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Non-semisimple 3-manifold invariants derived from the Kauffman bracket

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Abstract. We recover the family of non-semisimple quantum invariants of closed oriented 3-manifolds associated with the small quantum group of $\mathfrak{sl}_2$ using purely combinatorial methods based on Temperley–Lieb algebras and Kauffman bracket polynomials. These invariants can be understood as a first-order extension of Witten–Reshetikhin–Turaev invariants, which can be reformulated following our approach in the case of rational homology spheres.

1. Introduction

The distinction between *semisimple* and *non-semisimple* constructions in quantum topology refers to the properties of the algebraic ingredients involved. One of the most celebrated families of quantum invariants, known as Witten–Reshetikhin–Turaev (or WRT) invariants, is of the first kind. Indeed, if $r \geqslant 3$ is an integer called the *level* of the theory, then the WRT invariant τ_r can be constructed using a semisimple quotient of the category of representations of the small quantum group $\bar{U}_q \mathfrak{sl}_2$ at the r-th root of unity $q = e^{\frac{2\pi i}{r}}$, see [42]. (In this paper, the acronym WRT will not refer to the larger family of quantum invariants constructed by Reshetikhin and Turaev in terms of the representation theory of modular Hopf algebras, but rather to the specific subfamily recovering the topological invariants first obtained by Witten using Chern–Simons gauge theory and the Feynman path integral [49].) This invariant extends to a Topological Quantum Field Theory (TQFT for short), which can also be obtained using several different approaches based on methods ranging from combinatorics and skein theory [8] to geometric topology and conformal field theory [2]. On the other hand, the family of quantum invariants Z_r considered in this paper is of the second kind. It has already been defined using the non-semisimple representation theory of quantum groups (without quotient operation), as well as more general categorical methods. By contrast, the approach developed here relies uniquely on Temperley–Lieb algebras and Kauffman bracket polynomials. In particular, we provide the first reformulation of

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a non-semisimple quantum invariant of closed 3-manifolds that completely bypasses Hopf algebras and their representation theory. This is the first step towards a purely combinatorial construction of non-semisimple TQFTs which will naturally induce new families of representations of Kauffman bracket skein algebras of surfaces.

The invariant Z_r is defined for odd levels $r \geqslant 3$, it takes values in complex numbers, and it coincides with the renormalized Hennings invariant associated with $\bar{U}_q \mathfrak{sl}_2$ at $q = e^{\frac{2\pi i}{r}}$, as defined in [14]. Since the category of finite-dimensional representations of $\bar{U}_q \mathfrak{sl}_2$ is modular (in the non-semisimple sense of [34]), Z_r fits into the larger family of quantum invariants constructed in [13], and both of these approaches produce TQFT extensions whose properties are in sharp contrast with those of their semisimple counterparts, see for instance [12, Proposition 1.4]. It should also be noted that the family of invariants considered here is very closely related to the generalized Kashaev invariants of knots in 3-manifolds defined in [37], which have been extended to logarithmic Hennings invariants of links in 3-manifolds [4], although both constructions focus on a somewhat complementary case, namely when the level $r \geqslant 4$ is even. All these constructions build on the structure and properties of quantum groups and ribbon categories, and thus have a distinct algebraic flavor.

The goal of this paper is to reproduce the renormalized Hennings invariant associated with $\bar{U}_q \mathfrak{sl}_2$ relying exclusively on the technical setup used by Lickorish for the construction of WRT invariants [32]. One of the basic ingredients for this approach is given by the family of *Temperley–Lieb algebras*[1] $\mathrm{TL}(m)$ of parameter $\delta = -q - q^{-1}$, where m is a natural number, and by specific idempotent elements $f_m \in \mathrm{TL}(m)$ defined for $0 \leqslant m \leqslant r - 1$ called (*simple*) *Jones–Wenzl idempotents*. In particular, a leading role is played by a formal linear combination of simple Jones–Wenzl idempotents in the range $0 \leqslant m \leqslant r - 2$ called (*semisimple*) *Kirby color*, and denoted ω. The name comes from the fact that the scalar associated with an ω-labeled framed link by the graphical calculus based on the *Kauffman bracket polynomial* [28] with variable $A = q^{\frac{r+1}{2}}$ is invariant under Kirby II moves. Our main technical achievement is the introduction of a *non-semisimple Kirby color* Ω, which is given by Definition 3.1 in terms of *non-semisimple Jones–Wenzl idempotents* $g_m \in \mathrm{TL}(m)$ for $r \leqslant m \leqslant 2r - 2$, which are in turn given by equations (9)–(11). Although this generalization of simple Jones–Wenzl idempotents dates back to [21], the formulas reported here were found in [7], and were inspired by similar ones, for even values of the level r, due to

[1]The connection between the two approaches stems from the well-known equivalence between the Temperley–Lieb algebra $\mathrm{TL}(m)$ and the centralizer algebra for the m-th tensor power of the fundamental representation of a closely related Hopf algebra, *Lusztig's divided power quantum group* $U_q \mathfrak{sl}_2$, which contains the small quantum group $\bar{U}_q \mathfrak{sl}_2$ as a Hopf subalgebra. In light of this, we can say our purpose is to reformulate the renormalized Hennings invariant in diagrammatic terms, rather than algebraic ones.

Ibanez [25] and Moore [36]. It should also be noted that, when the level is a prime number p, then non-semisimple Jones–Wenzl idempotents recover p-Jones–Wenzl idempotents in the corresponding range, as defined in [9].

1.1. Outline of the construction

The topological notion underlying the graphical calculus developed in this paper is that of a *bichrome tangle*. Roughly speaking, a bichrome tangle is the union of a *blue* framed tangle, which is both oriented and labeled by idempotent morphisms of the *Temperley–Lieb category* TL of parameter $\delta = -q - q^{-1}$, and a *red* framed link, which carries neither orientations nor labels. When a bichrome tangle T is embedded inside a 3-manifold M, one should think of its blue part as an element of the corresponding Kauffman bracket skein module, and of its red part as a surgery prescription. *Bichrome links*, which are closed bichrome tangles, allow us to revisit, in Section 3.4, a construction due to Blanchet [5]. Indeed, the SO(3) version of the WRT invariant $\tau_r(M, T)$ can be defined for a closed oriented 3-manifold M decorated with a bichrome link $T \subset M$. This is done by means of a topological invariant F_ω of bichrome links, taking values in $\mathbb{C}$, which is constructed using the Kirby color ω and the Kauffman bracket polynomial. If M is a closed oriented 3-manifold, $T \subset M$ is a bichrome link, and $L \subset S^3$ is a red surgery presentation of M with positive signature σ_+ and negative signature σ_-, then

$$\tau_r(M, T) := \frac{F_\omega(L \cup T)}{\delta_+^{\sigma_+} \delta_-^{\sigma_-}}$$

is a topological invariant of the pair (M, T), where

$$\delta_+ := \frac{i^{-\frac{r-1}{2}} r^{\frac{1}{2}} q^{\frac{r-3}{2}}}{\{1\}},$$

$$\delta_- := -\frac{i^{\frac{r-1}{2}} r^{\frac{1}{2}} q^{\frac{r+3}{2}}}{\{1\}}.$$

Similarly, the non-semisimple invariant $Z_r(M, T)$ is defined for a closed oriented 3-manifold M decorated with a bichrome link $T \subset M$, but not an arbitrary one. Indeed, T needs to satisfy a certain *admissibility* condition which consists in requiring the presence, among the labels of its blue components, of an idempotent morphism of TL belonging to the ideal generated by f_{r-1}, which can be understood as the ideal of projective objects of the idempotent completion of TL. For instance, $g_r, \ldots, g_{2r-2}$ all correspond to projective objects, and the same holds for their tensor product with any other idempotent morphism of TL, but $f_0, \ldots, f_{r-2}$ do not. In particular, the red part of an admissible bichrome link is allowed to be empty, while the blue part is not.

In Section 3.4, we define a topological invariant F'_Ω of admissible bichrome links, with values in $\mathbb{C}$, using the non-semisimple Kirby color Ω, the Kauffman bracket polynomial, and the theory of *modified traces* [18]. We point out that the admissibility assumption is required precisely in order to use this last ingredient, without which non-semisimple quantum invariants essentially boil down to a reformulation of semisimple ones, as in [10]. Indeed, in the case of non-semisimple ribbon categories such as TL, non-degenerate modified traces can only be consistently defined on proper tensor ideals, and general existence results usually focus on the special case of the ideal of projective objects, see Section 2.2. Our main result can then be stated as follows.

Theorem 1.1. *If M is a closed oriented 3-manifold, $T \subset M$ is an admissible bichrome link, and $L \subset S^3$ is a red surgery presentation of M with positive signature σ_+ and negative signature σ_-, then*

$$Z_r(M,T) := \frac{F'_\Omega(L \cup T)}{\Delta_+^{\sigma_+} \Delta_-^{\sigma_-}}$$

is a topological invariant of the pair (M,T), where

$$\Delta_+ := i^{-\frac{r-1}{2}} r^{\frac{3}{2}} q^{\frac{r-3}{2}},$$

$$\Delta_- := i^{\frac{r-1}{2}} r^{\frac{3}{2}} q^{\frac{r+3}{2}}.$$

1.2. Strategy of the proof

Although the small quantum group $\bar{U} := \bar{U}_q \mathfrak{sl}_2$ and its category of finite-dimensional representations $\bar{U}$-mod do not appear in the definition of Z_r, they play an important role in the proof of its topological invariance. Indeed, a well-known faithful braided monoidal linear functor $F_{\mathrm{TL}} \colon \mathrm{TL} \to \bar{U}$-mod allows us to interpret morphisms of TL as intertwiners between tensor powers of the *fundamental representation* X of $\bar{U}$, as recalled in Section 5. Then, the idea is essentially to check that our definition of Z_r computes exactly the renormalized Hennings invariant associated with $\bar{U}$.

In Section 4 we prepare the ground for this comparison by introducing our algebraic setup. In particular, we fix a *left integral* of $\bar{U}$, which is a linear form $\lambda \in \bar{U}^*$ satisfying a crucial condition that can be understood as an algebraic version of the invariance under Kirby II moves, see Section 6.3. This provides the key ingredient for the definition of both the original Hennings invariant and its renormalized version.[2]

[2]Both [24] and [14] actually use a *right integral*, but the difference is simply a matter of conventions. Other related choices involve the use of top tangles, bottom tangles, or string links, the use of the adjoint, the coadjoint, or the regular representation, and so on. Changing one of these conventions requires changing accordingly all the others.

The left integral λ belongs to the space $\mathrm{QC}(\bar{U})$ of *quantum characters* of $\bar{U}$, which admits a basis composed of *quantum traces* and *pseudo quantum traces* corresponding to simple and indecomposable projective $\bar{U}$-modules. In Section 4.6 we adapt computations of Arike [3] to the odd level case, and obtain an explicit decomposition of λ into this basis of $\mathrm{QC}(\bar{U})$. Next, we use the fact that every quantum character can be interpreted as a $\bar{U}$-module morphism with target the trivial representation $\mathbb{C}$ and source the adjoint representation ad, which has been studied in detail by Ostrik [40]. An important property of ad is that it admits a $\mathbb{Z}$-grading, and that, as explained in Section 4.7, every quantum character is completely determined by its restriction to the subspace of degree 0 vectors of ad. Then, the rest of the paper is devoted to explain why and how the non-semisimple Kirby color Ω provides a diagrammatic implementation of the left integral λ, and for the proof it is sufficient to focus on the degree 0 part of ad.

The next step consists in reviewing the algorithm for the computation of the Hennings invariant. We point out that there exist already several places in the literature where different methods have been explained in detail. The interested reader can check [24] for the original definition, [29] for an improved construction that avoids the use of orientations, [22, 31, 34, 38, 47] for several reformulations, and [14] for the renormalized version involving modified traces. The common idea behind all these different approaches is essentially to get rid of the representation theory in the original construction of WRT invariants [42], to figure out explicitly the relevant combinatorics for elements of the quantum group, and to evaluate these using the left integral λ. We will briefly explain the procedure once again in Section 5 for convenience, but it should be noted that, once we establish that the non-semisimple Kirby color Ω implements the left integral λ, the rest of the proof should be regarded as a well-known consequence of the Hennings–Kauffman–Radford (or HKR) theory. More precisely, in Section 5.2 we introduce the *bead category* $\mathrm{TL}_{\bar{U}}$ by allowing elements of $\bar{U}$ to sit on strands of morphisms of the Temperley–Lieb category TL. This allows us to rephrase the HKR algorithm, and in particular the one presented by Kerler and Virelizier, as a procedure which, starting from a top tangle, returns a morphism of the bead category. Then, in Section 5.3 we prove our main technical result, which can be explained as follows: completing the HKR algorithm with the algebraic evaluation based on the left integral λ yields the same result as completing it with the diagrammatic evaluation based on the non-semisimple Kirby color Ω. The proof is obtained by explicit computation, and it is based on a series of formulas involving non-semisimple Jones–Wenzl idempotents which are established in Section 7.

1.3. Relation with WRT

As explained in Section 3.4, an invariant of closed oriented 3-manifolds decorated with non-admissible (possibly empty) bichrome links can be obtained by setting

$$Y_r(M,T) := Z_r(M \# S^3, T \cup O),$$

where $O \subset S^3$ is a blue unknot of framing 0 and label f_{r-1}. Then, using [10, Theorem 1], we can show that

$$Y_r(M,T) = h_1(M)\tau_r(M,T),$$

where $h_1(M) = |H_1(M)|$ if $|H_1(M)|$ is finite, and $h_1(M) = 0$ otherwise. Furthermore, if $T' \subset M'$ is an admissible bichrome link, then we have

$$Z_r(M \# M', T \cup T') = Y_r(M,T) Z_r(M',T).$$

Therefore, we can think of the invariant Z_r as a first-order extension of τ_r, at least for rational homology spheres, in the same spirit of [11, Section 1.3].

1.4. Future perspectives

An extended version $\overline{\mathrm{TL}}$ of the Temperley–Lieb category TL was introduced in [7] in order to recover a diagrammatic description of the full monoidal subcategory of $\bar{U}$-mod generated by the fundamental representation of $\bar{U}$. Indeed, roughly speaking, TL misses a few morphisms, since it is equivalent to the category of *tilting modules* of a different Hopf algebra, namely Lusztig's divided power quantum group $U_q\,\mathfrak{sl}_2$, of which the small quantum group $\bar{U}$ is a Hopf subalgebra. Now, although $\overline{\mathrm{TL}}$ can be avoided for the definition of Z_r, we expect it to play a major role in any purely diagrammatic proof of its topological invariance, as well as in the skein model for its TQFT extension based on the universal construction of [8].

Using $\overline{\mathrm{TL}}$, we can define appropriate bichrome versions of Kauffman bracket skein modules. State spaces of non-semisimple TQFTs are quotients of these bichrome skein modules, and they naturally carry actions of Kauffman bracket skein algebras. The prospect of obtaining new families of representations for these algebraic structures is especially interesting, since most geometric applications of WRT TQFTs exploit this technology. More generally, the development of alternative models of non-semisimple TQFTs is a crucial step for enhancing the flexibility of the theory, and for promoting its applications to the deep and mysterious questions concerning the geometric and dynamic content of quantum constructions in topology.

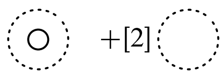
2. Temperley–Lieb category and modified trace

In this section we recall definitions for the main tools required by our construction: Temperley–Lieb algebras, Kauffman bracket skein relations, and modified traces. In order to do this, we fix once and for all an odd integer $3 \leqslant r \in \mathbb{Z}$, and we consider the primitive r-th root of unity $q = e^{\frac{2\pi i}{r}}$. For every natural number $k \in \mathbb{N}$ we adopt the notation

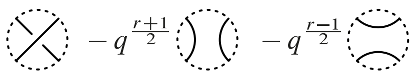
$$\{k\} := q^k - q^{-k}, \quad [k] := \frac{\{k\}}{\{1\}}, \quad [k]! := \prod_{j=1}^{k} [j], \quad \{k\}' := q^k + q^{-k}.$$

2.1. Temperley–Lieb category

For the definition of the Temperley–Lieb category, we will follow the approach of [32, Section 3.3] and [8, Section 3], which is based on the category of unoriented framed tangles [45, Section 7]. Let us consider the cube $I^3 \subset \mathbb{R}^3$, where $I \subset \mathbb{R}$ denotes the interval $[0, 1]$. An (m, m')-*tangle* is the unoriented image of a proper embedding into I^3 of a disjoint union of finitely many copies of I and S^1, whose boundary is composed of m points on the bottom line $I \times \{\frac{1}{2}\} \times \{0\} \subset I^3$ and m' points on the top line $I \times \{\frac{1}{2}\} \times \{1\} \subset I^3$. A *framed tangle* is a tangle equipped with a *framing*, that is, a transverse vector field along each of its components. We represent framed tangles in I^3 as planar diagrams projected orthogonally to $I \times \{0\} \times I$, and we adopt the blackboard framing convention, which means the framing always points to the reader. The *Temperley–Lieb category* TL is the linear category with set of objects $\mathbb{N}$, and with vector space of morphisms from $m \in \mathrm{TL}$ to $m' \in \mathrm{TL}$ denoted $\mathrm{TL}(m, m')$, and given by the quotient of the vector space generated by isotopy classes of framed (m, m')-tangles in I^3 modulo the subspace generated by vectors of the form



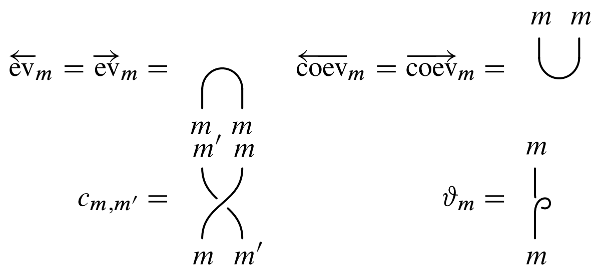
(S1)



(S2)

These pictures represent operations performed inside a disc D^3 embedded into I^3, and they leave tangles unchanged in the complement. Composition of morphisms of TL is given by gluing vertically two copies of I^3, in the opposite order with respect to [45], and then shrinking the result into I^3. Then, for every $m \in \mathrm{TL}$, the m-th *Temperley–Lieb algebra* is defined as $\mathrm{TL}(m) := \mathrm{TL}(m, m)$.

The Temperley–Lieb category TL can be given a *ribbon structure*, see [16, Section 8.10] for a definition. Tensor product of objects of TL is given by taking their sum, while tensor product of morphisms of TL is given by gluing horizontally two copies of I^3, and then shrinking the result into I^3. When representing graphically a morphism of TL, we will sometimes allow components (or their endpoints) to carry labels given by natural numbers, as a shorthand for the number of parallel strands with respect to the framing (although sometimes, when this information can be deduced from the rest of the diagram, labels can be omitted), and we will allow morphisms to be replaced by boxes containing their name. Then left and right evaluation and coevaluation, braiding, and twist morphisms are defined, for all $m, m' \in \mathrm{TL}$, by



(1)

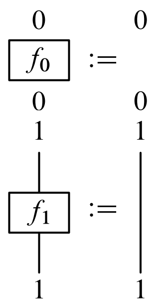
(2)

In particular, the dual $u^* \in \mathrm{TL}(m', m)$ of a morphism $u \in \mathrm{TL}(m, m')$ is obtained by a rotation of angle π. By abuse of notation, we still denote by TL the idempotent completion of TL. This means we promote idempotent endomorphisms $p \in \mathrm{TL}(m)$ to objects of TL, and for all $p \in \mathrm{TL}(m)$ and $p' \in \mathrm{TL}(m')$ we set

$$\mathrm{TL}(p, p') := \{u \in \mathrm{TL}(m, m') \mid up = u = p'u\}. \tag{3}$$

By the same abuse of notation, we sometimes consider natural numbers $m \in \mathrm{TL}$ as idempotent endomorphisms of TL.

Next, let us recall the definition of a special family of idempotents of TL, first defined in [26, 48], and later generalized in [21]. We will follow the approach of [7], where more details can be found. For every integer $0 \leqslant m \leqslant r - 1$ the *m-th simple Jones–Wenzl idempotent* $f_m \in \mathrm{TL}(m)$ is recursively defined as



(4)

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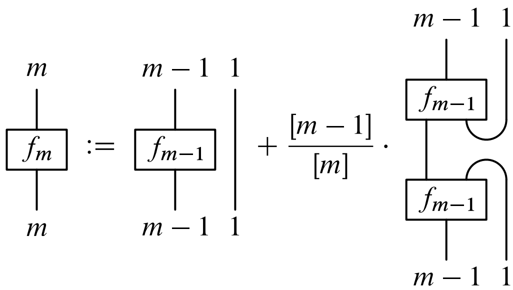
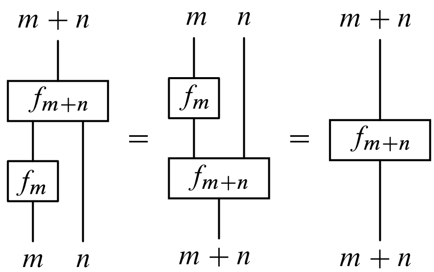


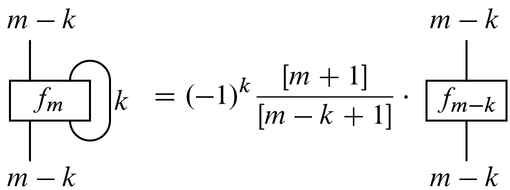
(6)

Let us recall some basic properties of simple Jones–Wenzl idempotents. We have



(7)

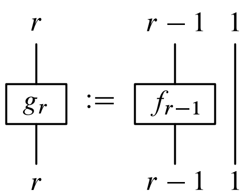
for all integers $0 \leqslant m \leqslant r-1$ and $0 \leqslant n \leqslant r-m-1$, and



(8)

for every integer $0 \leqslant k \leqslant m$. Simple Jones–Wenzl idempotents satisfy $f_m^* = f_m$ for every integer $0 \leqslant m \leqslant r-1$ with respect to the rigid structure determined by equation (1). In other words, a rotation of angle π fixes f_m.

