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Drinfeld centers of fusion categories arising from generalized Haagerup subfactors

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Abstract. We consider generalized Haagerup categories such that $1 \oplus X$ admits a Q-system for every non-invertible simple object X. We show that in such a category, the group of order two invertible objects has size at most four. We describe the simple objects of the Drinfeld center and give partial formulas for the modular data. We compute the remaining corner of the modular data for several examples and make conjectures about the general case. We also consider several types of equivariantizations and de-equivariantizations of generalized Haagerup categories and describe their Drinfeld centers.

In particular, we compute the modular data for the Drinfeld centers of a number of examples of fusion categories arising in the classification of small-index subfactors: the Asaeda–Haagerup subfactor; the $3^{\mathbb{Z}_4}$ and $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$ subfactors; the $2D2$ subfactor; and the 4442 subfactor.

The results suggest the possibility of several new infinite families of quadratic categories. A description and generalization of the modular data associated to these families in terms of pairs of metric groups is taken up in the accompanying paper [Comm. Math. Phys. 380 (2020), 1091–1150].

Contents

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1. Introduction

In the 1990s Asaeda and Haagerup discovered two "exotic" subfactors, known as the Haagerup subfactor and the Asaeda–Haagerup subfactor [2]. They called these subfactors *exotic* since they were the first examples of finite depth subfactors which did not arise from known symmetries of groups or quantum groups. The fusion category $\mathcal{C}$ which is the principal even part of the Haagerup subfactor has as its group of invertible objects

$$G = \mathrm{Inv}(\mathcal{C}) = \mathbb{Z}_3$$

and is tensor generated by a simple object X satisfying the fusion rules

$$g \otimes X \cong X \otimes g^{-1} \quad \text{and} \quad X \otimes X \cong 1 \oplus \bigoplus_{g \in G} g \otimes X.$$

The Haagerup subfactor corresponds to the algebra object $1 \oplus X$ in this category.

In [17] the second named author gave a new construction of the Haagerup subfactor in which the simple objects of $\mathcal{C}$ are realized by certain endomorphisms of the Cuntz C*-algebra $\mathcal{O}_4$ (extended to a von Neumann algebra closure). In this construction, the four Cuntz algebra generators correspond to embeddings of the four simple summands of $X \otimes X$. The structure constants for the action of these endomorphisms on the generators encode the associativity structure of the tensor category; these constants are in turn determined by polynomial equations.

More generally, it was shown that for an arbitrary finite Abelian group G of odd order, there is a similar system of polynomial equations whose solution gives an associativity structure for a tensor category satisfying the Haagerup fusion rules above, but with G replacing $\mathbb{Z}_3$. Such a generalized Haagerup category can then be realized via endomorphisms of the Cuntz algebra $\mathcal{O}_{|G|+1}$. There is an additional equation for the existence of a Q-system (an algebra structure with a unitarity condition [24]) on $1 \oplus X$; a subfactor corresponding to such a Q-system is called a *generalized Haagerup subfactor for* G, or a 3^G *subfactor* (after the shape of its principal graph). It was shown in [17] that there is a unique $3^{\mathbb{Z}_5}$ subfactor (up to isomorphism of the standard invariant), but the general existence question was left open.

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a fusion category is the category of half-braidings of $\mathcal{C}$ by objects $Y \in \mathcal{C}$. For a fusion category over $\mathbb{C}$, $\mathcal{Z}(\mathcal{C})$ is a non-degenerate braided fusion category. If $\mathcal{C}$ is spherical (in particular if $\mathcal{C}$ is unitary), then $\mathcal{Z}(\mathcal{C})$ has the structure of a modular tensor category [29]. A modular tensor category gives rise to a projective unitary representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$, with canonical generators mapped to a pair of matrices called the S and T *matrices,* also known as the *modular data.* The modular data encodes among other things the fusion rules of the category via the Verlinde formula [35].

Modular tensor categories appear in a variety of contexts, including conformal field theory [25], quantum topology [34], and topological quantum computing [12]. On the other hand, every fusion category can be realized as a category of modules over a commutative algebra in its Drinfeld center. Thus, the Drinfeld center construction provides a bridge between the theory of ordinary fusion categories and that of modular tensor categories.

A useful feature of the Cuntz algebra approach to the construction of subfactors is that it comes with a simple formalism for computing arbitrary tensor products and compositions of morphisms in the tensor category. This was exploited in [17] to give an explicit description of the tube algebra of the Haagerup category, and thereby compute the modular data of its Drinfeld center.

In [11], Evans and Gannon found simpler formulas for the modular data of the Drinfeld center of the Haagerup category. They generalized these formulas to an infinite family of modular data, which they conjectured were realized by Drinfeld centers of generalized Haagerup categories. They also computed solutions to the polynomial equations for generalized Haagerup categories for a number of odd groups, and found numerical evidence for the existence of (non-unique in some cases) generalized Haagerup subfactors for all odd cyclic groups up to order 19.

The fusion categories which are the even parts of the Asaeda–Haagerup subfactor have less symmetric fusion rules than the Haagerup category. The original construction of Asaeda and Haagerup used generalized open string bimodules, a generalization of Ocneanu's connection formalism for graphs, to describe the categories. Their calculations showed the existence of the categories, but did not give a practical framework for performing complicated computations within the category; in particular, a description of the Drinfeld center was not accessible.

In [14], the authors and Snyder gave a new construction of the Asaeda–Haagerup subfactor. The construction starts with a generalized Haagerup category for the group $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. The original Asaeda–Haagerup categories are then shown to be Morita equivalent to a category arising from this generalized Haagerup category via a construction called *de-equivariantization* (see [6, 8, 28]). The system of polynomial equations for generalized Haagerup categories associated to even groups is considerably more subtle than for the odd case, and involves a collection of characters $\epsilon_{(\cdot)}(g)$ of the group $G_2 \subseteq G$ of order elements, indexed by G, see [19].

Since the Drinfeld center is an invariant of Morita equivalence [33], this construction allows us to use the Cuntz algebra framework for generalized Haagerup categories to describe the Drinfeld center and compute the modular data of the Asaeda–Haagerup categories (which was first announced in [14]). The original motivation of this paper was to describe this computation.

However, it turns out that generalized Haagerup categories for groups of even order play a central role in the classification of small-index subfactors beyond the

Asaeda–Haagerup case. Subfactors with index less than 4 have index of the form $4\cos^2\frac{\pi}{k}, k = 3, 4, 5, \ldots$ (see [21]), and their principal graphs are simply laced Dynkin diagrams. In the 1990s Haagerup initiated the classification of subfactors with index slightly above 4 by searching for admissible principal graphs (which is how the Haagerup and Asaeda–Haagerup subfactors were discovered), see [15]. This classification has now been extended to index 5 (see [22]), and then to index 5.25 (see [1]), with only a small number of finite-depth examples appearing.

The most interesting index value between 5 and 5.25 is $3 + \sqrt{5}$, which is the first composite index above 4. There are exactly seven finite depth subfactor planar algebras at index $3 + \sqrt{5}$ up to duality [1]. Of these, two are the unique $3^{\mathbb{Z}_4}$ and $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$ subfactors; another one is the $2D2$ subfactor , which is related to the $3^{\mathbb{Z}_4}$ subfactor through a $\mathbb{Z}_2$-de-equivariantization; and another one is the 4442 subfactor, which is related to the $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$ subfactor through a $\mathbb{Z}_3$-equivariantization (and another one is related to the $3^{\mathbb{Z}_4}$ subfactor by Morita equivalence), see [19, 26, 27, 32].

In addition to the Asaeda–Haagerup subfactor, we compute the modular data for these four subfactors with index $3 + \sqrt{5}$, as well as for a $\mathbb{Z}_2$-de-equivariantization of a $3^{\mathbb{Z}_8}$ subfactor. We also numerically compute modular data for $3^{\mathbb{Z}_6}$, $3^{\mathbb{Z}_8}$, and $3^{\mathbb{Z}_{10}}$ subfactors (of which there are two each for $\mathbb{Z}_6$ and $\mathbb{Z}_{10}$).

More generally, we consider families of subfactors associated to these examples. The appearance of the characters $\epsilon_{(.)}(g)$ on G_2 makes the study of generalized Haagerup categories for even groups considerably more complicated than the odd case. Indeed, generalized Haagerup categories for odd groups can always be constructed as de-equivariantizations of near-group categories [18]; this is not true for even groups. However, our first main result restricts the groups for which generalized Haagerup subfactors can exist, under an additional assumption.

Theorem 1.1. *Let G be the group of invertible objects of a generalized Haagerup category, and suppose that $1 \oplus (g \otimes X)$ admits a Q-system for all $g \in G$. Let $G_2 = (\mathbb{Z}_2)^m \subseteq G$ be the elementary 2-group of order 2 elements. Then $m \leq 2$.*

In light of this result, we consider generalized Haagerup categories associated to a group $G = \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n} \times H$ with $m \geq n \geq 0$ and $|H|$ odd. By analyzing the tube algebra, we can describe the simple objects in the Drinfeld center. All of the simple objects in the center contain invertible simple summands except for

$$\mu = \bigoplus_{g\in G} g \otimes X$$

(which has $\frac{|G|^2}{2}$ half-braidings); and, in the case that $|G_2| = 2$, the objects $\nu_{\pm}$, where $\mu = \nu_+ \oplus \nu_-$ is the decomposition of μ into the direct sums of those $g \otimes X$ indexed by the two cosets of $2G$ in G (which have two half-braidings each).

We give formulas for the modular data in terms of the characters $\epsilon_{(.)}(g)$, except for the corner in which both indices are half-braidings of μ or $\nu_{\pm}$. In the case that $|G_2| = 4$, under the additional assumption that the braiding on G_2 is non-degenerate (which is a property of the characters $\epsilon_{(.)}(g)$), the modular data factors, and we get formulas for the modular data modulo the corner indexed by the $\frac{|G|^2}{8}$ half-braidings of μ in the Müger commutant of G_2 (see [30]).

To compute the missing corner of the modular data, it is necessary to perform detailed calculations in the tube algebra, and we do not have general formulas. However, we have conjectures based on computations for small examples.

For the Asaeda–Haagerup subfactor, we consider a generalized Haagerup category $\mathcal{C}$ for $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ with a certain form of ϵ, as in [14]. Then $\mathrm{Vec}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ lifts to the center as a non-degenerate subcategory, and its Müger commutant in $\mathcal{Z}(\mathcal{C})$ is exactly the Drinfeld center of the Asaeda–Haagerup categories. More generally, we can consider $\mathbb{Z}_2$-de-equivariantizations for generalized Haagerup categories for $\mathbb{Z}_{4m} \times \mathbb{Z}_2$ with a similar form of ϵ. We compute the missing corner of the modular data for the Asaeda–Haagerup case $m = 1$.

The $2D2$ subfactor arises from a $\mathbb{Z}_2$-de-equivariantization of a generalized Haagerup category for $\mathbb{Z}_4$ with non-trivial ϵ (which is uniquely determined up to gauge equivalence). We can similarly consider $\mathbb{Z}_2$-de-equivariantizations of generalized Haagerup categories for $\mathbb{Z}_{4m}$ with non-trivial ϵ. We obtain formulas for the modular data except for a corner, and compute the missing corner for the two known examples $m = 1, 2$.

Finally, we consider the 4442 subfactor, which comes from a $\mathbb{Z}_3$-equivariantization of a generalized Haagerup category $\mathcal{C}$ for $\mathbb{Z}_2 \times \mathbb{Z}_2$. We compute the modular data from the tube algebra of the Morita equivalent crossed product category, which is a $\mathbb{Z}_3$-graded extension of $\mathcal{C}$ and has a Cuntz algebra model. It may be possible to generalize this construction to generalized Haagerup categories of the form $\mathbb{Z}_2 \times \mathbb{Z}_2 \times H$ with $|H|$ odd, but we do not know of any examples for nontrivial H, and we do not attempt this generalization here.

In summary, these examples suggest several possibly infinite families of quadratic categories, which are distinguished by the 2-subgroups of the associated finite group.

- Generalized Haagerup categories for $G = \mathbb{Z}_{4m}$, and their $\mathbb{Z}_2$-de-equivariantizations (or more generally $\mathbb{Z}_{4m} \times H$ with H odd) - known to exist for $m = 1, 2$ and H trivial.
- Generalized Haagerup categories for $G = \mathbb{Z}_{4m+2}$ (or more generally $\mathbb{Z}_{4m+2} \times H$ with H odd) - known to exist for $m = 0, 1, 2$ and H trivial. (For $m = 1, 2$ these occur in pairs.)

- Generalized Haagerup categories for $G = \mathbb{Z}_{4m} \times \mathbb{Z}_2$ (or more generally $\mathbb{Z}_{4m} \times \mathbb{Z}_2 \times H$ with H odd), and their $\mathbb{Z}_2$-de-equivariantization – only known to exist when $m = 1$ and H is trivial.
- Generalized Haagerup categories for $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times H$, with H odd, and its $\mathbb{Z}_3$-equivariantization - both only known to exist when H is trivial.

An analysis of potential modular data associated to such families, as well as generalizations of this modular data, in terms of pairs of metric groups will appear in the accompanying paper [13].

To compute the missing corners of the modular data in examples, we use Mathematica to perform the arithmetic in the tube algebra. These calculations require formulas for the tube algebra operations. Such formulas are in principle straightforward to derive from the definition of the tube algebra, the Cuntz algebra model for generalized Haagerup categories, and the orbifold construction for (de)-equivariantization. In practice the formulas are laborious to write down, so we include them in an online appendix.

The paper is organized as follows.

In Section 2 we review some preliminaries on fusion categories, tube algebras, and generalized Haagerup subfactors.

In Section 3 we prove the restriction on G_2 for a generalized Haagerup category with a certain assumption. Then we describe the tube algebra and give partial formulas for the modular data.

In Section 4 we describe how to compute the remaining corner of the modular data, and discuss several examples with $|G_2|$=2.

In Section 5 we consider the Müger factorization of the Drinfeld center for the case that $|G_2| = 4$ and the braiding on G_2 is non-degenerate. This case includes the Asaeda–Haagerup subfactor (via de-equivariantization).

In Section 6 we consider a $\mathbb{Z}_2$-de-equivariantization of a generalized Haagerup for $\mathbb{Z}_{4m}$. This case includes the $2D2$ subfactor.

In Section 7 we describe the tube algebra and compute the modular data for the $\mathbb{Z}_3$-equivariantization of a generalized Haagerup category for $\mathbb{Z}_2 \times \mathbb{Z}_2$ (which is the even part of the 4442 subfactor).

In an online appendix, included with the `arxiv` source, we give formulas for multiplication, involution, and rotation in the tube algebras of generalized Haagerup categories and their (de)-equivariantizations.

In an accompanying Mathematica notebook `solutions.nb`, also included with the `arxiv` source, we give the structure constants for generalized Haagerup categories for $\mathbb{Z}_6$, $\mathbb{Z}_8$, and $\mathbb{Z}_{10}$.

This paper replaces an earlier arXiv preprint titled "Quantum doubles of generalized Haagerup subfactors and their orbifolds" (arXiv:1501.07679v1) which described the computations of most of the main examples in this paper but without discussing the general cases.

2. Preliminaries

2.1. Fusion categories and the category $\mathrm{End}_0(M)$

A fusion category over the complex numbers $\mathbb{C}$ is a rigid semisimple $\mathbb{C}$-linear tensor category with finitely many simple objects and finite-dimensional morphism spaces such that the identity object is simple (see [9]). In this paper we concentrate on unitary fusion categories embedded as full monoidal subcategories of the category of unital endomorphisms $\mathrm{End}(M)$ of a type III factor M. Note that such an embedding always exists under the assumption of unitarity (see [18, Theorem 2.1]). Our standard reference for the category $\mathrm{End}(M)$ is [4]. We follow the convention of using Greek letters for objects in $\mathrm{End}(M)$.

For a Hilbert space $\mathcal{H}$, we denote by $\mathbb{B}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$, and by $\mathcal{U}(\mathcal{H})$ the set of unitary operators on $\mathcal{H}$. The identity operator of $\mathcal{H}$ is denoted by $1_{\mathcal{H}}$ or simply by 1. For a unital C*-algebra A, we denote by $\mathcal{U}(A)$ the set of unitaries in A. The unit of A is denoted by 1_A or simply by 1.

Let M be a type III factor. Then the set of unital endomorphisms $\mathrm{End}(M)$ forms a category with the morphism space from an object ρ to another object σ given by

$$\mathrm{Hom}(\rho, \sigma) = (\rho, \sigma) = \{t \in M : t\rho(x) = \sigma(x)t \text{ for all } x \in M\}.$$

This category has a strict monoidal structure, with the monoidal product $\rho \otimes \sigma$ of two objects $\rho, \sigma \in \mathrm{End}(M)$ given by the composition $\rho \circ \sigma$, and the monoidal product between morphisms $t \in (\rho_1, \rho_2)$ and $s \in (\sigma_1, \sigma_2)$ given by

$$t \otimes s = t\rho_1(s) = \rho_2(s)t \in (\rho_1 \circ \sigma_1, \rho_2 \circ \sigma_2).$$

The monoidal unit of $\mathrm{End}(M)$ is the identity automorphism id of M. The endomorphism space (ρ, ρ) is just the relative commutant $M \cap \rho(M)'$; ρ is simple, or irreducible, if this space consists only of the scalars. When discussing the monoidal category $\mathrm{End}(M)$, we will generally suppress the "$\otimes$" symbol, and refer directly to multiplication in M and composition of endomorphisms.

The morphism space (ρ, σ) inherits a Banach space structure from M, and the $*$-operation of M maps (ρ, σ) to (σ, ρ), which makes $\mathrm{End}(M)$ a C*-tensor category (see [4, Section 1]). Moreover, if ρ is simple, the space (ρ, σ) is a Hilbert space with an inner product given by $t_1^* t_2 = \langle t_1, t_2 \rangle 1_M$ for $t_1, t_2 \in (\rho, \sigma)$.

For $\rho \in \mathrm{End}(M)$, its *dimension* $d(\rho)$ is defined by $[M : \rho(M)]_0^{1/2}$, where $[M : \rho(M)]_0 \in [1, \infty]$ is the minimum index of $\rho(M)$ in M. We denote by $\mathrm{End}_0(M)$ the set of $\rho \in \mathrm{End}(M)$ with finite $d(\rho)$. The dimension function $\mathrm{End}_0(M) \ni \rho \mapsto d(\rho)$ is additive with respect to the direct sum operation and multiplicative with respect to the monoidal product operation. The monoidal category $\mathrm{End}_0(M)$ is rigid, which means that objects have left and right duals. For any $\rho \in \mathrm{End}_0(M)$, there exists $\bar{\rho} \in \mathrm{End}_0(M)$, called a *dual,* or *conjugate,* endomorphism of ρ, and two isometries $r_\rho \in (\mathrm{id}, \bar{\rho} \circ \rho)$, $\bar{r}_\rho \in (\mathrm{id}, \rho \circ \bar{\rho})$ satisfying

$$\bar{r}_\rho^* \rho(r_\rho) = r_\rho^* \bar{\rho}(\bar{r}_\rho) = \frac{1}{d(\rho)}.$$

The evaluation morphism ev_ρ is identified with $\sqrt{d(\rho)}\bar{r}_\rho$, and the coevaluation morphism coev_ρ is identified with $\sqrt{d(\rho)} r_\rho^*$.

If ρ and σ are isomorphic objects in $\mathrm{End}_0(M)$, then there exists a unitary $u \in \mathcal{U}(M)$ satisfying $\rho = \mathrm{Ad}u \circ \sigma$, where $\mathrm{Ad}u$ is the inner automorphism of M given by $\mathrm{Ad}u(x) = uxu^*$; we then say that ρ and σ are *unitarily equivalent,* or simply *equivalent.* We denote by $\mathrm{Inn}(M)$ the group of inner automorphisms of M.

For a fusion category $\mathcal{C} \subset \mathrm{End}(M)$, the categorical dimension $d(\rho)$ coincides with the Frobenius-Perron dimension in $\mathcal{C}$. We denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects in $\mathcal{C}$. We often identify an element in $\mathrm{Irr}(\mathcal{C})$ with one of its representatives if there is no possibility of confusion. The global dimension of $\mathcal{C}$ is defined by

$$\dim \mathcal{C} = \sum_{\xi \in \mathrm{Irr}(\mathcal{C})} d(\xi)^2.$$

2.2. The Drinfeld center

Let $\mathcal{C}$ be a monoidal category. A half-braiding for an object $X \in \mathcal{C}$ is a natural isomorphism $\mathcal{E}_X : X \otimes (-) \to (-) \otimes X$, satisfying the hexagon and unit identities. If $\mathcal{C}$ is strict, these identities reduce to

$$\mathcal{E}_X(Y \otimes Z) = (\mathrm{id}_Y \otimes \mathcal{E}_X(Z)) \circ (e_X(Y) \otimes \mathrm{id}_Z) \quad \text{for all } Y, Z \in \mathcal{C}.$$

and

$$\mathcal{E}_X(1) = \mathrm{id}_X,$$

respectively.

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ is the category whose objects are half-braidings $(X, \mathcal{E}_X)$ of objects in $\mathcal{C}$ and whose morphisms are given by

$$\begin{aligned} &\mathrm{Hom}((X, \mathcal{E}_X), (Y, \mathcal{E}_Y)) \\ &\quad = \{t \in \mathrm{Hom}(X, Y) : (\mathrm{id}_Z \otimes t) \circ \mathcal{E}_X(Z) = \mathcal{E}_Y(Z) \circ (t \otimes \mathrm{id}_Z) \text{ for all } Z \in \mathcal{C}\}. \end{aligned}$$

The Drinfeld center is a braided monoidal category, with tensor product of $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ given by

$$(X \otimes Y, \mathcal{E}_{X\otimes Y})$$

with

$$\mathcal{E}_{X\otimes Y}(Z) = (\mathcal{E}_X(Z) \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \mathcal{E}_Y(Z)) \quad \text{for all } Z \in \mathcal{C},$$

and braiding given by

$$c_{X,Y} = \mathcal{E}_X(Y).$$

If $\mathcal{C}$ is semisimple, then a half-braiding $\mathcal{E}_X$ is determined by $\mathcal{E}_X(Y)$ as Y ranges over representatives of isomorphism classes of simple objects. If $\mathcal{C}$ is a fusion category, then $\mathcal{Z}(\mathcal{C})$ is a fusion category as well, and the braiding on $\mathcal{Z}(\mathcal{C})$ is non-degenerate. If $\mathcal{C}$ is a unitary fusion category, the unitary Drinfeld center (where the $e_X(Y)$ are required to be unitaries) is equivalent to the ordinary Drinfeld center [29]. We will therefore only consider unitary half-braidings in this paper.

A modular tensor category is a non-degenerate braided spherical fusion category $\mathcal{C}$. The S-matrix of a modular tensor category is defined by

$$S_{X,Y} = \frac{d_X d_Y}{\sqrt{\dim(\mathcal{C})}} \mathrm{tr}_{X\otimes Y}(c_{X,Y} \circ c_{Y,X}),$$

for simple objects X and Y, where c is the braiding on $\mathcal{C}$, d_X is the quantum dimension, $\dim(\mathcal{C})$ is the global dimension, and tr is the normalized spherical trace on $\mathrm{End}(X \otimes Y)$. The S-matrix is defined up to a choice of square root of the global dimension.

The T-matrix is defined by

$$T_{X,Y} = \delta_{X,Y} d_X \mathrm{tr}_{X\otimes X}(c_{X,X}),$$

and the conjugation matrix C is defined by

$$C_{X,Y} = \delta_{X,\overline{Y}},$$

where $\overline{Y}$ is the dual object of Y.

For a modular tensor category over $\mathbb{C}$, S is symmetric, T is diagonal with finite order, and S and T are unitary [10]. We have the relations

$$\alpha(ST)^3 = S^2 = C = T^{-1}CT$$

for a scalar α, see [3, 25].

If $\mathcal{C}$ is a spherical fusion category over $\mathbb{C}$, then $\mathcal{Z}(\mathcal{C})$ is a modular tensor category. We fix $\sqrt{\dim(\mathcal{Z}(\mathcal{C}))} = \dim(\mathcal{C})$, and then $\alpha = 1$, see [29].

We now return to a unitary fusion category $\mathcal{C}$ embedded in $\mathrm{End}_0(M)$. Let $\sigma \in \mathcal{C}$ be a (not necessarily simple) object in $\mathcal{C}$. Then the data of a half-braiding for σ is given by family of unitaries $\mathcal{E}_\sigma = \{\mathcal{E}_\sigma(\xi)\}_{\xi \in \mathrm{Irr}(\mathcal{C})}$ with $\mathcal{E}_\sigma(\xi) \in (\sigma\xi, \xi\sigma)$ such that for any $t \in (\zeta, \xi\eta)$ with $\xi, \eta, \zeta \in \mathrm{Irr}(\mathcal{C})$, we have

$$t\mathcal{E}_\sigma(\zeta) = \xi(\mathcal{E}_\sigma(\eta))\mathcal{E}_\sigma(\xi)\sigma(t). \tag{2.1}$$

In general, a single object σ may have several inequivalent half-braidings, and we introduce labeling to distinguish them and use the notation $\tilde{\sigma}^l = (\sigma, \mathcal{E}_\sigma^l)$ for simplicity. We denote by $\mathcal{F}$ the forgetful functor $\mathcal{F}\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$.

2.3. The tube algebra

We summarize the basics of tube algebras following [16].

The tube algebra for a fusion category $\mathcal{C} \subset \mathrm{End}(M)$ is a finite-dimensional C*-algebra with underlying vector space

$$\mathrm{Tube}\,\mathcal{C} = \bigoplus_{\xi,\eta,\zeta \in \mathrm{Irr}(\mathcal{C})} (\xi\zeta, \zeta\eta).$$

An element $x \in (\xi\zeta, \zeta\eta)$ is denoted as an element of $\mathrm{Tube}\,\mathcal{C}$ by $(\xi\ \zeta|x|\zeta\ \xi)$. The $*$-algebra operations of $\mathrm{Tube}\,\mathcal{C}$ are defined by

$$\begin{aligned}
&(\xi\ \zeta|x|\zeta\ \eta)(\xi'\ \zeta'|y|\zeta'\ \eta') \\
&\quad = \delta_{\eta,\xi'} \sum_{\nu \in \mathrm{Irr}(\mathcal{C})} \sum_{i=1}^{\dim(\nu,\zeta\zeta')} (\xi\ \nu|t\tbinom{\nu}{\zeta,\zeta'}_i^* \zeta(y) x \xi(t\tbinom{\nu}{\zeta,\zeta'}_i)|\nu\ \eta'), \\
&(\xi\ \zeta|x|\zeta\ \eta)^* = d(\zeta)(\eta\ \bar{\zeta}|\bar{\zeta}(\xi(\bar{r}_\zeta^*)x^*)r_\zeta|\bar{\zeta}\xi),
\end{aligned}$$

where $\{t\binom{\nu}{\zeta,\zeta'}_i\}_{i=1}^{\dim(\nu,\zeta\zeta')}$ is an orthonormal basis of the space $(\nu, \zeta\zeta')$. We denote

$$\mathcal{A}_{\xi,\eta} = \bigoplus_{\zeta \in \mathrm{Irr}(\mathcal{C})} (\xi\zeta, \zeta\eta),$$

which is a subspace of $\mathrm{Tube}\,\mathcal{C}$ satisfying

$$\mathcal{A}_{\xi,\eta}\mathcal{A}_{\xi',\eta'} \subseteq \delta_{\eta,\xi'}\mathcal{A}_{\xi,\eta'}, \quad \mathcal{A}_{\xi,\eta}^* = \mathcal{A}_{\eta,\xi}.$$

In particular, $\mathcal{A}_\xi := \mathcal{A}_{\xi,\xi}$ is a $*$-subalgebra of $\mathrm{Tube}\,\mathcal{C}$ with unit $1_\xi = (\xi\ 0|1|0\ \xi)$, where we denote id by 0 for simplicity. We have

$$\sum_{\xi \in \mathrm{Irr}(\mathcal{C})} 1_\xi = 1_{\mathrm{Tube}\,\mathcal{C}}.$$

The algebra $\mathcal{A}_\xi$ is a corner of Tube $\mathcal{C}$ in the sense that we have

$$1_\xi \operatorname{Tube}\mathcal{C} 1_\xi = \mathcal{A}_\xi.$$

In particular, every minimal projection in $\mathcal{A}_\xi$ is minimal in Tube $\mathcal{C}$. The space $\mathcal{A}_{\xi,\eta}$ is a $\mathcal{A}_\xi$-$\mathcal{A}_\eta$ bimodule with respect to the left and right multiplication of $\mathcal{A}_\xi$ and $\mathcal{A}_\eta$, respectively.

There is a one to one correspondence between the set of simple components of Tube $\mathcal{C}$ and $\operatorname{Irr}(\mathcal{Z}(\mathcal{C}))$, and we denote by $z(\tilde{\sigma}^l)$ the minimal central projection in Tube $\mathcal{C}$ corresponding to $\tilde{\sigma}^l$. Then the algebra $z(\tilde{\sigma}^l)$ Tube $\mathcal{C}$ is isomorphic to a full matrix algebra, and we can write down a system of matrix units for it in terms of $\mathcal{E}^l_\sigma$ as follows. We choose an orthonormal basis $\{w_\sigma(\xi)_i\}_{i=1}^{\dim(\xi,\sigma)}$ of (ξ, σ) for each $\xi \in \operatorname{Irr}(\mathcal{C})$, and set

$$\mathcal{E}^l_\sigma(\zeta)_{(\xi,i),(\eta,j)} = \zeta(w_\sigma(\eta)^*_j)\mathcal{E}^j_\sigma(\zeta)w_\sigma(\xi)_i \in (\xi\zeta, \zeta\eta),$$

$$e(\tilde{\sigma}^l)_{(\xi,i),(\eta,j)} = \frac{d(\sigma)}{\Lambda\sqrt{d(\xi)d(\eta)}} \sum_{\zeta\in\operatorname{Irr}(\mathcal{C})} d(\zeta)(\xi\ \zeta|\mathcal{E}_\sigma(\zeta)_{(\xi,i),(\eta,j)}|\zeta\ \eta) \in \operatorname{Tube}\mathcal{C}, \quad (2.2)$$

where Λ is the global dimension of $\mathcal{C}$. Then $\{e(\tilde{\sigma}^l)_{(\xi,i),(\eta,j)}\}_{(\xi,i),(\eta,j)}$ forms a system matrix units for the subalgebra $z(\tilde{\sigma}^l)$ Tube $\mathcal{C}$, and

$$z(\tilde{\sigma}^l) = \sum_{(\xi,i)} e(\tilde{\sigma}^l)_{(\xi,i),(\xi,i)}. \quad (2.3)$$

In particular, the rank of $z(\tilde{\sigma}^l)$ Tube $\mathcal{C}$ is

$$\sum_{\xi\in\operatorname{Irr}(\mathcal{C})} \dim(\xi, \sigma).$$

Conversely, every system of matrix units for the tube algebra arises this way (by varying the choices of orthonormal bases for each (ξ, σ)). This gives a very practical way to determine the data of half-braidings from the algebra structure of the tube algebra. Starting from a system of matrix units for Tube $\mathcal{C}$, one can use (2.2) to extract the components of the half-braidings $\mathcal{E}^l_\sigma(\zeta)_{(\xi,i),(\eta,j)}$.

Note that

$$\sum_i e(\tilde{\sigma}^l)_{(\xi,i),(\xi,i)}$$

is a minimal central projection in $\mathcal{A}_\xi$, and the corresponding simple component of $\mathcal{A}_\xi$ has rank $\dim(\xi, \sigma)$. Conversely, every minimal central projection in $\mathcal{A}_\xi$ is of this form. The element $e(\tilde{\sigma}^l)_{(\xi,i),(\xi,i)}$ acts on $\mathcal{A}_{\xi,\eta}$ as a projection of rank $\dim(\eta, \sigma)$.

