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Seifert hypersurfaces of 2-knots and Chern–Simons functional

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Abstract. For a given smooth 2-knot in S^4, we relate the existence of a smooth Seifert hypersurface of a certain class to the existence of irreducible SU(2)-representations of its knot group. For example, we see that any smooth 2-knot having the Poincaré homology 3-sphere as a Seifert hypersurface has at least four irreducible SU(2)-representations of its knot group. This result is false in the topological category. The proof uses a quantitative formulation of instanton Floer homology. Using similar techniques, we also obtain similar results about codimension-1 embeddings of homology 3-spheres into closed definite 4-manifolds and a fixed point type theorem for instanton Floer homology.

Contents

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1. Introduction

1.1. Chern–Simons functional for oriented 2-knots

A 2-knot is a smooth embedding from S^2 into S^4. The classification problem of isotopy classes of 2-knots has been studied since the introduction of the subject by Artin in 1925 ([2]). There are several diagrammatic approaches to the study of 2-knots including motion pictures ([18]), surface knot diagrams ([37]), and chart diagrams ([26]), as well as invariants of 2-knots, including (twisted) Alexander polynomials ([1]) and quandle cocycle invariants ([26]). We focus, instead, on *Seifert hypersurfaces* of 2-knots.

Definition 1.1. Let K be an oriented 2-knot in S^4. We call a closed oriented connected 3-manifold Y a *(smooth) Seifert hypersurface of K if there exists a smooth embedding $f: Y \setminus B^3 \to S^4$ such that $f|_{\partial(Y\setminus B^3)} = K$ as oriented manifolds, where B^3 is a small open 3-ball.*

We consider the following problem: what are the Seifert hypersurfaces for a given 2-knot? For a given 1-dimensional knot k in S^3, topological types of Seifert surfaces are determined by the Seifert genus $g(k)$ of k. In [36], Ozsváth and Szabó proved that the Seifert genus of a 1-knot can be computed from its knot Floer homology. However, in the case of 2-knots, the detection of topological types of Seifert hypersurfaces remains an open problem even for the unknot. One difficulty comes from the difference between smooth and topological Seifert hypersurfaces. For example, the Poincaré homology 3-sphere is a Seifert hypersurface of the unknot in the topological category but not in the smooth category. Our main result relates the existence of smooth Seifert hypersurfaces of a certain class to the existence of irreducible SU(2)-representations of its knot group. The main result is proved by using a *quantitative formulation* of instanton Floer homology. In order to state our main result, we introduce maps[1]

$$cs_{K,j} \colon R(K, j) := \operatorname{Hom}(G_j(K), \mathrm{SU}(2))/\,\mathrm{SU}(2) \to (0, 1],$$

where the group $G_j(K)$ is the kernel of the composite homomorphism

$$\psi_j \colon G(K) := \pi_1(S^4 \setminus K) \xrightarrow{\mathrm{Ab}} H_1(S^4 \setminus K; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\bmod j} \mathbb{Z}/j\mathbb{Z} \tag{1}$$

and the action of SU(2) on $\operatorname{Hom}(G_j(K), \mathrm{SU}(2))$ is given by the conjugation. The map $cs_{K,j}$ is an analog of the Chern–Simons functional; see Definition 5.1. The set

[1]The invariants $\{cs_{K,j}\}$ can be defined for every oriented null-homologous 2-knot K embedded into a fixed closed oriented 4-manifold X. See Remark 5.6.

$\operatorname{Im} cs_{K,j} \subset (0,1]$ is finite and is an invariant of the pair (K, j). First, we provide several fundamental properties of $\{cs_{K,j} : R(K,j) \to (0,1]\}_{j \in \mathbb{Z}_{>0}}$ containing a relationship to the Chern–Simons functionals of Seifert hypersurfaces:

Proposition 1.2. *The functionals $\{cs_{K,j} : R(K,j) \to (0,1]\}_{j \in \mathbb{Z}_{>0}}$ satisfy the following conditions.*

(1) *Let Y be a Seifert hypersurface of a given oriented 2-knot K. Then,*

$$\operatorname{Im} cs_{K,j} \subset \operatorname{Im} cs_Y$$

holds for any $j \in \mathbb{Z}_{>0}$, where $\operatorname{Im} cs_Y$ is given by

$$\operatorname{Im} cs_Y := \{cs_Y(\rho) : \rho \text{ is a flat } \mathrm{SU}(2)\text{-connection on } Y\} \cap (0,1]$$

and cs_Y is the $\mathrm{SU}(2)$-Chern–Simons functional for Y. Moreover, if Y is a Seifert 3-manifold, then

$$\operatorname{Im} cs_{K,j} \subset \mathbb{Q} \cap (0,1].$$

(2) *For any $j \in \mathbb{Z}_{>0}$,*

$$\operatorname{Im} cs_{K_1,j} \cup \operatorname{Im} cs_{K_2,j} \subset \operatorname{Im} cs_{K_1 \# K_2, j}.$$

(3) *For all positive integers m and j,*

$$\operatorname{Im} cs_{K,j} \subset \operatorname{Im} cs_{K,mj}.$$

(4) *The relation between $\operatorname{Im} cs_{K,j}$ and $\operatorname{Im} cs_{-K,j}$ is given by*

$$\operatorname{Im} cs_{K,j} = \{1 - r : r \in \operatorname{Im} cs_{-K,j} \cap (0,1)\} \cup \{1\}.$$

If K is reversible (i.e., K is isotopic to $-K$), then $1 - r \in \operatorname{Im} cs_{K,j}$ for any $r \in \operatorname{Im} cs_{K,j} \cap (0,1)$.

(5) *If $\operatorname{Im} cs_{K,j} \cap (0,1)$ is non-empty for $j \in \mathbb{Z}_{>0}$, there exist $2\#(\operatorname{Im} cs_{K,j} \cap (0,1))$ $\mathrm{SU}(2)$-irreducible representations of $G_j(K)$.*

We first calculate $\operatorname{Im} cs_{K,j}$ for *ribbon 2-knots*. Here, a ribbon 2-knot means a 2-knot obtained as the boundary of the union of disjoint embedded 3-disks in $\mathbb{R}^4$ with some number of disjoint 3-dimensional 1-handles attached. (For more details, see [46] and [26, Section 5.6].) Property (1) in Proposition 1.2 implies the following.

Corollary 1.3. *If K is a ribbon 2-knot, then $\operatorname{Im} cs_{K,j} = \operatorname{Im} cs_{U,j} = \{1\}$ for any $j \in \mathbb{Z}_{>0}$, where U is the 2-unknot.*

Next, we give calculations of $\{cs_{K,j}\}_{j \in \mathbb{Z}_{>0}}$ for a certain class of twisted spun knots.

Proposition 1.4. *Let $T(p,q)$ be the (p,q)-torus knot, $M(p,q,r)$ the Montesinos knot of type (p,q,r) for a pairwise relative prime tuple (p,q,r) of positive integers, and $k(p/q)$ any 2-bridge knot such that $\Sigma^2(k(p/q)) = L(p,q)$, where $\Sigma^2(k)$ is the double branched cover of $k \subset S^3$.*

(1) *For any $m \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{>0}$,*

$$\operatorname{Im} cs_{K(T(p,q),m),j} = \operatorname{Im} cs_{\Sigma(p,q,m)},$$

where $K(k,m)$ is the m-twisted spun knot of the knot k. For the definition of m-twisted spun knot, see [26, Section 6.1].

(2) *For any $j \in \mathbb{Z}_{>0}$,*

$$\operatorname{Im} cs_{K(M(p,q,r),2),j} = \operatorname{Im} cs_{\Sigma(p,q,r)}.$$

(3) *If p is odd and satisfies the condition*

$$\left\{ s \in \{2,\dots,p-2\} : \frac{s^2-1}{p} \in \mathbb{Z} \right\} = \emptyset,$$

then, for any $j \in \mathbb{Z}_{>0}$,

$$\operatorname{Im} cs_{K(k(p/q),2),j} = \left\{ -\frac{n^2 r}{p} \bmod 1 : 0 \le n \le \left\lceil \frac{p}{2} \right\rceil \right\},$$

where r is any integer satisfying $qr \equiv -1 \mod p$ and $\lceil - \rceil$ is the ceiling function.

In [16], Fintushel and Stern gave an algorithm to compute $\operatorname{Im} cs_{\Sigma(p,q,r)}$ when $\Sigma(p,q,r)$ is a homology 3-sphere. We give explicit calculations of $\operatorname{Im} cs_{K,j}$ for several 2-knots in Yoshikawa's table in [48] and twisted spun 2-knots of 3_1.

Example 1.5. We calculate $\operatorname{Im} cs_{K,j}$ for 8_1, 10_1 and 10_2, in Yoshikawa's table ([48]) and k-twisted spun 2-knots of 3_1.

- The 2-knots 8_1 and 10_1 are spun 2-knots $K(3_1,0)$ and $K(4_1,0)$. It is known that spun knots $K(k,0)$ are ribbon.[2] Therefore, $\operatorname{Im} cs_{8_1,j} = \operatorname{Im} cs_{10_1,j} = \{1\}$ for any $j \in \mathbb{Z}_{>0}$.
- The 2-knot 10_2 is the 2-twisted spun knot $K(3_1,2)$ of 3_1. The 2-knot $K(3_1,2)$ has $\Sigma(2,2,3) = L(3,1)$ as a Seifert hypersurface so we have $\operatorname{Im} cs_{10_2,j} = \{\frac{2}{3}, 1\}$ for any $j \in \mathbb{Z}_{>0}$ ([30]).

[2] It is known that $K(k,0)$ admits a surface diagram without triple intersections. For further details, see [41]. Such a 2-knot is known to be ribbon ([26, Section 4.5]).

- Since $3_1 = T(2,3)$, $-K(3_1, 6k-1)$ has $-\Sigma(2,3,6k-1)$ as a Seifert hypersurface for each $k \in \mathbb{Z}_{>0}$. Thus,

$$\begin{aligned}&\operatorname{Im} cs_{-K(3_1,6k-1)}\\&\quad= \left\{\frac{12(3k^2-k+3l^2)+1}{24(6k-1)} \bmod 1 \colon l \in \{k,\dots,5k-1\} \cap 2\mathbb{Z}\right\} \cup \{1\}.\end{aligned}$$

For example, $\operatorname{Im} cs_{-K(3_1,5)} = \{1/120, 49/120, 1\}$.

In order to state the main theorem, we also need two kinds of Floer theoretic invariants.

(i) In [11], Daemi introduced a sequence of invariants $\Gamma_Y(k) \in [0,\infty]$ of an oriented homology 3-sphere Y parametrized by $k \in \mathbb{Z}$. In [35], Nozaki, Sato, and the author introduced similar invariants $r_s(Y) \in (0,\infty]$ of oriented homology 3-spheres parametrized by $s \in [-\infty, 0]$. These invariants are defined by using a quantitative formulation of instanton Floer homology.

(ii) For an oriented homology 3-sphere Y, $s \in [-\infty,0]$ and $k \in \mathbb{Z}_{>0}$, we introduce invariants

$$l_Y^s, l_Y^k \in \mathbb{Z}_{>0} \cup \{\infty\} \quad \text{and} \quad l_Y \in \mathbb{Z}_{>0} \cup \{\infty\}$$

which satisfy the inequality

$$\max_{s\in[-\infty,0],k\in\mathbb{Z}_{>0}} \{l_Y^s, l_Y^k\} \le l_Y.$$

If Y is a Seifert homology 3-sphere, l_Y coincides with $2|\lambda(Y)|$, where $\lambda(Y)$ is the Casson invariant of Y.

In terms of $r_s(Y)$, $\Gamma_Y(k)$, l_Y^s and l_Y^k, our main result[3] is as follows.

Theorem 1.6. *Let Y be an oriented homology 3-sphere and K an oriented 2-knot.*

(1) *Suppose that $l_Y^s < \infty$ and $r_s(Y) < \infty$ for some $s \in [-\infty,0]$. If Y is a Seifert hypersurface of K, then*

$$r_s(Y) - \lfloor r_s(Y) \rfloor \in \bigcup_{1\le j\le l_Y^s} \operatorname{Im} cs_{K,j},$$

where[4]

$$\lfloor x \rfloor := \begin{cases} \max\{n \in \mathbb{Z} \colon n \le x\} & \text{if } x \notin \mathbb{Z}, \\ x-1 & \text{if } x \in \mathbb{Z}. \end{cases}$$

[3] Theorem 1.6 can be generalized to any oriented 2-knot K embedded into a fixed closed negative definite 4-manifold X with $0 = [K] \in H_2(X;\mathbb{Z})$.

[4] Note that our convention of $\lfloor - \rfloor$ is different from the usual floor function.

(2) *Suppose that* $l_Y^k < \infty$ *and* $\Gamma_{-Y}(k) < \infty$ *for some* $k \in \mathbb{Z}_{>0}$. *If* Y *is a Seifert hypersurface of* K, *then*

$$\Gamma_{-Y}(k) - \lfloor \Gamma_{-Y}(k) \rfloor \in \bigcup_{1 \le j \le l_Y^k} \operatorname{Im} cs_{K,j} .$$

We also prove a sufficient condition for $l_Y < \infty$.

Theorem 1.7. *If the Chern–Simons functional of* Y *is Morse–Bott,*[5] *then* $l_Y < \infty$.

For example, the Chern–Simons functionals of finite connected sums of Seifert homology 3-spheres are Morse–Bott. A sufficient condition for $r_s(Y) < \infty$ and $\Gamma_{-Y}(1) < \infty$ is given by $h(Y) < 0$. (See [11, 35]), where $h(Y)$ is the Frøyshov invariant of Y ([20]). Theorem 1.6 gives a relation between Seifert hypersurfaces and $\mathrm{SU}(2)$-representations of $G_j(K)$ in the following sense.

Theorem 1.8. *Let* Y *be an oriented homology* 3*-sphere and* K *an oriented* 2*-knot. Suppose* Y *is a Seifert hypersurface of* K.

(1) *If* $r_s(Y) < \infty$ *and* $l_Y^s < \infty$ *for some* $s \in [-\infty, 0]$ *and* Y *is a Seifert hypersurface of* K, *then there exists a positive integer* l *with* $l \le l_Y^s$ *such that there exists an irreducible representation*

$$\rho\colon G_l(K) \to \mathrm{SU}(2).$$

In particular, If $l_Y^s = 1$ *and* $r_s(Y) < \infty$ *for some* $s \in [-\infty, 0]$, *then there exists an irreducible representation* $\rho\colon G(K) \to \mathrm{SU}(2)$.

(2) *If* $\Gamma_{-Y}(k) < \infty$ *and* $l_Y^k < \infty$ *for some* $k \in \mathbb{Z}_{>0}$ *and* Y *is a Seifert hypersurface of* K, *then there exists a positive integer* l *with* $l \le l_Y^k$ *such that there exists an irreducible representation*

$$\rho\colon G_l(K) \to \mathrm{SU}(2).$$

In particular, if $\Gamma_{-Y}(k) < \infty$ *and* $l_Y^k = 1$ *for some* $k \in \mathbb{Z}_{>0}$, *then there exists an irreducible representation* $\rho\colon G(K) \to \mathrm{SU}(2)$.

As a corollary, we have the following result:

Corollary 1.9. *Let* n *be a positive integer. Then the knot group of any* 2*-knot having* $\Sigma(2, 3, 6n - 1)$ *as a Seifert hypersurface has at least two irreducible* $\mathrm{SU}(2)$*-representations. Moreover, when* $n = 1$, *the knot group admits at least four irreducible* $\mathrm{SU}(2)$*-representations.*

[5] For the definition of the Morse–Bott property, see Section 3.1.

Freedman [19] proved that, for any homology 3-sphere Y, there is a locally flat topological embedding from Y into S^4. This means that Y can be realized as a Seifert hypersurface of the unknot when we admit locally flat topological embeddings in the definition of Seifert hypersurfaces. Thus, Corollary 1.9 is false for topological Seifert hypersurfaces.

1.2. Embeddings of 3-manifolds into negative definite 4-manifolds

Existence of embeddings is a fundamental problem in differential topology. It is well known that every orientable closed 3-manifold can be embedded in S^5. However, the following problem is quite difficult in general.

Problem 1.10. For a given 4-manifold X and $c \in H_3(X;\mathbb{Z})$, which 3-manifold Y can be embedded in X with $[Y] = c \in H_3(X;\mathbb{Z})$?

This problem has been studied in several situations ([13, 22, 24, 25, 28]). For example, if p is an integer with $|p| > 1$, then $L(p,q)$ cannot be embedded into S^4 ([23]). As another example, by Donaldson's Theorem A (see [14]), the Poincaré homology 3-sphere cannot be smoothly embedded into S^4. However, by Freedman's result ([19]), it does admit locally flat embedding into S^4. Thus, we see that the smooth and locally flat topological embedding problems are different. Our main result relates the existence of embeddings of homology 3-spheres Y of a certain type into a definite 4-manifold X to the existence of irreducible representations $\pi_1(X) \to \mathrm{SU}(2)$. In order to state our main result, we recall from [44] the maps

$$cs^j_{X,c}\colon R(X_{j,c}) := \mathrm{Hom}(\pi_1(X_{j,c}), \mathrm{SU}(2))/\,\mathrm{SU}(2) \to (0,1]$$

defined for an oriented closed connected 4-manifold X and a class $c \in H_3(X;\mathbb{Z})$ having certain properties (see Section 2) with parameter $j \in \mathbb{Z}_{>0}$, where $\{X_{j,c}\}$ is the j-fold cyclic covering space of X corresponding to c. The functional $cs^j_{X,c}$ is an analog of the Chern–Simons functional. For the precise definition, see (7).

Theorem 1.11. *Let Y be an oriented homology 3-sphere and X be a closed connected oriented negative definite 4-manifold. Suppose that there exists a smooth embedding from Y to X with $0 \neq [Y] \in H_3(X;\mathbb{Z})$.*

(1) *If $r_s(Y) < \infty$ and $l^s_Y < \infty$ for some $s \in [-\infty, 0]$, then*

$$r_s(Y) - \lfloor r_s(Y) \rfloor \in \bigcup_{1 \le j \le l^s_Y} \mathrm{Im}\, cs^j_{X,[Y]}\,.$$

(2) *If $\Gamma_{-Y}(k) < \infty$ and $l^k_Y < \infty$ for some $k \in \mathbb{Z}_{>0}$, then*

$$\Gamma_{-Y}(k) - \lfloor \Gamma_{-Y}(k) \rfloor \in \bigcup_{1 \le j \le l^k_Y} \mathrm{Im}\, cs^j_{X,[Y]}\,.$$

If $[Y] = 0$, *then*

$$\infty = r_s(Y) = r_s(-Y) = \Gamma_{-Y}(k) = \Gamma_Y(k)$$

for any $s \in [-\infty, 0]$ *and* $k \in \mathbb{Z}_{>0}$.

The proposition below provides fundamental properties of $\{cs^j_{X,c}\}_{j\in\mathbb{Z}_{>0}}$.

Proposition 1.12. *Let* Y *be an oriented closed connected* 3*-manifold and* X *a closed connected oriented* 4*-manifold.*

(1) *If there exists an embedding from* Y *to* X *with* $0 \neq [Y] \in H_3(X;\mathbb{Z})$ *then*

$$\operatorname{Im} cs^j_{X,[Y]} \subset \operatorname{Im} cs_Y$$

for any j.

(2) *If* $\operatorname{Im} cs^j_{X,c} \cap (0,1) \neq \emptyset$ *for* $j \in \mathbb{Z}_{>0}$, *then there exist* $2\#(\operatorname{Im} cs^j_{X,c} \cap (0,1))$ *irreducible* $\mathrm{SU}(2)$*-representations of* $\pi_1(X_{j,c})$.

(3) *If* $R(X_{j,c})$ *is connected, then* $\operatorname{Im} cs^j_{X,c} = \{1\}$.

Since $\Sigma(2,3,5)$ satisfies a nice Floer theoretic condition (see Theorem 9.14), we can detect $\operatorname{Im} cs_{X,c}$ from the critical values of the Chern–Simons functional of $\Sigma(2,3,5)$ when X contains $\Sigma(2,3,5)$ as a smooth submanifold.

Theorem 1.13. *Suppose* X *is a negative definite* 4*-manifold containing* $\Sigma(2,3,5)$ *as a smooth submanifold. Then*

$$\operatorname{Im} cs^j_{X,[-\Sigma(2,3,5)]} = \left\{\frac{1}{120}, \frac{49}{120}, 1\right\} \subset (0,1]$$

for any $j \in \mathbb{Z}_{>0}$. *In particular,* $\pi_1(X)$ *admits at least four irreducible* $\mathrm{SU}(2)$*-representations.*

Note that $\Sigma(2,3,5) \times S^1$ satisfies the assumption of Theorem 1.13 and that there are exactly four irreducible $\mathrm{SU}(2)$-representations on $\Sigma(2,3,5) \times S^1$. Moreover, we prove the following existence result for $\mathrm{SU}(2)$-representations:

Theorem 1.14. *There is an* S^2*-component* C *in the* $\mathrm{SU}(2)$*-representation space* $R(\Sigma(2,3,5,7))$ *of* $\Sigma(2,3,5,7)$ *satisfying the following property: for any a closed definite* 4*-manifold* X *containing* $\Sigma(2,3,5,7)$ *as a smooth submanifold, all elements in* C *extend as* $\mathrm{SU}(2)$*-representations of* X. *Therefore,* $\pi_1(X)$ *admits an uncountable family of irreducible* $\mathrm{SU}(2)$*-representations. In particular, this implies the knot group of any* 2*-knot having* $\Sigma(2,3,5,7)$ *as a Seifert hypersurface has an uncountable family of irreducible* $\mathrm{SU}(2)$*-representations.*

Lastly, we state a non-existence result for embeddings of Seifert homology 3-spheres.

Theorem 1.15. *Let Y be a Seifert homology 3-sphere of a type $\Sigma(a_1, \dots, a_n)$. Suppose the Frøyshov invariant $h(Y)$ of Y is non-zero. Then Y cannot be smoothly embedded in any negative definite 4-manifold X such that the* SU(2)*-representation space $R(X_{j,c})$ of $X_{j,c}$ is connected for all j.*

If $\pi_1(X)$ is a free group or isomorphic to $\mathbb{Z}^l$ for some $l \in \mathbb{Z}_{>0}$, then $R(X)$ is connected. Moreover, if

$$R(a_1, \dots, a_n) = \frac{2}{a} - 3 + n + \sum_{i=1}^{n} \frac{2}{a_i} \sum_{k=1}^{a_i - 1} \cot\Big(\frac{a\pi k}{a_i}\Big) \cot\Big(\frac{\pi k}{a_i}\Big) \sin^2\Big(\frac{\pi k}{a_i}\Big) > 0,$$

then $h(\Sigma(a_1, \dots, a_n)) > 0$. (See [11].)

1.3. Fixed point theorems for SU(2)-representation spaces

Since instanton Floer homology is modeled on the infinite-dimensional Morse homology of the Chern–Simons functional of 3-manifolds, it is interesting to ask whether or not there is a Lefschetz type fixed point theorem for instanton Floer homology. Ruberman and Saveliev showed the following theorem.

Theorem 1.16 (Ruberman and Saveliev [39]). *Let h be an orientation preserving self-diffeomorphism on Y with some non-degenerate condition described in* [39, (3.7)]. *Then*

$$\lambda_{\mathrm{FO}}(X_h(Y)) = \frac{1}{2} L(h_*, I(Y)),$$

where $X_h(Y)$ is the mapping torus of $h: Y \to Y$, $\lambda_{\mathrm{FO}}(X)$ is the Furuta–Ohta invariant introduced in [21] *and $L(h_*, I_*(Y))$ is the Lefschetz number of $h_*: I_*(Y) \to I_*(Y)$ introduced in* [39].

Theorem 1.16 implies the following fixed point theorem for instanton Floer homology.

Corollary 1.17 (Ruberman and Saveliev [39]). *Under the same assumption of Theorem 1.16, if $L(h_*, I(Y)) \neq 0$, then $h^*: R^*(Y) \to R^*(Y)$ has a fixed point, where $R^*(Y)$ is the set of conjugacy classes of irreducible* SU(2)*-representations*[6] *of $\pi_1(Y)$.*

We prove a similar fixed point theorem by applying Theorem 1.11 to mapping tori of diffeomorphisms.

[6] In general, $\lambda_{\mathrm{FO}}(X_h(Y))$ can be defined for any orientation preserving diffeomorphism. Moreover, if $\lambda_{\mathrm{FO}}(X_h(Y)) \neq 0$, then $h^*: R^*(Y) \to R^*(Y)$ has a fixed point.

Theorem 1.18. *Let Y be an oriented homology 3-sphere and h an orientation preserving self-diffeomorphism of Y.*

(1) *If $r_s(Y) < \infty$ and $l^s_Y < \infty$ for some $s \in [-\infty, 0]$, then there exist a positive number $l \leq l^s_Y$ such that $(h^*)^l : R^*(Y) \to R^*(Y)$ has a fixed point.*

(2) *If $\Gamma_{-Y}(k) < \infty$ and $l^k_Y < \infty$ for some $k \in \mathbb{Z}_{>0}$, then there exist a positive number $l \leq l^k_Y$ such that $(h^*)^l : R^*(Y) \to R^*(Y)$ has a fixed point.*

Combining this with Theorem 1.14, we obtain the following result.

Theorem 1.19. *There exists an S^2-component C of $R^*(\Sigma(2,3,5,7))$ satisfying the following condition: for any orientation preserving diffeomorphism h on $\Sigma(2,3,5,7)$, the fixed point set of*

$$h^* : R^*(\Sigma(2,3,5,7)) \to R^*(\Sigma(2,3,5,7))$$

contains C.

This paper is organized as follows. In Section 2, we review the invariants $\{cs^j_{X,c}\}$, $\{r_s(Y)\}$ and $\{\Gamma_Y(k)\}$ appearing in the theorems in Section 1. In Section 3, we define $\{l^s_Y\}$, $\{l^k_Y\}$ and l_Y and show several properties of these invariants. In Section 4, we establish formal properties of the invariants $\{cs^j_{X,c}\}$ including a connected sum formula, the behavior of $\{cs^j_{X,c}\}$ for j, and a surgery formula. In Section 5, we introduce a family of invariants $\{cs_{K,j}\}$ of 2-knots and using the results of Section 4, we show Proposition 1.2. In Section 6, we give sufficient conditions (Theorem 1.7) for finiteness of l^s_Y, l^k_Y and l_Y. In Section 7, using a technique of instanton Floer theory, we show the existence of flat connections on 4-manifolds under assumptions of the existence of embeddings and prove Theorems 1.6 and 1.11. In Section 8, we prove Theorem 1.14. In Section 9, we prove Theorems 1.13 and 1.8. In this section, we also compare $\{cs_{K,j}\}$ with other 2-knot invariants which can be used to obstruct a certain class of Seifert hypersurfaces.

