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Quantum invariants of three-manifolds obtained by surgeries along torus knots

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Abstract. We study the asymptotic behavior of the Witten–Reshetikhin–Turaev invariant associated with the square of the n-th root of unity with odd n for a Seifert fibered space obtained by an integral Dehn surgery along a torus knot. We show that it can be described as a sum of the Chern–Simons invariants and the twisted Reidemeister torsions both associated with representations of the fundamental group to the two-dimensional complex special linear group.

Contents

1. Introduction

For a closed three-manifold M and an integer $r \geq 2$, we denote by

$$\tau_r(M; \exp(2\pi\sqrt{-1}/r))$$

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the Witten–Reshetikhin–Turaev quantum SU(2) invariant [54, 67]. Here, E. Witten introduced, in a physical way, the invariant by using the Chern–Simons action and the path integral, and N. Reshetikhin and V. Turaev defined it, in a mathematical way motivated by Witten's paper, by using quantum groups.

In [67, (2.23)], Witten suggests that, when $r \to \infty$, $\tau_r(M;\exp(2\pi\sqrt{-1}/r))$ splits into sums of terms of SU(2) representations of $\pi_1(M)$, and each term can be expressed in terms of the associated Chern–Simons invariant and the Reidemeister torsion. The asymptotic behavior of $\tau_r(M;\exp(2\pi\sqrt{-1}/r))$ are studied in [2,3,16,24,59]. In [16, (1.36)], D. Freed and R. Gompf gave a precise formula as a speculation and did computer calculation for the asymptotic behaviors of the invariants of some Seifert fibered three-manidolds including lens spaces. L. Jeffrey [24] confirmed the formula for lens spaces and for torus bundles over circles. L. Rozansky [59, 60] obtained the asymptotic expansion of $\tau_r(M;\exp(2\pi\sqrt{-1}/r))$ for Seifert fibered spaces $(O, g; 0 \mid 0; \alpha_1, \beta_1; \alpha_2, \beta_2, \dots, \alpha_k, \beta_k)$. See also [21].

The following conjecture is a part of the asymptotic expansion conjecture [1,2,4] by J. Andersen.

Conjecture 1 (Asymptotic expansion conjecture). *There exist constants $b_j \in \mathbb{C}$ and $d_j \in \mathbb{Q}$ such that*

$$\tau_r(M;\exp(2\pi\sqrt{-1}/r)) = \sum_{j=1}^{n} b_j e^{2\pi\sqrt{-1}rq_j} r^{d_j} + O(r^{-1})$$

for $r \to \infty$, where $0 = q_0 < q_1 < q_2 < \cdots < q_n$ are different values of the Chern–Simons invariants of M associated with representations of $\pi_1(M)$ to SU(2)*. See* [1] *for the original conjecture.*

The asymptotic expansion conjecture is proved for all finite order mapping tori in [2] (see also [4]), for three-manifolds obtained by rational Dehn surgeries along the figure-eight knot [3]. See [20] for more general Seifert fibered spaces. Note that the asymptotic expansion conjecture states that $\tau_r(M;\exp(2\pi\sqrt{-1}/r))$ grows at most polynomially when $r \to \infty$, which is true by topological quantum field theory.

For an odd integer $n \geq 3$, we can define another quantum invariant denoted by $\hat{\tau}_n(M;\exp(4\pi\sqrt{-1}/n))$. See Section 2.1 for the combinatorial definition together with that of $\tau_r(M;\exp(2\pi\sqrt{-1}/r))$. In [10], Q. Chen and T. Yang studies the asymptotic behavior of $\hat{\tau}_n(M;\exp(4\pi\sqrt{-1}/n))$ by using computer for closed three-manifolds obtained by integral Dehn surgeries along the knots 4_1 and 5_2, and proposed the following conjecture:

Conjecture 2 ([10, Conjecture 1.2]). *For a closed, hyperbolic three-manifold M, $\hat{\tau}_n(M;\exp(4\pi\sqrt{-1}/n))$ grows exponentially and*

$$4\pi\sqrt{-1}\lim_{n\to\infty}\frac{\log\hat{\tau}_n(M;\exp(4\pi\sqrt{-1}/n))}{n}=\mathrm{CS}(M)+\sqrt{-1}\,\mathrm{Vol}(M), \quad (1)$$

where $\mathrm{Vol}(M)$ *is the hyperbolic volume of M and* $\mathrm{CS}(M) := 2\pi\,\mathrm{cs}(M)$ *with* $\mathrm{cs}(M)$ *the Chern–Simons invariant of the complete hyperbolic metric of M* [9, 11].

