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The FKB invariant is the 3d index

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Abstract. We identify the q-series associated to an 1-efficient ideal triangulation of a cusped hyperbolic 3-manifold by Frohman and Kania-Bartoszynska with the 3D-index of Dimofte–Gaiotto–Gukov. This implies the topological invariance of the q-series of Frohman and Kania-Bartoszynska for cusped hyperbolic 3-manifolds. Conversely, we identify the tetrahedron index of Dimofte–Gaiotto–Gukov as a limit of quantum $6j$-symbols.

1. Introduction

In their seminal paper, Turaev and Viro [16] defined topological invariants of triangulated 3-manifolds using state sums whose building blocks are the quantum $6j$-symbols at roots of unity. An extension of the Turaev–Viro invariants to ideally triangulated 3-manifolds was given by Turaev [14, 15] and Benedetti and Petronio [1].

In [4] Frohman and Kania-Bartoszynska (abbreviated by FKB) aimed to construct topological invariants of ideally triangulated 3-manifolds away from roots of unity, and with this goal in mind, they studied some limits of quantum $6j$-symbols and associated analytic functions to suitable ideal triangulations. Their results apply to compact, oriented 3-manifolds with arbitrary boundary, but for simplicity, throughout our paper, we will assume that M is a compact, oriented 3-manifold with torus boundary components. In that case, FKB assigned to an 1-efficient ideal triangulation $\mathcal{T}$ of such a 3-manifold M a formal power series $I^{\mathrm{FKB}}_{\mathcal{T}}(q) \in \mathbb{Z}[[q]]$ which turns out to be analytic in the open unit disk $|q| < 1$ and which is a generating series of suitable closed oriented surfaces carried by the spine associated to $\mathcal{T}$. FKB did not prove that their building block satisfies the 2–3 Pachner moves of 1-efficient triangulations, although this, together with the conjectured topological invariance, is implicit in their work.

In a different direction, in [2, 3] Dimofte, Gaiotto, and Gukov (abbreviated DGG) studied the index of a superconformal $N = 2$ gauge theory via a 3d–3d correspondence. Using as a building block an explicit formula for the partition function I_Δ of an

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ideal tetrahedron, they associated an invariant $I_{\mathcal{T}}(m,e)(q) \in \mathbb{Z}[[q^{1/2}]]$ to a suitable ideal triangulation $\mathcal{T}$ of a 3-manifold M where the tuples of integers (m,e) parametrize $H_1(\partial M, \mathbb{Z})$, once one chooses a pair of meridian-longitude at each boundary component of M. The construction of DGG is predicted by physics to be a topological invariant, and indeed DGG proved that their invariant is unchanged under suitable 2–3 Pachner moves.

It turns out that the ideal triangulations with a well-defined 3D-index are exactly those that satisfy a combinatorial PL condition known as an index structure (see [5, Section 2.1]), and, equivalently, those that are 1-efficient (see [7, Theorem 1.2]), i.e., those that do not contain any normal 2-spheres or non-peripheral normal torii, see Jaco and Rubinstein [10], Kang and Rubinstein [11] and also [7, Section 1.1]. Moreover, in [7], it was shown that the 3D-index of an 1-efficient triangulation gives rise to an invariant of a cusped hyperbolic 3-manifold M (with nonempty boundary).

Thus, 1-efficient ideal triangulations is a common feature of the work of FKB and DGG. A second common feature is the presence of (generalized) normal surfaces, that is surfaces that intersect each tetrahedron in polygonal disks [6, Definition 10.3]. On the one hand, the FKB invariant is a generating series of suitable surfaces carried by the spine of an ideal triangulation $\mathcal{T}$, see [4, Section 2]. On the other hand, it was shown in [6] that the 3D-index can be written as the generating series of generalized spun normal surfaces, (these are surfaces that intersect each ideal tetrahedron in polygonal disks) where the latter are encoded by their quadrilateral coordinates.

Although generalized normal surfaces and spinal surfaces play an important role in the invariants of this paper, no drawing of them is given in this paper. One reason for this intentional omission is that these surfaces are uniquely encoded by triples of natural numbers at each tetrahedron such that the minimum of each triple is zero. In other words, a generalized normal surface is allowed to have at most two quad types in each tetrahedron described in detail in [6, Section 10]. What's more, the FKB invariant and the 3D-index are generating series of triples of natural numbers that satisfy the above-stated minimum condition.

Given these coincidences, it is not surprising that the invariants of ideal triangulations of [4] and [2, 3] coincide.