Similarly, thanks to [7, Lemma 3.2], for every integer $r \leqslant m \leqslant 2r-2$ the *m-th non-semisimple Jones–Wenzl idempotent* $g_m \in \mathrm{TL}(m)$ is recursively defined as



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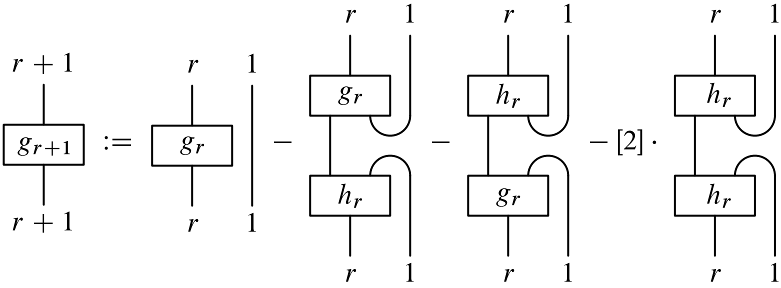

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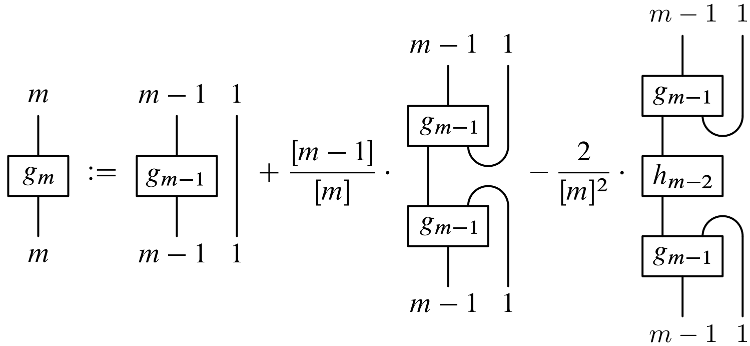
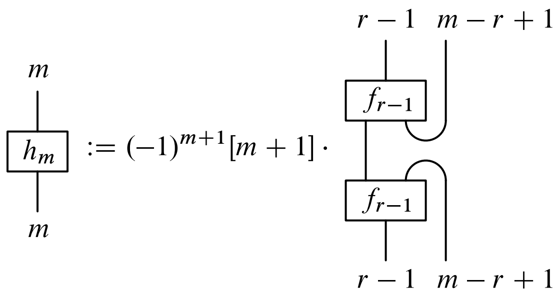


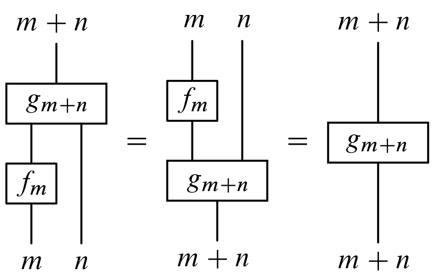
(11)

where $h_m \in \mathrm{TL}(m)$ is the nilpotent endomorphism defined as



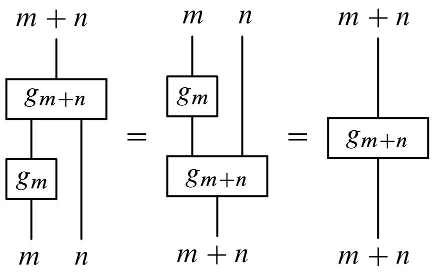
(12)

As explained in [7, Section 3], these endomorphisms satisfy



(13)

for all integers $0 \leqslant m \leqslant r-1$ and $r-m \leqslant n \leqslant 2r-m-2$,



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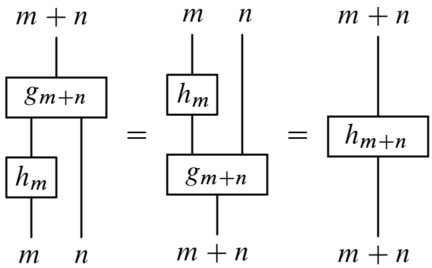

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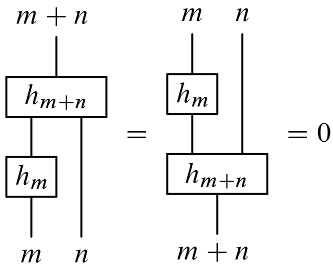
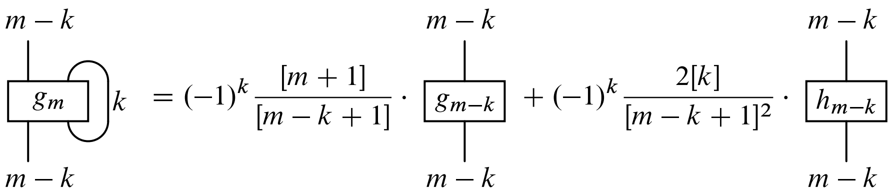


(16)

for all integers $r \leqslant m \leqslant 2r-2$ and $0 \leqslant n \leqslant 2r-m-2$,



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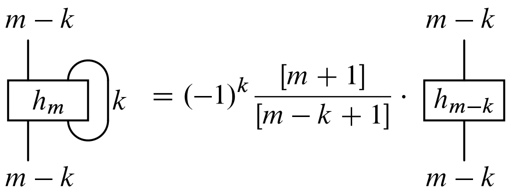
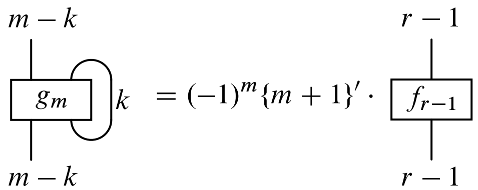


(18)

for every integer $0 \leqslant k \leqslant m - r$,



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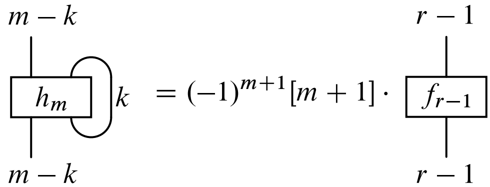
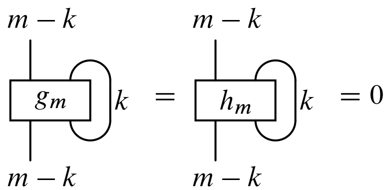


(20)

for $k = m - r + 1$, and

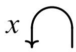


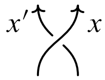
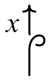
(21)

for every integer $m - r + 2 \leqslant k \leqslant m$, see [7, Lemma 3.1]. While for every integer $r \leqslant m \leqslant 2r - 2$ we have $h_m^* = h_m$, non-semisimple Jones–Wenzl idempotents do not meet, in general, this condition, with the only exception of $g_{2r-2}^* = g_{2r-2}$, as explained in [7, Remark 3.4]. For an explicit computation of non-semisimple Jones–Wenzl idempotents in the special case $r = 3$, see Section 3.5. For a representation theoretic interpretation of the recursive relations (4)–(6) and (9)–(11), compare with equations (49)–(50) using [7, Lemma 4.1].

2.2. Modified trace

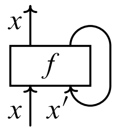
We recall now a few important definitions which will require the abstract language of *ribbon categories*, see again [16, Section 8.10] for a general definition. What is most important for our purposes is the fact that a ribbon category $\mathcal{C}$ comes equipped with a tensor product and a tensor unit, as well as structural data given by left and right evaluation and coevaluation, braiding, and twist morphisms, such as those introduced in equations (1) and (2) for TL. Using this structure, a diagrammatic calculus for morphisms of $\mathcal{C}$, based on the *Penrose graphical notation*, can be developed. We point out that our convention for orientations will be opposite with respect to the one of [46, Section I.1.6]. Consequently, if $\mathcal{C}$ is a ribbon category, then structure morphisms are represented, for all $x, x' \in \mathcal{C}$, by

$\overleftarrow{\mathrm{ev}}_x =$  $\overleftarrow{\mathrm{coev}}_x =$  $\overrightarrow{\mathrm{ev}}_x =$  $\overrightarrow{\mathrm{coev}}_x =$ 

$c_{x,x'} =$  $\vartheta_x =$ 

An *ideal* of a ribbon category $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ which is absorbent under tensor products and closed under retracts. In other words, if $x \in \mathcal{I}$, then for every $x' \in \mathcal{C}$ we have $x \otimes x' \in \mathcal{I}$, and for all $f \in \mathcal{C}(x', x)$ and $f' \in \mathcal{C}(x, x')$ satisfying $f' \circ f = \mathrm{id}_{x'}$ we have $x' \in \mathcal{I}$. The *ideal generated by an object* $x \in \mathcal{C}$ is the ideal of $\mathcal{C}$ whose objects x' satisfy $\mathrm{id}_{x'} = f' \circ f$ for some object $x'' \in \mathcal{C}$ and morphisms $f \in \mathcal{C}(x', x \otimes x'')$ and $f' \in \mathcal{C}(x \otimes x'', x')$. Remark that equations (9)–(11) immediately imply that, for every integer $r \leqslant m \leqslant 2r - 2$, the non-semisimple Jones–Wenzl idempotent g_m belongs to the ideal of TL generated by f_{r-1}.

The *partial trace* of an endomorphism $f \in \mathrm{End}_{\mathcal{C}}(x \otimes x')$ is the endomorphism $\mathrm{ptr}_{x'}(f) \in \mathrm{End}_{\mathcal{C}}(x)$ defined as

$\mathrm{ptr}_{x'}(f) :=$ 

Following [18], a *trace* t *on an ideal* $\mathcal{I}$ of a ribbon linear category $\mathcal{C}$ over $\mathbb{C}$ is a family of linear maps

$$\{\mathrm{t}_x \colon \mathrm{End}_{\mathcal{C}}(x) \to \mathbb{C} \mid x \in \mathcal{I}\}$$

satisfying:

(i) *cyclicity*: $\mathrm{t}_x(f' \circ f) = \mathrm{t}_{x'}(f \circ f')$ for all objects $x, x' \in \mathcal{I}$ and all morphisms $f \in \mathcal{C}(x, x')$, $f' \in \mathcal{C}(x', x)$;

(ii) *partial trace*: $\mathrm{t}_{x \otimes x'}(f) = \mathrm{t}_x(\mathrm{ptr}_{x'}(f))$ for all objects $x \in \mathcal{I}$, $x' \in \mathcal{C}$, and every endomorphism $f \in \mathrm{End}_{\mathcal{C}}(x \otimes x')$.

A trace t on $\mathcal{I}$ is *non-degenerate* if, for all $x \in \mathcal{I}$ and $x' \in \mathcal{C}$, the bilinear pairing $\mathrm{t}_x(\_ \circ \_) \colon \mathcal{C}(x', x) \times \mathcal{C}(x, x') \to \mathbb{C}$ determined by $(f', f) \mapsto \mathrm{t}_x(f' \circ f)$ is non-degenerate. Let us denote by Proj(TL) the ideal of *projective objects of* TL.

Proposition 2.1. *The ideal* Proj(TL) *is generated by* $f_{r-1} \in \mathrm{TL}$*, and there exists a unique trace* t^{TL} *on* Proj(TL) *satisfying*

$$\mathrm{t}^{\mathrm{TL}}_{f_{r-1}}(f_{r-1}) = 1. \tag{22}$$

Furthermore, t^{TL} *is non-degenerate.*

A proof of Proposition 2.1 is postponed to Section 6.1. For the moment, let us simply point out that Proposition 2.1 implies that every idempotent $p \in \mathrm{TL}(m)$ of $\mathrm{Proj}(\mathrm{TL})$ can be written as $p = u'u$ for some morphisms $u \in \mathrm{TL}(m, f_{r-1} \otimes m')$ and $u' \in \mathrm{TL}(f_{r-1} \otimes m', m)$. Furthermore, equations (19) and (20) imply

$$\mathrm{t}^{\mathrm{TL}}_{g_m}(g_m) = (-1)^m \{m+1\}', \quad \mathrm{t}^{\mathrm{TL}}_{g_m}(h_m) = (-1)^{m+1}[m+1] \tag{23}$$

for all $r \leqslant m \leqslant 2r-2$. See also [23, Corollary 3.4] for more existence results concerning modified traces, and [44, Proposition 3.12] for related formulas.

3. 3-Manifold invariant

In this section we introduce bichrome links, and we define a topological invariant Z_r of closed 3-manifolds decorated with admissible ones. We also explain how to use Z_r to obtain a topological invariant Y_r of closed 3-manifolds without decorations, and show that this invariant recovers the SO(3) WRT invariant τ_r for rational homology spheres. Every 3-manifold is assumed to be oriented.

3.1. Bichrome tangles

The notion of bichrome link will be based on a few preliminary definitions. First of all, the *category* $\mathcal{T}_{\mathrm{TL}}$ *of* TL*-labeled oriented framed tangles* is the category whose objects are finite sequences $(\underline{\varepsilon}, \underline{p}) = ((\varepsilon_1, p_1), \ldots, (\varepsilon_j, p_j))$, where $\varepsilon_i \in \{+,-\}$ is a sign and $p_i \in \mathrm{TL}(m_i)$ is an idempotent for every integer $1 \leqslant i \leqslant j$, and whose morphisms from $(\underline{\varepsilon}, \underline{p}) = ((\varepsilon_1, p_1), \ldots, (\varepsilon_j, p_j))$ to $(\underline{\varepsilon}', \underline{p}') = ((\varepsilon'_1, p'_1), \ldots, (\varepsilon'_{j'}, p'_{j'}))$ are isotopy classes of framed (j, j')-tangles whose components carry orientations and labels given by idempotents of TL matching those specified by $(\underline{\varepsilon}, \underline{p})$ and $(\underline{\varepsilon}', \underline{p}')$, with $+$ and $-$ corresponding to upward and downward orientation respectively. Composition of morphisms of $\mathcal{T}_{\mathrm{TL}}$ is given by gluing vertically two copies of I^3 and then shrinking the result into I^3. We will adopt a shorthand notation omitting signs for positive sequences. In other words, a sequence $\underline{p} = (p_1, \ldots, p_j)$ will stand for the object $((+, p_1), \ldots, (+, p_j))$ of $\mathcal{T}_{\mathrm{TL}}$. Furthermore, an object of $\mathcal{T}_{\mathrm{TL}}$ which is not underlined will stand for a sequence with a single entry, and $\varnothing$ will stand for the empty sequence.

The category $\mathcal{T}_{\mathrm{TL}}$ naturally supports a ribbon structure. Tensor product of objects of $\mathcal{T}_{\mathrm{TL}}$ is given by taking their concatenation, while tensor product of morphisms of $\mathcal{T}_{\mathrm{TL}}$ is given by gluing horizontally two copies of I^3 and then shrinking the result into I^3. When representing graphically a morphism of $\mathcal{T}_{\mathrm{TL}}$, we use the color blue. Then, left and right evaluation and coevaluation, braiding, and twist morphisms are

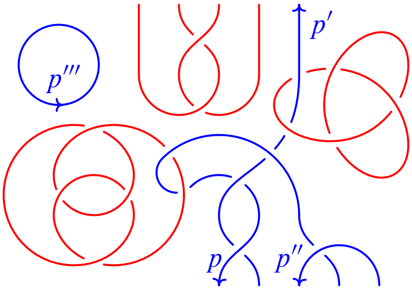
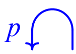
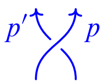
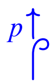


Figure 1. An example of a bichrome 2-top tangle with source $((-, p), (+, p'), (-, p''), (+, p), (+, p''))$ and target $(+, p')$.

defined, for all $p, p' \in \mathcal{T}_{\mathrm{TL}}$, by

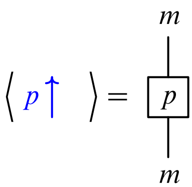
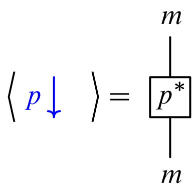
$\overleftarrow{\mathrm{ev}}_p =$  $\quad \overleftarrow{\mathrm{coev}}_p =$  $\quad \overrightarrow{\mathrm{ev}}_p =$  $\quad \overrightarrow{\mathrm{coev}}_p =$  (24)

$c_{p,p'} =$  $\quad \vartheta_p =$  (25)

There exists a ribbon functor

$$\langle \_ \rangle \colon \mathcal{T}_{\mathrm{TL}} \to \mathrm{TL},$$

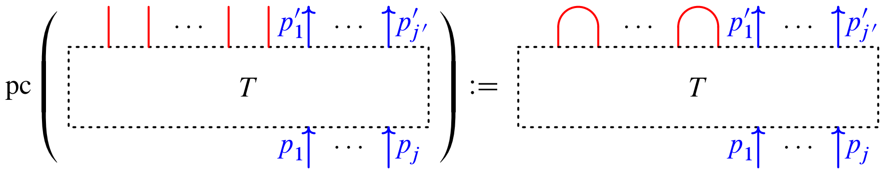
which we refer to as the *Kauffman bracket functor*, satisfying

  (26)

for every idempotent $p \in \mathrm{TL}(m)$, and sending structure morphisms of equations (1) and (2) to those of equations (24) and (25) respectively.

For every $k \in \mathbb{N}$ a *bichrome k-top tangle from* $(\underline{\varepsilon}, \underline{p})$ *to* $(\underline{\varepsilon}', \underline{p}')$, sometimes simply called a *top tangle*, is the union of a *blue* TL-labeled oriented framed tangle from $(\underline{\varepsilon}, \underline{p})$ to $(\underline{\varepsilon}', \underline{p}')$ and a *red* framed $(0, 2k)$-tangle satisfying the following condition: for every $1 \leqslant i \leqslant k$, the $2i$-th and $(2i - 1)$-th outgoing boundary points (starting from the left) are connected by a red component, while all the other incoming and outgoing boundary points belong to blue components. An example of a bichrome 2-top tangle is given in Figure 1. In order to distinguish red components from blue ones in black-and-white versions of this paper, remark that red components are unoriented and unlabeled, while blue ones are oriented and labeled.

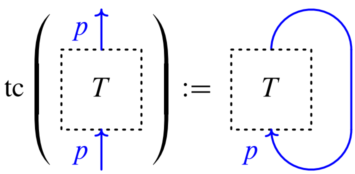
We denote by $\mathcal{T}_k(\underline{p}, \underline{p}')$ the set of isotopy classes of bichrome k-top tangles from $\underline{p}$ to $\underline{p}'$ featuring no closed red component, and we adopt the shorthand notation $\mathcal{T}_k(\underline{p})$ when $\underline{p} = \underline{p}'$. A bichrome k-top tangle is simply called a *bichrome tangle* if $k = 0$. We denote by $\mathcal{B}(\underline{p}, \underline{p}')$ the set of isotopy classes of bichrome tangles from $\underline{p}$ to $\underline{p}'$, and we adopt the shorthand notation $\mathcal{B}(\underline{p})$ when $\underline{p} = \underline{p}'$. Every bichrome k-top tangle $T \in \mathcal{T}_k(\underline{p}, \underline{p}')$ determines a bichrome tangle $\mathrm{pc}(T) \in \mathcal{B}(\underline{p}, \underline{p}')$ obtained by considering its *plat closure*, that is,



(27)

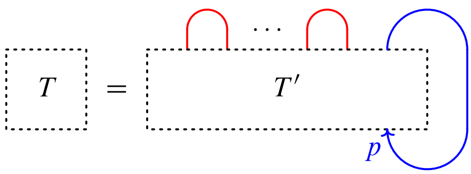
Then, a *top tangle presentation* of a bichrome tangle $T \in \mathcal{B}(\underline{p}, \underline{p}')$ is a bichrome top tangle $T' \in \mathcal{T}_k(\underline{p}, \underline{p}')$ whose plat closure is T. By definition, a top tangle presentation of a bichrome tangle has no closed red component.

A *bichrome link* is a bichrome tangle from $\varnothing$ to itself. A bichrome link is *admissible* if it features a *projective blue component*, which is a blue component labeled by an idempotent $p \in \mathrm{TL}(m)$ of $\mathrm{Proj}(\mathrm{TL})$. If $p \in \mathrm{TL}(m)$ is an idempotent of $\mathrm{Proj}(\mathrm{TL})$, then every bichrome tangle $T \in \mathcal{B}(p)$ determines an admissible bichrome link $\mathrm{tc}(T) \in \mathcal{B}(\varnothing)$ obtained by considering its *trace closure*, that is,



(28)

A *cutting presentation* of an admissible bichrome link $T \in \mathcal{B}(\varnothing)$ is a top tangle presentation $T'' \in \mathcal{T}_k(p)$ of a bichrome tangle $T' \in \mathcal{B}(p)$ whose trace closure is T, for some idempotent $p \in \mathrm{TL}(m)$ of $\mathrm{Proj}(\mathrm{TL})$. In other words, if $T \in \mathcal{B}(\varnothing)$ is an admissible bichrome link and $T' \in \mathcal{T}_k(p)$ is a cutting presentation of T, then



3.2. Ribbon Kauffman bracket

In order to define a topological invariant of admissible bichrome links, we first need to adjust the sign in front of the Kauffman bracket appropriately. The reason for this is rather subtle, and will be explained more carefully in Section 5.1, once the definition of the small quantum group $\bar{U} := \bar{U}_q\,\mathfrak{sl}_2$ will have been recalled. Very briefly, the problem originates from the comparison between the Temperley–Lieb category TL and the category of finite-dimensional representations $\bar{U}$-mod. Indeed, both are ribbon categories, and a dictionary between the two is provided by a faithful linear functor $F_{\mathrm{TL}} \colon \mathrm{TL} \to \bar{U}$-mod which preserves braided monoidal structures. However, F_{TL} does not preserve ribbon structures. In particular, partial traces and twist morphisms of TL are translated to those of $\bar{U}$-mod only up to a sign. If we want the diagrammatic construction based on TL to replicate the algebraic computation based on $\bar{U}$, as defined in [14], this sign needs to be controlled. This is precisely what we will do now, following [39, Theorem H.3].

First of all, we recall that, if K and K' are disjoint oriented knots in S^3, then their *linking number* $\mathrm{lk}(K, K')$ is defined as the transverse intersection number $S \pitchfork K' \in \mathbb{Z}$, where S is a Seifert surface for K. Alternatively, $\mathrm{lk}(K, K')$ can be computed as the difference between the number of positive and negative crossings of K over K' in a diagram of $K \cup K'$. We also recall that, if K is a framed oriented knot in S^3, its *framing number* $\mathrm{fr}(K)$ is defined as $\mathrm{lk}(K, \tilde{K}) \in \mathbb{Z}$, where $\tilde{K}$ is a parallel copy of K determined by its framing. The framing number is actually independent of the orientation of K.

A *blue link* is a bichrome link without red components. If T is a blue link with components $T_1, \ldots, T_\ell$ labeled by $p_1 \in \mathrm{TL}(m_1), \ldots, p_\ell \in \mathrm{TL}(m_\ell)$ respectively, we define its *ribbon number* $\mathrm{rb}(T)$ as the integer

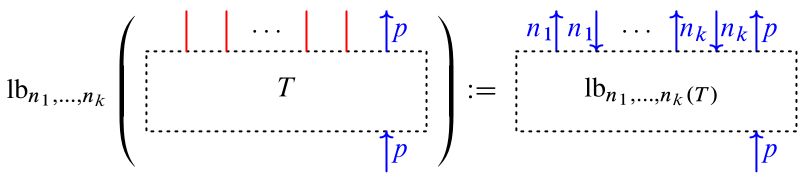
$$\mathrm{rb}(T) := \sum_{i=1}^{\ell} m_i(\mathrm{fr}(T_i) + 1). \tag{29}$$

The *ribbon Kauffman bracket of a blue link* T is the scalar $\langle T \rangle^{\mathrm{rb}} \in \mathrm{TL}(0) = \mathbb{C}$ defined by

$$\langle T \rangle^{\mathrm{rb}} := (-1)^{\mathrm{rb}(T)} \langle T \rangle, \tag{30}$$

as considered by Ohtsuki in [39, Theorem H.3].

Let now $r - 1 \leqslant n_1, \ldots, n_k \leqslant 2r - 2$ be integers. The *$(n_1, \ldots, n_k)$-labeling of a bichrome top tangle $T \in \mathcal{T}_k(p)$* is defined as the blue TL-labeled oriented framed tangle $\mathrm{lb}_{n_1,\ldots,n_k}(T)$ from $(+, p)$ to $((+, n_1), (-, n_1), \ldots, (+, n_k), (-, n_k), (+, p))$ obtained from T by turning all its red components blue, by orienting them from right to left, and by labeling the i-th one by n_i for all $1 \leqslant i \leqslant k$, that is,



(31)

Then the $(n_1,\dots,n_k)$*-bracket of a bichrome top tangle* $T \in \mathcal{T}_k(p)$ is the morphism $\langle T\rangle_{n_1,\dots,n_k} \in \mathrm{TL}(p, n_1 \otimes n_1 \otimes \dots \otimes n_k \otimes n_k \otimes p)$ defined by

$$\langle T\rangle_{n_1,\dots,n_k} := \langle \mathrm{lb}_{n_1,\dots,n_k}(T)\rangle. \tag{32}$$

We can extend the ribbon number defined in equation (29) for blue links to the $(n_1,\dots,n_k)$-labeling of a bichrome top tangle $T \in \mathcal{T}_k(p)$ with a single non-closed blue component B labeled by $p \in \mathrm{TL}(m)$, with closed blue components $B_1,\dots,B_j$ labeled by $p_1 \in \mathrm{TL}(m_1),\dots,p_j \in \mathrm{TL}(m_\ell)$, and with red components $R_1,\dots,R_k$, by setting, in the notation of equations (27) and (28) for plat and trace closures,

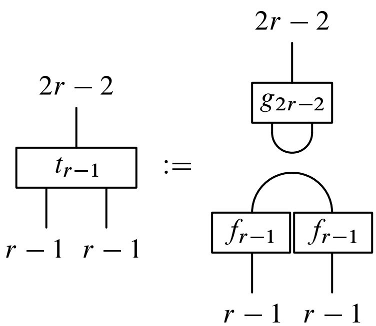
$$\mathrm{rb}(\mathrm{lb}_{n_1,\dots,n_k}(T)) := m\,\mathrm{fr}(\mathrm{tc}(B)) + \sum_{i=1}^{j} m_i(\mathrm{fr}(B_i)+1) + \sum_{i=1}^{k} n_i \mathrm{fr}(\mathrm{pc}(R_i)). \tag{33}$$

The *ribbon* $(n_1,\dots,n_k)$*-bracket of a bichrome top tangle* $T \in \mathcal{T}_k(p)$ is the morphism $\langle T\rangle^{\mathrm{rb}}_{n_1,\dots,n_k} \in \mathrm{TL}(p, n_1 \otimes n_1 \otimes \dots \otimes n_k \otimes n_k \otimes p)$ defined by

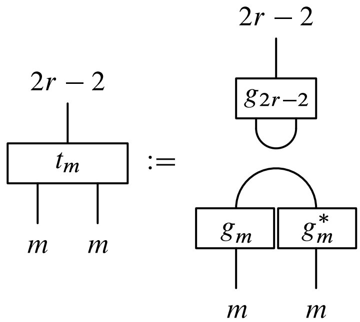
$$\langle T\rangle^{\mathrm{rb}}_{n_1,\dots,n_k} := (-1)^{\mathrm{rb}(\mathrm{lb}_{n_1,\dots,n_k}(T))}\langle T\rangle_{n_1,\dots,n_k}. \tag{34}$$

3.3. Bichrome link invariant

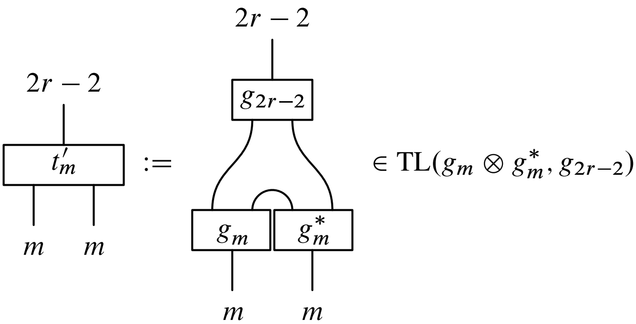
Let us define a topological invariant of admissible bichrome links. In order to do this, let us set



$\in \mathrm{TL}(f_{r-1} \otimes f_{r-1}, g_{2r-2})$,



$\in \mathrm{TL}(g_m \otimes g_m^*, g_{2r-2})$,



for every integer $r \leqslant m \leqslant 2r-2$. Remark that we are using the notation introduced in equation (3), which means we are considering Jones–Wenzl idempotents as objects of TL. See Section 3.5 for an explicit computation in the case $r = 3$.

