \lambda)$ whenever $\sum \nu_i = \lambda$ or equivalently $v_{\underline{w},\vec{\nu}} \in V^{\Bbbk}(\underline{w})[\lambda]$.

Definition 4.4. Recall that our choice of simple roots was $\Delta = \{\alpha_s, \alpha_t\}$. There is a partial order on the set of weights defined by $\mu \leq \nu$ if $\nu - \mu \in \mathbb{Z}_{\geq 0} \cdot \Delta$. If we restrict this partial order to the set $\operatorname{wt} V^{\mathcal{A}}(1) \cup \operatorname{wt} V^{\mathcal{A}}(2)$, the resulting order is

$$(-1,0) < (1,-1) < (-1,1) < (1,0), \tag{4.7}$$

$$(0,-1) < (-2,1) < (0,0) < (2,-1) < (0,1). \tag{4.8}$$

The lexicographic order gives a total order on the set $S(\underline{w})$. We will transport this total order to give a total order on the subsequence basis.

Example 4.5. In the image of $E(2121, (2,0)) \to V^{\Bbbk}(2121)[(2,0)]$ we have

$$v_{((0,1),(1,0),(2,-1),(-1,0))} > v_{((0,1),(1,0),(0,-1),(1,0))} > v_{((0,1),(1,-1),(0,0),(1,0))}.$$

Lemma 4.6. *If* $\operatorname{wt} \underline{w} \not\geq \chi$, *then* $V^{\Bbbk}(\underline{w})[\chi] = 0$.

Proof. If $\vec{\nu} \in S(\underline{w})$ is such that $\nu_i \in \operatorname{wt} V^{\Bbbk}(w_i)$, then $\sum \nu_i \leq \operatorname{wt} \underline{w}$. The subsequence basis spans $V^{\Bbbk}(\underline{w})$, so whenever $V^{\Bbbk}(\underline{w})[\chi] \neq 0$, we must have $\chi \leq \operatorname{wt} \underline{w}$. ∎

4.3. The evaluation functor and elementary diagrams

Notation 4.7. In the remainder of the section, we will use the same notation for diagrammatic morphisms and their image under the functor Ξ. But instead of saying diagram we will say map, for example the image of a light ladder diagram under Ξ will be referred to as a light ladder map.

To further simplify some of the statements below, our convention is that $\underline{w}$ and $\underline{u}$ are words in the alphabet $\{1, 2\}$ and ξ represents an invertible element of $\Bbbk$.

Recall that for each weight $\mu \in \operatorname{wt} V^{\Bbbk}(1) \cup \operatorname{wt} V^{\Bbbk}(2)$ there is an elementary light ladder diagram. The images of the elementary light ladder diagrams under the evaluation functor are the following elementary light ladder maps:

$$\begin{aligned}
L_{(1,0)} &= \mathrm{id}\colon V^{\Bbbk}(1) &&\to V^{\Bbbk}(1),\\
L_{(-1,1)} &= \mathbf{p}\colon V^{\Bbbk}(11) &&\to V^{\Bbbk}(2),\\
L_{(1,-1)} &= (\mathrm{id} \otimes \mathbf{cap}_1) \circ (\mathbf{i} \otimes \mathrm{id})\colon V^{\Bbbk}(21) &&\to V^{\Bbbk}(1),\\
L_{(-1,0)} &= \mathbf{cap}_1\colon V^{\Bbbk}(11) &&\to \Bbbk,\\
L_{(0,1)} &= \mathrm{id}\colon V^{\Bbbk}(2) &&\to V^{\Bbbk}(2),\\
L_{(2,-1)} &= (\mathrm{id} \otimes \mathbf{cap}_2 \otimes \mathrm{id}) \circ (\mathbf{i} \otimes \mathbf{i})\colon V^{\Bbbk}(22) &&\to V^{\Bbbk}(11),\\
L_{(0,0)} &= (\mathbf{cap}_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{i})\colon V^{\Bbbk}(12) &&\to V^{\Bbbk}(1),\\
L_{(-2,1)} &= \mathbf{p} \circ (\mathrm{id} \otimes \mathbf{cap}_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{i})\colon V^{\Bbbk}(112) &&\to V^{\Bbbk}(2),\\
L_{(0,-1)} &= \mathbf{cap}_2\colon V^{\Bbbk}(22) &&\to \Bbbk.
\end{aligned}$$

There are two simple neutral diagrams, and their images under the evaluation functor are the simple neutral maps:

$$N_{12}^{21} = (\mathbf{p} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{i}) : V^{\Bbbk}(12) \to V^{\Bbbk}(21),$$
$$N_{21}^{12} = (\mathrm{id} \otimes \mathbf{p}) \circ (\mathbf{i} \otimes \mathrm{id}) : V^{\Bbbk}(21) \to V^{\Bbbk}(12).$$

Lemma 4.8. *If $f : V^{\Bbbk}(\underline{w}) \to V^{\Bbbk}(\underline{u})$ is a morphism which is in the image of the functor Ξ, then $f : V^{\Bbbk}(\underline{w})[\chi] \to V^{\Bbbk}(\underline{u})[\chi]$, for all $\chi \in X$.*