Theorem 1.1. *If $\mathcal{T}$ is an 1-efficient triangulation, then for all elements $(m,e) \in H_1(\partial M, \mathbb{Z})$ we have*

$$I_{\mathcal{T}}^{\mathrm{FKB}}(m,e)(q) = I_{\mathcal{T}}(m,e)(q). \tag{1}$$

It follows that I^{FKB} is a topological invariant of cusped hyperbolic 3-manifolds.

Theorem 1.1 follows from the fact that both invariants can be expressed as generating series of surfaces with matching local weights (see Proposition 1.3 below).

Recall that the tetrahedron index is given by [3]

$$I_\Delta(m,e)(q) = \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)-(n+\frac{1}{2}e)m}}{(q;q)_n (q;q)_{n+e}} \tag{2}$$

where, for a natural number n, we define $(q;q)_n = \prod_{j=1}^n (1-q^j)$ and the summation in (2) is over the integers $n \geq \max\{0,-e\}$. Although the tetrahedron index is a function of a pair of integers, it can be presented as a function of three variables $a,b,c \in \mathbb{Z}$ [6, (8)] by

$$\begin{aligned} J_\Delta(a,b,c) &= (-q^{\frac{1}{2}})^{-b} I_\Delta(b-c,a-b) \\ &= (-q^{\frac{1}{2}})^{-c} I_\Delta(c-a,b-c) \\ &= (-q^{\frac{1}{2}})^{-a} I_\Delta(a-b,c-a). \end{aligned} \tag{3}$$

Then $J_\Delta(a,b,c)$ is invariant under all permutations of its arguments a,b,c and satisfies the translation property

$$J_\Delta(a,b,c) = (-q^{\frac{1}{2}})^s J_\Delta(a+s,b+s,c+s) \quad \text{for all } s \in \mathbb{Z}. \tag{4}$$

The leading term of $J_\Delta(a,b,c)$ is given by $(-q^{\frac{1}{2}})^{\nu(a,b,c)}$ (see [6, (8)]) where

$$\nu(a,b,c) = a^*b^* + a^*c^* + b^*c^* - \min\{a,b,c\} \tag{5}$$

where $a^* = a - \min\{a,b,c\}$, $b^* = b - \min\{a,b,c\}$ and $c^* = c - \min\{a,b,c\}$.

Consider the function

$$J_\Delta^{\mathrm{FKB}}(a,b,c) = (q;q)_\infty \sum_n (-1)^n \frac{q^{\frac{1}{2}n(3n+1)+n(a+b+c)+\frac{1}{2}(ab+bc+ca)}}{(q;q)_{n+a}(q;q)_{n+b}(q;q)_{n+c}} \tag{6}$$

for integers a,b and c, where the summation is over the integers (with the understanding that $(q;q)_m = \infty$ when $m<0$), or alternatively over the integers $n \geq -\min\{a,b,c\}$. FKB identify the above function as a limit of quantum $6j$-symbols. It turns out that the limit is equivalent to the stabilization of the coefficients of the quantum $6j$-symbols, and the latter follows from degree estimates. To state our result, consider the building blocks Θ and Tet (functions of three and six integer variables, respectively) whose definition is given explicitly in (10b) and (10c) of Section 2.1. Denote by $\widetilde{\Theta}$ and $\widetilde{\mathrm{Tet}}$ the shifted versions defined in Section 3.1. Then we have the following.

Proposition 1.2. *We have*

$$\lim_{N\to\infty} \widetilde{\Theta}(a+2N,b+2N,c+2N) = \frac{1}{1-q}\frac{1}{(q;q)_\infty^2} \tag{7}$$

and

$$\lim_{N\to\infty} \widetilde{\mathrm{Tet}}\begin{pmatrix} a+2N & b+2N & e+2N \\ d+2N & c+2N & f+2N \end{pmatrix}$$
$$= (-q^{-\frac{1}{2}})^{\nu(S_1^*,S_2^*,S_3^*)} \frac{1}{(1-q)(q;q)_\infty^4} J_\Delta^{\mathrm{FKB}}(S_1^*, S_2^*, S_3^*) \tag{8}$$

where S_i *are given in* (11), $S^* = \min\{S_1, S_2, S_3\}$ *and* $S_i^* = S_i - S^*$.

Observe that the quantum $6j$-symbols depend on six parameters (one per edge of the tetrahedron) while its limit given by (8) depends only on three parameters (one for each quadrilateral of the tetrahedron), and a further symmetry reduces the dependence to two parameters (obtained by ignoring one of the three quadrilateral types of the tetrahedron).

The next proposition identifies the tetrahedron index of [2,3] as a limit of quantum $6j$-symbols.