2. Preliminaries

2.1. Chern–Simons functional $\{cs^j_{X,c}\}_{j \in \mathbb{Z}_{>0}}$

We follow [44]. Let X be a closed connected oriented 4-manifold. We fix a class $c \in H_3(X;\mathbb{Z}) \cong H^1(X;\mathbb{Z}) \cong [X, B\mathbb{Z}]$. The class c determines a covering space

$$p_c : \widetilde{X}^c \to X$$

up to isomorphism. If c is not equal to 0, then $\widetilde{X}^c$ is connected. We impose the following condition on c:

Assumption 2.1. The class c can be represented by a connected oriented 3-manifold Y. (For example, $2[\{1\} \times S^3]$ in $H_3(S^1 \times S^3; \mathbb{Z})$ cannot be represented by a connected oriented 3-manifold.)

Suppose that c is not equal to 0. We fix a smooth classifying map $\tau\colon X \to S^1$ of p_c and a lift $\tilde{\tau}\colon \widetilde{X}^c \to \mathbb{R}$. Let $\widetilde{R}(X)$ be the set of SU(2)-connections on $X \times \mathrm{SU}(2)$ modulo null-homotopic SU(2)-gauge transformations. For $a \in \widetilde{R}(X)$, define

$$cs_{X,c}(a) := -\frac{1}{8\pi^2} \int_{\widetilde{X}^c} \mathrm{Tr}(F_{A_a} \wedge F_{A_a}),$$

where A_a is a smooth SU(2)-connection on $\widetilde{X}^c \times \mathrm{SU}(2)$ such that

$$A_a|_{\tilde{\tau}^{-1}(-\infty,-1]} = p_c^* a|_{\tilde{\tau}^{-1}(-\infty,-1]} \quad \text{and} \quad A_a|_{\tilde{\tau}^{-1}[1,\infty)} = 0.$$

The function $cs_{X,c}\colon \widetilde{R}(X) \to \mathbb{R}$ does not depend on the choices of additional data τ, $\tilde{\tau}$, A_a, representative of a and isomorphism class of $\widetilde{X}^c$. If $c = 0$, then we define $cs_{X,c}$ to be the zero map. Note that

$$\pi_0(\mathrm{Map}(X, \mathrm{SU}(2))) \cong [X, \mathrm{SU}(2)],$$

where $[X, \mathrm{SU}(2)]$ is the set of homotopy classes of maps from X to SU(2). Fix an oriented closed 3-manifold Y embedded in X such that $[Y] = c \in H_3(X; \mathbb{Z})$. If $[Y]$ is not zero in $H_3(X; \mathbb{Z})$, then $X \setminus Y$ is connected. Here, we suppose that Y is connected. In this case, every 0-dimensional framed submanifold in $Y \cup_{\mathrm{Id}} -Y$ bounds some 1-dimensional framed submanifold in X. By the Pontryagin construction, we see that every continuous map $Y \to \mathrm{SU}(2)$ can be extended to a continuous map $X \to \mathrm{SU}(2)$. Since c is not zero, then $X \setminus Y$ is connected. Let W_0 be the connected compact cobordism from Y to itself obtained by cutting X open along Y. We will use the following notations.

i. The manifold W_i is a copy of W_0 for $i \in \mathbb{Z}$.
ii. We denote $\partial(W_i)$ by $Y_+^i \cup Y_-^i$ where Y_+^i(resp. Y_-^i) is equal to Y(resp. $-Y$) as oriented manifolds.
iii. For $(m, n) \in (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{\infty\})$ with $m < n$, we set

$$W[m,n] := \coprod_{m \leq i \leq n} W_i / \{Y_-^j \sim Y_+^{j+1} j \in \{m, \dots, n\}\}. \tag{2}$$

Note that $W[-\infty, \infty] \to X$ is isomorphic to $\widetilde{X}^c \to X$ as $\mathbb{Z}$-covering spaces. Using the identification, we have

$$cs^j_{X,c}(a) = -\frac{1}{8\pi^2} \int_{W[-\infty,\infty]} \mathrm{Tr}(F_{A_a} \wedge F_{A_a}) \tag{3}$$

$$= cs(i_Y^* a) - cs(i_Y^* 0) = cs(i_Y^* a) \tag{4}$$

for every element $a \in \widetilde{R}(X_{j,c})$. Formula (3) gives the following:

Proposition 2.2. *For any element $g \in \pi_0(\mathrm{Map}(X, \mathrm{SU}(2)))$,*

$$cs_{X,c}(g^* a) = \deg(g|_Y) + cs_{X,c}(a). \tag{5}$$

By (5), we obtain a map

$$cs_{X,c}\colon R(X) := \widetilde{R}(X)/\pi_0(\mathrm{Map}(X, \mathrm{SU}(2))) \to \mathbb{R}/\mathbb{Z} \cong (0, 1].$$

We call $cs_{X,c}$ the *Chern–Simons functional for* (X, c). If we consider a C^∞-topology on $R(X)$, $cs_{X,c}$ is a continuous map. Note that $R(X)$ is compact, $\mathrm{Im}\, cs_{X,c}$ has a minimal value.

Lemma 2.3. *Suppose we have another pair (X', c') of a closed oriented 4-manifold X' with $c' \in H_3(X'; \mathbb{Z})$ and an orientation preserving diffeomorphism $f\colon X \to X'$ satisfying $f^* c = c'$. Then*

$$cs_{X',c'}(a) = cs_{X,c}(f^* a)$$

holds.

Proof. This follows from functoriality of integration and $\mathbb{Z}$-covering spaces. ∎

If c is not zero, the class c determines a homomorphism $\rho_c\colon H_1(X; \mathbb{Z}) \to \mathbb{Z}$. This gives us a surjective homomorphism

$$\phi_j := \rho_c \circ \mathrm{Ab}\colon \pi_1(X) \to \mathrm{Im}\, \rho_c \circ \mathrm{Ab} = i_c\mathbb{Z} \cong \mathbb{Z} \to \mathbb{Z}/j\mathbb{Z} \tag{6}$$

for some $i_c \in \mathbb{Z}_{>0}$. For each $j \in \mathbb{Z}_{>0}$, we denote by

$$p_j\colon X_{j,c} \to X$$

the covering space corresponding to $\mathrm{Ker}\, \phi_j$. Note that the closed 4-manifold obtained by identifying the boundary components of $W[0, j-1]$ is diffeomorphic to $X_{j,c}$. We also have a $\mathbb{Z}$-covering space

$$p_j^c\colon p_j^* \widetilde{X}^c \to X_{j,c}$$

for each $j \in \mathbb{Z}_{>0}$. This corresponds to the class $p_j^* c \in H^1(X_{j,c}; \mathbb{Z}) \cong [X_{j,c}, S^1]$.

Definition 2.4. Fix a pair (X, c) consisting of an oriented closed 4-manifold X and a class $c \in H_3(X;\mathbb{Z})$ satisfying Assumption 2.1. Suppose $c \neq 0$. Corresponding to the $(X_{j,c}, p_j^* c)$, we have a family of maps

$$\{cs^j_{X,c} := cs_{X_{j,c}, p_j^* c} \colon R(X_{j,c}) \to (0,1]\}_{j \in \mathbb{Z}_{>0}}. \tag{7}$$

When $c = 0$, we set $cs^j_{X,c} = 1$ for all j.

To compare $\{cs^j_{X,c}\}_{j\in\mathbb{Z}_{>0}}$ with the critical values of the Chern–Simons functional of oriented 3-manifolds, we will use the following definition.

Definition 2.5. For an oriented closed 3-manifold Y, we define

$$\Lambda_Y = \{cs_Y(a) \in \mathbb{R} \colon a \text{ is an SU(2)-flat connection on } Y\} \tag{8}$$

and

$$\Lambda_Y^* = \{cs_Y(a) \in \Lambda \colon a \text{ is an irreducible SU(2)-flat connection on } Y\}, \tag{9}$$

where cs_Y is the Chern–Simons functional of Y.

Proposition 2.6 (Proposition 1.12, 1). *For any oriented connected 4-manifold,* $0 \neq c \in H_3(X;\mathbb{Z})$ *with Assumption* 2.1 *and* $j \in \mathbb{Z}_{>0}$*,*

$$\operatorname{Im} cs^j_{X,c} \subset (\Lambda_Y \cap (0,1]).$$

If Y is a Seifert 3-manifold, then $\operatorname{Im} cs^j_{X,c} \subset \mathbb{Q} \cap (0,1]$.

Proof of Proposition 2.6. Suppose that Y is an oriented connected codimension 1 submanifold of X with $[Y] = c$. If c is not zero, then $X \setminus Y$ is connected. Let W_0 be the compact cobordism from Y to itself given by $\overline{X \setminus Y}$. Then, identifying boundaries of $W[0, j]$, gives us $X_{j,c}$ as in (2). Fix an element $a \in \widetilde{R}(X_{j,c})$ such that

$$cs^j_{X,c}(a) \leq 1.$$

Note that $W[-\infty, \infty] \to X_{j,c}$ is isomorphic to $p_j^c \colon p_j^* \widetilde{X}^c \to X_{j,c}$. Then

$$cs^j_{X,c}(a) = -\frac{1}{8\pi^2} \int_{W[-\infty,\infty]} \operatorname{Tr}(F_{A_a} \wedge F_{A_a}) \tag{10}$$

$$= cs_Y(i_Y^* a) - cs_Y(i_Y^* 0) = cs_Y(i_Y^* a). \tag{11}$$

This gives the conclusion. If Y is a Seifert 3-manifold, it is shown in [3] that $\Lambda_Y \subset \mathbb{Q}$. Therefore, we have $\operatorname{Im} cs^j_{X,c} \subset \mathbb{Q} \cap (0,1]$. ∎

Proposition 2.7. *The maps $cs^j_{X,c}$ are locally constant with respect to the C^∞-topology. In particular, for any $j \in \mathbb{Z}_{>0}$ and a pair (X, c) satisfying Assumption 2.1, $\operatorname{Im} cs^j_{X,c}$ is a finite set.*

Proof. Fix a closed oriented connected 3-manifold Y such that $[Y] = c \in H_3(X;\mathbb{Z})$. Using (11), we have $cs^j_{X,c}(a) = cs_Y(a|_Y)$ for any $j \in \mathbb{Z}_{>0}$. If ρ_t is a path of SU(2)-flat connections, then $cs^j_{X,c}(\rho_t) = cs_Y(\rho_t|_Y) = cs_Y(\rho_0|_Y)$ since cs_Y is locally constant. By compactness of $R(X_{j,c})$, $\operatorname{Im} cs^j_{X,c}$ is a finite set. ∎

The following gives us a sufficient condition for triviality of $\{cs^j_{X,c}\}_{j\in\mathbb{Z}_{>0}}$.

Proposition 2.8. *If X is connected and $\pi_1(X_{j,c})$ is isomorphic to $\mathbb{Z}^l$ or free for $j \in \mathbb{Z}_{>0}$, then*

$$\operatorname{Im} cs^j_{X,c} = \{1\}.$$

In particular, $\operatorname{Im} cs_{S^1\times S^3,c} = \operatorname{Im} cs_{T^4,c} = \operatorname{Im} cs_{T^2\times S^2,c} = \operatorname{Im} cs_{\#_m S^1\times S^3,c} = \{1\}$ *for any class c and $m \in \mathbb{Z}_{>0}$.*

Proof. Since

$$R(X_{j,c}) = \operatorname{Hom}(\pi_1(X_{j,c}), \mathrm{SU}(2))/\,\mathrm{SU}(2),$$

if $\pi_1(X_{j,c})$ is isomorphic to $\mathbb{Z}^l$ or free, then $R(X_{j,c})$ is connected. By Proposition 2.7, $cs^j_{X,c}(a) = cs^j_{X,c}(0) = 0$. ∎

The 4-manifolds below give non-trivial examples of $\{cs^j_{X,c}\}$.

Example 2.9. Let Y be an oriented closed connected 3-manifold. Then

$$\operatorname{Im} cs^j_{Y\times S^1,[Y]} = \Lambda_Y \cap (0,1]$$

for any $j \in \mathbb{Z}_{>0}$. This is a consequence of (11).

Lemma 2.10. *For any positive integer m,*

$$\operatorname{Im} cs^j_{X,c} \subset \operatorname{Im} cs^{mj}_{X,c}$$

for any $j, m \in \mathbb{Z}_{>0}$.

Proof. Note that $X_{mj,c}$ is the total space of a $\mathbb{Z}/m\mathbb{Z}$ covering space $p_{m,j}\colon X_{mj,c} \to X_{j,c}$. Choose $\rho \in \widetilde{R}^*(X_{j,c})$ such that $cs^j_{X,c}(\rho) < 1$. We put $\rho' := p^*_{m,j}\rho$. Then, one can check that

$$cs_{X_{j,c},c}(\rho) = cs_{(mj)_c X, p_j^* c}(\rho').$$

This completes the proof. ∎

Although there are many non-trivial examples of $cs^j_{X,c}$, we could not find the example of a pair (X,c) whose $cs^j_{X,c}$ is not constant with respect to j.

Question 2.11. Is there a 4-manifold X with a class $c \in H_3(X;\mathbb{Z})$ such that $\{\operatorname{Im} cs^j_{X,c}\}$ is not constant with respect to j?

The following tell us that non-triviality of $\operatorname{Im} cs^j_{X,c}$ implies the existence of irreducible $\mathrm{SU}(2)$-representations of $\pi_1(X_{j,c})$.

Lemma 2.12. *There exist* $2\#(\operatorname{Im} cs^j_{X,c} \cap (0,1))$ *irreducible* $\mathrm{SU}(2)$*-representations of* $\pi_1(X_{j,c})$.

Proof. The subspace of reducible $\mathrm{SU}(2)$-representations is connected. Thus, for a reducible representation ρ, one has $cs^j_{X,c}(\rho) = 1$ by Proposition 2.7. Therefore, for every element $r \in \operatorname{Im} cs^j_{X,c} \cap (0,1)$, we have an irreducible representation $\rho\colon \pi_1(X_{j,c}) \to \mathrm{SU}(2)$ such that $cs^j_{X,c}(\rho) = r$. Fix an oriented connected 3-manifold Y representing the class $c \in H_3(X;\mathbb{Z})$. As in the construction above, we can reconstruct $X_{j,c}$ by gluing

$$W[0, j-1] := W_0 \cup_Y W_1 \cup_Y \cdots \cup_Y W_{j-1}$$

along the boundaries. We regard ρ as an irreducible connection $\tilde{\rho}$ on $W[0, j-1]$. The restrictions of $\tilde{\rho}$ on the boundaries Y_0^+ and Y_j^- of $W[0,j]$ are isomorphic. Moreover, by the definition of $cs^j_{X,c}$, we have equalities

$$cs^j_{X,c}(\rho) = cs_Y(\tilde{\rho}|_{Y_0^+}) = -cs_Y(\tilde{\rho}|_{Y_j^-}) \in (0,1).$$

This implies that $\tilde{\rho}|_{Y_0^+}$ is an irreducible connection on Y_0^+.

There is a sign ambiguity when gluing $\tilde{\rho}$ along $Y_0^+ \cup Y_j^+$. We denote the two resulting glued connections by ρ_+ and ρ_-. One of ρ_+ and ρ_- is isomorphic to ρ. We see that ρ_+ and ρ_- are not gauge equivalent: suppose there is a gauge transformation g on $X_{j,c}$ such that $g^*\rho_+ = \rho_-$. Then by restricting $g^*\rho_+$ and ρ_- to $W[0, j-1]$, we obtain

$$g^*\tilde{\rho} = g^*\rho_+|_{W[0,j-1]} = \rho_-|_{W[0,j-1]} = \tilde{\rho}.$$

Since $\tilde{\rho}$ is irreducible, we conclude that $g = \pm 1$. We take a based loop $l \subset X_{j,c}$ such that $l \cdot [Y] = 1$. Then the holonomies of the connections $\rho_+|_l$ and $\rho_-|_l$ obtained by the pull-back satisfy

$$\mathrm{Hol}_{\rho_+|_l} = -\mathrm{Hol}_{\rho_-|_l} \in \mathrm{SU}(2)$$

by construction of the $\rho_\pm$. This contradicts the assumption that $\rho_+ = g^*\rho_+ = \rho_-$. This completes the proof. ∎

2.2. Holonomy perturbations

Our main tool of this paper is instanton Floer theory. We refer the reader to [15, 17] for the construction of instanton Floer homology. In instanton Floer theory, we consider perturbations of the Chern–Simons functional. First, we review the set $\mathcal{P}(Y)$ of perturbations used in ordinary Floer theory. Let $\mathcal{F}_d$ be the set of d embeddings of $S^1 \times D^2$ into Y for $d \in \mathbb{Z}_{>0}$. Fix $m \gg 2$, we denote by $C^m_{\mathrm{ad}}(\mathrm{SU}(2), \mathbb{R})$ the set of adjoint invariant real valued C^m-functions. The set of perturbations is

$$\mathcal{P}(Y) := \bigcup_{d \in \mathbb{Z}_{>0}} \mathcal{F}_d \times C^m_{\mathrm{ad}}(\mathrm{SU}(2), \mathbb{R})^d.$$

In this paper, we treat a slightly larger class $\mathcal{P}^*(Y)$ of perturbations than $\mathcal{P}(Y)$. The class $\mathcal{P}^*(Y)$ was used in [42] to calculate the instanton homologies of Seifert homology 3-spheres.

Definition 2.13. We define the set of perturbations[7] by

$$\mathcal{P}^*(Y) := \bigcup_{d \in \mathbb{Z}_{>0}} \mathcal{F}_d \times C^m_{\mathrm{ad}}(\mathrm{SU}(2), \mathbb{R})^d \times C^m(\mathbb{R}^d, \mathbb{R}).$$

We fix a volume form $d\mu$ on D^2 such that $\operatorname{supp} d\mu \subset \operatorname{int} D^2$ and $\int_{D^2} d\mu = 1$. For a triple $\pi = (f, h, q) \in \mathcal{P}^*(Y)$, we define the *perturbed Chern–Simons functional* by

$$cs_{Y,\pi} = cs_Y + h_\pi \colon \widetilde{\mathcal{B}}^*(Y) \to \mathbb{R},$$

where

- $\widetilde{\mathcal{B}}^*(Y)$ is the quotient set

  $$(\mathcal{A}^*(Y) := \{\text{irreducible SU(2)-connections on } Y \times \mathrm{SU}(2)\}) / \operatorname{Map}^0(Y, \mathrm{SU}(2)),$$

  where $\operatorname{Map}^0(Y, \mathrm{SU}(2))$ is the set of smooth maps whose mapping degrees are zero,
- cs_Y is the Chern–Simons functional given by

  $$cs_Y(a) := -\frac{1}{8\pi^2} \int_Y \operatorname{Tr}\Big(a \wedge da + \frac{2}{3} a \wedge a \wedge a\Big)$$

[7] We regard $\mathcal{P}(Y)$ as a subset of $\mathcal{P}^*(Y)$ via

$$(f, h) \mapsto \Big(f, h, (x_i)_{1 \le i \le d} \mapsto \sum_{1 \le i \le d} x_i\Big).$$

- and

$$h_\pi(a) := q\left(\left(\int_{x\in D^2} h_i \operatorname{Hol}_{f_i(s,x)}(a)\right)_{1\le i\le d}\right).$$

We often use an L^2_k-completion and a Banach manifold structure on $\widetilde{\mathcal{B}}^*(Y)$ for a fixed $k>2$. Note that the map $cs_{Y,\pi}$ descends to a map

$$cs_{Y,\pi}\colon \mathcal{B}^*(Y) := \mathcal{A}^*(Y)/\operatorname{Map}(Y,\mathrm{SU}(2)) \to \mathbb{R}/\mathbb{Z}.$$

We now write down the formal gradient vector field of $cs_{Y,\pi}$. Fix a Riemann metric g_Y on Y. Then, identifying $\mathfrak{su}(2)$ with its dual by the Killing form, we can regard the derivative h_i' as a map $h_i'\colon \mathrm{SU}(2)\to\mathfrak{su}(2)$. The holonomy of the loops $\{f_i(s,x)\colon s\in S^1\}$ gives us a section $\operatorname{Hol}_{f_i(s,x)}(a)$ of the bundle $\operatorname{Aut} P_Y$ over $\operatorname{Im} f_i$. The bundle map induced by $h_i'\colon \operatorname{Aut} P_Y\to \operatorname{ad} P_Y$, then gives us a section $h_i'(\operatorname{Hol}_{f_i(s,x)}(a))$ of $\operatorname{ad} P_Y$ over $\operatorname{Im} f_i$. We now describe the gradient-flow equation of $cs_{Y,\pi}$ with respect to the L^2-metric:

$$\begin{aligned}\frac{\partial}{\partial t}a_t &= -\operatorname{grad}_{a_t} cs_{Y,\pi}\\ &= *_{g_Y}\left(F(a_t)+\sum_{i=1}^{m}\partial_i q_{(h_i(\operatorname{Hol}(a_t)_{f_i(s,x)}))_{1\le i\le m}} h_i'(\operatorname{Hol}(a_t)_{f_i(s,x)})(f_i)_*\mathrm{pr}_2^* d\mu\right),\end{aligned} \tag{12}$$

where pr_2 is the projection $\mathrm{pr}_2\colon S^1\times D^2\to D^2$ and $*_{g_Y}$ is the Hodge star operator with respect to g_Y. (For the calculation of the gradient, see [5].) We denote $\mathrm{pr}_2^*\,d\mu$ by η. We set

$$\widetilde{R}(Y)_\pi := \{a\in\widetilde{\mathcal{B}}(Y)\colon \operatorname{grad}_a cs_{Y,\pi}=0\},$$

and

$$\widetilde{R}^*(Y)_\pi := \widetilde{R}(Y)_\pi\cap\widetilde{\mathcal{B}}^*(Y).$$

When we consider a smooth manifold structure on $\widetilde{R}^*(Y)_\pi$, we use an L^2_k-topology[8] for some $k>2$. The solutions of (12) correspond to connections A over $Y\times\mathbb{R}$ which satisfy the equation

$$F^+(A)+\pi(A)^+=0, \tag{13}$$

where

[8]These topologies on $R^*(Y)_\pi$ do not depend on the choice of $k>2$ if π is a smooth perturbation.

- the 2-form $\pi(A)$ is given by

$$\sum_{i=1}^{m} \partial_i q_{(h_i(\mathrm{Hol}(a_t)_{f_i(t,x,s)}))_{1\le i\le m}} h_i'(\mathrm{Hol}(A)_{\tilde{f}_i(t,x,s)}) \otimes (\tilde{f}_i)_*(\mathrm{pr}_1^* \eta),$$

- the map pr_1 is the projection map from $(S^1 \times D^2) \times \mathbb{R}$ to $S^1 \times D^2$,
- $F^+(A) = \frac{1}{2}(1 + *F(A))$ where $*$ is the Hodge star operator with respect to the product metric on $Y \times \mathbb{R}$, and similarly for $\pi(A)^+$, and
- $\tilde{f}_i\colon S^1 \times D^2 \times \mathbb{R} \to Y \times \mathbb{R}$ is the embedding given by $f_i \times id$ for each i.

We define $\|\pi\| = \|(f, h, q)\| := \|q \circ h\|_{C^m}$. We also define non-degenerate and regular perturbations for elements in $\mathcal{P}^*(Y)$ in the same way as in the case of $\mathcal{P}(Y)$. (See [44] for further details.) For an oriented homology 3-sphere Y and a fixed metric g_Y on Y, there exist a positive integer d, a collection of embeddings $f \in \mathcal{F}_d$ and a Baire subset Q of $C^m_{\mathrm{ad}}(\mathrm{SU}(2), \mathbb{R})^d \times C^m(\mathbb{R}^d, \mathbb{R})$ such that (f, u) is non-degenerate and regular for $u \in Q$.

For a 4-manifold W with cylindrical ends, we also use a large class of perturbations. Let $\mathcal{F}_d(W)$ be the set of d embeddings from $S^1 \times D^3$ to W for any $d \in \mathbb{Z}_{>0}$. Fix a volume form $d\nu$ on D^3 such that $\mathrm{supp}\, d\nu \subset \mathrm{int}\, D^3$ and $\int_{D^3} d\nu = 1$. We define

$$\mathcal{P}^*(W) := \bigcup_{d\in\mathbb{Z}_{>0}} \mathcal{F}_d(W) \times C^m_{\mathrm{ad}}(\mathrm{SU}(2), \mathbb{R})^d \times C^m(\mathbb{R}^d, \mathbb{R}).$$

For $\pi = (g, h, q) \in \mathcal{P}^*(W)$, we have the *perturbed ASD equation with respect to* π defined by

$$F^+(A) + \Big(\sum_{i=1}^{m} \partial_i q_{(h_i(\mathrm{Hol}(a_t)_{f_i(t,x,s)}))} dh_i\, \mathrm{Hol}_{g_i(t,x)}(A) \otimes (g_i)_* \,\mathrm{pr}_2^*\, d\nu\Big)^+ = 0, \tag{14}$$

where pr_2 is the projection $\mathrm{pr}_2\colon S^1 \times D^3 \to D^3$. We will often write the part

$$\sum_{i=1}^{m} \partial_i q_{(h_i(\mathrm{Hol}(a_t)_{f_i(t,x,s)}))} dh_i\, \mathrm{Hol}_{g_i(t,x)}(A) \otimes (g_i)_* \,\mathrm{pr}_2^*\, d\nu$$

by $\pi(A)$.

2.3. Moduli spaces of perturbed ASD equations

In this section, we review the construction of the cobordism map in instanton Floer theory. In this paper, we only use the moduli space of solutions to ASD equations on manifolds of the form $Y \times \mathbb{R}$ and W^*, where W is a negative definite cobordism from Y to itself and W^* is defined by

$$Y \times \mathbb{R}_{\leq 0} \cup_Y W \cup_Y Y \times \mathbb{R}_{\geq 0}. \tag{15}$$

We assume that $H_1(W;\mathbb{R}) = 0$. Fix a regular non-degenerate perturbation $\pi \in \mathcal{P}^*(Y)$. For two irreducible critical points a, $b \in \widetilde{R}^*(Y)_\pi$, we will define moduli spaces $M^Y(a,b)_\pi$ and $M(a, W^*, \theta)_{\pi_W}$.