Proof. It is well known that every $U_q^{\Bbbk}(\mathfrak{sp}_4)$ module homomorphism between finite-dimensional modules will preserve weight spaces. But we could also deduce this from observing that the maps id, $\mathbf{i}$, and the cap and cup maps all preserve weight spaces and that any map in the image of Ξ is a linear combination of vertical and horizontal compositions of these basic maps. ∎

Recall that to construct light ladder diagrams and double ladder diagrams we need to fix a word $\underline{x}_\lambda$ in 1 and 2 for all $\lambda \in X_+$, and we need to make choices of neutral diagrams in the algorithmic construction. We now fix an $\underline{x}_\lambda$ for all $\lambda \in X_+$ and fix a light ladder diagram $LL_{\underline{w},(\mu_1,\dots,\mu_m)}$ for all $\underline{w}$ and all $(\mu_1, \dots, \mu_m) \in E(\underline{w})$. This allows us to construct double ladder diagrams. The double ladder maps are the image of these double ladder diagrams under the evaluation functor.

Remark 4.9. The form of the arguments below do not depend on our choice of light ladder maps.

4.4. Pairing vectors and neutral maps

Lemma 4.10. *If $N : V^{\Bbbk}(\underline{w}) \to V^{\Bbbk}(\underline{u})$ is a neutral map, then $N(v_{\underline{w},+}) = \xi \cdot v_{\underline{u},+}$. Furthermore, if $(\mu_1, \dots, \mu_n)$ is a sequence of weights such that $\mu_i \in \mathrm{wt}\, V^{\Bbbk}(w_i)$, and $N(v_{\underline{w},(\mu_1,\dots,\mu_n)})$ has a non-zero coefficient for $v_{\underline{u},+}$ after being written in the subsequence basis, then $v_{\underline{w},(\mu_1,\dots,\mu_n)} = v_{\underline{w},+}$.*

Proof. Neutral maps are vertical and horizontal compositions of identity maps, and the basic neutral maps N_{12}^{21} and N_{12}^{21}. The lemma will follow from verifying its validity for the two basic neutral maps.

The following maps factor through $V^{\Bbbk}(1)$:

$$I_{12}^{21} := \mathbb{D}(L_{(1,-1)}) \circ L_{(0,0)} \quad \text{and} \quad I_{21}^{12} := \mathbb{D}(L_{(0,0)}) \circ L_{(1,-1)}. \tag{4.9}$$

Since $V^{\Bbbk}(1)$ contains no vectors of weight $\varpi_1 + \varpi_2$, it follows that

$$I_{12}^{21}(v_{(1,0)} \otimes v_{(0,1)}) = 0 \quad \text{and} \quad I_{21}^{12}(v_{(0,1)} \otimes v_{(1,0)}) = 0. \tag{4.10}$$

It is easy to use the diagrammatic relations to compute that the maps

$$b_{12}^{21} = qN_{12}^{21} + q^{-1}I_{12}^{21} \tag{4.11}$$

and

$$b_{21}^{12} = q^{-1}N_{21}^{12} + qI_{21}^{12} \tag{4.12}$$

are mutual inverses.

Both b_{12}^{21} and b_{21}^{12} are isomorphisms so they restrict to isomorphisms of weight spaces. Since the $\varpi_1 + \varpi_2$ weight spaces of $V^{\Bbbk}(12)$ and $V^{\Bbbk}(21)$ are one-dimensional, it follows that N_{12}^{21} sends the vector $v_{(1,0)} \otimes v_{(0,1)}$ to a non-zero scalar multiple of $v_{(0,1)} \otimes v_{(1,0)}$ and N_{12}^{21} sends $v_{(0,1)} \otimes v_{(1,0)}$ to a non-zero multiple of $v_{(1,0)} \otimes v_{(0,1)}$. Furthermore, the only subsequence basis vector which N_{12}^{21} sends to a non-zero multiple of $v_{(0,1)} \otimes v_{(1,0)}$ is $v_{(1,0)} \otimes v_{(0,1)}$, and the only subsequence basis vector which N_{21}^{12} sends to a non-zero multiple of $v_{(1,0)} \otimes v_{(0,1)}$ is $v_{(0,1)} \otimes v_{(1,0)}$. ■

4.5. Pairing vectors and light ladders

Lemma 4.11. *Let* $* \in \{1, 2\}$ *and* $\mu \in \mathrm{wt}\, V^{\Bbbk}(*)$. *Then the map*

$$\mathrm{id} \otimes L_\mu \colon V^{\Bbbk}(\underline{w}) \otimes V^{\Bbbk}(*) \to V^{\Bbbk}(\underline{u}),$$

is such that for all $\nu \in \mathrm{wt}(V^{\Bbbk}(*))$,

$$\mathrm{id} \otimes L_\mu(v_{\underline{w},+} \otimes v_\nu) = \begin{cases} 0 & \text{if } \nu > \mu, \\ \xi \cdot v_{\underline{u},+} & \text{if } \nu = \mu. \end{cases} \tag{4.13}$$