Proposition 1.3. *For integers* a, b *and* c *we have*

$$J_\Delta^{\mathrm{FKB}}(a,b,c) = J_\Delta(a,b,c). \tag{9}$$

2. A review of [16] and [4]

2.1. The building blocks

In this section we review the construction of the Turaev–Viro invariant and the results of [4]. Those invariants use some building blocks whose definition we recall now. Note that the normalization of the building blocks is not standard in the literature, and we will use the standard definitions of the building blocks that can be found in [12] and also in [13]. Recall the *quantum integer* $[n]$ and the *quantum factorial* $[n]!$ of a natural number n are defined by

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \qquad [n]! = \prod_{k=1}^{n} [k]!$$

with the convention that $[0]! = 1$. Let

$$\begin{bmatrix} a \\ a_1, a_2, \ldots, a_r \end{bmatrix} = \frac{[a]!}{[a_1]! \ldots [a_r]!}$$

denote the multinomial coefficient of natural numbers a_i such that $a_1 + \cdots + a_r = a$. We say that a triple (a, b, c) of natural numbers is *admissible* if $a + b + c$ is even and the triangle inequalities hold. In the formulas below, we use the following

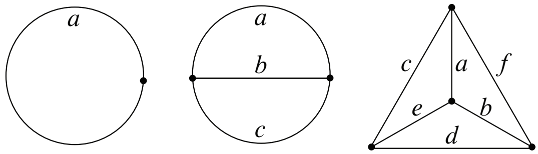

Figure 1. The Unknot, the Θ graph and the tetrahedron.

basic trivalent graphs $\mathrm{U}, \Theta, \mathrm{Tet}$ colored by one, three and six natural numbers (one in each edge of the corresponding graph) such that the colors at every vertex form an admissible triple shown in Figure 1.

Let us define the following functions:

$$\mathrm{U}(a) = (-1)^a [a+1], \tag{10a}$$

$$\Theta(a,b,c) = (-1)^{\frac{a+b+c}{2}} \left[\frac{a+b+c}{2} + 1\right] \left[\begin{matrix} \frac{a+b+c}{2} \\ \frac{-a+b+c}{2}, \frac{a-b+c}{2}, \frac{a+b-c}{2} \end{matrix}\right], \tag{10b}$$

$$\mathrm{Tet}\begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix} = \sum_{k=T^+}^{S^*} (-1)^k [k+1] \left[\begin{matrix} k \\ S_1 - k, S_2 - k, S_3 - k, k - T_1, k - T_2, k - T_3, k - T_4 \end{matrix}\right], \tag{10c}$$

where

$$S_1 = \frac{1}{2}(a + d + b + c), \tag{11a}$$

$$S_2 = \frac{1}{2}(a + d + e + f), \tag{11b}$$

$$S_3 = \frac{1}{2}(b + c + e + f), \tag{11c}$$

$$T_1 = \frac{1}{2}(a + b + e), \tag{12a}$$

$$T_2 = \frac{1}{2}(a + c + f), \tag{12b}$$

$$T_3 = \frac{1}{2}(c + d + e), \tag{12c}$$

$$T_4 = \frac{1}{2}(b + d + f), \tag{12d}$$

and

$$S^* = \min\{S_1, S_2, S_3\}, \tag{13a}$$

$$T^+ = \max\{T_1, T_2, T_3, T_4\}. \tag{13b}$$

2.2. The Turaev–Viro invariant

Suppose M is a 3-manifold as in our introduction, $\mathcal{T}$ is an ideal triangulation of M and X is the corresponding simple spine of $\mathcal{T}$, i.e., the dual 2-skeleton of $\mathcal{T}$. Let $V(X)$, $E(X)$, and $F(X)$ denote the vertices, edges, and faces of X.

An admissible coloring $c\colon F(X) \to \mathbb{N}$ of X is an assignment of natural numbers at each face of X such that at each edge of X the sum of the three colors are even, and they satisfy the triangle inequality. An admissible coloring c determines a 6-tuple $(a_v, b_v, c_c, d_v, e_v, f_v)$ of integers at each vertex v of X, a 3-tuple (a_e, b_e, c_e) of integers at each edge e of X and an integer u_f at each face f of X.