Remark 1.1. The quantity $\mathrm{Vol}(M)+\sqrt{-1}\,\mathrm{CS}(M)$ is often called *complex volume.* See [18, Theorem 2.8].

Remark 1.2. Note that this conjecture says that $\hat{\tau}_n(M;\exp(4\pi\sqrt{-1}/n))$ grows exponentially when $n\to\infty$ if M is hyperbolic. Note also that the first author calculated the asymptotic behavior of $\tau_n(M;\exp(2\pi\sqrt{-1}/n))$ by using computer for three-manifolds obtained by integral Dehn surgeries along the figure-eight knot 4_1, and observed that, possibly because of lack of precision, it grows exponentially and (1) holds if we replace $4\pi\sqrt{-1}$ with $2\pi\sqrt{-1}$, see [44].

In [50], T. Ohtsuki proved Conjecture 2 in the case of three-manifolds obtained from the figure-eight knot by integral surgeries. He also generalised the conjecture above as follows.

Conjecture 3. *For a closed, hyperbolic three-manifold M, we have*

$$\begin{aligned}&\hat{\tau}_n(M;\exp(4\pi\sqrt{-1}/n))\\&\quad\sim(\textit{some root of unity})\times\omega(M)n^{3/2}\exp\Big(\frac{n}{4\pi\sqrt{-1}}(\mathrm{CS}(M)+\sqrt{-1}\,\mathrm{Vol}(M))\Big)\end{aligned}$$

for $n\to\infty$, where $\omega(M)$ involves the square root of the twisted Reidemeister torsion associated with the holonomy representation (see for example [53]*), and we write $f(n)\sim g(n)$ for $n\to\infty$ if* $\lim_{n\to\infty}\frac{f(n)}{g(n)}=1$.

Note that the conjecture above implies Conjecture 2. Compare this with the complexified version [46] of Kashaev's conjecture for hyperbolic knots [28] in terms of the colored Jones polynomial [45], where a knot is called *hyperbolic* if its complement $S^3\setminus K$ possesses a unique complete hyperbolic structure with finite volume.

Conjecture 4 (Complexification of Kashaev's conjecture). *Let $K\subset S^3$ be a hyperbolic knot. Then we have*

$$2\pi\lim_{n\to\infty}\frac{\log J_n(K;\exp(2\pi\sqrt{-1}/n))}{n}=\mathrm{Vol}(S^3\setminus K)+\sqrt{-1}\,\mathrm{CS}(S^3\setminus K), \quad (2)$$

where $J_n(K;q)$ is the colored Jones polynomial of K associated with the n-dimensional representation of $\mathfrak{sl}(2;\mathbb{C})$ and CS *is the Chern–Simons invariant for a knot* [39].

Remark 1.3. R. Kashaev's original conjecture is that for any *hyperbolic* knot K, $\lim_{N\to\infty}\frac{2\pi\log|\langle K\rangle_N|}{N}=\mathrm{Vol}(S^3\setminus K)$, where $\langle K\rangle_N$ is his invariant defined in [27] depending on an integer $N\geq 2$.

Conjecture 4 is proved for 4_1 by T. Ekholm, 5_2 by T. Ohtsuki [49], $6_1, 6_2, 6_3$ by Ohtsuki and Y. Yokota [52]. For general knots, J. Murakami and the first author proposed the following conjecture generalizing Kashaev's conjecture, which was also complexified later.