Proof. It suffices to check the claim for L_μ and not all $\mathrm{id} \otimes L_\mu$. The claim is obvious for $L_{(1,0)}$ and $L_{(0,1)}$. For the rest of the cases, the claim follows from the calculation in Section 4.8. Note that in the L_μ step of the calculation, the first non-zero entry is $v_\mu \mapsto \xi \cdot v_{\underline{u},+}$. ■

Let $\vec{\mu} = (\mu_1, \dots, \mu_n) \in E(\underline{w}, \lambda)$. The light ladder map $LL_{\underline{w},\vec{\mu}} \colon V^{\Bbbk}(\underline{w}) \to V^{\Bbbk}(\underline{x}_\lambda)$ restricts to a map

$$LL_{\underline{w},\vec{\mu}} \colon V^{\Bbbk}(\underline{w})[\lambda] \to V^{\Bbbk}(\underline{x}_\lambda)[\lambda]. \tag{4.14}$$

Moreover, $V^{\Bbbk}(\underline{x}_\lambda)[\lambda] = \Bbbk \cdot v_{\underline{x}_\lambda,+}$. There is also a totally ordered set of linearly independent vectors in $V^{\Bbbk}(\underline{w})[\lambda]$, namely $v_{\underline{w},\vec{\nu}}$ for all $\vec{\nu} = (\nu_1, \dots, \nu_n) \in E(\underline{w}, \lambda)$.

Proposition 4.12. *One has*

$$LL_{\underline{w},\vec{\mu}}(v_{\underline{w},\vec{\nu}}) = \begin{cases} 0 & \text{if } \vec{\nu} > \vec{\mu}, \\ \xi \cdot v_{\underline{x}_\lambda,+} & \text{if } \vec{\nu} = \vec{\mu}. \end{cases} \tag{4.15}$$

Proof. By the inductive definition of the light ladder map $LL_{\underline{w},\vec{\mu}}$ and of the vector $v_{\underline{w},\vec{\nu}}$, this proposition follows from repeated use of Lemmas (4.11) and (4.10). ■

4.6. Pairing vectors and upside down light ladders

In the results of Section 4.5 we found the lexicographic order on sequences of weights was adapted to light ladders. There is another order on weights which is convenient for upside down light ladders.

Definition 4.13. Fix $\underline{w}$ and let $\vec{\mu} = (\mu_1, \dots, \mu_n)$ and $\vec{\nu} = (\nu_1, \dots, \nu_n)$ be sequences of weights such that $\mu_i, \nu_i \in \operatorname{wt} V^{\Bbbk}(w_i)$. Define a total order $<^{\mathbb{D}}$ on weight sequences by setting $\vec{\nu} <^{\mathbb{D}} \vec{\mu}$ if $(\nu_n, \dots, \nu_1) < (\mu_n, \dots, \mu_1)$ in the lexicographic order. We may also transport this order to give a total order on the subsequence basis.

Lemma 4.14. *Let $* \in \{1, 2\}$ and $\mu \in \operatorname{wt} V^{\Bbbk}(*)$. Then the map*

$$\operatorname{id} \otimes \mathbb{D}(L_\mu) \colon V^{\Bbbk}(\underline{w}) \to V^{\Bbbk}(\underline{u}) \otimes V^{\Bbbk}(*)$$

is such that

$$\operatorname{id} \otimes \mathbb{D}(L_\mu)(v_{\underline{w},+}) = \xi \cdot v_{\underline{u},+} \otimes v_\mu + \sum c_{\vec{\tau}} \cdot v_{\underline{u},\vec{\tau}} \otimes v_\nu, \quad c_{\vec{\tau}} \in \Bbbk, \tag{4.16}$$

where $v_{\underline{u},\vec{\tau}} \otimes v_\nu$ is a subsequence basis vector, $v_\nu > v_\mu$, and $v_{\underline{u},\vec{\tau}} < v_{\underline{u},+}$.

Proof. It suffices to check the claim for $\mathbb{D}(L_\mu)$ and not all $\operatorname{id} \otimes \mathbb{D}(L_\mu)$. The claim is obvious for $\mathbb{D}(L_{(1,0)})$ and $\mathbb{D}(L_{(0,1)})$. The rest of the cases follow from the calculation in Section 4.9. Note that the first line in the $\mathbb{D}(L_\mu)$ calculation is

$$v_{\underline{w},+} \mapsto \xi \cdot v_{\underline{u},+} \otimes v_\mu,$$

while the remaining terms are of the form $v_{\underline{u},\vec{\tau}} \otimes v_\nu$ where $\nu > \mu$. ■

Let $\vec{\mu} = (\mu_1, \dots, \mu_n) \in E(\underline{w}, \lambda)$. The associated upside down light ladder map $\mathbb{D}(LL_{\underline{w},\vec{\mu}}) \colon V^{\Bbbk}(\underline{x}_\lambda) \to V^{\Bbbk}(\underline{w})$ restricts to a map

$$\mathbb{D}(LL_{\underline{w},\vec{\mu}}) \colon V^{\Bbbk}(\underline{x}_\lambda)[\lambda] \to V^{\Bbbk}(\underline{w})[\lambda]. \tag{4.17}$$