If r is a natural number, a coloring c is r-admissible if the sum of the colors at each edge is $\leq 2(r-2)$. Let ζ_r denote a primitive rth root of unity. Turaev and Viro [16] define an invariant

$$\mathrm{TV}_X(\zeta_r) = \mathrm{ev}_{\zeta_r} \sum_c \prod_{v \in V(X)} \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ d_v & c_v & f_v \end{pmatrix} \prod_{e \in E(X)} \Theta(a_e, b_e, c_e)^{-1} \prod_{f \in F(X)} \mathrm{U}(u_f) \tag{14}$$

where ev_{ζ_r} denotes the evaluation of a rational function of q at $q = \zeta_r$, and the sum is over the set of r-admissible colorings Turaev and Viro prove that the above state-sum is a topological invariant of M, i.e., independent of the ideal triangulation $\mathcal{T}$. An extension of the above invariant $\mathrm{TV}_{(X,\gamma)}(\zeta_r)$ can be defined by fixing an element $\gamma \in H_1(\partial M, \mathbb{Z})$, which determines a spine $X(\gamma)$ (called an augmented spine in [4, Section 2.2]).

2.3. The FKB invariant

In [4] it was observed that an admissible coloring c of X gives rise to a surface $\Sigma(c)$ of M carried by X. These surfaces which follow the spine and resolve the singularities were called spinal surfaces in [4] and they are carried by the branched surface X. Spinal surfaces can be encoded by their weight coordinates, as is natural in normal surface theory, and their Haken sum can be defined in such a way that the sum of their weights is the weight of their sum. Thus, the weight coordinates of spinal surfaces generate a monoid $\mathcal{S}(X)$. There is a natural increasing filtration on $\mathcal{S}(X)$ where $\mathcal{S}(X)_N$ denotes the (finite set of) surfaces with maximum weight at each face at most N. The idea of [4] is to use the same building blocks where now q is a complex number inside the unit disk, and consider the sum

$$\mathrm{TV}_X^{(N)}(q) = \sum_{\Sigma \in \mathcal{S}(X)_N} \prod_{v \in V(F)} \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ d_v & c_v & f_v \end{pmatrix} \prod_{e \in E(X)} \Theta(a_e, b_e, c_e)^{-1} \prod_{f \in F(X)} \mathrm{U}(u_f). \quad (15)$$

Alas, $\mathrm{TV}_X^{(N)}(q)$ is not a topological invariant (see below). However, the following is true.

Theorem 2.1 ([4]). *Fix a* 1*-efficient ideal triangulation* $\mathcal{T}$ *of a* 3*-manifold* M *with torus boundary components and let* X *be the dual spine. Then, the following limit exists*

$$I_{\mathcal{T}}^{\mathrm{FKB}}(q) := \lim_{N \to \infty} \frac{2}{N} \mathrm{TV}_X^{(N)}(q) \in \mathbb{Z}[[q]]. \quad (16)$$

Remark 2.2. The limit in (16) is a correction of [4, Theorem 5.1 (ii)] where with the notation of [4], one has $k = 0, \dots, N/2$.

The existence of the above limit is only the beginning of a stability of the coefficients of the sequence $\mathrm{TV}_X^{(N)}(q)$ in the sense of asymptotic expansions of sequences in the Laurent polynomial ring $\mathbb{Z}((q^{\frac{1}{2}}))$ discussed in [8]. In examples, it appears that the sequence $\mathrm{TV}_X^{(N)}(q)$ stabilizes to a quasi-linear function, i.e., that we have

$$\lim_N \mathrm{TV}_X^{(N)}(q) - \frac{N}{2} I_{\mathcal{T}}^{\mathrm{FKB}}(q) = I_{(0),\mathcal{T}}^{\mathrm{FKB}}(q) + I_{(1),\mathcal{T}}^{\mathrm{FKB}}(q) \cdot \begin{cases} 0, & N \text{ even}, \\ 1, & N \text{odd}, \end{cases} \quad (17)$$

where $I_{\mathcal{T}}^{\mathrm{FKB}}(q)$, $I_{(0),\mathcal{T}}^{\mathrm{FKB}}(q)$ and $2I_{(1),\mathcal{T}}^{\mathrm{FKB}}(q) \in \mathbb{Z}[[q^{1/2}]]$. However, both $I_{(0),\mathcal{T}}^{\mathrm{FKB}}(q)$ and $I_{(1),\mathcal{T}}^{\mathrm{FKB}}(q)$ depend on the triangulation. For example, for the standard ideal triangulation of the figure eight knot complement $\mathcal{T}_{4_1,2}$ with two tetrahedra (and isometry signature `cPcbbbiht`) we have

$$\begin{aligned} I_{\mathcal{T}_{4_1,2}}^{\mathrm{FKB}}(q) &= 1 - 2q - 3q^2 + 2q^3 + 8q^4 + 18q^5 + \cdots, \\ I_{(0),\mathcal{T}_{4_1,2}}^{\mathrm{FKB}}(q) &= 1 + 4q^2 + 4q^3 - 6q^4 - 36q^5 + \cdots, \\ 2I_{(1),\mathcal{T}_{4_1,2}}^{\mathrm{FKB}}(q) &= -1 + 2q + 3q^2 - 2q^3 - 8q^4 - 18q^5 + \cdots, \end{aligned}$$