Definition 3.1. The *non-semisimple Kirby color* Ω is, by definition, the linear combination of morphisms

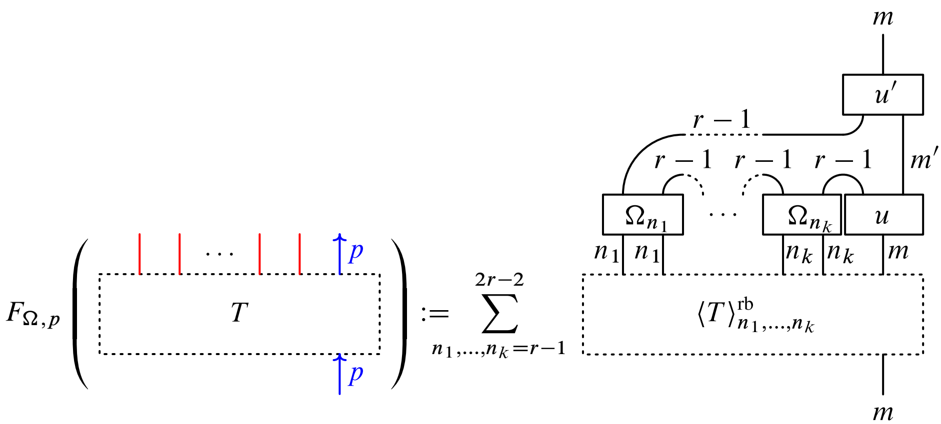
$$\Omega := \sum_{m=r-1}^{2r-2} \Omega_m \in \bigoplus_{m=r-1}^{2r-2} \mathrm{TL}(m \otimes m, 2r-2),$$

where $\Omega_m \in \mathrm{TL}(m \otimes m, 2r-2)$ is given by

$$\Omega_{r-1} := t_{r-1},$$

$$\Omega_m := (-1)^m \frac{\{m+1\}'}{2} t_m - (-1)^m [m+1] t'_m.$$

If $p \in \mathrm{TL}(m)$ is an idempotent of $\mathrm{Proj}(\mathrm{TL})$, the non-semisimple Kirby color Ω can be used to associate with every bichrome k-top tangle $T \in \mathcal{T}_k(p)$ an endomorphism $F_{\Omega,p}(T) \in \mathrm{TL}(p)$, in the notation of equation (3). Indeed, if $u \in \mathrm{TL}(m, f_{r-1} \otimes m')$ and $u' \in \mathrm{TL}(f_{r-1} \otimes m', m)$ are morphisms satisfying $p = u'u$, then, using the ribbon Kauffman bracket of equation (34), which coincides with the standard Kauffman bracket of equation (32) up to a sign, we can set



Proposition 3.2. *If $T \in \mathcal{B}(\varnothing)$ is an admissible bichrome link, $p \in \mathrm{TL}(m)$ is an idempotent of* $\mathrm{Proj}(\mathrm{TL})$*, and $T' \in \mathcal{T}_k(p)$ is a cutting presentation of T, then*

$$F'_\Omega(T) := (-1)^m \mathrm{t}_p^{\mathrm{TL}}(F_{\Omega,p}(T')) \tag{35}$$

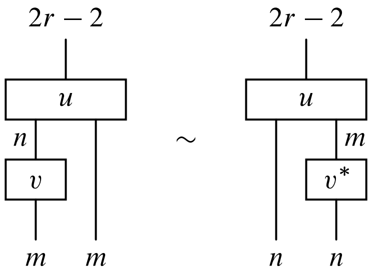
is a topological invariant of T, meaning it is independent of the choice of p and T'.

For a proof of Proposition 3.2, see Section 6.2.

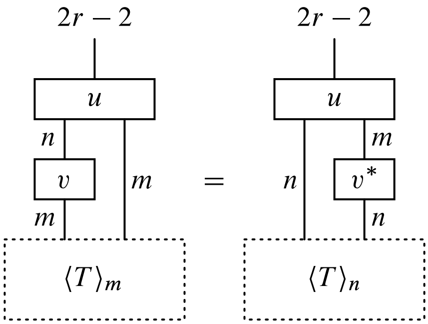
Remark 3.3. Let us consider the equivalence relation $\sim$ on the vector space

$$\bigoplus_{m=0}^{2r-2} \mathrm{TL}(m \otimes m, 2r-2)$$

determined, for all $0 \leqslant m,n \leqslant 2r-2$, $u \in \mathrm{TL}(n \otimes m, 2r-2)$, and $v \in \mathrm{TL}(m,n)$, by



Using the Kauffman bracket of equation (32), every $T \in \mathcal{T}_1(\varnothing)$ satisfies



Indeed, this can be shown using isotopy and the naturality of the braiding of TL. Furthermore, $\mathrm{TL}(m,n)$ is non-zero only if $m-n$ is even, in which case

$$(-1)^{m\,\mathrm{fr}(\mathrm{pc}(T))} = (-1)^{n\,\mathrm{fr}(\mathrm{pc}(T))}.$$

Then it is easy to see that every linear combination of morphisms which is equivalent to the non-semisimple Kirby color Ω determines the same topological invariant F'_Ω.

3.4. 3-Manifold invariant

We are now ready to define a topological invariant of closed 3-manifolds decorated with admissible bichrome links. The definition relies on the following computation.

Lemma 3.4. *For every admissible bichrome link $T \in \mathcal{B}(\varnothing)$ we have*

$F'_\Omega\Bigg($ 

$\Bigg) = \Delta_+ F'_\Omega(T),\quad F'_\Omega\Bigg($ 

$\Bigg) = \Delta_- F'_\Omega(T),$ (36)

where

$$\Delta_+ := i^{-\frac{r-1}{2}} r^{\frac{3}{2}} q^{\frac{r-3}{2}}, \quad \Delta_- := i^{\frac{r-1}{2}} r^{\frac{3}{2}} q^{\frac{r+3}{2}}.$$

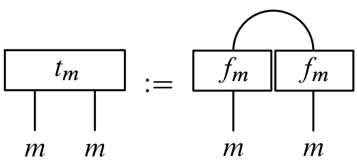
A proof of Lemma 3.4 will be given in Section 6.3. We are now ready to recall our main statement.

Theorem 3.5. *If M is a closed 3-manifold, $T \subset M$ is an admissible bichrome link, and $L \subset S^3$ is a red surgery presentation of M with positive signature σ_+ and negative signature σ_-, then*

$$Z_r(M,T) := \frac{F'_\Omega(L \cup T)}{\Delta_+^{\sigma_+} \Delta_+^{\sigma_-}} \tag{37}$$

is a topological invariant of (M,T), meaning it is independent of the choice of L.

For a proof of Theorem 3.5, see Section 6.3. In the meantime, let us relate Z_r to its semisimple counterpart by reviewing the construction of the SO(3) WRT invariant τ_r. In order to do this, let us set



$\in \mathrm{TL}(f_m \otimes f_m, f_0)$

for every integer $0 \leqslant m \leqslant r-2$. The *semisimple Kirby color* ω is the linear combination of morphisms

$$\omega := \sum_{m=0}^{r-2} \omega_m \in \bigoplus_{m=0}^{r-2} \mathrm{TL}(m \otimes m, 0),$$

where $\omega_m \in \mathrm{TL}(m \otimes m, 0)$ is given by

$$\omega_m := \begin{cases} [m+1]t_m & \text{if } m \equiv 0 \mod 2, \\ 0 & \text{if } m \equiv 1 \mod 2. \end{cases}$$

Remark 3.6. Another possibility is to consider the *double Kirby color* $\tilde{\omega}$, which is the linear combination of morphisms

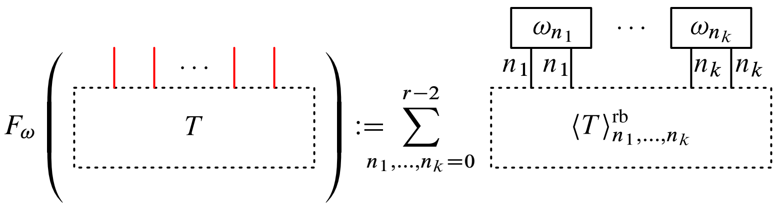
$$\tilde{\omega} := \sum_{m=0}^{r-2} \tilde{\omega}_m \in \bigoplus_{m=0}^{r-2} \mathrm{TL}(m \otimes m, 0),$$

where $\tilde{\omega}_m \in \mathrm{TL}(m \otimes m, 0)$ is given by

$$\tilde{\omega}_m := (-1)^m [m+1] t_m.$$

However, the invariant obtained from $\tilde{\omega}$ turns out to be essentially equivalent to the one obtained from ω. Furthermore, in order to extend τ_r to a TQFT, only the (idempotent completion of the) full subcategory of TL whose objects are even integers should be considered. A proof of this fact, in the equivalent language of the small quantum group $\bar{U}_q \mathfrak{sl}_2$, is given in [6, Section 5]. Therefore, in accordance with [10, Section 2.3], we use ω instead of $\tilde{\omega}$.

The semisimple Kirby color ω can be used to associate with every bichrome k-top tangle $T \in \mathcal{T}_k(\varnothing)$ a scalar $F_\omega(T) \in \mathrm{TL}(0) = \mathbb{C}$. Indeed, using the Kauffman bracket of equation (32), we can set

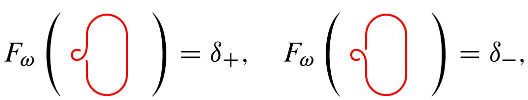


Remark that, since ω is a linear combination of even terms, replacing the Kauffman bracket of equation (32) with its ribbon version of equation (34) in the definition of F_ω yields the same result. Now, if $T \in \mathcal{B}(\varnothing)$ is a bichrome link and $T' \in \mathcal{T}_k(\varnothing)$ is a top tangle presentation of T, then

$$F_\omega(T) := F_\omega(T')$$

is a topological invariant of T, meaning it is independent of the choice of T'. Indeed, this is clear because the semisimple Kirby color ω has target $0 \in \mathrm{TL}$, which means $F_\omega(T)$ can be alternatively defined starting directly from a diagram of T, without ever choosing a top tangle presentation T'.

Lemma 3.7. *We have*



(38)

where

$$\delta_+ := \frac{i^{-\frac{r-1}{2}} r^{\frac{1}{2}} q^{\frac{r-3}{2}}}{\{1\}}, \quad \delta_- := -\frac{i^{\frac{r-1}{2}} r^{\frac{1}{2}} q^{\frac{r+3}{2}}}{\{1\}}.$$

A proof of Lemma 3.7 can be found in Section 6.3. The following result follows from [5, Theorem III.1], where τ_r is denoted by $\tilde{\vartheta}_A$.

Theorem 3.8 (Blanchet). *If M is a closed 3-manifold, $T \subset M$ is a bichrome link, and $L \subset S^3$ is a red surgery presentation of M with positive signature σ_+ and negative signature σ_-, then*

$$\tau_r(M,T) := \frac{F_\omega(L \cup T)}{\delta_+^{\sigma_+} \delta_-^{\sigma_-}} \tag{39}$$

is a topological invariant of (M,T), meaning it is independent of the choice of L.

If $\mathrm{b}_1(M)$ denotes the first Betti number of M, let us set

$$h_1(M) = \begin{cases} |H_1(M)| & \text{if } \mathrm{b}_1(M) = 0, \\ 0 & \text{if } \mathrm{b}_1(M) > 0. \end{cases}$$

Proposition 3.9. *If M and M' are closed 3-manifolds, $T \subset M$ is a bichrome link, and $T' \subset M'$ is an admissible bichrome link, then*

$$Z_r(M \# M', T \cup T') = h_1(M)\tau_r(M,T)Z_r(M',T').$$

For a proof of Proposition 3.9, see Section 6.3. For the moment, let us draw a few simple consequences from it. First of all, if both T and T' are admissible, then

$$Z_r(M \# M', T \cup T') = 0,$$

because τ_r vanishes against 3-manifolds decorated with admissible bichrome links, as a consequence of equation (8) with $m = r-1$. Next, if M is a closed 3-manifold and $O \subset S^3$ is a blue unknot of framing 0 and label f_{r-1}, then

$$Y_r(M) := Z_r(M \# S^3, O)$$

is a topological invariant of M. Furthermore, the chosen normalization of the modified trace t on Proj(TL) implies, by definition, that

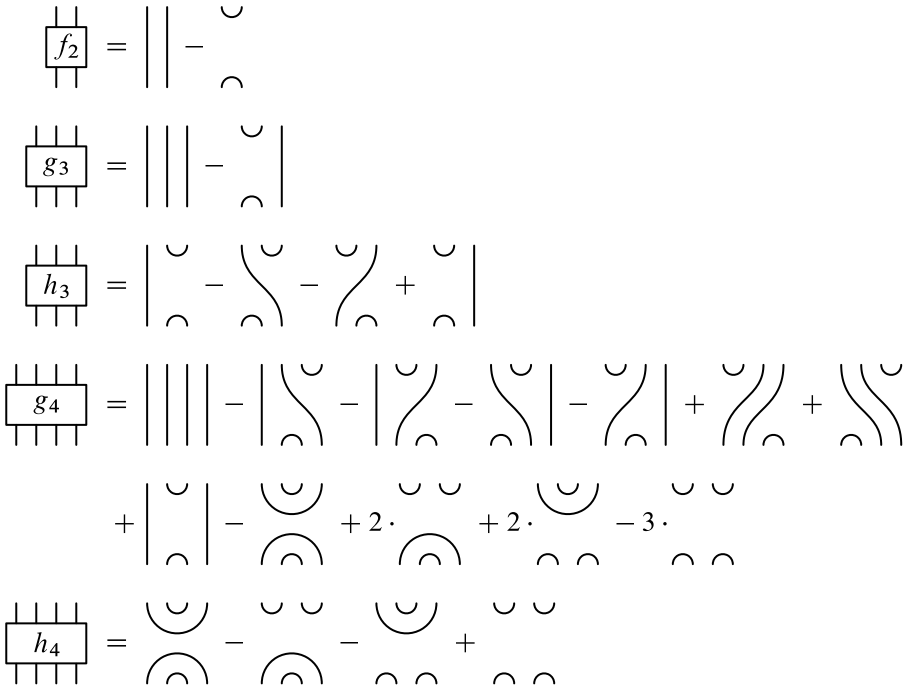
$$Z_r(S^3, O) = 1.$$

Then Proposition 3.9 immediately yields

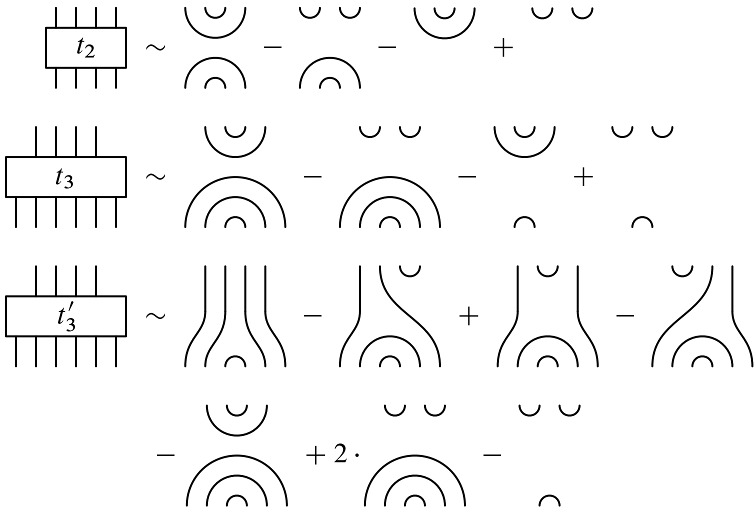
$$Y_r(M) = h_1(M)\tau_r(M).$$

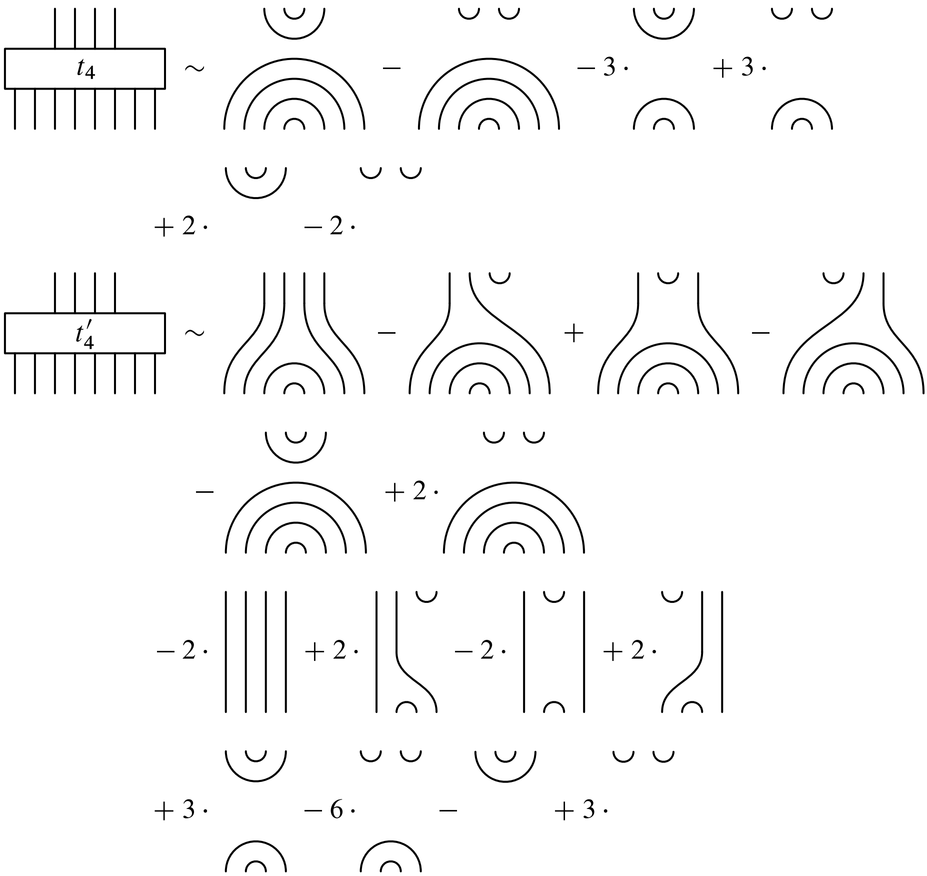
3.5. Level 3

Let us unpack some definitions for the first level $r = 3$. In this case, we have $q = e^{\frac{2\pi i}{3}}$, which means $[2] = -1$. For what concerns both simple and non-semisimple Jones–Wenzl idempotents, as well as their nilpotent endomorphisms, a direct computation gives



For what concerns components of the non-semisimple Kirby color, we have





where $\sim$ denotes the equivalence relation introduced in Remark 3.3.

4. Small quantum group

In this section we recall the definition of the small quantum group of $\mathfrak{sl}_2$ at odd roots of unity, which was first given by Lusztig in [33], as well as crucial results concerning its representation theory.

4.1. Definition

We denote by $\bar{U} = \bar{U}_q\,\mathfrak{sl}_2$ the *small quantum group of* $\mathfrak{sl}_2$, which is defined as the algebra over $\mathbb{C}$ with generators $\{E, F, K\}$ and relations

$$E^r = F^r = 0, \quad K^r = 1,$$

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

A Poincaré–Birkhoff–Witt basis of $\bar{U}$ is given by

$$\{E^a F^b K^c \mid 0 \leqslant a, b, c \leqslant r-1\}.$$

We make $\bar{U}$ into a Hopf algebra by setting

$$\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1}, \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF, \\ \Delta(K) &= K \otimes K, & \varepsilon(K) &= 1, & S(K) &= K^{-1}. \end{aligned}$$

We also fix an *R-matrix* $R = R' \otimes R'' \in \bar{U} \otimes \bar{U}$ given by

$$R := \frac{1}{r} \sum_{a,b,c=0}^{r-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2} - 2bc} K^b E^a \otimes K^c F^a,$$

whose inverse $R^{-1} = S(R') \otimes R'' \in \bar{U} \otimes \bar{U}$ is given by

$$R^{-1} = \frac{1}{r} \sum_{a,b,c=0}^{r-1} \frac{\{-1\}^a}{[a]!} q^{-\frac{a(a-1)}{2} + 2bc} E^a K^b \otimes F^a K^c.$$

Furthermore, we fix a *ribbon element* $v_+ \in \bar{U}$ given by

$$v_+ := \frac{i^{\frac{r-1}{2}}}{\sqrt{r}} \sum_{a,b=0}^{r-1} \frac{\{-1\}^a}{[a]!} q^{-\frac{a(a-1)}{2} + \frac{(r+1)(a-b-1)^2}{2}} F^a K^b E^a,$$

whose inverse $v_- \in \bar{U}$ is given by

$$v_- = \frac{i^{-\frac{r-1}{2}}}{\sqrt{r}} \sum_{a,b=0}^{r-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2} + \frac{(r-1)(a+b-1)^2}{2}} F^a K^b E^a.$$

These data make $\bar{U}$ into a ribbon Hopf algebra, and also determine further additional structures. For instance, the unique *pivotal element* $g \in \bar{U}$ compatible with the specified ribbon structure, in the sense that $g = S(R'')R'v_-$, is given by $g := K$, see [27, Proposition XIV.6.5]. Furthermore, the *M-matrix* $M = M' \otimes M'' \in \bar{U} \otimes \bar{U}$ defined by

$$M := R_{21} R_{12},$$

where $R_{12} = R$ is the R-matrix and R_{21} is obtained from R by reversing the order of its components, is given by

$$M = \frac{1}{r} \sum_{a,b,c,d=0}^{r-1} \frac{\{1\}^{a+b}}{[a]![b]!} q^{\frac{a(a-1)+b(b-1)}{2} - 2cd - (b+c)(b-d)} F^b K^c E^a \otimes E^b K^d F^a.$$

This determines a linear map $D\colon \bar{U}^* \to \bar{U}$ defined by

$$D(\varphi) := \varphi(M')M''$$

for every $\varphi \in \bar{U}^*$. Maps of this form were first considered in [15, Proposition 3.3], and D is therefore known as the *Drinfeld map*. The ribbon Hopf algebra $\bar{U}$ is *factorizable* in the sense that D is an isomorphism, as first shown in [34, Corollary A.3.3], see also [35, Example 3.4.3]. In particular, thanks to the construction of [14], the small quantum group $\bar{U}$ gives rise to a topological invariant of closed 3-manifolds, and more generally to a TQFT.