Proposition 4.15. *One has*

$$\mathbb{D}(LL_{\underline{w},\vec{\mu}})(v_{\underline{x}_\lambda,+}) = \xi \cdot v_{\underline{w},\vec{\mu}} + \sum c_{\vec{\tau}} \cdot v_{\underline{w},\vec{\tau}}, \quad c_{\vec{\tau}} \in \Bbbk, \tag{4.18}$$

where $v_{\underline{w},\vec{\mu}} <^{\mathbb{D}} v_{\underline{w},\vec{\tau}}$.

Proof. By the inductive definition of the light ladder map $LL_{\underline{w},(\mu_1,\dots,\mu_n)}$, this proposition follows from repeated use of Lemmas (4.14) and (4.10). ■

4.7. Proof of linear independence

Theorem 4.16. *The set*

$$\mathbb{LL}^{\underline{u}}_{\underline{w}} = \bigcup_{\lambda \in X_+} \mathbb{LL}^{\underline{u}}_{\underline{w}}(\lambda) \tag{4.19}$$

is a linearly independent subset of $\mathrm{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(V^{\Bbbk}(\underline{w}), V^{\Bbbk}(\underline{u}))$.

Proof. Let

$$\sum_{\lambda} \sum_{\substack{\vec{\mu} \in E(\underline{w},\lambda) \\ \vec{\nu} \in E(\underline{u},\lambda)}} {}^{\lambda}c^{\vec{\nu}}_{\vec{\mu}} \cdot \mathbb{LL}^{\underline{u},\vec{\nu}}_{\underline{w},\vec{\mu}} = 0, \quad {}^{\lambda}c^{\vec{\nu}}_{\vec{\mu}} \in \Bbbk \tag{4.20}$$

be a non-trivial linear relation. There is at least one $\lambda_0 \in X_+$ with ${}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu}} \neq 0$ such that if ${}^{\lambda}c^{\vec{\nu}}_{\vec{\mu}} \neq 0$ then $\lambda \not> \lambda_0$. Lemma (4.6) implies that for all $\lambda \neq \lambda_0$ with ${}^{\lambda}c^{\vec{\nu}}_{\vec{\mu}} \neq 0$, $V^{\Bbbk}(\underline{x}_{\lambda})[\lambda_0] = 0$. If $v_0 \in V^{\Bbbk}(\underline{w})[\lambda_0]$, then since light ladder maps preserve the weight of a vector (4.8)

$$0 = \sum_{\lambda} \sum_{\vec{\mu},\vec{\nu}} {}^{\lambda}c^{\vec{\nu}}_{\vec{\mu}} \cdot \mathbb{LL}^{\underline{u},\vec{\nu}}_{\underline{w},\vec{\mu}}(v_0) = \sum_{\vec{\mu},\vec{\nu}} {}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu}} \cdot \mathbb{LL}^{\underline{u},\vec{\nu}}_{\underline{w},\vec{\mu}}(v_0). \tag{4.21}$$

Note that for $\vec{\mu} \in E(\underline{w}, \lambda_0)$, $v_{\underline{w},\vec{\mu}} \in V^{\Bbbk}(\underline{w})[\lambda_0]$.

Let $\vec{\mu_0}$ be the largest $\vec{\mu}$, in the lexicographic order, such that ${}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu}} \neq 0$. Taking $v_0 = v_{\underline{w},\vec{\mu_0}}$ in (4.21) results in

$$0 = \sum_{\vec{\mu},\vec{\nu}} {}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu}} \cdot \mathbb{LL}^{\underline{u},\vec{\nu}}_{\underline{w},\vec{\mu}}(v_{\underline{w},\vec{\mu_0}}) = \sum_{\vec{\mu},\vec{\nu}} {}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu}} \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu}}) \circ LL_{\underline{w},\vec{\mu}}(v_{\underline{w},\vec{\mu_0}}). \tag{4.22}$$

Proposition (4.12) implies

$$0 = \sum_{\vec{\nu}} {}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu_0}} \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu}}) \circ LL_{\underline{w},\vec{\mu_0}}(v_{\underline{w},\vec{\mu_0}}) = \sum_{\vec{\nu}} {}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu_0}} \xi \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu}})(v_{\underline{x}_{\lambda},+}). \tag{4.23}$$