whereas for the geometric triangulation $\mathcal{T}_{4_1,3}$ of the figure eight knot complement with three tetrahedra (and isometry signature `dLQbcccdegj`) we have

$$I_{\mathcal{T}_{4_1,3}}^{\mathrm{FKB}}(q) = I_{\mathcal{T}_{4_1,2}}^{\mathrm{FKB}}(q)$$

as expected but

$$\begin{aligned} I_{(0),\mathcal{T}_{4_1,3}}^{\mathrm{FKB}}(q) &= 1 + 4q^2 + 4q^3 - 6q^4 - 36q^5 + \cdots, \\ 2I_{(1),\mathcal{T}_{4_1,3}}^{\mathrm{FKB}}(q) &= 1 + 2q + 2q^2 + 8q^3 - 12q^4 - 72q^5 + \cdots. \end{aligned}$$

The next result of [4] identifies the above limit with a generating series of the monoid of spinal surfaces, modulo the boundary torii. Such surfaces were called *unpeelable* in [4]. Define the weight $E_\infty(\Sigma)$ of a spinal surface Σ to be

$$E_\infty(\Sigma) = (-q^{\frac{1}{2}})^{-\chi(\Sigma)} \prod_f \frac{1}{1-q} \prod_v S_\infty \begin{pmatrix} a_v & b_v & e_v \\ d_v & c_v & f_v \end{pmatrix}, \tag{18}$$

where if $C_1 \geq C_2 \geq C_3$ are the sums of opposite edge weights of the tetrahedron,

$$\alpha = \frac{C_1 - C_3}{2}, \quad \beta = \frac{C_1 - C_2}{2},$$

and

$$\begin{aligned} S_\infty \begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix} &= (1-q)(q;q)_\infty \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{3}{2}n^2 + (\alpha+\beta+\frac{1}{2})n + \frac{1}{2}\alpha\beta}}{(q;q)_n (q;q)_{n+\alpha} (q;q)_{n+\beta}} \\ &= (1-q) J_\Delta^{\mathrm{FKB}}(S_1^*, S_2^*, S_3^*), \end{aligned} \tag{19}$$

where $S_1 \geq S_2 \geq S_3$, thus $S_3^* = 0$ and $\alpha = S_1 - S_3 = \frac{C_1 - C_3}{2}$ and $\beta = S_2 - S_3 = \frac{C_1 - C_2}{2}$. It follows that for a spinal surface Σ we have

$$E_\infty(\Sigma) = (-q^{\frac{1}{2}})^{-\chi(\Sigma)} J_\Delta^{\mathrm{FKB}}(\Sigma) \tag{20}$$

where $\alpha = S_1^*$ and $\beta = S_2^*$ and

$$J_\Delta^{\mathrm{FKB}}(\Sigma) = \prod_{j=1}^{t} J_\Delta^{\mathrm{FKB}}(a_j, b_j, c_j) \tag{21}$$

and $(a_1, b_1, c_1, \ldots, a_t, b_t, c_t)$ are the quad coordinates of Σ and t is the number of tetrahedra of $\mathcal{T}$. Note that Σ is unpeelable if and only if $\min\{a_j, b_j, c_j\} = 0$ for all $j = 1, \ldots, t$.

Theorem 2.3 ([4]). *Under the assumptions of Theorem 2.1, the limit coincides with the generating series of closed unpeelable surfaces carried by the spine of* $\mathcal{T}$

$$I_{\mathcal{T}}^{\mathrm{FKB}}(q) = \sum_{\Sigma:\ \text{unpeelable}} E_\infty(\Sigma). \tag{22}$$

It is possible to extend Theorems 2.1 and 2.3 using an element $\gamma \in H_1(\partial M, \mathbb{Z})$. Consider the augmented spine $X(\gamma)$. Then one can define $\mathrm{TV}_X^{(N)}(\gamma)(q)$ and the corresponding limit $I_{\mathcal{T}}^{\mathrm{FKB}}(\gamma)(q)$ exists and is identified with the generating series of unpeelable surfaces Σ with boundary γ.

3. Proofs

3.1. Stabilization of the building blocks

We prove some stabilization properties of the building blocks of quantum spin networks, using elementary degree estimates, in the spirit of [8], where the stabilization of the coefficients of the colored Jones polynomial of an alternating knot was proven, giving rise to a sequence of q-series, the first of which is known as the tail of the colored Jones polynomial.