Conjecture 5 (Complexification of the volume conjecture). *For any knot K, we have*

$$2\pi\lim_{n\to\infty}\frac{\log J_n(K;\exp(2\pi\sqrt{-1}/n))}{n}=\mathrm{Vol}(S^3\setminus K)+\sqrt{-1}\,\mathrm{CS}(S^3\setminus K). \quad (3)$$

Here we define $\mathrm{Vol}(S^3\setminus K):=v_3\left\|S^3\setminus K\right\|$ *with v_3 the volume of the regular, ideal, hyperbolic tetrahedron and $\|S^3\setminus K\|$ the simplicial volume (or Gromov's invariant) – see* [19] *and* [63, Chapter 6] *– and* CS *is a topological Chern–Simons invariant (defined by the left-hand side), which coincides with the Chern–Simons invariant when K is hyperbolic.*

As for non-hyperbolic knots, R. Kashaev and O. Tirkkonen proved the Volume Conjecture for torus knots. Note that since the simplicial volume equals the sum of those of hyperbolic pieces of the knot complement after the torus decomposition (also known as the *Jaco–Shalen–Johannson decomposition*) [23, 25], the simplicial volume of a torus knot vanishes. See [35, 68] for other cases, that is, for non-hyperbolic knots with non-zero volume.

It would be natural to study the asymptotic behavior of $\hat{\tau}_n(M;\exp(4\pi\sqrt{-1}/n))$ in the case where M is not hyperbolic, and derive a formula similar to Conjecture 1. In this paper, we calculate it for a certain family of Seifert fibered spaces with three singular fibers.

Let $T(a,b)$ be the torus knot of type (a,b) in the three-sphere S^3 for positive coprime integers a and b. Put $X:=S^3\setminus\mathrm{Int}\,N(T(a,b))$, where $N(T(a,b))$ is the regular neighborhood of $T(a,b)$ in S^3 and Int is the interior. Note that X is a compact three-manifold with boundary ∂X a torus $S^1\times S^1$. For an integer p we denote by X_p the closed three-manifold obtained from S^3 by p-Dehn surgery.

It can be shown that X_p is the Seifert fibered three-manifold of type $S(-a/c, b/d, p-ab)$, where c and d are integers such that $ad-bc=1$ [42] (see Figure 1 for a rational surgery description for $S(r_1,r_2,r_3)$ ($r_1,r_2,r_3\in\mathbb{Q}$). Note that $S(r_1,r_2,r_3)$ is the Seifert fibered space $(O,o;0\mid 0;\alpha_1,\beta_1;\alpha_2,\beta_2,\alpha_3,\beta_3)$ with $r_i=\alpha_i/\beta_i$ ($i=1,2,3$) in Seifert's notation [61, Satz 5] (see [62] for an English translation). We give a proof in Section 2.4 because we need to carefully choose the signs of the surgery coefficients.

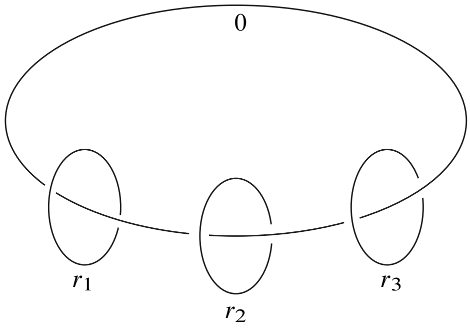

Figure 1. A surgery description for X_p, where 0, r_1, r_2, and r_3 are surgery coefficients.

Assume that n is odd, $p - ab > 0$, and $\gcd(p, ab) = 1$. Then we can express the asymptotic behavior of $\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n))$ in terms of the Chern–Simons invariants and the Reidemeister torsions associated with representations of $\pi_1(X_p)$ to $\mathrm{SL}(2;\mathbb{C})$. In fact we prove:

Main theorem (Theorem 9.8). *We have*

$$\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n)) = \frac{(-1)^{p+1} n^{3/2}}{2\pi}\big(A(n) + B(n)n^{-1/2} + O(n^{-1})\big),$$