The modular data (S, T) for the modular tensor category $\mathcal{Z}(\mathcal{C})$ can be computed in terms of Tube $\mathcal{C}$ as follows. For $\xi \in \mathrm{Irr}(\mathcal{C})$, we set

$$\mathbf{t}_\xi = d(\xi)(\xi\bar{\xi}|r_\xi \bar{r}_\xi^*|\bar{\xi}\xi).$$

Then $\mathbf{t}_\xi$ is a unitary central element of $\mathcal{A}_\xi$ with adjoint $\mathbf{t}_\xi^* = (\xi\ \xi|1|\xi\ \xi)$. Let

$$\mathbf{t} = \sum_{\xi \in \mathrm{Irr}(\mathcal{C})} \mathbf{t}_\xi.$$

Then $\mathbf{t}$ is a central unitary element in Tube $\mathcal{C}$, giving the T-matrix via

$$\mathbf{t} z(\tilde{\sigma}^l) = T_{\tilde{\sigma}^l, \tilde{\sigma}^l} z(\tilde{\sigma}^l).$$

We can compute T by computing the eigenvalues of $\mathbf{t}_\xi$ (or $\mathbf{t}_\xi^*$).

We introduce a linear transformation S_0 of

$$\bigoplus_\xi \mathcal{A}_\xi$$

by

$$S_0((\xi\ \eta|x|\eta\ \xi)) = d(\xi)(\bar{\eta}\ \xi|r_\eta^* \bar{\eta}(x\xi(\bar{r}_\eta))|\xi\bar{\eta}), \tag{2.4}$$

which can be thought of as rotation. Then S_0 preserves the center of Tube $\mathcal{C}$, and the S-matrix is given by the matrix coefficients of S_0 with respect to the basis $\{\frac{\sqrt{\Lambda}}{d(\sigma)} z(\tilde{\sigma}^l)\}$.

We can extract these matrix coefficients using the linear functional on Tube $\mathcal{C}$

$$\psi(\xi\ \zeta|x|\zeta\ \eta) = d(\xi)^2 \delta_{\xi,\eta} \delta_{\zeta,0} x.$$

Then

$$S_{\tilde{\sigma}^l, \tilde{\mu}^m} = \frac{d(\sigma)d(\mu)}{\Lambda} \psi(S_0(z(\tilde{\sigma}^l)), z(\tilde{\mu}^m)). \tag{2.5}$$

In practice, it is often easier to compute S and T from the following formulas via (2.2) (see [16, Lemma 5.3]). Let $\phi_\xi(x) = r_\xi^* \xi(x) r_\xi$ be the standard left inverse of $\xi \in \mathrm{Irr}(\mathcal{C})$. Fix a simple summand (η, j) of σ. Then

$$S_{\tilde{\sigma}^l, \tilde{\mu}^m} = \frac{d(\sigma)}{\Lambda} \sum_{\xi, i} d(\xi) \phi_\xi(\mathcal{E}_\mu^m(\eta)^*_{(\xi,i),(\xi,i)} \mathcal{E}_\sigma^l(\xi)^*_{(\eta,j),(\eta,j)}), \tag{2.6}$$

where the sum is taken over simple summands of μ, and

$$T_{\tilde{\sigma}^l, \tilde{\sigma}^l} = d(\eta)\phi_\eta(\mathcal{E}_\sigma^l(\eta)_{(\eta,j),(\eta,j)}). \tag{2.7}$$

Equation (2.2) implies the following observation, which allows us to determine $\mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$ from the algebra structure of Tube $\mathcal{C}$.

Lemma 2.1. *Let $\xi, \eta \in \mathrm{Irr}(\mathcal{C})$.*

1. *The $\mathcal{A}_\xi$-$\mathcal{A}_\eta$ bimodule $\mathcal{A}_{\xi,\eta}$ is decomposed as*

$$\mathcal{A}_{\xi,\eta} = \sum_{\mu \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))} z(\mu)\mathcal{A}_{\xi,\eta},$$

where each $z(\mu)\mathcal{A}_{\xi,\eta}$ is, if it is not 0, an irreducible $\mathcal{A}_\xi$-$\mathcal{A}_\eta$-bimodule of dimension $\dim(\xi, \mathcal{F}(\mu)) \dim(\eta, \mathcal{F}(\mu))$. If two objects $\mu, \nu \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$ are inequivalent, so are the corresponding bimodules $z(\mu)\mathcal{A}_{\xi,\eta}$ and $z(\nu)\mathcal{A}_{\xi,\eta}$.

2. *Let $\mu \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$ with $\dim(\xi, \mathcal{F}(\mu)) \neq 0$. Then every minimal projection in $z(\mu)\mathcal{A}_\xi$ acts on $\mathcal{A}_{\xi,\eta}$ as a projection of rank $\dim(\eta, \mathcal{F}(\mu))$.*

Apparently, the following property of simple objects of fusion categories has never been observed before except in [17, proof of Theorem 6.4; Lemma 8.1 (7)]. It often occurs in quadratic categories as a key feature that allows us to compute their tube algebras.

Definition 2.2. An simple object $\xi \in \mathcal{C}$ is said *multiplicity free* if the subalgebra $\mathcal{A}_\xi$ of Tube $\mathcal{C}$ is abelian.

Equation (2.2) and Lemma 2.1 imply the following.

Lemma 2.3. *Let $\xi, \eta \in \mathrm{Irr}(\mathcal{C})$.*

1. *The object η is multiplicity free if and only if for any $\mu \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$ the multiplicity of η in $\mathcal{F}(\mu)$ is at most one.*
2. *Assume that η is multiplicity free. Then the multiplicity of every irreducible $\mathcal{A}_\xi$-module contained in $\mathcal{A}_{\xi,\eta}$ as a left $\mathcal{A}_\xi$-module is at most one.*

2.4. Generalized Haagerup categories

We now introduce generalized Haagerup categories embedded in $\mathrm{End}(M)$ (see [19, Definition 2.7] for the definition without unitarity). Let $\mathcal{C} \subset \mathrm{End}(M)$ be a fusion category. Then the set of isomorphism classes of invertible objects forms a finite group, and we denote it by G. Throughout the paper we assume that G is abelian, and we use additive notation for G. We choose a representative $\alpha_g \in \mathcal{C}$ for each $g \in G$. Then by definition, we get

$$[\alpha_g][\alpha_h] = [\alpha_{g+h}].$$

Thus, the map $\alpha\colon G \to \mathrm{Aut}(M)$ is a G-kernel (see [20]), and it gives rise to an obstruction class in $H^3(G, \mathbb{T})$, which is identified with the associator of the pointed category $\mathrm{Inv}(\mathcal{C})$. Furthermore, we assume that there exists a self-dual object $\rho \in \mathcal{C}$ such that

$$\mathrm{Irr}(\mathcal{C}) = \{[\alpha_g]\}_{g \in G} \sqcup \{[\alpha_g \circ \rho]\}_{g \in G}, \tag{2.8}$$

with the fusion rules

$$[\alpha_g][\rho] = [\rho][\alpha_{-g}], \quad g \in G, \tag{2.9}$$

$$[\rho]^2 = [\mathrm{id}] + \sum_{g \in G} [\alpha_g \circ \rho]. \tag{2.10}$$

We consider a $\mathbb{Z}_2$-action on G defined by multiplying by -1. Then the third cohomology obstruction $c^{0,3}(\mathcal{C})$ for α actually belongs to $H^3(G, \mathbb{T})^{\mathbb{Z}_2}$ (see [19, Lemma 2.5]). Assume that it vanishes. Then we can choose α to be a G-action on M. From the fusion rules above, we see that there exist unitaries $u_g \in M$ for each $g \in G$ satisfying

$$\mathrm{Ad}u_g \circ \alpha_g \circ \rho = \rho \circ \alpha_{-g}.$$

Since α is a G-action, we get

$$\begin{aligned} \mathrm{Ad}u_{g+h} \circ \alpha_{g+h} \circ \rho = \rho \circ \alpha_{g+h} &= \rho \circ \alpha_g \circ \alpha_h \\ &= \mathrm{Ad}u_g \circ \alpha_g \circ \rho \circ \alpha_h \\ &= \mathrm{Ad}(u_g \alpha_g(u_h)) \circ \alpha_g \circ \alpha_h \circ \rho \\ &= \mathrm{Ad}(u_g \alpha_g(u_h)) \circ \alpha_{g+h} \circ \rho, \end{aligned}$$

and there exists a 2-cocycle $\omega \in Z^2(G, \mathbb{T})$ satisfying

$$u_g \alpha_g(u_h) = \omega(g, h) u_{g+h}.$$

The cohomology class $[\omega] \in H^2(G, \mathbb{T})$ in fact gives a class in $H^1(\mathbb{Z}_2, H^2(G, \mathbb{T}))$ (see [19, Lemma 2.6]), which we denote by $c^{1,2}(\mathcal{C})$.

Definition 2.4. Let $\mathcal{C} \subset \mathrm{End}(M)$ be a fusion category with abelian G satisfying (2.8)–(2.10). If the cohomology classes $c^{0,3}(\mathcal{C})$ and $c^{1,2}(\mathcal{C})$ vanish, we say that $\mathcal{C}$ is a *generalized Haagerup category.*

Generalized Haagerup categories are completely classified by the solutions of the following polynomial equations up to an appropriate equivalence relation (see [19, Theorem 5.7]). Let

$$d = \frac{|G| + \sqrt{|G|^2 + 4}}{2},$$

where $|G|$ is the order of G. We consider $\epsilon_h(g) \in \{1, -1\}$, $\eta_g \in \mathbb{T}$, and $A_g(h, k) \in \mathbb{C}$ for $g, h, k \in G$ satisfying the following conditions:

$$\epsilon_{h+k}(g) = \epsilon_h(g)\epsilon_k(g + 2h), \quad \epsilon_h(0) = 1, \tag{2.11}$$

$$\eta_{g+2h} = \eta_g, \quad \eta_g^3 = 1, \tag{2.12}$$

$$\sum_{h\in G} A_g(h,0) = -\frac{\overline{\eta_g}}{d}, \tag{2.13}$$

$$\sum_{h\in G} A_g(h-g,k)\overline{A_{g'}(h-g',k)} = \delta_{g,g'} - \frac{\overline{\eta_g}\eta_{g'}}{d}\delta_{k,0}, \tag{2.14}$$

$$A_{g+2h}(p,q) = \epsilon_h(g)\epsilon_h(g+p)\epsilon_h(g+q)\epsilon_h(g+p+q)A_g(p,q), \tag{2.15}$$

$$A_g(k,h) = \overline{A_g(h,k)}, \tag{2.16}$$

$$\begin{aligned} A_g(h,k) &= A_g(-k,h-k)\eta_g\epsilon_{-k}(g+h)\epsilon_{-k}(g+k)\epsilon_{-k}(g+h+k) \\ &= A_g(k-h,-h)\overline{\eta_g}\epsilon_{-h}(g+h)\epsilon_{-h}(g+k)\epsilon_{-h}(g+h+k), \end{aligned} \tag{2.17}$$

$$\begin{aligned} A_g(h,k) &= A_{g+h}(h,k)\eta_g\eta_{g+k}\overline{\eta_{g+h}}\,\overline{\eta_{g+h+k}}\epsilon_h(g)\epsilon_h(g+k) \\ &= A_{g+k}(h,k)\overline{\eta_g\eta_{g+h}}\eta_{g+k}\eta_{g+h+k}\epsilon_k(g)\epsilon_k(g+h), \end{aligned} \tag{2.18}$$

$$\begin{aligned} \sum_{l\in G} & A_g(x+y,l)A_{g-p+x}(-x,l+p)A_{g-q+x+y}(-y,l+q) \\ &= A_g(p+x,q+x+y)A_{g-p}(q+y,p+x+y) \\ &\qquad \times \eta_g\eta_{g+q+x}\eta_{g+p+q+y}\overline{\eta_{g+p}\eta_{g+x+y}\eta_{g+q+x+y}} \\ &\qquad \times \epsilon_p(g-p+x)\epsilon_{p+x}(g-p+q+y) \\ &\qquad \times \epsilon_q(g-q+x+y)\epsilon_{q+y}(g-q+x) \\ &\quad - \frac{\delta_{x,0}\delta_{y,0}}{d}\eta_g\eta_{g+p}\eta_{g+q}. \end{aligned} \tag{2.19}$$

These numerical invariants arise as follows (see [19, Section 3]). Let $\mathcal{C} \subset \mathrm{End}(M)$ be a generalized Haagerup category with G. Since $\mathrm{c}^{1,2}(\mathcal{C}) = 0$, we can choose ρ and α_g satisfying the relation

$$\alpha_g \circ \rho = \rho \circ \alpha_{-g}.$$

Let

$$G_2 = \{g \in G : 2g = 0\}.$$

Then thanks to the above relation, we have

$$\alpha_g((\rho,\rho^2)) = (\rho,\rho^2)$$

for $g \in G_2$. By replacing ρ with an equivalent endomorphism, we may further assume that α_g for $g \in G_2$ acts on (ρ,ρ^2) trivially. Then we can choose bases of intertwiner spaces consisting of isometries $s \in (\mathrm{id},\rho^2)$, $t_g \in (\alpha_g \circ \rho, \rho^2)$ satisfying

$$\alpha_h(s) = s, \quad \alpha_h(t_g) = \epsilon_h(g)t_{g+2h}, \tag{2.20}$$

$$\rho(s) = \frac{1}{d}s + \frac{1}{\sqrt{d}}\sum_{g\in G} t_g t_g, \tag{2.21}$$

$$\alpha_g \circ \rho(t_g) = \eta_g t_g s s^* + \frac{\overline{\eta_g}}{\sqrt{d}} s t_g^* + \sum_{h,k\in G} A_g(h,k) t_{g+h} t_{g+h+k} t_{g+k}^*. \tag{2.22}$$

Recall that the Cuntz algebra $\mathcal{O}_n$ for an integer n larger than 1 is the universal C*-algebra with generators $\{s_i\}_{i=1}^n$ satisfying the relations

$$s_i^* s_j = \delta_{i,j}1, \quad \sum_{i=1}^n s_i s_i^* = 1.$$

Note that the above isometries $\{s\} \cup \{t_g\}_{g\in G}$ satisfy the $\mathcal{O}_{|G|+1}$-relation.

Conversely, assume that we are given a solution $(\epsilon_h(g), \eta_g, A_g(h,g))$ of (2.11)–(2.19) (without knowing that it comes from a generalized Haagerup category). We consider the Cuntz algebra $\mathcal{O}_{|G|+1}$ with the canonical generators $\{s\} \cup \{t_g\}_{g\in G}$. Then we can introduce a G-action α on $\mathcal{O}_{|G|+1}$ and an endomorphism ρ of $\mathcal{O}_{|G|+1}$ by (2.20)–(2.22), which satisfy $\alpha_g \circ \rho = \rho \circ \alpha_{-g}$ and

$$\rho^2(x) = sxs^* + \sum_{g\in G} t_g \alpha_g \circ \rho(x) t_g^*.$$

Taking the weak closure of $\mathcal{O}_{|G|+1}$ in an appropriate representation, we get a type III factor M and a generalized Haagerup category $\mathcal{C} \subset \mathrm{End}(M)$ generated by (extensions of) α_g and ρ (see [19, Theorems 4.1 and Theorem 10.2]).

From now on, whenever we discuss a generalized Haagerup category $\mathcal{C}$ with a finite abelian group G, we choose and fix α_g, ρ, and $\{s\} \cup \{t_g\}_{g\in G} \subset M$ satisfying (2.20)–(2.22).

When a generalized Haagerup category comes from a generalized Haagerup subfactor (called a 3^G *subfactor* in [19]), the object $\mathrm{id} \oplus \rho$ has a Q-system. It is shown in [17, Section 7] that $\mathrm{id} \oplus \rho$ has a Q-system if and only if the following holds:

$$A_0(h,0) = \delta_{h,0} - \frac{1}{d-1}. \tag{2.23}$$

In concrete examples, it is often the case that a solution of (2.11)–(2.19) and (2.23) automatically satisfies

$$A_g(h,0) = \delta_{h,0} - \frac{1}{d-1}, \tag{2.24}$$

for any $g, h \in G$. In other words, once $\mathrm{id} \oplus \rho$ has a Q-system, so does any other $\mathrm{id} \oplus \alpha_g \circ \rho$ in known examples.

Under 2.24), we get $\eta_g = 1$ from (2.17), and (2.15) implies that the map

$$G \ni g \mapsto \epsilon_h(g) \in \{1, -1\}$$

is a character for any $h \in G_2$. In particular, it makes sense to say that $(\epsilon_h(g), A_g(h,k))$ is a solution of (2.11)–(2.19) and (2.24), omitting η_g.

Solutions to (2.11)–(2.19) and (2.24) have been computed for a number of small groups. We list the known solutions, up to group automorphism and gauge equivalence. For $G = \mathbb{Z}_2$, there is a unique solution, corresponding to the A_7 subfactor. For $\mathbb{Z}_3$, there is a unique solution, corresponding to the Haagerup subfactor. In [17] a unique solution was found for $\mathbb{Z}_5$. In [11] Evans and Gannon found a unique solution for $\mathbb{Z}_7$, exactly two solutions for $\mathbb{Z}_9$, and no solutions for $\mathbb{Z}_3 \times \mathbb{Z}_3$; they also found numerical evidence for solutions for several larger odd cyclic groups. In [19] unique solutions were found for $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. In [14] a solution was found for $\mathbb{Z}_4 \times \mathbb{Z}_2$. In working through the examples in this paper we found that there exactly two solutions each for $\mathbb{Z}_6$ and $\mathbb{Z}_{10}$, as well as at least one solution for $\mathbb{Z}_8$; these solutions are contained in the accompanying Mathematica notebook `solutions.nb`.

2.5. (De)-equivariantization and orbifolds

Let G be a finite group acting on a fusion category $\mathcal{C}$ by tensor autoequivalences. Then one can define the notion of a G-equivariant object in $\mathcal{C}$. The category of G-equivariant objects is a fusion category, called an *equivariantization* of $\mathcal{C}$ by G, and denoted $\mathcal{C}^G$. There is an inverse construction called *de-equivariantization,* by which $\mathcal{C}$ may be recovered from $\mathcal{C}^G$. We refer the reader to [8] for details. In addition to the equivariantization (which can be thought of as taking the "fixed points" of the G-action), one can also construct the crossed product $\mathcal{C} \rtimes G$. The crossed product is a quasi-trivial G-graded extenension of $\mathcal{C}$ which is Morita equivalent to $\mathcal{C}^G$, see [31].

For a fusion category $\mathcal{C}$ embedded in $\mathrm{End}(M)$, both equivariantization and de-equivariantization can sometimes be realized by an orbifold construction, in which the von Neumann M is enlarged to a crossed product by a group action, and the endomorphisms comprising the objects of $\mathcal{C}$ are extended to the larger algebra.

For generalized Haagerup categories, two specific types of orbifolds were constructed in [19, Section 8]. We briefly recall these constructions, and refer the reader there for more details.

2.5.1. De-equivariantization. Let $\mathcal{C}$ be a generalized Haagerup category embedded in $\mathrm{End}(M)$. Suppose there is a $z \in G_2$ such that $\epsilon_z(\cdot)$ is a character satisfying $\epsilon_z(z) = 1$. Let $P = M \rtimes_{\alpha_z} \mathbb{Z}_2$ be the crossed product of M by α_z. Then P is the von Neumann algebra generated by M and a unitary λ satisfying $\lambda^2 = 1$ and

$$\lambda x \lambda^{-1} = \alpha_z(x) \quad \text{for all } x \in M.$$

Each α_g can be extended to an automorphism $\tilde{\alpha}_g$ of P by setting

$$\tilde{\alpha}_g(\lambda) = \epsilon_z(g)\lambda.$$

Similarly, ρ can be extended to an endomorphism $\tilde{\rho}$ of P by setting

$$\tilde{\rho}(\lambda) = \lambda.$$

Then $g \mapsto \tilde{\alpha}_g$ defines an action of G on P, and we have

$$\tilde{\alpha}_g \circ \tilde{\rho} = \tilde{\rho} \circ \tilde{\alpha}_{-g} \quad \text{for all } g \in G.$$

Moreover,

$$[\tilde{\alpha}_g] = [\tilde{\alpha}_h] \iff g - h \in \{0, z\},$$

and if $G_0 \subset G$ is a set of representative elements for the $\{0, z\}$-cosets of G, we have

$$[\tilde{\rho}^2] = [\mathrm{id}] \bigoplus_{g \in G_0} 2[\tilde{\alpha}_g \tilde{\rho}].$$

The fusion category in $\mathrm{End}(P)$ tensor generated by $\tilde{\rho}$ is a $\mathbb{Z}_2$-de-equivariantization of $\mathcal{C}$.

2.5.2. Equivariantization. Let $\mathcal{C}$ again be a generalized Haagerup category, and let θ be an automorphism of G which preserves the structure of $\mathcal{C}$:

$$\epsilon_{\theta(h)}(\theta(g)) = \epsilon_h(g),$$
$$\eta_{\theta(g)} = \eta(g),$$
$$A_{\theta(g)}(\theta(h), \theta(k)) = A_g(h, k) \quad \text{for all } g, h, k \in G.$$

Define an automorphism γ on M by

$$\gamma(s) = s, \quad \gamma(t_g) = t_{\theta(g)}, \ g \in G.$$

Then

$$\gamma \circ \rho = \rho \circ \gamma$$

and

$$\gamma \circ \alpha_g = \alpha_{\theta(g)} \circ \gamma.$$

The automorphism γ thus induces an action of $\mathbb{Z}_m$ on $\mathcal{C}$, where m is the order of θ. Let $P = M \rtimes_\gamma \mathbb{Z}_m$ be the crossed product of M by γ. Then P is the von Neumann algebra generated by M and a unitary λ satisfying

$$\lambda^m = 1 \quad \text{and} \quad \lambda x \lambda^{-1} = \gamma(x) \text{ for all } x \in M.$$

We can extend ρ to an endomorphism $\tilde{\rho}$ of P by setting

$$\tilde{\rho}(\lambda) = \lambda.$$

Then the category $\mathcal{C}^\gamma$ in $\mathrm{End}(P)$ tensor generated by $\tilde{\rho}$ is a $\mathbb{Z}_m$-equivariantization of $\mathcal{C}$. The corresponding crossed product category is the category in $\mathrm{End}(M)$ tensor generated by $\mathcal{C}$ and γ.

2.6. Tube algebras for generalized Haagerup categories

In this section we describe a convenient basis for the tube algebra of a generalized Haagerup category. Let $\mathcal{C}$ be a generalized Haagerup category in $\mathrm{End}(M)$, with group of invertible objects $\{\alpha_g\}_{g\in G}$ and $s \in (\mathrm{id}, \rho^2)$, $t_g \in (\alpha_g, \rho^2)$, as in Section 2.4. Inside the tube algebra, let

$$\mathcal{A}_G = \bigoplus_{g,h\in G} \mathcal{A}_{\alpha_g,\alpha_h}, \qquad \mathcal{A}_{G,G\rho} = \bigoplus_{g,h\in G} \mathcal{A}_{\alpha_g,\alpha_h\rho},$$

$$\mathcal{A}_{G\rho,G} = \bigoplus_{g,h\in G} \mathcal{A}_{\alpha_g\rho,\alpha_h}, \qquad \mathcal{A}_{G\rho} = \bigoplus_{g,h\in G} \mathcal{A}_{\alpha_g\rho,\alpha_h\rho}.$$

Then

$$\mathrm{Tube}(\mathcal{C}) = \mathcal{A}_G \oplus \mathcal{A}_{G,G\rho} \oplus \mathcal{A}_{G\rho,G} \oplus \mathcal{A}_{G\rho}.$$

To simplify notation, when expressing elements of the tube algebra, we will suppress "α" and refer to the simple objects as g (instead of α_g) and ${}_g\rho$ (instead of $\alpha_g\rho$).

A basis for $\mathcal{A}_G$ is given by

$$\{(g\ k|1|k\ g)\}_{g,k\in G} \cup \{(g\ {}_k\rho|1|{}_k\rho\ \ -g\}_{g,k\in G};$$

a basis for $\mathcal{A}_{G,G\rho}$ is given by

$$\{(g\ {}_k\rho|t_{2k+g-h}|{}_k\rho\ {}_h\rho)\}_{g,h,k\in G};$$

a basis for $\mathcal{A}_{G\rho,G}$ is given by

$$\{({}_h\rho\ {}_k\rho|t^*_{h-g}|{}_k\rho\ g)\}_{g,h,k\in G};$$

and a basis for $\mathcal{A}_{G\rho}$ is given by

$$\{({}_{h_1}\rho\ {}_k\rho|t_{k-h_2+g}t^*_{h_1-k+g}|{}_k\rho\ {}_{h_2}\rho)\}_{h_1,h_2,k,g\in G}$$
$$\cup \{({}_h\rho\ {}_k\rho|ss^*|{}_k\rho\ {}_{2k-h}\rho)\}_{h,k\in G} \cup \{({}_h\rho\ k|1|k\ {}_{h-2k}\rho)\}_{h,k\in G}.$$

Multiplying two elements $(\xi\ \zeta|x|\zeta\ \eta)$ and $(\xi'\ \zeta'|y|\zeta'\ \eta')$ in a tube algebra requires summing over simple objects $\nu \in \mathrm{Irr}(\mathcal{C})$ and an orthonormal basis for each space $(\nu, \zeta\zeta')$. Note that for a generalized Haagerup category, any product of two objects in $\mathrm{Irr}(\mathcal{C})$ is of the form α_g, $\alpha_g\rho$, or $\alpha_g\rho^2$ for some $g \in G$. Moreover, we have

$$(\alpha_g, \alpha_h) = (\alpha_g\rho, \alpha_h\rho) = \delta_{g,h}\mathbb{C},$$

$$(\alpha_g, \alpha_h \rho^2) = \delta_{g,h} \mathbb{C} s, \quad (\alpha_g \rho, \alpha_h \rho^2) = \mathbb{C} t_{g+h}.$$

Therefore, all possible non-zero hom spaces $(\nu, \zeta\zeta')$ are 1-dimensional and spanned by either 1 or a Cuntz algebra generator. We can then take 1 or the appropriate Cuntz algebra generator in each case as a canonical isometry for computing the tube algebra multiplication.

3. Structure of the tube algebra for a generalized Haagerup category and restriction on the size of G_2

We fix a generalized Haagerup category $\mathcal{C} \subset \mathrm{End}(M)$ with a finite abelian group G. We choose α, ρ, s, and t_g satisfying (2.11)–(2.22) for $\mathcal{C}$. Recall that

$$G_2 = \{g \in G;\ 2g = 0\}.$$

In this section we will describe the structure of the tube algebra of $\mathcal{C}$, and show that this leads to a restriction on the size of G_2.

We denote by $\widehat{G}$ the dual group of G. Let

$$(\widehat{G})_2 = \{\chi \in \widehat{G};\ 2\chi = 0\}.$$

We choose $G_* \subset G$ and $\widehat{G}_* \subset \widehat{G}$ satisfying

$$G = G_2 \sqcup G_* \sqcup (-G_*),$$
$$\widehat{G} = (\widehat{G})_2 \sqcup \widehat{G}_* \sqcup (-\widehat{G}_*).$$

Let Λ be the global dimension of $\mathcal{C}$:

$$\Lambda = |G| + |G|d^2 = |G|(2 + |G|d) = |G| \frac{|G|^2 + 4 + |G|\sqrt{|G|^2 + 4}}{2}.$$

Letting $a = 1/|G|$ and $b = 1/\sqrt{|G|^2 + 4}$, we get the simple expression

$$\frac{1}{\Lambda} = \frac{a - b}{2}.$$

3.1. The non-invertible simple objects in $\mathcal{C}$ are multiplicity free

Lemma 3.1. *The object $\alpha_g \circ \rho$ is multiplicity free (i.e., the algebra $\mathcal{A}_{g\rho}$ is abelian) for any $g \in G$.*

Proof. It suffices to show the statement for $g = 0$ because the pair $(\mathcal{C}, \alpha_g \circ \rho)$ satisfies the same conditions as the pair $(\mathcal{C}, \rho)$. The map S_0^2 is an anti-isomorphism from each $\mathcal{A}_\xi$ in the tube algebra to $\mathcal{A}_{\bar{\xi}}$ (diagrammatically this corresponds to rotation by π). Since ρ is self-dual, the map S_0^2 restricted $\mathcal{A}_\rho$ is an anti-automorphism. On the other hand, we claim that this map is the identity, which would show that $\mathcal{A}_\rho$ is abelian.

Indeed, $\alpha_k(s) = s$ implies that S_0^2 acts on $(\rho\ k|1|k\ \rho)$ and $(\rho\ {}_k\rho|ss^*|_k\rho\ \rho)$ as the identity. We also have from (2.4)

$$\begin{aligned}
&S_0^2((\rho\ {}_g\rho|t_{h+g}t_{h-g}^*|_g\rho\ \rho))\\
&\quad = dS_0(({}_g\rho\rho|s^*\alpha_g \circ \rho(t_{h+g}t_{h-g}^*\rho(s))|\rho\ {}_g\rho))\\
&\quad = \sqrt{d}\,S_0(({}_g\rho\ \rho|s^*\alpha_g \circ \rho(t_{h+g}t_{h-g})|\rho\ {}_g\rho))\\
&\quad = d^{3/2}((\rho_g\rho|s^*\rho(s^*\alpha_g \circ \rho(t_{h+g}t_{h-g}s))|_g\rho\ \rho)).
\end{aligned}$$

Then using (2.21–2.22) gives

$$\begin{aligned}
&d^{3/2}s^*\rho(s^*\alpha_g \circ \rho(t_{h+g}t_{h-g}s))\\
&\quad = d^{3/2}\epsilon_{-h}(g+h)s^*\rho(s^*\alpha_{g-h} \circ \rho(t_{g-h})\alpha_g \circ \rho(t_{h-g}s))\\
&\quad = d\epsilon_{-h}(g+h)s^*\rho(t_{g-h}^*)\rho \circ \alpha_g \circ \rho(t_{h-g}s)\\
&\quad = d\epsilon_{-h}(g+h)\epsilon_{g-h}(h-g)s^*\alpha_{h-g} \circ \rho(t_{h-g}^*)\rho^2 \circ \alpha_{-g}(t_{h-g}s)\\
&\quad = d\epsilon_g(h-g)s^*t_{h-g}^*\rho^2 \circ \alpha_{-g}(t_{h-g}s)\\
&\quad = d\epsilon_g(h-g)s^*\alpha_h \circ \rho(t_{h-g}s)t_{h-g}^*\\
&\quad = ds^*\alpha_{h+g} \circ \rho(t_{h+g}s)t_{h-g}^*\\
&\quad = d^{1/2}t_{h+g}^*\rho(s)t_{h-g}^*\\
&\quad = t_{h+g}t_{h-g}^*.
\end{aligned}$$

Since elements of the above forms span $\mathcal{A}_\rho$, the claim is shown. ■

3.2. The structure of $\mathcal{A}_G$ and its action on $\mathcal{A}_{G,G\rho}$

In this section we describe the structure of the "group part" of the tube algebra $\mathcal{A}_G$, following [17] (where the computation was done for the odd case, i.e., $G_2 = \{0\}$). Then we describe the action of $\mathcal{A}_G$ on $\mathcal{A}_{G,G\rho}$, which will allow us to determine the simple objects in the Drinfeld center whose underlying objects contain invertible summands.

In what follows, we assume (2.24). It is straightforward to show the following, using (2.1).