Let $\vec{\nu_0}$ be the smallest $\vec{\nu}$, in the $<^{\mathbb{D}}$ order, such that ${}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu_0}} \neq 0$. Proposition (4.15) implies

$$\begin{aligned} 0 &= {}^{\lambda}c^{\vec{\nu_0}}_{\vec{\mu_0}} \xi \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu_0}})(v_{\underline{x}_{\lambda},+}) + \sum_{\vec{\nu_0} <^{\mathbb{D}} \vec{\nu}} {}^{\lambda_0}c^{\vec{\nu}}_{\vec{\mu_0}} \xi \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu}})(v_{\underline{x}_{\lambda},+}) \\ &= {}^{\lambda_0}c^{\vec{\nu_0}}_{\vec{\mu_0}} \xi \cdot v_{\underline{u},\vec{\nu_0}} + \text{"higher terms,"} \end{aligned} \tag{4.24}$$

where "higher terms" is a linear combination of subsequence basis vectors all of which are greater than $v_{\underline{u},\vec{\nu_0}}$ in the $<^{\mathbb{D}}$ order. Since the subsequence basis vectors are linearly independent, we must have ${}^{\lambda_0}c^{\vec{\nu_0}}_{\vec{\mu_0}} \xi = 0$, which is a contradiction. ∎

4.8. Elementary light ladder calculations

One has

$$L_{(-1,1)}(v_{(1,0)} \otimes (-)) \colon \begin{cases} v_{(1,0)} \mapsto 0, \\ v_{(-1,1)} \mapsto -v_{(0,1)}, \\ v_{(1,-1)} \mapsto -v_{(2,-1)}, \\ v_{(-1,0)} \mapsto \dfrac{-q}{[2]_q} v_{(0,0)}; \end{cases} \tag{4.25}$$

$$L_{(1,-1)}(v_{(0,1)} \otimes (-)) \colon \begin{cases} v_{(1,0)} \mapsto, 0 \\ v_{(-1,1)} \mapsto 0, \\ v_{(1,-1)} \mapsto -v_{(1,0)}, \\ v_{(-1,0)} \mapsto -v_{(-1,1)}; \end{cases} \tag{4.26}$$

$$L_{(-1,0)}(v_{(1,0)} \otimes (-)) \colon \begin{cases} v_{(1,0)} \mapsto 0, \\ v_{(-1,1)} \mapsto 0, \\ v_{(1,-1)} \mapsto 0, \\ v_{(-1,0)} \mapsto 1; \end{cases} \tag{4.27}$$

$$L_{(2,-1)}(v_{(0,1)} \otimes (-)) \colon \begin{cases} v_{(0,1)} \mapsto 0, \\ v_{(2,-1)} \mapsto v_{(1,0)} \otimes v_{(1,0)}, \\ v_{(0,0)} \mapsto v_{(1,0)} \otimes v_{(-1,1)} + q^{-1} v_{(-1,1)} \otimes v_{(1,0)}, \\ v_{(-2,1)} \mapsto v_{(-1,1)} \otimes v_{(-1,1)}, \\ v_{(0,-1)} \mapsto -v_{(1,0)} \otimes v_{(-1,0)} + v_{(-1,1)} \otimes v_{(1,-1)}; \end{cases} \tag{4.28}$$

$$L_{(0,0)}(v_{(1,0)} \otimes (-)) \colon \begin{cases} v_{(0,1)} \mapsto 0, \\ v_{(2,-1)} \mapsto 0, \\ v_{(0,0)} \mapsto -q^{-1} v_{(1,0)}, \\ v_{(-2,1)} \mapsto -v_{(-1,1)}, \\ v_{(0,-1)} \mapsto -v_{(1,-1)}; \end{cases} \tag{4.29}$$

$$L_{(-2,1)}(v_{(1,0)} \otimes v_{(1,0)} \otimes (-)) \colon \begin{cases} v_{(0,1)} \mapsto 0, \\ v_{(2,-1)} \mapsto 0, \\ v_{(0,0)} \mapsto 0, \\ v_{(-2,1)} \mapsto v_{(0,1)}, \\ v_{(0,-1)} \mapsto v_{(2,-1)}; \end{cases} \tag{4.30}$$

$$L_{(0,-1)}(v_{(0,1)} \otimes (-))\colon \begin{cases} v_{(0,1)} & \mapsto 0, \\ v_{(2,-1)} & \mapsto 0, \\ v_{(0,0)} & \mapsto 0, \\ v_{(-2,1)} & \mapsto 0, \\ v_{(0,-1)} & \mapsto 1. \end{cases} \tag{4.31}$$

4.9. Upside down elementary light ladder calculations

One has

$$\mathbb{D}(L_{(-1,1)})\colon v_{(0,1)} \mapsto q^{-1} v_{(1,0)} \otimes v_{(-1,1)} - v_{(-1,1)} \otimes v_{(1,0)}, \tag{4.32}$$

$$\mathbb{D}(L_{(1,-1)})\colon v_{(1,0)} \mapsto -q^{-3} v_{(0,1)} \otimes v_{(1,-1)} + q^{-1} v_{(2,-1)} \otimes v_{(-1,1)} - \frac{q}{[2]_q} v_{(0,0)} \otimes v_{(1,0)}, \tag{4.33}$$

$$\mathbb{D}(L_{(-1,0)})\colon 1 \mapsto -q^{-4} v_{(1,0)} \otimes v_{(-1,0)} + q^{-3} v_{(-1,1)} \otimes v_{(1,-1)} - q^{-1} v_{(1,-1)} \otimes v_{(-1,1)} + v_{(-1,0)} \otimes v_{(1,0)}, \tag{4.34}$$