Fix a positive integer $q \geq 3$. Let $A_{a,b}$ be an SU(2)-connection on $Y \times \mathbb{R}$ satisfying $A_{a,b}|_{Y\times(-\infty,1]} = p^*a$ and $A_{a,b}|_{Y\times[1,\infty)} = p^*b$ where p is the projection $Y \times \mathbb{R} \to Y$. We then define

$$M^Y(a,b)_\pi := \{A_{a,b} + c \colon c \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_q} \text{ satisfying (13)}\}/\mathcal{G}(a,b), \tag{16}$$

where $\mathcal{G}(a,b)$ is given by

$$\mathcal{G}(a,b) := \{g \in \mathrm{Aut}(P_{Y\times\mathbb{R}}) \subset \mathrm{End}(\mathbb{C}^2)_{L^2_{q+1,\mathrm{loc}}} \colon \nabla_{A_{a,b}}(g) \in L^2_q\}.$$

The action of $\mathcal{G}(a,b)$ on $\{A_{a,b} + c \colon c \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_q}$ satisfying (13)$\}$ is given by pull-backs of connections. The space $\mathbb{R}$ acts on $M^Y(a,b)_\pi$ by translation. We denote by θ the product SU(2)-connection on Y. We also have moduli spaces $M^Y(a,\theta)_\pi$ defined by similar way as $M^Y(a,b)_\pi$ but we use a weighted norm to define $M^Y(a,\theta)_\pi$. (See [35].)

Next, for two irreducible critical points $a, b \in \widetilde{R}(Y)_\pi$, let $A_{a,b}$ be an SU(2)-connection on W^* satisfying $A_{a,b}|_{Y\times(-\infty,1]} = p^*a$ and $A_{a,b}|_{Y\times[1,\infty)} = p^*b$ where p is the projection $Y \times \mathbb{R} \to Y$. We define

$$M(a, W^*, b)_\pi := \{A_{a,b} + c \colon c \in \Omega^1(W^*) \otimes \mathfrak{su}(2)_{L^2_q} \text{ satisfying (14)}\}/\mathcal{G}(a,b), \tag{17}$$

where $\mathcal{G}(a,b)$ is given by the same formula as in the case of $Y \times \mathbb{R}$.

2.4. Invariants $\{r_s(Y)\}_{s\in[-\infty,0]}$ and Daemi's invariants $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}}$

In this section, we review of two families of $(0,\infty]$-valued homology cobordism invariants $\{r_s(Y)\}_{s\in[-\infty,0]}$ and $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}}$ of homology 3-spheres. To define $\{r_s(Y)\}_{s\in[-\infty,0]}$, we use $\mathbb{Z}$-graded filtered instanton Floer homology whose filtration comes from the Chern–Simons functional. On the other hand, Daemi used $\mathbb{Z}/8\mathbb{Z}$-graded instanton homology with some local coefficient coming from Chern–Simons functional to define $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}}$. For more details on $\{r_s(Y)\}$ and $\{\Gamma_Y(k)\}$, see [11, 35].

2.4.1. Invariants $\{r_s(Y)\}_{s\in[-\infty,0]}$. For an oriented homology 3-sphere Y, we review the definition of $\{r_s(Y)\}_{s\in[-\infty,0]}$. This invariant was defined in [35] to analyze the structure of the homology cobordism group of homology 3-spheres. For r and s in $[-\infty,\infty)$ satisfying $-\infty \le s \le 0 \le r < \infty$ such that r is a regular value of cs_Y, we have a filtered instanton Floer cohomology $I^1_{[s,r]}(Y)$. In this paper, we use the class of perturbations $\mathcal{P}^*(Y)$ instead of $\mathcal{P}(Y)$ used in [35].

Definition 2.14. Let Y be an oriented homology 3-sphere and g_Y be a Riemannian metric on Y. For $\varepsilon > 0$, we define a class of perturbations $\mathcal{P}^*_\varepsilon(Y,g)$ as a subset of $\mathcal{P}^*(Y)$ consisting of elements which satisfy

(1) $|h_\pi(a)| < \varepsilon$ for all $a \in \widetilde{\mathcal{B}}(Y)$ and

(2) $\|\operatorname{grad}_g h_\pi(a)\|_{L^4} < \frac{\varepsilon}{2}$, $\|\operatorname{grad}_g h_\pi(a)\|_{L^2} < \frac{\varepsilon}{2}$ for all $a \in \widetilde{\mathcal{B}}(Y)$.

We choose a suitable small ε by the following argument. Let $\{R_\alpha\}$ be the connected components of $R^*(Y)$. Let U_α be a neighborhood of R_α in $\mathcal{B}(Y)$ with respect to the C^∞-topology such that $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$ and $\{U_\alpha\}$ is a covering of $R^*(Y)$. We take all lifts of U_α with respect to

$$\mathrm{pr}\colon \widetilde{\mathcal{B}}_Y \to \mathcal{B}_Y.$$

Since $\mathrm{Map}(Y,\mathrm{SU}(2))/\mathrm{Map}_0(Y,\mathrm{SU}(2))$ is isomorphic to $\mathbb{Z}$, we denote all lifts by $\{U^i_\alpha\}_{i\in\mathbb{Z}}$. In addition, we assume the following conditions on U^i_α.

- If $a \in U^i_\alpha$, then $|cs(a) - cs(R_\alpha)| < \min\{\frac{d(r,\Lambda_Y)}{8}, \frac{d(s,\Lambda_Y)}{8}\}$, where $d(r,\Lambda_Y)$ is given by
$$d(r,\Lambda_Y) := \min\{|r-a| \in \mathbb{R}_{>0} \colon a \in \Lambda_Y\}.$$
- U^i_α has no reducible connections.

Note that, for any element $\rho \in \widetilde{R}(Y)$, we have unique α and $i \in \mathbb{Z}$ such that $\rho \in U^i_\alpha$.

By the Uhlenbeck compactness theorem, we can take a sufficiently small real number $\varepsilon_1(Y,g,\{U_\alpha\}) > 0$ satisfying the following condition:

$$\big((a \in \mathcal{B}^*(Y) \text{ and } \|F(a)\|_{L^2} \le \varepsilon_1(Y,g,\{U_\alpha\})\big) \implies a \in U_\alpha \text{ for some } \alpha. \tag{18}$$

Definition 2.15. Now, we take the supremum value

$$\varepsilon_1(Y,g,r,s) := \frac{1}{2}\sup_{\{U_\alpha\}} \varepsilon_1(Y,g,\{U_\alpha\}),$$

where $\{U_\alpha\}$ runs over all coverings of $\{R_\alpha\}$ given as above method. We define

$$\varepsilon(Y,r,s,g) := \begin{cases} \min\Big\{\varepsilon_1(Y,g), \dfrac{d(s,\Lambda_Y)}{8}, \dfrac{d(r,\Lambda_Y)}{8}, \dfrac{\lambda_Y}{32}\Big\} & \text{if } s \in \mathbb{R}_Y, \\ \min\Big\{\varepsilon_1(Y,g), \dfrac{d(r,\Lambda_Y)}{8}, \dfrac{\lambda_Y}{32}\Big\} & \text{if } s \in \Lambda_Y, \end{cases}$$

where $\lambda_Y := \min\{|a-b| \colon a, b \in \Lambda_Y \text{ with } a \neq b\}$. We then define

$$\mathcal{P}^*(Y, g, r, s) := \mathcal{P}^*_{\varepsilon(Y,g,r,s)}(Y, g).$$

We also use the notation $\lambda_Y := \min\{|a-b| \colon a, b \in \Lambda_Y \text{ with } a \neq b\}$. Then we define a class of perturbations which we will use later. For a non-degenerate perturbation $\pi \in \mathcal{P}^*(Y)$, we consider a map

$$\text{ind} \colon \widetilde{R}(Y)_\pi \to \mathbb{Z} \tag{19}$$

called the *Floer index* in order to construct $\mathbb{Z}$-gradings on Floer's chain complexes. Fix two elements $r, s \in [-\infty, \infty)$ satisfying $-\infty \leq s \leq 0 \leq r < \infty$ and $r \in \Lambda^*_Y$. For a metric g on Y, a non-degenerate regular perturbation $\pi \in \mathcal{P}(Y, r, s, g)$, the (co)chains of the filtered instanton Floer (co)homologies are defined by

$$\begin{aligned}
&\mathrm{CI}^{[s,r]}_i(Y, \pi) \\
&:= \begin{cases} \mathbb{Z}\left\{[a] \in \widetilde{R}^*(Y)_\pi \colon \text{ind}(a) = i, s < cs_{Y,\pi}(a) < r\right\} & \text{if } s \in \mathbb{R}_Y, \\ \mathbb{Z}\left\{[a] \in \widetilde{R}^*(Y)_\pi \colon \text{ind}(a) = i, s - \dfrac{\lambda_Y}{2} < cs_{Y,\pi}(a) < r\right\} & \text{if } s \in \Lambda_Y, \end{cases}
\end{aligned}$$

and

$$\mathrm{CI}^i_{[s,r]}(Y, \pi) := \mathrm{Hom}(\mathrm{CI}^{[s,r]}_i(Y, \pi), \mathbb{Z}),$$

where $\lambda_Y := \min\{|a-b| \colon a \neq b, a, b \in \Lambda_Y\}$. The (co)boundary maps

$$\partial^{[s,r]} \colon \mathrm{CI}^{[s,r]}_i(Y, \pi) \to \mathrm{CI}^{[s,r]}_{i-1}(Y, \pi) \quad (\text{resp. } \delta^r \colon \mathrm{CI}^i_{[s,r]}(Y) \to \mathrm{CI}^{i+1}_{[s,r]}(Y))$$

are given by the restriction of Floer's usual differential

$$\partial(a) := \sum_{b \in \widetilde{R}^*(Y)_\pi \colon \text{ind}(b) = i-1} \#(M^Y(a, b)_\pi / \mathbb{R}) b$$

(resp. $\delta^{[s,r]} := (\partial^{[s,r]})^*$). For further details of ∂, see [15, Section 5.2].

There is a cohomology class $\theta^{[s,r]}_Y \in I^1_{[s,r]}(Y)$ defined by

$$\theta^{[s,r]}_Y([a]) := \#(M^Y(a, \theta)_\pi / \mathbb{R}). \tag{20}$$

As in the discussion in [15, Section 3.3.1], one can see that $I^{[s,r]}_*(Y)$ and $\theta^{[s,r]}_Y \in I^1_{[s,r]}(Y)$ do not depend on the choice of $\pi \in \mathcal{P}^*(Y, g, r, s)$. Therefore, $I^{[s,r]}_*(Y)$ and $\theta^{[s,r]}_Y \in I^1_{[s,r]}(Y)$ are equivalent to the original definitions in [35].

Definition 2.16 ([35, Definition 3.1]). We define

$$r_s(Y) := \sup\{r \in \mathbb{R}_{\geq 0} \colon 0 = \theta^{[s,r]}_Y \in I^1_{[s,r]}(Y)\}.$$

In this paper, we will use the following property of the class $\theta_Y^{[s,r]}$. Let W be a negative definite cobordism such that $\partial W = Y_0 \cup (-Y_1)$ satisfying $H_1(W;\mathbb{R}) = 0$. Let $I(W)\colon I^1_{[s,r]}(Y_0) \to I^1_{[s,r]}(Y_1)$ be the cobordism map introduced in [35].

Proposition 2.17 ([35, Lemma 2.12]). *Suppose that $H_*(W;\mathbb{R}) \cong H_*(S^3;\mathbb{R})$. For two real numbers $r, s \in \mathbb{R}$ satisfying $s \leq 0 \leq r$ and r is regular value of cs_Y,*

$$I(W)(\theta_{Y_1}^{[s,r]}) = c(W)\theta_{Y_0}^{[s,r]},$$

where $c(W) = \#H_1(W;\mathbb{Z})$.

Theorem 2.18 ([35, Theorem 1.1]). *The invariants $\{r_s(Y)\}_{s\in[-\infty,0]}$ satisfy the following conditions.*

- *For s, s_1, $s_2 \in [-\infty, 0]$ with $s = s_1 + s_2$,*

  $$r_s(Y_1\#Y_2) \geq \min\{r_{s_1}(Y_1) + s_2, r_{s_2}(Y_2) + s_1\} \tag{21}$$

  holds.
- *If there exists a negative definite cobordism W with $\partial W = Y_1 \amalg -Y_2$, then the inequality*

  $$r_s(Y_2) \leq r_s(Y_1) \tag{22}$$

  holds for any $s \in [-\infty, 0]$.

2.4.2. Daemi's invariants $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}}$. Let Λ be the Novikov ring

$$\Lambda := \left\{\sum_{i=1}^{\infty} a_i\lambda^{r_i} : a_i \in \mathbb{Q}, r_i \in \mathbb{R}, \lim_{i\to\infty} r_i = \infty\right\},$$

where λ is a formal variable. We have an evaluating function mdeg: $\Lambda \to \mathbb{R}$ defined by

$$\mathrm{mdeg}\Big(\sum_{i=1}^{\infty} a_i\lambda^{r_i}\Big) := \min_{i\in\mathbb{Z}_{>0}}\{r_i : a_i \neq 0\}.$$

Fix a non-degenerate regular perturbation π and orientations of the determinant line bundles $\mathbb{L}_a$ in [35, Section 2]. Note that the Floer index (19) descends to a map

$$\mathrm{ind}\colon R(Y)_\pi \to \mathbb{Z}/8\mathbb{Z}.$$

Then define a $\mathbb{Z}/8\mathbb{Z}$-graded chain complex $C_*^\Lambda(Y)$ over Λ by

$$C_i^\Lambda(Y) := C_i(Y) \otimes \Lambda = \Lambda\{[a] \in R^*(Y)_\pi : \mathrm{ind}(a) = i \mod 8\}$$

with the differential

$$d^{\Lambda}([a]) := \sum_{\substack{\mathrm{ind}(a)-\mathrm{ind}(b)\equiv 1 (\mathrm{mod}\ 8)\\ A\in M^Y([a],[b])_\pi}} \#(M^Y([a],[b])_\pi/\mathbb{R})\cdot\lambda^{\mathcal{E}(A)}[b],$$

where ind is the function (19),

$$\mathcal{E}(A) := \frac{1}{8\pi^2}\int_{Y\times\mathbb{R}} \mathrm{Tr}((F(A)+\pi(A))\wedge(F(A)+\pi(A)))$$

and $M^Y([a],[b])_\pi$ denotes $M^Y(a,b)_\pi$ for some representatives a and b of $[a]$ and $[b]$ satisfying $\mathrm{ind}(a)-\mathrm{ind}(b)=1$. Extend the function mdeg to C^{Λ}_* by

$$\mathrm{mdeg}\Big(\sum_{1\le k\le n}\eta_k[a_k]\Big) = \min_{1\le k\le n}\{\mathrm{mdeg}(\eta_k)\}.$$

In addition, we define two maps:

i. The map $D_1\colon C^{\Lambda}_1(Y)\to\Lambda$ is given by

$$D_1([a]) = (\#M^Y([a],[\theta])_\pi/\mathbb{R})\cdot\lambda^{\mathcal{E}(A)},$$

where $A\in M^Y([a],[\theta])_\pi$ and $M^Y([a],[\theta])_\pi$ denotes $M^Y(a,\theta^i)_\pi$ for some lifts a and θ^i of $[a]$ and $[\theta]$ satisfying $\mathrm{ind}(a)-\mathrm{ind}(\theta^i)=1$.

ii. The map $U\colon C^{\Lambda}_*(Y)\to C^{\Lambda}_{*-4}(Y)$ is defined by

$$U([a]) := \sum_{\substack{[b]\in\widetilde{R}(Y)_\pi\\ \mathrm{ind}([b])-\mathrm{ind}([a])=4}} -\frac{1}{2}\#N(a,b)[b]\cdot\lambda^{\mathcal{E}(A)},$$

where the space $N^Y(a,b)$ is the codimension 4-submanifold of $M^Y(a,b)$ given by

$$\begin{aligned}N(a,b) := \{[A]\in M^Y(a,b)\colon & s_1(r([A]))\text{ and}\\ & s_2(r([A]))\text{ are linearly dependent}\}\end{aligned}$$

and A is an element in $N(a,b)$. Here $r\colon M^Y(a,b)\to\mathcal{B}^*(Y\times(-1,1))$ is the restriction map and s_1 and s_2 are generic sections of the bundle $\mathbb{E}\otimes\mathbb{C}\to\mathcal{B}^*(Y\times(-1,1))$. The SO(3)-bundle $\mathbb{E}$ is given by a basepoint fibration of $\mathcal{B}^*(Y\times(-1,1))$.

Now, in our conventions, $\Gamma_{-Y}(k)$ is given by

$$\Gamma_{-Y}(k) = \lim_{|\pi|\to 0}\inf_{\substack{\alpha\in C^{\Lambda}_*(Y),\, d^{\Lambda}(\alpha)=0\\ D_1U^j(\alpha)=0 (1\le j<k-1)\\ D_1U^{k-1}(\alpha)\ne 0}}\{\mathrm{mdeg}(D_1U^k(\alpha))-\mathrm{mdeg}(\alpha)\} \tag{23}$$

for $k \in \mathbb{Z}_{>0}$. In [11], Daemi also introduced $\Gamma_Y(k)$ for any negative $k \in \mathbb{Z}$. The invariants $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}_{\le 0}}$ use information of D_2 and U. However, in this paper, we only use the positive part $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}_{>0}}$.

Theorem 2.19 ([11]). *The sequence of invariants $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}_{>0}}$ has the following properties:*

(i) *if there exists a negative definite cobordism W with $\partial W = Y_1 \amalg -Y_2$, then the inequality*

$$\Gamma_{Y_1}(k) \le \Gamma_{Y_2}(k) \tag{24}$$

holds for any $k \in \mathbb{Z}_{>0}$;

(ii) *the invariant $\Gamma_Y(k) < \infty$ for $k \in \mathbb{Z}_{>0}$ if and only if $k \le 2h(Y)$.*

2.5. Relations between $\{r_s(Y)\}$ and $\{\Gamma_Y(k)\}$

It is natural to ask if there is a relation between $\{r_s(Y)\}_{s\in[-\infty,0]}$ and $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}_{>0}}$. In [35], the following equality is showed.

Theorem 2.20 ([35]). *For any oriented homology 3-sphere Y,*

$$r_{-\infty}(Y) = \Gamma_{-Y}(1).$$

Therefore, $\{r_s(Y)\}_{s\in[-\infty,0]}$ and $\{\Gamma_Y(k)\}_{k\in\mathbb{Z}_{>0}}$ satisfy the following inequalities:

$$r_0(Y) \le \cdots \le r_s(Y) \le \cdots \le r_{-\infty}(Y) = \Gamma_{-Y}(1) \le \cdots \le \Gamma_{-Y}(k).$$

It is also natural to ask if there is an oriented homology 3-sphere Y such that $\{r_s(Y)\}$ and $\{\Gamma_{-Y}(k)\}$ do not coincide. In [35], we proved that

$$\{-\Sigma(2,3,6k+1)\#\Sigma(2,3,5)\}_{k\in\mathbb{Z}_{>0}}$$

gives examples whose $\{r_s(Y)\}$ and $\Gamma_{-Y}(k)$ do not coincide. In this case, $\{r_s(Y)\}$ is not a constant with respect to s. Our connected sum formula implies

$$r_0(-\Sigma(2,3,6k+1)\#\Sigma(2,3,5)) = \frac{1}{24(6k+1)}.$$

On the other hand, since $h(-\Sigma(2,3,6k+1)\#\Sigma(2,3,5)) = 0$,

$$\Gamma_{\Sigma(2,3,6k+1)\#(-\Sigma(2,3,5))}(1) = \infty.$$

There is also an example of Y such that $\Gamma_Y(k)$ is not constant with respect to k: in [11], Daemi calculated

$$\Gamma_{\Sigma(2,3,5)}(k) = \begin{cases} \dfrac{1}{120} & \text{if } k = 1, \\ \dfrac{49}{120} & \text{if } k = 2, \\ \infty & \text{if } k \geq 3. \end{cases}$$

For $-\Sigma(2,3,5)$, we have $r_s(-\Sigma(2,3,5)) = \Gamma_{\Sigma(2,3,5)}(1)$ for any $s \in [-\infty, 0]$.

Remark 2.21. In [12], we will give a generalization $\mathcal{J}_Y(k,s)$ of both of $r_s(Y)$ and $\Gamma_Y(k)$. A theorem similar to Theorem 1.8 can be proven for $\mathcal{J}_Y(k,s)$.

3. The invariants l_Y^s, l_Y^k, and l_Y

3.1. Perturbations and invariants l_Y^s, l_Y^k, and l_Y

Let Y be an oriented homology 3-sphere and g_Y a Riemann metric on Y, and fix a perturbation $\pi \in P_Y^*$. Suppose that $R_\pi^*(Y)$ is a submanifold of $\mathcal{B}^*(Y)$ as the zero set of the gradient vector field of $cs_{Y,\pi}$. For any point $a \in R_\pi^*(Y)$, we have the operator

$$\mathrm{Hess}_a(cs_{Y,\pi}) = *d_a + \mathrm{Hess}_a h_\pi \colon \mathrm{Ker}\, d_a^* \to \mathrm{Ker}\, d_a^*,$$

where $\mathrm{Ker}\, d_a^*$ is a model of $T_a\mathcal{B}^*(Y)$ and $d_a^* \colon \Omega_Y^1 \otimes \mathfrak{su}(2) \to \Omega_Y^0 \otimes \mathfrak{su}(2)$. Note that

$$\mathrm{Hess}_a(cs_{Y,\pi}) \colon \mathrm{Ker}\, d_a^* \to \mathrm{Ker}\, d_a^*$$

is a self adjoint elliptic operator.

Definition 3.1. We call π a *Morse–Bott perturbation* if

$$\mathrm{Hess}_a(cs_{Y,\pi}) \colon (T_aR(Y))^{\perp_{L^2}} \cap \mathrm{Ker}\, d_a^* \to (T_aR(Y))^{\perp_{L^2}} \cap \mathrm{Ker}\, d_a^* \tag{25}$$

is invertible for any $a \in R_\pi^*(Y)$.

If $cs_{Y,\pi}$ is Morse–Bott for a perturbation π with $h_\pi = 0$, then we call cs_Y *Morse–Bott*. In this paper, we set

$$H_a^1(\pi) := \mathrm{Ker}\,\mathrm{Hess}_a(cs_{Y,\pi})|_{\mathrm{Ker}\, d_a^*}.$$

If $h = 0$, then we write H_a^1. If we use this notation, $cs_{Y,\pi}$ is Morse–Bott if and only if $H_a^1(\pi) = T_aR_\pi^*(Y)$ for each $a \in R_\pi^*(Y)$. (In general, $T_aR_\pi^*(Y) \subset H_a^1(\pi)$ holds.) Note that the condition $H_a^1 = T_aR^*(Y)$ does not depend on the choice of metric. Next, we define the notion of *Morse–Bott perturbation at level r*.

Definition 3.2. We say $cs_{Y,\pi}$ is *Morse–Bott at the level r* if

$$\mathrm{Hess}_a(cs_{Y,\pi})\colon (T_a R(Y))^{\perp_{L^2}} \cap \operatorname{Ker} d_a^* \to (T_a R(Y))^{\perp_{L^2}} \cap \operatorname{Ker} d_a^* \tag{26}$$

is invertible for any $a \in R_\pi^*(Y) \cap cs_{Y,\pi}^{-1}(r)$.

If cs_Y is Morse–Bott at the level r, one can show cs_Y is Morse–Bott at the level $r+m$ for any $m \in \mathbb{Z}$. Set $\Lambda_Y := \operatorname{Im} cs_Y|_{\widetilde{R}(Y)}$. If 0 is a Morse–Bott perturbation for any element $r \in \Lambda_Y$, then cs_Y is Morse–Bott.

Lemma 3.3. *If the Chern–Simons functionals for Y_1 and Y_2 are Morse–Bott, then the Chern–Simons functional for $Y_1\#Y_2$ is also Morse–Bott.*

Proof. Note that

$$R^*(Y_1\#Y_2) = R^*(Y_1) \times R^*(Y_2) \times \mathrm{SO}(3) \amalg R^*(Y_1) \amalg R^*(Y_2).$$

There are three patterns $a_1 *_h a_2$ ($h \in \mathrm{SO}(3)$), $a_1 * \theta$ and $\theta * a_2$ of elements in $R^*(Y_1\#Y_2)$, where $[a_1] \in R^*(Y_1)$ and $[a_2] \in R^*(Y_2)$. Suppose that

$$H^1_{[a_1]}(Y_1) = T_{[a_1]}R^*(Y_1) \quad \text{and} \quad H^1_{[a_2]}(Y_2) = T_{[a_2]}R^*(Y_2).$$

It is sufficient to prove

$$\dim H^1_{a_\#}(Y_1\#Y_2) = \dim T_{a_\#}R^*(Y_1\#Y_2) \quad \text{for any } a_\# \in R^*(Y_1\#Y_2).$$

Fix critical points $a_1 \in R^*(Y_1)$, $a_2 \in R^*(Y_2)$ and $a_\# \in R^*(Y_1\#Y_2)$. The Mayer–Vietoris sequence of the local coefficient cohomology implies the existence of the following exact sequence:

$$0 \to H^0_{a_1}(Y_1') \oplus H^0_{a_2}(Y_2') \to H^0(S^2) \to H^1_{a_\#}(Y_1\#Y_2) \to H^1_{a_1}(Y_1') \oplus H^1_{a_2}(Y_2') \to 0$$

where Y_i' is a punctured Y_i for $i = 1$ and 2. The other sequence implies

$$H^0_{a_i}(Y_i) \to H^0_{a_i}(Y_i') \oplus \mathbb{R}^3 \to H^0_\theta(S^2) \to H^1_{a_i}(Y_i) \to H^1_{a_i}(Y_i') \to 0$$

is also exact for $i = 1$ and 2. If both of a_i are irreducible, then

$$H^1_{a_\#}(Y_1\#Y_2) \cong H^1_{a_1}(Y_1) \oplus H^1_{a_2}(Y_2) \oplus \mathbb{R}^3.$$

If a_1 is irreducible and $a_2 = \theta$, then

$$H^1_{a_\#}(Y_1\#Y_2) \cong H^1_{a_1}(Y_1) \oplus H^1_{a_2}(Y_2).$$

This proves the desired result. ∎

Corollary 3.4. *Let Y be a finite connected sum of Seifert homology 3-spheres. Then the Chern–Simons functional of Y is Morse–Bott.*

Proof. It is shown in [16] that the Chern–Simons functional of any Seifert homology 3-sphere is Morse–Bott. Lemma 3.3 then gives the conclusion. ■

Here, we will introduce the invariant l_Y. For any Riemann metric g_Y on Y, there exists a sequence of non-degenerate regular perturbations $\{\pi_n\}$ such that $\|\pi_n\| \to 0$. We define two quantities

$$l(Y, g) := \min\Big\{ \sup_{n\in\mathbb{Z}_{>0}} \# R^*_{\pi_n}(Y) \colon \{\pi_n\} \text{ is non-deg regular}, \|\pi_n\| \to 0 \Big\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

and

$$l(Y, g, r, i) := \min\Big\{ \sup_{n\in\mathbb{Z}_{>0}} \#\{a \in \widetilde{R}^*_{\pi_n}(Y) \colon |\, cs_{\pi_n}(a) - r| < \lambda_Y, \operatorname{ind}(a) = i\} \colon$$
$$\{\pi_n\} \text{ is non-deg regular}, \|\pi_n\| \to 0 \Big\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

for a given $r \in \Lambda_Y$, where $\lambda_Y = \frac{1}{2}\min\{|a-b| \colon a, b \in \Lambda_Y\}$. We now give two invariants for homology 3-spheres.