Lemma 3.2. *For $k \in G_2$, the object α_k has a unique half-braiding $\mathcal{E}_k$, and it is given by $\mathcal{E}_k(g) = \epsilon_k(g)$ and $\mathcal{E}_k({}_g\rho) = \epsilon_k(g)$.*

The above lemma implies that α_k for $k \in G_2$ has a unique extension $\widetilde{\alpha_k}$ to the Drinfeld center $\mathcal{Z}(\mathcal{C})$, which we often again denote by k for simplicity. Thanks to (2.2) and (2.3), the corresponding minimal central projection in Tube $\mathcal{C}$ is

$$z(k) = \frac{1}{\Lambda} \sum_{g \in G} (\epsilon_k(g)(k\ g|1|g\ k) + d\epsilon_k(g)(k\ {}_g\rho|1|{}_g\rho\ k)) \in \mathcal{A}_k.$$

3.2.1. Structure of $\mathcal{A}_G$. Note that $\mathcal{A}_{g,h} = 0$ unless $g = -h$. Therefore, we can write

$$\mathcal{A}_G = \bigoplus_{k \in G_2} \mathcal{A}_k \oplus \bigoplus_{g \in G_*} \mathcal{B}_g,$$

where

$$\mathcal{B}_g := \mathcal{A}_g \oplus \mathcal{A}_{g,-g} \oplus \mathcal{A}_{-g,g} \oplus \mathcal{A}_{-g}.$$

We now determine the algebra structure of the $\mathcal{A}_k$ and $\mathcal{B}_g$. For $(g,\tau) \in G \times \widehat{G}$, let

$$p(g,\tau) = \frac{1}{|G|} \sum_{h \in G} \langle h, \tau\rangle (g\ h|1|h\ g).$$

Then the $p(g,\tau)$ are mutually orthogonal projections which sum to the identity in $\mathcal{A}_G$, and we have

$$p(g,\tau)(g\ \rho|1|\rho\ -g) = (g\ \rho|1|\rho\ -g)p(-g,-\tau).$$

For $(g,\tau) \in (G \times \widehat{G}) \backslash (G_2 \times (\widehat{G})_2)$, set

$$\begin{aligned}
E(g,\tau)_{11} &= p(g,\tau), E(g,\tau)_{22} = p(-g,-\tau),\\
E(g,\tau)_{12} &= p(g,\tau)(g\ \rho|1|\rho\ -g) = (g\ \rho|1|\rho\ -g)p(-g,-\tau),\\
E(g,\tau)_{21} &= p(-g,-\tau)(-g\ \rho|1|\rho\ g) = (-g\ \rho|1|\rho\ g)p(g,\tau).
\end{aligned}$$

Then since $(g,\tau) \neq (-g,-\tau)$, it is easy to see that the $E(g,\tau)_{ij}$ are matrix units for a 2×2 subalgebra of $\mathcal{A}_G$. For $(k,\tau) \in G_2 \times \widehat{G}_*$, this gives a subalgebra of $\mathcal{A}_k$, which we denote by $\mathcal{A}_k^\tau$. For $(g,\tau) \in G_* \times G$, this gives a subalgebra of $\mathcal{B}_g$ which we denote by $\mathcal{B}_g^\tau$.

What remains is to decompose the projections $p(k,\chi)$ for $(k,\chi) \in G_2 \times (\widehat{G})_2$.

Recall that for $k \in G_2$, the map

$$G \ni g \mapsto \epsilon_k(g) \in \{1,-1\}$$

is a character, and we can regard ϵ_k as an element in $(\widehat{G})_2$. Thus, we have the following expression

$$z(k) = \frac{|G|}{\Lambda} p(k, \epsilon_k)(1_k + d(k\ \rho|1|\rho\ k)) = \frac{|G|}{\Lambda}(1_k + d(k\ \rho|1|\rho\ k))p(k, \epsilon_k).$$

For $\chi \in (\widehat{G})_2$, the projection $p(k, \chi)$ is central in $\mathcal{A}_0$, and $p(k, \chi)\mathcal{A}_0$ is a 2-dimensional algebra spanned by $p(k, \chi)$ and $p(k, \chi)(k\ \rho|1|\rho\ k)$. We have

$$\begin{aligned}
&(p(k, \chi)(k\ \rho|1|\rho\ k))^2 \\
&\quad = p(k, \chi) + \sum_{g \in G} p(0, \chi)(k\ {}_g\rho|1|_g\rho\ k) \\
&\quad = p(k, \chi) + \sum_{g \in G} \langle g, \chi \rangle p(k, \chi)(k\ \rho|1|\rho\ k) \\
&\quad = \begin{cases} p(k, \epsilon_k) + |G| p(k, \epsilon_k)(k\ \rho|1|\rho\ k), & \chi = \epsilon_k, \\ p(k, \chi), & \chi \neq \epsilon_k. \end{cases}
\end{aligned}$$

Set

$$E(k, k) = \frac{|G| d}{\Lambda} p(k, \epsilon_k)(d 1_k - (k\ \rho|1|\rho\ k))$$

and

$$E(k, \chi)_\pm = \frac{1}{2} p(k, \chi)(1_k \pm (k\ \rho|1|\rho\ k))$$

for $\chi \in (\widehat{G})_2 \setminus \{\epsilon_k\}$.

Then $E(k, k)$ is a minimal central projection in $\mathcal{A}_k$ orthogonal to $z(k)$, and the pair $E(k, \chi)_\pm$ are mutually orthogonal minimal central projections in $\mathcal{A}_k$.

To summarize, we have the following decompositions.

Lemma 3.3. 1. *For $g \in G_*$, we have*

$$\mathcal{B}_g = \bigoplus_{\tau \in \widehat{G}} \mathcal{B}_g^\tau.$$

2. *For $k \in G_2$, we have*

$$\mathcal{A}_k = \mathbb{C} z(k) \oplus \mathbb{C} E(k, k) \oplus \bigoplus_{\chi \in (\widehat{G})_2 \setminus \{k\}} (\mathbb{C} E(k, \chi)_+ \oplus \mathbb{C} E(k, \chi)_-) \oplus \bigoplus_{\tau \in G_*} \mathcal{A}_k^\tau.$$

3.2.2. Action of $\mathcal{A}_G$ on $\mathcal{A}_{G, {}_G\rho}$. In this section we determine the action of each $\mathcal{A}_g$ on each $\mathcal{A}_{g, {}_h\rho}$, using Lemma 2.3. Since, each $\mathcal{A}_{{}_h\rho}$ is multiplicity free by Lemma 3.1, each irreducible $\mathcal{A}_g$-module can appear in $\mathcal{A}_{g, {}_h\rho}$ with multiplicity at most one. Thus, it suffices to known whether each simple component of $\mathcal{A}_g$ acts on $\mathcal{A}_{h,{}_g\rho}$ trivially or not.

For $g \in G \setminus G_2$, we can easily determine the $\mathcal{A}_g$-action on $\mathcal{A}_{g,{}_h\rho}$, as we have $\dim \mathcal{A}_g = \dim \mathcal{A}_{g,{}_h\rho} = |G|$. Namely, every simple component $\mathbb{C}p(g,\tau)$ acts on $\mathcal{A}_{g,{}_h\rho}$ with multiplicity one.

Since

$$\{(k\ h|1|h\ k)(k\ \rho|t_{k-g}|\rho\ {}_g\rho)\}_{h\in G},$$

is a basis for $\mathcal{A}_{k,{}_g\rho}$, so is

$$\{p(k,\tau)(k\ \rho|t_{k-g}|\rho\ {}_g\rho)\}_{\tau\in\widehat{G}}.$$

Since $z(k)$ is central in Tube $\mathcal{C}$, it acts on $\mathcal{A}_{k,{}_g\rho}$ trivially. Then the algebras $\mathbb{C}E(k,k)$ and $\mathcal{A}_k^\tau$ for $\tau \in \widehat{G}_*$ must act on $\mathcal{A}_{k,{}_g\rho}$ non-trivially, since each $p(k,\tau)$ acts non-trivially on the basis above.

It remains to show how $\mathbb{C}E(k,\chi)_\pm$ acts on $p(k,\chi)(0\ \rho|t_{k-g}|\rho\ {}_g\rho)$ for $\chi \in (\widehat{G})_2 \setminus \{\epsilon_k\}$.

Lemma 3.4. *For* $\chi \in (\widehat{G})_2 \setminus \{\epsilon_k\}$*, we have*

$$(k\ \rho|1|\rho\ k)p(k,\chi)(k\ \rho|t_{k-g}|\rho\ {}_g\rho) = \langle g+k, \chi+\epsilon_k\rangle p(k,\chi)(k\ \rho|t_{k-g}|\rho\ {}_g\rho).$$

Proof. Since $(k\ \rho|1|\rho\ k)$ commutes with $p(k,\chi)$ as $\chi \in (\widehat{G})_2$, we have

$$\begin{aligned}
&(k\ \rho|1|\rho\ k)p(k,\chi)(k\ \rho|t_{k-g}|\rho\ {}_g\rho)\\
&\quad= p(k,\chi)(k\ \rho|1|\rho\ k)(k\ \rho|t_{k-g}|\rho\ {}_g\rho)\\
&\quad= p(k,\chi)\sum_{h\in G}\epsilon_k(h)(k\ {}_h\rho|t_h^*\rho(t_{k-g})t_h|{}_h\rho\ {}_g\rho)\\
&\quad= p(k,\chi)\sum_{h\in G}\epsilon_k(h)A_{g-k}(h+k-g,h+k-g)\\
&\qquad\times\epsilon_{k-g}(g-k)(k\ {}_h\rho|t_{2h+k-g}|{}_h\rho\ {}_g\rho)\\
&\quad= p(k,\chi)\sum_{h\in G}\epsilon_k(h)A_{g-k}(h+k-g,h+k-g)\epsilon_{k-g}(g-k)\\
&\qquad\times\epsilon_h(k-g)(k\ h|1|h\ k)(k\ \rho|t_{k-g}|\rho\ {}_g\rho)\\
&\quad= \Big(\sum_{h\in G}A_{g-k}(h+k-g,h+k-g)\epsilon_{h+k-g}(g-k)\langle h,\chi+\epsilon_k\rangle\Big)\\
&\qquad\times p(k,\chi)(k\ \rho|t_{k-g}|\rho\ {}_g\rho)\\
&\quad= \langle g+k,\chi+\epsilon_k\rangle\Big(\sum_{h\in G}A_{g-k}(h,h)\epsilon_h(g-k)\langle h,\chi+\epsilon_k\rangle\Big)\\
&\qquad\times p(k,\chi)(k\ \rho|t_{k-g}|\rho\ {}_g\rho).
\end{aligned}$$

Equations (2.17) and (2.24) then imply

$$\sum_{h\in G} A_{g-k}(h,h)\epsilon_h(g-k)\langle h,\chi+\epsilon_k\rangle = \sum_{h\in G}\Big(\delta_{h,0}-\frac{1}{d-1}\Big)\langle h,\chi+\epsilon_k\rangle = 1. \quad \blacksquare$$

The above lemma implies that

$$E(k,\chi)_{\langle g+k,\chi+\epsilon_k\rangle} p(k,\chi)(k\ \rho|t_{k-g}|\rho\ {}_g\rho) = p(k,\chi)(k\ \rho|t_{k-g}|\rho\ {}_g\rho),$$

and

$$E(k,\chi)_{-\langle g+k,\chi+\epsilon_k\rangle}\mathcal{A}_{k,{}_g\rho} = \{0\}.$$

In conclusion, we have that every simple component of every $\mathcal{A}_g$ acts nontrivially on every $\mathcal{A}_{g,{}_h\rho}$, except that for $\mathcal{A}_k$ for $k \in G_2$:

1. the components $\mathbb{C}z(k)$ act trivially on each $\mathcal{A}_{k,{}_h\rho}$;
2. the components $\mathbb{C}E(k,\chi)_{-\langle h+k,\chi+\epsilon_k\rangle}$, with $\chi \in \widehat{G}_2 \setminus \{\epsilon_k\}$ act trivially on $\mathcal{A}_{k,{}_h\rho}$.

3.3. The simple objects in $\mathcal{Z}(\mathcal{C})$ and restriction on G_2

In this section we use the structure of the tube algebra to describe the simple objects of $\mathcal{Z}(\mathcal{C})$ and prove the restriction on G_2.

The minimal central projections in $\mathcal{A}_k$, $k \in G_2$ and in $\mathcal{B}_g$, $g \in G_*$ correspond to the simple objects $\kappa \in \mathcal{Z}(\mathcal{C})$ such that $\mathcal{F}(\kappa) \in \mathcal{C}$ contains an invertible summand. Since the non-invertible simple objects $\alpha_h \circ \rho$ of $\mathcal{C}$ are all multiplicity free, for each simple $\kappa \in \mathcal{Z}(\mathcal{C})$, the object $\mathcal{F}(\kappa)$ contains each of $\alpha_h \circ \rho$ with multiplicity at most one. If $\mathcal{F}(\kappa)$ contains α_g, whether the multiplicity of $\alpha_h\rho$ in $\mathcal{F}(\kappa)$ is 0 or 1 is determined by whether the action of $z(\kappa)\mathcal{A}_g$ on $\mathcal{A}_{g,{}_h\rho}$ is trivial or not, which we have just determined above.

Summing up, we get the following list of simple objects in $\mathcal{Z}(\mathcal{C})$ whose underlying objects in $\mathcal{C}$ contain invertible summands. In each case, the parts of the half-braidings $\mathcal{E}_{(\cdot)}(\cdot)_{g,g}$ (or $\mathcal{E}_{(\cdot)}(\cdot)_{(g,i),(g,i)}$ when there is multiplicity) can be read off using (2.2) and the formulas obtained above for the minimal projections in $\mathcal{A}_g$ and $\mathcal{B}_g$, as in [17].

Lemma 3.5. *Let the notation be as above, and let $k \in G_2$.*

1. *Let*

$$\pi = \mathrm{id} \oplus \bigoplus_{g\in G} \alpha_g \circ \rho.$$

The object π has a unique half-braiding, which gives $e(\tilde{\pi})_{0,0} = E(0,0)$, and

$$\mathcal{E}_\pi(h)_{0,0} = 1, \quad \mathcal{E}_\pi({}_h\rho)_{0,0} = -\frac{1}{d^2}.$$

More generally, $\alpha_k\pi$ has a unique half-braiding $\mathcal{E}_{k\pi}(\xi) = \mathcal{E}_k(\xi)\alpha_k(\mathcal{E}_\pi(\xi))$.

2. *For $\chi \in (\hat{G})_2 \setminus \{0\}$, let*

$$\varphi_{\chi,\pm} = \mathrm{id} \oplus \bigoplus_{\langle g,\chi\rangle=\pm 1} \alpha_g \circ \rho.$$

The object $\varphi_{\chi,+}$ has a unique half-braiding, which gives

$$e(\widetilde{\varphi_{\chi,+}})_{0,0} = E(0,\chi)_+,$$

and $\varphi_{\chi,-}$ has a unique half-braiding, which gives

$$e(\widetilde{\varphi_{\chi,-}})_{0,0} = E(0,\chi)_-.$$

The corresponding half-braidings are

$$\mathcal{E}_{\varphi_{\chi,\pm}}(h)_{0,0} = \langle h,\chi\rangle, \quad \mathcal{E}_{\varphi_{\chi,\pm}}({}_h\rho)_{0,0} = \frac{\pm\langle h,\chi\rangle}{d}.$$

More generally, $\alpha_k\varphi_{\chi,\pm}$ has a unique half-braiding

$$\mathcal{E}_{k\varphi_{\xi,\pm}}(\xi) = \mathcal{E}_k(\xi)\alpha_k(\mathcal{E}_{\varphi_{\chi,\pm}}(\xi)).$$

3. *Let*

$$\sigma_k = \alpha_k \oplus \alpha_k \oplus \bigoplus_{g\in G} \alpha_g \circ \rho.$$

The object σ_k has exactly $(|G|-|G_2|)/2$ half-braidings parametrized by $\tau \in \hat{G}_*$, which give $e(\widetilde{\sigma_k}^\tau)_{(k,i),(k,j)} = E(k,\tau)_{ij}$, and

$$\mathcal{E}^\tau_{\sigma_k}(h)_{(k,1),(k,1)} = \langle h,\tau\rangle, \quad \mathcal{E}^\tau_{\sigma_k}({}_h\rho)_{(k,i),(k,i)} = 0.$$

(4) *For $g \in G_*$, let*

$$\sigma_g = \alpha_g \oplus \alpha_{-g} \oplus \bigoplus_{h\in G} \alpha_h \circ \rho.$$

The object σ_g has exactly $|G|$ half-braidings parametrized by $\tau \in \hat{G}$, which give $e(\widetilde{\sigma_g}^\tau)_{g,g} = p(g,\tau)$, and

$$\mathcal{E}^\tau_{\sigma_g}(h)_{g,g} = \langle h,\tau\rangle, \quad \mathcal{E}^\tau_{\sigma_g}({}_h\rho)_{g,g} = 0.$$

Since the algebra $\mathcal{A}_\rho$ has the basis

$$\{(\rho\ k|1|k\ \rho)\}_{k\in G_2} \cup \{(\rho\ {}_k\rho|ss^*|{}_k\rho\ \rho)\}_{k\in G_2} \cup \{(\rho\ {}_g\rho|t_{h+g}t^*_{h-g}|{}_g\rho\ \rho)\}_{g,h\in G},$$

we see that $\dim \mathcal{A}_\rho = |G|^2 + 2|G_2|$. In a similar way, we can see that

$$\dim \mathcal{A}_{{}_g\rho,\rho} = \begin{cases} |G|^2 + 2|G_2|, & g \in 2G, \\ |G|^2, & g \in G \setminus 2G. \end{cases}$$

Let Z_G be the sum of the central projections in the tube algebra that have a non-zero subprojection in some $\mathcal{A}_g$. In other words,

$$Z_G = \sum_{k\in G_2}\Big(z(k) + z(\widetilde{k\pi}) + \sum_{\chi\in((\widehat{G})_2)\setminus\{0\}}\big(z(\widetilde{k\varphi_{\chi,+}}) + z(\widetilde{k\varphi_{\chi,-}})\big) + \sum_{\tau\in\widehat{G}_*} z(\widetilde{\sigma_k}^{\tau})\Big) + \sum_{g\in G_*}\sum_{\tau\in\widehat{G}} z(\widetilde{\sigma_g}^{\tau}).$$

Theorem 3.6. *Let $\mathcal{C}\subset \mathrm{End}(M)$ be a generalized Haagerup category with a finite abelian group G satisfying (2.24). Then $|G_2|\le 4$.*

Proof. If G is an odd group, we have nothing to prove, and we assume that G is an even group. Then

$$\begin{aligned}\dim(1-Z_G)\mathcal{A}_\rho &= \dim\mathcal{A}_\rho - |G_2|(1+|(\widehat{G})_2|-1+|\widehat{G}_*|) - |G_*||\widehat{G}|\\ &= |G|^2+2|G_2| - \Big(|G_2|^2 + (|G|+|G_2|)\frac{|G|-|G_2|}{2}\Big)\\ &= |G|^2+2|G_2| - \frac{|G|^2+|G_2|^2}{2}\\ &= \frac{|G|^2}{2} - \frac{|G_2|^2}{2} + 2|G_2|.\end{aligned}$$

On the other hand, if $g\in G\setminus 2G$, we get

$$\begin{aligned}&\dim(1-Z_G)\mathcal{A}_{g\rho,\rho}\\ &= \dim\mathcal{A}_{g\rho,\rho} - |G_2|(1+\#\{\chi\in(\widehat{G})_2\setminus\{0\};\ \langle g,\chi\rangle = \langle 0,\chi\rangle\} + |\widehat{G}_*|) - |G_*||\widehat{G}|\\ &= |G|^2 - |G_2|(1+\#\{\chi\in(\widehat{G})_2\setminus\{0\};\ \langle g,\chi\rangle = 1\} - \frac{|G|^2-|G_2|^2}{2}\\ &= \frac{|G|^2+|G_2|^2}{2} - |G_2|(1+\#\{\chi\in(\widehat{G})_2\setminus\{0\};\ \langle g,\chi\rangle = 1\}.\end{aligned}$$

Note that the group $(\widehat{G})_2$ is identified with the dual group of $G/2G\cong\mathbb{Z}_2^m$, and $g+2G$ in $G/2G$ is not 0. Thus, we get

$$\#\{\chi\in(\widehat{G})_2\setminus\{0\};\ \langle g,\chi\rangle = 1\} = \frac{|(\widehat{G})_2|}{2} - 1 = \frac{|G_2|}{2} - 1,$$

and

$$\dim(1-Z_G)\mathcal{A}_{g\rho,\rho} = \frac{|G|^2}{2}.$$

Since ρ and $\alpha_g\circ\rho$ are multiplicity free, Lemma 2.3 implies that

$$\dim(1-Z_G)\mathcal{A}_{g\rho,\rho} \le \dim(1-Z_G)\mathcal{A}_\rho.$$

This is possible only if $|G_2|\le 4$. ∎

The above computation shows the following. If $|G_2| = 2$, we have

$$\dim(1 - Z_G)\mathcal{A}_\rho = \frac{|G|^2}{2} + 2,$$

and for $g \in G \setminus 2G$, we have

$$\dim(1 - Z_G)\mathcal{A}_{g\rho,\rho} = \frac{|G|^2}{2}.$$

This means that if we set

$$\mu = \bigoplus_{g \in G} \alpha_g \rho, \quad \nu_+ = \bigoplus_{g \in 2G} \alpha_g \rho, \quad \nu_- = \bigoplus_{g \in G \setminus 2G} \alpha_g \rho,$$

then the object μ has exactly $|G|^2/2$ half-braidings $\{\mathcal{E}^i_\mu\}_{i \in I}$, and each of ν_+ and ν_- has exactly two half-braidings $\{\mathcal{E}^j_{\nu_\pm}\}_{j=0,1}$.

If $|G_2| = 4$, we have

$$\dim(1 - Z_G)\mathcal{A}_\rho = \dim(1 - Z_G)\mathcal{A}_{g\rho,\rho} = \frac{|G|^2}{2},$$

for any $g \in G$. Thus, μ has exactly $|G|^2/2$ half-braidings $\{\mathcal{E}^i_\mu\}_{i \in I}$.

3.4. Modular data

So, far we have determined the algebra structure of Tube $\mathcal{C}$. We can now compute the modular data S and T.

The basic idea, following [17], is that to apply the formula (2.6) to a pair of objects $\widetilde{\kappa_1}^l, \widetilde{\kappa_2}^m$ in $\mathcal{Z}(\mathcal{C})$, it is enough to know the components of the half-braidings $\mathcal{E}^m_{\kappa_2}(\eta)_{(\xi,i),(\xi,i)}$ and $\mathcal{E}^l_{\kappa_1}(\xi)_{(\eta,j),(\eta,j)}$ for a fixed simple summand (η, j) of κ_1, as (ξ, i) varies over the simple summands of κ_2.

In particular, in our situation where $\mathcal{C}$ is a generalized Haagerup category, if κ_1 contains an invertible summand α_g, then we can choose $\eta = \alpha_g$. Then we only need to know the components of the half-braidings of the form $\mathcal{E}^l_{\kappa_1}(\xi)_{(g,j),(g,j)}$, which are given by Lemma 3.5; and $\mathcal{E}^m_{\kappa_2}(g)_{(\xi,i),(\xi,i)}$. If $g = 0$, then by the definition of a half-braiding, we have $\mathcal{E}^m_{\kappa_2}(g)_{(\xi,i),(\xi,i)} = 1$ for any (ξ, i). Moreover, if $g \notin G_2$, then $\mathcal{E}^m_{\kappa_2}(g)_{(h,i),(h,i)}$ can be computed from Lemma 3.5, and $\mathcal{E}^m_{\kappa_2}(g)_{h\rho,h\rho} = 0$, since $(\alpha_h \rho \alpha_g, \alpha_g \alpha_h \rho) = 0$.

Therefore, we can compute all entries of the S-matrix involving 0, $\tilde{\pi}$, $\widetilde{\varphi_{\chi,\pm}}$, and $\widetilde{\sigma_g}^\tau$, for $g \in G_*$. For $k, l \in G_2$, the computation of $S_{\widetilde{\sigma_k}^\tau, \widetilde{\sigma_l}^\theta}$ using (2.6) would in principle require determining $\mathcal{E}_{\sigma_k^\tau}(l)_{h\rho,h\rho}$. However, since in the formula $\mathcal{E}_{\sigma_k^\tau}(l)_{h\rho,h\rho}$ multiplies $\mathcal{E}_{\sigma_l^\theta}(h\rho)_{(l,1),(l,1)}$, which is 0 by Lemma 3.5, we don't actually need to know $\mathcal{E}_{\sigma_k^\tau}(l)_{h\rho,h\rho}$.

The computations are summarized in the following lemma.

Lemma 3.7. *We have the following formulas for entries of S and T:*

$$S_{0,0} = \frac{1}{\Lambda} = \frac{a-b}{2}, \quad S_{0,\tilde{\pi}} = \frac{1+|G|d}{\Lambda} = \frac{a+b}{2},$$

$$S_{0,\widetilde{\varphi_{\chi,\pm}}} = \frac{1+\frac{|G|}{2}d}{\Lambda} = \frac{a}{2}, \quad S_{0,\widetilde{\sigma_g}^\tau} = \frac{2+|G|d}{\Lambda} = a,$$

$$S_{\tilde{\pi},\tilde{\pi}} = \frac{1}{\Lambda} = \frac{a-b}{2}, \quad S_{\tilde{\pi},\widetilde{\varphi_{\chi,\pm}}} = \frac{a}{2}, \quad S_{\tilde{\pi},\widetilde{\sigma_g}^\tau} = a,$$

$$S_{\widetilde{\varphi_{\chi_1,\varepsilon_1}},\widetilde{\varphi_{\chi_2,\varepsilon_2}}} = \frac{a}{2} + \frac{\varepsilon_1\varepsilon_2\delta_{\chi_1,\chi_2}}{4}, \quad S_{\widetilde{\varphi_{\chi,\pm}},\widetilde{\sigma_g}^\tau} = a\langle g,\chi\rangle,$$

$$S_{\widetilde{\sigma_g}^\tau,\widetilde{\sigma_h}^\theta} = a(\langle h,\tau\rangle\langle g,\theta\rangle + \overline{\langle h,\tau\rangle\langle g,\theta\rangle}).$$

$$T_{0,0} = T_{\tilde{\pi},\tilde{\pi}} = T_{\widetilde{\varphi_{\chi,\pm}},\widetilde{\varphi_{\chi,\pm}}} = 1, \quad T_{\widetilde{\sigma_g}^\tau,\widetilde{\sigma_g}^\tau} = \langle g,\tau\rangle.$$

Proof. The computation not involving $\varphi_{\chi,\pm}$ is the same as in [17], so we compute only those involving $\varphi_{\chi,\pm}$, using (2.6).

1. Computation of $S_{\widetilde{\varphi_{\chi,\pm}},\tilde{\pi}}$:

$$\begin{aligned} S_{\widetilde{\varphi_{\chi,\pm}},\tilde{\pi}} &= \frac{d(\varphi_{\chi,\pm})}{\Lambda}\sum_{\xi} d(\xi)\phi_\xi(\mathcal{E}_\pi(0)^*_{\xi,\xi}\mathcal{E}_{\varphi_{\chi,\pm}}(\xi)^*_{0,0}) \\ &= \frac{a}{2}\Big(1 + d\sum_{g\in G}\phi_{g\rho}(\mathcal{E}_\pi(0)^*_{g\rho,g\rho}\mathcal{E}_{\varphi_{\chi,\pm}}(g\rho)^*_{0,0})\Big) \\ &= \frac{a}{2}\Big(1 + d\sum_{g\in G}\frac{\pm\langle g,\chi\rangle}{d}\Big) \\ &= \frac{a}{2}. \end{aligned}$$

2. Computation of $S_{\widetilde{\varphi_{\chi_1,\varepsilon_1}},\widetilde{\varphi_{\chi_2,\varepsilon_2}}}$:

$$\begin{aligned} S_{\widetilde{\varphi_{\chi_1,\varepsilon_1}},\widetilde{\varphi_{\chi_2,\varepsilon_2}}} &= \frac{d(\varphi_{\chi_1,\varepsilon_1})}{\Lambda}\sum_{\xi} d(\xi)\phi_\xi\big(\mathcal{E}_{\varphi_{\chi_2,\varepsilon_2}}(0)^*_{\xi,\xi}\mathcal{E}_{\varphi_{\chi_1,\varepsilon_1}}(\xi)^*_{0,0}\big) \\ &= \frac{a}{2}\Big(1 + d\sum_{\langle g,\chi_2\rangle=\varepsilon_2}\phi_{g\rho}\big(\mathcal{E}_{\varphi_{\chi_2,\varepsilon_2}}(0)^*_{g\rho,g\rho}\mathcal{E}_{\varphi_{\chi_1,\varepsilon_1}}(g\rho)^*_{0,0}\big)\Big) \\ &= \frac{a}{2}\Big(1 + \varepsilon_1\sum_{\langle g,\chi_2\rangle=\varepsilon_2}\langle g,\chi_1\rangle\Big). \end{aligned}$$

Note that we have

$$\sum_{\langle g,\chi_2\rangle=1}\langle g,\chi_1\rangle + \sum_{\langle g,\chi_2\rangle=-1}\langle g,\chi_1\rangle = 0,$$

and so

$$\varepsilon_1 \sum_{\langle g,\chi_2\rangle=\varepsilon_2} \langle g,\chi_1\rangle = \varepsilon_1\varepsilon_2 \sum_{\langle g,\chi_2\rangle=1} \langle g,\chi_1\rangle = \varepsilon_1\varepsilon_2\delta_{\chi_1,\chi_2}\frac{|G|}{2}.$$

3. Computation of $S_{\widetilde{\varphi_{\chi,\pm}},\widetilde{\sigma_0}^\tau}$:

$$\begin{aligned}
S_{\widetilde{\varphi_{\chi,\pm}},\widetilde{\sigma_0}^\tau} &= \frac{d(\varphi_{\chi,\pm})}{\Lambda}\sum_{\xi,i} d(\xi)\phi_\xi\big(\mathcal{E}^\tau_{\sigma_0}(0)^*_{(\xi,i),(\xi,i)}\mathcal{E}_{\varphi_{\chi,\pm}}(\xi)^*_{0,0}\big)\\
&= \frac{a}{2}\Big(2 + d\sum_{g\in G}\phi_{g\rho}\big(\mathcal{E}^\tau_{\sigma_0}(0)^*_{g\rho,g\rho}\mathcal{E}_{\varphi_{\chi,\pm}}(g\rho)^*_{0,0}\big)\Big)\\
&= \frac{a}{2}\Big(2 + d\sum_{g\in G}\frac{\pm\langle g,\chi\rangle}{d}\Big) = a.
\end{aligned}$$

4. Computation of $S_{\widetilde{\varphi_{\chi,\pm}},\widetilde{\sigma_g}^\tau}$:

$$\begin{aligned}
S_{\widetilde{\varphi_{\chi,\pm}},\widetilde{\sigma_g}^\tau} &= \frac{d(\varphi_{\chi,\pm})}{\Lambda}\sum_{\xi} d(\xi)\phi_\xi\big(\mathcal{E}^\tau_{\sigma_g}(0)^*_{\xi,\xi}\mathcal{E}_{\varphi_{\chi,\pm}}(\xi)^*_{0,0}\big)\\
&= \frac{a}{2}\Big(\mathcal{E}^\tau_{\sigma_g}(0)^*_{g,g}\mathcal{E}_{\varphi_{\chi,\pm}}(g)^*_{0,0} + \mathcal{E}^\tau_{\sigma_g}(0)^*_{-g,-g}\mathcal{E}_{\varphi_{\chi,\pm}}(-g)^*_{0,0}\\
&\quad + d\sum_{h\in G}\phi_{h\rho}\big(\mathcal{E}^\tau_{\sigma_0}(0)^*_{h\rho,h\rho}\mathcal{E}_{\varphi_{\chi,\pm}}(h\rho)^*_{0,0}\big)\Big)\\
&= \frac{a}{2}\Big(\mathcal{E}_{\varphi_{\chi,\pm}}(g)^*_{0,0} + \mathcal{E}_{\varphi_{\chi,\pm}}(-g)^*_{0,0} + d\sum_{h\in G}\frac{\pm\langle h,\chi\rangle}{d}\Big)\\
&= a\langle g,\chi\rangle.
\end{aligned}$$

■

We have determined the entries of the modular data where both indices are among the objects 0, $\tilde{\pi}$, $\widetilde{\varphi_{\chi,\pm}}$, and $\widetilde{\sigma_g}^\tau$. We would like to extend the formulas in Lemma 3.7 to the objects k, $k\tilde{\pi}$, and $k\widetilde{\varphi_{\chi,\pm}}$ for $k \in G_2$.