$$\mathbb{D}(L_{(2,-1)})\colon v_{(1,0)} \otimes v_{(1,0)} \mapsto -q^{-2} v_{(0,1)} \otimes v_{(2,-1)} + v_{(2,-1)} \otimes v_{(0,1)}, \tag{4.35}$$

$$\mathbb{D}(L_{(0,0)})\colon v_{(1,0)} \mapsto \frac{-q^{-3}}{[2]_q} v_{(1,0)} \otimes v_{(0,0)} + q^{-2} v_{(-1,1)} \otimes v_{(2,-1)} - v_{(1,-1)} \otimes v_{(0,1)}, \tag{4.36}$$

$$\begin{aligned} \mathbb{D}(L_{(-2,1)})\colon v_{(0,1)} \mapsto & -q^{-4} v_{(1,0)} \otimes v_{(1,0)} \otimes v_{(-2,1)} \\ & + \frac{q^{-2}}{[2]_q} v_{(1,0)} \otimes v_{(-1,1)} \otimes v_{(0,0)} \\ & + \frac{q^{-3}}{[2]_q} v_{(-1,1)} \otimes v_{(1,0)} \otimes v_{(0,0)} \\ & - q^{-2} v_{(-1,1)} \otimes v_{(-1,1)} \otimes v_{(2,-1)} \\ & - q^{-1} v_{(1,0)} \otimes v_{(-1,0)} \otimes v_{(0,1)} \\ & + v_{(-1,1)} \otimes v_{(1,-1)} \otimes v_{(0,1)}, \end{aligned} \tag{4.37}$$

$$\begin{aligned}\mathbb{D}(L_{(0,-1)})\colon 1 \mapsto q^{-6} v_{(0,1)} \otimes v_{(0,-1)} - q^{-4} v_{(2,-1)} \otimes v_{(-2,1)} \\ + \frac{q^{-2}}{[2]_q} v_{(0,0)} \otimes v_{(0,0)} \\ - q^{-2} v_{(-2,1)} \otimes v_{(2,-1)} \\ + v_{(0,-1)} \otimes v_{(0,1)}.\end{aligned} \tag{4.38}$$

5. Characters of tilting modules

5.1. Object adapted cellular category structure

We refer to [10, Definition 2.4] for the definition of a strictly object adapted cellular category or SOACC.

Let $\mathbb{k}$ be a field and let $q \in \mathbb{k}^\times$ such that $q + q^{-1} \neq 0$. In this section we will show that $\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}$ is an SOACC. It follows that the endomorphism algebras in $\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}$ are cellular algebras. Since we proved that $\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}$ is equivalent to $\mathbf{Fund}(U_q^{\mathbb{k}}(\mathfrak{sp}_4))$, the result about cellular algebras also follows from [3] and the result about $\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}$ being an SOACC follows from [1, Proposition 2.4]. For more discussion about the relation between our work and [3] we recommend [8, p. 6] (but replace $\mathfrak{sl}_n$ webs with $\mathcal{D}_{\mathfrak{sp}_4}$).

For each $\lambda \in X_+$, choose an object $\underline{x}_\lambda$ in $\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}$ such that $\operatorname{wt} \underline{x}_\lambda = \lambda$. The set $\Lambda = \{\underline{x}_\lambda\}_{\lambda \in X_+}$ is in bijection with X_+, and we define a partial order on Λ by setting $\underline{x}_\lambda \leq \underline{x}_\mu$ whenever $\lambda \leq \mu$, i.e., $\mu - \lambda \in \mathbb{Z}_{\geq 0}\Phi_+$.

For any object $\underline{w}$ in $\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}$ and for all $\vec{\mu} \in E(\underline{w}, \lambda)$ we fix a light ladder diagram $LL_{\vec{\mu}} := LL_{\underline{w},\vec{\nu}} \in \operatorname{Hom}_{\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x}_\lambda)$ and an upside down light ladder diagram $\mathbb{D}(LL_{\vec{\nu}}) := \mathbb{D}(LL_{\underline{w},\vec{\nu}}) \in \operatorname{Hom}_{\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}}(\underline{x}_\lambda, \underline{w})$.

If $\underline{x}_\lambda = x_1 x_2 \ldots x_n$ where $x_i \in \{1, 2\}$, then the set $E(\underline{x}_\lambda, \lambda)$ contains a single element, $\vec{\lambda} = (\operatorname{wt} x_1, \operatorname{wt} x_2 \ldots \operatorname{wt} x_n)$. Recall that in our definition of double ladder diagrams we choose $LL_{\vec{\lambda}} = \mathrm{id}_{\underline{x}_\lambda} = \mathbb{D}(LL_{\vec{\lambda}})$.