4.2. Center

Let us describe the center $Z(\bar{U})$ of the algebra $\bar{U}$, which has been studied in detail by Kerler in [30] starting from the *quantum Casimir element*

$$C := EF + \frac{q^{-1}K + qK^{-1}}{\{1\}^2} = FE + \frac{qK + q^{-1}K^{-1}}{\{1\}^2} \in Z(\bar{U}). \tag{40}$$

The minimal polynomial of C is

$$\Psi(X) = \prod_{m=0}^{r-1}(X - \beta_m) = (X - \beta_{r-1}) \prod_{m=0}^{\frac{r-3}{2}}(X - \beta_m)^2,$$

where

$$\beta_m := \frac{\{m+1\}'}{\{1\}^2}.$$

Indeed, $\beta_{r-m-2} = \beta_m$ for every integer $0 \leqslant m \leqslant r-2$. If we set

$$\Psi_{r-1}(X) = \frac{\Psi(X)}{(X - \beta_{r-1})}, \qquad \Psi_m(X) = \frac{\Psi(X)}{(X - \beta_m)^2}$$

for every integer $0 \leqslant m \leqslant r-2$, then we can define the central elements

$$e_{r-1} := \frac{\Psi_{r-1}(C)}{\Psi_{r-1}(\beta_{r-1})} \in Z(\bar{U}), \tag{41}$$

$$e_m := \frac{\Psi_m(C)}{\Psi_m(\beta_m)} - \frac{\Psi'_m(\beta_m)(C - \beta_m)\Psi_m(C)}{\Psi_m(\beta_m)^2} \in Z(\bar{U}), \tag{42}$$

$$w_m := \frac{(C - \beta_m)\Psi_m(C)}{\Psi_m(\beta_m)} \in Z(\bar{U}). \tag{43}$$

Furthermore, if we consider the non-central projector

$$v_m := \frac{1}{r}\sum_{a=0}^{r-1} q^{-am}K^a \in \bar{U} \tag{44}$$

on the eigenspace of eigenvalue q^m for the regular action of K on $\bar{U}$, then setting

$$T_m := \sum_{j=0}^{m} v_{m-2j} \in \bar{U} \tag{45}$$

for every integer $0 \leqslant m \leqslant r-2$ allows us to decompose

$$w_m = w_m^+ + w_m^-$$

for the central elements

$$w_m^+ := T_m \frac{(C-\beta_m)\Psi_m(C)}{\Psi_m(\beta_m)} \in Z(\bar{U}), \tag{46}$$

$$w_m^- := (1-T_m) \frac{(C-\beta_m)\Psi_m(C)}{\Psi_m(\beta_m)} \in Z(\bar{U}). \tag{47}$$

It is proven in [30, Lemma 14] that a basis of $Z(\bar{U})$ is given by

$$\{e_{r-1}\} \cup \left\{ e_m, w_m^+, w_m^- \;\middle|\; 0 \leqslant m \leqslant \frac{r-3}{2} \right\},$$

and that basis vectors satisfy

$$e_m e_{m'} = \delta_{m,m'} e_m, \quad w_m^\varepsilon e_{m'} = \delta_{m,m'} w_m^\varepsilon, \quad w_m^\varepsilon w_{m'}^{\varepsilon'} = 0. \tag{48}$$

Remark that $e_{r-m-2} = e_m$ and $w_{r-m-2}^\varepsilon = w_m^{-\varepsilon}$ for every integer $0 \leqslant m \leqslant r-2$. For future convenience, we also set

$$e_m := e_{2r-m-2}, \quad w_m^\varepsilon := w_{2r-m-2}^\varepsilon, \quad w_m := w_{2r-m-2},$$

for every integer $r \leqslant m \leqslant 2r-2$.

Lemma 4.1. *The ribbon element and its inverse $v_+, v_- \in Z(\bar{U})$ admit the Jordan decompositions*

$$v_\pm = q^{\frac{r\pm1}{2}} e_{r-1} + \sum_{m=0}^{\frac{r-3}{2}} q^{\frac{r\mp1}{2}m^2 \mp m} \left(e_m \mp \frac{(m+1)\{1\}}{[m+1]} w_m^+ \mp \frac{(m-r+1)\{1\}}{[m+1]} w_m^- \right).$$

Proof. The formula for v_- is obtained from [30, Lemma 15] by carefully comparing Kerler's conventions with ours. Indeed, what Kerler refers to as the ribbon element is actually v_- in our notation. Furthermore, Kerler uses a different set of generators for the definition of $\bar{U}_q\,\mathfrak{sl}_2$, which can be obtained from ours by replacing E with $\{1\}E$. Consequently, Kerler's quantum Casimir element is equal to $\{1\}C$, and his basis of $Z(\bar{U})$ is

$$\{e_{r-1}\} \cup \left\{ e_m, \{1\}w_m^+, \{1\}w_m^- \;\middle|\; 0 \leqslant m \leqslant \frac{r-3}{2} \right\}.$$

The formula for v_+ is obtained from the one for v_- using equation (48). ■

4.3. Finite-dimensional representations

The *category* $\bar{U}$-mod *of finite-dimensional representations of* $\bar{U}$ supports the structure of a ribbon category. In order to recall it, let us adopt Sweedler's notation for iterated coproducts, that is,

$$\Delta^{(m)}(x) = x_{(1)} \otimes \cdots \otimes x_{(m+1)} \in \bar{U}^{\otimes m+1}$$

for all $x \in \bar{U}$ and $m \in \mathbb{N}$. Then the coproduct Δ is used to define, for all $V, W \in \bar{U}$-mod, the tensor product $V \otimes W$, which is determined by

$$x \cdot v \otimes w := (x_{(1)} \cdot v) \otimes (x_{(2)} \cdot w)$$

for all $x \in \bar{U}$, $v \in V$, and $w \in W$. The counit ε is used to define the tensor unit $\mathbb{C} \in \bar{U}$-mod, which is determined by the trivial representation over $\mathbb{C}$ given by

$$x \cdot 1 := \varepsilon(x)$$

for every $x \in \bar{U}$. The antipode S is used to define, for every $V \in \bar{U}$-mod, the two-sided dual V^*, which is determined by

$$(x \cdot \varphi)(v) := \varphi(S(x) \cdot v)$$

for all $x \in \bar{U}$, $\varphi \in V^*$, and $v \in V$. The pivotal element g and its inverse are used to define, for every $V \in \bar{U}$-mod, left and right evaluation and coevaluation morphisms $\overleftarrow{\mathrm{ev}}_V \colon V^* \otimes V \to \mathbb{C}$, $\overleftarrow{\mathrm{coev}}_V \colon \mathbb{C} \to V \otimes V^*$, $\overrightarrow{\mathrm{ev}}_V \colon V \otimes V^* \to \mathbb{C}$, and $\overrightarrow{\mathrm{coev}}_V \colon \mathbb{C} \to V^* \otimes V$, which are determined by

$$\begin{aligned} \overleftarrow{\mathrm{ev}}_V(\varphi \otimes v) &:= \varphi(v), & \overleftarrow{\mathrm{coev}}_V(1) &:= \sum_{i=1}^{n} v_i \otimes \varphi^i, \\ \overrightarrow{\mathrm{ev}}_V(v \otimes \varphi) &:= \varphi(g \cdot v), & \overrightarrow{\mathrm{coev}}_V(1) &:= \sum_{i=1}^{n} \varphi^i \otimes (g^{-1} \cdot v_i), \end{aligned}$$

for all $v \in V$ and $\varphi \in V^*$, where $\{v_i \in V \mid 1 \leqslant i \leqslant n\}$ and $\{\varphi^i \in V^* \mid 1 \leqslant i \leqslant n\}$ are dual bases. The R-matrix R is used to define, for all $V, W \in \bar{U}$-mod, the braiding morphism $c_{V,W} \colon V \otimes W \to W \otimes V$, which is determined by

$$c_{V,W}(v \otimes w) := (R'' \cdot w) \otimes (R' \cdot v)$$

for all $v \in V$ and $w \in W$. The inverse ribbon element v_- is used to define, for every $W \in \bar{U}$-mod, the twist morphism $\vartheta_W \colon W \to W$, which is determined by

$$\vartheta_W(w) := v_- \cdot w$$

for every $w \in W$.

4.4. Simple and projective modules

Let us recall the classification of *simple* and *indecomposable projective* $\bar{U}$-modules. For every integer $0 \leqslant m \leqslant r-1$ we denote by X_m the simple $\bar{U}$-module with basis

$$\{a_j^m \mid 0 \leqslant j \leqslant m\}$$

and action given, for all integers $0 \leqslant j \leqslant m$, by

$$\begin{aligned} K \cdot a_j^m &= q^{m-2j} a_j^m, \\ E \cdot a_j^m &= [j][m-j+1] a_{j-1}^m, \\ F \cdot a_j^m &= a_{j+1}^m, \end{aligned}$$

where $a_{-1}^m := a_{m+1}^m := 0$. Among these, we highlight the *fundamental representation* $X := X_1$, which is a monoidal generator for the family of $\bar{U}$-modules considered in this paper (in the sense that every simple and every indecomposable projective $\bar{U}$-module is a direct summand of a tensor power of X), and the *Steinberg module* X_{r-1}, which is the only $\bar{U}$-module which is both simple and projective. Every simple object of $\bar{U}$-mod is isomorphic to X_m for some integer $0 \leqslant m \leqslant r-1$. Next, for every integer $r \leqslant m \leqslant 2r-2$ we denote by P_m the indecomposable projective $\bar{U}$-module with basis

$$\{a_j^m, x_k^m, y_k^m, b_j^m \mid 0 \leqslant j \leqslant 2r-m-2, 0 \leqslant k \leqslant m-r\}$$

and action given, for all integers $0 \leqslant j \leqslant 2r-m-2$ and $0 \leqslant k \leqslant m-r$, by

$$\begin{aligned} K \cdot a_j^m &= q^{-m-2j-2} a_j^m, \\ E \cdot a_j^m &= -[j][m+j+1] a_{j-1}^m, \\ F \cdot a_j^m &= a_{j+1}^m, \\ K \cdot x_k^m &= q^{m-2k} x_k^m, \\ E \cdot x_k^m &= [k][m-k+1] x_{k-1}^m, \\ F \cdot x_k^m &= \begin{cases} x_{k+1}^m & \text{if } 0 \leqslant k < m-r, \\ a_0^m & \text{if } k = m-r, \end{cases} \\ K \cdot y_k^m &= q^{m-2k} y_k^m, \\ E \cdot y_k^m &= \begin{cases} a_{2r-m-2}^m & \text{if } k = 0, \\ [k][m-k+1] y_{k-1}^m & \text{if } 0 < k \leqslant m-r, \end{cases} \\ F \cdot y_k^m &= y_{k+1}^m, \\ K \cdot b_j^m &= q^{-m-2j-2} b_j^m, \end{aligned}$$

$$E \cdot b_j^m = \begin{cases} x_{m-r}^m & \text{if } j = 0, \\ a_{j-1}^m - [j][m+j+1]b_{j-1}^m & \text{if } 0 < j \leqslant 2r-m-2, \end{cases}$$

$$F \cdot b_j^m = \begin{cases} b_{j+1}^m & \text{if } 0 \leqslant j < 2r-m-2, \\ y_0^m & \text{if } j = 2r-m-2, \end{cases}$$

where $a_{-1}^m := a_{2r-m-1}^m := x_{-1}^m := y_{m-r+1}^m := 0$. We point out that P_m is sometimes denoted by P_{2r-m-2}, because it is the projective cover of X_{2r-m-2} for every integer $r \leqslant m \leqslant 2r-2$. Every indecomposable projective object of $\bar{U}$-mod is isomorphic to either X_{r-1} or P_m for some integer $r \leqslant m \leqslant 2r-2$.

Fusion formulas for decompositions of tensor products in $\bar{U}$-mod are given by

$$X_{m-1} \otimes X_1 \cong \begin{cases} X_1 & \text{if } m = 1, \\ X_m \oplus X_{m-2} & \text{if } 1 < m \leqslant r-1, \\ P_r & \text{if } m = r, \end{cases} \tag{49}$$

$$P_{m-1} \otimes X_1 \cong \begin{cases} P_{r+1} \oplus X_{r-1} \oplus X_{r-1} & \text{if } m = r+1, \\ P_m \oplus P_{m-2} & \text{if } r+1 < m \leqslant 2r-2, \\ X_{r-1} \oplus X_{r-1} \oplus P_{2r-3} & \text{if } m = 2r-1. \end{cases} \tag{50}$$

These formulas should be compared with equations (4)–(6) and (9)–(11) using [7, Lemma 4.1].

As proved in [20, Theorem 4.7.1], the ideal $\mathrm{Proj}(\bar{U}\text{-mod})$ of projective objects of $\bar{U}$-mod is generated by the Steinberg module X_{r-1}, and there exists a unique trace $\mathrm{t}^{\bar{U}}$ on $\mathrm{Proj}(\bar{U}\text{-mod})$ satisfying

$$\mathrm{t}_{X_{r-1}}^{\bar{U}}(\mathrm{id}_{X_{r-1}}) = 1. \tag{51}$$

Furthermore, $\mathrm{t}^{\bar{U}}$ is non-degenerate.

4.5. Regular representation

The regular representation of $\bar{U}$, which is determined by the regular action of $\bar{U}$ onto itself by left multiplication, decomposes into a direct sum of indecomposable projective $\bar{U}$-modules. Explicit bases for indecomposable projective factors of $\bar{U}$ are described in [3, Section 4] when r is even, but can be easily generalized to our case. Basis vectors are defined in terms of the non-central projectors $v_m \in \bar{U}$ given by equation (44) for $m \in \mathbb{Z}$. If, for every integer $0 \leqslant j \leqslant r-1$, we set

$$a_j^{r-1,n} := F^j E^{r-1} F^{r-n-1} v_{-2n-1},$$

then

$$\{a_j^{r-1,n} \in \bar{U} \mid 0 \leqslant j \leqslant r-1\}$$

is a basis for a submodule $\bar{X}_{r-1,n}$ of $\bar{U}$ which is isomorphic to X_{r-1}. Similarly, if, for all integers $r \leqslant m \leqslant 2r-2$, $0 \leqslant j \leqslant 2r-m-2$, and $0 \leqslant k \leqslant m-r$, we set

$$\begin{aligned}
a_j^{m,n} &:= F^j E^{r-1} F^{r-n-1} v_{-m-2n-2}, \\
x_k^{m,n} &:= \sum_{h=0}^{m-r} \frac{[k]![h]!}{[m-k-r]![m-h-r]!} E^{m-k-h-1} F^{r-n-h-2} v_{-m-2n-2}, \\
y_k^{m,n} &:= \sum_{h=0}^{m-r} \frac{[m-r]![h]!}{[m-h-r]!} F^{2r-m+k-1} E^{r-h-2} F^{r-n-h-2} v_{-m-2n-2}, \\
b_j^{m,n} &:= \sum_{h=0}^{m-r} \frac{[m-r]![h]!}{[m-h-r]!} F^j E^{r-h-2} F^{r-n-h-2} v_{-m-2n-2},
\end{aligned}$$

then

$$\{a_j^{m,n}, x_k^{m,n}, y_k^{m,n}, b_j^{m,n} \in \bar{U} \mid 0 \leqslant j \leqslant 2r-m-2, 0 \leqslant k \leqslant m-r\}$$

is a basis for a submodule $\bar{P}_{m,n}$ of $\bar{U}$ which is isomorphic to P_m. If we set

$$\bar{X}_{r-1} := \bigoplus_{n=0}^{r-1} \bar{X}_{r-1,n}, \quad \bar{P}_m := \bigoplus_{n=0}^{2r-m-2} \bar{P}_{m,n},$$

then it can be easily checked that

$$\bar{U} \cong \bar{X}_{r-1} \oplus \bigoplus_{m=r}^{2r-2} \bar{P}_m.$$

4.6. Integral

A key ingredient for the HKR approach to the construction of 3-manifold invariants are integrals, whose theory is well established [41, 43]. A *left integral* $\lambda \in \bar{U}^*$ and a *right integral* $\mu \in \bar{U}^*$ are linear forms satisfying

$$\lambda(x_{(2)})x_{(1)} = \lambda(x)1 \in \bar{U}, \quad \mu(x_{(1)})x_{(2)} = \mu(x)1 \in \bar{U}^*$$

for every $x \in \bar{U}^*$. Since $\bar{U}$ is finite-dimensional, left integrals and right integrals span one-dimensional vector spaces. If λ is a left integral, then $\lambda \circ S$ is a right integral, and similarly, if μ is a right integral, then $\mu \circ S$ is a left integral. Every left integral $\lambda \in \bar{U}^*$ satisfies

$$\lambda(xy) = \lambda(yS^2(x)) \tag{52}$$

for all $x, y \in \bar{U}^*$, which means it can be regarded as an intertwiner between the trivial representation $\mathbb{C}$ and the *adjoint representation* ad, which is determined by the action of $\bar{U}$ onto itself given by

$$\mathrm{ad}_x(y) := x_{(1)} y S(x_{(2)})$$

for all $x, y \in \bar{U}^*$. Every left integral $\lambda \in \bar{U}^*$ is of the form

$$\lambda(E^a F^b K^c) := \xi \delta_{a,r-1} \delta_{b,r-1} \delta_{c,r-1}$$

for some $\xi \in \mathbb{C}$. For the purpose of our construction, it will be convenient to fix the left integral $\lambda \in \bar{U}^*$ determined by the coefficient

$$\xi := r([r-1]!)^2 = \frac{r^3}{\{1\}^{2r-2}} \in \mathbb{C}^*.$$

The *stabilization parameters* corresponding to this normalization are

$$\lambda(v_+) = i^{\frac{r-1}{2}} r^{\frac{3}{2}} q^{\frac{r+3}{2}}, \quad \lambda(v_-) = i^{-\frac{r-1}{2}} r^{\frac{3}{2}} q^{\frac{r-3}{2}}. \tag{53}$$

Following the approach of [3, Section 5], we give an explicit decomposition of the left integral $\lambda \in \bar{U}^*$ into a linear combination of *quantum traces* and *pseudo quantum traces* corresponding to indecomposable projective $\bar{U}$-modules. This requires a few preliminary definitions. First of all, for every Hopf algebra H, we denote by $\mathrm{QC}(H)$ the space of *quantum characters* of H, which are linear forms $\varphi \in H^*$ satisfying $\varphi(xy) = \varphi(yS^2(x))$ for all $x, y \in H$. Thanks to equation (52), left integrals are quantum characters. Similarly, for every algebra A, we denote by $\mathrm{SLF}(A)$ the space of *symmetric linear functions* on A, which are linear forms $\varphi \in A^*$ satisfying $\varphi(xy) = \varphi(yx)$ for all $x, y \in A$. A pivotal element $g \in H$, which is a group-like element satisfying $gxg^{-1} = S^2(x)$ for every $x \in H$, defines an isomorphism between $\mathrm{QC}(H)$ and $\mathrm{SLF}(H)$ sending every quantum character φ to the symmetric linear function $\varphi(\_g^{-1})$. For a finite-dimensional ribbon Hopf algebra H and a non-zero left integral $\lambda \in H^*$, [24, Proposition 4.2] gives

$$\mathrm{SLF}(H) = \{\lambda(\_g^{-1}z) \in H^* \mid z \in Z(H)\}.$$

Therefore, the idea is to use the basis of $Z(\bar{U})$ given in Section 4.2 in order to find a basis of $\mathrm{QC}(\bar{U})$, and then to compute the corresponding coefficients of λ.

If, for every integer $r-1 \leqslant m \leqslant 2r-2$, we denote by $\bar{Q}_m$ the generalized eigenspace of eigenvalue β_m for the regular action of the quantum Casimir element $C \in Z(\bar{U})$ on the regular representation $\bar{U}$, then the same argument of [17, Proposition D.1.1] shows

$$\bar{Q}_m = \begin{cases} \bar{X}_{r-1} & \text{if } m = r-1, \\ \bar{P}_m \oplus \bar{P}_{3r-m-2} & \text{if } r \leqslant m \leqslant 2r-2. \end{cases}$$

The regular action of the canonical central element $e_m \in Z(\bar{U})$ recovers the projector onto $\bar{Q}_m$ with respect to the decomposition

$$\bar{U} = \bigoplus_{m=r-1}^{3\frac{r-1}{2}} \bar{Q}_m,$$

and a basis for $Z(\bar{Q}_m)$ is given by $\{e_{r-1}\}$, if $m = r-1$, and by $\{e_m, w_m^+, w_m^-\}$, if $r \leqslant m \leqslant 2r-2$. We will first decompose the symmetric linear form

$$\lambda(\_K^{-1}e_m) \in \mathrm{SLF}(\bar{Q}_m)$$

with respect to the corresponding basis of $\mathrm{SLF}(\bar{Q}_m)$.

If, for every integer $0 \leqslant n \leqslant r-1$, we set

$$A_j^{r-1,n} := \frac{1}{([r-1]!)^2} a_j^{r-1,n} \in \bar{X}_{r-1},$$

then

$$\{A_j^{r-1,n} \in \bar{X}_{r-1} \mid 0 \leqslant j, n \leqslant r-1\}$$

is a basis of $\bar{X}_{r-1}$, and we can denote by

$$\{\psi_{r-1,n}^j \in \bar{X}_{r-1}^* \mid 0 \leqslant j, n \leqslant r-1\}$$

the dual basis. Then, in the standard basis of X_{r-1}, the action of $\bar{X}_{r-1}$ is represented by the matrix

$$\psi_{r-1} \in M_{r\times r}(\bar{X}_{r-1}^*)$$

whose (j,n)-th entry is given by $\psi_{r-1,n}^j$. Therefore, if we set

$$\tau_{r-1} := \mathrm{tr}(\psi_{r-1}) = \sum_{n=0}^{r-1} \psi_{r-1,n}^n,$$

then $\{\tau_{r-1}\}$ is a basis of $\mathrm{SLF}(\bar{Q}_{r-1})$. We call τ_{r-1} the $r-1$-*trace*.

Lemma 4.2. *We have*

$$\lambda(\_K^{-1}e_{r-1}) = \tau_{r-1}.$$

Proof. On the one hand, we have

$$\lambda(a_j^{r-1,n}K^{-1}e_{r-1}) = \lambda(F^jE^{r-1}F^{r-n-1}v_{-2n-1}K^{-1}) = ([r-1]!)^2\delta_{j,n}.$$

This implies

$$\lambda(A_j^{r-1,n}K^{-1}e_{r-1}) = \delta_{j,n}.$$

On the other hand,

$$\tau_{r-1}(A_j^{r-1,n}) = \delta_{j,n}.$$

∎

Next, if, for all integers $r \leqslant m \leqslant 2r-2$ and $0 \leqslant n \leqslant 2r-m-2$, we set

$$\begin{aligned} A_j^{m,n} &:= \frac{[m+1]^2}{([r-1]!)^2} a_j^{m,n} \in \bar{P}_m, \\ X_k^{m,n} &:= \frac{[m+1]^2}{([r-1]!)^2} x_k^{m,n} \in \bar{P}_m, \\ Y_k^{m,n} &:= \frac{[m+1]^2}{([r-1]!)^2} y_k^{m,n} \in \bar{P}_m, \\ B_j^{m,n} &:= \frac{[m+1]^2}{([r-1]!)^2} b_j^{m,n} + \frac{\{m+1\}'}{([r-1]!)^2} a_j^{m,n} \in \bar{P}_m, \end{aligned}$$

then

$$\{A_j^{m,n}, X_k^{m,n}, Y_k^{m,n}, B_j^{m,n} \in \bar{P}_m \mid 0 \leqslant j,n \leqslant 2r-m-2, 0 \leqslant k \leqslant m-r\}$$

is a basis of $\bar{P}_m$, and we can denote by

$$\{\psi_{m,n}^j, \xi_{m,n}^k, \zeta_{m,n}^k, \varphi_{m,n}^j \in \bar{P}_m^* \mid 0 \leqslant j,n \leqslant 2r-m-2, 0 \leqslant k \leqslant m-r\}$$

the dual basis. Then, in the standard bases of P_m and of P_{3r-m-2}, the action of $\bar{P}_m$ is represented by the matrices

$$\begin{pmatrix} \varphi_m & 0 & 0 & \psi_m \\ 0 & 0 & 0 & \xi_m \\ 0 & 0 & 0 & \zeta_m \\ 0 & 0 & 0 & \varphi_m \end{pmatrix}, \begin{pmatrix} 0 & \zeta_m & \xi_m & 0 \\ 0 & \varphi_m & 0 & 0 \\ 0 & 0 & \varphi_m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_{2r \times 2r}(\bar{P}_m^*),$$

where

$$\psi_m, \varphi_m \in M_{(2r-m-1)\times(2r-m-1)}(\bar{P}_m^*)$$

denote the matrices whose (j,n)-th entries are given by $\psi_{m,n}^j$, $\varphi_{m,n}^j$, and where

$$\xi_m, \zeta_m \in M_{(m-r+1)\times(2r-m-1)}(\bar{P}_m^*)$$

denote the matrices whose (k,n)-th entries are given by $\xi_{m,n}^k$, $\zeta_{m,n}^k$ respectively. Therefore, if we set

$$\tau_m := \operatorname{tr}(\varphi_m) = \sum_{n=0}^{2r-m-2} \varphi_{m,n}^n, \quad \tau'_m := \operatorname{tr}(\psi_m) = \sum_{n=0}^{2r-m-2} \psi_{m,n}^n,$$

then $\{\tau_m, \tau_{3r-m-2}, \tau'_m + \tau'_{3r-m-2}\}$ is a basis of $\mathrm{SLF}(\bar{Q}_m)$. We call τ_m the *m-trace* and τ'_m the *pseudo m-trace*. Remark that neither τ'_m nor τ'_{3r-m-2} is a symmetric linear function, only their sum $\tau'_m + \tau'_{3r-m-2}$ is.