First, note that for $k, l \in G_2$, we have

$$S_{k,l} = \frac{\epsilon_k(l)\epsilon_l(k)}{\Lambda} = \frac{a-b}{2}\epsilon_k(l)\epsilon_l(k), \quad T_{k,k} = \epsilon_k(k).$$

Next, for any invertible object g and any simple object X in a modular tensor category $\mathcal{D}$, $\mathcal{E}_X(g)\circ\mathcal{E}_g(X)$ is an automorphism of $g\otimes X$, and hence a scalar. Therefore,

$$|S_{g,X}| = \frac{d_X}{\sqrt{\dim(\mathcal{D})}},$$

and in particular $|S_{g,X}| \neq 0$. We then have, for any X and Y,

$$S_{g\otimes X,Y} = \frac{\sqrt{\dim(\mathcal{D})}}{d_X}S_{k,Y}S_{X,Y} = \frac{S_{k,Y}}{|S_{k,Y}|}S_{X,Y}.$$

Returning to our generalized Haagerup category, for $k \in G_2$ and a simple object X in $\mathcal{Z}(\mathcal{C})$, we set

$$s(k, X) = \frac{S_{k,X}}{|S_{k,X}|}.$$

Then we have

$$\begin{aligned} S_{kX,Y} &= s(k, Y) S_{X,Y}, \\ T_{kX,kX} &= \overline{s(k, X)} T_{k,k} T_{X,X}, \end{aligned}$$

and so

$$S_{kX,lY} = s(k,l)s(k,Y)s(l,X)S_{X,Y} = \epsilon_k(l)\epsilon_l(k)s(k,Y)s(l,X)S_{X,Y}, \tag{3.1}$$

for $l \in G_2$ and $Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$.

Note that we have, for any $\tilde{\gamma}^i \in \mathcal{Z}(\mathcal{C})$ with simple summand (η, j),

$$s(k, \tilde{\gamma}^i) = \mathcal{E}_k(\eta)^* \mathcal{E}^i_\gamma(k)^*_{(\eta,j),(\eta,j)}.$$

In particular, when γ contains $g \in G$, we get

$$s(k, \tilde{\gamma}^i) = \epsilon_k(g) \mathcal{E}^i_\gamma(k)^*_{g,g},$$

and when γ contains $\alpha_g \circ \rho$,

$$s(k, \tilde{\gamma}^i) = \epsilon_k(g) \mathcal{E}^i_\gamma(k)^*_{g\rho,g\rho}.$$

These facts imply the following formulas.

Lemma 3.8. *For $k \in G_2$, we have*

$$\begin{aligned} s(k, \tilde{\pi}) &= 1, \\ s(k, \widetilde{\varphi_{\chi,\pm}}) &= \langle k, \chi \rangle, \\ s(k, \widetilde{\sigma_g}^\tau) &= \epsilon_k(g) \langle k, \tau \rangle. \end{aligned}$$

Lemma 3.8, together with Lemma 3.7 and (3.1), determines the entries of the modular data where both indices are among the objects k, $k\tilde{\pi}$, $k\widetilde{\varphi_{\chi,\pm}}$, and $\widetilde{\sigma_g}^\tau$. It remains to determine the entries of the modular data which involve the $\tilde{\mu}^i$ and $\widetilde{\nu_\varepsilon}^j$.

Let

$$(G \times \widehat{G})_* = (G_2 \times (\widehat{G})_*) \sqcup (G_* \times \widehat{G}).$$

Then we have

$$G \times \widehat{G} = (G \times \widehat{G})_2 \sqcup (G \times \widehat{G})_* \sqcup -(G \times \widehat{G})_*.$$

Theorem 3.9. *Let $\mathcal{C} \subset \mathrm{End}(M)$ be a generalized Haagerup category with a finite group G satisfying (2.24). Assume $|G_2| = 2$. The following set exhausts the equivalence classes of the simple objects in $\mathcal{Z}(\mathcal{C})$:*

$$\{k\}_{k\in G_2} \cup \{k\tilde{\pi}\}_{k\in G_2} \cup \{k\widetilde{\varphi_{\chi_0,\varepsilon}}\}_{k\in G_2,\,\varepsilon\in\{1,-1\}}$$
$$\cup \{\widetilde{\sigma_g}^{\tau}\}_{(g,\tau)\in(G\times\hat{G})_*} \cup \{\widetilde{\nu_\varepsilon}^{j}\}_{\varepsilon\in\{1,-1\},\, j\in\{0,1\}} \cup \{\tilde{\mu}^i\}_{i\in I},$$

where we use the notation $(\widehat{G})_2 = \{0, \chi_0\}$ and I is an index set with $|I| = |G|^2/2$. Every object in $\mathcal{Z}(\mathcal{C})$ is self-dual.

There exist characters $G_2 \ni k \mapsto s(k, \widetilde{\nu_\pm}^j) \in \{1,-1\}$ and $G_2 \ni k \mapsto s(k, \tilde{\mu}^i) \in \{1,-1\}$ satisfying

$$S_{k,l} = s(k,l)\frac{a-b}{2}, \quad S_{k.l\tilde{\pi}} = s(k,l)\frac{a+b}{2}, \quad S_{k,l\widetilde{\varphi_{\chi_0,\pm}}} = s(k,l)\frac{a}{2}\langle k,\chi_0\rangle,$$

$$S_{k,\widetilde{\sigma_g}^\tau} = a\langle k,\tau\rangle\epsilon_k(g), \quad S_{k,\widetilde{\nu_\pm}^j} = s(k,\widetilde{\nu_\pm}^j)\frac{b}{2}, \quad S_{k,\tilde{\mu}^i} = s(k,\tilde{\mu}^i)b,$$

$$S_{k\tilde{\pi},l\tilde{\pi}} = s(k,l)\frac{a-b}{2},$$
$$S_{k\tilde{\pi},l\widetilde{\varphi_{\chi_0,\pm}}} = s(k,l)\frac{a}{2}\langle k,\chi_0\rangle,$$
$$S_{k\tilde{\pi},\widetilde{\sigma_g}^\tau} = a\langle k,\tau\rangle\epsilon_k(g),$$
$$S_{k\tilde{\pi},\widetilde{\nu_\pm}^j} = -s(k,\widetilde{\nu_\pm}^j)\frac{b}{2}, \quad S_{k\tilde{\pi},\tilde{\mu}^i} = -s(k,\tilde{\mu}^i)b,$$
$$S_{k\widetilde{\varphi_{\chi_0,\varepsilon_1}},l\widetilde{\varphi_{\chi_0,\varepsilon_2}}} = s(k,l)\langle k,\chi_0\rangle\langle l,\chi_0\rangle\Big(\frac{a}{2} + \frac{\varepsilon_1\varepsilon_2}{4}\Big),$$
$$S_{k\widetilde{\varphi_\varepsilon},\widetilde{\sigma_g}^\tau} = a\langle k,\tau\rangle\epsilon_k(g)\langle g,\chi_0\rangle,$$
$$S_{k\widetilde{\varphi_{\chi_0,\varepsilon_1}},\widetilde{\nu_{\varepsilon_2}}^j} = s(k,\widetilde{\nu_{\varepsilon_2}}^j)\frac{\varepsilon_1\varepsilon_2}{4},$$
$$S_{k\widetilde{\varphi_{\chi_0,\pm}},\tilde{\mu}^j} = 0,$$

$$S_{\widetilde{\sigma_g}^\tau,\widetilde{\sigma_h}^\theta} = a(\langle h,\tau\rangle\langle g,\theta\rangle + \overline{\langle h,\tau\rangle\langle g,\theta\rangle}), \quad S_{\widetilde{\sigma_g}^\tau,\widetilde{\nu_\pm}^k} = 0, \quad S_{\widetilde{\sigma_g}^\tau,\tilde{\mu}^j} = 0,$$
$$T_{k,k} = T_{k\tilde{\pi},k\tilde{\pi}} = \epsilon_k(k), \quad T_{k\widetilde{\varphi_{\chi_0,\pm}},k\widetilde{\varphi_\pm}} = \epsilon_k(k)\langle k,\chi\rangle, \quad T_{\widetilde{\sigma_g}^\tau,\widetilde{\sigma_g}^\tau} = \langle g,\tau\rangle.$$

Proof. Most of the computation is similar to that in [17], except for

1. computation of $S_{\widetilde{\varphi_{\chi_0,\varepsilon}},\widetilde{\nu_+}^i}$:

$$S_{\widetilde{\varphi_{\chi_0,\varepsilon}},\widetilde{\nu_+}^i} = \frac{d(\varphi_{\chi_0,\varepsilon})}{\Lambda}\sum_{g\in 2G} d\phi_{g\rho}(\mathcal{E}^i_{\nu_+}(0)^*_{g\rho,g\rho}\mathcal{E}_{\varphi_{\chi_0,\varepsilon}}(g\rho)^*_{0,0})$$
$$= \frac{\varepsilon}{2|G|}\sum_{g\in 2G}\langle g,\chi_0\rangle = \frac{\varepsilon}{4};$$

2. computation of $S_{\widetilde{\varphi_{\chi_0,\varepsilon}},\widetilde{\nu_-}^i}$:

$$
\begin{aligned}
S_{\widetilde{\varphi_{\chi_0,\varepsilon}},\widetilde{\nu_-}^i} &= \frac{d(\varphi_{\chi_0,\varepsilon})}{\Lambda}\sum_{g\in G\setminus 2G} d\phi_{g\rho}(\mathcal{E}^i_{\nu_-}(0)^*_{g\rho,g\rho}\mathcal{E}_{\varphi_{\chi_0,\varepsilon}}(g\rho)^*_{0,0}) \\
&= \frac{\varepsilon}{2|G|}\sum_{g\in G\setminus 2G}\langle g,\chi_0\rangle = -\frac{\varepsilon}{4};
\end{aligned}
$$

3. computation of $S_{\widetilde{\varphi_{\chi_0,\varepsilon}},\tilde{\mu}^j}$:

$$
\begin{aligned}
S_{\widetilde{\varphi_{\chi_0,\varepsilon}},\tilde{\mu}^j} &= \frac{d(\varphi_{\chi_0,\varepsilon})}{\Lambda}\sum_{g\in G} d\phi_{g\rho}(\mathcal{E}^j_{\mu}(0)^*_{g\rho,g\rho}\mathcal{E}_{\varphi_{\chi_0,\varepsilon}}(g\rho)^*_{0,0}) \\
&= \frac{1}{2|G|}\sum_{g\in G}\varepsilon\langle g,\chi_0\rangle = 0.
\end{aligned}
$$

We have already seen in the proof of Lemma 3.1 that S_0^2 restricted to $\mathcal{A}_{g\rho}$ is the identity, which implies that every simple object $X\in\mathcal{Z}(\mathcal{C})$ with $\mathcal{F}(X)$ containing $\alpha_g\circ\rho$ for some $g\in G$ is self-dual. The only simple objects not satisfying this condition are those in G_2, and they are again self-dual. ∎

In a similar way, we can show the following.

Theorem 3.10. *Let $\mathcal{C}\subset\mathrm{End}(M)$ be a generalized Haagerup category with a finite group G satisfying (2.24). Assume $|G_2|=4$. The following set exhausts the equivalence classes of the simple objects in $\mathcal{Z}(\mathcal{C})$:*

$$
\begin{aligned}
&\{k\}_{k\in G_2}\cup\{k\tilde{\pi}\}_{k\in G_2}\cup\{k\widetilde{\varphi_{\chi,\varepsilon}}\}_{k\in G_2,\ \chi\in(\hat{G})_2\setminus\{0\},\ \varepsilon\in\{1,-1\}} \\
&\quad\cup\{\widetilde{\sigma_g}^{\tau}\}_{(g,\tau)\in(G\times\hat{G})_*}\cup\{\tilde{\mu}^i\}_{i\in I}
\end{aligned}
$$

where I is an index set with $|I|=|G|^2/2$. Every object in $\mathcal{Z}(\mathcal{C})$ is self-dual.

Let $k,l\in G_2$, and let

$$s(k,l)=\epsilon_k(l)\epsilon_l(k).$$

There exist characters

$$G_2\ni k\mapsto s(k,\tilde{\mu}^i)\in\{1,-1\}$$

satisfying

$$S_{k,l}=s(k,l)\frac{a-b}{2},\quad S_{k.l\tilde{\pi}}=s(k,l)\frac{a+b}{2},\quad S_{k,l\widetilde{\varphi_{\chi,\pm}}}=s(k,l)\frac{a}{2}\langle k,\chi\rangle,$$

$$S_{k,\widetilde{\sigma_g}^{\tau}}=a\langle k,\tau\rangle\epsilon_k(g),\quad S_{k,\tilde{\mu}^i}=s(k,\tilde{\mu}^i)b,$$

$$S_{k\tilde{\pi},l\tilde{\pi}}=s(k,l)\frac{a-b}{2},\quad S_{k\tilde{\pi},l\widetilde{\varphi_{\chi,\pm}}}=s(k,l)\frac{a}{2}\langle k,\chi\rangle,\quad S_{k\tilde{\pi},\widetilde{\sigma_g}^{\tau}}=a\langle k,\tau\rangle\epsilon_k(g),$$

S						
l	$s(k,l)\frac{a-b}{2}$					
$l\tilde{\pi}$	$s(k,l)\frac{a+b}{2}$	$s(k,l)\frac{a-b}{2}$				
$l\widetilde{\varphi_{\chi_2,\varepsilon_2}}$	$s(k,1)\frac{a}{2}\langle k,\chi_2\rangle$	$s(k,l)\frac{a}{2}\langle k,\chi_2\rangle$	$s(k,l)\langle k,\chi_2\rangle\langle l,\chi_1\rangle(\frac{a}{2}+\frac{\varepsilon_1\varepsilon_2\delta_{\chi_1,\chi_2}}{4})$			
$\widetilde{\sigma_h}^\theta$	$a\langle k,\theta\rangle\epsilon_k(h)$	$a\langle k,\theta\rangle\epsilon_k(h)$	$a\langle k,\theta\rangle\epsilon_k(h)\langle h,\chi_1\rangle$	$a(\langle h,\tau\rangle\langle g,\theta\rangle+\overline{\langle h,\tau\rangle\langle g,\theta\rangle}$		
$\widetilde{\nu_{\varepsilon_2}}^j$	$s(k,\widetilde{\nu_{\varepsilon_2}}^j)\frac{b}{2}$	$-s(k,\widetilde{\nu_{\varepsilon_2}}^j)\frac{b}{2}$	$s(k,\widetilde{\nu_{\varepsilon_2}}^j)\frac{\varepsilon_1\varepsilon_2}{4}$	0	?	
$\tilde{\mu}^i$	$s(k,\tilde{\mu}^i)b$	$-s(k,\tilde{\mu}^i)b$	0	0	?	?
	k	$k\tilde{\pi}$	$k\widetilde{\varphi_{\chi_1,\varepsilon_1}}$	$\widetilde{\sigma_g}^\tau$	$\widetilde{\nu_{\varepsilon_1}}^{j'}$	$\tilde{\mu}^{i'}$
T	$\epsilon_k(k)$	$\epsilon_k(k)$	$\epsilon_k(k)\langle k,\chi_1\rangle$	$\langle g,\tau\rangle$	?	?

Table 1. Partial modular data for generalized Haagerup categories for even groups satisfying (2.24). Undetermined entries are labeled by "?". (The index set for $\widetilde{\nu_\varepsilon}$ is empty when $|G_2|=4$.)

$$S_{k\tilde{\pi},\tilde{\mu}^i}=-s(k,\tilde{\mu}^i)b,\quad S_{k\widetilde{\varphi_{\chi_1,\varepsilon_1}},l\widetilde{\varphi_{\chi_2,\varepsilon_2}}}=s(k,l)\langle k,\chi_2\rangle\langle l,\chi_1\rangle\Big(\frac{a}{2}+\frac{\varepsilon_1\varepsilon_2\delta_{\chi_1,\chi_2}}{4}\Big),$$

$$S_{k\widetilde{\varphi_{\chi,\varepsilon}},\widetilde{\sigma_g}^\tau}=a\langle k,\tau\rangle\epsilon_k(g)\langle g,\chi\rangle,\quad S_{k\widetilde{\varphi_{\chi,\pm}},\tilde{\mu}^j}=0,$$

$$S_{\widetilde{\sigma_g}^\tau,\widetilde{\sigma_h}^\theta}=a(\langle h,\tau\rangle\langle g,\theta\rangle+\overline{\langle h,\tau\rangle\langle g,\theta\rangle}),\quad S_{\widetilde{\sigma_g}^\tau,\tilde{\mu}^j}=0,$$

$$T_{k,k}=T_{k\tilde{\pi},k\tilde{\pi}}=\epsilon_k(k),\quad T_{k\widetilde{\varphi_{\chi,\pm}},k\widetilde{\varphi_{\chi,\pm}}}=\epsilon_k(k)\langle k,\chi\rangle,\quad T_{\widetilde{\sigma_g}^\tau,\widetilde{\sigma_g}^\tau}=\langle g,\tau\rangle.$$

The results of Theorems 3.9 and 3.10 are summarized in Table 1.

We now show how to determine $s(k,\tilde{\mu}^i)$ and $s(k,\widetilde{\nu_\pm}^i)$ for $k\in G_2$. For $g\in G$ and $k\in G_2$, we set

$$U_g(k)=\epsilon_k(g)({}_g\rho\, k|1|k\ {}_g\rho).$$

Then $\{U_g(k)\}_{k\in G_2}$ forms a representation of G_2 in $\mathcal{A}_{g\rho}$. Let $X=\tilde{\gamma}^i$ be a simple object in $\mathcal{Z}(\mathcal{C})$ such that γ contains ${}_g\rho$ and ${}_h\rho$. Since $\mathcal{A}_{g\rho}$ is abelian, $U_g(k)e(X)_{g\rho,h\rho}$ is a scalar multiple of $e(X)_{g\rho,h\rho}$.

Lemma 3.11. *Let the notation be as above.*

$$U_g(k)e(X)_{g\rho,h\rho}=s(k,X)e(X)_{g\rho,h\rho}.$$

Proof. It suffices to show the statement in the case of $h=g$. We have already seen

$$s(k,X)=\epsilon_k(g)\mathcal{E}^i_\gamma(k)^*_{g\rho,g\rho},$$

and so

$$\mathcal{E}^i_\gamma(k)_{g\rho,g\rho}=s(k,X)\epsilon_k(g)\in(\alpha_g\rho\alpha_k,\alpha_k\alpha_g\rho)=\mathbb{C}.$$

Thus

$$
\begin{aligned}
&e(X)_{g\rho,g\rho}\\
&= \frac{d(X)}{\Lambda d}\sum_{\xi} d(\xi)({}_{g\rho}\,\xi|\mathcal{E}^i_\gamma(\xi)_{g\rho,g\rho}|\xi\,{}_{g\rho})\\
&= \frac{d(X)}{\Lambda d}\Big(\sum_{k\in G_2}({}_{g\rho}\,k|\mathcal{E}^i_\gamma(k)_{g\rho,g\rho}|k\,{}_{g\rho}) + \sum_{h\in G} d({}_{g\rho}\,{}_{h\rho}|\mathcal{E}^i_\gamma({}_{h\rho})_{g\rho,g\rho}|{}_{h\rho}\,{}_{g\rho})\Big)\\
&= \frac{d(X)}{\Lambda d}\Big(\sum_{k\in G_2} s(k,X)U_g(k) + \sum_{h\in G} d({}_{g\rho}\,{}_{h\rho}|\mathcal{E}^i_\gamma({}_{h\rho})_{g\rho,g\rho}|{}_{h\rho}\,{}_{g\rho})\Big),
\end{aligned}
$$

which shows the statement. ∎

From the definition of $U_g(k)$, we can see that the representation $\{U_g(k)\}_{k\in G_2}$ of G_2 in $\mathcal{A}_{g\rho,g\rho}$ is a multiple of the regular representation - that is, each character of G_2 occurs with the same multiplicity. For the same reason, we see that the left multiplication of $U_g(k)$ on $\mathcal{A}_{g\rho,h\rho}$ gives rise to a representation of G_2, which is a multiple of the regular representation. On the other hand, the above formulas tell us the multiplicity of each character in $Z_G\mathcal{A}_{g\rho}$ and $Z_G\mathcal{A}_{g\rho,h\rho}$. Thus, we also know the multiplicity of each character in $(1-Z_G)\mathcal{A}_{g\rho}$ and $(1-Z_G)\mathcal{A}_{g\rho,h\rho}$.

If $G_2=\mathbb{Z}_2$, and a_0 is the nontrivial element, then this amounts to saying that for each $g\in G$, the function $s(a_0,\kappa)$ takes on the value 1 for exactly half of the simple objects $\kappa\in\mathcal{Z}(\mathcal{C})$ such that $\mathcal{F}(\kappa)$ contains $\alpha_g\rho$; and also for each $g,h\in G$, the function $s(a_0,\kappa)$ takes on the value 1 for exactly half of the simple objects $\kappa\in\mathcal{Z}(\mathcal{C})$ such that $\mathcal{F}(\kappa)$ contains both $\alpha_g\rho$ and $\alpha_h\rho$. For $g\in 2G$, and $h\in G\backslash 2G$, the set of simple objects in $\mathcal{Z}(\mathcal{C})$ whose underlying object in $\mathcal{C}$ contains $\alpha_g\rho$ is

$$\{k\tilde{\pi}\}_{k\in G_2}\cup\{k\widetilde{\varphi_{\chi_0,+}}\}_{k\in G_2}\cup\{\widetilde{\sigma_g}^\tau\}_{(g,\tau)\in(G\times\hat{G})_*}\cup\{\widetilde{\nu_+}^j\}_{j\in\{0,1\}}\cup\{\tilde{\mu}^i\}_{i\in I};$$

contains α_h is

$$\{k\tilde{\pi}\}_{k\in G_2}\cup\{k\widetilde{\varphi_{\chi_0,-}}\}_{k\in G_2}\cup\{\widetilde{\sigma_g}^\tau\}_{(g,\tau)\in(G\times\hat{G})_*}\cup\{\widetilde{\nu_-}^j\}_{j\in\{0,1\}}\cup\{\tilde{\mu}^i\}_{i\in I};$$

and contains both is

$$\{k\tilde{\pi}\}_{k\in G_2}\cup\{\widetilde{\sigma_g}^\tau\}_{(g,\tau)\in(G\times\hat{G})_*}\cup\{\tilde{\mu}^i\}_{i\in I}.$$

Thus, we can determine the signs of $s(a_0,\widetilde{\nu_\varepsilon}^j)$ and $s(a_0,\tilde{\mu}^i)$ from those of $s(a_0,k\tilde{\pi})$, $s(a_0,k\widetilde{\varphi_{\chi_0,\varepsilon}})$, and $s(a_0,\widetilde{\sigma_g}^\tau)$, which we have already computed above.

Recall that we can canonically decompose G as $G_e\times G_o$ where G_e is a 2-group and G_o is an odd group. We can consider two cases: $G_e=\mathbb{Z}_2$ or $G_e=\mathbb{Z}_{2^m}$ for some $m>1$. Let a_0 be the non-trivial element of G_2, and let χ_0 be the nontrivial element of $\widehat{G}_2$. If $G_e = \mathbb{Z}_2$, then $\chi_0(a_0) = -1$, and exactly half of the characters in G_* take the value 1 on a_0. On the other hand, if $G_e = \mathbb{Z}_{2m}$ with $m > 1$, then $\chi_0(a_0) = 1$, and there is one more character in G_* which takes the value -1 on a_0 than the number of characters which take the value 1 on a_0.

Corollary 3.12. *Assume $|G_2| = 2$, and let a_0 be the unique non-trivial element of G_2. Let I be the index set of the half-braidings of μ, and let*

$$I_\pm = \{i \in I;\ s(a_0, \tilde{\mu}^i) = \pm 1\}.$$

1. *Assume that G_e is $\mathbb{Z}_2$. Then $|I_+| = |G|^2/4 - 1$, $|I_-| = |G|^2/4 + 1$, and $s(a_0, \widetilde{\nu_\pm}^j) = 1$.*
2. *Assume that G_e is not $\mathbb{Z}_2$. Then $|I_\pm| = |G|^2/4$ and $s(a_0, \widetilde{\nu_\pm}^j) = -1$.*

Proof. If G_e is $\mathbb{Z}_2$, then we have $s(a_0, k\widetilde{\varphi_{\chi_0,\varepsilon}}) = \chi_0(a_0) = -1$, for $k \in G_2$. Therefore, by the discussion above we must have $s(a_0, \widetilde{\nu_\pm}^j) = 1$. Also, since $s(a_0, k\tilde{\pi}) = 1$ for $k \in G_2$, and $s(a_0, \widetilde{\sigma_g}^\tau) = \epsilon_{a_0}(g)\langle a_0, \tau\rangle$ is equal to 1 for exactly half of $(g, \tau) \in (G \times \widehat{G})_*$, we must have $|I_-| = |I_+| + 2$.

Similarly, if G_e is not $\mathbb{Z}_2$, then we have $s(a_0, k\widetilde{\varphi_{\chi_0,\varepsilon}}) = \chi_0(a_0) = 1$, so $s(a_0, \widetilde{\nu_\pm}^j) = -1$. Since $s(a_0, k\tilde{\pi}) = 1$ for $k \in G_2$; $s(a_0, \widetilde{\sigma_g}^\tau)$ is equal to 1 for exactly half of $(g, \tau) \in G_* \times \widehat{G}$; and the set $(g, \tau) \in G_2 \times \widehat{G}_*$ for which $s(a_0, \widetilde{\sigma_g}^\tau) = 1$ has size two less than that of the set for which $s(a_0, \widetilde{\sigma_g}^\tau) = -1$, we must have $|I_+| = |I_-|$. ■

4. Computing the remaining corner of the modular data

Table 1 gives formulas for the modular data of the center of a generalized Haagerup category for an even group satisfying (2.24), except for the corners of S and T indexed by $\widetilde{\nu_\varepsilon}^j$ (for $|G_2| = 2$) and $\tilde{\mu}^i$. Since $\mathcal{F}(\nu)$ and $\mathcal{F}(\mu)$ do not have any invertible simple summands, we are unable to prove a general formula for the modular data in terms of the characters ϵ. However, for specific examples we can compute the missing corner directly from the tube algebra using the full data of the category (ϵ, A). In this section we outline the method of computation . We also look at some examples with $|G_2| = 2$, and formulate conjectures for the general case. .

4.1. Outline of the method

To perform the computation of the modular data, we need to find the minimal projections $e(\widetilde{\nu_\varepsilon}^j)_{g\rho,g\rho}$ and $e(\widetilde{\mu}^i)_{g\rho,g\rho}$ in $\mathcal{A}_{g\rho}$ for each g. The projections $e(\cdot)_{g\rho,g\rho}$ for different g can then be added together to find the minimal central projections $z(\widetilde{\nu_\varepsilon}^j)$ and $z(\tilde{\mu}^i)$ in Tube $\mathcal{C}$. Then the modular data can be computed using (2.5) or (2.6).

First, we find formulas for the minimal central projections $z(\cdot)$ corresponding to each of $k\tilde{\pi}$, $k\widetilde{\varphi_{\chi,\pm}}$, and $\widetilde{\sigma_{g,\tau}}$ as follows. Let

$$\mathcal{A}_G = \bigoplus_{g,h\in G} \mathcal{A}_{g,h},$$

and similarly for $\mathcal{A}_{G\rho}$, etc. For each minimal central projection p in $\mathcal{A}_G$ and each $h \in G$ such that $p\mathcal{A}_{G,h\rho}$ is nontrivial, we choose a minimal subprojection p' of p and a partial isometry $j(p',h)$ in $\mathcal{A}_{G,h\rho}$ such that $j(p',h)j^*(p',h) = p'$; then $j(p',h)j^*(p',h)$ is the corresponding minimal projection in $\mathcal{A}_{h\rho}$. In this way we can find all of the minimal central projections $z(\cdot)$ in the tube algebra such that $1_{\mathcal{A}_G} z(\cdot) \neq 0$. Summing the $z(\cdot)$ for all of the k, $k\tilde{\pi}$, $k\widetilde{\varphi_{\chi,\pm}}$, and $\widetilde{\sigma_{g,\tau}}$, we obtain Z_G.

Next, we diagonalize the action of $\mathbf{t}_{g\rho}$ on $\mathcal{A}_{g\rho}$ for a representative g in each coset of $2G$ (since $1_{g\rho}$ is equivalent to $1_{g+2h\rho}$). We use Mathematica to perform the calculations. It turns out that even for relatively small examples, it is difficult to calculate the eigenvalues directly, since the arithmetic takes place in a complicated number field. So, instead we first find the eigenvalues numerically and guess their exact values. Then we use this guess to construct the minimal polynomial q of $\mathbf{t}_{g\rho}$. We then verify that the guess is correct by showing through exact calculation that $q(\mathbf{t}_{g\rho})$ vanishes, and that no proper factor of q vanishes on $\mathbf{t}_{g\rho}$.

Once we have the eigenvalues of $\mathbf{t}_{g\rho}$, we can compute the projections onto the eigenspaces. For each eigenvalue λ, set

$$p_g^\lambda = \frac{q_\lambda(\mathbf{t}_{g\rho})}{q_\lambda(\lambda)}, \quad \text{where } q_\lambda \text{ is the polynomial } q_\lambda(z) = \frac{q(z)}{z-\lambda}.$$

For each eigenvalue λ, let $(p_g^\lambda)' = (1 - Z_G)p_g^\lambda$.

If the number of non-zero $(p_g^\lambda)'$ in $\mathcal{A}_{g\rho}$ is equal to the dimension of $(1-Z_G)\mathcal{A}_{g\rho}$, then the $(p_g^\lambda)'$ must be exactly the $e(\widetilde{\nu_\varepsilon}^j)_{g\rho,g\rho}$ and $e(\tilde{\mu}^i)_{g\rho,g\rho}$ that we are looking for.

If the number of non-zero $(p_g^\lambda)'$ in $\mathcal{A}_{g\rho}$ is less than the dimension of $(1-Z_G)\mathcal{A}_{g\rho}$, then some of the $(p_g^\lambda)'$ have rank greater than 1 and we need to split them up. In the small examples that we consider, the highest rank that comes up for $(p_g^\lambda)'$ is 2. We can then split up a given projection $(p_g^\lambda)'$ by finding an element x in $\mathcal{A}_{g\rho}$ such that $x(p_g^\lambda)'$ is not a scalar multiple of $(p_g^\lambda)'$. Then we can write down a linear relation among $(p_g^\lambda)'$, $x(p_g^\lambda)'$, and $x^2(p_g^\lambda)'$ and find the minimal subprojections of $(p_g^\lambda)'$.

Once we have found all of the $e(\cdot)_{g\rho,g\rho}$, we need to match them up and add them together and find the $z(\cdot)$. Matching minimal projections of $\mathcal{A}_{g\rho}$ and $\mathcal{A}_{h\rho}$ which share a common T-eigenvalue can be done by looking at the action of $\mathcal{A}_{g\rho,h\rho}$. However, in the small examples we consider, there is only one matching that gives consistent modular data.

Remark 4.1. 1. The first step, finding an expression for Z_G, is not strictly necessary if one is only interested in finding the missing corner of modular data. If we skip this step, there may be a bit more work when we diagonalize the action of $\mathbf{t}_{g\rho}$ in figuring out which eigenvalues correspond to the $\widetilde{\nu_\varepsilon}^j$ and the $\tilde{\mu}^i$, and in splitting up the corresponding eigenprojections if some of the $\widetilde{\nu_\varepsilon}^j$ or $\tilde{\mu}^i$ share eigenvalues with subprojections of Z_G.

2. To find the partial isometries $j(p', h)$ in $\mathcal{A}_{g,h\rho}$, we can simply multiply p' by elements of a basis for $\mathcal{A}_{g,h\rho}$ until we find something non-zero. Since p' is minimal and $\mathcal{A}_{h\rho}$ is Abelian, any non-zero element in $p'\mathcal{A}_{g,h\rho}$ can be rescaled to a partial isometry.

3. If we use (2.5) to compute the S-matrix, we can take advantage of the fact that the expression

$$\psi(S_0(\cdot), \cdot)$$

is bilinear to simplify the calculation in the case that the $\mathbf{t}$-eigenvalues for the $\tilde{\mu}^i$ are multiplicity free. In this case, we can write each $\tilde{\mu}^i$ as a linear combination of powers of $\mathbf{t}$. Then we can find the corresponding values of the S-matrix by first computing

$$\psi(S_0(\mathbf{t}^m), \mathbf{t}^n)$$

for various m and n, and then taking an appropriate linear combination of those values. The advantage is that the tube algebra calculations now take place with simpler numbers, and the more complicated coefficients are only introduced at the last step.

4.2. Examples with $|G_2| = 2$

Example 4.2. For $G = \mathbb{Z}_2$ there is a unique generalized Haagerup category $\mathcal{C}$, which is the even part of the A_7 subfactor (or the quantum group category PSU(2) at level 6). The structure constants (ϵ, A) for this category are given in [19, Section 9.1].