For $\vec{\mu} \in E(\underline{w}, \lambda)$ and $\vec{\nu} \in E(\underline{u}, \lambda)$ we set

$$\mathbb{LL}^{\lambda}_{\vec{\mu},\vec{\nu}} := \mathbb{D}(LL_{\vec{\nu}}) \circ LL_{\vec{\mu}} \in \operatorname{Hom}_{\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u}). \tag{5.1}$$

It follows from our main theorem that $\{\mathbb{LL}^{\lambda}_{\vec{\mu},\vec{\nu}}\}_{\lambda \in X_+}$ forms a basis for $\operatorname{Hom}_{\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u})$.

Remark 5.1. In the definition of an SOACC, one fixes the data of two sets, $E(\underline{w}, \lambda)$ and $M(\underline{w}, \lambda)$, which are in a fixed bijection. We are choosing to ignore the set $M(\underline{w}, \lambda)$.

Definition 5.2. Fix $\lambda \in X_+$. Let $(\mathcal{D}^{\mathbb{k}}_{\mathfrak{sp}_4})_{<\lambda}$ be the $\mathbb{k}$-linear subcategory whose morphisms are spanned by $\mathbb{LL}^{\chi}_{\vec{\mu},\vec{\nu}}$ with $\chi < \lambda$.

Lemma 5.3. *Let* $f \in \mathrm{Hom}_{\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u})$ *and let* $\vec{\mu} \in E(\underline{u}, \lambda)$. *Then*

$$LL_{\vec{\mu}} \circ f \equiv \sum_{\vec{\nu} \in E(\underline{w}, \lambda)} * \cdot LL_{\vec{\nu}} \mod (\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4})_{<\lambda}, \tag{5.2}$$

where $*$ *represents an element of* $\Bbbk$.

Proof. Writing $LL_{\vec{\mu}} \circ f$ in the double ladder basis, we find that

$$\begin{aligned} LL_{\vec{\mu}} \circ f &= \sum_{\substack{\chi \in X_+ \\ \vec{\nu} \in E(\underline{w}, \chi) \\ \vec{\tau} \in E(\underline{x}_\lambda, \chi)}} * \cdot \mathbb{LL}^{\chi}_{\vec{\nu}, \vec{\tau}} \\ &\equiv \sum_{\substack{\vec{\mu} \in E(\underline{w}, \lambda) \\ \vec{\tau} \in E(\underline{x}_\lambda, \lambda)}} * \cdot \mathbb{LL}^{\lambda}_{\vec{\nu}, \vec{\tau}} \mod (\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4})_{<\lambda} \\ &\equiv \sum_{\vec{\nu} \in E(\underline{w}, \lambda)} * \cdot LL_{\vec{\nu}} \mod (\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4})_{<\lambda} \end{aligned} \tag{5.3}$$

The second equality follows from the observation that if $\chi \in X_+$ and $E(\underline{x}_\lambda, \chi) \neq \emptyset$, then $\chi \leq \lambda$. The third equality follows from recalling that $E(\underline{x}_\lambda, \lambda) = \{\vec{\lambda}\}$ and $LL_{\vec{\lambda}} = \mathrm{id}_{\underline{x}_\lambda}$. ■

Corollary 5.4. *The category* $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}$ *with fixed choices of* $\underline{x}_\lambda$ *and light ladder diagrams is an SOACC.*

5.2. Tilting character algorithm

We will describe a way to compute the filtration multiplicities $(T^{\Bbbk}(\lambda), V^{\Bbbk}(\mu))$ for all $\lambda, \mu \in X_+$ using the light ladder diagrams in $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}$. The ideas in this section are standard, and we follow [12, 16]. The reader may also wish to consult [3, Appendix 4B] and compare our discussion with the theory of cell modules for cellular algebras.

Lemma 5.5. *The indecomposable tilting module* $T^{\Bbbk}(\lambda)$ *has a local endomorphism ring, and if* J *is the Jacobson radical of the ring* $\mathrm{End}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(T^{\Bbbk}(\lambda))$, *then*

$$\mathrm{End}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(T^{\Bbbk}(\lambda))/J \xrightarrow{\sim} \Bbbk \cdot \mathrm{id}, \tag{5.4}$$

where $\varphi + J = c_\varphi \,\mathrm{id} + J \mapsto c_\varphi \,\mathrm{id}$.

Proof. Restriction to $T^{\Bbbk}(\lambda)[\lambda]$ is a $\Bbbk$-linear ring homomorphism

$$\mathrm{End}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(T^{\Bbbk}(\lambda)) \to \mathrm{End}_{\Bbbk}(T^{\Bbbk}(\lambda)[\lambda]) = \Bbbk \cdot \mathrm{id}\,. \tag{5.5}$$

Since id acts on the λ weight space as multiplication by 1, this ring homomorphisms is surjective. Also, $T^{\Bbbk}(\lambda)$ is indecomposable so its endomorphism ring is local, and therefore the kernel of the ring homomorphism in (5.5) is J. ■