with

$$A(n) := 2e^{\frac{n+1}{4}\pi\sqrt{-1}}\Bigg(\sum_{\substack{(h,k,l)\in\mathcal{H}_+^{\triangle} \\ h\underset{(2)}{\equiv}k\underset{(2)}{\equiv}l\underset{(2)}{\equiv}1}} - \sum_{\substack{(h,k,l)\in\mathcal{H}_+^{\nabla} \\ h\underset{(2)}{\equiv}k\underset{(2)}{\equiv}l\underset{(2)}{\equiv}0}}\Bigg)\mathcal{T}_+^{\mathrm{Irr}}(h,k,l)e^{n\mathcal{CS}_+^{\mathrm{Irr}}(h,k,l)\pi\sqrt{-1}}$$
$$+ 2e^{\frac{n+1}{4}\pi\sqrt{-1}}\Bigg(\sum_{\substack{(h,k,l)\in\mathcal{H}_-^{\triangle} \\ h\underset{(2)}{\equiv}k\underset{(2)}{\equiv}l\underset{(2)}{\equiv}1}} - \sum_{\substack{(h,k,l)\in\mathcal{H}_-^{\nabla} \\ h\underset{(2)}{\equiv}k\underset{(2)}{\equiv}l\underset{(2)}{\equiv}0}}\Bigg)\mathcal{T}_-^{\mathrm{Irr}}(h,k,l)e^{n\mathcal{CS}_-^{\mathrm{Irr}}(h,k,l)\pi\sqrt{-1}}$$

and

$$B(n) := \frac{1}{2}\sqrt{-1}(-1)^{a+b+ab}e^{n(1-p)\pi\sqrt{-1}/4}\sum_{0<l<(p-1)/2}\mathcal{T}^{\mathrm{Abel}}(l)e^{n\mathcal{CS}^{\mathrm{Abel}}(l)\pi\sqrt{-1}},$$

where $\mathcal{T}_\pm^{\mathrm{Irr}}(h,k,l)$ and $\mathcal{T}^{\mathrm{Abel}}(l)$ are related to the twisted Reidemeister torsions, and $\mathcal{CS}_\pm^{\mathrm{Irr}}(h,k,l)$ and $\mathcal{CS}^{\mathrm{Abel}}(l)$ are related to the Chern–Simons invariant as described in the following:

- $(\mathcal{T}^{\mathrm{Abel}}(l))^{-2} = \pm\,\mathrm{Tor}(X_p; \tilde{\rho}_l^{\mathrm{Abel}})$, *where*

$$\mathrm{Tor}(X_p; \tilde{\rho}_l^{\mathrm{Abel}})$$

is the homological Reidemeister torsion of X_p twisted by the Abelian representation

$$\tilde{\rho}_l^{\text{Abel}}\colon \pi_1(X_p) \to \mathrm{SL}(2;\mathbb{C});$$

- $\mathcal{CS}^{\text{Abel}}(l) = \mathrm{CS}(X_p;\tilde{\rho}_l^{\text{Abel}}) \pmod{\mathbb{Z}}$, *where*

$$\mathrm{CS}(X_p;\tilde{\rho}_l^{\text{Abel}})$$

is the Chern–Simons invariant of X_p associated with $\tilde{\rho}_l^{\text{Abel}}$;

- $(\mathcal{T}_{\pm}^{\text{Irr}}(h,k,l))^{-2} = |\operatorname{Tor}(X_p;\tilde{\rho}^{\text{Irr}}_{p-ab-h,a-k,b-l})|$, *where*

$$\operatorname{Tor}(X_p;\tilde{\rho}^{\text{Irr}}_{p-ab-h,a-k,b-l})$$

is the homological Reidemeister torsion of X_p twisted by the irreducible representation

$$\tilde{\rho}^{\text{Irr}}_{p-ab-h,a-k,b-l}\colon \pi_1(X_p) \to \mathrm{SL}(2;\mathbb{C});$$

- $\mathcal{CS}(h,k,l) \equiv \mathrm{CS}(X_p;\tilde{\rho}^{\text{Irr}}_{p-ab-h,a-k,b-l}) \pmod{2\mathbb{Z}}$, *where*

$$\mathrm{CS}(X_p;\tilde{\rho}^{\text{Irr}}_{p-ab-h,a-k,b-l})$$

is the Chern–Simons invariant of X_p associated with $\tilde{\rho}^{\text{Irr}}_{p-ab-h,a-k,b-l}$;

- $\mathcal{H}_{\pm}^{\Delta} \subset \mathcal{H}$ *and* $\mathcal{H}_{\pm}^{\nabla} \subset \mathcal{H}$ *are certain index sets, where*

$$\mathcal{H} := \{(h,k,l) \in \mathbb{Z}^3 \mid 0 < h < p-ab, 0 < k < a, 0 < l < b, h \equiv k \equiv l \pmod{2}\}.$$

Remark 1.4. Let $\mathrm{TV}_n(M)$ be the Turaev–Viro invariant of a three-manifold M, see [65]. It is known that $\mathrm{TV}_n(M) = |\hat{\tau}_n(M;\exp(4\pi\sqrt{-1}/n))|^2$ [6, 55], where we choose the parameter of $\mathrm{TV}_n(M)$ so that this equality holds.