Then $G = G_2$, $(G \times \widehat{G})_*$ is empty, $|I| = |I_+| = 2$, and $\mathcal{Z}(\mathcal{C})$ has rank 14. The T-eigenvalues for the six objects $(\widetilde{\nu_+}^0, \widetilde{\nu_+}^1, \widetilde{\nu_-}^0, \widetilde{\nu_-}^1, \tilde{\mu}^0, \tilde{\mu}^1)$ are given by

$$\left(i, -i, i, -i, \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}\right),$$

and the corresponding block of the S-matrix is

$$\frac{1}{8}\begin{pmatrix} -2+\sqrt{2} & 2+\sqrt{2} & 2+\sqrt{2} & -2+\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} \\ 2+\sqrt{2} & -2+\sqrt{2} & -2+\sqrt{2} & 2+\sqrt{2} & 2\sqrt{2} & -2\sqrt{2} \\ 2+\sqrt{2} & -2+\sqrt{2} & -2+\sqrt{2} & 2+\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} \\ -2+\sqrt{2} & 2+\sqrt{2} & 2+\sqrt{2} & -2+\sqrt{2} & 2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & 2\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} & 0 & 0 \\ 2\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 \end{pmatrix}.$$

Let a_0 be the non-trivial element of G, which we also use to label the corresponding invertible object in $\mathcal{Z}(\mathcal{C})$. Since $\epsilon_{a_0}(a_0) = -1$, a_0 is a fermion (which means its twist is -1), and $(\mathcal{Z}(\mathcal{C}), k_0)$ is a spin-modular category in the sense of [5].

The trivial component $\mathcal{Z}(\mathcal{C})_0$ with respect to the $\mathbb{Z}_2$-grading associated to a_0 is the supermodular tensor subcategory generated by the objects k, $k\tilde{\pi}$, and $\widetilde{\nu_\epsilon}^j$. Then $\mathcal{Z}(\mathcal{C})$ is a modular closure of $\mathcal{Z}(\mathcal{C})_0$, and by [23, Theorem 5.4] there are exactly 16 different modular closures of $\mathcal{Z}(\mathcal{C})_0$. The modular data for 8 of these can be computed by the zesting formula in [7, Theorem 3.15].

Example 4.3. For $G = \mathbb{Z}_4$, there is a unique generalized Haagerup category satisfying (2.24). The structure constants (ϵ, A) for this category are given in [19, Section 9.3]. In this case $|I_+| = |I_-| = 4$. We index the $\tilde{\mu}^i$ by $\{+,-\} \times \{1,2\} \times \{-1,1\}$ (with the sign corresponding to $I_\pm$).

We can compute the missing corner of the modular data following the outline in Section 4.1. Some of the $\mathbf{t}$-eigenvalues for the $\tilde{\mu}^i$ have multiplicity, so it is necessary to split the corresponding eigenprojections by brute force as explained above.

Then the 12×12 block corresponding to the

$$(\widetilde{\nu_+}, \widetilde{\nu_-}, \widetilde{\mu_+}, \widetilde{\mu_-})$$

is as follows. The eigenvalues of the T-matrix are

$$(i, -i, i, -i, \zeta_5^2, \zeta_5^2, \zeta_5^{-2}, \zeta_5^{-2}, i\zeta_5^2, -i\zeta_5^2, i\zeta_5^{-2}, -i\zeta_5^{-2}),$$

where $\zeta_r^n = e^{\frac{2\pi i n}{r}}$, and the corresponding block of the S-matrix is

$$\begin{pmatrix}
c_3 & c_2 & c_1 & c_4 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
c_2 & c_3 & c_4 & c_1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
c_1 & c_4 & c_3 & c_2 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
c_4 & c_1 & c_2 & c_3 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & -1 & -1 & -1 & c_3 & c_3 & c_1 & c_1 & c_3 & c_3 & c_1 & c_1 \\
1 & 1 & 1 & 1 & c_3 & c_3 & c_1 & c_1 & c_2 & c_2 & c_4 & c_4 \\
-1 & -1 & -1 & -1 & c_1 & c_1 & c_3 & c_3 & c_1 & c_1 & c_3 & c_3 \\
1 & 1 & 1 & 1 & c_1 & c_1 & c_3 & c_3 & c_4 & c_4 & c_2 & c_2 \\
1 & -1 & 1 & -1 & c_3 & c_2 & c_1 & c_4 & c_2 & c_3 & c_4 & c_1 \\
-1 & 1 & -1 & 1 & c_3 & c_2 & c_1 & c_4 & c_3 & c_2 & c_1 & c_4 \\
1 & -1 & 1 & -1 & c_1 & c_4 & c_3 & c_2 & c_4 & c_1 & c_2 & c_3 \\
-1 & 1 & -1 & 1 & c_1 & c_4 & c_3 & c_2 & c_1 & c_4 & c_3 & c_2
\end{pmatrix}$$

where $c_k = 2\cos\frac{k\pi}{5} \in \frac{1}{2}\{\pm 1 \pm \sqrt{5}\}$.

This information can be summarized by Table 2.

$\sqrt{n^2+4}\cdot\mathbf{S}$			
$\widetilde{\nu_{\varepsilon_1'}}^{\varepsilon_2'}$	$\frac{\varepsilon_2\varepsilon_2'}{2} - \frac{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}{4}\sqrt{n^2+4}$		
$\widetilde{\mu_+}^{l',\varepsilon'}$	$-\varepsilon'$	$-2\cos\frac{4\pi a l l'}{r}$	
$\widetilde{\mu_-}^{l',\varepsilon'}$	$\varepsilon_2'\varepsilon$	$-\varepsilon 2\cos\frac{4\pi a l l'}{r}$	$\varepsilon\varepsilon' 2\cos\frac{4\pi a l l'}{r}$
	$\widetilde{\nu_{\varepsilon_1}}^{\varepsilon_2}$	$\widetilde{\mu_+}^{l,\varepsilon}$	$\widetilde{\mu_-}^{l,\varepsilon}$
$\mathbf{T}$	$\varepsilon_2 i$	$\zeta_r^{al^2}$	$\varepsilon i\zeta_r^{al^2}$

Table 2. The missing corner of modular data for the generalized Haagerup category for $\mathbb{Z}_4$. Here $n = |G| = 4$, $r = (n^2+4)/4 = 5$, l ranges from 1 to $(r-1)/2 = 2$, and $a = 2$ satisfies $(\frac{a}{r}) = -1$, where $(\frac{a}{r})$ is the Jacobi symbol.

Example 4.4. For $G = \mathbb{Z}_8$, there is a generalized Haagerup category satisfying (2.24), whose data (ϵ, A) is given in the accompanying Mathematica notebook `solutions.nb`. It is too difficult to compute the modular data exactly, but we have checked numerically that the modular data appears to conform Table 2 as well (for $n = 8$).

Example 4.5. For each of $G = \mathbb{Z}_6$ and $G = \mathbb{Z}_{10}$, there are exactly two generalized Haagerup categories which satisfy (2.24). The data (ϵ, A) for these categories is given in the accompanying Mathematica notebook `solutions.nb`. We did not compute the modular data exactly, but numerical calculations led to a conjecture summarized by Table 3. In each case, the two generalized Haagerup categories for G correspond to the two different possible values of the Jacobi symbol $(\frac{a}{r})$.

For these examples, we also have a fermion in $\mathcal{Z}(\mathcal{C})$, so in each case there are 16 different modular closures of the supermodular subcategory, as above.

$\sqrt{n^2+4}\cdot\mathbf{S}$			
$\widetilde{\nu_{\varepsilon_1'}}^{\varepsilon_2'}$	$\frac{\varepsilon_2\varepsilon_2'}{2} - \frac{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}{4}\sqrt{n^2+4}$		
$\widetilde{\mu_+}^{l',s'}$	$(-1)^{s'+1}$	$(-1)^{ss'+1}2\cos\frac{4\pi a l l'}{r}$	
$\widetilde{\mu_-}^{m',\varepsilon'}$	$-(\frac{a}{r})\varepsilon'\varepsilon_2$	$f(s)\varepsilon^s 2\cos\frac{4\pi a m' l}{r}$	$-(\frac{a}{r})(-1)^{\frac{(1-\varepsilon)(1-\varepsilon')}{4}}2\sin\frac{4\pi a m m'}{r}$
	$\widetilde{\nu_{\varepsilon_1}}^{\varepsilon_2}$	$\widetilde{\mu_+}^{l,s}$	$\widetilde{\mu_-}^{m,\varepsilon}$
$\mathbf{T}$	$\varepsilon_2 i$	$\zeta_4^s\zeta_r^{al^2}$	$\zeta_8^{(\frac{a}{m}+2)\varepsilon}\zeta_r^{am^2}$

Table 3. Conjecture for the missing corner of modular data for the generalized Haagerup category for $\mathbb{Z}_{4m+2}$. Here $n = |G|$; $r = (n^2+4)/8$; $1 \le l \le (r-1)/2$; $0 \le s \le 3$; $0 \le m \le r-1$; $f(0) = -1$, $f(1) = -(\frac{a}{r})$, $f(2) = 1$, and $f(3) = (\frac{a}{r})$.

5. Tensor product factorization

In this section we consider a generalized Haagerup category satisfying (2.24) such that $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, the $\mathrm{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ subcategory lifts to the center by Lemma 3.2, and if the braiding on this subcategory is non-degenerate, we can apply Müger factorization.

5.1. Müger factorization

We fix a generalized Haagerup category $\mathcal{C} \subset \mathrm{End}(M)$ with a finite abelian group G satisfying (2.24). We assume that $|G_2| = 4$ (i.e., $G_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$), and the symmetric bicharacter $s(k,l) = \epsilon_k(l)\epsilon_l(k)$ on $G_2 \times G_2$ is non-degenerate. Then the subcategory of $\mathcal{Z}(\mathcal{C})$ generated by G_2, which we still denote by G_2 for simplicity, is a modular tensor category, whose modular data are $S^{G_2}_{k,l} = \frac{s(l,k)}{2}$, $T^{G_2}_{k,k} = \epsilon_k(k)$. Let

$$\mathcal{Z}(\mathcal{C}) \cap G_2' = \{X \in \mathcal{Z}(\mathcal{C})\colon \mathcal{E}(k,X) = \mathcal{E}(X,k)^{-1} \text{ for all } k \in G_2\},$$

where $\mathcal{E}$ is the braiding of $\mathcal{Z}(\mathcal{C})$. Then Müger's factorization theorem [30, Theorem 4.2]) says that this is a modular tensor category with

$$\mathcal{Z}(\mathcal{C}) \cong G_2 \boxtimes (\mathcal{Z}(\mathcal{C}) \cap G_2').$$

We denote by (S', T') the modular data of $\mathcal{Z}(\mathcal{C}) \cap G_2'$. Then we have the tensor product factorization $S = S^{G_2} \otimes S'$, $T = T^{G_2} \otimes T'$.

A simple object $X \in \mathcal{Z}(\mathcal{C})$ belongs to $\mathrm{Irr}(\mathcal{Z}(\mathcal{C}) \cap G_2')$ if and only if $s(k,X) = 1$ for any $k \in G_2$. For $X, Y \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}) \cap G_2')$, we have $S'_{X,Y} = 2S_{X,Y}$, $T'_{X,X} = T_{X,X}$.

Let

$$J_1 = \{(k,\chi) \in G_2 \times ((\hat{G})_2 \setminus \{0\})\colon s(l,k)\langle l,\chi\rangle = 1 \text{ for all } l \in G_2\},$$
$$J_2 = \{(g,\tau) \in (G \times \hat{G})_*\colon \langle k,\tau\rangle\epsilon_k(g) = 1 \text{ for all } k \in G_2\},$$
$$I_0 = \{i \in I\colon s(k,\tilde{\mu}^i) = 1 \text{ for all } k \in G_2\}.$$

Then Theorem 3.10 implies

$$\mathrm{Irr}(\mathcal{Z}(\mathcal{C}) \cap G_2') = \{0,\tilde{\pi}\} \cup \{\widetilde{k\varphi_{\chi,\varepsilon}}\}_{(k,\chi)\in J_1,\varepsilon\in\{1,-1\}} \cup \{\widetilde{\sigma_g}^{\tau}\}_{(g,\tau)\in J_2} \cup \{\tilde{\mu}^i\}_{i\in I_0}.$$

Since $|I| = 4|I_0|$, we have $|I_0| = |G|^2/8$.

The modular data (S,T) are determined by (S',T'), and the latter have been already decided except for the $I_0 \times I_0$ entries. In concrete examples, we can often compute them by diagonalizing the multiplication operators of $U_g(k)$ and $\mathbf{t}_{g\rho}$ on $\mathcal{A}_{g\rho}$.

5.2. When $G_e = \mathbb{Z}_2 \times \mathbb{Z}_2$

In this section, we assume that $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times G_o$ with odd G_o, and that the symmetric bicharacter $s(k, l)$ on $G_2 \times G_2$ is non-degenerate. In this case, we can easily identify the index sets J_1 and J_2 as follows.

We identify $\widehat{G}$ with $(\widehat{G})_2 \times \widehat{G_o}$. Then for each $l \in G_2$, there exists a unique $\chi_l \in (\widehat{G})_2$ satisfying $s(k,l) = \langle k, \chi_l \rangle$ for any $k \in G_2$. We set $\Phi_{k,\varepsilon} = k\widetilde{\varphi_{\chi_k,\varepsilon}}$ for $k \in G_2 \setminus \{0\}$.

For the sets G_* and $\widehat{G}_*$, we may assume that there exist subsets $G_{o*} \subset G_o$ and $\widehat{G}_{o*} \subset \widehat{G}_o$ satisfying $G_* = (G_2 \times G_{o*})$ and $\widehat{G}_* = (\widehat{G}_2 \times \widehat{G}_{o*})$. Let $\widetilde{G} = G_2 \times G_o \times \widehat{G}_o$, and let

$$\widetilde{G}_* = (G_2 \times \{0\} \times \widehat{G}_{0*}) \sqcup (G_2 \times G_{o*} \times \widehat{G}_o) \subset G_2 \times G_o \times \widehat{G}_o,$$

which satisfies

$$\widetilde{G} = \widetilde{G}_2 \sqcup \widetilde{G}_* \sqcup -\widetilde{G}_*.$$

We identify G_2 with $\widetilde{G}_2$.

For $(k, h, \tau) \in \widetilde{G}_*$, we set

$$\Sigma_{k,h,\tau} = \widetilde{\sigma_{(k,h)}}^{(\chi_k + \epsilon_k, \tau)}.$$

We set

$$a' = 2a = 1/2|G_o| = 1/\sqrt{|\widetilde{G}|}, \quad b' = 2b = 1/\sqrt{4|G_o|^2 + 1}.$$

Theorem 5.1. *Let the notation be as above.The set*

$$\{0, \tilde{\pi}\} \cup \{\Phi_{k,\varepsilon}\}_{k \in \widetilde{G}_2 \setminus \{0\}, \varepsilon \in \{1,-1\}} \cup \{\Sigma_{k,h,\tau}\}_{(k,h,\tau) \in \widetilde{G}_*} \cup \{\tilde{\mu}^i\}_{i \in I_0}$$

exhausts all simple objects in $\mathcal{Z}(\mathcal{C}) \cap G_2'$. Except for $\tilde{\mu}^i$-$\tilde{\mu}^{i'}$ entries, the modular data of $\mathcal{Z}(\mathcal{C}) \cap G_2'$ are given by

$$S'_{0,0} = S^{\tilde{\mathcal{C}}_1}_{\tilde{\pi},\tilde{\pi}} = \frac{a' - b'}{2}, \quad S'_{0,\tilde{\pi}} = \frac{a' + b'}{2},$$

$$S'_{0,\Phi_{k,\varepsilon}} = S'_{\tilde{\pi},\Phi_{k,\varepsilon}} = \frac{a'}{2}, \quad S'_{0,\Sigma_{k,h,\tau}} = S'_{\tilde{\pi},\Sigma_{k,h,\tau}} = a',$$

$$S'_{0,\tilde{\mu}^j} = b', \quad S'_{\tilde{\pi},\tilde{\mu}^j} = -b',$$

$$S'_{\Phi_{k,\varepsilon},\Phi_{k',\varepsilon'}} = s(k,k')\frac{a' + \varepsilon\varepsilon'\delta_{k,k'}}{2}, \quad S'_{\Phi_{k,\varepsilon},\Sigma_{l,h,\tau}} = s(k,l)a', \quad S'_{\Phi_{k,\varepsilon},\tilde{\mu}^j} = 0,$$

$$S'_{\Sigma_{k,h,\tau},\Sigma_{k',h',\tau'}} = s(k,k')(\langle h, \tau' \rangle \langle h', \tau \rangle + \overline{\langle h, \tau' \rangle \langle h', \tau \rangle})a', \quad S'_{\Sigma_{k,h,\tau},\tilde{\mu}^j} = 0,$$

$$T'_{0,0} = T'_{\tilde{\pi},\tilde{\pi}} = 1, \quad T'_{\Phi_{k,\varepsilon},\Phi_{k,\varepsilon}} = \epsilon_k(k), \quad T'_{\Sigma_{k,h,\tau},\Sigma_{k,h,\tau}} = \epsilon_k(k)\langle h, \tau \rangle.$$

The data in Theorem 5.1 are summarized in Table 4.

S					
0	$\frac{a'-b'}{2}$				
$\tilde{\pi}$	$\frac{a'+b'}{2}$	$\frac{a'-b'}{2}$			
$\Phi_{k',\varepsilon'}$	$\frac{a'}{2}$	$\frac{a'}{2}$	$s(k,k')\frac{a'+\varepsilon\varepsilon'\delta_{k,k'}}{2}$		
$\Sigma_{l',h',\tau'}$	a'	a'	$s(k,l')a'$	$s(l,l')(\langle h,\tau'\rangle\langle h,\tau'\rangle+\overline{\langle h,\tau'\rangle\langle h,\tau'\rangle})a'$	
$\tilde{\mu}^{i'}$	b'	$-b'$	0	0	?
	0	$\tilde{\pi}$	$\Phi_{k,\varepsilon}$	$\Sigma_{l,h,\tau}$	$\tilde{\mu}^i$
T	1	1	$\epsilon_k(k)$	$\epsilon_l(l)(\langle h,\tau\rangle$	?

Table 4. Modular data for the commutant of G_2 for $G_e = G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, with the entries labeled by "?" undetermined.

Example 5.2. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. There is a unique generalized Haagerup category for G. The structure constants (ϵ, A) are given in [19, Section 9.4]. Looking at ϵ, we find that

$$S^{\mathbb{Z}_2\times\mathbb{Z}_2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

and

$$T^{\mathbb{Z}_2\times\mathbb{Z}_2} = \mathrm{Diag}(1,-1,-1,-1).$$

Since we have $(S^{\mathbb{Z}_2\times\mathbb{Z}_2})^2 = I$ and $(S^{\mathbb{Z}_2\times\mathbb{Z}_2}T^{\mathbb{Z}_2\times\mathbb{Z}_2})^3 = -I$, the modular group relation for S' and T' takes the form $S'^2 = I$, $(S'T')^3 = -I$.

We can compute the $I_0 \times I_0$ entries from the tube algebra, following the outline in Section 4.1. We have $|I_0| = 2$. The two eigenvalues for $\tilde{\mu}^i$ are $\zeta_5^{\pm 1}$, and the corresponding block of the S-matrix is

$$\frac{1}{10}\begin{pmatrix} 5+\sqrt{5} & -5+\sqrt{5} \\ -5+\sqrt{5} & 5+\sqrt{5} \end{pmatrix}.$$

It was pointed out to us by Marcel Bischoff that this example is related to a simple current extension of $\mathrm{SU}(5)_5$, see [36, Section 3.3].

5.3. When $G = \mathbb{Z}_{2m} \times \mathbb{Z}_2$ with $\epsilon_{(km,l)}(i,j) = (-1)^{kj}$

In this section we assume that $G = \mathbb{Z}_{2m} \times \mathbb{Z}_2$ with $m \geq 2$ and $\epsilon_{(km,l)}(i,j) = (-1)^{kj}$. In this case, we have

$$G_2 = \{(0,0),(m,0),(0,1),(m,1)\},$$

$$s((mi,j),(mi',j')) = \epsilon_{(mi,j)}((mi',j'))\epsilon_{(mi',j')}((mi,j)) = (-1)^{ij'+i'j},$$

which is non-degenerate on $G_2 \times G_2$, and

$$T_{(mi,j),(mi,j)} = (-1)^{ij}.$$

Thus, the modular data (S^{G_2}, T^{G_2}) are those of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_2})$. We will show below that m must be even if such a generalized Haagerup category $\mathcal{C}$ exists.

We can identify the two index sets J_1 and J_2 as follows. We identify $\widehat{\mathbb{Z}_n}$ with $\mathbb{Z}_n$ via $\langle j, k \rangle = \zeta_n^{jk}$ where $\zeta_n = e^{2\pi i/n}$. Let $\widetilde{G} = \mathbb{Z}_{2m}^2$, and let

$$\widetilde{G}_* = \{(i,j) \in \mathbb{Z}_{2m}^2;\ i \in \{0,m\}, 0 < j < m\} \cup \{(i,j) \in \mathbb{Z}_{2m}^2;\ 0 < i < m\},$$

which satisfies

$$\widetilde{G} = \widetilde{G}_2 \sqcup \widetilde{G}_* \sqcup -\widetilde{G}_*.$$

For $(i,j) \in \mathbb{Z}_2^2 \setminus \{0\}$, we set

$$\Phi_{(i,j),\varepsilon} = (mj, mi)\widetilde{\varphi_{(mi,j),\varepsilon}}.$$

For $(i,j) \in \widetilde{G}_*$, we set

$$\Sigma_{(i,j)} = \tilde{\sigma}_{(i,j)}^{(j,0)}.$$

Let

$$a' = 2a = \frac{1}{2m} = \frac{1}{\sqrt{|\widetilde{G}|}},$$

$$b' = 2b = \frac{1}{\sqrt{4m^2+1}}.$$

Theorem 5.3. *With the above assumptions the natural number m is always even. The following set exhausts the simple objects of $\mathcal{Z}(\widetilde{\mathcal{C}} \cap G_2')$:*

$$\{0,\ \tilde{\pi}\} \cup \{\Phi_{(i,j,\varepsilon)}\}_{(i,j)\in\mathbb{Z}_2^2\setminus\{0\},\varepsilon\in\{1,-1\}} \cup \{\Sigma_{(i,j)}\}_{(i,j)\in\widetilde{G}_*} \cup \{\tilde{\mu}^i\}_{i\in I_0},$$

and they are all self-conjugate. Except for $\tilde{\mu}^i$-$\tilde{\mu}^{i'}$ entries, the modular data are given as

$$S'_{0,0} = S'_{\tilde{\pi},\tilde{\pi}} = \frac{a'-b'}{2}, \quad S'_{0,\tilde{\pi}} = \frac{a'+b'}{2}, \quad S'_{0,\Phi_{(i,j),\varepsilon}} = S'_{\tilde{\pi},\Phi_{(i,j),\varepsilon}} = \frac{a'}{2},$$

$$S'_{0,\Sigma_{(i,j)}} = S'_{\tilde{\pi},\Sigma_{(i,j)}} = a', \quad S'_{0,\tilde{\mu}^i} = b', \quad S'_{\tilde{\pi},\tilde{\mu}^j} = -b',$$

$$S'_{\Phi_{(i,j),\varepsilon},\Phi_{(i',j'),\varepsilon'}} = \frac{a' + \delta_{i,i'}\delta_{j,j'}\varepsilon\varepsilon'}{2},$$

$$S'_{\Phi_{(i,j),\varepsilon},\Sigma_{(i',j')}} = (-1)^{ii'+jj'}a',$$

$$S'_{\Phi_{(i,j),\varepsilon},\tilde{\mu}^k} = 0,$$

$$S'_{\Sigma_{(i,j)},\Sigma_{(i',j')}} = 2a' \cos\frac{(ij'+i'j)\pi}{m}, \quad S'_{\Sigma_{(i,j)},\tilde{\mu}^k} = 0,$$

$$T'_{0,0} = T'_{\tilde{\pi},\tilde{\pi}} = T'_{\Phi_{(i,j),\varepsilon},\Phi_{(i,j),\varepsilon}} = 1, \quad T'_{\Sigma_{(i,j)},\Sigma_{(i,j)}} = \zeta_{2m}^{ij}.$$

The data in Theorem 5.3 is summarized in Table 5.

Proof. The above formulas for the modular data follow from Theorem 3.10 except that it instead gives

$$S'_{\Phi_{(i,j),\varepsilon},\Phi_{(i',j'),\varepsilon'}} = (-1)^{m(ij'+i'j)} \frac{a' + \delta_{i,i'}\delta_{j,j'}\varepsilon\varepsilon'}{2},$$

$$T'_{\Phi_{(i,j),\varepsilon},\Phi_{(i,j),\varepsilon}} = (-1)^{mij}.$$

Note that we have $(ST)^3 = I$ because every object in $\mathcal{Z}(\mathcal{C})$ is self-conjugate thanks to Theorem 3.10. Since $(S^{G_2}T^{G_2})^3 = I$, we get $(S'T')^3 = I$. To show that m is even, we compute the $\Phi_{(i,j),\varepsilon}$-$\Phi_{(i',j'),\varepsilon'}$ entries of

$$S'T'S' = (T'S'T')^{-1} = \overline{T'S'T'}.$$

We have

$$\begin{aligned}(S'T'S')_{\Phi_{(i,j),\varepsilon},\Phi_{(i',j'),\varepsilon'}} &= \sum_x S_{\Phi_{(i,j),\varepsilon},x} T'_{x,x} S_{\Phi_{(i',j'),\varepsilon'},x} \\ &= \frac{a'^2}{4} + \frac{a'^2}{4} + \sum_{(i'',j'',\varepsilon'')\in J_1\times\{1,-1\}} (-1)^{m(ij''+ji''+i'j''+j'i''+i''j'')} \\ &\qquad \times \frac{(a' + \varepsilon\varepsilon''\delta_{i,i''}\delta_{j,j''})(a' + \varepsilon'\varepsilon''\delta_{i',i''}\delta_{j',j''})}{4}\end{aligned}$$

S					
0	$\frac{a'-b'}{2}$				
$\tilde{\pi}$	$\frac{a'+b'}{2}$	$\frac{a'-b'}{2}$			
$\Phi_{(h',j'),\varepsilon'}$	$\frac{a'}{2}$	$\frac{a'}{2}$	$\frac{a'+\delta_{(h,j),(h',j')}\varepsilon\varepsilon'}{2}$		
$\Sigma_{(k',l')}$	a'	a'	$(-1)^{hk'+jl'}a'$	$2a'\cos\frac{(kl'+k'l)\pi}{m}$	
$\tilde{\mu}^{i'}$	b'	$-b'$	0	0	?
	0	$\tilde{\pi}$	$\Phi_{(h,j),\varepsilon}$	$\Sigma_{(k,l)}$	$\tilde{\mu}^i$
T	1	1	1	ζ_{2m}^{ij}	?

Table 5. Modular data for the commutant of $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ for $G = \mathbb{Z}_{2m} \times \mathbb{Z}_2$ with $\epsilon_{(km,l)}(i,j) = (-1)^{kj}$, with the entries labeled by "?" undetermined.

$$
\begin{aligned}
&+ a'^2 \sum_{(p,q)\in J_2} (-1)^{ip+jq+i'p+j'q} \zeta_{2m}^{pq} \\
&= \frac{a'^2}{2} + \sum_{(i'',j'')\in \mathbb{Z}_2^2\setminus\{0\}} (-1)^{m(ij''+ji''+i'j''+j'i''+i''j'')} \\
&\qquad\qquad \times \frac{a'^2 + \varepsilon\varepsilon'\delta_{i,i''}\delta_{j,j''}\delta_{i',i''}\delta_{j',j''}}{2} \\
&+ \frac{a'^2}{2} \sum_{(p,q)\in \widetilde{G}\setminus\widetilde{G}_2} (-1)^{ip+jq+i'p+j'q} \zeta_{2m}^{pq} \\
&= \sum_{(i'',j'')\in \mathbb{Z}_2^2} (-1)^{m((i+i')j''+(j+j')i''+i''j'')} \\
&\qquad\qquad \times \frac{a'^2 + \varepsilon\varepsilon'\delta_{i,i''}\delta_{j,j''}\delta_{i',i''}\delta_{j',j''}}{2} \\
&+ \frac{a'^2}{2}\Big(-\sum_{(p,q)\in\widetilde{G}_2} (-1)^{(i+i')p+(j+j')q}\zeta_{2m}^{pq} \\
&\qquad + \sum_{(p,q)\in\widetilde{G}} (-1)^{(i+i')p+(j+j')q}\zeta_{2m}^{pq}\Big) \\
&= \frac{(-1)^{mij}\varepsilon\varepsilon'\delta_{i,i'}\delta_{j,j'}}{2} \\
&+ \frac{a'^2(-1)^{m(i+i')(j+j')}}{2} \sum_{(p,q)\in\widetilde{G}} \zeta_{2m}^{(p+m(j+j'))(q+m(i+i'))} \\
&= (-1)^{m(i+i')(j+j')} \frac{a' + (-1)^{mij}\varepsilon\varepsilon'\delta_{i,i'}\delta_{j,j'}}{2}.
\end{aligned}
$$

This coincides with $(\overline{T'S'T'})_{\Phi_{(i,j),\varepsilon},\Phi_{(i',j'),\varepsilon'}}$ if and only if m is even. ∎

Since $\epsilon_{(0,1)}((0,1)) = 1$, we can de-equivariantize $\mathcal{C}$ to get another fusion category $\widetilde{\mathcal{C}}_1$ realized as endomorphisms of $P = M \rtimes_{\alpha_{(0,1)}} \mathbb{Z}_2$, with implementing unitary $\lambda \in P$. Then the set

$$
\{\alpha'_{(g,0)}\}_{g\in\mathbb{Z}_{2m}} \cup \{\alpha'_{(g,0)} \circ \rho'\}_{g\in\mathbb{Z}_{2m}}
$$

exhausts all simple objects in $\widetilde{\mathcal{C}}_1$, as we have $\alpha'_{(0,1)} = \mathrm{Ad}\lambda$. We have

$$
\alpha'_{(g,0)} \circ \alpha'_{(h,0)} = \alpha'_{(g+h,0)},
$$

$$
\alpha'_{(-g,0)} \circ \rho' = \rho' \circ \alpha'_{(g,0)},
$$

$$
\rho'^2(x) = sxs^* + \sum_{g\in\mathbb{Z}_{2m}} \big(t_{(g,0)}\alpha'_{(g,0)} \circ \rho'(x) t^*_{(g,0)} + t_{(g,1)}\lambda\alpha'_{(g,0)} \circ \rho'(x)(t_{(g,1)}\lambda)^*\big).
$$

We denote by $\mathcal{P}_{4m+1}$ the C^*-algebra generated by

$$\{s\} \cup \{t_{(g,0)}\}_{g\in\mathbb{Z}_{2m}} \cup \{t_{(g,1)}\lambda\}_{g\in\mathbb{Z}_{2m}},$$

which satisfies the Cuntz algebra relations. Note that α' and ρ' globally preserve $\mathcal{P}_{4m+1}$.

Let β be the dual action of $\alpha_{(0,1)}$, which is a period 2 automorphism of P satisfying $\beta(x) = x$ for any $x \in M$, and $\beta(\lambda) = -\lambda$. Let $\gamma = \alpha'_{(m,0)} \circ \beta$. Then γ commutes with α' and ρ', and therefore induces a $\mathbb{Z}_2$-action on $\widetilde{\mathcal{C}}_1$. The equivariantization of $\widetilde{\mathcal{C}}_1$ with respect to this action is equivalent to $\mathcal{C}$.

Let $\widetilde{\mathcal{C}}_2$ the fusion category generated by γ, which is equivalent to $\mathrm{Vec}_{\mathbb{Z}_2}$, and let $\widetilde{\mathcal{C}}$ be the fusion category generated by $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$. Then $\widetilde{\mathcal{C}}$ is the crossed product category for the $\mathbb{Z}_2$-action on $\widetilde{\mathcal{C}}_1$ induced by γ. Therefore, $\widetilde{\mathcal{C}}$ is Morita equivalent to $\mathcal{C}$, and their Drinfeld centers are braided equivalent.