Lemma 4.3. *For every integer $r \leqslant m \leqslant 2r-2$ we have*

$$\lambda(\_K^{-1}e_m) = \{m+1\}'(\tau_m + \tau_{3r-m-2}) + [m+1]^2(\tau'_m + \tau'_{3r-m-2}).$$

Proof. On one hand, we have

$$\begin{aligned}
&\lambda(a_j^{m,n}K^{-1}e_m) \\
&\quad = \lambda(F^j E^{r-1} F^{r-n-1} v_{-m-2n-2} K^{-1}) \\
&\quad = ([r-1]!)^2 \delta_{j,n}, \\
&\lambda(x_k^{m,n}K^{-1}e_m) \\
&\quad = \sum_{h=0}^{m-r} \frac{[k]![h]!}{[m-k-r]![m-h-r]!} \lambda(E^{m-k-h-1} F^{r-n-h-2} v_{-m-2n-2} K^{-1}) \\
&\quad = 0, \\
&\lambda(y_k^{m,n}K^{-1}e_m) \\
&\quad = \sum_{h=0}^{m-r} \frac{[m-r]![h]!}{[m-h-r]!} \lambda(F^{2r-m+k-1} E^{r-h-2} F^{r-n-h-2} v_{-m-2n-2} K^{-1}) \\
&\quad = 0, \\
&\lambda(b_j^{m,n}K^{-1}e_m) \\
&\quad = \frac{[m+1]^2}{([r-1]!)^2} \sum_{h=0}^{m-r} \frac{[m-r]![h]!}{[m-h-r]!} \lambda(F^j E^{r-h-2} F^{r-n-h-2} v_{-m-2n-2} K^{-1}) \\
&\quad = 0.
\end{aligned}$$

This implies

$$\begin{aligned}
\lambda(A_j^{m,n}K^{-1}e_m) &= [m+1]^2 \delta_{j,n}, \\
\lambda(X_k^{m,n}K^{-1}e_m) &= 0, \\
\lambda(Y_k^{m,n}K^{-1}e_m) &= 0, \\
\lambda(B_j^{m,n}K^{-1}e_m) &= \{m+1\}' \delta_{j,n}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tau_m(A_j^{m,n}) &= 0, & \tau'_m(A_j^{m,n}) &= \delta_{j,n}, \\
\tau_m(X_k^{m,n}) &= 0, & \tau'_m(X_k^{m,n}) &= 0, \\
\tau_m(Y_k^{m,n}) &= 0, & \tau'_m(Y_k^{m,n}) &= 0, \\
\tau_m(B_j^{m,n}) &= \delta_{j,n}, & \tau'_m(B_j^{m,n}) &= 0.
\end{aligned}$$

■

The previous discussion implies a basis of $\mathrm{QC}(\bar{U})$ is given by

$$\{\tau_m(\_K) \mid r-1 \leqslant m \leqslant 2r-2\} \cup \left\{\tau'_m(\_K) + \tau'_{3r-m-2}(\_K) \;\middle|\; r \leqslant m \leqslant 3\frac{r-1}{2}\right\}.$$

For every integer $r-1 \leqslant m \leqslant 2r-2$ we call $\tau_m(\_K)$ the *quantum m-trace*, and for every integer $r \leqslant m \leqslant 2r-2$ we call $\tau'_m(\_K)$ the *pseudo quantum m-trace*. We are now ready to decompose the left integral.

Proposition 4.4. *The left integral $\lambda \in \bar{U}^*$ can be written as*

$$\lambda = \tau_{r-1}(\_K) + \sum_{m=r}^{2r-2} \{m+1\}' \tau_m(\_K) + [m+1]^2 \tau'_m(\_K). \tag{54}$$

Proof. The formula follows from

$$\lambda = \sum_{m=r-1}^{3\frac{r-1}{2}} \lambda(\_e_m).$$ ∎

4.7. Adjoint representation

Let us highlight an important property of the adjoint representation ad of $\bar{U}$ which follows directly from the explicit description provided in [40]. The adjoint action of $\bar{U}$ onto itself is determined by

$$\mathrm{ad}_E(x) = [E,x]K^{-1}, \quad \mathrm{ad}_F(x) = K^{-1}[KF,x], \quad \mathrm{ad}_K(x) = KxK^{-1}$$

for every $x \in \mathrm{ad}$. We can define a $\mathbb{Z}$-grading on ad by setting

$$\deg(E) = 1, \quad \deg(F) = -1, \quad \deg(K) = 0.$$

Remark that for every integer $0 \leqslant m \leqslant r-1$ the generalized eigenspace $\bar{Q}_m$ defines a subrepresentation ad_m of ad, because C is central. In other words, we have

$$\mathrm{ad} = \bigoplus_{m=r-1}^{3\frac{r-1}{2}} \mathrm{ad}_m.$$

If ad^0_m denotes the space of degree 0 vectors of ad_m, a basis of ad^0_{r-1} is given by

$$\{K^a e_{r-1} \mid 0 \leqslant a \leqslant r-1\},$$

and similarly, for every integer $r \leqslant m \leqslant 2r-2$, a basis of ad^0_m is given by

$$\{K^a e_m, K^a w_m \mid 0 \leqslant a \leqslant r-1\}.$$

Lemma 4.5. *Every $\bar{U}$-module morphism from* ad_m *to* P_{2r-2} *is uniquely determined by its restriction to* ad^0_m.

Proof. As mentioned earlier, the proof follows from the explicit description of ad given in [40]. Indeed, if $m = r - 1$, the $\bar{U}$-module ad_{r-1} is projective, and it decomposes as

$$\underline{X}_{r-1,r-1} \oplus \bigoplus_{n=1}^{\frac{r-1}{2}} \underline{P}_{r-1,r+2n-1},$$

where $\underline{X}_{r-1,r-1}$ is isomorphic to X_{r-1}, and $\underline{P}_{r-1,r+2n-1}$ is isomorphic to P_{r+2n-1}. Each of these submodules is generated by some vector in ad^0_{r-1}. On the other hand, if $r \leqslant m \leqslant 3\frac{r-1}{2}$, the $\bar{U}$-module ad_m is not projective, and it decomposes as the direct sum of a projective submodule

$$\underline{X}_{m,r-1} \oplus \underline{X}'_{m,r-1} \oplus \left(\bigoplus_{n=1}^{3\frac{r-1}{2}-m} \underline{P}_{m,r+2n-1} \oplus \underline{P}'_{m,r+2n-1} \right) \oplus \bigoplus_{n=r}^{m} \underline{P}_{m,2(r-m+n-1)}$$

and a non-projective submodule

$$\bigoplus_{n=r}^{m} \underline{X}^{\downarrow}_{m,2(m-n)} \oplus \underline{X}^{+}_{m,r-2(m-n+1)} \oplus \underline{X}^{-}_{m,r-2(m-n+1)} \oplus \underline{X}^{\uparrow}_{m,2(m-n)},$$

where $\underline{X}_{m,r-1}$ and $\underline{X}'_{m,r-1}$ are isomorphic to X_{r-1}, $\underline{P}_{m,r+2n-1}$ and $\underline{P}'_{m,r+2n-1}$ are isomorphic to P_{r+2n-1}, $\underline{P}_{m,2(r-m+n-1)}$ is isomorphic to $P_{2(r-m+n-1)}$, $\underline{X}^{\downarrow}_{m,2(m-n)}$ and $\underline{X}^{\uparrow}_{m,2(m-n)}$ are isomorphic to $X_{2(m-n)}$, and $\underline{X}^{+}_{m,r-2(m-n+1)}$ and $\underline{X}^{-}_{m,r-2(m-n+1)}$ are isomorphic to $X_{r-2(m-n+1)}$. Each of these submodules is generated by some vector in ad^0_m, with the exception of $\underline{X}^{+}_{m,r-2(m-n+1)}$ and $\underline{X}^{-}_{m,r-2(m-n+1)}$ for every integer $r \leqslant n \leqslant m$. Remark however that these $\bar{U}$-modules admit no non-trivial morphism to P_{2r-2}, which is the projective cover of $X_0 = \mathbb{C}$. ■

5. Beads

In this section, we set up the technology for the comparison between the non-semisimple invariant Z_r and the renormalized Hennings invariant associated with $\bar{U}$. In order to establish their equivalence, we first need to relate the Temperley–Lieb category TL to the category $\bar{U}$-mod of finite-dimensional left $\bar{U}$-modules.

5.1. Ribbon structures

Let us consider the monoidal linear functor

$$F_{\mathrm{TL}} \colon \mathrm{TL} \to \bar{U}\text{-mod}$$

which sends the monoidal generator $1 \in \mathrm{TL}$ to the fundamental representation $X \in \bar{U}$-mod defined in Section 4.4, the evaluation $\mathrm{ev}_1 \in \mathrm{TL}(2,0)$ to the morphism $e \in \mathrm{Hom}_{\bar{U}}(X \otimes X, \mathbb{C})$ defined by

$$e(a_0^1 \otimes a_0^1) := 0, \quad e(a_0^1 \otimes a_1^1) := -1, \quad e(a_1^1 \otimes a_0^1) := q^{-1}, \quad e(a_1^1 \otimes a_1^1) := 0, \tag{55}$$

and the coevaluation $\mathrm{coev}_1 \in \mathrm{TL}(0,2)$ to the morphism $c \in \mathrm{Hom}_{\bar{U}}(\mathbb{C}, X \otimes X)$ defined by

$$c(1) := qa_0^1 \otimes a_1^1 - a_1^1 \otimes a_0^1. \tag{56}$$

As a consequence of the skein relation (S1), the functor F_{TL} is braided. This means it sends the braiding $c_{1,1} \in \mathrm{TL}(2,2)$ to the morphism

$$q^{\frac{r+1}{2}} \mathrm{id}_{X \otimes X} + q^{\frac{r-1}{2}} c \circ e \in \mathrm{End}_{\bar{U}}(X \otimes X), \tag{57}$$

which coincides with the braiding $c_{X,X} \in \mathrm{End}_{\bar{U}}(X \otimes X)$ determined by the R-matrix $R \in \bar{U} \otimes \bar{U}$.

However, F_{TL} does not behave well with respect to the other structure morphisms of the ribbon categories TL and $\bar{U}$-mod. Indeed, for every $u \in \mathrm{TL}(m)$, it sends the partial trace $\mathrm{ptr}_1(u) \in \mathrm{TL}(m-1)$ to the morphism

$$(\mathrm{id}_{X^{\otimes m-1}} \otimes e) \circ (F_{\mathrm{TL}}(u) \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^{\otimes m-1}} \otimes c) \in \mathrm{End}_{\bar{U}}(X^{\otimes m-1}), \tag{58}$$

which is obtained from the partial trace $\mathrm{ptr}_X(F_{\mathrm{TL}}(u)) \in \mathrm{End}_{\bar{U}}(X^{\otimes m-1})$ determined by the pivotal element $K \in \bar{U}$ by a change of sign. In particular, F_{TL} sends the twist $\vartheta_1 \in \mathrm{TL}(1,1)$ to the morphism

$$-q^{\frac{r+3}{2}} \mathrm{id}_X \in \mathrm{End}_{\bar{U}}(X), \tag{59}$$

which coincides with the twist $\vartheta_X \in \bar{U}$-mod determined by the inverse ribbon element $v_- \in \bar{U}$ only up to the sign.

Remark 5.1. More generally, for all $m, m' \in \mathrm{TL}$ we have

$$F_{\mathrm{TL}}(c_{m,m'}) = c_{F_{\mathrm{TL}}(m), F_{\mathrm{TL}}(m')}, \tag{60}$$

but for every $u \in \mathrm{TL}(m)$ and $0 \leqslant k \leqslant m$ we have

$$F_{\mathrm{TL}}(\mathrm{ptr}_k(u)) = (-1)^k \mathrm{ptr}_{F_{\mathrm{TL}}(k)}(F_{\mathrm{TL}}(u)). \tag{61}$$

In particular, this implies

$$F_{\mathrm{TL}}(\vartheta_m) = (-1)^m \vartheta_{F_{\mathrm{TL}}(m)}. \tag{62}$$

This sign discrepancy in the comparison between the ribbon structure of TL and that of $\bar{U}$-mod is the reason why we need to use the ribbon Kauffman bracket of equation (34), instead of the standard one, in the definition of the admissible bichrome link invariant F'_Ω of Proposition 3.2.

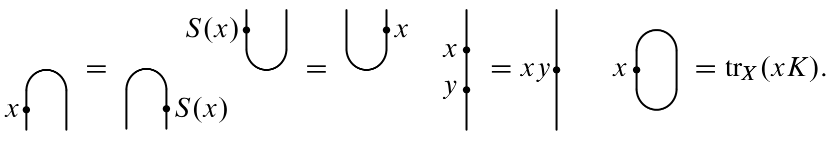
Although it does not preserve ribbon structures, the functor F_{TL} is faithful. Indeed, this can be shown by considering Lusztig's divided power quantum group of $\mathfrak{sl}_2$, which is denoted by U, and which contains $\bar{U}$ as a Hopf subalgebra. Then, the functor F_{TL} can be written as the composition of two faithful functors: the first one is the equivalence from TL to the full monoidal subcategory of U-mod generated by the fundamental representation of U, while the second one is the restriction functor from U-mod to $\bar{U}$-mod. By abuse of notation, we still denote by $F_{\mathrm{TL}} \colon \mathrm{TL} \to \mathrm{Vect}_{\mathbb{C}}$ the composition of F_{TL} with the forgetful functor from $\bar{U}$-mod to $\mathrm{Vect}_{\mathbb{C}}$.

5.2. Bead category

Let us now revisit the HKR algorithm, which we will adapt to our purposes. This requires a few preliminary definitions. First of all, let us denote by $\mathcal{B}$ the ribbon linear category obtained from TL by allowing strands of morphisms to carry beads labeled by elements of $\bar{U}$. Remark that there exists a unique monoidal linear functor $\mathcal{F} \colon \mathcal{B} \to \mathrm{Vect}_{\mathbb{C}}$ extending F_{TL} and sending every x-labeled bead to the linear endomorphism of X determined by the action of x. The *bead category* $\mathrm{TL}_{\bar{U}}$ is the ribbon linear category defined as the quotient of $\mathcal{B}$ with respect to the kernel of $\mathcal{F}$, and we denote by

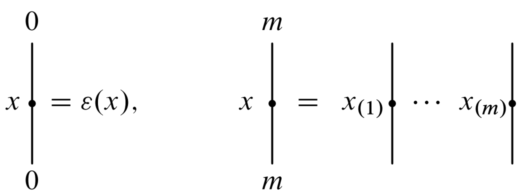
$$F_{\bar{U}} \colon \mathrm{TL}_{\bar{U}} \to \mathrm{Vect}_{\mathbb{C}}$$

the faithful monoidal linear functor induced by $\mathcal{F}$ on the quotient. In the bead category $\mathrm{TL}_{\bar{U}}$ we have



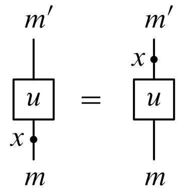
(63)

When m parallel strands are represented graphically by a single strand with label m, we adopt the convention



(64)

Remark that

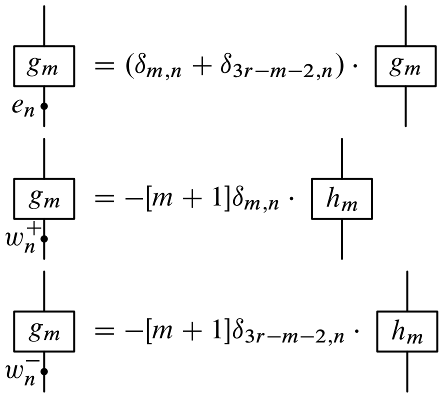


(65)

for every $x \in \bar{U}$ and every $u \in \mathrm{TL}(m, m')$. Furthermore, as a consequence of [7, Lemma 4.1], we also have

(66)

for every integer $0 \leqslant m \leqslant r-1$, and



(67)

(68)

(69)

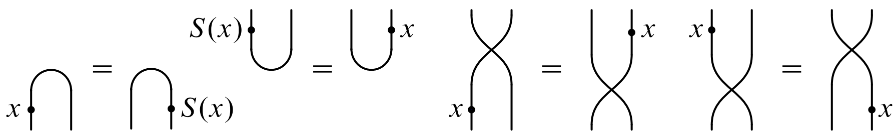
for every integer $r \leqslant m \leqslant 2r-2$.

Let now $p \in \mathrm{TL}(m)$ be an idempotent and $T \in \mathcal{T}_k(p)$ be a bichrome k-top tangle. If $\mathrm{lb}_{n_1,\ldots,n_k}(T)$ denotes the $(n_1,\ldots,n_k)$-labeling of T introduced in equation (31) for integers $r-1 \leqslant n_1,\ldots,n_k \leqslant 2r-2$, then let us define a morphism

$$B_{n_1,\ldots,n_k}(T) \in \mathrm{TL}_{\bar{U}}(p, n_1 \otimes n_1 \otimes \cdots \otimes n_k \otimes n_k \otimes p),$$

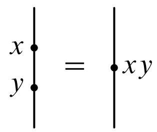
called the *bead presentation of* $\mathrm{lb}_{n_1,\ldots,n_k}(T)$, which will be obtained through the following version of the HKR algorithm based on *singular diagrams* [30, Section 3.3], also known as *flat diagrams* [47, Section 4.8]. A singular diagram of a framed tangle is obtained from a regular diagram by discarding framings and forgetting the difference between overcrossings and undercrossings. On the set of singular diagrams, we consider the equivalence relation generated by all singular versions of the usual local moves corresponding to ambient isotopies of tangles, except for the first Reidemeister move. In particular, two equivalent singular diagrams represent homotopic tangles, but not all homotopies are allowed. Then, let us explain how to define the bead presentation of $\mathrm{lb}_{n_1,\ldots,n_k}(T)$ (the reader is also invited to check the references above, or indeed any of those listed in Section 1.2, where more details about the HKR algorithm can be found). We start from a regular diagram of $\mathrm{lb}_{n_1,\ldots,n_k}(T)$, and we pass to its singular version while also inserting beads labeled by components of the R-matrix around crossings as shown:

Next, we need to collect all beads sitting on the same strand in one place, which has to be next to an upward oriented endpoint, for components which are not closed. As we slide beads past maxima, minima, and crossings, we change their labels according to the rule

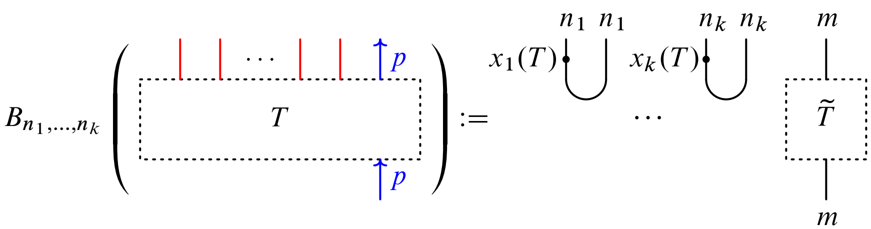


Next, we pass from our singular diagram to an equivalent one whose singular crossings all belong to singular versions of twist morphisms, and we replace them with beads labeled by pivotal elements according to the rule

This is indeed possible, because we started from a bichrome top tangle featuring a unique blue incoming boundary point and a unique outgoing one. Finally, we collect all remaining beads, changing their labels along the way as before, and we multiply everything together according to the rule



In the end, we are left with a planar tangle carrying at most a single bead on each of its components. Inserting idempotents of TL as shown in equation (26) gives



(70)

for some $x_1(T), \dots, x_k(T) \in \bar{U}$ and some $\widetilde{T} \in \mathrm{TL}_{\bar{U}}(p)$. This defines the bead presentation of the $(n_1, \dots, n_k)$-labeling $\mathrm{lb}_{n_1,\dots,n_k}(T)$ of T.

Proposition 5.2. *The ribbon Kauffman bracket of equation* (34) *yields*

$$F_{\bar{U}}(B_{n_1,\dots,n_k}(T)) = F_{\mathrm{TL}}(\langle T \rangle^{\mathrm{rb}}_{n_1,\dots,n_k}) \tag{71}$$

for all $T \in \mathcal{T}_k(p)$ and $r-1 \leqslant n_1, \dots, n_k \leqslant 2r-2$.

Remark that equation (71) uses the abusive notation $F_{\mathrm{TL}} \colon \mathrm{TL} \to \mathrm{Vect}_{\mathbb{C}}$ introduced at the end of Section 5.1. In other words, it should be read as an equality between linear maps, where we are omitting the forgetful functor from $\bar{U}$-mod to $\mathrm{Vect}_{\mathbb{C}}$. The proof of Proposition 5.2 follows directly from the construction, which is due to Hennings, Kauffman, and Radford, and will not be given here. We only stress once again the fact that the use of the ribbon version of the Kauffman bracket of equation (34) is required by Remark 5.1. Indeed, the ribbon number of equation (33) measures precisely the sign difference between $F_{\bar{U}}(B_{n_1,\dots,n_k}(T))$ and $F_{\mathrm{TL}}(\langle T \rangle_{n_1,\dots,n_k})$, as it counts the total number of times (weighted by the label) partial traces and twist morphisms appear in $\mathrm{lb}_{n_1,\dots,n_k}(T)$.

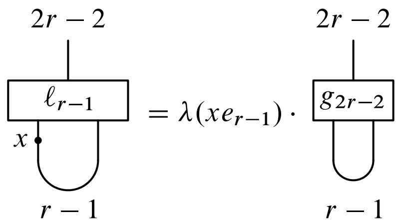
5.3. Diagrammatic integral

Let us introduce a key definition for the diagrammatic translation of Hennings' construction.

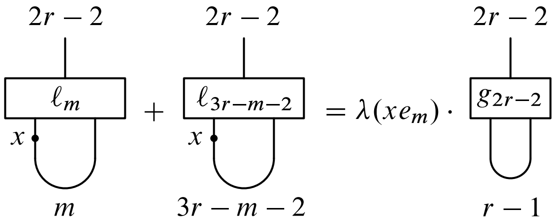
Definition 5.3. A *diagrammatic integral* ℓ of TL is a family of morphisms

$$\ell_{r-1} \in \mathrm{TL}(f_{r-1} \otimes f_{r-1}, g_{2r-2}), \quad \ell_m \in \mathrm{TL}(g_m \otimes g_m^*, g_{2r-2})$$

with $r \leqslant m \leqslant 2r-2$ satisfying



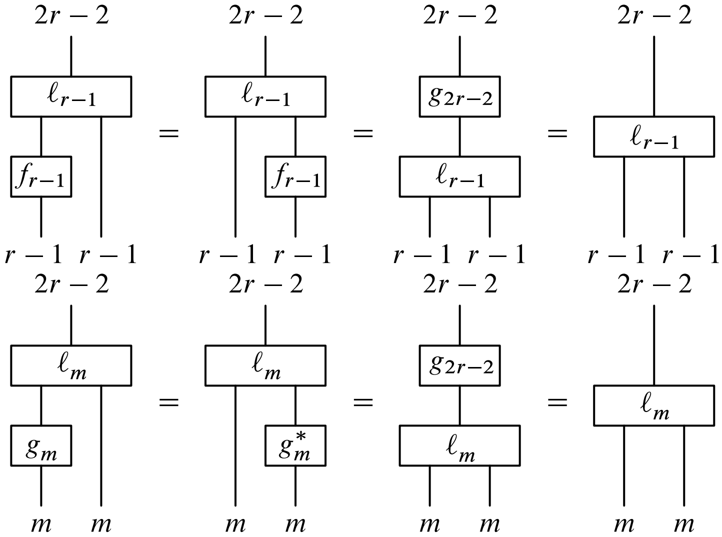
(72)



(73)

for every $x \in \bar{U}$.

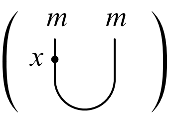
We point out that, by definition, a diagrammatic integral ℓ satisfies



for all integers $r \leqslant m \leqslant 2r-2$.

Now, despite the fact that Definition 5.3 determines a system of $\frac{r-1}{2}$ equations for each element of the quantum group $\bar{U}$, which is a vector space of dimension r^3, the actual number of conditions we need to verify in order to check whether a family ℓ of morphisms of TL provides a diagrammatic integral or not can be drastically reduced. Indeed, this will follow essentially from Lemma 4.5. In order to explain how, let us start with a quick remark.

Remark 5.4. The linear map sending every $x \in \bar{U}$ to



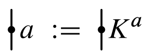
$$F_{\bar{U}}\left(\quad \right)(1) \in X^{\otimes m} \otimes X^{\otimes m}$$

defines a $\bar{U}$-module morphism $j_m \colon \mathrm{ad} \to X^{\otimes m} \otimes X^{\otimes m}$ for every $m \in \mathbb{N}$. Indeed,

$$j_m(\mathrm{ad}_x(y)) = x \cdot j_m(y)$$

follows from equation (63) for every $x \in \bar{U}$ and $y \in \mathrm{ad}$.