Definition 3.5. We define invariants l_Y and $l_{Y,r,i}$ by

$$l_Y := \min\{l(Y, g_Y) \colon g_Y \text{ is a Riemann metric}\} \in \mathbb{Z}_{>0} \cup \{\infty\}$$

and

$$l_{Y,r,i} := \min\{l(Y, g_Y, r, i) \colon g_Y \text{ is a Riemann metric}\} \in \mathbb{Z}_{>0} \cup \{\infty\}$$

for given $r \in \Lambda^*_Y$ and $i \in \mathbb{Z}$.

Note that $l_Y = l_{-Y}$ and $l_{Y,r,i} = l_{-Y,-r,-i-3}$ by definition. We combine $l_{Y,r,i}$, $r_s(Y)$ and $\Gamma_Y(k)$ and define l^s_Y and l^k_Y.

Definition 3.6. We set

$$l^s_Y := \begin{cases} 1 & \text{if } r_s(Y) = \infty, \\ l_{Y, r_s(Y), 1} & \text{if } r_s(Y) < \infty, \end{cases}$$

and

$$l^k_Y := \begin{cases} 1 & \text{if } \Gamma_Y(k) = \infty, \\ \sum_{m\in\mathbb{Z}} l_{Y,\Gamma_Y(k),1+8m} & \text{if } \Gamma_Y(k) < \infty \text{ and } k \in 2\mathbb{Z}+1, \\ \sum_{m\in\mathbb{Z}} l_{Y,\Gamma_Y(k),5+8m} & \text{if } \Gamma_Y(k) < \infty \text{ and } k \in 2\mathbb{Z}, \end{cases}$$

for $s \in [-\infty, 0]$ and $k \in \mathbb{Z}_{>0}$.

Lemma 3.7. *For any homology 3-sphere Y, $l^s_Y \geq 1$ and $l^k_Y \geq 1$.*

Proof. Suppose $r = r_s(Y) < \infty$ for $s \in [-\infty, 0]$. By the definition of $r_s(Y)$, for a sequence $\{r_n\}_{n\in\mathbb{Z}_{>0}}$ such that $r_n > r$, $\lim_{n\to\infty} r_n = r$ and $r_n \in \mathbb{R} \setminus \Lambda_Y$, we have $0 \neq \theta^{[s,r_n]}_Y \in I^1_{[s,r_n]}(Y)$. If $r' < r$, then $0 = \theta^{[s,r']}_Y \in I^1_{[s,r']}(Y)$. Then, for any Riemann metric g_Y on Y and any sequence of perturbations $\{\pi_n\}_{n\in\mathbb{Z}_{>0}}$ with $\|\pi_n\| \to 0$, there is a sequence $\{c_n\}_{n\in\mathbb{Z}_{>0}}$ of critical points of cs_{Y,π_n} such that $M^Y(c_n,\theta)_{\pi_n} \neq \emptyset$, $\mathrm{ind}(c_n) = 1$ for all n and

$$\lim_{n\to\infty} cs_Y(c_n) = r_s(Y).$$

Therefore, $l^s_Y \geq 1$. Next we see $l^k_Y \geq 1$. Suppose $r = \Gamma_Y(k) < \infty$ for $k \in \mathbb{Z}_{>0}$. Suppose that k is odd and $l^k_Y = 0$. The assumption $l^k_Y = 0$ implies that there exist a Riemann metric g_Y on Y and a sequence of perturbations $\{\pi_n\}_{n\in\mathbb{Z}_{>0}}$ with $\|\pi_n\| \to 0$ such that

$$\emptyset = \bigcup_{m\in\mathbb{Z}} \{a \in \widetilde{R}^*_{\pi_n}(Y) \colon |cs_{\pi_n}(a) - \Gamma_Y(k)| < \lambda_Y, \mathrm{ind}(a) = 1 + 8m\}. \tag{27}$$

Then, for g and $\{\pi_n\}_{n\in\mathbb{Z}_{>0}}$,

$$\Gamma_Y(k) = \lim_{n\to\infty} \inf_{\substack{\alpha\in C^\Lambda_*(Y,\pi_n), d^\Lambda(\alpha)=0 \\ D_1U^j(\alpha)=0 (1\leq j<k-1) \\ D_1U^{k-1}(\alpha)\neq 0}} \{\mathrm{mdeg}(D_1U^k(\alpha)) - \mathrm{mdeg}(\alpha)\}.$$

This implies there is a sequence $\{\alpha_n\}_{n\in\mathbb{Z}_{>0}}$ of elements in $C^\Lambda_1(Y,\pi_n)$ such that

$$\lim_{n\to\infty} \big(\mathrm{mdeg}(D_1U^k(\alpha_n)) - \mathrm{mdeg}(\alpha_n)\big) = \Gamma_Y(k). \tag{28}$$

We write $\alpha_n = \sum_{i\in\mathbb{Z}_{>0}} q^n_i [c^n_i] \lambda^{s^n_i}$, where $q^n_i \in \mathbb{Q}$, $[c^n_i] \in R^*(Y)_{\pi_n}$ and $s^n_i \in \mathbb{R}$ with $\lim_{i\to\infty} s^n_i = \infty$. Then, by taking suitable lift $c^n_{i_n}$ of $[c^n_{i_n}]$ for each n, (28) implies

$$\lim_{n\to\infty} cs_Y(c^n_{i_n}) = \Gamma_Y(k).$$

This contradicts (27). The proof for $k \in 2\mathbb{Z}$ is the same. ∎

The following proposition provides a relation between l_Y and $l_{Y,r,i}$:

Proposition 3.8. *We write $\{a_1, \dots, a_n\} = (0,1] \cap \Lambda^*_Y$. Then, we have*

$$\sum_{\substack{1\leq i\leq n \\ j\in\mathbb{Z}}} l_{Y,a_i,j} = l_Y.$$

Proof. For any metric g_Y, there exists $\varepsilon > 0$ such that any perturbation π with $\|\pi\| < \varepsilon$, we have

$$\bigcup_{\substack{1\leq i\leq n \\ j\in\mathbb{Z}}} \{a \in \widetilde{R}^*_\pi(Y) \colon |cs_\pi(a) - a_i| < \lambda_Y, j = \mathrm{ind}(a)\} \cong R^*_\pi(Y).$$

This implies the conclusion. ∎

We will see a connected sum formula of l_Y and $l_{Y,r,i}$ under some assumptions in Section 6.

3.2. Calculation of l_Y

In this section, we give several ways to calculate l_Y or $l_{Y,r,i}$.

Lemma 3.9 (Morse inequality for cs). *For an oriented homology 3-sphere Y, the inequality $l_Y \geq \sum_{i=0}^{7} \text{Rank}\, I_i(Y)$ holds.*

Proof. By the definition of instanton homology, we have

$$\#R^*_\pi(Y) \geq \sum_{i=0}^{7} \text{Rank}\, I_i(Y)$$

for every non-degenerate regular perturbation. This completes the proof. ∎

The following lemma give explicit calculations:

Lemma 3.10. *For a Seifert homology 3-sphere of type $\Sigma(p,q,r)$,*

$$l_{\Sigma(p,q,r)} = 2|\lambda(\Sigma(p,q,r))|,$$

where $\lambda(Y)$ is the Casson invariant of Y.

Proof. For a Seifert homology 3-sphere $\Sigma(p,q,r)$, it is shown in [16] that $cs_{\Sigma(p,q,r)}$ is non-degenerate and Floer indices of all of its critical points are even. Therefore, $\pi_n = (\emptyset, 0, 0)$ gives a sequence of non-degenerate regular perturbations. This implies the conclusion. ∎

We give calculations of l_Y for the degenerate case $\Sigma(a_1, \dots, a_n)$ in Theorem 6.5.

Lemma 3.11. *For an oriented homology 3-sphere Y and $r \in \Lambda_Y$,*

$$l_{Y,r,i} \geq \text{Rank}\, I_i^{[r+\lambda_Y, r-\lambda_Y]}(Y)$$

holds, where $\lambda_Y = \frac{1}{2}\min\{|a-b| : a, b \in \Lambda_Y\}$.

Proof. Take a Riemann metric g_Y and a sequence $\{\pi_n\}$ of non-degenerate regular perturbations such that

$$l(Y, g, r, i) = \sup_{n \in \mathbb{Z}_{>0}} \#\{a \in \widetilde{R}^*_{\pi_n}(Y) : |\, cs_{\pi_n}(a) - r| < \lambda_Y, \text{ind}(a) = i\}$$

and $\|\pi_n\| \to 0$. Then the chain complex of $I_i^{[r+\lambda_Y, r-\lambda_Y]}(Y)$ is generated by the elements $\{a \in \widetilde{R}^*_{\pi_n}(Y) : r + \lambda_Y < cs_{\pi_n}(a) < r - \lambda_Y\}$. By the definition of $l_{Y,r,i}$, we have

$$l(Y, g, r, i) \geq \text{Rank}\, I_i^{[r+\lambda_Y, r-\lambda_Y]}(Y).$$

∎

4. Some remarks for $\{cs^j_{X,c}\}_{j\in\mathbb{Z}_{>0}}$

In this section, we give several properties of $\{cs^j_{X,c}\}_{j\in\mathbb{Z}_{>0}}$ including the connected sum formula and the surgery formula.

4.1. Connected sum formula

First we show a connected sum formula for $\{cs^j_{X,c}\}_{j\in\mathbb{Z}_{>0}}$. Let X_1 and X_2 be oriented closed 4-manifolds with fixed classes $c_1 \in H^1(X_1;\mathbb{Z})$ and $c_2 \in H^1(X_2;\mathbb{Z})$. Fix embeddings $l_i\colon S^1 \times D^3 \to X_i$ with $c_i(l_i(S^1 \times *)) = 1$. For a diffeomorphism $\psi\colon \partial(\operatorname{Im} l_1) \to \partial(\operatorname{Im} l_2)$, one can define a connected sum $X_1\#_\psi X_2$ of X_1 and X_2 along ψ by

$$X_1\#_\psi X_2 := (X_1 \setminus \operatorname{int}\operatorname{Im} l_1) \cup_\psi (X_2 \setminus \operatorname{int}\operatorname{Im} l_2).$$

The class c_1 determines a class $c_\# \in H_1(X_1\#_\psi X_2;\mathbb{Z})$. We write the j-covering space corresponding to

$$\pi_1(X_i) \xrightarrow{\mathrm{Ab}} H_1(X_i;\mathbb{Z}) \to \mathbb{Z}/j\mathbb{Z}$$

by $p^j_i\colon (X_i)_{j,c} \to X_i$ for $i = 1$ and $i = 2$. We fix lifts $\tilde{l}_i\colon S^1 \times D^3 \to (X_i)_{j,c}$ of $l_i\colon S^1 \times D^3 \to X_i$ for $i = 1, 2$.

Proposition 4.1 (connected sum formula). $\operatorname{Im} cs_{X_1,c_1} \cup \operatorname{Im} cs_{X_2,c_2} \subset \operatorname{Im} cs_{X_1\#_\psi X_2,c_\#}$.

Proof. Choose $\rho \in \widetilde{R}^*((X_1)_{j,c_1})$ such that $cs^j_{X_1,c_1}(\rho) < 1$. We can see that $j_{c_\#}X_1\#_\psi X_2$ is obtained by gluing of $(X_1)_{j,c_1} \setminus \operatorname{Im} \widetilde{l_1}$ and $(X_2)_{j,c_2} \setminus \operatorname{Im} \widetilde{l_2}$ along $S^1 \times S^2$. The restriction of ρ to $\partial(\operatorname{Im} \widetilde{l_1})$ determines an element of $\operatorname{Hom}(\mathbb{Z}, \mathrm{SU}(2))/\mathrm{SU}(2)$. Then the flat connection $\rho|_{\partial(\operatorname{Im}\widetilde{l_1})=\partial(\operatorname{Im}\widetilde{l_2})}$ can be extended whole of $(X_2)_{j,c_2} \setminus \operatorname{Im}\widetilde{l_2}$ using a homomorphism

$$\pi_1((X_2)_{j,c_2} \setminus \operatorname{Im}\widetilde{l_2}) \to H_1((X_2)_{j,c_2} \setminus \operatorname{Im}\widetilde{l_2};\mathbb{Z}) \to \mathbb{Z} \subset \mathrm{SU}(2).$$

We denote this extension of $\rho|_{\partial(\operatorname{Im}\widetilde{l_1})=\partial(\operatorname{Im}\widetilde{l_2})}$ by $\tilde{\rho}$. By construction, we have

$$cs^j_{X_1,c_1}(\rho) = cs^j_{X_1\#_\psi X_2,c_\#}(\tilde{\rho}).$$

This completes the proof. ∎

4.2. Mapping tori

Let $X_h(Y)$ be the mapping torus of a fixed orientation preserving diffeomorphism $h\colon Y \to Y$ on an oriented 3-manifold Y and $P = Y \times \mathrm{SU}(2)$. The map h gives an action $h^*\colon R^*(Y) \to R^*(Y)$. We put

$$R^h(Y) := \{\rho \in R^*(Y)\colon h^*\rho = \rho\}.$$

For the mapping torus, we have a convenient formula for $cs_{X,[c]}$.

Proposition 4.2. *Suppose* $c = [Y] \in H_3(X_h(Y); \mathbb{Z})$. *Then*

$$\operatorname{Im} cs^j_{X_h(Y),[Y]} = \operatorname{Im} cs_{X_{h^j}(Y),[Y]} = \{cs_Y(a) \in (0,1] : a \in R^{(h^j)}(Y)\}$$

holds for any $j \in \mathbb{Z}_{>0}$.

Proof. In [39], it is shown that the inclusion $i: Y \to X_h(Y)$ induces a two-to-one correspondence

$$i^*: R^*(X_h(Y)) \to R^h(Y). \tag{29}$$

We have the following commutative diagram:

$$\begin{tikzcd} R^*(X_h(Y)) \arrow[r, "cs_{X_h(Y),[Y]}"] \arrow[d, "i_Y^*"'] & (0,1] \arrow[d, "\mathrm{Id}"] \\ R^h(Y) \arrow[r, "cs_Y"'] & (0,1] \end{tikzcd}$$

This completes the proof. ∎

4.3. Surgery along $S^1 \times D^3$

In order to prove Theorem 1.11, we need a surgery formula for $cs^j_{X,c}$. Let X be a closed connected oriented 4-manifold with $0 \neq c \in H_3(X;\mathbb{Z})$ and $0 \neq d \in H_1(X;\mathbb{Z})$ satisfying $c \cdot d = 0$. Let $l: S^1 \times D^3 \to X$ be an embedding such that $[l|_{S^1 \times \{0\}}] = d$ and Y be a closed connected oriented 3-submanifold of X such that $[Y] = c$ and $Y \cap l = \emptyset$. We define the 4-manifold X_l obtained by surgery along l by

$$X_l := X \setminus \operatorname{int}(\operatorname{Im} l) \cup_{S^1 \times S^2} D^2 \times S^2. \tag{30}$$

We have two inclusion maps $i_1: X \setminus \operatorname{Im} l \to X$ and $i_2: X \setminus \operatorname{Im} l \to X_l$. Set $c' := i_1^*(\mathrm{PD}(c)) \in H^1(X \setminus \operatorname{Im} l, \mathbb{Z})$. Then we can take $0 \neq c^* \in H^1(X_l; \mathbb{Z})$ such that $i_2^*(c') = c^*$.

Proposition 4.3. *For any* $j \in \mathbb{Z}_{>0}$

$$\operatorname{Im} cs^j_{X_l,c^*} \subset \operatorname{Im} cs^j_{X,c}.$$

Proof. Suppose that $\rho \in \widetilde{R}^*((X_l)_{j,c^*})$ such that $cs^j_{X_l,c^*}(\rho) < 1$. Note that $(X_l)_{j,c^*}$ has a decomposition

$$W[0, j-1] = W_0 \cup_Y W_1 \cup_Y \cdots \cup_Y W_{j-1},$$

where W_i is a copy of $W_0 = \overline{X_l \setminus Y}$. Since Y and $\operatorname{Im} l$ are disjoint, W_i can be written as

$$\big(X_l \setminus (Y \cup D^2 \times S^2)\big) \cup D^2 \times S^2.$$

We denote by $V_0 \subset W_0$ the submanifold corresponding to $D^2 \times S^2$. We define

$$W_*[0, j-1] := (W_0 \setminus \operatorname{int} V_0) \cup_Y (W_1 \setminus \operatorname{int} V_1) \cup_Y \cdots \cup_Y (W_j \setminus \operatorname{int} V_j),$$

where (W_i, V_i) is a copy of (W_0, V_0) for $i \in \mathbb{Z}_{>0}$. The flat connection ρ determines flat connections ρ^j and ρ_*^j on $W[0, j-1]$ and $W_*[0, j-1]$ via pull-back. Note that, by identifying the boundaries of $W_0 \setminus \operatorname{int} V_0 \cup_{S^1 \times S^2} S^1 \times D^3$, we recover X. Here we see that the flat connection $\rho_*^j|_{W_0 \setminus \operatorname{int} V_0}$ extends to a flat connection $W_0 \setminus \operatorname{int} V_0 \cup_{S^1 \times S^2} S^1 \times D^3$. To see this, we consider the restriction $(\rho_*^j)|_{\partial(V_0) = S^1 \times S^2}$. Since the flat connection $(\rho_*^j)|_{\partial(V_0)}$ extends to V_0, the holonomies for all loops in $\partial V_0 = S^1 \times S^2$ are zero. Thus, $(\rho_*^j)|_{\partial(V_0) = S^1 \times S^2}$ is isomorphic to trivial flat connection. This fact allows us to extend the flat connection $\rho_*^j|_{W_0 \setminus \operatorname{int} V_0}$ to a flat connection on $W_0 \setminus \operatorname{int} V_0 \cup_{S^1 \times S^2} S^1 \times D^3$ trivially. Therefore, the connection ρ_*^j extends to a connection on

$$(W_0 \setminus \operatorname{int} V_0 \cup_{S^1 \times S^2} S^1 \times D^3) \cup_Y \cdots \cup_Y (W_j \setminus \operatorname{int} V_j \cup_{S^1 \times S^2} S^1 \times D^3). \tag{31}$$

Identifying the boundaries of (31) gives a construction of $X_{j,c}$. Therefore, the extension on (31) gives a connection ρ_j' on $X_{j,c}$. By the construction of ρ_j', we conclude

$$cs^j_{X,c}(\rho') = cs_{X_l, c^*}(\rho).$$

This completes the proof. ∎

4.4. Calculations

We give explicit calculations for several mapping tori of Seifert manifolds. Regarding $\Sigma(p,q,r)$ as $\{(x,y,z) \in \mathbb{C}^3 : x^p + y^q + z^r = 0\} \cap S^5$, we define

$$\tau \colon \Sigma(p,q,r) \to \Sigma(p,q,r),$$
$$\iota \colon \Sigma(p,q,r) \to \Sigma(p,q,r)$$

by

$$\tau \colon (x,y,z) \mapsto (x, y, e^{\frac{2\pi i}{r}} z), \tag{32}$$
$$\iota \colon (x,y,z) \mapsto (\bar{x}, \bar{y}, \bar{z}). \tag{33}$$

This gives the following calculations:

Proposition 4.4. *Let (p,q,r) be a relatively prime triple of positive integers. For any $j \in \mathbb{Z}_{>0}$,*

$$\operatorname{Im} cs_{X_\tau(\Sigma(p,q,r)),[\Sigma(p,q,r)]} = \Lambda_{\Sigma(p,q,r)} \cap (0,1],$$
$$\operatorname{Im} cs_{X_\iota(\Sigma(p,q,r)),[\Sigma(p,q,r)]} = \Lambda_{\Sigma(p,q,r)} \cap (0,1].$$

Proof. By applying Proposition 4.2, we have

$$\operatorname{Im} cs_{X_{\tau^j},[Y]} = \{cs(a) \in (0,1] \colon a \in R^*(Y)(\tau^j)^* a = a\} = \Lambda_Y \cap (0,1].$$

It is shown in [10,43] that

$$\tau^* \colon R^*(\Sigma(p,q,r)) \to R^*(\Sigma(p,q,r))$$

and

$$\iota^* \colon R^*(\Sigma(p,q,r)) \to R^*(\Sigma(p,q,r))$$

are equal to the identity. This completes the proof. ∎

This property of $\iota^* \colon R^*(\Sigma(p,q,r)) \to R^*(\Sigma(p,q,r))$ is shown for any homology 3-sphere of type $\Sigma(a_1,\dots,a_n)$. Therefore, Proposition 4.4 can be proved for such Seifert homology 3-spheres.

5. Invariants of 2-knots

In this section, by the use of $cs_{X,c}$, we will introduce an invariant of oriented 2-knots.

5.1. Invariants $\{cs_{K,j}\}_{j\in\mathbb{Z}_{>0}}$

Let K be an oriented 2-knot in S^4. It is known that the normal bundle ν_K of K is always trivial. Moreover, trivializations of ν_K are unique up to isotopy. Therefore, we fix such a tubular neighborhood $\nu_K \colon S^2 \times D^2 \to S^4$. Then we have an oriented homology $S^1 \times S^3$ defined by

$$X(K) := D^3 \times S^1 \cup_{\nu_K} S^4 \setminus \nu_K.$$

Note that an orientation of S^4 gives an orientation of $X(K)$. If we have a Seifert hypersurface Y_K of K, then we can regard Y_K as a generator of $H_3(X(K);\mathbb{Z})$ by the following discussion. First we regard $Y_K \setminus B(\varepsilon)$ as a submanifold in $S^4 \setminus \nu_K$. Then we cap off $Y_K \setminus B^3$ in $X(F)$. The orientation of K determines a generator $[1_K]$ of $H_3(X(K);\mathbb{Z})$ such that $[* \times S^1] \cdot [Y_K] = 1$. Note that the class $[Y_K] \in H_3(X(K);\mathbb{Z})$

satisfies Assumption 2.1. The class $[1_K]$ gives an isomorphism class of $\mathbb{Z}$-covering space

$$p_K \colon \widetilde{X}(K) \to X(K). \tag{34}$$

We denote by $(X(K))_{j,1_K}$ the total space of the $\mathbb{Z}/j\mathbb{Z}$ covering space corresponding to the kernel of the composite homomorphism $\pi_1(X(K)) \to \mathbb{Z} \to \mathbb{Z}/j\mathbb{Z}$ for $j \in \mathbb{Z}_{>0}$. The map (34) gives a covering map

$$p^j_K \colon \widetilde{X}(K) \to (X(K))_{j,1_K}$$

for each $j \in \mathbb{Z}_{>0}$.

Definition 5.1. For an oriented knot K and $j \in \mathbb{Z}_{>0}$, we define

$$cs_{K,j} := cs^j_{X(K),1_K} \colon R(X(K)_{j,1_K}) \to (0,1].$$

Example 5.2. For the unknot U, $X(K) \cong S^1 \times S^3$. Therefore, $\operatorname{Im} cs_{U,j} = \{1\}$ for any $j \in \mathbb{Z}$.

The lemma below is an easy consequence of Van Kampen's theorem.

Lemma 5.3. *For any* 2*-knot* K*,*

$$\pi_1(S^4 \setminus K) \cong \pi_1(X(K)).$$

Moreover, $\operatorname{Ker}\psi_j \cong \operatorname{Ker}\phi_j$*, where* ψ_j *and* ψ_j *are introduced in* (1) *and* (6)*. In particular,* $G_j(K) \cong \pi_1(X(K)_{j,1_K})$*.*

Via Lemma 5.3, we regard $cs_{K,j}$ as maps from $R(K,j)$ to $(0,1]$. When $j=1$, we write cs_K instead of $cs_{K,1}$. By Lemma 2.3, we see that $\operatorname{Im} cs_{K,j}$ is an isotopy invariant for each $j \in \mathbb{Z}_{>0}$. If we have a Seifert hypersurface Y, then we have the following formula:

Proposition 5.4 (Proposition 1.2 (1))**.** *For any oriented* 2*-knot* K *and* $j \in \mathbb{Z}_{>0}$*,*

$$\operatorname{Im} cs_{K,j} \subset (0,1] \cap \Lambda_Y,$$

where $\Lambda_Y = \operatorname{Im} cs|_{\widetilde{R}(Y)}$*.*

Proof. This is an immediate corollary of Proposition 2.6. ∎

Lemma 5.5 (Proposition 1.2 (3))**.** *For any positive integer* m*,*

$$\operatorname{Im} cs_{K,j} \subset \operatorname{Im} cs_{K,mj}$$

for any $j \in \mathbb{Z}_{>0}$*.*

Proof. This is a corollary of Lemma 2.10. ∎

Remark 5.6. We can generalise $\{cs_{K,j}\}_{j\in\mathbb{Z}_{>0}}$ to functionals $\{cs_{(K,X),j}\}_{j\in\mathbb{Z}_{>0}}$ for any 2-knot K embedded into any closed oriented 4-manifold X with $0 = [K] \in H_2(X;\mathbb{Z})$. Moreover, Proposition 5.4 and Lemma 5.5 hold for $\{cs_{K,j}\}_{j\in\mathbb{Z}_{>0}}$.

5.2. Connected sum formula

Let K_1 and K_2 be 2-knots. We denote by $K_1\#K_2$ the connected sum of K_1 and K_2.

Proposition 5.7 (Proposition 1.2 (2)). $\operatorname{Im} cs_{K_1,j} \cup \operatorname{Im} cs_{K_2,j} \subset \operatorname{Im} cs_{K_1\#K_2,j}$.

Proof. Since $X(K_1\#K_2)_{j,1_{K_1\#K_2}}$ is obtained the connected sum of $X(K_1)_{j,1_{K_1}}$ and $X(K_2)_{j,1_{K_2}}$ along some embedded $S^1\times D^3$, this is also a corollary of Proposition 4.1. ∎

5.3. Calculation for $cs_{K,j}$

First, we give simplest examples.