Theorem 5.4. *The fusion category $\widetilde{\mathcal{C}}$ is equivalent to $\widetilde{\mathcal{C}}_1 \boxtimes \widetilde{\mathcal{C}}_2$. In consequence,*

$$\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\widetilde{\mathcal{C}}_1) \boxtimes \mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_2}),$$

and the modular data of $\mathcal{Z}(\widetilde{\mathcal{C}}_1)$ are (S', T').

Proof. Every intertwiner between products of objects of objects $\alpha'_{(g,0)}$ and ρ' belongs to $\mathcal{P}_{4m+1}$, and γ acts on $\mathcal{P}_{4m+1}$ trivially. On the other hand, every intertwiner between products of γ is a scalar, and so we get the splitting

$$\widetilde{\mathcal{C}} \cong \widetilde{\mathcal{C}}_1 \boxtimes \widetilde{\mathcal{C}}_2. \qquad \blacksquare$$

As a consequence of Theorem 5.4, the modular data of the Drinfeld center of the de-equivariantization $\widetilde{\mathcal{C}}_1$ is determined by Theorem 5.3, except for the $I_0 \times I_0$ corner.

Example 5.5. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. It was shown in [14] that there is a generalized Haagerup category $\mathcal{C}$ for G with $\epsilon_{(2k,l)}(i,j) = (-1)^{kj}$ such that the even parts of the Asaeda–Haagerup subfactor are Morita equivalent to the $\mathbb{Z}_2$-de-equivariantization $\widetilde{\mathcal{C}}_1$. Therefore the modular data for the Drinfeld center of the Asaeda–Haagerup categories is given in part by the table in Table 5. To find the missing corner, we can work in the tube algebra of the generalized Haagerup category $\mathcal{C}$, or directly in the tube algebra of the de-equivariantization $\widetilde{\mathcal{C}}_1$.

We have $|I_0| = 8$, and we find that the T-eigenvalues for $\tilde{\mu}^i$ are of the form $\zeta_{17}^{3i^2}$, for $1 \le i \le 8$. Since these numbers are distinct (and also different from the T-eigenvalues for the other objects), we can compute the corresponding projections in the tube algebra as eigenprojections of $\mathbf{t}$. Then the corresponding block of the S-matrix is given by the formula

$$S_{\tilde{\mu}^i,\tilde{\mu}^{i'}} = -\frac{2}{\sqrt{17}} \cos\frac{12\pi i i'}{17}.$$

We do not know whether there exist further examples of "generalized Asaeda–Haagerup categories," i.e., generalized Haagerup categories for $\mathbb{Z}_{4m} \times \mathbb{Z}_2$ with

$$\epsilon_{(2mk,l)}(i,j) = (-1)^{kj}$$

for $m > 1$.

6. $\mathbb{Z}_2$-de-equivariantization

In this section we consider $\mathbb{Z}_2$-de-equivariantizations of generalized Haagerup categories for cyclic groups. For a generalized Haagerup category with group G, the gauge equivalence class of the cocycle $\epsilon_g(h)$ is determined by its restriction to $(g,h) \in G_2 \times G$, which is a bicharacter under the assumption (2.24), see [19]. For an even cyclic group $G = \mathbb{Z}_{2n}$, this bicharacter is determined by the value $\epsilon_n(1)$. In all known examples, we have $\epsilon_n(1) = -1$, and hence $\epsilon_n(n) = (-1)^n$. To perform de-equivariantization with respect to n as in Section 2.5.1, we require $\epsilon_n(n) = 1$. This means that n must be even, i.e., $|G|$ is divisible by 4.

Therefore, we consider a generalized Haagerup category $\mathcal{C}$ for $G = \mathbb{Z}_{4m}$ and assume $\epsilon_{2m}(g) = (-1)^g$, meaning $\alpha_{2m}(t_g) = (-1)^g t_g$. Then we may assume that

$$\alpha_h(t_0) = t_{2h}, \quad \alpha_h(t_1) = t_{1+2h} \quad \text{for } 0 \le h < 2m.$$

We extend $\rho \in \mathrm{End}(M)$ and $\alpha_g \in \mathrm{Aut}(M)$ to the crossed product

$$M \rtimes_{\alpha_{2m}} \mathbb{Z}_2 = M \vee \{\lambda\}$$

by

$$\rho'(\lambda) = \lambda, \quad \alpha'_g(\lambda) = (-1)^g \lambda.$$

We denote by $\mathcal{D}$ the fusion category generated by ρ', which is a $\mathbb{Z}_2$-de-equivariantization of $\mathcal{C}$. Note that we have $\alpha'_{2m} = \mathrm{Ad}\lambda$, and the set

$$\{\alpha'_g\}_{0 \le g < 2m} \cup \{\alpha'_g \circ \rho'\}_{0 \le g < 2m}$$

exhausts the equivalence classes of simple objects in $\mathcal{D}$. We have the decomposition

$$\rho'^2(x) = sxs^* + \sum_{g=0}^{2m-1} (t_g \alpha'_g \circ \rho'(x) t_g^* + t_{g+2m}\lambda \alpha'_g \circ \rho'(x)(t_{g+2m}\lambda)^*).$$

Let

$$d = d(\rho') = 2m + \sqrt{4m^2 + 1},$$

and let Λ' be the global dimension of $\mathcal{D}$:

$$\begin{aligned}\Lambda' &= 2m(1+d^2)\\ &= 2m(2+4md)\\ &= 4m(4m^2+1+2m\sqrt{4m^2+1}).\end{aligned}$$

Setting $a' = 1/2m$ and $b' = 1/\sqrt{4m^2+1}$, we get

$$\frac{1}{\Lambda'} = \frac{a'-b'}{2}.$$

To represent an element in the cyclic group $\mathbb{Z}_{2m}$, we always use a number $0 \le g < 2m$. However, note that indices of the Cuntz algebra generators t_g live in the cyclic group $\mathbb{Z}_{4m}$). As for Tube($\mathcal{C}$), we use the shorthand notation

$$(g\ h|1|h\ g) = (\alpha'_g\ \alpha'_h|1|\alpha'_h\ \alpha'_g).$$

We can decompose Tube($\mathcal{D}$) in a similar way as Tube($\mathcal{C}$):

$$\mathrm{Tube}(\mathcal{D}) = \mathcal{A}_G \oplus \mathcal{A}_{G,G\rho'} \oplus \mathcal{A}_{G\rho',G} \oplus \mathcal{A}_{G\rho'}$$

where a basis for $\mathcal{A}_G$ is given by

$$\begin{aligned}&\{(g\ k|1|k\ g)\}_{0\le g,k<2m} \cup \{(0\ {}_k\rho'|1|{}_k\rho'\ 0\}_{0\le k<2m}\}\\ &\quad\cup \{(g\ {}_k\rho'|\lambda|{}_k\rho'\ 2m-g\}_{0<g<2m,\ 0\le k<2m};\end{aligned}$$

a basis for $\mathcal{A}_{G,G\rho'}$ is given by

$$\{(g\ {}_k\rho'|t_{2k+g-h}|{}_k\rho'\ {}_h\rho), (g\ {}_k\rho'|t_{2k+g-h+2m}\lambda|{}_k\rho'\ {}_h\rho')\}_{0\le g,h,k<2m};$$

a basis for $\mathcal{A}_{G\rho',G}$ is given by

$$\{({}_h\rho'\ {}_k\rho'|t^*_{h-g}|{}_k\rho'\ g), ({}_h\rho'\ {}_k\rho'|t^*_{h-g+2m}\lambda|{}_k\rho'\ g)\}_{g,h,k\in G};$$

and a basis for $\mathcal{A}_{G\rho'}$ is given by

$$\begin{aligned}&\{({}_{h_1}\rho'\ {}_k\rho'|t_{k-h_2+g}t^*_{h_1-k+g}|{}_k\rho'\ {}_{h_2}\rho'),\\ &\ ({}_{h_1}\rho'\ {}_k\rho'|t_{k-h_2+g+2m}t^*_{h_1-k+g+2m}|{}_k\rho'\ {}_{h_2}\rho'),\\ &\ ({}_{h_1}\rho'\ {}_k\rho'|t_{k-h_2+g+2m}t^*_{h_1-k+g}\lambda|{}_k\rho'\ {}_{h_2}\rho'),\\ &\ ({}_{h_1}\rho'\ {}_k\rho'|t_{k-h_2+g}t^*_{h_1-k+g+2m}\lambda|{}_k\rho'\ {}_{h_2}\rho')\}_{0\le h_1,h_2,k,g<2m}\\ &\quad\cup \{({}_h\rho'\ {}_k\rho|ss^*|{}_k\rho'\ {}_{2k-h}\rho')\}_{0\le k<2m,\ 0\le h\le\min\{2k,2m-1\}}\\ &\quad\cup \{({}_h\rho'\ {}_k\rho'|ss^*\lambda|{}_k\rho'\ {}_{2k-h+2m}\rho')\}_{0\le k<2m,\ \max\{0,2k+1\}\le h<2m}\\ &\quad\cup \{({}_h\rho'\ k|1|k\ {}_{h-2k}\rho')\}_{0\le k<2m,\ \max\{0,2k\}\le h<2m}\\ &\quad\cup \{({}_h\rho'\ k|\lambda|k\ {}_{h-2k+2m}\rho')\}_{0\le k<2m,\ 0\le h\le\min\{2k,2m-1\}}.\end{aligned}$$

To compute multiplication in the tube algebra, we need to choose orthonormal bases for morphism sets of the form $(\nu, \zeta\zeta')$, where $\nu, \zeta, \zeta' \in \mathrm{Irr}(\mathcal{D})$. Note that each product $\zeta\zeta'$ is of the form α'_g, $\alpha'_g\rho'$, or $\alpha'_g\rho'^2$ for some $g \in \mathbb{Z}_{4m}$, so we can always choose our basis from the set

$$\{1, \lambda, s, s\lambda\} \cup \{t_g, t_g\lambda\}_{g\in\mathbb{Z}_{4m}}.$$

To describe the Drinfeld center and compute the modular data, we will follow a similar outline as in Section 3. Namely, we first describe the structure of $\mathcal{A}_G$ and its action on $\mathcal{A}_{G,G\rho'}$. Then we use this to describe the simple objects in $\mathcal{Z}(\mathcal{D})$ whose underlying objects in $\mathcal{D}$ contain invertible summands, as well as the parts of the corresponding half-braidings needed to compute the modular data. Finally, we deduce the remaining simple objects in $\mathcal{Z}(\mathcal{D})$ and compute the modular data, except for the corner where both indices correspond to objects without invertible simple summands.

While the overall approach is the same as for a generalized Haagerup category, working in a de-equivariantization makes some of the arguments more difficult.

6.1. The structure of $\mathcal{A}_G$ and its action on $\mathcal{A}_{G,G\rho'}$

The multiplication in $\mathcal{A}_G$ is given by

$$(g\ h|1|h\ g)(g\ k|1|k\ g) = \begin{cases} (g\ h+k|1|h+k\ g), & h+k < 2m, \\ (-1)^g(g\ h+k-2m|1|h+k-2m\ g), & h+k \geq 2m, \end{cases}$$

so the map

$$\mathbb{Z}_{2m} \ni h \mapsto \zeta_{4m}^{gh}(g\ h|1|h\ g) \in \mathcal{A}_g,$$

where $\zeta_k = e^{\frac{2\pi i}{k}}$, is a representation. Let

$$p(g,k) = \frac{1}{2m}\sum_{h=0}^{2m-1}\zeta_{4m}^{gh}\zeta_{2m}^{hk}(g\ h|1|h\ g).$$

Then $p(g,k) \in \mathcal{A}_g$ is a projection.

Note that $\mathcal{A}_{g,h} = \{0\}$ unless $g = h = 0$ or $g + h = 2m$. The latter case can be further subdivided according to whether $g = h = m$, or one of g, h is less than m. We consider these three cases separately.

6.1.1. The structure of $\mathcal{A}_0$. We have

$$(0\ h|1|h\ 0)(0\ \rho'|1|\rho'\ 0) = (0\ \rho'|1|\rho'\ 0)(0\ 2m-h|1|2m-h\ 0),$$

$$(0\ \rho'|1|\rho'\ 0)^2 = 1_{\rho'} + \sum_{h=0}^{2m-1}2(0\ h|1|h\ 0)(0\ \rho'|1|\rho'\ 0) = 1_{\rho'} + 4mp(0,0)(0\ \rho'|1|\rho'\ 0)).$$

Thus, we have

$$p(0,k)(0\ \rho'|1|\rho'\ 0) = (0\ \rho'|1|\rho'\ 0)p(0,2m-k) \text{ for } 0 < k < m,$$

and the linear span of

$$\{p(0,0),\ p(0,m),\ p(0,0)(0\ \rho'|1|\rho'\ 0),\ p(0,m)(0\ \rho'|1|\rho'\ 0)\}$$

is a subalgebra of $\mathcal{A}_0$ isomorphic to $\mathbb{C}^4$. Let

$$\begin{aligned}
z(\widetilde{\mathrm{id}}) &= \frac{1}{\Lambda'}\sum_{g\in G}((0\ g|1|g\ 0) + d(0\ {}_g\rho'|1|{}_g\rho'\ 0)) \\
&= \frac{2m}{\Lambda'}p(0,0)(1_0 + d(0\ \rho'|1|\rho'\ 0)), \\
E(0,0) &= \frac{2md}{\Lambda'}p(0,0)(d1_0 - (0\ \rho'|1|\rho'\ 0)), \\
E(0,m)_\pm &= \frac{1}{2}p(0,m)(1_0 \pm (0\ \rho'|1|\rho'\ 0)).
\end{aligned}$$

Then these are the minimal projections of the subalgebra. For $0 < k < m$, we set

$$\begin{aligned}
E(0,k)_{11} &= p(0,k), \\
E(0,k)_{22} &= p(0,2m-k), \\
E(0,k)_{12} &= p(0,k)(0\ \rho'|1|\rho'\ 0), \\
E(0,k)_{21} &= p(0,2m-k)(0\ \rho'|1|\rho'\ 0),
\end{aligned}$$

and set $\mathcal{A}_0^k = \mathrm{span}\{E(0,k)_{ij}\}_{1\le i,j,\le 2}$. Then $\mathcal{A}_0^k$ is isomorphic to the 2 by 2 matrix algebra with a system of matrix units $\{E(0,k)_{ij}\}_{1\le i,j\le 2}$. Now, we have

$$\mathcal{A}_0 = \mathbb{C}z(\widetilde{\mathrm{id}}) \oplus \mathbb{C}E(0,0) \oplus \mathbb{C}E(0,m)_+ \oplus \mathbb{C}E(0,m)_- \oplus \bigoplus_{0<k<m}\mathcal{A}_0^k.$$

6.1.2. The structure of $\mathcal{A}_{g,h}$ for $g \neq 0$. Note that $p(g,2m)$ is well defined and equal to $p(g,0)$.

Lemma 6.1. *For* $0 < g < 2m$ *and* $0 \le k \le 2m$*,*

$$p(g,k)(g\ \rho|\lambda|\rho\ 2m-g) = (g\ \rho|\lambda|\rho\ 2m-g)p(2m-g,2m-k).$$

Proof. On one hand, we have

$$\begin{aligned}
&p(g,k)(g\ \rho'|\lambda|\rho'\ 2m-g) \\
&= \frac{1}{2m}\sum_{h=0}^{2m-1}\zeta_{4m}^{gh}\zeta_{2m}^{hk}(g\ h|1|h\ g)(g\ \rho'|\lambda|\rho'\ 2m-g)
\end{aligned}$$

$$= \frac{1}{2m}\sum_{h=0}^{2m-1}\zeta_{4m}^{gh}\zeta_{2m}^{hk}(g\ {}_h\rho'|\alpha'_h(\lambda)|_h\rho'\ 2m-g)$$
$$= \frac{1}{2m}\sum_{h=0}^{2m-1}\zeta_{4m}^{gh}\zeta_{2m}^{hk}(-1)^h(g\ {}_h\rho'|\lambda|_h\rho'\ 2m-g).$$

On the other hand,

$$\begin{aligned}
&(g\ \rho'|\lambda|\rho'\ 2m-g)p(2m-g,2m-k)\\
&= \frac{1}{2m}\sum_{h=0}^{2m-1}\zeta_{4m}^{(2m-g)h}\zeta_{2m}^{h(2m-k)}(g\ \rho'|\lambda|\rho'\ 2m-g)(2m-g\ h|1|h\ 2m-g)\\
&= \frac{1}{2m}(g\ \rho'|\lambda|\rho'\ 2m-g)1_{2m-g}\\
&\quad + \frac{1}{2m}\sum_{h=1}^{2m-1}\zeta_{4m}^{(2m-g)h}\zeta_{2m}^{h(2m-k)}(g\ {}_{2m-h}\rho'|\lambda^{-1}\lambda\alpha'_g(\lambda)|_{2m-h}\rho'\ 2m-g)\\
&= \frac{1}{2m}(g\ \rho'|\lambda|\rho'\ 2m-g)\\
&\quad + \frac{1}{2m}\sum_{h=1}^{2m-1}\zeta_{4m}^{(2m-g)(2m-h)}\zeta_{2m}^{(2m-h)(2m-k)}(-1)^g(g\ {}_h\rho'|\lambda|_h\rho'\ 2m-g)\\
&= \frac{1}{2m}\sum_{h=0}^{2m-1}\zeta_{4m}^{gh}\zeta_{2m}^{hk}(-1)^h(g\ {}_h\rho'|\lambda|_h\rho'\ 2m-g),
\end{aligned}$$

which shows the statement. ■

For $g = m$, the linear span of

$$\{p(m,0),\ p(m,m),\ p(m,0)(m\ \rho'|\lambda|\rho'\ m),\ p(m,m)(m\ \rho'|\lambda|\rho'\ m)\},$$

is a commutative subalgebra of $\mathcal{A}_m$ isomorphic to $\mathbb{C}^4$. Note that we have

$$\begin{aligned}
&(m\ \rho'|\lambda|\rho'\ m)^2\\
&= (m\ 0|s^*\rho'(\lambda)\lambda\alpha_m(s)|0\ m)\\
&\quad + \sum_{h=0}^{2m-1}(m\ {}_h\rho'|t_h^*\rho'(\lambda)\lambda\alpha_m(t_h) + \lambda^{-1}t_{h+2m}^*\rho'(\lambda)\lambda\alpha'_m(t_{h+2m}\lambda)|_h\rho'\ m)\\
&= 1_m + \sum_{h=0}^{2m-1}(m\ {}_h\rho'|t_h^*\alpha_m(t_h) + \lambda^{-1}t_{h+2m}^*\alpha_m(t_{h+2m})\lambda|_h\rho'\ m)\\
&= 1_m.
\end{aligned}$$

Let

$$E(m,0)_\pm = \frac{1}{2}p(m,0)(1_m \pm (m\ \rho'|\lambda|\rho'\ m)),$$
$$E(m,m)_\pm = \frac{1}{2}p(m,m)(1_m \pm (m\ \rho'|\lambda|\rho'\ m)).$$

Then they are the minimal projections of the subalgebra. For $0 < k < m$, we set

$$\begin{aligned} E(m,k)_{11} &= p(m,k),\\ E(m,k)_{22} &= p(m,2m-k),\\ E(m,k)_{12} &= p(m,k)(m\ \rho|\lambda|\rho\ m),\\ E(m,k)_{21} &= p(m,2m-k)(m\ \rho|\lambda|\rho\ m). \end{aligned}$$

and set $\mathcal{A}_m^k = \mathrm{span}\{E(m,k)_{ij}\}_{1\le i,j,\le 2}$. Then $\mathcal{A}_m^k$ is isomorphic to the 2 by 2 matrix algebra with a system of matrix units $\{E(m,k)_{ij}\}_{1\le i,j\le 2}$. Now, we have

$$\mathcal{A}_m = \mathbb{C}E(m,0)_+ \oplus \mathbb{C}E(m,0)_- \oplus \mathbb{C}E(m,m)_+ \oplus \mathbb{C}E(m,m)_- \oplus \bigoplus_{0<k<m} \mathcal{A}_m^k.$$

Finally, for $0 < g < m$ and $0 \le k < 2m$, we set

$$\begin{aligned} E(g,k)_{11} &= p(g,k),\\ E(g,k)_{22} &= p(2m-g,2m-k),\\ E(g,k)_{12} &= p(g,k)(g\ \rho'|\lambda|\rho'\ 2m-g),\\ E(g,k)_{21} &= p(2m-g,2m-k)(2m-g\ \rho'|\lambda|\rho'\ g), \end{aligned}$$

and

$$\mathcal{B}_g^k = \mathrm{span}\{E(g,k)_{ij}\}_{1\le i,j,\le 2}.$$

Then $\mathcal{B}_g^k$ is isomorphic to the 2 by 2 matrix algebra with a system of matrix units $\{E(g,k)_{ij}\}_{1\le i,j\le 2}$. Now, we have

$$\mathcal{A}_g \oplus \mathcal{A}_{g,2m-g} \oplus \mathcal{A}_{2m-g,g} \oplus \mathcal{A}_{2m-g} = \bigoplus_{0\le k<2m} \mathcal{B}_g^k.$$

6.1.3. The action of $\mathcal{A}_G$ on $\mathcal{A}_{G,{}_G\rho'}$. We now determine the action of each simple component of each $\mathcal{A}_g$ on each $\mathcal{A}_{g,{}_h\rho'}$. Note that unlike in the case of a generalized Haagerup category, $\alpha_h\rho'$ is not necessarily multiplicity free (and in fact it is not, as we will see shortly), and therefore irreducible modules over simple components of $\mathcal{A}_g$ in $\mathcal{A}_{g,{}_h\rho'}$ can appear with multiplicity.

Note that for each $0 \le g,k,h < 2m$, the space $p(g,k)\mathcal{A}_{g,{}_h\rho'}$ is 2-dimensional, with basis

$$\{p(g,k)(g\ \rho'|t_{g-h}|\rho'\ {}_h\rho'),\ p(g,k)(g\ \rho'|t_{g-h+2m}\lambda|\rho'\ {}_h\rho')\}.$$

For each g, k such that at least one of g or k does not belong to $\{0, m\}$, the projection $p(g,k)$ is minimal in $\mathcal{A}_g$, so the irreducible modules over the corresponding simple components in each $\mathcal{A}_{g,{}_h\rho'}$ have multiplicity 2.

Similarly, since $\mathbb{C}z(\widetilde{\mathrm{id}})$ acts trivially on each $p(0,0)\mathcal{A}_{0,{}_h\rho'}$, we have that $\mathbb{C}E(0,0)$ acts as the identity on $p(0,0)\mathcal{A}_{0,{}_h\rho'}$, and therefore the irreducible $\mathbb{C}E(0,0)$-module in each $\mathcal{A}_{0,{}_h\rho'}$ has multiplicity 2.

It remains only to determine the actions of $\mathbb{C}E(0,m)_\pm$, $\mathbb{C}E(m,0)_\pm$, and $\mathbb{C}E(m,m)_\pm$.

We first look at $\mathbb{C}E(0,m)_\pm$. Note that we have

$$(p(0,m)(0\ \rho'|1|\rho'\ 0))^2 = p(0,m).$$

Lemma 6.2. *The element* $(0\ \rho'|1|\rho'\ 0)$ *acts on* $p(0,m)\mathcal{A}_{0,{}_g\rho'}$ *as multiplying by* $(-1)^g$.

Proof. Since the two elements

$$\{p(0,m)(0\ \rho'|t_{-g}|\rho'\ {}_g\rho'), p(0,m)(0\ \rho'|t_{2m-g}\lambda|\rho'\ {}_g\rho')\},$$

are exchanged, up to scalar multiple, by right multiplication of $({}_g\rho'\ m|\lambda|m\ {}_g\rho')$, it suffices to show

$$(0\ \rho'|1|\rho'\ 0)p(0,m)(0\ \rho'|t_{-g}|\rho'\ {}_g\rho') = (-1)^g(0\ \rho'|t_{-g}|\rho'\ {}_g\rho').$$

Indeed,

$$\begin{aligned}
&(0\ \rho'|1|\rho'\ 0)p(0,m)(0\ \rho'|t_{-g}|\rho'\ {}_g\rho') \\
&= p(0,m)(0\ \rho'|1|\rho'\ 0)(0\ \rho'|t_{-g}|\rho'\ {}_g\rho') \\
&= p(0,m)\sum_{h=0}^{2m-1}\big(0\ {}_h\rho'|t_h^*\rho(t_{-g})t_h + \lambda^{-1}t_{h+2m}^*\rho(t_{-g})t_{h+2m}\lambda|{}_h\rho'\ {}_g\rho'\big) \\
&= p(0,m)\sum_{h=0}^{2m-1}(0\ h|1|h\ 0)\big(0\ \rho'|\alpha_{-h}(t_h^*\rho(t_{-g})t_h \\
&\qquad\qquad + (-1)^g t_{h+2m}^*\rho(t_{-g})t_{h+2m})|\rho'\ {}_g\rho'\big) \\
&= p(0,m)\sum_{h=0}^{2m-1}(-1)^h\epsilon_h(-g)\big(0\ \rho'|t_{-h}^*\rho(t_{2h-g})t_{-h} \\
&\qquad\qquad + (-1)^g t_{-h+2m}^*\rho(t_{2h-g})t_{-h+2m}|\rho'\ {}_g\rho'\big) \\
&= p(0,m)\Big(0\ \rho'|\sum_{h=0}^{4m-1}(-1)^h\epsilon_h(-g)t_{-h}^*\rho(t_{2h-g})t_{-h}|\rho'\ {}_g\rho'\Big).
\end{aligned}$$

Here we have

$$\sum_{h=0}^{4m-1}(-1)^h\epsilon_h(-g)t_{-h}^*\rho(t_{2h-g})t_{-h}$$

$$= \sum_{h=0}^{4m-1} (-1)^h \epsilon_h(-g)\epsilon_{2h-g}(g-2h) t_{-h}^* \alpha_{g-2h}\rho(t_{g-2h}) t_{-h}$$
$$= \sum_{h=0}^{4m-1} (-1)^h \epsilon_h(-g)\epsilon_{2h-g}(g-2h) A_{g-2h}(h-g, h-g) t_{-g}.$$

Since

$$A_{g-2h}(h-g, h-g) = \epsilon_{g-h}(-h) A_{g-2h}(g-h, 0) = \delta_{g,h} - \frac{\epsilon_{g-h}(-g)}{d-1},$$

we get

$$\sum_{h=0}^{4m-1} (-1)^h \epsilon_h(-g)\epsilon_{2h-g}(g-2h) A_{g-2h}(h-g, h-g)$$
$$= (-1)^g - \frac{1}{d-1} \sum_{h=0}^{4m-1} (-1)^h \epsilon_h(-g)\epsilon_{2h-g}(g-2h)\epsilon_{g-h}(-g)$$
$$= (-1)^g - \frac{1}{d-1} \sum_{h=0}^{4m-1} (-1)^h \epsilon_h(-g)\epsilon_h(g-2h)\epsilon_{h-g}(g)\epsilon_{g-h}(-g)$$
$$= (-1)^g - \frac{\epsilon_{-g}(g)\epsilon_g(-g)}{d-1} \sum_{h=0}^{4m-1} (-1)^h$$
$$= (-1)^g,$$

which shows the statement. ∎

It follows from the lemma that $\mathbb{C}E(0,m)_+$ acts nontrivially $\mathcal{A}_{0,g\rho'}$ if and only if g is even, and $\mathbb{C}E(0,m)_-$ acts nontrivially for g odd. Since $p(0,m)\mathcal{A}_{0,g\rho'}$ is 2-dimensional, the irreducible $\mathbb{C}E(0,m)_\pm$-modules in $\mathcal{A}_{0,g\rho'}$ occur with multiplicity 2.

Finally, we consider $\mathbb{C}E(m,0)_\pm$ and $\mathbb{C}E(m,m)_\pm$.

Lemma 6.3. *Let the notation be as above:*

1. *the action of* $(m\ \rho'|\lambda|\rho'\ m)$ *on* $p(m,0)\mathcal{A}_{m,g\rho'}$ *has eigenvalues both* 1 *and* -1*;*
2. *the action of* $(m\ \rho'|\lambda|\rho'\ m)$ *on* $p(m,m)\mathcal{A}_{m,g\rho'}$ *has eigenvalues both* 1 *and* -1.

Proof. Since $(m\rho'|\lambda|\rho' m)$ acts as an invertible transformation of period 2, it suffices to show that it is not a scalar. Indeed, it is easy to show that $(m\ \rho'|\lambda|\rho'\ m)$ switches the two basis elements (up to scalar multiple). ∎

It follows from the lemma that each of $\mathbb{C}E(m,0)_\pm$ and $\mathbb{C}E(m,m)_\pm$ act nontrivially on each $\mathcal{A}_{m,g\rho'}$, and each irreducible $\mathbb{C}E(m,0)_\pm$-module and $\mathbb{C}E(m,m)_\pm$-module in each $\mathcal{A}_{m,g\rho'}$ occurs with multiplicity 1.

6.2. The simple objects in $\mathcal{Z}(\mathcal{D})$ and the modular data

Now, that we have determined the structure of $\mathcal{A}_G$ and the action of $\mathcal{A}_G$ on $\mathcal{A}_{G,G\rho'}$, we can describe all of the simple objects in $\mathcal{Z}(\mathcal{D})$ whose underlying objects in $\mathcal{D}$ contain an invertible summand, and then compute the corresponding modular data, as in Section 3.

Let

$$\pi = \mathrm{id} \oplus 2 \bigoplus_{g=0}^{2m-1} \alpha'_g \circ \rho',$$

$$\varphi_+ = \mathrm{id} \oplus 2 \bigoplus_{g=0}^{m-1} \alpha'_{2g} \circ \rho',$$

$$\varphi_- = \mathrm{id} \oplus 2 \bigoplus_{g=0}^{m-1} \alpha'_{2g+1} \circ \rho',$$

$$\psi = \alpha'_m \oplus \bigoplus_{g=0}^{2m-1} \alpha'_g \circ \rho',$$

$$\sigma_0 = \mathrm{id} \oplus \mathrm{id} \oplus 2 \bigoplus_{g=0}^{2m-1} \alpha'_g \circ \rho'.$$

For $0 < g \leq m$, let

$$\sigma_g = \alpha'_g \oplus \alpha'_{2m-g} \oplus 2 \bigoplus_{h=0}^{2m-1} \alpha'_h \circ \rho'.$$

Then these objects have half-braidings given by the central projections in Tube $\mathcal{D}$ which have nontrivial components in $\mathcal{A}_G$. We can read off the components of the half-braidings $\mathcal{E}_{(\cdot)}(\cdot)_{(g,i),(g,i)}$ from the matrix units for A_G computed above. We also have $\mathcal{E}_{(\cdot)}(0)_{g\rho',g\rho'} = 1$ and $\mathcal{E}_{(\cdot)}(h)_{g\rho',g\rho'} = 1$ for $h \neq 0, m$.

One complication due to the de-equivariantization is that we will now also need $\mathcal{E}_{\psi}^{(\varepsilon_1,\varepsilon_2)}(m)_{g\rho',g\rho'}$ for computing the modular data. In the following lemma we determine $\mathcal{E}_{\psi}^{(\varepsilon_1,\varepsilon_2)}(m)_{g\rho',g\rho'}$ in terms of a number $a^{\varepsilon_1}(g)$. This number will in turn be determined later by using properties of the overall modular data.

Lemma 6.4. 1. id *has a unique half-braiding* $\mathcal{E}_0(\xi) = 1$.