As we will show now, it is actually sufficient to restrict ourselves to beads labeled by $K^a \in \bar{U}$ with $a \in \mathbb{Z}$ in the range $0 \leqslant a \leqslant r-1$. Therefore, from now on, for every integer $a \in \mathbb{Z}$ we adopt the shorthand notation



Furthermore, let us set

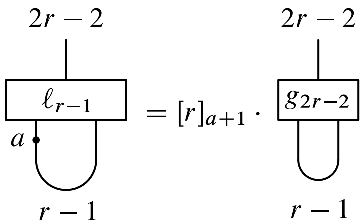
$$[k]_a := \begin{cases} \dfrac{[ak]}{[a]} & \text{if } a \not\equiv 0 \mod r, \\ k & \text{if } a \equiv 0 \mod r, \end{cases} \qquad \{k\}'_a := \{ak\}',$$

for all integers $a, k \in \mathbb{Z}$. Remark that $[k]_a$ and $\{k\}'_a$ are obtained from $[k]$ and $\{k\}'$ by a change of variable replacing q with q^a.

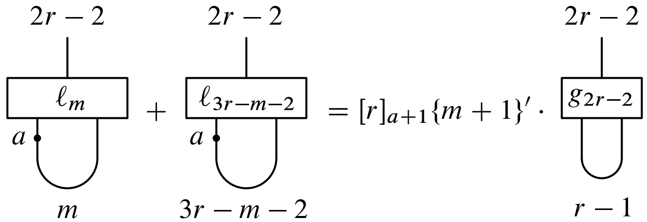
Lemma 5.5. *A family ℓ of morphisms*

$$\ell_{r-1} \in \mathrm{TL}(f_{r-1} \otimes f_{r-1}, g_{2r-2}), \quad \ell_m \in \mathrm{TL}(g_m \otimes g_m^*, g_{2r-2})$$

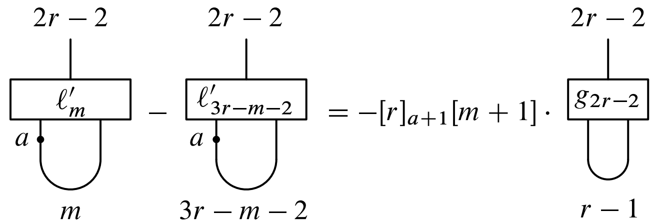
with $r \leqslant m \leqslant 2r-2$ is a diagrammatic integral of TL *if and only if*



(74)



(75)

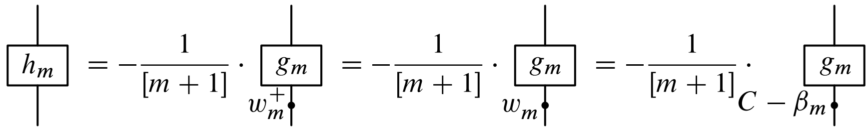


(76)

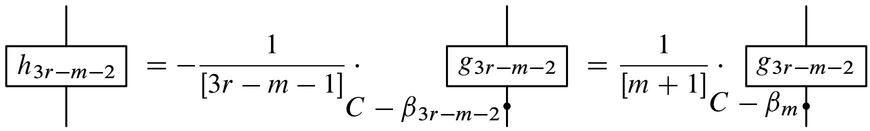
for every $a \in \mathbb{Z}$, where

$$\ell'_m := \ell_m(h_m \otimes g_m^*) \in \mathrm{TL}(g_m \otimes g_m^*, g_{2r-2}).$$

Proof. Thanks to Remark 5.4, the left-hand sides of equations (72) and (73) determine $\bar{U}$-module morphisms from ad_{r-1} to P_{2r-2} and from ad_m to P_{2r-2} respectively. Thanks to Lemma 4.5, every morphism of this type is uniquely determined by its restriction to ad^0_{r-1} and ad^0_m respectively. Then, we simply need to check that equations (74), (75), and (76) are equivalent to equations (72) and (73) for $x = K^a$. On one hand, thanks to equations (67)–(69), we have



This implies



On the other hand, thanks to Lemma 4.2 we have

$$\lambda(K^a e_{r-1}) = \tau_{r-1}(K^{a+1}) = [r]_{a+1},$$

and thanks to Lemma 4.3 for every integer $r \leqslant m \leqslant 2r-2$ we have

$$\begin{aligned}
\lambda(K^a e_m) &= \{m+1\}'(\tau_m(K^{a+1}) + \tau_{3r-m-2}(K^{a+1})) \\
&= \{m+1\}'([2r-m-1]_{a+1} + [m-r+1]_{a+1}) \\
&= [r]_{a+1}\{m+1\}', \\
\lambda(K^a w_m) &= [m+1]^2\big(\tau'_m(K^{a+1}(C-\beta_m)) + \tau'_{3r-m-2}(K^{a+1}(C-\beta_m))\big) \\
&= [m+1]^2([2r-m-1]_{a+1} + [m-r+1]_{a+1}) \\
&= [r]_{a+1}[m+1]^2. \qquad \blacksquare
\end{aligned}$$

Proposition 5.6. *The non-semisimple Kirby color of Definition 3.1 is a diagrammatic integral of* TL.

Proof. Equation (74) follows from equation (78), equation (75) follows from equations (79) and (95) with $k = 0$, and equation (76) follows from equations (80) and (92) with $k = 0$. ■

6. Proofs

Let us prove all the results we claimed in Sections 2 and 3.

6.1. Proof of results from Section 2.2

Proof of Proposition 2.1. As we recalled at the beginning of Section 5, if U denotes Lusztig's divided power quantum group of $\mathfrak{sl}_2$, then TL is equivalent, as a braided monoidal category, to the full monoidal subcategory of U-mod generated by the fundamental representation. Under this equivalence, f_{r-1} is sent to the Steinberg module, which is projective [1, Theorem 9.8]. This means f_{r-1} is projective too, and thus it generates Proj(TL) [18, Lemma 4.4.1]. The rest of the statement follows from [19, Theorem 5.5 and Corollary 5.6]. ∎

Remark 6.1. It was already observed in Section 5.1 that the ribbon structure of TL does not agree with the one of $\bar{U}$-mod, see for instance Remark 5.1. This implies in particular that, for every idempotent $p \in \mathrm{TL}(m)$ of Proj(TL) and every endomorphism $u \in \mathrm{TL}(p)$, we have

$$\mathrm{t}^{\mathrm{TL}}_p(u) = (-1)^m \mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p)}(F_{\mathrm{TL}}(u)). \tag{77}$$

This sign discrepancy in the comparison between the modified trace of TL and the one of $\bar{U}$-mod is the reason behind the sign in equation (35).

6.2. Proof of results from Section 3.3

Proof of Proposition 3.2. First of all, we need to show that a cutting presentation of T exists. In order to construct one, let us orient red components of T, let us consider disjoint paths γ_i for every $1 \leqslant i \leqslant k$, each joining a basepoint p_i on the i-th red component T_i to a basepoint q_i on the top line $I \times \{\frac{1}{2}\} \times \{1\} \subset I^3$, and let us choose a projective blue component of T, meaning a blue component labeled by an idempotent $p \in \mathrm{TL}(m)$ of Proj(TL). Let us cut open the projective blue component and all red ones following the specified paths and orientations, thus obtaining the bichrome k-top tangle T' represented in Figure 2. By construction, T' is a cutting presentation of T.

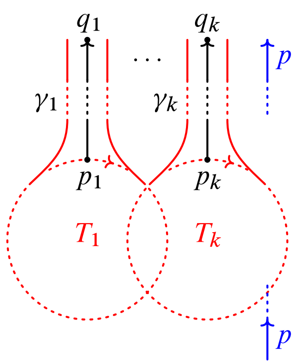
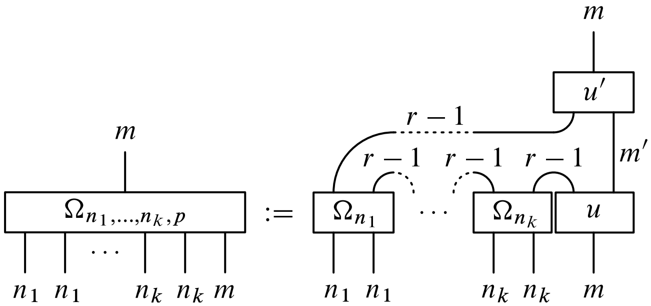


Figure 2. Cutting presentation T' of T.

We need to show $F_\Omega(T)$ does not depend on the choice of the cutting presentation of T. Thanks to equation (77), we have

$$\mathrm{t}^{\mathrm{TL}}_p(F_{\Omega,p}(T')) = (-1)^m \mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p)}(F_{\mathrm{TL}}(F_{\Omega,p}(T'))).$$

The advantage of looking at $F_{\mathrm{TL}}(F_{\Omega,p}(T'))$ rather than $F_{\Omega,p}(T')$ is that the former can be computed using a different approach. In order to do this, it will be convenient to fix some additional notation, so let us set



This means morphisms $\Omega_{n_1,\dots,n_k,p} \in \mathrm{TL}(n_1 \otimes n_1 \otimes \cdots \otimes n_k \otimes n_k \otimes m, m)$ satisfy

$$F_{\Omega,p}(T') = \sum_{n_1,\dots,n_k=r-1}^{2r-2} \Omega_{n_1,\dots,n_k,p} \langle T' \rangle^{\mathrm{rb}}_{n_1,\dots,n_k}.$$

Thanks to Proposition 5.2, we have

$$\begin{aligned} F_{\mathrm{TL}}(F_{\Omega,p}(T')) &= \sum_{n_1,\dots,n_k=r-1}^{2r-2} F_{\mathrm{TL}}(\Omega_{n_1,\dots,n_k,p}) \circ F_{\mathrm{TL}}(\langle T' \rangle^{\mathrm{rb}}_{n_1,\dots,n_k}) \\ &= \sum_{n_1,\dots,n_k=r-1}^{2r-2} F_{\mathrm{TL}}(\Omega_{n_1,\dots,n_k,p}) \circ F_{\bar{U}}(B_{n_1,\dots,n_k}(T')). \end{aligned}$$

Then, thanks to Proposition 5.6, we have

$$\sum_{n_1,\dots,n_k=r-1}^{2r-2} F_{\mathrm{TL}}(\Omega_{n_1,\dots,n_k,p}) \circ F_{\bar{U}}(B_{n_1,\dots,n_k}(T')) = \Big(\prod_{i=1}^{k} \lambda(x_i(T'))\Big) F_{\bar{U}}(\tilde{T}'),$$

where $x_i(T') \in \bar{U}$ and $\tilde{T}' \in \mathrm{TL}_{\bar{U}}(p)$ are given by equation (70). Summing up

$$F_\Omega(T) = \mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p)}\Big(\Big(\prod_{i=1}^{k} \lambda(x_i(T'))\Big) F_{\bar{U}}(\tilde{T}')\Big).$$

We will show now that the fact that $F_\Omega(T)$ is independent of the cutting presentation T' of T follows essentially from the fact that λ is a quantum character, and that $\mathrm{t}^{\bar{U}}$ is a modified trace.

First of all, we claim $F_\Omega(T)$ does not depend on the choice of the path γ_i. Indeed, we can decompose $x_i(T')$ as $x(\gamma_i)_{(1)} x(T_i) S(x(\gamma_i)_{(2)})$, where $x(\gamma_i)$ is collected traveling along γ_i, and $x(T_i)$ is collected traveling along T_i, as shown in Figure 3. This means

$$\begin{aligned}\lambda(x(\gamma_i)_{(1)} x(T_i) S(x(\gamma_i)_{(2)})) &= \lambda(x(T_i) S(x(\gamma_i)_{(2)}) S^2(x(\gamma_i)_{(1)})) \\ &= \lambda(x(T_i) S(S(x(\gamma_i)_{(1)}) x(\gamma_i)_{(2)})) \\ &= \varepsilon(x(\gamma_i)) \lambda(x(T_i)),\end{aligned}$$

where the first equality follows from the fact that λ is a quantum character. Then, since $x(\gamma_i)$ is a product of copies of components of the R-matrix and copies of the pivotal element, which satisfy

$$\varepsilon(R')R'' = \varepsilon(R'')R' = 1, \quad \varepsilon(g) = 1,$$

the contribution of the framed path γ_i is trivial, both for the computation of $\lambda(x_i(T'))$ and for its effect on other components of T'.

Next, we claim $F_\Omega(T)$ does not depend on the choice of the orientation of T_i. Indeed, we can switch between the two possible ones by adding a braiding and a twist, as shown in Figure 4. This means

$$\begin{aligned}\lambda(R'' u^{-1} S(x(T_i)) S(R')) &= \lambda(S^{-1}(R') R'' u^{-1} S(x(T_i))) \\ &= \lambda(S(u) u^{-1} S(x(T_i))) \\ &= \lambda(g^{-2} S(x(T_i))) \\ &= \lambda(x(T_i)),\end{aligned}$$

where we are using the identities

$$S(R) \otimes S(R'') = R' \otimes R'', \quad S(u) = g^{-1} v_+, \quad u^{-1} = g^{-1} v_-,$$

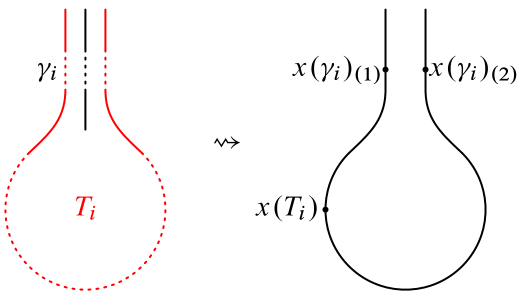


Figure 3. Independence of path.

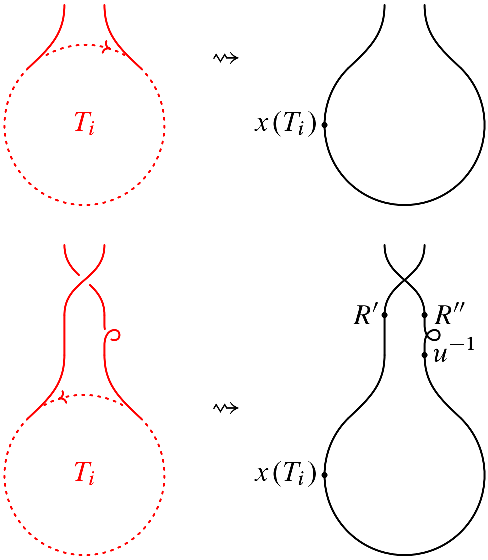


Figure 4. Independence of orientation.

as well as [24, Proposition 4.2], which gives, for every $x \in \bar{U}$, the identity

$$\lambda(g^{-2}S(x)) = \lambda(x).$$

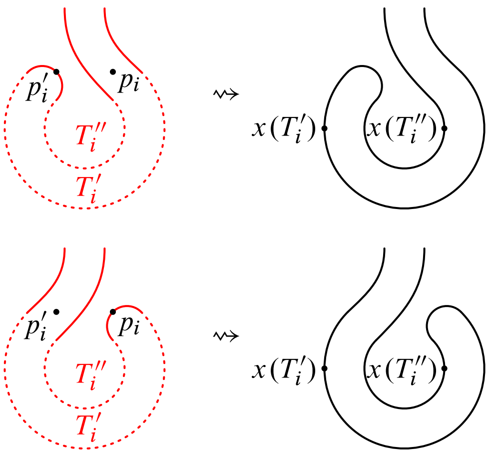
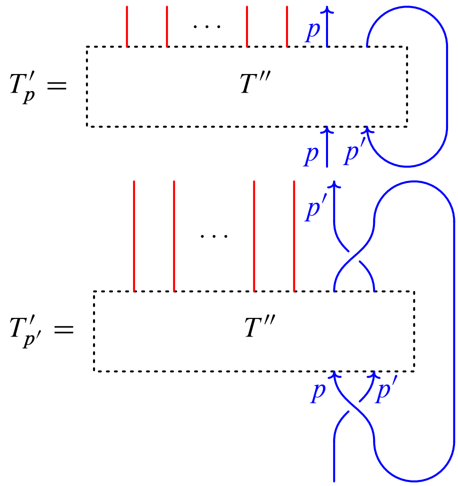


Figure 5. Independence of basepoint.

Now, we claim $F_\Omega(T)$ does not depend on the choice of the basepoint p_i. Indeed, if p'_i is another basepoint, we can decompose $x(T_i)$ as $x(T''_i)x(T'_i)$, where $x(T'_i)$ is collected traveling from p_i to p'_i, and $x(T''_i)$ is collected traveling from p'_i to p_i, as shown in Figure 5. This means

$$\lambda(x(T''_i)x(T'_i)) = \lambda(x(T'_i)S^2(x(T''_i))).$$

Finally, we claim $F_\Omega(T)$ does not depend on the choice of the projective blue component of T. Indeed, if



are different cutting presentations of T, then

$$\begin{aligned}
\mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p)}(F_{\bar{U}}(\widetilde{T}'_p)) &= \mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p)}(\mathrm{ptr}_{F_{\mathrm{TL}}(p')}(F_{\bar{U}}(\widetilde{T}''))) \\
&= \mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p\otimes p')}(F_{\bar{U}}(\widetilde{T}'')) \\
&= \mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p'\otimes p)}(F_{\mathrm{TL}}(c_{p,p'}) \circ F_{\bar{U}}(\widetilde{T}'') \circ F_{\mathrm{TL}}(c^{-1}_{p,p'})) \\
&= \mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p')}(\mathrm{ptr}_{F_{\mathrm{TL}}(p)}(F_{\mathrm{TL}}(c_{p,p'}) \circ F_{\bar{U}}(\widetilde{T}'') \circ F_{\mathrm{TL}}(c^{-1}_{p,p'}))) \\
&= \mathrm{t}^{\bar{U}}_{F_{\mathrm{TL}}(p)}(F_{\bar{U}}(\widetilde{T}'_{p'})),
\end{aligned}$$

because $\mathrm{t}^{\bar{U}}$ is a modified trace. ∎

6.3. Proof of results from Section 3.4

Proof of Lemma 3.4. It follows from the proof of Proposition 3.2 and from equation (53) that

$$\Delta_\pm = \lambda(v_\mp) = i^{\mp\frac{r-1}{2}} r^{\frac{3}{2}} q^{\frac{r\mp 3}{2}}. \qquad ∎$$

Proof of Theorem 3.5. The proof of the invariance of $Z_r(M,T)$ under Kirby moves follows the same argument of [14, Proposition 2.13]. Indeed, if the bead collected while traveling along a red component has label x, then the operation of sliding a strand, either blue or red, adds a bead with label $R'x_{(1)}$ on the sliding component, and changes the label of the red component to $R''_{(1)}x_{(2)}S(R''_{(2)})$, as shown in Figure 6. This means

$$\begin{aligned}
\lambda(R''_{(1)}x_{(2)}S(R''_{(2)}))R'x_{(1)} &= \lambda(x_{(2)}S(R''_{(2)})S^2(R''_{(1)}))R'x_{(1)} \\
&= \lambda(x_{(2)}S(S(R''_{(1)})R''_{(2)}))R'x_{(1)} \\
&= \varepsilon(R'')\lambda(x_{(2)})R'x_{(1)} \\
&= \lambda(x_{(2)})x_{(1)} \\
&= \lambda(x)1,
\end{aligned}$$

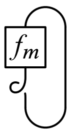
because λ is a left integral. Therefore, $Z_r(M,T)$ is invariant under Kirby II moves.

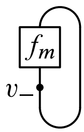
Furthermore, it follows from Lemma 3.4 that adding an unknotted red component of framing ± 1 contributes a factor of

$$\lambda(v_{\mp}) = \Delta_{\pm}.$$

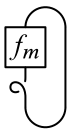
Therefore, $Z_r(M,T)$ is also invariant under Kirby I moves. ∎

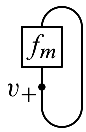
Proof of Lemma 3.7. On one hand, thanks to Lemma 4.1 and equations (66) and (8), we have

$$\delta_{+,m} :=$$


$$=$$


$$= (-1)^m q^{\frac{r+1}{2}m^2+m}[m+1],$$

$$\delta_{-,m} :=$$


$$=$$


$$= (-1)^m q^{\frac{r-1}{2}m^2-m}[m+1].$$

Remark that for every integer $0 \leqslant m \leqslant r-2$ we have

$$\delta_{\pm,m} = \delta_{\pm,r-m-2}, \quad (-1)^m[m+1] = (-1)^{r-m-2}[r-m-1].$$

Therefore, we have

$$\delta_{\pm} = \sum_{m=0}^{\frac{r-3}{2}} [2m+1]\delta_{\pm,2m} = \sum_{m=0}^{\frac{r-3}{2}} (-1)^m[m+1]\delta_{\pm,m} = \sum_{m=0}^{\frac{r-3}{2}} q^{\frac{r\pm 1}{2}m^2 \pm m}[m+1]^2.$$

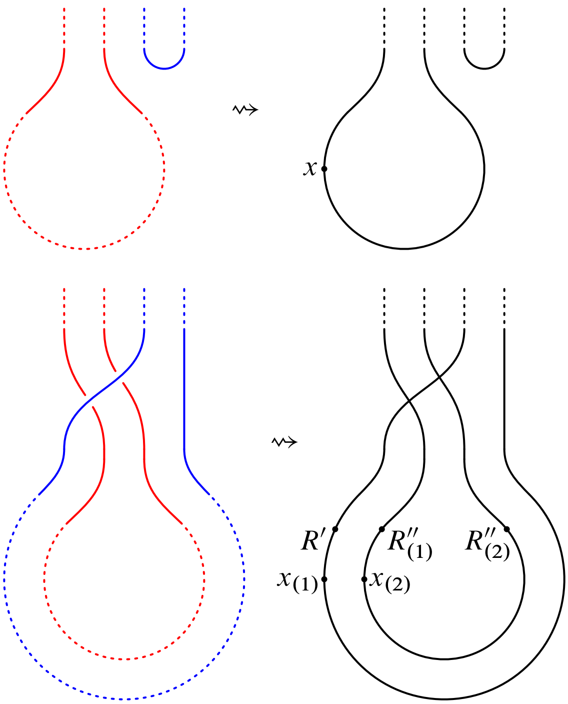


Figure 6. Invariance under Kirby II moves.

On the other hand, equation (54) can be rearranged as

$$\begin{aligned}\lambda = \tau_{r-1}(\_K) &+ \sum_{m=0}^{\frac{r-3}{2}} \{m+1\}' \big(\tau_{2r-m-2}(\_K) + \tau_{m+r}(\_K)\big) \\ &+ \sum_{m=0}^{\frac{r-3}{2}} [m+1]^2 \big(\tau'_{2r-m-2}(\_K) + \tau'_{m+r}(\_K)\big),\end{aligned}$$

and Lemma 4.1 can be rewritten as

$$\begin{aligned}v_{\mp} = q^{\frac{r\mp1}{2}} e_{r-1} &+ \sum_{m=0}^{\frac{r-3}{2}} q^{\frac{r\pm1}{2}m^2 \pm m} e_m \\ &\pm \sum_{m=0}^{\frac{r-3}{2}} q^{\frac{r\pm1}{2}m^2 \pm m} \Big(\frac{(m+1)\{1\}}{[m+1]} w_m^+ + \frac{(m-r+1)\{1\}}{[m+1]} w_m^- \Big).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\lambda(v_\mp) &= \sum_{m=0}^{\frac{r-3}{2}} q^{\frac{r\pm1}{2}m^2\pm m}\{m+1\}'\big([m+1]+[r-m-1]\big)\\ &\quad \pm \sum_{m=0}^{\frac{r-3}{2}} q^{\frac{r\pm1}{2}m^2\pm m}[m+1]^2\big((m+1)\{1\}-(m-r+1)\{1\}\big)\\ &= \pm r\{1\}\sum_{m=0}^{\frac{r-3}{2}} q^{\frac{r\pm1}{2}m^2\pm m}[m+1]^2.\end{aligned}$$

This means

$$\delta_\pm = \pm\frac{\lambda(v_\mp)}{r\{1\}} = \pm\frac{i^{\mp\frac{r-1}{2}} r^{\frac{1}{2}} q^{\frac{r\mp3}{2}}}{\{1\}}. \qquad \blacksquare$$

Proof of Proposition 3.9. It follows from the construction that

$$Z_r(M \mathbin{\#} M', T \cup T') = \psi_r(M,T) Z_r(M',T'),$$

where ψ_r is the Hennings invariant associated with $\bar{U}$, compare with [14, Proposition 2.11] and [13, Proposition 3.11]. Now, the result follows directly from [10, Theorem 1]. $\blacksquare$

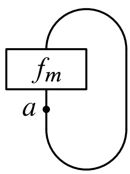
7. Computations

In this section we collect formulas which are used in the proof of Proposition 5.6. Everything is based on equations (63)–(65).