Proposition 5.8. *For any ribbon 2-knot K, $\operatorname{Im} cs_{K,j} = \{1\}$ holds.*

Proof. In [46], it is shown that the ribbon 2-knots have the finite connected sums of $S^1\times S^2$'s as Seifert hypersurfaces. By Proposition 5.4, $\operatorname{Im} cs_{K,j} \in (0,1]\cap\Lambda_Y$ for any $j\in\mathbb{Z}_{>0}$. If Y is a finite connected sum of $S^1\times S^2$'s, then the space of SU(2)-representation of Y is connected. This implies $\Lambda_Y = \mathbb{Z}$. This completes the proof. ∎

Next, we give non-trivial calculations for twisted spun knots. Let k be an oriented knot in S^3. We denote by $K(k,m)$ the *m-twisted spun knot* of k. In [49], Zeeman showed m-fold branched covering space $\Sigma^m(k)$ gives a Seifert hypersurface of $K(k,m)$. When $m\neq 0$, $K(k,m)$ is a fibered 2-knot with fiber $\Sigma^m(k)$. Moreover, the monodromy is given by the covering transformation of $\Sigma^m(k)$. Now, we give a proof of Proposition 1.4.

Proposition 5.9 (Proposition 1.4). *Let $T(p,q)$ be the (p,q)-torus knot and let $M(p,q,r)$ be the Montesinos knot of type (p,q,r) for a pairwise relative prime tuple (p,q,r) of positive integers. Let $k(p/q)$ be a 2-bridge knot such that $\Sigma^2(k(p/q)) = L(p,q)$, where $\Sigma^2(k)$ is the double branched cover of $k\subset S^3$.*

(1) *For any $m\in\mathbb{Z}_{>0}$ and $j\in\mathbb{Z}_{>0}$,*

$$\operatorname{Im} cs^j_{K(T(p,q),m)} = \operatorname{Im} cs_{\Sigma(p,q,m)},$$

where $K(k,m)$ is the m-twisted spun knot of the knot k.

(2) *For any* $j \in \mathbb{Z}_{>0}$,

$$\operatorname{Im} cs^j_{K(M(p,q,r),2)} = \operatorname{Im} cs_{\Sigma(p,q,r)}.$$

(3) *Here we also suppose that* p *is odd and satisfies the condition:*

$$\left\{s \in \{2, \dots, p-2\} \colon \frac{s^2-1}{p} \in \mathbb{Z}\right\} = \emptyset.$$

For any $j \in \mathbb{Z}_{>0}$,

$$\operatorname{Im} cs^j_{K(k(p/q),2)} = \left\{-\frac{n^2 r}{p} \bmod 1 \colon 1 \leq n \leq \left\lceil \frac{p}{2} \right\rceil\right\} \cup \{1\},$$

where r *is any integer satisfying* $qr \equiv -1 \mod p$.

Proof of Proposition 1.4. It is known that the m-fold branched cover of $T(p,q)$ (resp. the double branched cover of $M(p,q,r)$) is $\Sigma(p,q,m)$ (resp. $\Sigma(p,q,r)$), see [6]. Moreover, the covering transformations are given by τ and ι in (32) and (33). As mentioned in the proof of Proposition 4.4, ι and τ induce bijection between $R^*(\Sigma(p,q,m) \setminus B)$ and $R^*(\Sigma(p,q,r) \setminus B)$, where B are small open balls. On the other hand, if $m > 0$, $K(T(p,q),m)$ is a fibered knot whose fiber is $\Sigma(p,q,m)$ and the monodromy is given by τ. Similarly, $K(M(p,q,r),2)$ is also a fibered knot whose fiber is $\Sigma(p,q,r)$ and the monodromy is given by ι. Next, we prove the statement for a rational knot. It is known that the double branched cover of $k(p/q)$ is given by $L(p,q)$. The condition

$$\left\{s \in \{2, \dots, p-2\} \colon \frac{s^2-1}{p} \in \mathbb{Z}\right\} = \emptyset.$$

implies that every order 2-diffeomorphism on $L(p,q) \setminus B$ induces maps Id or ψ_{p-1} on $\pi_1(L(p,q) \setminus B) \cong \mathbb{Z}/p\mathbb{Z}$, where B are small open balls and ψ_{p-1} is given by $1 \mapsto p-1$. Therefore, every order 2-diffeomorphism h on $L(p,q) \setminus B$ satisfies $h^* = 1 \colon R^*(L(p,q)) \to R^*(L(p,q))$. In [31], Kirk and Klassen showed

$$\left\{-\frac{n^2 r}{p} \bmod 1 \colon 1 \leq n \leq \left\lceil \frac{p}{2} \right\rceil\right\} \cup \{1\} = \Lambda_{L(p,q)} \cap (0,1],$$

where r is any integer satisfying $qr \equiv -1 \mod p$. This completes the proof. ∎

In [16], for a Seifert homology 3-sphere of type $\Sigma(a_1, \dots, a_n)$, Fintushel and Stern gave an algorithm to compute Λ_Y.

6. Morse type perturbation

In this section, we will prove the following theorem:

Theorem 6.1. *Let r be an element of Λ^*_Y. Suppose that the Chern–Simons functional of Y is Morse–Bott at the level r. Then*

$$l_{Y,r,i} < \infty$$

for any $i \in \mathbb{Z}$. In particular, if cs is Morse–Bott, then

$$l_Y < \infty.$$

The lemma below is key to proving Theorem 6.1. The proof uses the essentially the same technique as in the proof of [42, Theorem 5.11] and in [4].

Lemma 6.2. *Suppose that the Chern–Simons functional of Y is Morse–Bott at the level $r \in \mathbb{R}/\mathbb{Z}$. We set $C_r := R^*(Y) \cap cs^{-1}(r)$. Let $g_r \colon C_r \to \mathbb{R}$ be a Morse function. Then there exists a smooth family of perturbations $\{\pi_\varepsilon\}$ parametrized by $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ and a small neighborhood of U of C_r such that the critical points of $cs_{\pi_\varepsilon}|_U$ correspond to* $\mathrm{Crit}(g_r)$*, π_ε is non-degenerate on U for any $\varepsilon \in (0, \varepsilon_0)$, $\lim_{\varepsilon \to 0} \|\pi_\varepsilon\| \to 0$ and $\mathrm{supp}\, h \subset U$. Moreover, there is an embedding*

$$L \colon \mathrm{Crit}(g_r) \times [0, \varepsilon_0) \to \mathcal{B}^*(Y)$$

such that $\mathrm{Im}\, L|_{\mathrm{Crit}(g_r) \times \{\varepsilon\}}$ coincides with the critical point set of $cs_{\pi_\varepsilon}|_U$ and that $\mathrm{Im}\, L|_{\mathrm{Crit}(g_r) \times \{0\}} = \mathrm{Crit}(g_r)$. Here, we consider the L^2_k-topology on $\mathcal{B}^*(Y)$ for a fixed $k > 2$.

We define a nice class of perturbations called *Morse–Bott type perturbations*. We can take orientation preserving diffeomorphisms $f_i^r \colon S^1 \times D^2 \to Y$ $(1 \le i \le N)$ such that the smooth map $\psi \colon \mathcal{B}^*(Y) \to \mathbb{R}^N$ defined by

$$\psi([A]) := \left(\int_{D^2} \mathrm{Tr}(\mathrm{Hol}(A)_{f_i^r(-,x)} d\mu) \right)_{1 \le i \le N}$$

gives a diffeomorphism from C_r to its image ([15]). We fix a closed tubular neighborhood $p_r \colon N_r \to \psi(C_r)$ of $\psi(C_r)$ and a Euclidean metric on N_r. Fix a smooth bump function $\rho_r \colon \mathbb{R}^N \to \mathbb{R}$ such that $\rho_r(v) = 1$ if $|v| < 1$, $\rho(v) = 0$ if $|v| > 2$ and ρ_r depends only on $|(x,t)|$, where $|-|$ is a metric on N_r. We consider the pull-back $p^*(g_r \circ \psi^{-1}) \colon N_r \to \mathbb{R}$.

Definition 6.3 (Morse–Bott type perturbation). Now, we define $h\colon \mathcal{B}^*(Y) \to \mathbb{R}$ by the composition of $\psi\colon B^*(Y) \to \mathbb{R}^N$ and the function $q\colon \mathbb{R}^N \to \mathbb{R}$ given by

$$q_r(x) := \begin{cases} \rho_r(x) g_r \circ \psi^{-1} \circ p(x) & \text{if } x \in N_r, \\ 0 & \text{if } x \notin N_r. \end{cases}$$

We call a pair $\pi^r(g) := ((f_i^r), (h_i = \mathrm{Tr}), q_r) \in \mathcal{P}^*(Y)$ a *Morse–Bott type perturbation* at C_r. When cs_Y is Morse–Bott, depending on the choice of $g\colon R^*(Y) \to \mathbb{R}$, we can define Morse–Bott perturbations $\pi(g) \in \mathcal{P}^*(Y)$ in the same way.

Now, we give a proof of Lemma 6.2.

Proof of Lemma 6.2. We take a Morse–Bott type perturbation $\pi^r(g)$ for C_r. Put

$$cs_{Y,\pi_t} := cs_Y + t h_{\pi^r(g)}\colon \mathcal{B}^*(Y) \to \mathbb{R}.$$

For a fixed element $x \in \mathcal{B}^*(Y)$, we consider an essentially self-adjoint elliptic operator

$$\mathrm{Hess}_x(cs_Y) = *d_x\colon \mathrm{Ker}\, d_x^* \cap L_k^2(\Omega_Y^1 \otimes \mathfrak{su}(2)) \to \mathrm{Ker}\, d_x^* \cap L_{k-1}^2(\Omega_Y^1 \otimes \mathfrak{su}(2)).$$

The formal tangent bundle $T^k\mathcal{B}^*(Y)$ of $\mathcal{B}^*(Y)$ is defined as the quotient bundle of

$$\mathcal{A}^*(Y) \times \bigcup_{x \in \mathcal{A}^*(Y)} \big(\mathrm{Ker}\, d_x^* \cap L_k^2(\Omega_Y^1 \otimes \mathfrak{su}(2))\big)$$

by the action of $\mathcal{G}(Y)$. Then, the operator $\mathrm{Hess}_x(cs)$ defines a bundle map

$$\mathrm{Hess}_x(cs)\colon T^k\mathcal{B}^*(Y) \to T^{k-1}\mathcal{B}^*(Y).$$

We define $\varepsilon_0 > 0$ by

$$\begin{aligned} \varepsilon_0 := \min\big\{|\lambda|\colon & \lambda \text{ is a non-zero eigenvalue of} \\ & \mathrm{Hess}_x(cs)\colon \mathrm{Ker}\, d_x^* \cap L^2(\Omega_Y^1 \otimes \mathfrak{su}(2)) \to \mathrm{Ker}\, d_x^* \cap L^2(\Omega_Y^1 \otimes \mathfrak{su}(2)), \\ & x \in C_r\big\}. \end{aligned}$$

We take an open neighborhood U of C_r in $\mathcal{B}^*(Y)$ such that

$$\begin{aligned} \left\{\frac{1}{2}\varepsilon_0\right\} \cap \{|\lambda|\colon & \lambda \text{ is an eigenvalue of} \\ & \mathrm{Hess}_x(cs)\colon \mathrm{Ker}\, d_x^* \cap L^2(\Omega_Y^1 \otimes \mathfrak{su}(2)) \to \mathrm{Ker}\, d_x^* \cap L^2(\Omega_Y^1 \otimes \mathfrak{su}(2)), \\ & x \in U\} = \emptyset. \end{aligned}$$

Then,

$$L_0^k := \bigcup_{x \in U} \left\{L^2\text{-eigenspaces of } \mathrm{Hess}_x(cs) \text{ whose eigenvalues } \lambda \text{ satisfy } |\lambda| < \frac{1}{2}\varepsilon_0\right\} \to U$$

gives a finite rank subbundle of $T^kU := T^kB^*(Y)|_U$. This gives a decomposition $T^kU = L_0^k \oplus L_1^k$. We have the following section:

$$\mathbf{grad}_1 \colon U \times (-\varepsilon, \varepsilon) \to L_1^{k-1}, \quad (c,t) \mapsto \mathrm{pr}_{L_1^{k-1}} \mathrm{grad}_x(cs_{Y,\pi_t}).$$

For a point $(c,t) \in U \times (-\varepsilon, \varepsilon)$ and a small neighborhood V_x, of x, by taking a trivialization, we can regard the section $\mathbf{grad}_1$ as a smooth map

$$\mathbf{grad}_1' \colon V_x \times (-\varepsilon, \varepsilon) \to (L_1^{k-1})_x.$$

Since cs_Y is Morse–Bott function, $d\,\mathbf{grad}_1'|_{V_x \times \{0\}}$ is surjective for $x \in C_r$. Therefore, if $|t|$ is sufficiently small, $d\,\mathbf{grad}_1'|_{V_x \times \{t\}}$ is sujective. Thus $\mathbf{grad}_1^{-1}(0)$ is a smooth submanifold and diffeomorphic to $C_r \times (-\varepsilon, \varepsilon)$ for a small $\varepsilon > 0$. We take such a diffeomorphism $H \colon C_r \times (-\varepsilon, \varepsilon) \to \mathbf{grad}_1^{-1}(0)$. Then, the critical points of cs_{Y,π_t} can be seen as the zero set of the following map:

$$\mathbf{grad} := \mathbf{grad}_0 \circ H \colon C_r \times (-\varepsilon, \varepsilon) \to L_0^{k-1}.$$

For $x \in C_r$ and its neighborhood V_x in C_r, we regard $\mathbf{grad}$ as a smooth map

$$\mathbf{grad}' = \text{trivialization} \circ \mathbf{grad}_1 \circ H \colon V_x \times (-\varepsilon, \varepsilon) \to (L_1^{k-1})_x.$$

We have the following Taylor expansion of $\mathbf{grad}'$ with respect to t at $t = 0$: for $x = (c,t) \in C_r \times (-\varepsilon, \varepsilon)$,

$$\begin{aligned}
\mathbf{grad}'(c,t) &= \mathbf{grad}'_{(c,0)} + t\frac{d}{dt}(\mathbf{grad}'_{(c,t)})|_{t=0} + O(t^2) \\
&= t\frac{d}{dt}(\mathrm{pr}_{(L_1)_x} \mathrm{grad}_x(cs_{Y,\pi_t}) \circ H(c,t))|_{t=0} + O(t^2) \\
&= t\frac{d}{dt}(\mathrm{pr}_{(L_1)_x} \mathrm{grad}_x(t\rho_r(x) g_r \circ \psi^{-1} \circ p \circ \psi(x))) \circ H(c,t))|_{t=0} \\
&\quad + O(t^2) \\
&= t\,\mathrm{pr}_{(L_1)_x} \mathrm{grad}_x(g_r \circ \psi^{-1} \circ p \circ \psi) \circ d_t H(c,t)) + O(t^2).
\end{aligned}$$

Now, we have a C^2 section $\mathbf{grad}^* \colon C_r \times (-\varepsilon, \varepsilon) \to L_1$ given by

$$\mathbf{grad}^*(c,t) := \begin{cases} 1/t\,\mathbf{grad}'(c,t) & \text{if } t \neq 0, \\ \mathrm{pr}_{L_1} \mathrm{grad}_x(g_r \circ \psi^{-1} \circ p \circ \psi) \circ d_t H(c,t)) & \text{if } t = 0. \end{cases}$$

Then $\mathbf{grad}^*(c,t)$ is transverse when $t = 0$. Therefore, for t sufficiently small, $\mathbf{grad}^*|_{C_r \times \{t\}}$ is transverse. One can see that the zero set of $\mathbf{grad}^*(c,0)$ corresponds to the set

$$\left\{ c \in C_r \colon \mathrm{pr}_{L_1} \mathrm{grad}_x(g_r \circ \psi^{-1} \circ p \circ \psi) \circ d_t H(c,0) = 0 \right\} \cong \mathrm{Crit}(g_r) \subset C_r.$$

We construct an embedding $L\colon \mathrm{Crit}(g_r) \times [0, \varepsilon_0) \to \mathcal{B}^*(Y)$. Note that for ε sufficiently small, $\mathbf{grad}^*\colon C_r \times (-\varepsilon, \varepsilon) \to L_1$ is transverse to the zero section. By the implicit function theorem, one can see that $(\mathbf{grad}^* |_{C_r\times(-\varepsilon,\varepsilon)})^{-1}(0)$ is a compact 1-dimensional manifold. If we take ε sufficiently small, then

$$(\mathbf{grad}^* |_{C_r\times\{0\}})^{-1}(0) \to (\mathbf{grad}^* |_{C_r\times(-\varepsilon,\varepsilon)})^{-1}(0) \to (-\varepsilon, \varepsilon)$$

gives a fiber bundle.

Note that $(\mathbf{grad}^* |_{C_r\times\{0\}})^{-1}(0)$ is a finite set and $(\mathbf{grad}^* |_{C_r\times(-\varepsilon,\varepsilon)})^{-1}(0)$ is a union of 1-dimensional open intervals. As a trivialization of the above bundle on $[0, \varepsilon)$, we have a diffeomorphism

$$L'\colon (\mathbf{grad}^* |_{C_r\times\{0\}})^{-1}(0) \times [0, \varepsilon) \to \bigcup_{t\in[0,\varepsilon)} (\mathbf{grad}^* |_{C_r\times\{t\}})^{-1}(0).$$

Therefore, the set of critical points of $(cs + th_{\pi^r(g)})|_U$ corresponds to $\mathrm{Crit}(g_r)$ for small $t > 0$. The embedding L is given by the composition of L' and the following map:

$$\bigcup_{t\in[0,\varepsilon)} (\mathbf{grad}^* |_{C_r\times\{t\}})^{-1}(0) \xrightarrow{\text{inclusion}} C_r \times [0, \varepsilon) \xrightarrow{\mathrm{pr}_{C_r}} C_r \to \mathcal{B}^*(Y).$$

One can easily see that $(cs_Y + th_{\pi^r(g)})|_U$ is non-degenerate at $x \in U$ if and only if $\mathbf{grad}^* |_{U\times\{t\}}$ is transverse to the zero section. Since $\mathbf{grad}^* |_{U\times\{0\}}$ is transverse to the zero section and transversality is an open condition, $\mathbf{grad}^* |_{U\times\{t\}}$ is transverse to the zero section for sufficiently small $t > 0$. ∎

Using Lemma 6.2 and some technique of [40], we will show the following lemma:

Lemma 6.4. *Suppose that the Chern–Simons functional of Y is Morse–Bott at the level $r \in \Lambda^*$. There exists a family of perturbations $\pi_n = (f, h_n, q_n) \in \mathcal{P}^*(Y, g)$ such that $\|\pi_n\| \to 0$, the perturbations π_n are non-degenerate and regular and*

$$\sup \#\{a \in \widetilde{R}^*_{\pi_n}(Y)\colon r - \lambda_Y < cs_{Y,\pi_n}(a) < r + \lambda_Y\} < \infty,$$

where $\lambda_Y = \frac{1}{2}\min\{|a - b|\colon a, b \in \Lambda^*_Y, a \neq b\}$. In particular,

$$\sum_{i\in\mathbb{Z}} l_{Y,r,i} < \infty.$$

Proof. We regard r as an element of $\mathbb{R}/\mathbb{Z}$ via mapping $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$. Suppose that $C_r = R^*(Y) \cap cs_Y^{-1}(r)$ is a submanifold in $\mathcal{B}^*(Y)$. We take a Morse function $g_r\colon C_r \to \mathbb{R}$. Then, by taking a Morse–Bott perturbation $\pi^r(g)$, we obtain a sequence of perturbation $\{\pi_t\}_{t\in(0,\varepsilon_0)} = \{(f, h, \varepsilon q)\}$ such that

$$cs_{Y,\pi_t}\colon \mathcal{B}^*(Y) \to \mathbb{R}/\mathbb{Z}$$

satisfying the conclusion of Lemma 6.2.

Next, we perturb the other part $R^*(Y) \setminus cs^{-1}(r)$. By the argument to construct perturbations as in [15, Section 5.5.1], one can take a sequence of perturbations

$$\{\pi'_n\}_{n\in\mathbb{Z}_{>0}} = \{(g, h_n, q_n)\}_{n\in\mathbb{Z}_{>0}}$$

supported in a small neighborhood of $R^*(Y) \setminus cs_Y^{-1}(r)$ such that $cs_{\pi'_n}$ is non-degenerate for each n and $\lim_{n\to\infty} \|\pi'_n\| \to 0$. Then, $\pi''_n := (f \cup g, h'_n + h, q'_n + \frac{1}{n}q)$ satisfies the following conditions: for a large $n > 0$,

$$\#\{a \in \widetilde{R}^*_{\pi_n}(Y) \colon r - \lambda_Y < cs_{Y,\pi_n}(a) < r + \lambda_Y\} = \#\mathrm{Crit}(g),$$

where $\lambda_Y = \frac{1}{2}\min\{|a-b| \colon a, b \in \Lambda_Y\}$. Using the technique of [40, Section 8], one can add a small perturbation $\{\pi^*_n\} = \{(f, h^*_n + h_n, q^*_n + q_n)\}$ such that the perturbation π^*_n is non-degenerate and regular for each $n \in \mathbb{Z}_{>0}$ and the critical point sets of cs_{π_n} and $cs_{\pi^*_n}$ coincide. This completes the proof. ■

Theorem 6.5. *For a Seifert homology* 3*-sphere of type* $\Sigma(a_1, \dots, a_n)$*,*

$$l_{\Sigma(a_1,\dots,a_n)} = 2|\lambda(\Sigma(a_1, \dots, a_n))|.$$

Proof. Saveliev showed that $R^*(\Sigma(a_1, \dots, a_n))$ has a perfect Morse function whose critical points have odd Floer indices in [43]. By the use of a Morse-type perturbation as in Lemma 6.2, one can see that there is no differential for such perturbations. This completes the proof. ■

The following theorem provides a connected sum formula of l_Y under suitable assumptions:

Theorem 6.6 (connected sum formula of l_Y). *Suppose that* Y_1 *and* Y_2 *are Seifert homology* 3*-spheres, then*

$$l_{Y_1\#Y_2} \le 4 l_{Y_1} l_{Y_2} + l_{Y_1} + l_{Y_2}.$$

Proof. The critical submanifold of $cs_{Y_1\#Y_2}$ in $\mathcal{B}^*(Y_1\#Y_2)$ is given by

$$R^*(Y_1\#Y_2) = R^*(Y_1) \amalg R^*(Y_2) \amalg R^*(Y_1) \times R^*(Y_2) \times \mathrm{SO}(3).$$

Since Y_1 and Y_2 are Seifert homology 3-spheres, the Chern–Simons functionals of Y_1 and Y_2 have Morse–Bott type perturbations such that the critical points correspond to $\mathrm{Crit}(f_1)$ and $\mathrm{Crit}(f_2)$, where $f_1 \colon R^*(Y_1) \to \mathbb{R}$ and $f_2 \colon R^*(Y_2) \to \mathbb{R}$ are perfect Morse function such that $l_{Y_1} = \#\mathrm{Crit}(f_1)$ and $l_{Y_2} = \#\mathrm{Crit}(f_2)$. Note that SO(3) has a Morse function s whose critical point set is the four point set. Then $R^*(Y_1\#Y_2)$ has a Morse function such that the number of critical points is $4 l_{Y_1} l_{Y_2} + l_{Y_1} + l_{Y_2}$. Thus, one can take Morse–Bott type perturbations whose critical points correspond to $4 l_{Y_1} l_{Y_2} + l_{Y_1} + l_{Y_2}$ points. This completes the proof. ■

The connected sum formula below is useful to calculate l_Y^s and l_Y^k.

Theorem 6.7. *Let r be an element in $\Lambda_{Y_1\#Y_2}$. Suppose that Y_1 and Y_2 are Seifert homology 3-spheres. Then*

$$l_{Y_1\#Y_2,r,i} \leq 4\sum_{\substack{r=r_1+r_2\\ i=i_1+i_2}} l_{Y_1,r_1,i_1} l_{Y_2,r_2,i_2} + l_{Y_1,r,i} + l_{Y_2,r,i}.$$

Proof. We decompose the set

$$R^*(Y_1\#Y_2) \cap \{a\colon \mathrm{ind}(a) = i, cs_{Y_1\#Y_2}(a) = r\}$$

as the union of

$$\coprod_{\substack{r=r_1+r_2\\ i=i_1+i_2}} \{a \in R^*(Y_1)\colon \mathrm{ind}(a) = i_1, cs_{Y_1}(a) = r_1\} \\ \times \{a \in R^*(Y_2)\colon \mathrm{ind}(a) = i_2, cs_{Y_2}(a) = r_2\} \times \mathrm{SO}(3)$$

and

$$\{a \in R^*(Y_1)\colon \mathrm{ind}(a) = i, cs_{Y_1}(a) = r\} \amalg \{a \in R^*(Y_2)\colon \mathrm{ind}(a) = i, cs_{Y_2}(a) = r\}.$$

Then, the same proof as for Theorem 6.6 can be used to prove the desired result. ∎

We will calculate l_Y, l_Y^s and l_Y^k for a certain class of homology 3-spheres in Section 9.

7. Convergence theorem and proof of main theorems

7.1. Convergence theorem

In this section, we fix an oriented homology 3-sphere Y embedded in an oriented negative definite 4-manifold X. In this section, we assume that $H_1(X;\mathbb{R}) \cong \mathbb{R}$ and $0 \neq [Y] = c \in H_3(X;\mathbb{Z})$. Let W_0 be the oriented compact 4-manifold with $\partial W_0 = Y \cup (-Y)$ obtained by taking the closure of $X \setminus Y$. For a positive integer l, the manifold $W[0,l]$ is the compact oriented 4-manifold obtained from X defined in (2) and $W^*[0,l]$ is the cylindrical end 4-manifold written by $(W[0,l])^*$ in (15). Here we recall $\partial W_i = Y_i^+ \cup Y_i^-$. Fix a small collar neighborhood of Y_i^+ in $W^*[0,l]$ denoted by

$$Y_i^+ \times I \subset W^*[0,l] \tag{35}$$

and a Riemann metric g_Y on Y. We also fix a Riemann metric $g_{W^*[0,l]}$ on $W^*[0,l]$ such that the restrictions of $g_{W^*[0,l]}$ to $Y \times (-\infty, 0]$, $Y \times [0,\infty)$ and $Y_j^+ \times I$ for $j \in \{0,\dots,l_Y^s\}$ equal $g_Y \times dt^2$.

In this setting, we will show the following existence theorem:

Theorem 7.1. (1) *Suppose $r_s(Y) < \infty$ for some $s \in [-\infty, 0]$. Then, for any $l \in \mathbb{Z}_{\geq 0}$, a sequence of non-degenerate regular perturbations $\{\pi_n\}$ with $\|\pi_n\| \to 0$ and a sequence of perturbations $\{\pi^n_{W^*[0,l-1]}\}$ of ASD equations on $W^*[0,l]$ compatible with $\{\pi_n\}$ on $Y \times \mathbb{R}_{\leq 0}$ and $Y \times \mathbb{R}_{\geq 0}$, there exist sequences of critical points $\{a_n\}_{n \in \mathbb{Z}_{>0}}$ and $\{b_n\}_{n\in\mathbb{Z}_{>0}}$ of cs_{π_n} such that*

$$M(a_n, W^*[0, l-1], b_n) \neq \emptyset, \quad \lim_{n\to\infty} cs_{\pi_n}(a_n) = r_s(Y),$$

and $cs_{\pi_n}(a_n) - cs_{\pi_n}(b_n) \to 0$ as $n \to \infty$.