2. π *has a unique half-braiding, which gives* $e(\tilde{\pi})_{0,0} = E(0,0)$*, and*

$$\mathcal{E}_\pi(h)_{0,0} = 1, \quad \mathcal{E}_\pi({}_h\rho')_{0,0} = -\frac{1}{d^2}.$$

3. *Each of* φ_+ *and* φ_- *has a unique half-braiding, which gives*

$$e(\widetilde{\varphi_\pm})_{0,0} = E(0,m)_\pm$$

and

$$\mathcal{E}_{\varphi_\pm}(h)_{0,0} = (-1)^h, \quad \mathcal{E}_{\varphi_\pm}({}_h\rho')_{0,0} = \frac{\pm(-1)^h}{d}.$$

4. *ψ has exactly 4 half-braidings parametrized by the set*

$$\{(+,+), (+,-), (-,+), (-,-)\},$$

which gives

$$e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{m,m} = E\Big(m, \frac{1-\varepsilon_1}{2}m\Big)_{\varepsilon_2},$$

and

$$\begin{aligned}
\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(h)_{m,m} &= (\varepsilon_1 i)^h, \\
cE_\psi^{(\varepsilon_1,\varepsilon_2)}({}_h\rho')_{m,m} &= \frac{\varepsilon_2(-\varepsilon_1 i)^h}{d}\lambda, \\
\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{g\rho',g\rho'} &= (-1)^m \varepsilon_2 a^{\varepsilon_1}(g)(\varepsilon_1 i)^{m+g}\lambda.
\end{aligned}$$

Here we identify the symbols $+$ with 1 and $-$ with -1 in an appropriate way. The number $a^{\varepsilon_1}(g) \in \{1,-1\}$ satisfies $a^\varepsilon(g+2) = a^\varepsilon(g)$.

5. *σ_0 has exactly $m-1$ half-braidings parametrized by $0 < k < m$, which gives*

$$e(\widetilde{\sigma_0}^k)_{(0,s),(0,t)} = E(0,k)_{st},$$

and

$$\mathcal{E}_{\sigma_0}^k(h)_{(0,1),(0,1)} = \zeta_{2m}^{kh}, \quad \mathcal{E}_{\sigma_0}^k({}_h\rho')_{(0,s),(0,s)} = 0.$$

6. *σ_m has exactly $m-1$ half-braidings parametrized by $0 < k < m$, which give*

$$e(\widetilde{\sigma_m}^k)_{(m,s),(m,t)} = E(m,k)_{st},$$

and

$$\begin{aligned}
\mathcal{E}_{\sigma_m}^k(h)_{(m,1),(m,1)} &= i^h \zeta_{2m}^{kh}, \\
\mathcal{E}_{\sigma_m}^k({}_h\rho')_{(m,s),(m,s)} &= 0.
\end{aligned}$$

7. *For $0 < g < m$, σ_g has exactly $2m$ half-braidings parametrized by $0 \le k < 2m$, which give*

$$e(\widetilde{\sigma_g}^k)_{g,g} = p(g,k), \quad e(\widetilde{\sigma_g}^k)_{2m-g,2m-g} = p(2m-g, 2m-k),$$

and

$$\begin{aligned}
\mathcal{E}_{\sigma_g}^k(h)_{g,g} &= \zeta_{4m}^{gh}\zeta_{2m}^{kh}, \\
\mathcal{E}_{\sigma_g}^k(h)_{2m-g,2m-g} &= \zeta_{4m}^{(2m-g)h}\zeta_{2m}^{(2m-k)h}, \\
\mathcal{E}_{\sigma_g}^k({}_h\rho')_{g,g} &= 0.
\end{aligned}$$

Proof. The only statement that we have not shown yet is about $\mathcal{E}_{\psi}^{(\varepsilon_1,\varepsilon_2)}(m)_{g\rho',g\rho'}$. We first note that $i^{m+g}({}_g\rho'\ m|\lambda|m\ {}_g\rho')$ is a period two unitary in $\mathcal{A}_{g\rho'}$ satisfying

$$\begin{aligned}
&p\Big(m,\frac{1-\varepsilon}{2}m\Big)(m\ \rho'|t_{m-g}|\rho'\ {}_g\rho')i^{m+g}({}_g\rho'\ m|\lambda|m\ {}_g\rho')\\
&= i^{m+g}p\Big(m,\frac{1-\varepsilon}{2}m\Big)(m\ {}_m\rho'|\lambda^{-1}\rho'(\lambda)t_{m-g}\alpha'_m(\lambda)|{}_m\rho'\ {}_g\rho')\\
&= (-1)^m i^{m+g}p\Big(m,\frac{1-\varepsilon}{2}m\Big)(m\ {}_m\rho'|t_{m-g}\lambda|{}_m\rho'\ {}_g\rho')\\
&= (-1)^m i^{m+g}p\Big(m,\frac{1-\varepsilon}{2}m\Big)(m\ m|1|m\ m)(m\ \rho'|\alpha'^{-1}_m(t_{m-g}\lambda)|\rho'\ {}_g\rho')\\
&= (-1)^m i^{m+g}p\Big(m,\frac{1-\varepsilon}{2}m\Big)(-i)^m(-1)^{\frac{1-\varepsilon}{2}}\\
&\quad\times(m\ \rho'|(-1)^m\epsilon_{3m}(m-g)t_{3m-g}\lambda|\rho'\ {}_g\rho')\\
&= \varepsilon\epsilon_{3m}(m-g)i^g p\Big(m,\frac{1-\varepsilon}{2}m\Big)(m\ \rho'|t_{3m-g}\lambda|\rho'\ {}_g\rho').
\end{aligned}$$

Since the right multiplication of $i^{m+g}({}_g\rho'\ m|\lambda|m\ {}_g\rho')$ and the left multiplication of $(m\ \rho'|\lambda|\rho'\ m)$ on the 2-dimensional space $p(m,\frac{1-\varepsilon}{2})\mathcal{A}_{m,g\rho'}$ are commuting period two transformations that are not scalars, they coincide up to sign, and

$$\begin{aligned}
&(m\ \rho'|\lambda|\rho'\ m)p\Big(m,\frac{1-\varepsilon}{2}m\Big)(m\ \rho'|t_{m-g}|\rho'\ {}_g\rho')\\
&\quad= i^g b^{\varepsilon}(g)p\Big(m,\frac{1-\varepsilon}{2}m\Big)(m\ \rho'|t_{3m-g}\lambda|\rho'\ {}_g\rho'),
\end{aligned}$$

with $b^{\varepsilon}(g)\in\{1,-1\}$.

Since $e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{m,m}(m\ \rho'|t_{m-g}|\rho'\ {}_g\rho')$ is a multiple of a partial isometry with range projection $e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{m,m}$, there exists a positive number c satisfying

$$\begin{aligned}
&e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{g\rho',g\rho'}\\
&\quad= c(m\ \rho'|t_{m-g}|\rho'\ {}_g\rho'))^* p\Big(m,\frac{1-\varepsilon_1}{2}m\Big)\frac{1}{2}(1_m+\varepsilon_2(m\ \rho'|\lambda|\rho'\ m))\\
&\quad\quad\times(m\ \rho'|t_{g-m}|\rho'\ {}_g\rho')\\
&\quad= \frac{c\epsilon_{m-g}(g-m)}{2}({}_g\rho'\ \rho'|t^*_{g-m}|\rho'\ m)p\Big(m,\frac{1-\varepsilon_1}{2}m\Big)\\
&\quad\quad\times(m\ \rho'|t_{g-m}+\varepsilon_2 b^{\varepsilon_1}(g)i^g t_{3m-g}\lambda|\rho'\ {}_g\rho')\\
&\quad= \frac{c\epsilon_{m-g}(g-m)}{4m}\sum_{h=0}^{2m-1} i^h\varepsilon_1^h({}_g\rho'\ \rho'|t^*_{m-g}|\rho'\ m)\\
&\quad\quad\times(m\ {}_h\rho'|\alpha'_h(t_{m-g}+\varepsilon_2 b^{\varepsilon_1}(g)i^g t_{3m-g}\lambda)|{}_h\rho'\ {}_g\rho').
\end{aligned}$$

On the other hand, we have

$$e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{g\rho',g\rho'} = \frac{1}{4md}1_{g\rho'} + \frac{1}{4md}({}_g\rho'\ m|\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{g\rho',g\rho'}|m\ {}_g\rho') + \frac{1}{4m}\sum_{h=0}^{2m-1}({}_g\rho'\ {}_h\rho'|\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}({}_h\rho')_{g\rho',g\rho'}|{}_h\rho'\ {}_g\rho')),$$

and so,

$$\begin{aligned}1 &= cd\epsilon_{m-g}(g-m)s^*\rho'(t_{g-m}+\varepsilon_2 b^{\varepsilon_1}(g)i^g t_{3m-g}\lambda)t_{m-g}^*\alpha'_g\circ\rho'(s)\\ &= cds^*\alpha'_{m-g}\circ\rho'(t_{m-g})t_{m-g}^*\circ\rho'(s) = c,\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{g\rho',g\rho'} &= cd\epsilon_{m-g}(g-m)i^m\varepsilon_1^m\lambda^{-1}s^*\\ &\quad\times\rho'\big(\alpha'_m(t_{m-g}+\varepsilon_2 b^{\varepsilon_1}(g)i^g t_{3m-g}\lambda)\big)t_{g-m}^*\alpha'_g\circ\rho'(s\lambda)\\ &= d\epsilon_{m-g}(g-m)\epsilon_m(3m-g)i^m\varepsilon_1^m\varepsilon_2 b^{\varepsilon_1}(g)i^g(-1)^{m+g}\\ &\quad\times\lambda^{-1}s^*\rho'(t_{m-g})\lambda t_{g-m}^*\rho(s)\lambda\\ &= i^{m+g}\varepsilon_1^m\varepsilon_2 b^{\varepsilon_1}(g)\epsilon_m(3m-g)\lambda.\end{aligned}$$

Setting $a^\varepsilon(g) = (-1)^m b^\varepsilon(g)\varepsilon^g\epsilon_m(3m-g)$, we get

$$\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{g\rho',g\rho'} = (-1)^m(\varepsilon_1 i)^{m+g}\varepsilon_2 a^{\varepsilon_1}(g)\lambda.$$

Let $0\le g<m$. Since

$$({}_{2g}\rho'\ g|1|g\ \rho')^*({}_{2g}\rho'\ g|1|g\ \rho') = 1_{\rho'}$$

and

$$({}_{2g}\rho'\ g|1|g\ \rho')({}_{2g}\rho'\ g|1|g\ \rho')^* = 1_{2g\rho'},$$

we have

$$e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{2g\rho',2g\rho'} = ({}_{2g}\rho'\ g|1|g\ \rho')e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{\rho',\rho'}({}_{2g}\rho'\ g|1|g\ \rho')^*,$$

and so

$$\begin{aligned}&({}_{2g}\rho'\ m|\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{2g\rho',2g\rho'}|m\ {}_{2g}\rho')\\ &= ({}_{2g}\rho'\ g|1|g\ \rho')(\rho'\ m|\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{\rho',\rho'}|m\ \rho')(\rho'\ 2m-g|1|2m-g\ {}_{2g}\rho')\\ &= ({}_{2g}\rho'\ g+m|\alpha'_g(\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{\rho',\rho'})|g\ \rho')(\rho'\ 2m-g|1|2m-g\ {}_{2g}\rho')\\ &= ({}_{2g}\rho'\ m|\lambda^{-1}\alpha'_g(\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{\rho',\rho'})\alpha'_{2g}\circ\rho'(\lambda)|m\ {}_{2g}\rho')\\ &= ({}_{2g}\rho'\ m|(-1)^g\mathcal{E}_\psi^{(\varepsilon_1,\varepsilon_2)}(m)_{\rho',\rho'}|m\ {}_{2g}\rho').\end{aligned}$$

This shows $a^\varepsilon(2g) = a^\varepsilon(0)$. In the same way, we can show $a^\varepsilon(2g+1) = a^\varepsilon(1)$. ∎

In view of the above result, we introduce the following index sets:

$$J_1 = \{+, -\},$$

to index the objects $\phi_\pm$, and

$$J_2 = \{(+,+),\ (+,-),\ (-,+),\ (-,-)\},$$

to index the half-braidings of ψ.

We will now determine the remaining simple objects in $\mathcal{Z}(\mathcal{D})$.

Let $\widetilde{G} = \mathbb{Z}_{2m} \times \mathbb{Z}_{2m}$, and let

$$\widetilde{G}_* = \{(j,k) \in \widetilde{G};\ j \in \{0,m\},\ 0 < k < m\} \cup \{(g,k) \in \widetilde{G};\ 0 < g < m\}.$$

Then we have

$$\widetilde{G} = \widetilde{G}_2 \sqcup \widetilde{G}_* \sqcup -\widetilde{G}_*.$$

Lemma 6.5. *For any* $0 \leq k < 2m$*, we have*

$$\mathcal{A}_{k\rho'} = \bigoplus_{(\varepsilon_1,\varepsilon_2)\in J_2} \mathbb{C}e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{k\rho',k\rho'} \oplus M_2(\mathbb{C})^{\oplus 4m^2}.$$

Proof. We show the statement for $k = 0$ as our standing assumptions for ρ and $\alpha_k \circ \rho$ are equivalent.

Let

$$\mathbf{t}_{\rho'} = d(\rho'\ \rho'|ss^*|\rho'\ \rho'), \quad U = i^m(\rho'\ m|\lambda|m\ \rho'),$$

and

$$x_{g,h} = (\rho'\ {}_g\rho'|t_{h+g}t_{h-g}^*|_g\rho'\ \rho') \quad \text{for } 0 \leq g < 2m,\ 0 \leq h < 4m.$$

Then the set

$$\{1_{\rho'},\ U,\ \mathbf{t}_{\rho'},\ U\mathbf{t}_{\rho'}\} \cup \{x_{g,h}, Ux_{g,h}\}_{0\leq g<2m,\ 0\leq h<4m}$$

forms a basis of $\mathcal{A}_{\rho'}$, and we have $\dim \mathcal{A}_{\rho'} = 4 + 16m^2$. Note that $\mathbf{t}_{\rho'}$ is central, and U is a unitary of period two. Let $\mathcal{A}^0_{\rho'}$ be the linear span of

$$\{1_{\rho'},\ \mathbf{t}_{\rho'}\} \cup \{x_{g,h}\}_{0\leq g<2m,\ 0\leq h<4m}.$$

Then $\mathcal{A}^0_{\rho'}$ is a $*$-subalgebra of $\mathcal{A}_{\rho'}$. Since

$$\begin{aligned} &Ux_{g,h}U^{-1} \\ &= (-1)^m(\rho'\ m|\lambda|m\ \rho')(\rho'\ {}_g\rho'|t_{h+g}t_{h-g}^*|_g\rho'\ \rho')(\rho'\ m|\lambda|m\ \rho') \\ &= \begin{cases} (-1)^m(\rho'\ {}_{g+m}\rho'|\alpha_m(t_{h+g}t_{h-g}^*)\lambda|_{g+m}\rho'\ \rho')(\rho'\ m|\lambda|m\ \rho'), & 0 \leq g < m, \\ (-1)^{m+g}(\rho'\ m|\lambda|m\ \rho')(\rho'\ {}_{g-m}\rho'|\lambda t_{h+g}t_{h-g}^*|_{g-m}\rho'\ \rho'), & m \leq g < 2m, \end{cases} \end{aligned}$$

$$= (-1)^g (\rho' \ {}_g\rho' | \lambda \alpha_m (t_{h+g} t_{h-g}^*) \lambda |_g \rho' \ \rho')$$
$$= (-1)^g \epsilon_m (h+g) \epsilon_m (h-g) x_{g,h+2m},$$

we see that U normalizes $\mathcal{A}^0_{\rho'}$.

In the same way as in the proof of Lemma 3.1, we can prove that $\mathcal{A}^0_{\rho'}$ is abelian by showing that the restriction of S_0^2 to $\mathcal{A}^0_{\rho'}$ is the identity. It is easy to show $S_0^2(1_{\rho'}) = 1_{\rho'}$ and $S_0^2(\mathbf{t}_{\rho'}) = \mathbf{t}_{\rho'}$. For $(\rho' \ {}_g\rho' | x_{g,h} |_g \rho' \ \rho')$, we have

$$S_0^2((\rho' \ {}_g\rho' | x_{g,h} |_g \rho' \ \rho')) = S_0((_g\rho' \ \rho' | ds^* \alpha'_g \circ \rho'(x_{g,h} \rho'(s)) | \rho' \ {}_g\rho'))$$
$$= (\rho' \ {}_g\rho' | d^2 s^* \rho'(s^* \alpha'_g \circ \rho'(x_{g,h} \rho'(s)) \alpha'_g \circ \rho'(s)) |_g \rho' \ \rho')$$
$$= (\rho' \ {}_g\rho' | d^2 s^* \rho'(s^* \alpha'_g \circ \rho'(x_{g,h} \rho'(s) s)) |_g \rho' \ \rho'),$$

which is equal to $x_{g,h}$ thanks to the proof of Lemma 3.1. Thus, the claim is shown.

Since $\mathcal{A}^0_{\rho'}$ is abelian and normalized by U, and $\mathcal{A}_{\rho'} = \mathcal{A}^0_{\rho'} + U \mathcal{A}^0_{\rho'}$, any simple component of $\mathcal{A}_{\rho'}$ is either $\mathbb{C}$ or $M_2(\mathbb{C})$. We already known that

$$\bigoplus_{(\varepsilon_1,\varepsilon_2) \in J_2} \mathbb{C} e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{k\rho',k\rho'} \cong \mathbb{C}^4$$

is a direct summand of $\mathcal{A}_{\rho'}$. On the other hand, let

$$\mathcal{A}^{\varepsilon_1,\varepsilon_2}_{\rho'} = \{x \in \mathcal{A}_{\rho'};\ Ux = \varepsilon_1 x,\ xU = \varepsilon_2 x\} = \frac{1+\varepsilon_1 U}{2} \mathcal{A}_{\rho'} \frac{1+\varepsilon_2 U}{2}.$$

Then it is easy to show

$$\dim \mathcal{A}^{+,+}_{\rho'} = \dim \mathcal{A}^{-,-}_{\rho'} = 2 + 4m^2,$$

$$\dim \mathcal{A}^{+,-}_{\rho'} = \dim \mathcal{A}^{-,+}_{\rho'} = 4m^2,$$

which shows that $\mathcal{A}_{\rho'}$ contains $M_2(\mathbb{C})^{\oplus 4m^2}$ as a direct summand. Thus, we get the statement. ∎

We can now conclude that the remaining simple objects of $\mathcal{Z}(\mathcal{D})$ are all given by half-braidings of the object

$$\mu = 2 \bigoplus_{g=0}^{2m-1} \alpha'_g \rho'.$$

Lemma 6.6. *The object μ has exactly $2m^2$ half-braidings $\{\mathcal{E}^j_\mu\}_{j=1}^{2m^2}$, which give the remaining simple objects in $\mathcal{Z}(\mathcal{D})$.*

Proof. Let

$$z = z(\widetilde{\mathrm{id}}) + z(\tilde{\pi}) + \sum_{\varepsilon \in J_1} z(\widetilde{\varphi_\varepsilon}) + \sum_{(\varepsilon_1,\varepsilon_2) \in J_2} z(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}) + \sum_{(g,k) \in \tilde{G}_*} z(\widetilde{\sigma_g}^k),$$

which is a central projection of Tube($\mathcal{D}$). Note that we have $|\widetilde{G}_*| = 2(m^2-1)$. Thus, thanks to the previous lemma. we have

$$(1-z)\mathcal{A}_{k\rho'} \cong M_2(\mathbb{C})^{\oplus 2m^2}.$$

Since

$$({}_{2k}\rho'\ k|1|k\ \rho')^*({}_{2k}\rho'\ k|1|k\ \rho') = 1_{\rho'}$$

and

$$({}_{2k}\rho'\ k|1|k\ \rho')({}_{2k}\rho'\ k|1|k\ \rho')^* = 1_{{}_{2k}\rho'}$$

for any $0 < k < m$, we have

$$\dim(1-z)\mathcal{A}_{{}_{2k}\rho',\rho'} = \dim(1-z)\mathcal{A}_{\rho'} = 8m^2.$$

In the same way,

$$\dim(1-z)\mathcal{A}_{{}_{2k+1}\rho',{}_1\rho'} = \dim(1-z)\mathcal{A}_{{}_1\rho'} = 8m^2.$$

Direct counting shows $\dim \mathcal{A}_{{}_1\rho',\rho'} = 16m^2$. On the other hand, we can write down the basis of $z\mathcal{A}_{{}_1\rho',\rho'}$ coming from $\tilde{\pi}$, $\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}$, and $\widetilde{\sigma_g}^k$, showing $\dim z\mathcal{A}_{{}_1\rho',\rho'} = 8m^2$. Thus, we get $\dim(1-z)\mathcal{A}_{{}_1\rho',\rho'} = 8m^2$, and

$$(1-z)\mathrm{Tube}(\mathcal{D}) \cong M_{4m^2}(\mathbb{C})^{\oplus 2m^2}.$$

This shows the statement. ∎

Let

$$I = \{1, 2, \ldots, 2m^2\},$$

which we use to index the half-braidings of μ.

We would like to determine duality for the different half-braidings of ψ and μ.

Lemma 6.7. *With the above notation, we have*

$$\overline{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}} = \widetilde{\psi}^{(\varepsilon_1,(-1)^m\varepsilon_2)},$$
$$\overline{\tilde{\mu}^i} = \tilde{\mu}^i.$$

Proof. Direct computation shows

$$S_0^2(p(m,k)) = p(m, 2m-k),$$
$$S_0^2((m\ \rho'|\lambda|\rho'\ m)) = (-1)^m(m\ \rho'|\lambda|\rho'\ m),$$

which implies that

$$S_0^2(z(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})) = z(\widetilde{\psi}^{(\varepsilon_1,(-1)^m\varepsilon_2)}),$$

and the first statement.

Recall that we have

$$\mathcal{A}_{\rho'} = \bigoplus_{(\varepsilon_1,\varepsilon_2)\in J_2} \mathbb{C}e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{\rho',\rho'} \oplus \mathcal{A}^1_{\rho'}, \quad \mathcal{A}^1_{\rho'} \cong M_2(\mathbb{C})^{\oplus 4m^2}.$$

We already know

$$S_0^2(e(\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)})_{\rho,\rho}) = e(\widetilde{\psi}^{(\varepsilon_1,(-1)^m\varepsilon_2)})_{\rho,\rho}.$$

In the proof of the previous lemma, we saw that the subalgebra $\mathcal{A}^0_{\rho'}$ includes all the central projection of $\mathcal{A}^1_{\rho'}$, and S_0^2 acts on $\mathcal{A}^0_{\rho'}$ trivially. Thus, we get

$$S_0^2(z(\tilde{\mu}^i)) = z(\tilde{\mu}^i). \qquad \blacksquare$$

We can now compute the modular data for $\mathcal{Z}(\mathcal{D})$. The main difficulty is determining the numbers $a^\varepsilon(g)$, which were introduced in the formulas for $\mathcal{E}_{\psi}^{(\varepsilon_1,\varepsilon_2)}(m)_{g\rho',g\rho'}$ in Lemma 6.4, and we need to use the modular relation for this.

Theorem 6.8. *Let the notation be as above. The following set exhausts the simple objects of the Drinfeld center $\mathcal{Z}(\mathcal{D})$:*

$$\{0,\ \tilde{\pi}\} \cup \{\widetilde{\varphi_\varepsilon}\}_{\varepsilon\in J_1} \cup \{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}\}_{(\varepsilon_1,\varepsilon_2)\in J_2} \cup \{\widetilde{\sigma_g}^k\}_{(g,k)\in\tilde{G}_*} \cup \{\tilde{\mu}^i\}_{i\in I}.$$

We have $\overline{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}} = \widetilde{\psi}^{(\varepsilon_1,(-1)^m\varepsilon_2)}$, and the others are self-conjugate. Except for $\tilde{\mu}^i$-$\tilde{\mu}^{i'}$ entries, the S-matrix and T-matrix are given as

$$S_{0,0} = S_{\tilde{\pi},\tilde{\pi}} = \frac{a'-b'}{2}, \quad S_{0,\tilde{\pi}} = \frac{a'+b'}{2},$$

$$S_{0,\widetilde{\varphi_\pm}} = S_{0,\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}} = S_{\tilde{\pi},\widetilde{\varphi_\pm}} = S_{\tilde{\pi},\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}} = \frac{a'}{2},$$

$$S_{0,\widetilde{\sigma_g}^k} = S_{\tilde{\pi},\widetilde{\sigma_g}^\tau} = a', \quad S_{0,\tilde{\mu}^i} = b', \quad S_{\tilde{\pi},\tilde{\mu}^i} = -b',$$

$$S_{\widetilde{\varphi_\varepsilon},\widetilde{\varphi_{\varepsilon'}}} = \frac{a'+\varepsilon\varepsilon'}{2}, \quad S_{\widetilde{\varphi_\pm},\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}} = \frac{(-1)^m a'}{2},$$

$$S_{\widetilde{\varphi_\pm},\widetilde{\sigma_g}^k} = (-1)^g a', \quad S_{\widetilde{\varphi_\pm},\tilde{\mu}^k} = 0,$$

$$S_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\widetilde{\psi}^{(\varepsilon_1',\varepsilon_2')}} = \frac{(-\varepsilon_1\varepsilon_1')^m a' + \varepsilon_2\varepsilon_2'(\varepsilon_1 i)^m \delta_{\varepsilon_1,\varepsilon_1'}}{2},$$

$$S_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\widetilde{\sigma_g}^k} = (-\varepsilon_1)^g(-1)^k a', \quad S_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\mu}^i} = 0,$$

$$S_{\widetilde{\sigma_g}^k,\widetilde{\sigma_{g'}}^{k'}} = 2a'\cos\frac{(gg'+gk'+g'k)\pi}{m}, \quad S_{\widetilde{\sigma_g}^k,\tilde{\mu}^i} = 0,$$

$$T_{0,0} = T_{\tilde{\pi},\tilde{\pi}} = T_{\widetilde{\varphi_\pm},\widetilde{\varphi_\pm}} = 1, \quad T_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}} = (\varepsilon_1 i)^m, \quad T_{\widetilde{\sigma_g}^k,\widetilde{\sigma_g}^k} = \zeta_{4m}^{g^2+2kg}.$$

The data in Theorem 6.8 are summarized in Table 6.

S						
0	$\frac{a'-b'}{2}$					
$\tilde{\pi}$	$\frac{a'+b'}{2}$	$\frac{a'-b'}{2}$				
$\widetilde{\phi'_\varepsilon}$	$\frac{a'}{2}$	$\frac{a'}{2}$	$\frac{a'+\varepsilon\varepsilon'}{2}$			
$\tilde{\psi}^{(\varepsilon'_1,\varepsilon'_2)}$	$\frac{a'}{2}$	$\frac{a'}{2}$	$\frac{(-1)^m a'}{2}$	$\frac{(-\varepsilon_1\varepsilon'_1)ma'+\varepsilon_2\varepsilon'_2(\varepsilon_1 i)^m\delta_{\varepsilon_1,\varepsilon'_1}}{2}$		
$\widetilde{\sigma'_g}^{k'}$	a'	a'	$(-1)^{g'}$	$(-\varepsilon_1)^{g'}(-1)^{k'}a'$	$2a'\cos\left(\frac{(gg'+gk'+g'k)\pi}{m}\right)$	
$\tilde{\mu}^{i'}$	b'	$-b'$	0	0	0	?
	0	$\tilde{\pi}$	$\widetilde{\phi_\varepsilon}$	$\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)}$	$\widetilde{\sigma_g}^k$	$\tilde{\mu}^i$
T	1	1	1	$(\varepsilon_1 i)^m$	$\zeta_{4m}^{g^2+2kg}$	?

Table 6. Modular data for the $\mathbb{Z}_2$-de-equivariantization of a generalized Haagerup category for $G = \mathbb{Z}_{4m}$ with $\epsilon_{2m}(g) = (-1)^g$, with entries labeled by "?" undetermined.

Proof. The only statements that do not directly follow from the previous arguments are about $S_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\psi}^{(\varepsilon'_1,\varepsilon'_2)}}$ and $S_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\mu}^i}$. Direct computation shows

$$S_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\psi}^{(\varepsilon'_1,\varepsilon'_2)}} = \frac{(-\varepsilon_1\varepsilon'_1)^m}{4m} + \varepsilon_2\varepsilon'_2(\varepsilon'_1 i)^m \frac{a^{\varepsilon'_1}(0) + a^{\varepsilon'_1}(1)\varepsilon_1\varepsilon'_1}{4}.$$

Since S is a symmetric matrix, we have

$$S_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\psi}^{(\varepsilon'_1,\varepsilon'_2)}} = S_{\tilde{\psi}^{(\varepsilon'_1,\varepsilon'_2)},\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)}},$$

and

$$(\varepsilon'_1)^m(a^{\varepsilon'_1}(0) + a^{\varepsilon'_1}(1)\varepsilon_1\varepsilon'_1) = (\varepsilon_1)^m(a^{\varepsilon_1}(0) + a^{\varepsilon_1}(1)\varepsilon_1\varepsilon'_1).$$

This is equivalent to

$$a^+(0) - a^+(1) = (-1)^m(a^-(0) - a^-(1)).$$

Thus, either $a^+(0) = a^+(1)$, $a^-(0) = a^-(1)$, or

$$-a^+(1) = -(-1)^m a^-(1) = (-1)^m a^-(0) = a^+(0).$$

Assume $a^+(0) = a^+(1)$, $a^-(0) = a^-(1)$ first. Then we get

$$S_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\psi}^{(\varepsilon'_1,\varepsilon'_2)}} = \frac{(-\varepsilon_1\varepsilon'_1)^m}{4m} + \frac{\varepsilon_2\varepsilon'_2(\varepsilon_1 i)^m a^{\varepsilon_1}(0)\delta_{\varepsilon_1,\varepsilon'_1}}{2}.$$

Since S is a unitary,

$$\begin{aligned}
1 &= \sum_a |S_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},a}|^2 \\
&= \frac{1}{16m^2} + \frac{1}{16m^2} + \frac{1}{16m^2} \\
&\quad + \frac{1}{16m^2} + \sum_{s,t\in\{1,-1\}} \Big| \frac{(-\varepsilon_1 s)^m}{4m} + \frac{\varepsilon_2 t(\varepsilon_1 i)^m a^{\varepsilon_1}(0)\delta_{\varepsilon_1,s}}{2} \Big|^2 \\
&\quad + \frac{|\widetilde{G}_*|}{(2m)^2} + \sum_{i\in I} |S_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\widetilde{\mu}^j}|^2 \\
&= \frac{1}{2} - \frac{1}{4m^2} + 2\sum_{s\in\{1,-1\}} \Big(\frac{1}{16m^2} + \frac{\delta_{\varepsilon_1,s}}{4} \Big) + \sum_{i\in I} |S_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\widetilde{\mu}^j}|^2 \\
&= 1 + \sum_{i\in I} |S_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\widetilde{\mu}^j}|^2,
\end{aligned}$$

showing $S_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\widetilde{\mu}^j} = 0$.