7.1. Traces

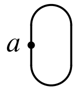
Let us start by computing traces of K^a in $\mathrm{TL}_{\bar{U}}$ for every $a \in \mathbb{Z}$.

Lemma 7.1. *For every integer* $1 \leqslant m \leqslant r-1$ *we have*



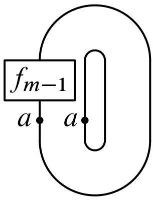
$$= (-1)^m[m+1]_{a+1}. \tag{78}$$

Proof. Equation (78) is proved by induction on $1 \leqslant m \leqslant r-1$. If $m = 1$ then, thanks to equation (63),

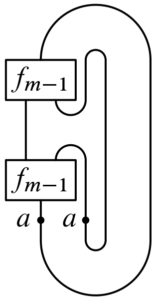


$$= -[2]_{a+1}.$$

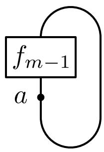
If $2 \leqslant m \leqslant r-1$, then, by induction hypothesis, thanks to equations (63)–(65),



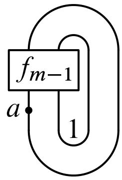
$$+ \frac{[m-1]}{[m]} \cdot$$



$$= -[2]_{a+1} \cdot$$



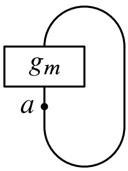
$$+ \frac{[m-1]}{[m]} \cdot$$



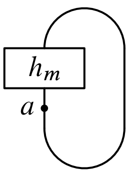
$$= (-1)^m([m]_{a+1}[2]_{a+1} - [m-1]_{a+1})$$
$$= (-1)^m[m+1]_{a+1}.$$

■

Lemma 7.2. *For every integer $r \leqslant m \leqslant 2r-2$, we have*

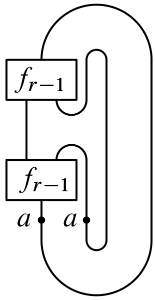


$$= (-1)^m[r]_{a+1}\{m-r+1\}'_{a+1}, \tag{79}$$



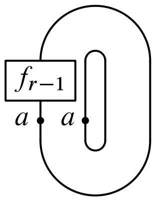
$$= 0. \tag{80}$$

Proof. Equation (80) follows from equations (63)–(65), which give

$(-1)^{m+1}[m+1]\cdot$ 

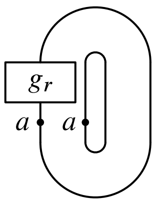
$= 0.$

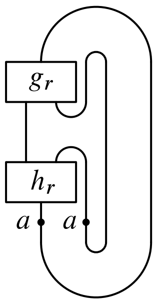
Equation (79) is proved by induction on $r \leqslant m \leqslant 2r-2$. If $m = r$ then, thanks to equation (63),

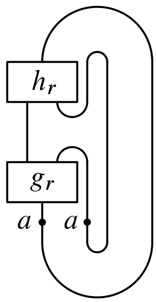


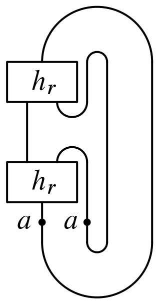
$= -[r]_{a+1}\{1\}'_{a+1}.$

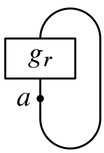
If $m = r+1$ then, thanks to equations (63)–(65),

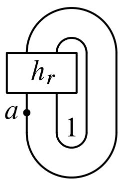


$-$ 

$-$ 

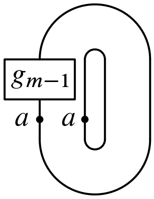
$-[2]\cdot$ 

$= -[2]_{a+1}\cdot$ 

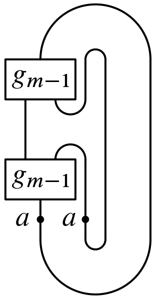
$-2\cdot$ 

$$= [r]_{a+1}\big(\{1\}'_{a+1}[2]_{a+1} - 2\big)$$
$$= [r]_{a+1}\{2\}'_{a+1}.$$

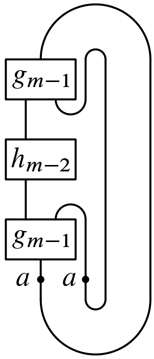
If $r+2 \leqslant m \leqslant 2r-2$ then by induction hypothesis, thanks to equations (63)–(65),



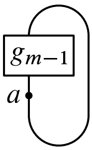
$+ \frac{[m-1]}{[m]} \cdot$



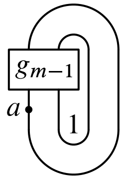
$- \frac{2}{[m]^2} \cdot$



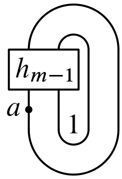
$= -[2]_{a+1} \cdot$



$+ \frac{[m-1]}{[m]} \cdot$



$- \frac{2}{[m]^2} \cdot$



$$
\begin{aligned}
&= (-1)^m [r]_{a+1} \big(\{m-r\}'_{a+1} [2]_{a+1} - \{m-r-1\}'_{a+1} \big) \\
&= (-1)^m [r]_{a+1} \{m-r+1\}'_{a+1}.
\end{aligned}
$$

■

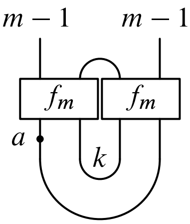
7.2. Partial traces

Next, let us tackle some harder computations. In order to do this, we will make extensive use of the identity

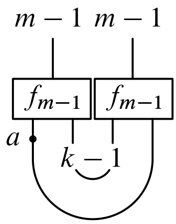
$$[a][b-c] + [b][c-a] + [c][a-b] = 0, \tag{81}$$

which holds for all integers $a, b, c \in \mathbb{Z}$.

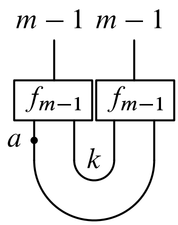
Lemma 7.3. *For all integers $1 \leqslant m \leqslant r-1$ and $0 \leqslant k \leqslant m$ we have*

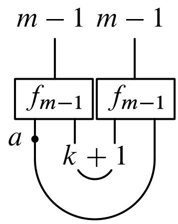


$= \left(-\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2} \right) \cdot$

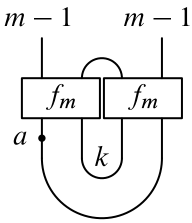
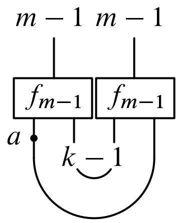


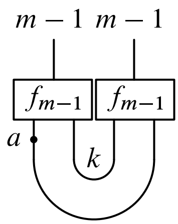
$- \frac{[2]_{a+1}[m-k]^2}{[m]^2} \cdot$

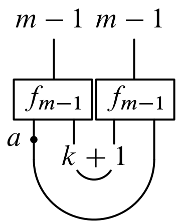


$$+\frac{[m-k][m-k-1]}{[m]^2}\cdot$$  (82)

Proof. Equation (82) is proved by induction on $1 \leqslant m \leqslant r-1$. In order to do this, let us set

 $$= \alpha_{m,k}^{k-1}\cdot$$ 

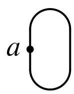
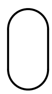
$$+\,\alpha_{m,k}^{k}\cdot$$ 

$$+\,\alpha_{m,k}^{k+1}\cdot$$ 

Remark that we have

$$\alpha_{m,0}^{-1} = \alpha_{m,m-1}^{m} = \alpha_{m,m}^{m} = \alpha_{m,m}^{m+1} = 0$$

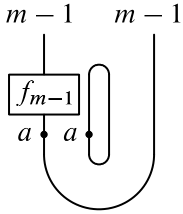
for every integer $1 \leqslant m \leqslant r-1$. If $m = 1$, we have

 $$= -[2]_{a+1},$$  $$= -[2].$$

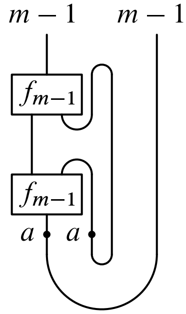
This gives the condition

$$\alpha_{1,0}^{0} = -[2]_{a+1}, \quad \alpha_{1,1}^{0} = -[2].$$

If $1 < m \leqslant r-1$ and $k = 0$, let us consider



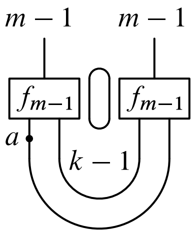
$$+ \frac{[m-1]}{[m]} \cdot$$



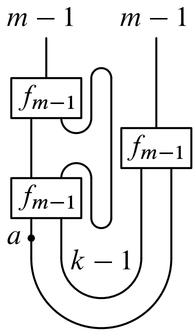
This gives the condition

$$\alpha^0_{m,0} = -[2]_{a+1}, \quad \alpha^1_{m,0} = \frac{[m-1]}{[m]}.$$

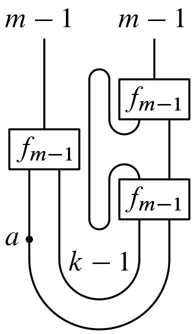
If $1 < m \leqslant r-1$ and $1 \leqslant k \leqslant m$, let us consider



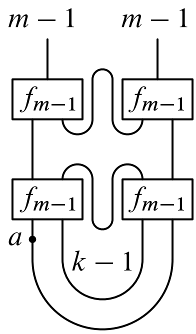
$$+ \frac{[m-1]}{[m]} \cdot$$



$$+ \frac{[m-1]}{[m]} \cdot$$



$$+ \frac{[m-1]^2}{[m]^2} \cdot$$



This gives the condition

$$\alpha^{k-1}_{m,k} = -\frac{\{m\}'}{[m]} + \frac{[m-1]^2}{[m]^2}\alpha^{k-2}_{m-1,k-1},$$
$$\alpha^{k}_{m,k} = \frac{[m-1]^2}{[m]^2}\alpha^{k-1}_{m-1,k-1},$$

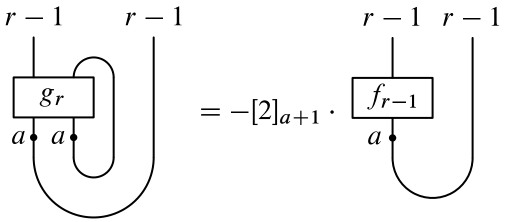
$$\alpha_{m,k}^{k+1} = \frac{[m-1]^2}{[m]^2}\alpha_{m-1,k-1}^{k}.$$

Thanks to equation (81), the solution is

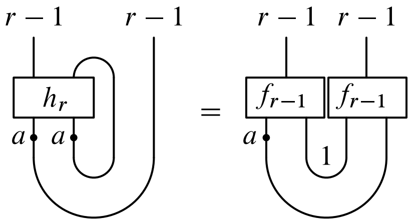
$$\begin{aligned}
\alpha_{m,k}^{k-1} &= -\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}, \\
\alpha_{m,k}^{k} &= -\frac{[2]_{a+1}[m-k]^2}{[m]^2}, \\
\alpha_{m,k}^{k+1} &= \frac{[m-k][m-k-1]}{[m]^2}.
\end{aligned}$$

■

Lemma 7.4. *We have*

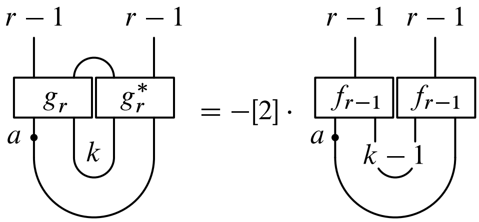


(83)

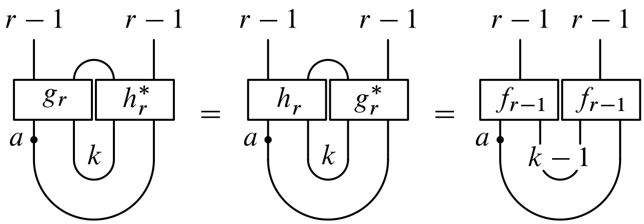


(84)

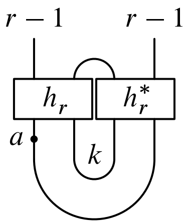
Furthermore, for every integer $1 \leqslant k \leqslant r$, *we have*



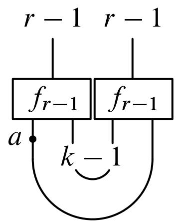
(85)



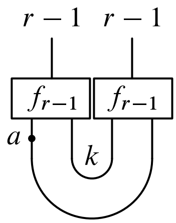
(86)



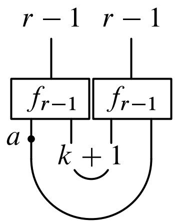
$= [k][k-1]\cdot$



$- [2]_{a+1}[k]^2\cdot$



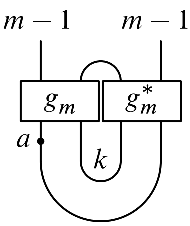
$+ [k+1][k]\cdot$



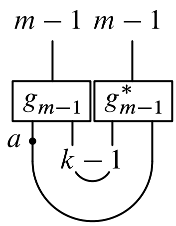
(87)

Proof. Equations (83)–(86) are easy to prove, and left to the reader, while equation (87) follows from equation (82). ∎

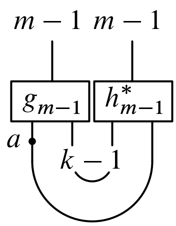
Lemma 7.5. *For all integers $r+1 \leqslant m \leqslant 2r-2$, if $0 \leqslant k \leqslant m-r$ we have*



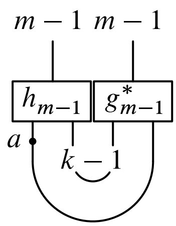
$$= \left(-\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}\right)\cdot$$



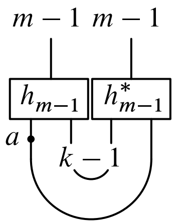
$$-\frac{2[k][m-k+1]}{[m]^3}\cdot\Bigg($$

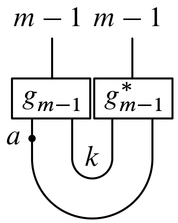


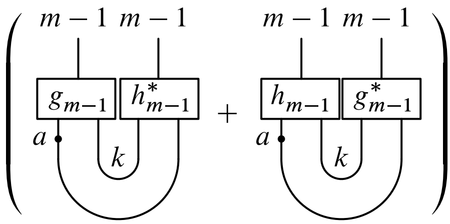
$+$

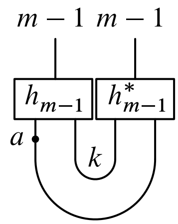


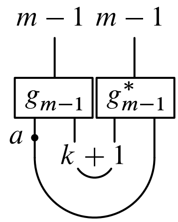
$$\Bigg)$$

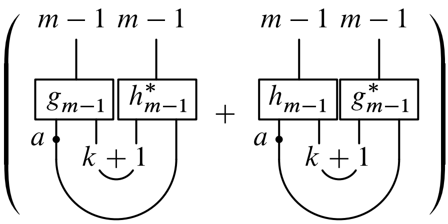
$+ \dfrac{4[k][k-1]}{[m]^4} \cdot$ 

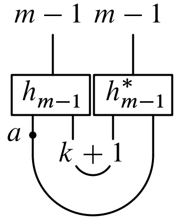
$- \dfrac{[2]_{a+1}[m-k]^2}{[m]^2} \cdot$ 

$+ \dfrac{2[2]_{a+1}[k][m-k]}{[m]^3} \cdot$ 

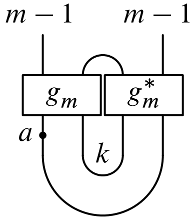
$+ \left(\dfrac{3[2]_{a+1}}{[m]^2}\delta_{k,m-r} - \dfrac{4[2]_{a+1}[k]^2}{[m]^4}\right) \cdot$ 

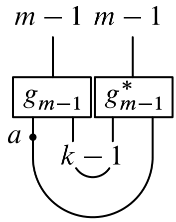
$+ \dfrac{[m-k][m-k-1]}{[m]^2} \cdot$ 

$+ \left(-\dfrac{1}{[m]^2} - \dfrac{2[k][m-k-1]}{[m]^3}\right) \cdot$ 

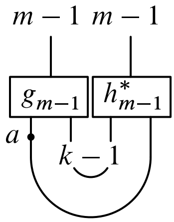
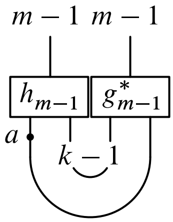
$$+\left(-\frac{[2]}{[m]^2}\delta_{k,m-r-1}-\frac{2[2]}{[m]^2}\delta_{k,m-r}+\frac{4[k+1][k]}{[m]^4}\right)\cdot$$  (88)

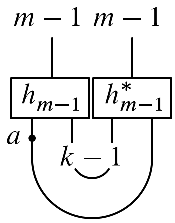
and if $m-r+1\leqslant k\leqslant m$ *we have*

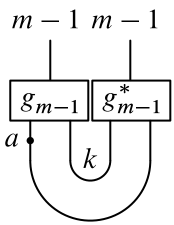


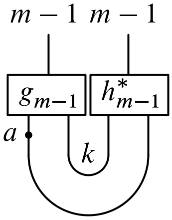
$$=\left(-\frac{[m+1]}{[m]}+\frac{[m-k+1][m-k]}{[m]^2}\right)\cdot$$ 

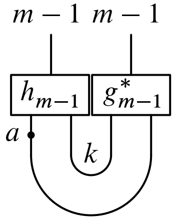
$$+\left(-\frac{1}{[m]^2}+\frac{\{m\}'[m-k+1][m-k]}{[m]^3}\right)$$

$\cdot$ ( $+$ )

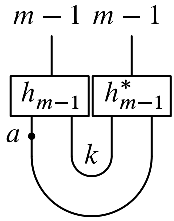
$$+\left(\frac{[2]}{[m]^2}+\frac{(\{m\}')^2[m-k+1][m-k]}{[m]^4}\right)\cdot$$ 

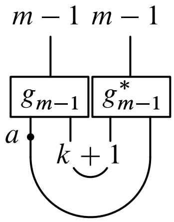
$$-\frac{[2]_{a+1}[m-k]^2}{[m]^2}\cdot$$ 

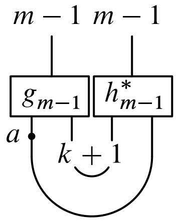
$$-\frac{[2]_{a+1}\{m\}'[m-k]^2}{[m]^3}\cdot\Bigg($$


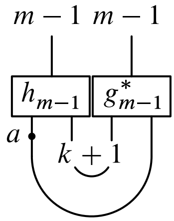
$$+$$


$$\Bigg)$$

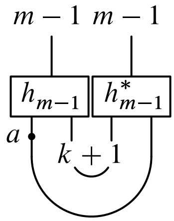
$$-\frac{[2]_{a+1}(\{m\}')^2[m-k]^2}{[m]^4}\cdot$$


$$+\frac{[m-k][m-k-1]}{[m]^2}\cdot$$


$$+\frac{\{m\}'[m-k][m-k-1]}{[m]^3}\cdot\Bigg($$


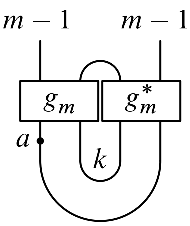
$$+$$


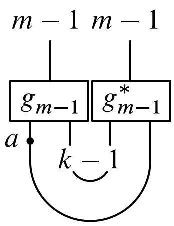
$$\Bigg)$$

$$+\frac{(\{m\}')^2[m-k][m-k-1]}{[m]^4}\cdot$$


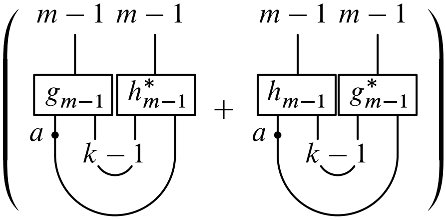
(89)

Proof. Equations (88) and (89) are proved by induction on $r+2 \leqslant m \leqslant 2r-2$. In order to do this, let us set

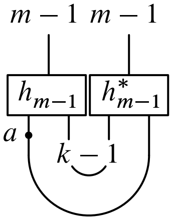


$$= \alpha_{m,k}^{k-1}\cdot$$


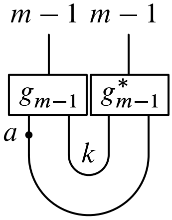
$+\ \beta_{m,k}^{k-1}\ \cdot$



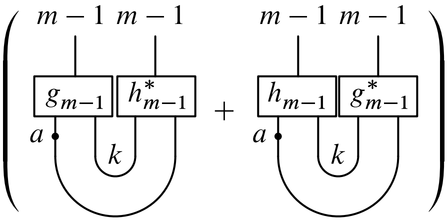
$+\ \gamma_{m,k}^{k-1}\ \cdot$



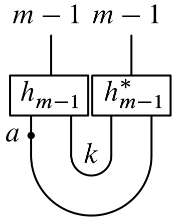
$+\ \alpha_{m,k}^{k}\ \cdot$



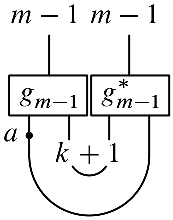
$+\ \beta_{m,k}^{k}\ \cdot$

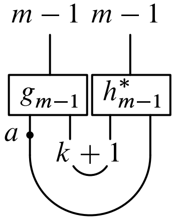


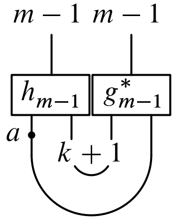
$+\ \gamma_{m,k}^{k}\ \cdot$



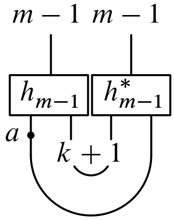
$+\ \alpha_{m,k}^{k+1}\ \cdot$



$+\,\beta^{k+1}_{m,k}\cdot\Bigg($ 

$+$ 

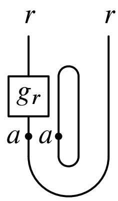
$\Bigg)$

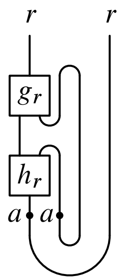
$+\,\gamma^{k+1}_{m,k}\cdot$ 

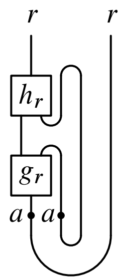
Remark that we have

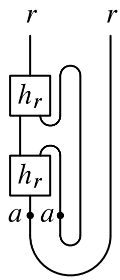
$$\begin{aligned}
\alpha^{-1}_{m,0} = \beta^{-1}_{m,0} = \gamma^{-1}_{m,0} &= 0,\\
\alpha^{m}_{m,m-1} = \beta^{m}_{m,m-1} = \gamma^{m}_{m,m-1} &= 0,\\
\alpha^{m}_{m,m} = \beta^{m}_{m,m} = \gamma^{m}_{m,m} &= 0,\\
\alpha^{m+1}_{m,m} = \beta^{m+1}_{m,m} = \gamma^{m+1}_{m,m} &= 0
\end{aligned}$$

for every integer $r+1 \leqslant m \leqslant 2r-2$. If $m = r+1$ and $k = 0$, let us consider