(2) *Suppose that $\Gamma_{-Y}(k) < \infty$ for some $k \in \mathbb{Z}_{>0}$. Then, for any $l \in \mathbb{Z}_{\geq 0}$, a sequence of non-degenerate regular perturbations $\{\pi_n\}$ with $\|\pi_n\| \to 0$ and a sequence of perturbations $\{\pi^n_{W^*[0,l-1]}\}$ of ASD equations on $W^*[0,l]$ compatible with $\{\pi_n\}$ on $Y \times \mathbb{R}_{\leq 0}$ and $Y \times \mathbb{R}_{\geq 0}$, there exist sequences of critical points $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ and $\{b_n\}_{n\in\mathbb{Z}_{>0}}$ of cs_{π_n} such that*

$$M(a_n, W^*[0, l-1], b_n) \neq \emptyset, \quad \lim_{n\to\infty} cs_{\pi_n}(a_n) = \Gamma_{-Y}(k),$$

and $cs_{\pi_n}(a_n) - cs_{\pi_n}(b_n) \to 0$ as $n \to \infty$.

Remark 7.2. Daemi [11] and Nozaki, Sato, and the author [35] proved similar existence results for solutions of perturbed ASD-equations. The author expects that Theorem 7.1 can be proved for $\mathcal{J}_Y(k, s)$ which will be defined in [12] for positive $k \in \mathbb{Z}$.

Proof of Theorem 7.1. The proof consists of two parts.

(1) Put $r_s(Y) = r$. Take a sequence of positive $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ and $\pi_n \in \mathcal{P}^*(Y, s, r + \varepsilon_n, g)$ for each n. We consider the cobordism map

$$\mathrm{CW}[0, l] \colon \mathrm{CI}^1_{[s,r+\varepsilon_n]}(Y) \to \mathrm{CI}^1_{[s,r+\varepsilon_n]}(Y)$$

given by counting the moduli spaces $M(a, W^*[0, l], b)$. In [35], we showed the counts of the following oriented 0-dimensional compact manifolds:

$$\bigcup_{\substack{b\in\widetilde{R}^*(Y)_{\pi_n}\\ \mathrm{ind}(b)=0, cs_{\pi_n}(b)<r+\varepsilon_n}} M^Y(a, b)_{\pi_n}/\mathbb{R} \times M(b, W^*[0, l], \theta),$$

$$M^{Y_1}(a, \theta)_{\pi_n}/\mathbb{R} \times M(\theta, W^*[0, l], \theta),$$

$$-\bigcup_{\substack{c\in\widetilde{R}^*(Y_2)_{\pi_2}\\ \mathrm{ind}(b)=1, cs_{\pi_2}(c)<r+\varepsilon_n}} M(a, W^*[0, l], c) \times M(c, \theta)_{\pi_n}/\mathbb{R}$$

are zero for each generator of $a \in C_1^{[s,r+\varepsilon_n]}(Y)$. This implies that

$$n^{[s,r+\varepsilon_n]}\partial^{[s,r+\varepsilon_n]}(a) + c(W)\theta_Y^{[s,r+\varepsilon_n]}(a) = \theta_Y^{[s,r+\varepsilon_n]}\mathrm{CW}^{[s,r+\varepsilon_n]}(a) \tag{36}$$

for any $a \in \mathrm{CI}_1^{[s,r+\varepsilon_n]}(Y)$ for some $c(W) > 0$. By the choice of r, for an element $a \in \mathrm{CI}^{[s,r-\lambda_Y]}(Y)$ with $\partial^{[s,r-\lambda_Y]}(a) = 0$, the equation

$$\theta_Y^{[s,r-\lambda_Y]}(a) = 0$$

holds. However, since $[\theta_Y^{[s,r+\varepsilon_n]}] \neq 0$ for any n, we have a sequence $\{\zeta_n\}$ of elements in $\mathrm{CI}_1^{[r-\lambda,r+\varepsilon_n]}(Y)$ such that $\partial^{[s,r+\varepsilon_n]}(\zeta_n) = 0$ and $\theta_Y^{[s,r+\varepsilon_n]}(\zeta_n) \neq 0$. Then we put $a = \zeta_n$ and obtain the following equations:

$$c(W)\theta_Y^{[s,r+\varepsilon_n]}(\zeta_n) = \theta_Y^{[s,r+\varepsilon_n]}(\mathrm{CW}^{[s,r+\varepsilon_n]}(\zeta_n)) \tag{37}$$

for each n.

Since the left-hand side of (37) is non-zero, $\theta_Y^{[s,r+\varepsilon_n]}\mathrm{CW}^{[s,r+\varepsilon_n]}(\zeta_n)$ is also non-zero. Since $\mathrm{CW}^{[s,r+\varepsilon_n]}(\zeta_n)$ is a cycle, if we write

$$\zeta_n = \sum_i s_i^n a_i^n \quad \text{and} \quad \mathrm{CW}^{[s,r+\varepsilon_n]}(\zeta_n) = \sum_j r_j^n b_j^n$$

($s_i^n, r_j^n \in \mathbb{Q}$), then there exist i_0 and j_0 such that $r + \varepsilon_n > cs_{\pi_n}(b_{j_0}^n) > r - \lambda_Y$ and

$$M(a_{i_0}^n, W^*[0,l-1], b_{j_0}^n) \neq \emptyset.$$

Put $a_n := a_{i_0}^n$ and $b_n := b_{j_0}^n$. By the choices of $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ and $\{b_n\}_{n\in\mathbb{Z}_{>0}}$, we conclude that

$$\lim_{n\to\infty} cs_{Y,\pi_n}(a_n) = \lim_{n\to\infty} cs_{Y,\pi_n}(b_n) = r_s(Y).$$

(2) Fix $1 \leq l \leq l_Y^k$. We will also use the formula essentially showed in [20]. There exists a sequence $\{b_j\}$ of rational numbers such that

$$D_1^Y U_Y^{k-1}\mathrm{CW}[0,l](a) = c(W[0,l])(D_1^Y U_Y^{k-1}(a)) + \sum_{1\leq j<k-1} b_j D_1^Y U_Y^j(a), \tag{38}$$

where U_Y and D_1^Y (resp. U_Y and D_1^Y) are U-map and D_1 map for Y (resp. for Y) and

$$\mathrm{CW}[0,l]\colon C_i^\Lambda(Y) \to C_i^\Lambda(Y)$$

is a cobordism a map as in the same in [11]. We take a sequence $\{\zeta_n\} \subset C_*^\Lambda(Y)$ such that

$$d^\Lambda(\zeta_n) = 0, \quad D_1U^j(\zeta_n) = 0 (1 \leq j < k-1), \quad D_1U^{k-1}(\zeta_n) \neq 0,$$

and

$$\Gamma_{-Y}(k) = \lim_{|\pi_n|\to 0} \mathrm{mdeg}(D_1 U^{k-1}(\zeta_n)) - \mathrm{mdeg}(\zeta_n). \tag{39}$$

By combining (38) and (39), we have

$$D_1^Y U_Y^{k-1} \mathrm{CW}[0,l](\zeta_n) = c(W[0,l])(D_1^Y U_Y^{k-1}(\zeta_n)).$$

Thus,

$$\mathrm{mdeg}(D_1^Y U_Y^{k-1} \mathrm{CW}[0,l](\zeta_n)) = \mathrm{mdeg}(D_1^Y U_Y^{k-1}(\zeta_n)).$$

Note that the elements $\{\mathrm{CW}[0,l](\zeta_n)\}_{n\in\mathbb{Z}_{>0}}$ also satisfy the following three equations

$$d^\Lambda(c) = 0, \quad D_1 U^j(c) = 0 \quad (1 \le j < k-1), \quad D_1 U^{k-1}(c) \neq 0.$$

This implies

$$\Gamma_{-Y}(k) \le \mathrm{mdeg}(D_1^Y U_Y^{k-1} \mathrm{CW}[0,l](\zeta_n)) - \mathrm{mdeg}(\mathrm{CW}[0,l](\zeta_n)).$$

Since

$$\mathrm{mdeg}(\zeta_n) \le \mathrm{mdeg}(\mathrm{CW}[0,l](\zeta_n)) + \delta_n$$

is proved in [11], where $\{\delta_n\}$ is a sequence of positive number with $\delta_n \to 0$, we have

$$\lim_{n\to\infty} \mathrm{mdeg}(D_1^Y U_Y^{k-1} \mathrm{CW}[0,l](\zeta_n)) - \mathrm{mdeg}(\mathrm{CW}[0,l](\zeta_n)) = \Gamma_Y(k).$$

This implies that

$$\lim_{n\to\infty} (\mathrm{mdeg}(\mathrm{CW}[0,l](a_n)) - \mathrm{mdeg}(a_n)) = 0. \tag{40}$$

We write ζ_n by

$$\sum_{1\le i\le N} m^n(i)\lambda^{r_i^n} a_i^n$$

and

$$\mathrm{CW}[0,l](\zeta_n) = \sum_{i,j} M(b_j^n, W^*[0,l], a_i^n) m^n(i) \lambda^{r_i^n - \mathcal{E}(b_j^n, a_i^n)} b_j^n.$$

This implies for some $i_0, j_0 \in \mathbb{Z}_{>0}$,

$$M(b_{j_0}^n, W^*[0,l], a_{i_0}^n) \neq \emptyset$$

and (40) implies $\lim_{n\to\infty} \mathcal{E}(b_{j_0}^n, a_{i_0}^n) \to 0$. Put $a_n := a_{i_0}^n$ and $b_n := b_{j_0}^n$. By the choices of $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ and $\{b_n\}_{n\in\mathbb{Z}_{>0}}$, we conclude that

$$\lim_{n\to\infty} cs_{Y,\pi_n}(a_n) = \lim_{n\to\infty} cs_{Y,\pi_n}(b_n) = \Gamma_{-Y}(k). \qquad \blacksquare$$

The following theorem is a key theorem to prove Theorem 1.11. In the proof of the next theorem, we use the finiteness of l^s_Y or l^k_Y.

Theorem 7.3. (1) *Suppose that $l^s_Y < \infty$, $r_s(Y) < \infty$ for some $s \in [-\infty, 0]$. Then, there exist a positive integer $l \le l^s_Y$ and a flat connection A_∞ on $W[0, l-1]$ such that A_∞ is an irreducible flat connection on $W[0, l-1]$ with equal boundary flat connections and $cs^l_{X,[Y]}(A_\infty) = r_s(Y)$ when we regard A_∞ as a connection on $X_{l,[Y]}$.*

(2) *Suppose that $l^k_Y < \infty$, $\Gamma_Y(k) < \infty$ for some $k \in \mathbb{Z}_{>0}$. Then, there exist a positive integer $l \le l_Y$ and a flat connection A_∞ on $W[0, l-1]$ such that A_∞ is an irreducible flat connection on $W[0, l-1]$ with equal boundary flat connections and $cs^l_{X,[Y]}(A_\infty) = \Gamma_Y(k)$ when we regard A_∞ as a connection on $X_{l,[Y]}$.*

Proof. The proof is comprised of two parts.

(1) Suppose that $\{\pi_n\}_{n\in\mathbb{Z}_{>0}}$ and g_Y are a sequence of non-degenerate regular holonomy perturbations of cs_Y and a Riemann metric on Y such that

$$l^s_Y = \# \sup_{n\in\mathbb{Z}_{>0}} \big\{[a] \in \widetilde{R}^*_{\pi_n}(Y), \mathrm{ind}([a]) = 1, \\ r_s(Y) - \lambda_Y < cs_{\pi_n}([a]) < r_s(Y) + \lambda_Y\big\}. \tag{41}$$

Let $\{\pi^n_{W^*[0,l^s_Y-1]}\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of regular perturbations of the ASD-equations on $W^*[0, l^s_Y - 1]$ such that the perturbed equation of $\pi^n_{W^*[0,l^s_Y-1]}$ coincides with $F^+(A) + \pi^+_n(A) = 0$ on $Y \times (-\infty, 0]$, $Y \times [0, \infty)$ and $Y^+_j \times I$ for $j \in \{0, \dots, l^s_Y\}$ for any n. Here, we apply Theorem 7.1 (1). Then there exist sequences of critical points $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ and $\{b_n\}_{n\in\mathbb{Z}_{>0}}$ of cs_{π_n} such that

$$M(a_n, W^*[0, l^s_Y - 1], b_n) \neq \emptyset, \quad \lim_{n\to\infty} cs_{\pi_n}(a_n) = r_s(Y),$$

and

$$cs_{\pi_n}(a_n) - cs_{\pi_n}(b_n) \to 0 \quad \text{as } n \to \infty.$$

Take an element $A_n \in M(a_n, W^*[0, l^s_Y - 1], b_n)$ for every $n \in \mathbb{Z}_{>0}$. The conditions $cs_{\pi_n}(a_n) - cs_{\pi_n}(b_n) \to 0$, $\|\pi_n\| \to 0$, and $\|\pi^n_{W^*[0,l^s_Y-1]}\| \to 0$ as $n \to \infty$ imply

$$\|F(A_n) + \pi^n_{W^*[0,l^s_Y-1]}(A)\|_{L^2(W^*[0,l^s_Y-1])} \to 0. \tag{42}$$

This implies that the connection A_n is close to some critical point of cs_{Y,π_n} near $Y^+_j \times I$. On the other hand, we have gauge transformations $\{g^j_n\}$ on $Y^+_j \times I$ and critical points $\{c^j_n\}$ of cs_{Y,π_n} such that

$$\|(g^j_n)^* A_n|_{Y^+_j\times I} - p^* c^j_n\|_{C^l} \le c(\|F(A_n|_{Y^+_j\times I})\|_{L^2(Y^+_j\times I)} + \|\pi_n\|), \tag{43}$$

where the constant c depends only on the metric g_Y and l, which we take to be a positive integer greater 2, and $p\colon Y_j^+ \times I \to Y$ is the projection. Since cs_{Y,π_n} is non-degenerate, c_n^j in (43) is unique up to gauge transformations. Note that if n is large, then

$$c_n^j \in \left\{[a] \in \widetilde{R}^*_{\pi_n}(Y), \mathrm{ind}([a]) = 1, r_s(Y) - \lambda_Y < cs_{\pi_n}([a]) < r_s(Y) + \lambda_Y\right\}.$$

By (41), we can use the Pigeonhole principle. Then, for each n, there exist j_n and j_n' such that $c_n^{j_n} \cong c_n^{j_n'}$ as connections on Y. Moreover, such patterns of choices of j_n and j_n' are finite. By changing gauge transformations g_n^j, we assume $c_n^{j_n} = c_n^{j_n'}$ Thus, after taking a subsequence of $\{A_n\}_{n\in\mathbb{Z}_{>0}}$, we can assume that j_n and j_n' do not depend on the choices of n. By summarizing the above discussion, we obtained integers $0 \leq j$ and $j' \leq l_Y^s$, gauge transformations g_n^j and $g_n^{j'}$ on $Y_j^+ \times I$ and $Y_{j'}^+ \times I$, and critical points $\{c_n\}$ of cs_{Y,π_n} such that

$$\|(g_n^j)^* A_n|_{Y_j^+\times I} - p^* c_n\|_{C^l} \leq c(\|F(A_n|_{Y_j^+\times I})\|_{L^2(Y_j^+\times I)} + \|\pi_n\|), \tag{44}$$

$$\|(g_n^{j'})^* A_n|_{Y_{j'}^+\times I} - p^* c_n\|_{C^l} \leq c(\|F(A_n|_{Y_{j'}^+\times I})\|_{L^2(Y_j^+\times I)} + \|\pi_n\|) \tag{45}$$

hold. After taking a subsequence of $\{c_n\}$ and pull-back by gauge transformations $\{h_n\}$, we can assume that $\{h_n^* c_n\}$ is C^∞-convergent to c_∞. Then

$$\{h_n^*(g_n^j)^* A_n|_{Y_j^+\times I}\} \quad \text{and} \quad \{h_n^*(g_n^{j'})^* A_n|_{Y_{j'}^+\times I}\}$$

are convergent sequences on $Y_j^+ \times I$ and $Y_{j'}^+ \times I$.

We can extend gauge transformations $g_n^j \circ h_n$ and $g_n^{j'} \circ h_n$ to gauge transformations $\{g_n\}$ on $W^*[0, l_Y^s - 1]$ such that $\{g_n^* A_n\}$ converges to $\{g_n^* A_n\}$ on all of $Y \times \mathbb{R}$. We set $\lim_{n\to\infty} g_n^* A_n = A_\infty$. The limit A_∞ has the following properties: there are $j, j' \leq l_Y^s$ such that A_∞ is an SU(2)-irreducible flat connection on $W^*[0, l_Y^s - 1]$ with $A_\infty|_{Y_j^+} \cong A_\infty|_{Y_{j'}^+}$ and

$$cs_Y(A|_{Y_j^+}) = r_s(Y) \tag{46}$$

for any j. Suppose $j < j'$. Then $A_\infty|_{W[j,j']}$ satisfies the conclusion. By (46), we conclude that

$$r = \lim_{n\to\infty} cs_Y(a_n) = cs_{\overline{W[j,j']},[Y_j^+]}(A_\infty) = cs_{X,[Y]}^{j'-j+1}(A_\infty),$$

where $\overline{W[j,j']}$ is the oriented closed 4-manifold obtained by identifying the boundaries Y_j^+ and $Y_{j'}^-$ of $W[j,j']$.

Here, since A_∞ is a connection on $W[j,j']$ such that $A_\infty|_{Y_j^+} \cong A_\infty|_{Y_{j'}^-}$, we regard A_∞ as a connection on $\overline{W[j,j']}$. Moreover, $\overline{W[j,j']}$ can be identified with $X_{j'-j+1,[Y]}$. This completes the proof.

(2) The second proof is essentially the same as the first. Suppose $\{\pi_n\}_{n\in\mathbb{Z}_{>0}}$ and g_Y are a sequence of non-degenerate regular holonomy perturbations of cs_Y and Riemann metric on Y such that

$$l_Y^k = \# \sup_{n\in\mathbb{Z}_{>0}} \left\{ [a] \in R_\pi^*(Y) \colon \operatorname{ind}([a]) \equiv \begin{cases} 1 & \text{if } k \text{ is odd} \\ 5 & \text{if } k \text{ is even} \end{cases} \mod 8, \right.$$
$$\left. |\Gamma_{-Y}(k) - cs_{\pi_n}([a])| < \lambda_Y \right\}$$

and $\{\pi^n_{W^*[0,l_Y^k-1]}\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of regular perturbations of ASD-equations on $W^*[0,l_Y^k-1]$ such that the perturbed equation of $\pi^n_{W^*[0,l_Y^k-1]}$ coincide with $F^+(A)+\pi_n^+(A)=0$ on $Y\times(-\infty,0]$, $Y\times[0,\infty)$ and $Y_j^+\times I$ for $j\in\{0,\dots,l_Y^k\}$ for any n. Then we can do the same discussion as in the first proof. ■

7.2. Proof of main theorems

In this section, we prove several theorems in Section 1.

7.2.1. Proof of Theorem 1.11. In this section, we will give a proof of Theorem 1.11.

Theorem 7.4 (Theorem 1.11). *Let Y be an oriented homology 3-sphere and X be an oriented negative definite 4-manifold. Suppose there exists an embedding from Y to X with $0 \neq [Y] \in H_3(X;\mathbb{Z})$.*

(1) *If $l_Y^s < \infty$ and $r_s(Y) < \infty$ for some $s \in [-\infty, 0]$,*

$$r_s(Y) - \lfloor r_s(Y) \rfloor \in \bigcup_{1\le j\le l_Y^s} \operatorname{Im} cs_{j_{[Y]}X,[Y]}.$$

(2) *If $l_Y^k < \infty$ and $\Gamma_{-Y}(k) < \infty$ for some $k \in \mathbb{Z}_{>0}$,*

$$\Gamma_{-Y}(k) - \lfloor \Gamma_{-Y}(k) \rfloor \in \bigcup_{1\le j\le l_Y^k} \operatorname{Im} cs_{j_{[Y]}X,[Y]}.$$

If $[Y]=0$, then

$$r_s(Y) = r_s(-Y) = \Gamma_Y(k) = \Gamma_{-Y}(k) = \infty$$

for any s and k.

The proof of Theorem 1.11 is decomposed into two cases: $[Y] \neq 0$ and $[Y] = 0$. The first case gives an extension of the main theorem in [44]. However, the proof is not the same. In [44], the author used ASD moduli spaces for 4-manifolds with periodic ends to prove the main result. However, in this paper, we only use ASD moduli spaces for 4-manifolds with cylindrical ends. The essential part of the proof is contained in the proof of Theorem 7.1.

Proof of Theorem 1.11. First, we assume $[Y] \neq 0$. Suppose that Y is a homology 3-sphere Y embedded in a negative definite 4-manifold X with $0 \neq [Y] \in H_3(X;\mathbb{Z})$. We assume that X is connected. Put

$$c := PD([Y]) \in H^1(X;\mathbb{Z}) = \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}).$$

By the following discussion, we may assume that $H_1(X;\mathbb{R}) \cong \mathbb{R}$ to prove Theorem 1.11. Suppose Rank $H_1(X;\mathbb{R}) \geq 2$. We have an exact sequence:

$$0 \to \bigoplus_{\mathrm{Rank}\, H_1(X;\mathbb{Z})} \mathbb{Z} \oplus \text{torsion} \cong \mathrm{Ker}\, c \to H_1(X;\mathbb{Z}) \xrightarrow{c} \mathbb{Z}.$$

We take a generator $d_1 \in H_1(X;\mathbb{Z})$ of the free part of $\mathrm{Ker}\, c$. Then $d_1 \cdot [Y] = 0$. We also have the following exact sequence:

$$H_1(X \setminus Y;\mathbb{Z}) \to H_1(X;\mathbb{Z}) \to H_1(Y \times I; \partial(Y \times I);\mathbb{Z}) \cong H^3(Y;\mathbb{Z}) \cong \mathbb{Z}.$$

The map $H_1(X;\mathbb{Z}) \to H_1(Y \times I, \partial(Y \times I);\mathbb{Z}) \cong H^3(Y;\mathbb{Z}) \cong \mathbb{Z}$ corresponds to the pairing of c. As $d_1 \in \mathrm{Ker}\, c$, the class d_1 is represented by a closed oriented manifold $l_1' \subset X \setminus Y$. If $[Y] \neq 0$, $X \setminus Y$ is connected. Therefore, by considering the connected sum, we can assume l_1' is connected. We extend l_1' to a framed loop $l_1 \colon S^1 \times D^3 \to X$ such that $\mathrm{Im}\, l \cap Y = \emptyset$. Then by considering surgery along l_1, we obtain an oriented connected 4-manifold X_{l_1}. It is shown in [27] that the 4-manifold X_{l_1} obtained by surgery along l_1 has the same intersection form as X so X_{l_1} is also negative definite. By Proposition 4.3, there exists a class $0 \neq c_1 \in H^1(X_{l_1};\mathbb{Z})$ such that

$$\mathrm{Im}\, cs^j_{X_{l_1}, PD(c_1)} \subset \mathrm{Im}\, cs^j_{X,[Y]}.$$

Note that $b_1(X) = b_1(X_{l_1}) - 1$. By induction, there exists a sequence of negative definite 4-manifolds $\{X_{l_j}\}$ and classes $\{c_j\}$ for $1 \leq j \leq \mathrm{Rank}\,\mathrm{Ker}\, c$ such that

$$\mathrm{Im}\, cs^j_{X_{l_N}, PD(c_N)} \subset \cdots \subset \mathrm{Im}\, cs^j_{X_{l_1}, PD(c_1)} \subset \mathrm{Im}\, cs^j_{X,[Y]}$$

for any j, where l_j are embeddings $S^1 \times D^3 \to X_{l_{j-1}}$ satisfying $[l_{S^1 \times \{0\}}] = c_j$. This implies that we can suppose that $H_1(X_{l_N};\mathbb{R}) = \mathbb{R}$. Here we put $W_0 := \overline{X \setminus Y}$ and suppose $H_*(W_0;\mathbb{R}) \cong H_*(S^3;\mathbb{R})$. Suppose that $r_s(Y) < \infty$ and $l^s_Y < \infty$ for

$s \in [-\infty, 0]$ (resp. $\Gamma_Y(k) < 1$ and $l_Y^k < \infty$). Theorem 7.3 implies that there exist an integer $j \le l_Y^s$ (resp. $j \le l_Y^k$) and a representation $\rho\colon \pi_1(W[0, j-1]) \to \mathrm{SU}(2)$ satisfying the following conditions:

- the restrictions of ρ to the components of $\partial(W[0, j-1]) = Y_0^+ \amalg Y_{j-1}^-$ coincide via the identification $Y_0^+ \to Y_{j-1}^-$;
- $cs^j_{X,c}(\rho) = r_s(Y)$.

This implies that $r_s(Y) \in \operatorname{Im} cs^j_{X,c}$(resp. $\Gamma_Y(k) \in \operatorname{Im} cs^j_{X,c}$).

If $[Y] = 0$, since X is connected, X is decomposed into two parts: $X_1 \cup_Y X_2 = X$. Since X is negative definite, X_1 and X_2 are negative definite 4-manifolds. Therefore, both Y and $-Y$ bound a negative definite 4-manifold. By using (22) and (24), we have $\infty = \Gamma_{-Y}(k) = \Gamma_Y(k) = r_s(Y) = r_s(-Y)$. ■

In [44], the author constructed an obstruction class to the existence of embeddings by developing gauge theory for 4-manifolds with periodic ends. The main theorem of [44] is as follows.

Theorem 7.5 ([44]). *Let Y be an oriented homology 3-sphere and X an oriented homology $S^1 \times S^3$. Suppose that the Chern–Simons functional of Y is non-degenerate and there exists an embedding from Y to X such that $[Y]$ generates $H_3(X;\mathbb{Z})$. Then*

$$0 = \theta_Y^{[-\infty,r]} \in I^1_{[-\infty,r]}(Y) \quad \textit{for } 0 \le r \le \min_{1 \le j \le 2\#R(Y)+3} \min\{r \in \operatorname{Im} cs^j_{X,[Y]}\},$$

where $R(Y)$ is the quotient set of $\mathrm{Hom}(\pi_1(Y), \mathrm{SU}(2))$ by the conjugation of $\mathrm{SU}(2)$.

Now, we prove Theorem 7.5 without using any gauge theory on 4-manifolds with periodic ends.