Recall the modular group relation $(S)^2 = (ST)^3 = C$, $TC = CT$, where $C_{a,b} = \delta_{a,\bar{b}}$, and $\bar{b}$ is determined by $\overline{S_{a,b}} = S_{a,\bar{b}}$. Recall $\overline{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}} = \widetilde{\psi}^{(\varepsilon_1,(-1)^m\varepsilon_2)}$. We compare the $\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)}$-$\widetilde{\psi}^{(\varepsilon'_1,\varepsilon'_2)}$ entries of the both sides of

$$STS = CT^{-1}S^{-1}T^{-1} = C\overline{TST} = \overline{T}S\overline{T}.$$

On the one hand,

$$\begin{aligned}
&(STS)_{\widetilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\widetilde{\psi}^{(\varepsilon'_1,\varepsilon'_2)}} \\
&\quad = \frac{1}{16m^2} + \frac{1}{16m^2} + \frac{1}{16m^2} + \frac{1}{16m^2} \\
&\qquad + \sum_{s,t\in\{1,-1\}} \Big(\frac{(-\varepsilon_1 s)^m}{4m} + \frac{\varepsilon_2 t(\varepsilon_1 i)^m a^{\varepsilon_1}(0)\delta_{\varepsilon_1,s}}{2} \Big) \\
&\qquad\qquad \times \Big(\frac{(-\varepsilon'_1 s)^m}{4m} + \frac{\varepsilon'_2 t(\varepsilon'_1 i)^m a^{\varepsilon'_1}(0)\delta_{\varepsilon'_1,s}}{2} \Big)(si)^m \\
&\qquad + \sum_{(g,k)\in\widetilde{G}_*} \frac{(-\varepsilon_1)^g(-1)^k}{2m} \frac{(-\varepsilon'_1)^g(-1)^k}{2m} \zeta_{4m}^{g^2+2gk} \\
&\quad = \frac{1}{4m^2} + 2\sum_{s\in\{1,-1\}} \Big(\frac{(\varepsilon_1\varepsilon'_1)^m}{16m^2} + \frac{\varepsilon_2\varepsilon'_2(-\varepsilon_1\varepsilon'_1)^m a^{\varepsilon_1}(0) a^{\varepsilon'_1}(0)\delta_{\varepsilon_1,s}\delta_{\varepsilon'_1,s}}{4} \Big)(si)^m \\
&\qquad + \sum_{k=1}^{m-1} \frac{1}{4m^2} + \sum_{k=1}^{m-1} \frac{(\varepsilon_1\varepsilon'_1)^m}{4m^2} i^{m+2k} + \sum_{g=1}^{m-1}\sum_{k=0}^{2m-1} \frac{(\varepsilon_1\varepsilon'_1)^g}{4m^2}\zeta_{4m}^{g^2+2gk}
\end{aligned}$$

$$= \frac{1}{4m^2} + (\varepsilon_1\varepsilon_2 i)^m \frac{1+(-1)^m}{8m^2}$$
$$+ \frac{\varepsilon_2\varepsilon_2'(-\varepsilon_1 i)^m \delta_{\varepsilon_1,\varepsilon_1'}}{2} + \frac{m-1}{4m^2} - (\varepsilon_1\varepsilon_2 i)^m \frac{1+(-1)^m}{8m^2}$$
$$= \frac{1}{4m} + \frac{\varepsilon_2\varepsilon_2'(-\varepsilon_1 i)^m \delta_{\varepsilon_1,\varepsilon_1'}}{2}.$$

On the other hand,

$$(\bar{T}S\bar{T})_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)}} = \frac{1}{4m} + \frac{\varepsilon_2\varepsilon_2'(-\varepsilon_1 i)^m a^{\varepsilon_1}(0)\delta_{\varepsilon_1,\varepsilon_1'}}{2}.$$

Thus, we get $a^\varepsilon(0) = 1$.

Assume now that the second case

$$-a^+(1) = -(-1)^m a^-(1) = (-1)^m a^-(0) = a^+(0),$$

occurs. Then

$$S_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\psi}^{(\varepsilon_1',\varepsilon_2')}} = \frac{(-\varepsilon_1\varepsilon_1')^m}{4m} + \frac{\varepsilon_2\varepsilon_2' a^+(0) i^m \delta_{\varepsilon_1,-\varepsilon_1'}}{2}.$$

In the same way as above, we get $S_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\mu}^j} = 0$, and

$$(STS)_{\tilde{\psi}^{(\varepsilon_1,\varepsilon_2)},\tilde{\psi}^{(\varepsilon_1',\varepsilon_2')}}$$
$$= \frac{1}{16m^2} + \frac{1}{16m^2} + \frac{1}{16m^2} + \frac{1}{16m^2}$$
$$+ \sum_{s,t\in\{1,-1\}} \left(\frac{(-\varepsilon_1 s)^m}{4m} + \frac{\varepsilon_2 t i^m a^+(0)\delta_{\varepsilon_1,-s}}{2} \right)$$
$$\times \left(\frac{(-\varepsilon_1' s)^m}{4m} + \frac{\varepsilon_2' t i^m a^+(0)\delta_{\varepsilon_1',-s}}{2} \right)(si)^m$$
$$+ \sum_{(g,k)\in\tilde{G}_*} \frac{(-\varepsilon_1)^g(-1)^k}{2m} \frac{(-\varepsilon_1')^g(-1)^k}{2m} \zeta_{4m}^{g^2+2gk}$$
$$= \frac{1}{4m^2} + 2\sum_{s\in\{1,-1\}} \left(\frac{(\varepsilon_1\varepsilon_1')^m}{16m^2} + \frac{\varepsilon_2\varepsilon_2'(-1)^m \delta_{\varepsilon_1,-s}\delta_{\varepsilon_1',-s}}{4} \right)(si)^m$$
$$+ \sum_{k=1}^{m-1} \frac{1}{4m^2} + \sum_{k=1}^{m-1} \frac{(\varepsilon_1\varepsilon_1')^m}{4m^2} i^{m+2k} + \sum_{g=1}^{m-1}\sum_{k=0}^{2m-1} \frac{(\varepsilon_1\varepsilon_1')^g}{4m^2} \zeta_{4m}^{g^2+2gk}$$
$$= \frac{1}{4m^2} + (\varepsilon_1\varepsilon_2 i)^m \frac{1+(-1)^m}{8m^2} + \frac{\varepsilon_2\varepsilon_2'(\varepsilon_1 i)^m \delta_{\varepsilon_1,\varepsilon_1'}}{2}$$
$$+ \frac{m-1}{4m^2} - (\varepsilon_1\varepsilon_2 i)^m \frac{1+(-1)^m}{8m^2}$$

$$= \frac{1}{4m} + \frac{\varepsilon_2 \varepsilon_2' (\varepsilon_1 i)^m \delta_{\varepsilon_1, \varepsilon_1'}}{2}.$$

On the other hand,

$$(\bar{T} S \bar{T})_{\tilde{\psi}^{(\varepsilon_1, \varepsilon_2)}, \tilde{\psi}^{(\varepsilon_1, \varepsilon_2)}} = \frac{1}{4m} + \frac{\varepsilon_2 \varepsilon_2' i^m a^+(0) \delta_{\varepsilon_1, -\varepsilon_1'}}{2},$$

which is a contradiction. ■

6.3. Examples

To compute the missing corner in examples, we use Mathematica and the formulas for tube algebras for a de-equivariantization of a generalized Haagerup category, included in the online appendix.

Example 6.9. For $G = \mathbb{Z}_4$, let $\mathcal{C}$ be the generalized Haagerup category satisfying (2.24). Then $\epsilon_2(g) = (-1)^g$, and we have the $\mathbb{Z}_2$-de-equivariantization $\mathcal{D}$, which is the principal even part of the $2D2$ subfactor [19, 26].

We can compute the $I \times I$-corner of the modular data of the Drinfeld center by diagonalizing the action of $\mathbf{t}$ on the tube algebra, using a similar method to that outlined in Section 4.1. We have $|I_0| = 2$. The two T-eigenvalues for $\tilde{\mu}^i$ are $\zeta_5^{\pm 2}$, and the corresponding block of the S-matrix is

$$\frac{1}{10} \begin{pmatrix} -5 + \sqrt{5} & 5 + \sqrt{5} \\ 5 + \sqrt{5} & -5 + \sqrt{5} \end{pmatrix}.$$

This S-matrix looks similar to that of the commutant of G_2 in the center of the generalized Haagerup category for $\mathbb{Z}_2 \times \mathbb{Z}_2$, which also has rank 10, but with differences in several blocks.

Example 6.10. The generalized Haagerup category $\mathcal{C}$ for $\mathbb{Z}_8$ with (ϵ, A) given in the Mathematica notebook `solutions.nb` satisfies $\epsilon_4(g) = (-1)^g$. Again, we can compute the $I \times I$-corner of the modular data of the Drinfeld center of the $\mathbb{Z}_2$-de-equivariantization $\mathcal{D}$ by diagonalizing the action of $\mathbf{t}$ on the tube algebra. We have $|I| = 8$, and we find that the missing corner is the same as that of the Drinfeld center of the Asaeda–Haagerup categories: the eigenvalues of the $\tilde{\mu}^i$ are $\zeta_{17}^{3i^2}$, for $1 \le i \le 8$, with

$$S_{\tilde{\mu}^i, \tilde{\mu}^{i'}} = -\frac{2}{\sqrt{17}} \cos \frac{12\pi i i'}{17}.$$

7. $\mathbb{Z}_3$-equivariantization

In this section we compute the modular data for the Drinfeld center of the even part of the 4442 subfactor. The 4442 subfactor was first constructed in [27]. It is self dual, and its even part is a $\mathbb{Z}_3$-equivariantization of the generalized Haagerup category for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, see [19, Corollary 9.5].

The structure constants (ϵ, A) of the generalized Haagerup category $\mathcal{C}$ are given in [19, Theorem 9.4] in terms of a sign s and a fourth root of unity $\frac{z}{\sqrt{d}}$. We fix $s = 1$ and $z = \sqrt{d}$. We denote the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\{0, a, b, c\}$, in the same order as the matrices (ϵ, A).

Let θ be the automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ satisfying

$$\theta(a) = b,\ \theta(b) = c, \theta(c) = a.$$

Then ϵ and A are invariant under θ, so we can define an automorphism $\gamma(s) = s$ and $\gamma(t_g) = t_{\theta(g)}$, $g \in G$. Then the even part of the 4442 subfactor is equivalent to the equivariantization $\mathcal{C}^{\mathbb{Z}_3}$ with respect to the action generated by γ.

To describe the Drinfeld center of $\mathcal{C}^{\mathbb{Z}_3}$, it is easier to work instead with the Morita equivalent category $\mathcal{C} \rtimes \mathbb{Z}_3$, which is generated by $\mathcal{C}$ and γ. The category $\mathcal{C} \rtimes \mathbb{Z}_3$ leaves the Cuntz algebra generated by s and t_g, $g \in G$ invariant, so we can do all of our calculations in terms of this Cuntz algebra.

Note that $H = \mathrm{Inv}(\mathcal{C} \rtimes \mathbb{Z}_3) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\theta \mathbb{Z}_3$ is isomorphic to the alternating group on four letters. We denote a typical element of H by

$$(i, g) = \gamma^i \circ \alpha_g,\ i \in \{0, 1, 2\},\ g \in \{0, a, b, c\}.$$

The tube algebra of $\mathcal{C} \rtimes_\gamma \mathbb{Z}_3$ inherits the $\mathbb{Z}_3$-grading of the category, and so we look for the minimal central projections in the graded components of the tube algebra separately. For detailed calculations within the tube algebra, such as diagonalization of **t**, we use Mathematica and the tube algebra formulas for an equivariantization of a generalized Haagerup category, which are included in the online appendix.

We first look at the trivially-graded component of $\mathrm{Tube}(\mathcal{C} \rtimes_\gamma \mathbb{Z}_3)$, which contains $\mathrm{Tube}\,\mathcal{C}$. The group part $\mathcal{A}_G$ of $\mathrm{Tube}\,\mathcal{C}$ is Abelian, and following the notation of Section 3, its minimal projections are $z(k)$ and $E(k, \epsilon_k)$, for $k \in G$; and $E(k, \epsilon_l)_\pm$, for $k \neq l \in G$.

In the larger algebra $\mathrm{Tube}(\mathcal{C} \rtimes_\gamma \mathbb{Z}_3)$, we can break up the trivially-graded group part as

$$\mathcal{A}_{(0,G)} = \mathcal{A}_{(0,0)} \oplus \mathcal{A}_{(0,G\backslash\{0\})},$$

with $\dim(\mathcal{A}_{(0,0)}) = 24$ and $\dim(\mathcal{A}_{(0,G\backslash\{0\})}) = 72$.

For $g, h \neq 0 \in G$, we have that 1_g is equivalent to 1_h, since g and h are in the same conjugacy class in H. Therefore, there are 8 minimal central projections in $\mathcal{A}_{(0,G\setminus\{0\})}$ which all have rank three. They are

$$z(a)+z(b)+z(c), \quad E(a,\epsilon_a)+E(b,\epsilon_b)+E(c,\epsilon_c)$$

and

$$E(a,\epsilon_0)_\varepsilon+E(b,\epsilon_0)_\varepsilon+E(c,\epsilon_0)_\varepsilon, \quad E(a,\epsilon_b)_\varepsilon+E(b,\epsilon_c)_\varepsilon+E(c,\epsilon_a)_\varepsilon,$$

$$E(a,\epsilon_c)_\varepsilon+E(b,\epsilon_a)_\varepsilon+E(c,\epsilon_b)_\varepsilon, \quad \varepsilon\in\{\pm\}.$$

To find the center of $\mathcal{A}_{(0,0)}$, we also consider the projections

$$p^\omega = \frac{1}{3}\sum_{i=0}^{2}\omega^i((0,0)\ (i,0)|1|(i,0)\ (0,0)),$$

for ω a cube root of unity.

Then the minimal central projections of $\mathcal{A}_{(0,0)}$ are

$$p^\omega z(0) \text{ and } p^\omega E(0,0), \quad \omega\in\{1,\zeta_3,\zeta_3^{-1}\},$$

which each have rank one, and

$$E(0,\epsilon_a)_\varepsilon+E(0,\epsilon_b)_\varepsilon+E(0,\epsilon_c)_\varepsilon,\ \varepsilon\in\{\pm\},$$

which each have rank three.

Therefore, there are 16 minimal central projections in $\mathcal{A}_{(0,G)}$. To find the corresponding minimal central projections in Tube($\mathcal{C}\rtimes_\gamma \mathbb{Z}_3$), we follow a similar procedure as in Section 4. Namely, for each minimal central projection p in $\mathcal{A}_{(0,G)}$, we choose a minimal subprojection p' and a basis $\{j_s\}_{s\in S}$ of mutually orthogonal partial isometries for $p'\mathcal{A}_{(0,G),(0,G)\rho}$ (this is not difficult since for a fixed p' and $h\in G$, the space $p'\mathcal{A}_{(0,G),(0,h)\rho}$ turns out to be at most 3-dimensional). Then the corresponding minimal central projection in the tube algebra is

$$p+\sum_{s\in S} j_s^* j_s.$$

After computing the 16 minimal central projections of Tube($\mathcal{C}\rtimes_\gamma \mathbb{Z}_3$) which have nontrivial component in $\mathcal{A}_{(0,G)}$, we can list the corresponding objects in the Drinfeld center $\mathcal{Z}(\mathcal{C}\rtimes_\gamma \mathbb{Z}_3)$.

Lemma 7.1. 1. *The (identity) object*

$$(0,0)$$

and the object

$$(0,0) \oplus \bigoplus_{g \in G} {}_{(0,g)}\rho$$

each have three irreducible half-braidings.

2. *The objects*

$$\bigoplus_{0 \neq g \in G} (0,g)$$

and

$$\bigoplus_{0 \neq g \in G} \big((0,g) \oplus 3_{(0,g)}\rho\big) \oplus 3_{(0,0)}\rho$$

each have a unique irreducible half-braiding.

3. *The objects*

$$\bigoplus_{0 \neq g \in G} \big((0,g) \oplus {}_{(0,g)}\rho\big) \oplus 3_{(0,0)}\rho$$

and

$$\bigoplus_{0 \neq g \in G} \big((0,g) \oplus 2_{(0,g)}\rho\big)$$

each have three irreducible half-braidings.

4. *The objects*

$$3(0,0) \oplus 2 \bigoplus_{0 \neq g \in G} {}_{(0,g)}\rho$$

and

$$3\big((0,0) \oplus {}_{(0,0)}\rho\big) \oplus \bigoplus_{0 \neq g \in G} {}_{(0,g)}\rho$$

each have a unique irreducible half-braiding.

We can now find the remaining minimal central projections in the 0-graded component of $\mathrm{Tube}(\mathcal{C} \rtimes_\gamma \mathbb{Z}_3)$ as follows. Let $Z_{(0,G)}$ be the sum of the 16 minimal central projections with non-trivial component in $\mathcal{A}_{(0,G)}$. The dimension of $\mathcal{A}_{{}_{(0,0)}\rho}$ is 72, and for $h \neq 0$ the dimensions of $\mathcal{A}_{{}_{(0,h)}\rho}$ and $\mathcal{A}_{{}_{(0,0)}\rho, {}_{(0,h)}\rho}$ are 56 and 48, respectively. On the other hand, the dimensions of $Z_{(0,G)}\mathcal{A}_{{}_{(0,0)}\rho}$, $Z_{(0,G)}\mathcal{A}_{{}_{(0,h)}\rho}$, and $Z_{(0,G)}\mathcal{A}_{{}_{(0,0)}\rho, {}_{(0,h)}\rho}$ are 48, 32, and 24, respectively. This implies that

$$\begin{aligned}\dim((1 - Z_{(0,G)})\mathcal{A}_{{}_{(0,0)}\rho}) &= \dim((1 - Z_{(0,G)})\mathcal{A}_{{}_{(0,h)}\rho}) \\ &= \dim((1 - Z_{(0,G)})\mathcal{A}_{{}_{(0,0)}\rho, {}_{(0,h)}\rho}) = 24.\end{aligned}$$

Therefore, all of the subalgebras $(1 - Z_{(0,G)})\mathcal{A}_{{}_{(0,h)}\rho})$ are 24-dimensional, and the corresponding projections $(1 - Z_{(0,G)})1_{{}_{(0,h)}\rho}$ are equivalent in the tube algebra.

To find the minimal central projections in $(1 - Z_{(0,G)})\mathcal{A}_{(0,h)\rho}$, we diagonalize $\mathbf{t}_{(0,h)\rho}$. For each $h \in G$ the minimal polynomial of $\mathbf{t}_{(0,h)\rho}$ is

$$q(x) = (x^2 - 1)(x^2 - \zeta_5^2)(x^2 - \zeta_5^{-2}).$$

We let

$$q_\lambda(x) = \frac{q(x)}{x - \lambda} \quad \text{and} \quad p_h^\lambda = \frac{q_\lambda(\mathbf{t}_{(0,h)\rho})}{q_\lambda(\lambda)}, \quad \lambda \in \{\pm\zeta_5^{\pm 1}\}.$$

We find that each p_h^λ is a rank three projection, which is a minimal central projection in $\mathcal{A}_{(0,h)\rho}$ for $\lambda = -\zeta_5^{\pm 1}$, but splits as a sum of three minimal central projections for $\lambda = \zeta_5^{\pm 1}$. Then we can match up the minimal subprojections of the p_h^λ for $\lambda = \zeta_5^{\pm 1}$ for different h to find the corresponding minimal central projections in Tube($\mathcal{C} \rtimes_\gamma \mathbb{Z}_3$).

Lemma 7.2. 1. *The object*

$$\bigoplus_{h \in G} {}_{(0,h)}\rho$$

has 6 *irreducible half-braidings.*

2. *The object*

$$3\bigoplus_{h \in G} {}_{(0,h)}\rho$$

has 2 *irreducible half-braidings.*

Next we consider the non-trivially-graded components of the tube algebra. Let $\tau \in \{1, 2\}$. Then we have $\mathcal{A}_{(\tau,g)}$ is isomorphic to $\mathcal{A}_{(\tau,h)}$ and similarly $\mathcal{A}_{(\tau,g)\rho}$ is isomorphic to $\mathcal{A}_{(\tau,h)\rho}$ for all g and h in G, so it suffices to consider $\mathcal{A}_{(\tau,0)}$ and $\mathcal{A}_{(\tau,0)\rho}$, which have dimensions 6 and 54 respectively. The space $\mathcal{A}_{(\tau,0),(\tau,0)\rho}$ has dimension 12.

The algebra $\mathcal{A}_{(\tau,0)}$ is Abelian, and the ω-eigenspace of $\mathbf{t}_{(\tau,0)}$ is 2-dimensional for each cube root of unity ω. For each ω, let

$$r_\tau^\omega = \frac{1}{3}\sum_{i=0}^{2} \omega^i ((\tau, 0)\ (i, 0)|1|(i, 0)\ (\tau, 0))$$

and

$$s_\tau^\omega = \frac{1}{3}\sum_{i=0}^{2} \omega^i ((\tau, 0)\ {}_{(i,0)}\rho|1|{}_{(i,0)}\rho\ (\tau, 0)).$$

Then $(s_\tau^\omega)^2 = r_\tau^\omega + s_\tau^\omega$ and the six minimal projections of $\mathcal{A}_{(\tau,0)}$ are given by

$$p_\tau^{\omega,0} = \frac{5+\sqrt{5}}{10} r_\tau^\omega - \frac{1}{\sqrt{5}} s_\tau^\omega \quad \text{and} \quad p_\tau^{\omega,1} = \frac{5-\sqrt{5}}{10} r_\tau^\omega + \frac{1}{\sqrt{5}} s_\tau^\omega,$$

for the three choices of cube root of unity ω.

The $\mathbf{t}_{(\tau,0)}$-eigenvalue for each $p_\tau^{\omega,i}$ is ω. We can find the corresponding minimal central projections in $\mathrm{Tube}(\mathcal{C} \rtimes_\gamma \mathbb{Z}_3)$ by looking at the action of $\mathcal{A}_{(\tau,0)}$ on $\mathcal{A}_{(\tau,0),{}_{(\tau,0)}\rho}$, in a similar way to the 0-graded case above.

Lemma 7.3. *For each τ in $\{1,2\}$, the objects*

$$\bigoplus_{g\in G} \big((\tau,g) \oplus {}_{(\tau,g)}\rho\big) \quad \textit{and} \quad \bigoplus_{g\in G} \big((\tau,g) \oplus 3_{(\tau,g)}\rho\big)$$

each have three irreducible half-braidings.

Finally, we can determine the remaining minimal central projections in $\mathcal{A}_{(\tau,G)\rho}$ by diagonalizing $\mathbf{t}_{(\tau,0)\rho}$, in a similar way to the 0-graded case. We find that $\mathbf{t}_{(\tau,0)\rho}$ has six additional eigenvalues, which are $\{\omega\zeta_5^{\pm 1}\}$, for ω a cube root of unity. The corresponding eigenprojections all have rank two and are central.

Lemma 7.4. *For each τ in $\{1,2\}$, the object*

$$2\bigoplus_{g\in G} {}_{(\tau,g)}\rho$$

has six irreducible half-braidings.

Now, that we have found the 48 minimal central projections of $\mathrm{Tube}(\mathcal{C} \rtimes_\gamma \mathbb{Z}_3)$ and their T-eigenvalues, we can compute the S-matrix using (2.5). To display the modular data, we group the simple objects in $\mathcal{Z}(\mathcal{C} \rtimes_\gamma \mathbb{Z}_3)$ into eight blocks of sizes 6, 2, 6, 2, 6, 2, 12, and 12, respectively, corresponding to the enumerations in Lemmas 7.1–7.4. Within each block we use the following indexing convention. We factor the block size into a product of a power of two and a power of three. Then we index each factor of size three by a cube root of unity ω and each factor of size two by a sign ε (or ε_1,ε_2).

Theorem 7.5. *With notation as above and appropriate ordering within each block, the modular data for the Drinfeld center of the even part of the* 4442 *subfactor is given by the table in Table 7.*

S								
	$\alpha - \varepsilon\varepsilon'\beta$							
3	$\alpha - \varepsilon\varepsilon'\beta$	$\varepsilon\varepsilon'\beta - \alpha$						
$\frac{1}{8}$	1	-1	$-1 - 2\varepsilon\varepsilon' + 8\delta_{\varepsilon,\varepsilon'}\delta_{\omega,\omega'}$					
$\frac{1}{8}$	1	3	$-1 - 2\varepsilon\varepsilon'$	$2\varepsilon, \varepsilon' - 1$				
$\frac{1}{6\sqrt{5}}$	ε	3ε	0	0	$2\cos\frac{(2-\varepsilon\varepsilon')\pi}{5}$			
$\frac{1}{2\sqrt{5}}$	ε	$-\varepsilon$	0	0	$2\cos\frac{(2-\varepsilon\varepsilon')\pi}{5}$	$2\cos\frac{(3+\varepsilon\varepsilon')\pi}{5}$		
$\frac{1}{3\sqrt{5}}$	$2\cos\frac{(3-\varepsilon\varepsilon'_1)\pi}{10}\omega^{\varepsilon'_2}$	0	0	0	$-\varepsilon'_1\omega^{-\varepsilon'_2}$	0	$2\cos\frac{(3-\varepsilon_1\varepsilon'_1)\pi}{10}(\omega\omega')^{-\varepsilon_2\varepsilon'_2}$	
$\frac{1}{3\sqrt{5}}$	$\varepsilon\omega^{\varepsilon'_2}$	0	0	0	$2\cos\frac{(2-\varepsilon'_1\varepsilon)\pi}{5}\omega^{-\varepsilon'_2}$	0	$\varepsilon_1(\omega\omega')^{-\varepsilon'_2\varepsilon_2}$	$2\cos\frac{(3+\varepsilon_1\varepsilon'_1)\pi}{5}(\omega\omega')^{-\varepsilon_2\varepsilon'_2}$
T	1	-1	1	-1	ζ_5^{ε}	$-\zeta_5^{\varepsilon}$	ω	$\omega\zeta_5^{\varepsilon_1}$

Table 7. The modular data for the Drinfeld center of the 4442 fusion category. Here $\alpha = \frac{5}{120}$ and $\beta = \frac{2\sqrt{5}}{120}$. The eight blocks have sizes 6, 2, 6, 2, 6, 2, 12, and 12. The indexing of each block is as indicated in the text and primes are used for indices corresponding to rows. The number to the left of each row is a multiplicative factor which applies to each entry in that row.

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References

[1] N. Afzaly, S. Morrison, and D. Penneys, The classification of subfactors with index at most $5\frac{1}{4}$. 2015, arXiv:1509.00038. To appear in *Mem. Amer. Math. Soc.*

[2] M. Asaeda and U. Haagerup, Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$. *Comm. Math. Phys.* **202** (1999), no. 1, 1–63 Zbl 1014.46042 MR 1686551

[3] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*. Univ. Lect. Ser. 21, American Mathematical Society, Providence, R.I., 2001 Zbl 0965.18002 MR 1797619

[4] M. Bischoff, Y. Kawahigashi, R. Longo, and K.-H. Rehren, *Tensor categories and endomorphisms of von Neumann algebras.* With applications to quantum field theory. SpringerBriefs Math. Phys. 3, Springer, Cham, 2015 Zbl 1328.46047 MR 3308880

[5] C. Blanchet, A spin decomposition of the Verlinde formulas for type A modular categories. *Comm. Math. Phys.* **257** (2005), no. 1, 1–28 Zbl 1079.81061 MR 2163566

[6] A. Bruguières, Catégories prémodulaires, modularisations et invariants des variétés de dimension 3. *Math. Ann.* **316** (2000), no. 2, 215–236 Zbl 0943.18004 MR 1741269

[7] P. Bruillard, C. Galindo, T. Hagge, S.-H. Ng, J. Y. Plavnik, E. C. Rowell, and Z. Wang, Fermionic modular categories and the 16-fold way. *J. Math. Phys.* **58** (2017), no. 4, article id. 041704 Zbl 1365.81098 MR 3641612

[8] V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, On braided fusion categories. I. *Selecta Math. (N.S.)* **16** (2010), no. 1, 1–119 Zbl 1201.18005 MR 2609644

[9] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*. Math. Surv. Monogr. 205, American Mathematical Society, Providence, RI, 2015 Zbl 1365.18001 MR 3242743

[10] P. Etingof, D. Nikshych, and V. Ostrik, On fusion categories. *Ann. of Math. (2)* **162** (2005), no. 2, 581–642 Zbl 1125.16025 MR 2183279

[11] D. E. Evans and T. Gannon, The exoticness and realisability of twisted Haagerup–Izumi modular data. *Comm. Math. Phys.* **307** (2011), no. 2, 463–512 Zbl 1236.46055 MR 2837122

[12] M. H. Freedman, A. Kitaev, and Z. Wang, Simulation of topological field theories by quantum computers. *Comm. Math. Phys.* **227** (2002), no. 3, 587–603 Zbl 1014.81006 MR 1910832

[13] P. Grossman and M. Izumi, Infinite families of potential modular data related to quadratic categories. *Comm. Math. Phys.* **380** (2020), no. 3, 1091–1150 Zbl 1467.18039 MR 4179724

[14] P. Grossman, M. Izumi, and N. Snyder, The Asaeda–Haagerup fusion categories. *J. Reine Angew. Math.* **743** (2018), 261–305 Zbl 1405.46046 MR 3859276

[15] U. Haagerup, Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$. In *Subfactors (Kyuzeso, 1993)*, pp. 1–38, World Scientific, River Edge, N.J., 1994 Zbl 0933.46058 MR 1317352

[16] M. Izumi, The structure of sectors associated with Longo–Rehren inclusions. I. General theory. *Comm. Math. Phys.* **213** (2000), no. 1, 127–179 Zbl 1032.46529 MR 1782145

[17] M. Izumi, The structure of sectors associated with Longo–Rehren inclusions. II. Examples. *Rev. Math. Phys.* **13** (2001), no. 5, 603–674 Zbl 1033.46506 MR 1832764

[18] M. Izumi, A Cuntz algebra approach to the classification of near-group categories. In *Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday*, pp. 222–343, Proc. Centre Math. Appl. Austral. Nat. Univ. 46, Australian Natural University, Canberra, 2017 Zbl 1403.46049 MR 3635673

[19] M. Izumi, The classification of 3^n subfactors and related fusion categories. *Quantum Topol.* **9** (2018), no. 3, 473–562 Zbl 1403.46048 MR 3827808

[20] V. F. R. Jones, Actions of finite groups on the hyperfinite type II_1 factor. *Mem. Amer. Math. Soc.* **28** (1980) **237** Zbl 0454.46045 MR 587749

[21] V. F. R. Jones, Index for subfactors. *Invent. Math.* **72** (1983), no. 1, 1–25 Zbl 0508.46040 MR 696688

[22] V. F. R. Jones, S. Morrison, and N. Snyder, The classification of subfactors of index at most 5. *Bull. Amer. Math. Soc. (N.S.)* **51** (2014), no. 2, 277–327 Zbl 1301.46039 MR 3166042

[23] T. Lan, L. Kong, and X.-G. Wen, Modular extensions of unitary braided fusion categories and 2+1D topological/SPT orders with symmetries. *Comm. Math. Phys.* **351** (2017), no. 2, 709–739 Zbl 1361.81184 MR 3613518

[24] R. Longo, A duality for Hopf algebras and for subfactors. I. *Comm. Math. Phys.* **159** (1994), no. 1, 133–150 Zbl 0802.46075 MR 1257245

[25] G. Moore and N. Seiberg, Lectures on RCFT. In *Physics, geometry, and topology (Banff, AB, 1989)*, pp. 263–361, NATO Adv. Sci. Inst. Ser. B: Phys. 238, Plenum, New York, 1990 Zbl 0985.81740 MR 1153682

[26] S. Morrison and D. Penneys, 2-supertransitive subfactors at index $3 + \sqrt{5}$. *J. Funct. Anal.* **269** (2015), no. 9, 2845–2870 Zbl 1341.46034 MR 3394622

[27] S. Morrison and D. Penneys, Constructing spoke subfactors using the jellyfish algorithm. *Trans. Amer. Math. Soc.* **367** (2015), no. 5, 3257–3298 Zbl 1360.46054 MR 3314808

[28] M. Müger, Conformal field theory and Doplicher–Roberts reconstruction. In *Mathematical physics in mathematics and physics (Siena, 2000)*, pp. 297–319, Fields Inst. Commun. 30, Amer. Math. Soc., Providence, RI, 2001 Zbl 1058.81066 MR 1867563

[29] M. Müger, From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors. *J. Pure Appl. Algebra* **180** (2003), no. 1–2, 159–219 Zbl 1033.18003 MR 1966525

[30] M. Müger, On the structure of modular categories. *Proc. London Math. Soc. (3)* **87** (2003), no. 2, 291–308 Zbl 1037.18005 MR 1990929

[31] D. Nikshych, Non-group-theoretical semisimple Hopf algebras from group actions on fusion categories. *Selecta Math. (N.S.)* **14** (2008), no. 1, 145–161 Zbl 1177.16019 MR 2480712

[32] D. Penneys and E. Peters, Calculating two-strand jellyfish relations. *Pacific J. Math.* **277** (2015), no. 2, 463–510 Zbl 1339.46060 MR 3402358

[33] P. Schauenburg, The monoidal center construction and bimodules. *J. Pure Appl. Algebra* **158** (2001), no. 2–3, 325–346 Zbl 0984.18006 MR 1822847

[34] V. G. Turaev, Modular categories and 3-manifold invariants. *Internat. J. Modern Phys. B* **6** (1992), no. 11–12, 1807–1824. Zbl 0798.57002 MR 1186845

[35] E. Verlinde, Fusion rules and modular transformations in 2D conformal field theory. *Nuclear Phys. B* **300** (1988), no. 3, 360–376 Zbl 1180.81120 MR 954762

[36] F. Xu, Examples of subfactors from conformal field theory. *Comm. Math. Phys.* **357** (2018), no. 1, 61–75 Zbl 1386.81136 MR 3764563

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