We begin by expressing the building blocks of Section 2.1 in terms of the quantum factorial $(q;q)_n$ where $(qx;q)_n = \prod_{j=1}^n (1-q^j x)$ for n a nonnegative integer. We have

$$[n] = q^{-\frac{n-1}{2}} \frac{1-q^n}{1-q}, \quad [n]! = q^{-\frac{n(n-1)}{4}} \frac{(q;q)_n}{(1-q)^n},$$

$$\begin{bmatrix} a \\ a_1, a_2, \dots, a_r \end{bmatrix} = \frac{[a]!}{[a_1]! \dots [a_r]!} = q^{-\frac{1}{4}(a^2 - \sum_{j=1}^r a_j^2)} \frac{(q;q)_a}{(q;q)_{a_1} \dots (q;q)_{a_r}}, \tag{23}$$

and

$$(\mathrm{Tet})\begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix} = \sum_{k=T^+}^{S^*} (-1)^k \frac{1-q^k}{1-q} q^{\delta(\mathrm{Tet})\left(\begin{smallmatrix} a & b & e \\ d & c & f \end{smallmatrix}\right)}$$
$$\times \frac{(q;q)_k}{\prod_{i=1}^3 (q;q)_{S_i - k} \prod_{j=1}^4 (q;q)_{k-T_j}} \tag{24}$$

where $\delta(\mathrm{Tet})\left(\begin{smallmatrix} a & b & e \\ d & c & f \end{smallmatrix}\right)$ is defined in Lemma 3.1 below and S_i and T_j are given in (11) and (12).

Since $(q;q)_n \in \mathbb{Z}[q]$ is a polynomial with constant term 1, it follows that one has $1/(q;q)_n \in \mathbb{Q}(q) \cap \mathbb{Z}[[q]]$. The building blocks are rational functions of q with denominators products of cyclotomic polynomials, hence they are well-defined elements of the Laurent polynomial ring $\mathbb{Z}((q))$. If $f(q) \in \mathbb{Z}((q))$ we will denote by $\mathrm{lt}(f) q^{\delta(f)}$ the monomial with the lowest power of q appearing in the Laurent expansion of $f(q)$, and we will denote $\tilde{f}(q) = \mathrm{lt}(f)^{-1} q^{-\delta(f)} f(q)$ the shifted series, which, when $\mathrm{lt}(f) = \pm 1$, is an element of $1 + q\mathbb{Z}[[q]]$.

Note that our notation differs slightly from [9, Section 2], where we studied the leading terms of the building blocks with the aim of computing the degree of the colored Jones polynomial.

The next lemma is elementary (see [9, Lemma 2.4]).

Lemma 3.1. *For all admissible colorings we have*

$$\mathrm{lt}(\mathrm{U})(a) = (-1)^a, \quad \mathrm{lt}(\Theta)(a,b,c) = (-1)^{\frac{a+b+c}{2}}, \quad \mathrm{lt}(\mathrm{Tet})\begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix} = (-1)^{T^+},$$

and

$$
\begin{aligned}
\delta(\mathrm{U})(a) &= \frac{a}{2},\\
\delta(\Theta)(a,b,c) &= -\frac{1}{8}(a^2+b^2+c^2)+\frac{1}{4}(ab+ac+bc)+\frac{1}{4}(a+b+c),\\
\delta(\mathrm{Tet})\begin{pmatrix} a & b & c\\ d & e & f\end{pmatrix} &= \frac{1}{4}\Big(-(T^+)^2+\sum_i (S_i-T^+)^2+\sum_j (T^+-T_j)^2\Big)-\frac{T^+}{2},
\end{aligned}
$$

where S_j and T_i are given in (11) *and* (12)*.*

We have all the ingredients to give a proof of Proposition 1.2.

Proof of Proposition 1.2. The first identity follows from the fact that

$$
\widetilde{\Theta}(a,b,c) = \frac{1-q^{\frac{a+b+c}{2}+1}}{1-q}\,\frac{(q;q)_{\frac{a+b+c}{2}}}{(q;q)_{\frac{-a+b+c}{2}}(q;q)_{\frac{a-b+c}{2}}(q;q)_{\frac{a+b-c}{2}}}
$$

and the fact that

$$
\lim_{N\to\infty} q^{\kappa+\lambda N} = 0, \qquad \lim_{N\to\infty} (q;q)_{\kappa'+\lambda N} = (q;q)_\infty \tag{25}
$$

for integers κ, κ' and λ with $\lambda > 0$.