Theorem 7.6. *Theorem 1.11 implies Theorem 7.5.*

Proof. The inequality

$$r_0(Y) \le r_{-\infty}(Y) \tag{47}$$

was proved in [35]. Moreover, when the Chern–Simons functional of Y is non-degenerate, by a similar discussion given in [40, Theorem 8.4], we can take a sequence of non-degenerate regular perturbations $\{\pi_n\}$ such that

- $\|\pi_n\| \to 0$,
- π_n is zero near a neighborhood of $R(Y)$, and
- $R_{\pi_n}(Y) = R(Y)$ for any n.

This proves

$$l_Y \leq \# R(Y). \tag{48}$$

Inequalities (47) and (48) imply that Theorem 1.11 recovers Theorem 7.5. ■

Theorem 1.6 is a corollary of Theorem 1.11.

Theorem 7.7 (Theorem 1.6). *Let Y be an oriented homology 3-sphere and K an oriented 2-knot. The invariants $\{cs_{K,j}\}_{j\in\mathbb{Z}_{>0}}$ satisfy the following properties.*

- *Suppose that l^s_Y is finite and $r_s(Y) < \infty$ for some $s \in [-\infty, 0]$. If Y is a Seifert hypersurface of K, then*

$$r_s(Y) - \lfloor r_s(Y) \rfloor \in \bigcup_{1\leq j\leq l^s_Y} \operatorname{Im} cs_{K,j}$$

  holds.
- *Suppose that l^k_Y is finite and $\Gamma_{-Y}(k) < \infty$ for some $k \in \mathbb{Z}_{>0}$. If Y is a Seifert hypersurface of K, then*

$$\Gamma_{-Y}(k) - \lfloor \Gamma_{-Y}(k) \rfloor \in \bigcup_{1\leq j\leq l^k_Y} \operatorname{Im} cs_{K,j}$$

  holds.

Proof of Theorem 1.6. The Seifert hypersurface Y determines a codimension 1-submanifold of $X(K)$ with $1_K = [Y] \in H_3(X(K);\mathbb{Z})$. Since $l^s_Y < \infty$, we can apply Theorem 1.11. Then, we obtain

$$r_s(Y) \in \bigcup_{1\leq j\leq l^s_Y} \operatorname{Im} cs^j_{X(K),1_K} = \bigcup_{1\leq j\leq l^s_Y} \operatorname{Im} cs_{K,j}\,.$$

This completes the proof. The proof of the second statement is the same. ■

7.2.2. Proof of Theorem 1.8 .

In this section, we prove Theorem 1.8.

Theorem 7.8 (Theorem 1.8). *Let Y be an oriented homology 3-sphere and K an oriented 2-knot.*

(1) *If $r_s(Y) < \infty$, $l^s_Y < \infty$ for some $s \in [-\infty, 0]$ and Y is a Seifert hypersurface of K, then there exists a positive integer $l \leq l^s_Y$ such that there exists an irreducible representation*

$$\rho\colon G_l(K) \to \mathrm{SU}(2).$$

(2) *If* $\Gamma_{-Y}(k) < \infty$, $l^k_Y < \infty$ *for some* $k \in \mathbb{Z}_{>0}$ *and* Y *is a Seifert hypersurface of* K, *then there exists a positive integer* $l(\leq l^k_Y)$ *such that there exists an irreducible representation*

$$\rho\colon G_l(K) \to \mathrm{SU}(2).$$

Proof of Theorem 1.8. The proofs of (1) and (2) are the same. We show (1). Theorem 1.6 implies

$$r_s(Y) \in \operatorname{Im} cs_{K,j} \cap (0,1) \tag{49}$$

for given $s \in [-\infty, 0]$ and some $j(\leq l_Y)$. We combine (49) and Lemma 2.12 and obtain the conclusion. ∎

8. Extendability of SU(2)-representations

Theorem 1.11 shows existence of finite irreducible SU(2)-representations. In this section, we provide a method to prove the existence of infinitely many irreducible SU(2)-representations.

Definition 8.1. Let (X, Y) be a pair consisting of a closed 4-manifold X and a codimension-1 smooth submanifold Y of X and let $\rho\colon \pi_1(Y) \to \mathrm{SU}(2)$ be a representation. If ρ is extended to a representation $\tilde{\rho}\colon \pi_1(X) \to \mathrm{SU}(2)$, then we call ρ an *extendable representation* for (X, Y).

By using our method, one can give a partial answer to the following question:

Question 8.2. For a given pair (X, Y), which SU(2)-representations of $\pi_1(Y)$ are extendable for (X, Y)?

The easiest examples are $(X = Y \times S^1, Y)$. In this case, all representations are extendable. For example, Theorem 1.13 proves that if X is negative definite and $Y = -\Sigma(2,3,5)$, then all SU(2)-representations of $\pi_1(Y)$ are extendable. The goal of this section is to prove the following theorem:

Theorem 8.3. *Let* Y *be an oriented homology* 3*-sphere and let* $\rho\colon \pi_1(Y) \to \mathrm{SU}(2)$ *be an* SU(2)*-representation of* $\pi_1(Y)$.

- *Suppose* $cs_Y(\rho) = r_s(Y) < \infty$ *and* $l^s_Y < \infty$ *for some* $s \in [-\infty, 0]$, *the Chern–Simons functional of* Y *is Morse–Bott at the level* $r_s(Y)$, *and for each component* $\bigcup_\alpha C_\alpha = cs_Y^{-1}(r_s(Y)) \cap R^*(Y)$, *we assume there exist Morse functions* $g_\alpha\colon C_\alpha \to \mathbb{R}$ *such that*

$$\{p \in C_\alpha \colon \mathrm{ind}_{g_\alpha}(p) + \mathrm{ind}(C_\alpha) = 1, d(g_\alpha)_p = 0\} = \emptyset$$

if $\rho \notin C_\alpha$ *and*

$$1 = \#\{p \in C_{\alpha_0} \colon \mathrm{ind}_{g_{\alpha_0}}(p) + \mathrm{ind}(C_{\alpha_0}) = 1, d(g_{\alpha_0})_p = 0\},$$

where C_{α_0} *is the component which contains* ρ *and* $\mathrm{ind}(C_\alpha)$ *is the Morse–Bott index of* C_α. *Let* X *be a closed negative definite* 4*-manifold containing* Y *as a submanifold. Then* ρ *is extendable for the pair* (X, Y). *Moreover, all* SU(2)*-representations which lie the same component of* ρ *are extendable for* (X, Y).

- *Suppose* $cs_Y(\rho) = \Gamma_{-Y}(k) < \infty$ *and* $l_Y^k < \infty$ *for some* $k \in \mathbb{Z}_{>0}$, *the Chern–Simons functional of* Y *is Morse–Bott at the level* $\Gamma_{-Y}(k)$, *and for each component* $\bigcup_\alpha C_\alpha = cs_Y^{-1}(\Gamma_{-Y}(k)) \cap R^*(Y)$, *we assume there exist Morse functions* $g_\alpha \colon C_\alpha \to \mathbb{R}$ *such that*

$$\left\{ p \in C_\alpha \colon \mathrm{ind}_{g_\alpha}(p) + \mathrm{ind}(C_\alpha) = \begin{cases} 1 & \textit{if } k \textit{ is odd} \\ 5 & \textit{if } k \textit{ is even} \end{cases} \mod 8, \right.$$
$$\left. d(g_\alpha)_p = 0 \right\} = \emptyset$$

if $\rho \notin C_\alpha$ *and*

$$1 = \#\left\{ p \in C_{\alpha_0} \colon \mathrm{ind}_{g_{\alpha_0}}(p) + \mathrm{ind}(C_{\alpha_0}) = \begin{cases} 1 & \textit{if } k \textit{ is odd} \\ 5 & \textit{if } k \textit{ is even} \end{cases} \mod 8, \right.$$
$$\left. d(g_{\alpha_0})_p = 0 \right\},$$

where C_{α_0} *is the component which contains* ρ *and* $\mathrm{ind}(C_\alpha)$ *is the Morse–Bott index of* C_α. *Let* X *be a closed negative definite* 4*-manifold containing* Y *as a submanifold. Then* ρ *is extendable for the pair* (X, Y). *Moreover, all* SU(2)*-representations which lie the same component of* ρ *are extendable for* (X, Y).

Proof. Put $r = r_s(Y) = cs_Y(\rho)$. Since $r_s(Y) < \infty$, by Theorem 1.11 we can suppose $0 \neq [Y] \in H_3(X; \mathbb{Z})$. First, we can assume the class $[Y]$ generates $H_3(X; \mathbb{R}) \cong \mathbb{R}$ by the same argument as in the proof of Theorem 7.1. Put $g' := \coprod g_\alpha \colon \coprod C_\alpha = cs_Y^{-1}(r) \cap R^*(Y) \to \mathbb{R}$. We fix another Morse function $g \colon cs_Y^{-1}(r) \cap R^*(Y) \to \mathbb{R}$ obtained by deforming the Morse function g' satisfying the following conditions:

- the point $\rho \in C_\alpha$ is a critical point of g whose Morse index is $1 - \mathrm{ind}(C_{\alpha_0})$,
- $\{p \in C_\alpha \colon \mathrm{ind}_{g_{C_\alpha}}(p) + \mathrm{ind}(C_\alpha) = 1, d(g_{C_\alpha})_p = 0\} = \emptyset$ if $\alpha \neq \alpha_0$ and
- $1 = \#\{p \in C_{\alpha_0} \colon \mathrm{ind}_{g_{C_{\alpha_0}}}(p) + \mathrm{ind}(g_{C_{\alpha_0}}) = 1, d(g_{C_{\alpha_0}})_p = 0\}$.

By using Lemma 6.2, we take a Morse–Bott type perturbation $\pi_\varepsilon(g)$ for $\varepsilon \in (0, \varepsilon_0)$. We take a sequence of non-generate regular perturbations $\{\pi_n\}$ such that

$$cs_{\pi_{1/n}(g)}|_U = cs_{\pi_n}|_U \quad \text{and} \quad \|\pi_n\| \to 0,$$

where U is a neighborhood of $cs_Y^{-1}(r) \cap R^*(Y)$ in $B^*(Y)$. By Lemma 6.2, we can see $\{\pi_n\}$ satisfies the following conditions:

- there is a correspondence

$$L_n\colon \{\text{the critical point set of } g\colon cs_Y^{-1}(r) \cap R^*(Y) \to \mathbb{R}\}$$
$$\to \{\text{critical point set of } cs_{\pi_n}|_U\}$$

  for each n such that $\lim_{n\to\infty} L_n(a) = a$,
- $\#\{a \in \widetilde{R}^*_{\pi_n}(Y)\colon \mathrm{ind}(a) = 1, |\, cs_{\pi_n}(a) - r| < \frac{1}{2}\lambda_Y\} = 1$, and
- $\{\rho\} = \{p\colon \text{the critical point set of } g\colon \lim_{n\to\infty} L_n(p) \in C_{\alpha_0}\}$.

Thus, we conclude that $l_Y^s = 1$. Then we apply Theorem 7.3 and obtain an irreducible flat connection A_∞ on $W[0,0]$ such that

$$cs^1_{X,[Y]}(A_\infty) = r_s(Y).$$

Moreover, by the proof of Theorem 7.3, we can see that

$$A_\infty|_{Y_0^+} \cong A_\infty|_{Y_0^-} \cong \rho.$$

Thus, A_∞ gives an extension of ρ. Moreover, for another point ρ' which lies in the same component of ρ, we can take another Morse function g with the following conditions:

- the point $\rho' \in C_\alpha$ is a critical point of g whose Morse index is $1 - \mathrm{ind}(C_{\alpha_0})$,
- $\{p \in C_\alpha\colon \mathrm{ind}_{g_{C_\alpha}}(p) + \mathrm{ind}(C_\alpha) = 1, d(g_{C_\alpha})_p = 0\} = \emptyset$ if $\alpha \neq \alpha_0$ and
- $1 = \#\{p \in C_{\alpha_0}\colon \mathrm{ind}_{g_{C_{\alpha_0}}}(p) + \mathrm{ind}(g_{C_{\alpha_0}}) = 1, d(g_{C_{\alpha_0}})_p = 0\}$

by modifying g'. Then, by the same discussion, we have an extension A_∞ of ρ'. ■

We give a sufficient condition for the assumptions of Theorem 8.3 to hold.

Corollary 8.4. *Let Y be an oriented homology 3-sphere and let $\rho\colon \pi_1(Y) \to \mathrm{SU}(2)$ be an $\mathrm{SU}(2)$-representation of $\pi_1(Y)$.*

(1) *Let l be a positive integer. Suppose $cs_Y(\rho) = r_s(Y) < \infty$ and $l_Y^s = 1$ for some $s \in [-\infty, 0]$, the Chern–Simons functional of Y is Morse–Bott at the level $r_s(Y)$, $H_*(cs_Y^{-1}(r_s(Y)) \cap R^*(Y); \mathbb{Z}) \leq 1$ for any $*$ and $cs_Y^{-1}(r_s(Y)) \cap R^*(Y)$ has a perfect Morse function. Let X be a closed negative definite 4-manifold containing Y as a submanifold. Then ρ is extendable for the pair (X, Y).*

Moreover, all $\mathrm{SU}(2)$*-representations which lie the same component of* ρ *are extendable for* (X, Y).

(2) *Let* l *be a positive integer. Suppose* $cs_Y(\rho) = \Gamma_{-Y}(k) < \infty$ *and* $l_Y^k = 1$ *for some* $k \in \mathbb{Z}_{>0}$*, the Chern–Simons functional of* Y *is Morse–Bott at the level* $\Gamma_{-Y}(k)$*,*

$$H_*\big(cs_Y^{-1}(\Gamma_{-Y}(k)) \cap R^*(Y); \mathbb{Z}\big) \leq 1$$

for any $*$ *and* $cs_Y^{-1}(r_s(Y)) \cap R^*(Y)$ *has a perfect Morse function.Let* X *be a closed negative definite* 4*-manifold containing* Y *as a submanifold. Then* ρ *is extendable for the pair* (X, Y)*. Moreover, all* $\mathrm{SU}(2)$*-representations which lie the same component of* ρ *are extendable for* (X, Y).

Proof. We check that the assumptions of Theorem 8.3 are satisfied. By assumption, we have a Morse function $g: cs_Y^{-1}(r_s(Y)) \cap R^*(Y) \to \mathbb{R}$ such that the differentials of the Morse complex with respect to g are zero. Since

$$H_*(cs_Y^{-1}(r_s(Y)) \cap R^*(Y); \mathbb{Z}) \leq 1,$$

we have

$$\#\{p \in cs_Y^{-1}(r_s(Y)) \cap R^*(Y) \colon \mathrm{ind}(p) = *, dg_p = 0\} \leq 1 \quad \text{for all } * \in \mathbb{Z}_{\geq 0}.$$

Note that $cs_Y^{-1}(r_s(Y)) \cap R^*(Y)$ is connected. Then we use a Morse–Bott type perturbation $\pi_\varepsilon(g)$ for $\varepsilon \in (0, \varepsilon_0)$ given in Lemma 6.2 to define filtered instanton chain complexes $\mathrm{CI}_*^{[s,r]}(Y)$ given in Section 2.4.1. Since the value of $r_s(Y)$ depends only on the index-1-critical points of $cs_{Y,\pi_\varepsilon(g)}$ in $cs_Y^{-1}(r_s(Y)) \cap R^*(Y)$, we have

$$\begin{aligned}\#\big\{p \in cs_Y^{-1}(r_s(Y)) \cap R^*(Y) \colon\ & \mathrm{ind}_g(p) + \mathrm{ind}_{cs_Y}\big(cs_Y^{-1}(r_s(Y)) \cap R^*(Y)\big) = 1, \\ & p \in C_\alpha, dg_p = 0\big\} \geq 1.\end{aligned}$$

We suppose that g has ρ as a critical point satisfying $\mathrm{ind}_g(p) + \mathrm{ind}_{cs_Y}(C_\alpha) = 1$. Thus, the assumptions of 1 of Theorem 8.3 are satisfied. A similar argument proves 2. ∎

9. Examples and applications

In this section, we give several examples of computations of l_Y, l_Y^s and l_Y^k and applications of main theorems in Section 1.

9.1. Calculations of l_Y, l_Y^s, and l_Y^k

In this section, we give several computations of l_Y, l_Y^s and l_Y^k.

9.1.1. Seifert homology 3-spheres. In this section, we discuss the invariants l^s_Y, l^k_Y and l_Y for Seifert homology 3-spheres of type $\Sigma(a_1, \dots, a_n)$. In [16], the R-invariant of $\Sigma(a_1, \dots, a_n)$:

$$R(a_1, \dots, a_n) = \frac{2}{a} - 3 + n + \sum_{i=1}^{n} \frac{2}{a_i} \sum_{k=1}^{a_i-1} \cot\Big(\frac{a\pi k}{a_i}\Big) \cot\Big(\frac{\pi k}{a_i}\Big) \sin^2\Big(\frac{\pi k}{a_i}\Big) \qquad (50)$$

is introduced, where $a = a_1 \cdots a_n$. In [16], Fintushel and Stern gave an algorithm to calculate SU(2)-representation spaces $R(\Sigma(a_1, \dots, a_n))$. Fix a sequence of integers $b, b_1, \dots, b_n \in \mathbb{Z}$ such that

$$a \sum_{1 \le k \le n} \frac{b_k}{a_k} = 1 + ab.$$

We call $(b, (a_1, b_1), \dots, (a_n, b_n))$ a *Seifert invariant*. For a such sequence, the fundamental group of $\Sigma(a_1, \dots, a_n)$ is given by

$$\pi_1(\Sigma(a_1, \dots, a_n)) = \{x_1, \dots, x_n, h \colon [x_i, h] = 1, x_i^{a_i} = h^{-b_i}, x_1 \cdots x_n = 1\}.$$

Note that if $\rho\colon \pi_1(\Sigma(a_1, \dots, a_n)) \to \mathrm{SU}(2)$ is an irreducible representation, then $\rho(h) = \pm 1$. Suppose that a_1 is even. We choose $\{b_i\}$ so that b_j is even for $j \neq 1$. The number l_i is even if and only if $\rho(h)^{b_i} = 1$. Then connected components of the SU(2)-representations ρ of $\pi_1(\Sigma(a_1, \dots, a_n))$, imposing some technical conditions, are parametrized by the rotation numbers $(l_1, \dots, l_n)$ given by

$$g_i^{-1} \rho(x_i) g_i = \begin{pmatrix} e^{\pi i l_i/a_i} & 0 \\ 0 & e^{-\pi i l_i/a_i} \end{pmatrix} \quad \text{for some } g_i \in \mathrm{SU}(2)\ (1 \le i \le n)$$

and $0 \le l_i \le a_i$. For this parametrization, the value of the Chern–Simons functional is then given by

$$cs(\rho(l_1, \dots, l_n)) = \Big(\sum_{i=1}^{n} a l_i/a_i\Big)^2 / 4a \mod 1. \qquad (51)$$

Moreover, the Floer index[9] $\mathrm{ind}(\rho_{(l_1,\dots,l_n)})$ is given by

$$\begin{aligned} \mathrm{ind}(\rho_{(l_1,\dots,l_n)}) = {} & \frac{2e^2}{a} + 3 - m \\ & + \sum_{i=1}^{m} \frac{2}{a_i} \sum_{k=1}^{a_i-1} \cot\Big(\frac{a\pi k}{a_i}\Big) \cot\Big(\frac{\pi k}{a_i}\Big) \sin^2\Big(\frac{\pi e k}{a_i}\Big) \mod 8. \end{aligned}$$

[9]When the Chern–Simons functional of an oriented homology 3-sphere Y is Morse–Bott, we can also define a Morse-index. (See [16].)

Example 9.1. Suppose that $n = 3$. Since $R(2, 3, 6k - 1) = 1$, it is proved in [11, Theorem 6] and [35, Corollary 1.4] that

$$\Gamma_{\Sigma(2,3,6k-1)}(1) = r_s(-\Sigma(2, 3, 6k - 1)) = \frac{1}{24(6k - 1)}$$

for any $s \in [-\infty, 0]$. Since $\lambda(\Sigma(2, 3, 6k - 1)) = k$,

$$l_{\Sigma(2,3,6k-1)} = 2k$$

by Theorem 6.5. Moreover, it is shown in [16] that the Chern–Simons functional of $\Sigma(2, 3, 6k - 1)$ is Morse. Then (l_1, l_2, l_3) determines a representation if and only if

$$|l_1/a_1 - l_2/a_2| < l_3/a_3 < 1 - |1 - (l_1/a_1 - l_2/a_2)|.$$

Here, we choose $a_1 = 2$, $a_2 = 3$, $a_3 = 6k - 1$. Then, the space of the representations are parametrized by

$$R^*(\Sigma(2, 3, 6k - 1)) = \big\{(1, 2, l_3) \colon l_3 \in \{k, k + 1, \ldots, 5k - 1\} \cap 2\mathbb{Z}\big\}. \tag{52}$$

Then the Chern–Simons functional is given by

$$cs(\rho_{(1,2,l_3)}) = \frac{12(3k^2 - k + 3l_3^2) + 1}{24(6k - 1)} \bmod 1.$$

If k is odd, $\rho_{(1,2,5k-1)}$ has the unique minimal value $\frac{1}{6(6k-1)}$ of cs in $(0, 1]$ and if k is even, $\rho_{(1,2,k)}$ has the unique minimal value $\frac{1}{6(6k-1)}$ of cs in $(0, 1]$. Therefore, we can conclude that

$$l^1_{\Sigma(2,3,6k-1)} = l^s_{\Sigma(2,3,6k-1)} = 1$$

for any $s \in [-\infty, 0]$. In the case of $k = 1$, we can also see that $l^2_{\Sigma(2,3,5)} = 1$.

Example 9.2. Suppose $n = 4$, $a_1 = 2$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 7$. One can check that $R(2, 3, 5, 7) = 1$. It is shown that

$$\Gamma_{\Sigma(2,3,5,7)}(1) = r_s(-\Sigma(2, 3, 5, 7)) = \frac{1}{840}$$

for any $s \in [-\infty, 0]$. The space $R^*(\Sigma(2, 3, 5, 7))$ has six S^2-components and 16 points. We put critical values of the Chern–Simons functional for $-\Sigma(2, 3, 5, 7)$ in Table 1 which are given in [16, 42]. In Table 1, the Floer indices for Morse–Bott components with dimension greater than 0 are described as bold numbers. Table 1 implies that $\nu(-\Sigma(2, 3, 5, 7)) = r_s(-\Sigma(2, 3, 5, 7))$ for any $s \in [-\infty, 0]$.

Next, we consider the finite connected sums of Seifert homology 3-spheres.

(l_1, l_2, l_3, l_4)	cs	ind	(l_1, l_2, l_3, l_4)	cs	ind
$(1, 0, 2, 2)$	$681/840$	7	$(0, 2, 4, 2)$	$184/840$	3
$(1, 0, 2, 4)$	$561/840$	5	$(2, 2, 2, 2)$	$436/840$	5
$(1, 0, 2, 6)$	$81/840$	1	$(2, 2, 2, 4)$	$316/840$	3
$(1, 0, 4, 4)$	$729/840$	7	$(2, 2, 4, 4)$	$484/840$	5
$(1, 2, 0, 2)$	$625/840$	7	$(2, 2, 4, 6)$	$4/840$	1
$(1, 2, 0, 4)$	$505/840$	5	$(1, 2, 2, 2)$	$121/840$	1
$(1, 2, 2, 0)$	$721/840$	7	$(1, 2, 2, 4)$	$1/840$	7
$(1, 2, 4, 0)$	$49/840$	1	$(1, 2, 2, 6)$	$361/840$	3
$(0, 2, 2, 2)$	$16/840$	1	$(1, 2, 4, 2)$	$289/840$	3
$(0, 2, 2, 4)$	$736/840$	7	$(1, 2, 4, 4)$	$169/840$	1
$(0, 2, 2, 6)$	$256/840$	3	$(1, 2, 4, 6)$	$529/840$	5

Table 1. Critical values of $-\Sigma(2,3,5,7)$. The usual instanton group of $-\Sigma(2,3,5,7)$ is isomorphic to $\mathbb{Z}^{28}$ and the Casson invariant is 14. This implies that $l_{-\Sigma(2,3,5,7)} = 28$. On the other hand, for $s \in [-\infty, 0]$, $l^s_{-\Sigma(2,3,5,7)} = 1$. Moreover, if we let Y be the connected sum $\#_n(-\Sigma(2,3,5,7))$ for a positive n, then the connected sum formula of $r_0(Y)$ implies that $r_0(Y) = \frac{1}{840}$. Thus, we can also see that $l^0_Y = 1$.

Example 9.3. Fix $k \in \mathbb{Z}_{>0}$. Let Y be the linear combination

$$n\Sigma(2,3,6k-1)\#(\#_{6(6k-1)<a_1\cdots a_n} m_{(a_1,\ldots,a_n)}\Sigma(a_1,\ldots,a_n)),$$

where $n < 0$, $k \in \mathbb{Z}_{>0}$ and $m_{(a_1,\ldots,a_n)}$ is a sequence of integer parametrized by the list $(a_1,\ldots,a_n)$ with

$$a_1 \cdots a_n < 24(6k-1).$$

Then Theorem 2.18 (21) implies

$$\begin{aligned}\frac{1}{24(6k-1)} &= r_0(n\Sigma(2,3,6k-1)) \\ &\geq \min\{r_0(Y), r_0(\#_{6(6k-1)<a_1\cdots a_n} m_{(a_1,\ldots,a_n)}\Sigma(a_1,\ldots,a_n))\}.\end{aligned}$$

By the condition $6(6k-1) < a_1 \cdots a_n$ and formula (51), we conclude that

$$\frac{1}{24(6k-1)} = r_0(Y).$$

Moreover, we can see that

$$\# cs_Y^{-1}\Big(\frac{1}{24(6k-1)}\Big) = 1.$$

Therefore,

$$l_Y^0 = 1.$$

Because of the choice of $(a_1, \dots, a_n)$, we have

$$\frac{1}{24(6k-1)} = r_0(Y).$$

9.1.2. A hyperbolic homology 3-sphere. In [35], Nozaki, Sato, and the author tried to calculate $r_s(Y)$ for the hyperbolic 3-manifold obtained along $1/2$-surgery along the mirror image of 5_2.

Example 9.4. Let Y be $S^3_{1/2}(5_2^*)$. The Casson invariant of Y is 4. In [35], it is checked that all critical points of cs of Y are non-degenerate and there are eight SU(2)-representations. Moreover, approximate critical values of cs are given in [35, Table 2]. We put the minimal value of cs of $S^3_{1/2}(5_2^*)$ in $(0, 1]$ by t_0. Then, $r_s(Y) = \Gamma_{-Y}(1) = t_0$ for any $s \in [-\infty, 0]$. Since $\# cs^{-1}(t_0) = 1$, we can conclude that

$$l_Y^s = l_Y^1 = 1$$

for any $s \in [-\infty, 0]$. The manifold $S^3_{1/2}(5_2^*)$ also satisfies $\nu(S^3_{1/2}(5_2^*)) = r_s(S^3_{1/2}(5_2^*))$ for any $s \in [-\infty, 0]$.