$-$ 

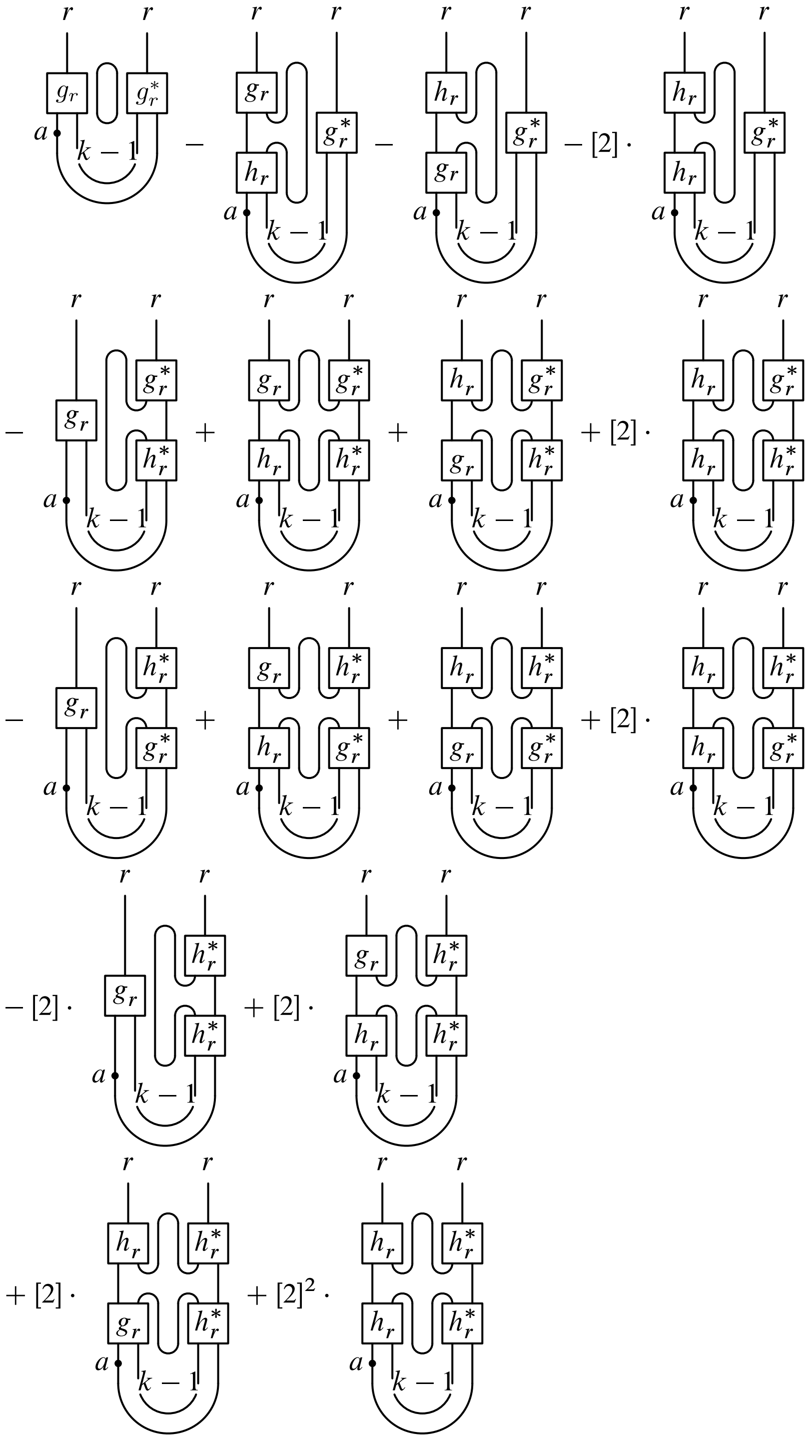
$-$ 

$-\,[2]\cdot$ 

This gives the condition

$$\begin{aligned}
&\alpha^{0}_{r+1,0} = -[2]_{a+1}, && \beta^{0}_{r+1,0} = 0, && \gamma^{0}_{r+1,0} = 0,\\
&\alpha^{1}_{r+1,0} = 0, && \beta^{1}_{r+1,0} = -1, && \gamma^{1}_{r+1,0} = -[2].
\end{aligned}$$

If $m = r + 1$ and $1 \leqslant k \leqslant r + 1$, let us consider



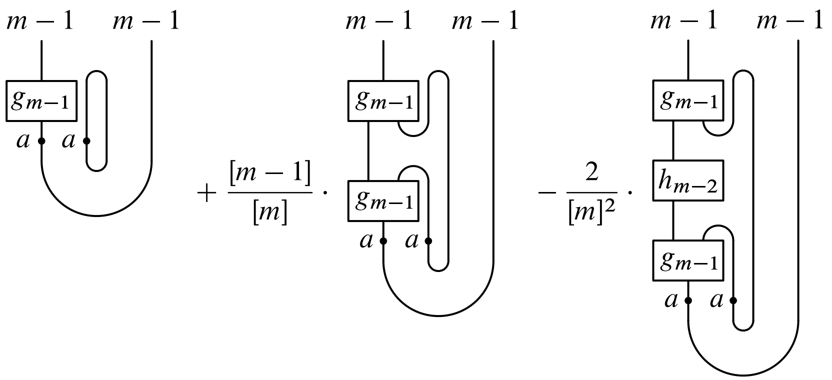
When $k = 1$, this gives the condition

$$
\begin{aligned}
\alpha^0_{r+1,1} &= -\{1\}', & \beta^0_{r+1,1} &= -2, & \gamma^0_{r+1,1} &= 0, \\
\alpha^1_{r+1,1} &= 0, & \beta^1_{r+1,1} &= 0, & \gamma^1_{r+1,1} &= -[2]_{a+1}, \\
\alpha^2_{r+1,1} &= 0, & \beta^2_{r+1,1} &= 1, & \gamma^2_{r+1,1} &= 2[2],
\end{aligned}
$$

thanks to equations (83) and (84), and when $2 \leqslant k \leqslant r+1$, this gives the condition

$$
\begin{aligned}
\alpha^{k-1}_{r+1,k} &= -[2] + [k-1][k-2], \\
\beta^{k-1}_{r+1,k} &= -1 + \{1\}'[k-1][k-2], \\
\gamma^{k-1}_{r+1,k} &= [2] + (\{1\}')^2[k-1][k-2], \\
\alpha^{k}_{r+1,k} &= -[2]_{a+1}[k-1]^2, \\
\beta^{k}_{r+1,k} &= -[2]_{a+1}\{1\}'[k-1]^2, \\
\gamma^{k}_{r+1,k} &= -[2]_{a+1}(\{1\}')^2[k-1]^2, \\
\alpha^{k+1}_{r+1,k} &= [k][k-1], \\
\beta^{k+1}_{r+1,k} &= \{1\}'[k][k-1], \\
\gamma^{k+1}_{r+1,k} &= (\{1\}')^2[k][k-1],
\end{aligned}
$$

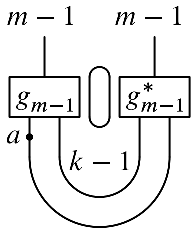
thanks to equations (85), (86), and (87). If $r+2 \leqslant m \leqslant 2r-2$ and $k = 0$, let us consider



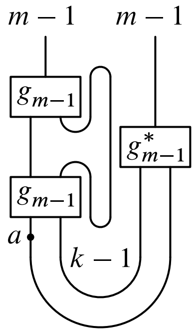
This gives the condition

$$
\begin{aligned}
\alpha^0_{m,0} &= -[2]_{a+1}, & \beta^0_{m,0} &= 0, & \gamma^0_{m,0} &= 0, \\
\alpha^1_{m,0} &= \frac{[m-1]}{[m]}, & \beta^1_{m,0} &= -\frac{1}{[m]^2}, & \gamma^1_{m,0} &= 0.
\end{aligned}
$$

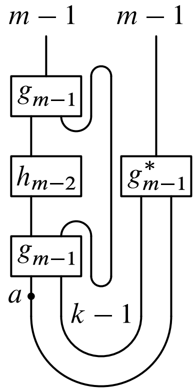
If $r + 2 \leqslant m \leqslant 2r - 2$ and $1 \leqslant k \leqslant m$, let us consider



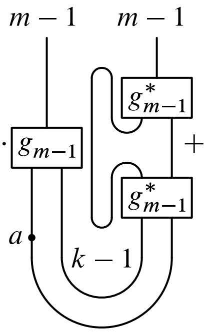
$+ \dfrac{[m-1]}{[m]} \cdot$



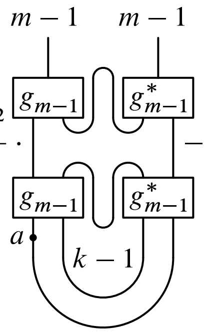
$- \dfrac{2}{[m]^2} \cdot$



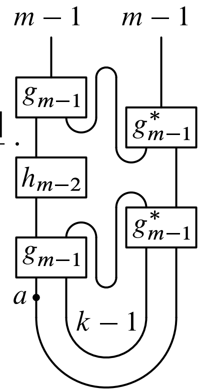
$+ \dfrac{[m-1]}{[m]} \cdot$



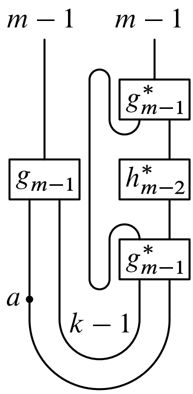
$+ \dfrac{[m-1]^2}{[m]^2} \cdot$



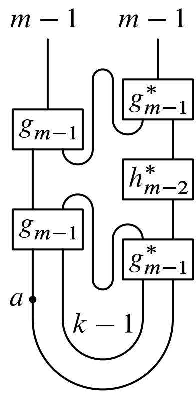
$- \dfrac{2[m-1]}{[m]^3} \cdot$



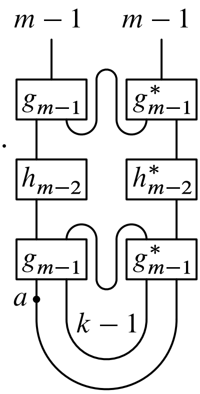
$- \dfrac{2}{[m]^2} \cdot$



$- \dfrac{2[m-1]}{[m]^3} \cdot$



$+ \dfrac{4}{[m]^4} \cdot$



This gives the condition

$$
\begin{aligned}
\alpha_{m,k}^{k-1} &= -\frac{\{m\}'}{[m]} + \frac{[m-1]^2}{[m]^2}\alpha_{m-1,k-1}^{k-2}, \\
\beta_{m,k}^{k-1} &= -\frac{2}{[m]^2} + \frac{[m-1]^2}{[m]^2}\beta_{m-1,k-1}^{k-2} - \frac{2[m-1]}{[m]^3}\alpha_{m-1,k-1}^{k-2},
\end{aligned}
$$

$$\begin{aligned}
\gamma_{m,k}^{k-1} &= \frac{[m-1]^2}{[m]^2}\gamma_{m-1,k-1}^{k-2} - \frac{4[m-1]}{[m]^3}\beta_{m-1,k-1}^{k-2} + \frac{4}{[m]^4}\alpha_{m-1,k-1}^{k-2}, \\
\alpha_{m,k}^{k} &= \frac{[m-1]^2}{[m]^2}\alpha_{m-1,k-1}^{k-1}, \\
\beta_{m,k}^{k} &= \frac{[m-1]^2}{[m]^2}\beta_{m-1,k-1}^{k-1} - \frac{2[m-1]}{[m]^3}\alpha_{m-1,k-1}^{k-1}, \\
\gamma_{m,k}^{k} &= \frac{[m-1]^2}{[m]^2}\gamma_{m-1,k-1}^{k-1} - \frac{4[m-1]}{[m]^3}\beta_{m-1,k-1}^{k-1} + \frac{4}{[m]^4}\alpha_{m-1,k-1}^{k-1}, \\
\alpha_{m,k}^{k+1} &= \frac{[m-1]^2}{[m]^2}\alpha_{m-1,k-1}^{k}, \\
\beta_{m,k}^{k+1} &= \frac{[m-1]^2}{[m]^2}\beta_{m-1,k-1}^{k} - \frac{2[m-1]}{[m]^3}\alpha_{m-1,k-1}^{k}, \\
\gamma_{m,k}^{k+1} &= \frac{[m-1]^2}{[m]^2}\gamma_{m-1,k-1}^{k} - \frac{4[m-1]}{[m]^3}\beta_{m-1,k-1}^{k} + \frac{4}{[m]^4}\alpha_{m-1,k-1}^{k}.
\end{aligned}$$

Thanks to equation (81), when $1 \leqslant k \leqslant m-r$ the solution is

$$\begin{aligned}
\alpha_{m,k}^{k-1} &= -\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}, \\
\beta_{m,k}^{k-1} &= -\frac{2[k][m-k+1]}{[m]^3}, \\
\gamma_{m,k}^{k-1} &= \frac{4[k][k-1]}{[m]^4}, \\
\alpha_{m,k}^{k} &= -\frac{[2]_{a+1}[m-k]^2}{[m]^2}, \\
\beta_{m,k}^{k} &= \frac{2[2]_{a+1}[k][m-k]}{[m]^3}, \\
\gamma_{m,k}^{k} &= \frac{3[2]_{a+1}}{[m]^2}\delta_{k,m-r} - \frac{4[2]_{a+1}[k]^2}{[m]^4}, \\
\alpha_{m,k}^{k+1} &= \frac{[m-k][m-k-1]}{[m]^2}, \\
\beta_{m,k}^{k+1} &= -\frac{1}{[m]^2} - \frac{2[k][m-k-1]}{[m]^3}, \\
\gamma_{m,k}^{k+1} &= -\frac{[2]}{[m]^2}\delta_{k,m-r-1} - \frac{2[2]}{[m]^2}\delta_{k,m-r} + \frac{4[k+1][k]}{[m]^4},
\end{aligned}$$

and when $m-r+1 \leqslant k \leqslant m$ the solution is

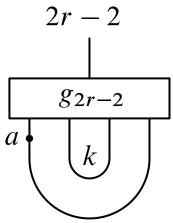
$$\alpha_{m,k}^{k-1} = -\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2},$$

$$\begin{aligned}
\beta_{m,k}^{k-1} &= -\frac{1}{[m]^2} + \frac{\{m\}'[m-k+1][m-k]}{[m]^3}, \\
\gamma_{m,k}^{k-1} &= \frac{[2]}{[m]^2} + \frac{(\{m\}')^2[m-k+1][m-k]}{[m]^4}, \\
\alpha_{m,k}^{k} &= -\frac{[2]_{a+1}[m-k]^2}{[m]^2}, \\
\beta_{m,k}^{k} &= -\frac{[2]_{a+1}\{m\}'[m-k]^2}{[m]^3}, \\
\gamma_{m,k}^{k} &= -\frac{[2]_{a+1}(\{m\}')^2[m-k]^2}{[m]^4}, \\
\alpha_{m,k}^{k+1} &= \frac{[m-k][m-k-1]}{[m]^2}, \\
\beta_{m,k}^{k+1} &= \frac{\{m\}'[m-k][m-k-1]}{[m]^3}, \\
\gamma_{m,k}^{k+1} &= \frac{(\{m\}')^2[m-k][m-k-1]}{[m]^4}.
\end{aligned}$$

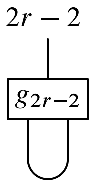
■

7.3. Pseudo traces

Finally, let us move on to the most complicated case, and let us start by remarking that, if $1 \leqslant k \leqslant r-1$, then we have

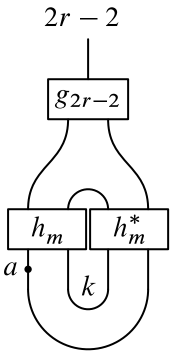


$$= -\frac{[r-k]_{a+1}}{[k]} \cdot$$

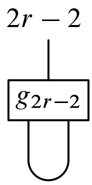


(90)

Lemma 7.6. *For all integers $r \leqslant m \leqslant 2r-2$ and $0 \leqslant k \leqslant m$ we have*



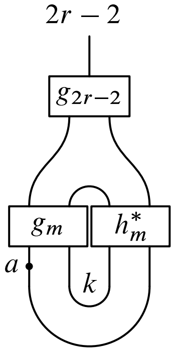
$$= \delta_{k,m-r+1}(-1)^{m+1}[r]_{a+1}[m+1] \cdot$$



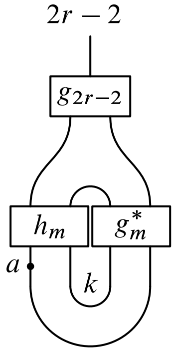
(91)

Proof. The computation is easy, and left to the reader. ■

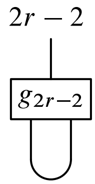
Lemma 7.7. *For every integer $r \leqslant m \leqslant 2r-2$, if $0 \leqslant k \leqslant m-r$ we have*



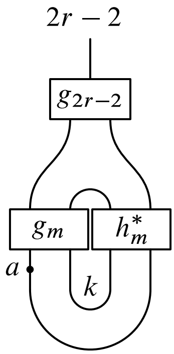
$=$



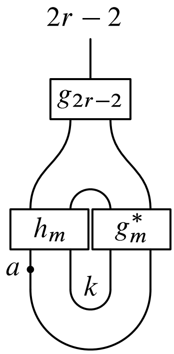
$$= (-1)^m \frac{[2r-m+k-1]_{a+1}[m+1]}{[m-k+1]} \cdot \quad (92)$$



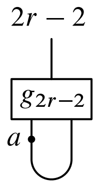
and if $m-r+1 \leqslant k \leqslant m$ we have



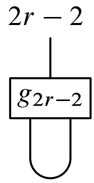
$=$



$$= \delta_{k,m-r+1}(-1)^{m+1}[m+1] \cdot$$

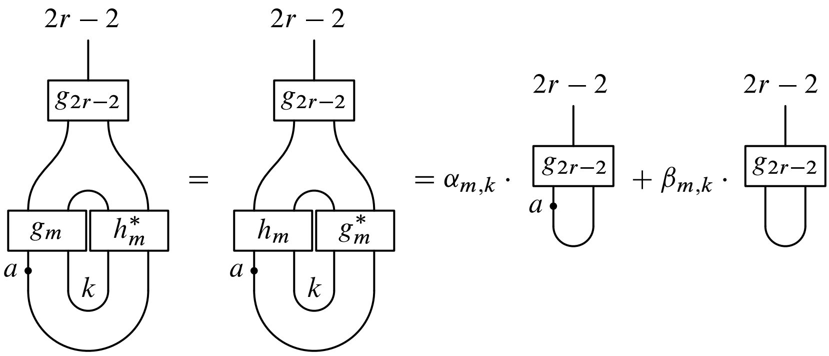


$$+ (1-\delta_{k,m-r+1})(-1)^{m+1} \frac{[m-k+1]_{a+1}[m+1]}{[m-k+1]} \cdot$$



(93)

Proof. Equations (92) and (93) are proved by induction on $r \leqslant m \leqslant 2r-2$. In order to do this, let us set



It is also convenient to set

$$\alpha_{m,-1} = \beta_{m,-1} = \alpha_{m,m+1} = \beta_{m,m+1} = \alpha_{m,m+2} = \beta_{m,m+2} = 0.$$

If $m = r$ then equations (84), (86), and (90) give the condition

$$\begin{aligned} \alpha_{r,0} &= 0, \qquad \beta_{r,0} = -[r-1]_{a+1}, \\ \alpha_{r,k} &= \delta_{k,1}, \quad \beta_{r,k} = -(1-\delta_{k,1})\frac{[r-k+1]_{a+1}}{[k-1]}. \end{aligned}$$

If $r+1 \leqslant m \leqslant 2r-2$ and $0 \leqslant k \leqslant m-r$, then equations (88), (90), and (91) give the condition

$$\begin{aligned} \alpha_{m,k} = &\left(-\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}\right)\alpha_{m-1,k-1} \\ &- \frac{[2]_{a+1}[m-k]^2}{[m]^2}\alpha_{m-1,k} + \frac{[m-k][m-k-1]}{[m]^2}\alpha_{m-1,k+1}, \\ \beta_{m,k} = &\left(-\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}\right)\beta_{m-1,k-1} \\ &- \frac{[2]_{a+1}[m-k]^2}{[m]^2}\beta_{m-1,k} + \frac{[m-k][m-k-1]}{[m]^2}\beta_{m-1,k+1} \\ &- \delta_{k,m-r-1}(-1)^m\frac{[r]_{a+1}}{[m]}. \end{aligned}$$

Thanks to equation (81), the solution is

$$\alpha_{m,k} = 0, \quad \beta_{m,k} = (-1)^m\frac{[2r-m+k-1]_{a+1}[m+1]}{[m-k+1]}.$$

If $r+1 \leqslant m \leqslant 2r-2$ and $m-r+1 \leqslant k \leqslant m$, then equations (89), (90), and (91) give the condition

$$\begin{aligned}
\alpha_{m,k} &= \Big(-\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}\Big)\alpha_{m-1,k-1} \\
&\quad - \frac{[2]_{a+1}[m-k]^2}{[m]^2}\alpha_{m-1,k} + \frac{[m-k][m-k-1]}{[m]^2}\alpha_{m-1,k+1}, \\
\beta_{m,k} &= \Big(-\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}\Big)\beta_{m-1,k-1} \\
&\quad - \frac{[2]_{a+1}[m-k]^2}{[m]^2}\beta_{m-1,k} + \frac{[m-k][m-k-1]}{[m]^2}\beta_{m-1,k+1} \\
&\quad - \delta_{k,m-r+1}(-1)^m \frac{[r]_{a+1}}{[m]}.
\end{aligned}$$

Thanks to equation (81), the solution is

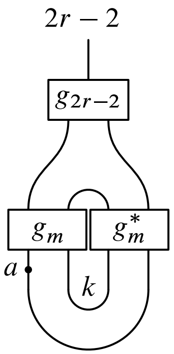
$$\begin{aligned}
\alpha_{m,k} &= \delta_{k,m-r+1}(-1)^{m+1}[m+1], \\
\beta_{m,k} &= (1-\delta_{k,m-r+1})(-1)^{m+1}\frac{[m-k+1]_{a+1}[m+1]}{[m-k+1]}.
\end{aligned}$$ ■

Lemma 7.8. *For every integer $0 \leqslant h \leqslant r$ we have*

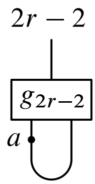
$$2[r-h]_{a+1} = 2[r]_{a+1} - 2[h]_{a+1}. \tag{94}$$

Proof. The computation is easy, and left to the reader. ■

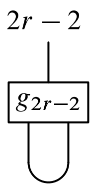
Lemma 7.9. *For all integers $r \leqslant m \leqslant 2r-2$ and $0 \leqslant k \leqslant m-r$ we have*



$$= (-1)^m \frac{\{m-k-r+1\}'_{a+1}[m+1]}{[m-k+1]} \cdot$$

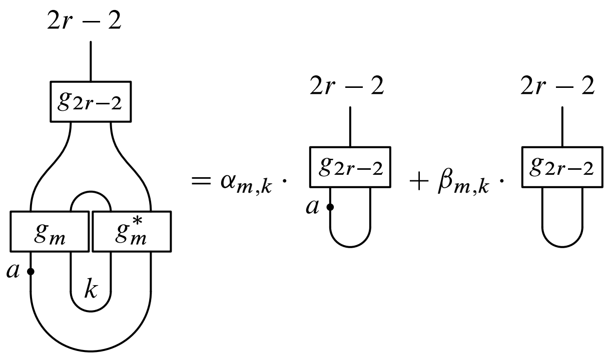


$$- (-1)^m\Big(\frac{2[m-k-r+1]_{a+1}[k]}{[m-k+1]^2} + \frac{[r]_{a+1}[m+1]}{[m-k+1]}\sum_{h=1}^{m-k-r}\frac{\{h\}'}{[h]}\Big) \cdot$$



(95)

Proof. Equation (95) is proved by induction on $r \leqslant m-k \leqslant 2r-2$. In order to do this, let us set



It is also convenient to set

$$\alpha_{m,-1} = \beta_{m,-1} = \alpha_{m,m+1} = \beta_{m,m+1} = \alpha_{m,m+2} = \beta_{m,m+2} = 0.$$

If $m-k = r$ and $m = r$, then equations (83) and (90) give the condition

$$\alpha_{r,0} = -[2]_{a+1}, \quad \beta_{r,0} = 0.$$

If $r \leqslant m-k \leqslant 2r-2$ and $r+1 \leqslant m \leqslant 2r-2$, then equations (88) and (90) give the condition

$$\begin{aligned}
\alpha_{m,k} = {} & \Big(-\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}\Big)\alpha_{m-1,k-1} \\
& - \frac{[2]_{a+1}[m-k]^2}{[m]^2}\alpha_{m-1,k} + \frac{[m-k][m-k-1]}{[m]^2}\alpha_{m-1,k+1} \\
& - \delta_{k,m-r-1}(-1)^m \frac{2}{[m]}, \\
\beta_{m,k} = {} & \Big(-\frac{[m+1]}{[m]} + \frac{[m-k+1][m-k]}{[m]^2}\Big)\beta_{m-1,k-1} \\
& - \frac{[2]_{a+1}[m-k]^2}{[m]^2}\beta_{m-1,k} + \frac{[m-k][m-k-1]}{[m]^2}\beta_{m-1,k+1} \\
& + (-1)^m \frac{2[2r-m+k+1]_{a+1}}{[m][m-k-1]}.
\end{aligned}$$

Thanks to equations (81) and (94) the solution is

$$\begin{aligned}
\alpha_{m,k} &= (-1)^m \frac{\{m-k-r+1\}'_{a+1}[m+1]}{[m-k+1]}, \\
\beta_{m,k} &= (-1)^{m+1}\Big(\frac{2[m-k-r+1]_{a+1}[k]}{[m-k+1]^2} + \frac{[r]_{a+1}[m+1]}{[m-k+1]}\sum_{h=1}^{m-k-r}\frac{\{h\}'}{[h]}\Big).
\end{aligned}$$

■

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