For the second identity, the sum over k (with $T^+ \le k \le S^*$) in (24) achieves the minimum q-degree uniquely at $k = T^+$. After changing variables to $k = S^* - \ell$ it follows that

$$
\begin{aligned}
\widetilde{\mathrm{Tet}}\begin{pmatrix} a & b & c\\ d & e & f\end{pmatrix} &= \frac{1}{1-q}\sum_{\ell=0}^{S^*-T^+}(-1)^\ell(1-q^{S^*-\ell})q^{\frac{1}{2}\ell(3\ell+1)+\ell(S_1^*+S_2^*+S_3^*)}\\
&\quad\times\frac{(q;q)_{S^*-\ell}}{\prod_i (q;q)_{S_i^*+\ell}\prod_j (q;q)_{T_j^*-\ell}},
\end{aligned}
$$

where $S_i^* = S_i - S^*$ and $T_j^* = S^* - T_j$. It follows that for all natural numbers N, we have

$$
\begin{aligned}
&\widetilde{\mathrm{Tet}}\begin{pmatrix} a+2N & b+2N & c+2N\\ d+2N & e+2N & f+2N\end{pmatrix}\\
&= \frac{1}{1-q}\sum_{\ell=0}^{N+S^*-T^+}(-1)^\ell(1-q^{4N+S^*-\ell})q^{\frac{1}{2}\ell(3\ell+1)+\ell(S_1^*+S_2^*+S_3^*)}\\
&\quad\times\frac{(q;q)_{4N+S^*-\ell}}{\prod_i (q;q)_{S_i^*+\ell}\prod_j (q;q)_{N+T_j^*-\ell}}.
\end{aligned} \tag{26}
$$

Equation (25) applied to each fixed ℓ implies that

$$\lim_{N\to\infty} \widetilde{\mathrm{Tet}}\begin{pmatrix} a+2N & b+2N & c+2N \\ d+2N & e+2N & f+2N \end{pmatrix}$$
$$= \frac{1}{1-q}\sum_{\ell=0}^{\infty}(-1)^\ell q^{\frac{1}{2}\ell(3\ell+1)+\ell(S_1^*+S_2^*+S_3^*)} \frac{(q;q)_\infty}{\prod_i (q;q)_{S_i^*+\ell}\prod_j (q;q)_\infty} \tag{27}$$

and this concludes the proof. ■

3.2. The tetrahedron index as a limit of q-$6j$-symbols

In this section we give a proof of Proposition 1.3. Observe that $J_\Delta^{\mathrm{FKB}}(a,b,c)$ is symmetric under all permutations of (a,b,c). Moreover, we claim that it satisfies the translation property (4). Indeed, using the definition of J_Δ^{FKB} as a sum over the integers (6), it follows that

$$\begin{aligned} &J_\Delta^{\mathrm{FKB}}(a+s,b+s,c+s) \\ &\quad= (q;q)_\infty \sum_n (-1)^n \frac{\mathfrak{Q}}{(q;q)_{n+a+s}(q;q)_{n+b+s}(q;q)_{n+c+s}} \\ &\quad= (-q^{\frac{1}{2}})^s (q;q)_\infty \sum_m (-1)^m \frac{q^{\frac{1}{2}m(3m+1)+m(a+b+c)+\frac{1}{2}(ab+ac+bc)}}{(q;q)_{m+a}(q;q)_{m+b}(q;q)_{m+c}} \end{aligned}$$

where

$$\mathfrak{Q} := q^{\frac{1}{2}n(3n+1)+n(a+s+b+s+c+s)+\frac{1}{2}((a+s)(b+s)+(b+s)(c+s)+(c+s)(a+s))}$$

and in the first equality we shifted variables to $n+s=m$.

Since both sides of (9) satisfy the translation property (4) and are symmetric in (a,b,c), to prove the said equation, it suffices to assume that $a\ge b\ge c=0$. We will use the following identities

$$(qx;q)_\infty = \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(n+1)}{2}}x^n}{(q;q)_n}, \tag{28}$$

$$\frac{1}{(q;q)_m(q;q)_n} = \sum_{\substack{r,s,t\ge 0\\ r+s=m,s+t=n}} \frac{q^{rt}}{(q;q)_r(q;q)_s(q;q)_t}, \tag{29}$$

whose proofs may be found, for example, in [17, (7) and (13) of Section D]. We have