9.2. Applications

In this section, we give several applications of the main theorems Proposition 1.2, Theorem 1.6, Theorem 1.8, Theorem 1.11 and Theorem 1.15.

9.2.1. Ribbon 2-knots. The class of ribbon 2-knots is one of the fundamental classes of 2-knots. There are several studies of the properties of ribbon 2-knots [29, 45–47]. For example, it is shown in [45] that the knot group $G(K)$ is torsion free for any ribbon 2-knot K. In [29], Kawauchi gave a characterization of Alexander modules of ribbon 2-knots. Moreover, every ribbon 2-knot has a finite connected sum of copies of $S^1 \times S^2$ ([46]) as a Seifert hypersurface. However, the classification problem for ribbon 2-knots is open. There several invariants which obstruct the ribbon property of 2-knots. Ruberman ([38]) introduced the *Gromov norm* $|K| \in [0, \infty)$ of 2-knots K by using the Gromov norm of Seifert hypersurfaces. This invariant satisfies the inequality

$$|K| \leq |Y|$$

for any Seifert hypersurface Y of K, where $|Y|$ is the Gromov norm of Y. If K is a ribbon 2-knot, then $|K| = 0$. Moreover, the following gauge theoretic invariants can be useful in obstructing the ribbon property of 2-knots:

(1) Mrowka, Ruberman, and Saveliev ([34]) constructed a version of Seiberg–Witten invariant $\lambda_{\mathrm{MRS}}(X)$ for homology $S^1 \times S^3$'s. Then one define

$$\lambda_{\mathrm{MRS}}(K) := \lambda_{\mathrm{MRS}}(X(K)).$$

Moreover, if K is ribbon, then $\lambda_{\mathrm{MRS}}(K) = 0$ ([33]).

(2) Levine and Ruberman ([32]) defined a $\mathbb{Z}$-valued invariant $\tilde{d}(X, y)$ of homology $S^1 \times S^3$ by considering some variant of d-invariant of cross sections. Using $\tilde{d}(X, y)$, Levine and Ruberman defined invariants of 2-knots $\tilde{d}(K)$, $\tilde{d}(-K)$, $\tilde{d}(\bar{K})$ and $\tilde{d}(-\bar{K})$. They show that

$$\tilde{d}(K) = \tilde{d}(-K) = \tilde{d}(\bar{K}) = \tilde{d}(-\bar{K}) = 0$$

if K is ribbon.

Remark 9.5. Furuta and Ohta ([21]) also constructed a Casson type invariant $\lambda_{\mathrm{FO}}(X)$ of homology $S^1 \times S^3$'s satisfying the condition $H_*(\widetilde{X};\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$. Such homology $S^1 \times S^3$'s are called $\mathbb{Z}[\mathbb{Z}]$ *homology* $S^1 \times S^3$. It is conjectured that $\lambda_{\mathrm{FO}}(X) = -\lambda_{\mathrm{MRS}}(X)$ for any $\mathbb{Z}[\mathbb{Z}]$ homology $S^1 \times S^3 X$. In [33], this conjecture is checked for a certain class of mapping tori. One can also define $\lambda_{\mathrm{FO}}(K)$ for 2-knots K satisfying the condition $\Delta_K(t) = 1$. By construction, if $\lambda_{\mathrm{FO}}(K) \neq 0$, then $G(K)$ has an $\mathrm{SU}(2)$-irreducible representation.

Our invariants $\{\mathrm{Im}\, cs_{K,j}\}_{k \in \mathbb{Z}_{>0}}$ are also useful for obstructing the ribbon property of 2-knots.

Corollary 9.6. *Let K be an oriented 2-knot and Y a Seifert hypersurface of K.*

(1) *If Y is an oriented homology 3-sphere, $l_Y^s < \infty$, $r_s(Y) < \infty$ and $r_s(Y) \notin \mathbb{Z}$ for some $s \in [-\infty, 0]$, then K is not ribbon.*

(2) *If Y is an oriented homology 3-sphere, $l_Y^k < \infty$, $\Gamma_{-Y}(k) < \infty$ and $\Gamma_{-Y}(k) \notin \mathbb{Z}$ for some $k \in \mathbb{Z}_{>0}$, then K is not ribbon.*

Proof. We only prove (1) since (2) is similar. Suppose such a Seifert hypersurface Y of K exists. Then Theorem 1.6 implies that

$$1 \neq r_s(Y) - \lfloor r_s(Y) \rfloor \in \mathrm{Im}\, cs_{K,j_0}$$

for some $j_0 \in \mathbb{Z}_{>0}$. If K is a ribbon 2-knot, then $\mathrm{Im}\, cs_{K,j} = \{1\}$ for any j by Corollary 1.3. This contradicts to $r_s(Y) - \lfloor r_s(Y) \rfloor \in \mathrm{Im}\, cs_{K,j_0} \cap (0, 1) \neq \emptyset$. ∎

Remark 9.7. Note that if K admits a homology 3-sphere as a Seifert hypersurface, then $\Delta_K(t) = 1$.

The following corollary seems difficult to show using $|K|$, $\lambda_{\mathrm{MRS}}(K)$, $\lambda_{\mathrm{FO}}(K)$ and $\tilde{d}(K)$:

Corollary 9.8. *Let k be a positive integer. Any 2-knot K having*

$$(-\Sigma(2,3,6k+5))\#\Sigma(2,3,5)$$

as a Seifert hypersurface is not ribbon.

Proof. By Example 9.3,

$$r_0((-\Sigma(2,3,6k+5))\#\Sigma(2,3,5)) < 1 \quad \text{and} \quad l^0_{(-\Sigma(2,3,6k+5))\#\Sigma(2,3,5)} = 1.$$

Therefore, Corollary 9.6 tells us that such a knot K cannot be a ribbon. ∎

9.2.2. Rigidity of $\{\mathrm{Im}\, cs_{K,j}\}_{j\in\mathbb{Z}_{>0}}$. The invariant $\tilde{d}(K)$ is determined by the d-invariant of a Seifert hypersurface. We will see that similar properties hold for $\mathrm{Im}\, cs_{K,j}$ of a certain class of 2-knots. We give a sufficient condition on Seifert hypersurfaces to determine $\mathrm{Im}\, cs_{K,j}$.

Theorem 9.9. *Suppose that an oriented homology 3-sphere Y is a Seifert hypersurface of a given oriented 2-knot K and there exist $k_1,\dots,k_m \in \mathbb{Z}_{>0}$ and $s_1,\dots,s_n \in [-\infty, 0]$ such that*

$$\bigcup_{\substack{k_1,\dots,k_m,\\ s_1,\dots,s_n}} (r_s(Y) \cup \Gamma_{-Y}(k)) = \Lambda_Y \cap (0,1)$$

and $l_Y^{s_i} = l_Y^{l_i} = 1$ for any i. Then

$$\mathrm{Im}\, cs_{K,j} = \Lambda_Y \cap (0,1)$$

for any $j \in \mathbb{Z}_{>0}$. In particular, for any Seifert hypersurface Y' and $r \in \Lambda_Y \cap (0,1)$, there exists an $\mathrm{SU}(2)$-representation ρ_r on Y' such that $cs_{Y'}(\rho_r) = r$.

Proof. By Proposition 1.2 and Theorem 1.6,

$$\Lambda_Y \cap (0,1] = \bigcup_{\substack{k_1,\dots,k_m\\ s_1,\dots,s_n}} (r_s(Y) \cup \Gamma_{-Y}(k)) \subset \mathrm{Im}\, cs_{K,1} \subset \Lambda_Y \cap (0,1].$$

This implies that $\mathrm{Im}\, cs_{K,1} = \Lambda_Y \cap (0,1]$. We use Proposition 1.2 again and obtain

$$\Lambda_Y \cap (0,1] = \mathrm{Im}\, cs_{K,1} \subset \mathrm{Im}\, cs_{K,j} \subset \Lambda_Y \cap (0,1].$$

This completes the proof. ∎

Corollary 9.10. *For any 2-knot having $-\Sigma(2,3,5)$ as a Seifert hypersurface,*

$$\operatorname{Im} cs_{K,j} = \left\{\frac{1}{120}, \frac{49}{120}, 1\right\}$$

for any $j \in \mathbb{Z}_{>0}$. For such a 2-knot K and its Seifert hypersurface Y', Y' has at least two $\mathrm{SU}(2)$-representations of $\pi_1(Y')$.

Proof. We check that $-\Sigma(2,3,5)$ satisfy the assumption of Theorem 9.9. In [11], Daemi proved

$$\Gamma_{\Sigma(2,3,5)}(1) = \frac{1}{120} \quad \text{and} \quad \Gamma_{\Sigma(2,3,5)}(2) = \frac{49}{120}.$$

Note that $\Lambda_{-\Sigma(2,3,5)} \cap (0,1] = \{\frac{1}{120}, \frac{49}{120}, 1\}$ and the two elements in $R^*(\Sigma(2,3,5))$ are non-degenerate. Therefore, $l^1_{\Sigma(2,3,5)} = l^2_{\Sigma(2,3,5)} = 1$. ■

9.2.3. Seifert hypersurfaces of 2-knots. In this section, we treat the following problem: what are the Seifert hypersurfaces for a given 2-knot? Solving this problem has two parts. The first part is the construction of Seifert hypersurfaces of a given oriented 2-knot. For twisted spun 2-knots, Zeeman ([49]) constructed Seifert hypersurfaces. In general, for a given 2-knot, there are several formulations of diagrams of them containing the motion picture, the ch-diagram and the surface diagram ([26]). For such diagrams, there are several ways to construct Seifert hypersurfaces ([7,8]). The second part is to obstruct the existence of a certain class of 3-manifolds as Seifert hypersurfaces.

Theorem 9.11. *Let (p,q) be a relative prime pair. Let $k(p/q)$ be a 2-bridge knot such that $\Sigma^2(k(p/q)) = L(p,q)$, where $\Sigma^2(k)$ is the double branched cover of $k \subset S^3$.*

(1) *For an oriented 3-manifold Y, if the twisted spun knot $K(k(p/q),2)$ has Y as a Seifert hypersurface, then*

$$\left\{-\frac{n^2 r}{p} \bmod 1 : 0 \le n \le \left\lceil \frac{p}{2} \right\rceil\right\} \subset \Lambda_Y \cap (0,1],$$

where r is any integer satisfying $qr \equiv -1 \mod p$.

(2) *If Y is an oriented homology 3-sphere and $r_s(Y) < \infty$, $l^s_Y = 1$ for some $s \in [-\infty, 0]$ and*

$$r_s(Y) - \lfloor r_s(Y) \rfloor \notin \left\{-\frac{n^2 r}{p} \bmod 1 : 0 \le n \le \left\lceil \frac{p}{2} \right\rceil\right\},$$

then $K(k(p/q),2)$ does not have Y as a Seifert hypersurface, where r is any integer satisfying $qr \equiv -1 \mod p$.

(3) *If Y is an oriented homology 3-sphere, $\Gamma_{-Y}(k) < 1$,*

$$\Gamma_{-Y}(k) - \lfloor \Gamma_{-Y}(k) \rfloor \notin \left\{ -\frac{n^2 r}{p} \bmod 1 \colon 0 \le n \le \left\lceil \frac{p}{2} \right\rceil \right\},$$

and $l_Y^k = 1$ for some $k \in \mathbb{Z}_{>0}$, then $K(k(p/q), 2)$ does not have Y as a Seifert hypersurface, where r is any integer satisfying $qr \equiv -1 \mod p$.

Proof. This is a corollary of Proposition 1.4, Proposition 1.2 and Theorem 1.6. ■

As a corollary, we have the following result:

Corollary 9.12. *Let (p, q) be a relative prime pair. Let $k(p/q)$ be a 2-bridge knot such that $\Sigma^2(k(p/q)) = L(p, q)$, where $\Sigma^2(k)$ is the double branched cover of $k \subset S^3$. Then any oriented homology 3-sphere given in Example 9.3 with $6n - 1 > p$ cannot be a Seifert hypersurface of $K(k(p/q), 2)$.*

Proof. This is a corollary of Example 9.3 and Theorem 9.11. ■

9.2.4. Embedding from homology 3-spheres into 4-manifolds. In this section, we treat the existence problem of embeddings from 3-manifolds into 4-manifolds. This problem has been studied in several situations [13, 22, 24, 25, 28]). Here, we develop a gauge theoretic method and focus on a certain class of homology 3-spheres and negative definite 4-manifolds. We give a relationship between existence of embeddings and SU(2)-representations of fundamental groups. First, we prove Theorem 1.13.

Theorem 9.13 (Theorem 1.13). *Suppose X is a negative definite 4-manifold containing $\Sigma(2, 3, 5)$ as a submanifold. Then*

$$\operatorname{Im} cs_{X,[-\Sigma(2,3,5)]} = \left\{ \frac{1}{120}, \frac{49}{120}, 1 \right\} \subset (0, 1].$$

In particular, there exist at least four irreducible $\mathrm{SU}(2)$-representations of $\pi_1(X)$.

To prove Theorem 1.13, we will show the following theorem:

Theorem 9.14. *Let X be a negative definite 4-manifold. Suppose that an oriented homology 3-sphere Y is embedded in X and there exists $k_1, \dots, k_m \in \mathbb{Z}_{>0}$ and $s_1, \dots, s_n \in [-\infty, 0]$ such that*

$$\bigcup_{\substack{k_1,\dots,k_m, \\ s_1,\dots,s_n}} (r_s(Y) \cup \Gamma_{-Y}(k)) = \Lambda_Y \cap (0, 1)$$

and $l_Y^{s_i} = l_Y^{l_i} = 1$ for any i. Then

$$\operatorname{Im} cs^j_{X,[Y]} = \Lambda_Y \cap (0, 1)$$

for any $j \in \mathbb{Z}_{>0}$*. In particular, for any other embedding* $Y' \subset X$ *and* $r \in \Lambda_Y \cap (0,1)$ *with* $[Y'] = [Y]$*, there exists an* $\mathrm{SU}(2)$*-representation* ρ_r *on* Y' *such that* $cs_{Y'}(\rho_r) = r$*.*

Proof. By Theorem 1.11 and Proposition 1.12,

$$\Lambda_Y \cap (0,1] = \bigcup_{\substack{k_1,\dots,k_m\\ s_1,\dots,s_n}} (r_s(Y) \cup \Gamma_{-Y}(k)) \subset \operatorname{Im} cs^1_{X,[Y]} \subset \Lambda_Y \cap (0,1].$$

This implies that $\operatorname{Im} cs^1_{X,[Y]} = \Lambda_Y \cap (0,1]$. We use Lemma 2.10 and obtain

$$\Lambda_Y \cap (0,1] = \operatorname{Im} cs^1_{X,[Y]} \subset \operatorname{Im} cs^j_{X,[Y]} \subset \Lambda_Y \cap (0,1].$$

This completes the proof. ∎

Here we give a proof of Theorem 1.13.

Proof of Theorem 1.13. We check that $-\Sigma(2,3,5)$ satisfies the assumption of Theorem 9.14. The proof is written in Corollary 9.10. ∎

Theorem 9.15 (Theorem 1.15). *Let* Y *be a Seifert homology* 3*-sphere of type* $\Sigma(a_1, \dots, a_n)$ *with*[10]

$$\Lambda^*_{\Sigma(a_1,\dots,a_n)} \cap \mathbb{Z} = \emptyset.$$

Suppose the Frøyshov invariant $h(Y)$ *of* Y *is non-zero. Then* Y *cannot be embedded in any negative definite* 4*-manifold* X *such that the* $\mathrm{SU}(2)$*-representation space* $R(X_{j,c})$ *of* $X_{j,c}$ *is connected for any* j*.*

Proof of Theorem 1.15. It is shown that $\Gamma_{\Sigma(a_1,\dots,a_n)}(1) < \infty$ if $h(\Sigma(a_1,\dots,a_n)) > 0$ in [11]. Since the $\mathrm{SU}(2)$-Chern–Simons functional of $Y = \Sigma(a_1,\dots,a_n)$ is Morse–Bott, $l_Y < \infty$. By Theorem 1.11, there exists $l \leq l_Y$ such that

$$r_{-\infty}(Y) - \lfloor r_{-\infty}(Y) \rfloor = \Gamma_Y(1) - \lfloor \Gamma_Y(1) \rfloor \in \operatorname{Im} cs^l_{X,[\Sigma(a_1,\dots,a_n)]}.$$

Here, $\Gamma_Y(1) \in \Lambda^*_Y$ and the formula (51) implies that $1 \neq \Gamma_Y(1) - \lfloor \Gamma_Y(1) \rfloor \in (0,1]$. This implies that

$$\operatorname{Im} cs^l_{X,[\Sigma(a_1,\dots,a_n)]} \cap (0,1) \neq \emptyset.$$

Suppose that $R(X_{j,c})$ is connected. Then, since $cs^j_{X,c}$ is locally constant, we see that $\operatorname{Im} cs^j_{X,c} = \{1\}$. ∎

As a corollary of Corollary 8.4, we can show that several $\mathrm{SU}(2)$-representations on some classes of Seifert homology 3-spheres are extendable.

[10] This condition can be seen as a combinatorial condition via (51).

Corollary 9.16. *Let (X, Y) be a couple consisting of a closed connected negative definite 4-manifold X and an oriented codimension-1-submanifold Y of X with $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$ and $\rho\colon \pi_1(Y) \to \mathrm{SU}(2)$ be an $\mathrm{SU}(2)$-representation of $\pi_1(Y)$.*

- *Suppose k is odd and n is a positive integer. If $Y = -\Sigma(2,3,6k-1)$ and ρ is a representation corresponding to $\rho_{(1,2,5k-1)}$ in (52), then ρ is extendable for (X, Y).*
- *Suppose k is even n is a positive integer. If $Y = -\Sigma(2,3,6k-1)$ and ρ is a representation corresponding to $\rho_{(1,2,k)}$ in (52), then ρ is extendable for (X, Y).*
- *If $Y = -\Sigma(2,3,5,7)$ and ρ is any representation corresponding to $(1,1,2,4)$, then ρ is extendable for (X, Y).*

Proof. The following calculations have been checked in Section 9:

- if k is odd, then $l^s_{-\Sigma(2,3,6k-1)} = 1$,
$$cs_Y(\rho_{(1,2,5k-1)}\#\theta\#\cdots\#\theta) = \frac{1}{24(6k-1)} = r_s(-\Sigma(2,3,6k-1)),$$
and $cs^{-1}(\frac{1}{24(6k-1)})$ is the one point;
- if k is even, then $l^s_{-\Sigma(2,3,6k-1)} = 1$,
$$cs_Y(\rho_{(1,2,k)}\#\theta\cdots\#\theta) = \frac{1}{24(6k-1)} = r_s(-\Sigma(2,3,6k-1)),$$
and $cs^{-1}(\frac{1}{24(6k-1)})$ is the one point;
- $l^s_{-\Sigma(2,3,5,7)} = 1$,
$$cs_Y(\rho_{(1,1,2,4)}\#\theta\#\cdots\#\theta) = \frac{1}{840} = r_s(-\Sigma(2,3,5,7)),$$
and $cs_Y^{-1}(\frac{1}{840}) \cap \widetilde{R}^*(Y) \cong S^2$. Of course, S^2 has a perfect Morse function.

Therefore, in each case, we can apply Corollary 8.4. ∎

In the case of $Y = -\Sigma(2,3,5,7)$, the SU(2)-representations corresponding to $(1,1,2,4)$ are parametrized by S^2. Therefore, one can prove Theorem 1.14.

Theorem 9.17 (Theorem 1.14). *Let X be a closed definite 4-manifold X containing $\Sigma(2,3,5,7)$ as a submanifold. Then there is an S^2-component C in $R^*(\Sigma(2,3,5,7))$ such that all elements in C can be extended to X. In particular, there exists an uncountable family of irreducible $\mathrm{SU}(2)$-representations of $\pi_1(X)$.*

On the other hand, there are several homology 3-spheres Y such that no irreducible representation of Y is extendable for (S^4, Y).

Example 9.18. Let Y be an oriented homology 3-sphere embedded into S^4. Then no irreducible representation is extendable for (S^4, Y). For example, if p is odd and k is a positive integer, it is shown in [9] that $\Sigma(p, pk+1, pk+2)$ can be embedded in to S^4. Thus, every irreducible SU(2)-representation of $\Sigma(p, pk+1, pk+2)$ is not extendable for (S^4, Y).

9.2.5. Fixed point theorems. We first prove Theorem 1.18.

Theorem 9.19 (Theorem 1.18). *Let Y be an oriented homology 3-sphere and let h be an orientation preserving self-diffeomorphism of Y.*

(1) *If $r_s(Y) < \infty$ and $l_Y^s < \infty$ for some $s \in [-\infty, 0]$, then there exists a positive number $l \leq l_Y^s$ such that*

$$(h^*)^l\colon R^*(Y) \to R^*(Y)$$

has a fixed point.

(2) *If $\Gamma_{-Y}(k) < \infty$ and $l_Y^k < \infty$ for some $k \in \mathbb{Z}_{>0}$, then there exists a positive number $l \leq l_Y^k$ such that*

$$(h^*)^l\colon R^*(Y) \to R^*(Y)$$

has a fixed point.

Proof of Theorem 1.18. We only prove (1) since (2) is similar. We apply Theorem 1.11 for the mapping torus $X_h(Y)$ of $h\colon Y \to Y$. Then we obtain the inclusion

$$r_s(Y) \in \bigcup_{1\leq j\leq l_Y^s} \operatorname{Im} cs^j_{X_h(Y),[Y]}.$$

Therefore, there exists some $l \in \{1, \dots, l_Y^s\}$ such that $R^*(X_h(Y)_{l,[Y]})$ is non-empty. Note that $X_h(Y)_{l,[Y]}$ is diffeomorphic to $X_{h^l}(Y)$ so we can identify $R^*(X_h(Y)_{l,[Y]})$ with $R^*(X_{h^l}(Y))$. Since $R^*(X_h(Y)_{l,[Y]})$ is non-empty, we obtain an element of $R^*(X_{h^l}(Y))$. This gives a fixed point of $(h^l)^*$. ∎

Corollary 9.20. *Fix a Seifert homology 3-sphere $\Sigma(a_1, \dots, a_n)$ with*

$$R(a_1, \dots, a_n) > 0 \quad \text{and} \quad \Lambda^*_{-\Sigma(a_1,\dots,a_n)} \cap \mathbb{Z} = \emptyset.$$

Let Y be an oriented homology 3-sphere such that

$$\min\{a+b\colon a \in \Lambda^*_{-\Sigma(a_1,\dots,a_n)} \cup \mathbb{Z}, b \in \Lambda^*_Y, a+b \geq 0\} > \frac{1}{4a_1\cdots a_n} \tag{53}$$

and

$$\Lambda^*_Y \cap \mathbb{Z} = \emptyset. \tag{54}$$

Then, for any orientation preserving diffeomorphism h on $-\Sigma(a_1,\dots,a_n)\#Y$, there exists some $l \in \mathbb{Z}_{>0}$ such that

$$(h^l)^*\colon R^*(-\Sigma(a_1,\dots,a_n)\#Y) \to R^*(-\Sigma(a_1,\dots,a_n)\#Y)$$

has a fixed point.

Proof. It is proved in [35, Corollary 1.4] that for a Seifert homology 3-sphere $\Sigma(a_1, \dots, a_n)$ satisfying $R(a_1,\dots,a_n) > 0$, one has $r_0(-\Sigma(a_1,\dots,a_n)) = \frac{1}{4a_1\cdots a_n}$. In particular, $\frac{1}{4a_1\dots a_n} \in \Lambda^*_{-\Sigma(a_1,\dots,a_n)}$. Set $M := -\Sigma(a_1,\dots,a_n)\#Y$. First, we use the connected sum formula for r_0:

$$\frac{1}{4a_1\cdots a_n} = r_0(-\Sigma(a_1,\dots,a_n)) \geq \min\{r_0(M), r_0(-Y)\}. \tag{55}$$

Since r_0 is contained in the set of critical values of irreducible SU(2)-flat connections,

$$r_0(M) \in \{a+b\colon a \in \Lambda_{-\Sigma(a_1,\dots,a_n)}, b \in \Lambda_Y, a+b > 0\}. \tag{56}$$

Here we used $cs_{Y_1\#Y_2}(\rho_1\#\rho_2) = cs_{Y_1}(\rho_1) + cs_{Y_2}(\rho_2)$ for oriented 3-manifolds Y_1 and Y_2 and SU(2)-representations ρ_1 and ρ_2. On the other hand, by formula (51) for critical values of $cs_{-\Sigma(a_1,\dots,a_n)}$ and $\Lambda^*_{-\Sigma(a_1,\dots,a_n)} \cap \mathbb{Z} = \emptyset$, we see

$$\min\{a\colon a \in \Lambda^*_{-\Sigma(a_1,\dots,a_n)} \cap \mathbb{R}_{\geq 0}\} = \frac{1}{4a_1\cdots a_n}. \tag{57}$$

If $r_0(M) = a+b \leq \frac{1}{4a_1\cdots a_n}$ (we used (55) and (56)) for $a \in \Lambda^*_{-\Sigma(a_1,\dots,a_n)} \cup \mathbb{Z}$, $b \in \Lambda^*_Y \cup \mathbb{Z}$, then $b = 0$ by our assumption (53). Then (57) implies $a = \frac{1}{4a_1\cdots a_n}$. Moreover, by the condition (54), we have

$$cs_M^{-1}\left(\frac{1}{4a_1\cdots a_n}\right) = \Big\{\rho\#\theta_{-Y} \in \widetilde{R}(M)\colon cs_{-\Sigma(a_1,\dots,a_n)}(\rho) = \frac{1}{4a_1\cdots a_n},$$
$$\rho \in \widetilde{R}(-\Sigma(a_1,\dots,a_n))\Big\},$$

where θ_{-Y} is the product connection on $-Y$. Thus the Chern–Simons functional of M is Morse–Bott at the level $r_0(M) = \frac{1}{4a_1\cdots a_n}$. By using Theorem 6.1, we conclude l^0_M is finite. One can then apply Theorem 1.18 to complete the proof. ∎

At the end of this section, we prove Theorem 1.19.

Theorem 9.21 (Theorem 1.19). *For any orientation preserving diffeomorphism h on $\Sigma(2,3,5,7)$, the fixed point set of*

$$h^*\colon R^*(\Sigma(2,3,5,7)) \to R^*(\Sigma(2,3,5,7))$$

is uncountable.

Proof of Theorem 1.19. Theorem 1.14 implies that, for any orientation preserving diffeomorphism h, $X_h(Y)$ has an uncountable family of irreducible SU(2)-representations. Thus (29) implies the conclusion. ■

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