Lemma 5.6. *Let $\underline{w}$ be an object in $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}$ such that* $\operatorname{wt}\underline{w} = \lambda$. *Then*

$$V^{\Bbbk}(\underline{w}) = T^{\Bbbk}(\lambda) \oplus \bigoplus_{\mu<\lambda} T^{\Bbbk}(\mu)^{r_{\underline{w},\mu}} \tag{5.6}$$

and $r_{\underline{w},\mu}$ is the rank of the pairing

$$\kappa_{\underline{w},\mu} \colon \operatorname{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(V^{\Bbbk}(\underline{w}), T^{\Bbbk}(\mu)) \times \operatorname{Hom}_{U_q^{\Bbbk}(\mathfrak{sp}_4)}(T^{\Bbbk}(\mu), V^{\Bbbk}(\underline{w})) \to \Bbbk \cdot \mathrm{id}, \quad (f,g) \mapsto c_{f\circ g} \cdot \mathrm{id}\,.$$

Proof. The claim about the decomposition of $V^{\Bbbk}(\underline{w})$ follows from character considerations. The second claim about computing multiplicities using the rank of the composition pairing is standard [12, Lemma 11.65]. ■

Remark 5.7. The pairing $\kappa_{\underline{w},\operatorname{wt}\underline{w}}$ will always have rank 1.

Lemma 5.8. *The light ladder diagrams $\{LL_{\vec{\nu}}\}_{\vec{\nu}\in E(\underline{w},\mu)}$ form a basis for*

$$\operatorname{Hom}_{\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x}_\mu)/(\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4})_{<\mu},$$

and the upside down light ladder diagrams $\{\mathbb{D}(LL_{\vec{\nu}})\}_{\vec{\nu}\in E(\underline{w},\mu)}$ form a basis for

$$\operatorname{Hom}_{\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}}(\underline{x}_\mu, \underline{w})/(\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4})_{<\mu}.$$

For all pairs $\vec{\chi}, \vec{\nu} \in E(\underline{w},\mu)$ there is a scalar $c^{\vec{\nu}}_{\vec{\chi}} \in \Bbbk$ such that

$$LL_{\vec{\nu}} \circ \mathbb{D}(LL_{\vec{\chi}}) = c^{\vec{\nu}}_{\vec{\chi}}\, \mathrm{id}_{\underline{x}_\mu} + (\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4})_{<\mu},$$

which is computed as the coefficient of the identity in the double ladder basis. The rank of the matrix $(c^{\vec{\nu}}_{\vec{\chi}})_{\vec{\chi},\vec{\nu}\in E(\underline{w},\mu)}$ is equal to the rank of the pairing $\kappa_{\underline{w},\mu}$.

Proof. Since $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}$ is an object adapted cellular category, this follows from the discussion in [12, Appendix 11.5]. ■

Proposition 5.9. *The character of the indecomposable tilting module with highest weight $\lambda \in X_+$ is*

$$[T^{\Bbbk}(\lambda)] = [V^{\Bbbk}(\lambda)] + \sum_{\mu<\lambda} \#E(\underline{x}_\lambda,\mu)[V^{\Bbbk}(\mu)] - \sum_{\mu<\lambda} \operatorname{rk}_{\Bbbk}(c^{\vec{\nu}}_{\vec{\chi}})_{\vec{\chi},\vec{\nu}\in E(\underline{x}_\lambda,\mu)}[T^{\Bbbk}(\mu)]. \tag{5.7}$$

Proof. Since $(V^{\Bbbk}(\underline{x}_{\lambda}) : V^{\Bbbk}(\mu)) = \#E(\underline{x}_{\lambda}, \mu)$, the claim follows from Lemma (5.6) and Lemma (5.8). ∎

Remark 5.10. Since the sums on the right-hand side of (5.7) are indexed over $\mu < \lambda$, and the partially ordered set $(X_{+}, <)$ has the descending chain condition, one can determine $[T^{\Bbbk}(\lambda)]$ by computing $\#E(\underline{x}_{\chi}, \nu)$ and $r_{\underline{x}_{\chi},\nu}$ for all $0 \leq \nu \leq \chi \leq \lambda$.

Remark 5.11. Calculations of tilting module characters can be made completely within the diagrammatic category $\mathcal{D}^{\Bbbk}_{\mathfrak{sp}_4}$. The quantity $\#E(\underline{x}_{\lambda}, \mu)$ is equal to the number of light ladder diagrams from $\underline{x}_{\lambda}$ to $\underline{x}_{\mu}$, and $r_{\underline{x}_{\lambda},\mu}$ is equal to the rank of the matrix $(c^{\vec{\nu}}_{\vec{\chi}})_{\vec{\chi},\vec{\nu} \in E(\underline{w},\mu)}$. Moreover, these matrices can be computed in $\mathcal{D}_{\mathfrak{sp}_4}$. If $M \subset \mathcal{A}$ is a maximal ideal and $\Bbbk = \mathcal{A}/M$, then the rank of the mod M reduction of the matrix $(c^{\vec{\nu}}_{\vec{\chi}})_{\vec{\chi},\vec{\nu} \in E(\underline{w},\mu)}$ is equal to $(V^{\Bbbk}(\underline{w}) : T^{\Bbbk}(\lambda))$.

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