$$\begin{aligned}
(q;q)_\infty \sum_{k=0}^{\infty}(-1)^k & \frac{q^{\frac{3}{2}k^2+(a+b+\frac{1}{2})k}}{(q;q)_k(q;q)_{k+a}(q;q)_{k+b}} \\
&= \sum_k (-1)^k \frac{q^{\frac{3}{2}k^2+(a+b+\frac{1}{2})k}}{(q;q)_k(q;q)_{k+a}}(q^{k+b+1};q)_\infty \\
&= \sum_{k,\ell}(-1)^{k+\ell}\frac{q^{\frac{3}{2}k^2+(a+b+\frac{1}{2})k+\frac{\ell(\ell+1)}{2}}}{(q;q)_k(q;q)_{k+a}(q;q)_\ell}q^{(k+b)\ell} \\
&= \sum_n \sum_{\substack{k+a+\ell=n+a \\ n=k+\ell}} (-1)^n \frac{q^{k(k+a)}q^{\frac{1}{2}n^2+\frac{n}{2}+bn}}{(q;q)_k(q;q)_\ell(q;q)_{k+a}} \\
&= \sum_n (-1)^n \frac{q^{\frac{1}{2}n^2+\frac{n}{2}+bn}}{(q;q)_n(q;q)_{n+a}} \\
&= q^{-\frac{1}{2}ab} I_\Delta(-b,a).
\end{aligned}$$

It follows that $I^{\mathrm{FKB}}(a,b,0) = I_\Delta(-b,a) = J_\Delta(b,a,0) = J_\Delta(a,b,0)$, which concludes the proof of the proposition.

3.3. Proof of Theorem 1.1

Fix a 1-efficient ideal triangulation $\mathcal{T}$ with spine X. Recall the generalized normal surfaces of [7] and [6, Section 10]. Each generalized normal surface S has weight $I(S)$ given by [6, (25)]

$$I(S) = (-q^{\frac{1}{2}})^{-\chi(\Sigma)} \prod_{j=1}^{t} J_\Delta(a_j,b_j,c_j) \tag{30}$$

where $(a_1,b_1,c_1,\ldots,a_t,b_t,c_t)$ are the quad coordinates of Σ and t is the number of tetrahedra of $\mathcal{T}$.

Lemma 3.2. *There is a bijection between the closed generalized normal surfaces of $\mathcal{T}$ and the closed unpeelable spinal surfaces of X. If S is a generalized normal surface and Σ is the corresponding unpeelable surface, then*

$$I(S) = E_\infty(\Sigma). \tag{31}$$

Proof. Using the notation of [6, Section 7], the closed generalized normal surfaces of $\mathcal{T}$ are given by $Q_0(\mathcal{T},\mathbb{Z})/\mathbb{T} = (\mathbb{E}+\mathbb{T})/\mathbb{T}$ where $\mathbb{E}$ and $\mathbb{T}$ are the subspaces of integer solutions to the normal surface equations generated by the edges and the tetrahedra of $\mathcal{T}$, respectively. Every element of $Q_0(\mathcal{T}, \mathbb{Z})$ is encoded by a vector $(a_1, b_1, c_1, \ldots, a_t, b_t, c_t) \in \mathbb{Z}^{3t}$ of quadrilateral coordinates where t is the number of tetrahedra of $\mathcal{T}$. Moreover, the tetrahedral solution to the gluing equations corresponding to the ℓ-th tetrahedron is the $3t$ vector of integers with coordinates $(a_j, b_j, c_j) = \delta_{j,\ell}(1,1,1)$ where $\delta_{j,\ell} = 1$ if $j = \ell$ and 0 otherwise. Thus, every generalized normal surface $S \in (\mathbb{E} + \mathbb{T})/\mathbb{T}$ has coordinate vector $(a_1, b_1, c_1, \ldots, a_t, b_t, c_t) \in \mathbb{N}^{3t}$ satisfying $\min\{a_j, b_j, c_j\} = 0$ for all $j = 1, \ldots, t$. And conversely, every such vector corresponds to a unique generalized normal surface. On the other hand, every unpeelable closed surface is uniquely described by its quad coordinate vector $(a_1, b_1, c_1, \ldots, a_t, b_t, c_t) \in \mathbb{N}^{3t}$ satisfying $\min\{a_j, b_j, c_j\} = 0$ for all $j = 1, \ldots, t$, and all such vectors give rise to unpeelable surfaces. This concludes the first part of the lemma. The second part, i.e., equation (31) follows from equations (20) and (21) and Proposition 1.3. ∎

When $(m, e) = 0$, Theorem 1.1 follows from Theorem 2.3, Lemma 3.2 and the fact that the 3D-index is given by [6, Corollary 8.2]

$$I_{\mathcal{T}}(0,0)(q) = \sum_S I(S) \tag{32}$$

where the sum is over the set of generalized normal surfaces. When $\gamma \in H_1(\partial M, \mathbb{Z})$, one uses the obvious extension of Lemma 3.2 along with the extension of Theorem 2.3 combined with [6, Definition 8.1]. This concludes the proof of the theorem.

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