Chen and Yang proposed the following conjecture:

Conjecture 6 ([10, Conjecture 1.1]). *For a hyperbolic three-manifold M with possibly non-empty boundary, we have*

$$2\pi \lim_{n\to\infty} \frac{\log \mathrm{TV}_n(M)}{n} = \mathrm{Vol}(M),$$

where n runs over all odd integers.

For a link complement, the following conjecture was proposed by R. Detcherry, E. Kalfagianni and Yang.

Conjecture 7 ([14, Conjecture 5.1]). *For any link L in S^3, we have*

$$2\pi \lim_{n\to\infty} \frac{\log \mathrm{TV}_n(S^3 \setminus L)}{n} = v_3 \|S^3 \setminus L\|,$$

where n runs over all odd integers.

Note that they also proved that the Turaev–Viro invariant of a link complement can be calculated as a sum of the squares of the absolute values of the colored Jones polynomials. They also proved Conjecture 7 in the case where L is a knot with volume zero.

In [13], Detcherry and Kalfagianni proposed a similar conjecture for the Turaev–Viro invariant:

Conjecture 8 ([13, Conjecture 8.1]). *For any compact orientable three-manifold M with empty or toroidal boundary, we have*

$$2\pi \limsup_{n\to\infty} \frac{\log \mathrm{TV}_n(M)}{n} = v_3 \|M\|,$$

where n runs over all odd integers.

Compare this with Conjecture 6; they replace lim with lim sup.

See [38] for computer calculations of the asymptotic behaviors of the Turaev–Viro invariants.

Finally, note that if we could show that $A(n)$ or $B(n)$ does not vanish, then Conjecture 8 holds for X_p.

For the asymptotic behavior of $\hat{\tau}_n(X_p; \exp(2\pi\sqrt{-1}/n))$ evaluated at the point $\exp(2\pi\sqrt{-1}/n)$, see [16, 24], [48, Section 7.2], and [2].

For coprime, odd integers p_1 and p_2 greater than or equal to three, put $S(-2, p_1, p_2)$ the Seifert fibered space with three singular fibers with index -2, p_1 and p_2. Then the fundamental group $\pi_1(S(-2, p_1, p_2))$ has the following presentation:

$$\begin{aligned}\pi_1(M_{p_1,p_2}) = \langle \alpha, \beta, \gamma, f \mid [\alpha, f] = [\beta, f] = [\gamma, f] = \alpha^2 f = \beta^{p_1} f = \gamma^{p_2} f \\ = \alpha\beta\gamma = 1\rangle.\end{aligned}$$

For coprime, odd integers k_1 and k_2 with $0 < k_1 < p_1$ and $0 < k_2 < p_2$, let

$$\hat{\rho}_{k_1,k_2} \colon \pi_1(S(-2, p_1, p_2)) \to \mathrm{SL}(2; \mathbb{C})$$

be an irreducible representation such that the eigenvalues of $\hat{\rho}_{k_1,k_2}(\beta)$ ($\hat{\rho}_{k_1,k_2}(\gamma)$, respectively) are $\exp(\pm\frac{k_1\pi\sqrt{-1}}{p_1})$ ($\exp(\pm\frac{k_2\pi\sqrt{-1}}{p_1})$, respectively). Then in [51, Proposition 4.17], it is shown that the conjugacy class of $\hat{\rho}_{k_1,k_2}$ is unique.

Put

$$\widehat{\mathcal{CS}}(S(-2, p_1, p_2); \hat{\rho}_{k_1,k_2}) := \frac{1}{4}\Bigl(\frac{1}{2} - \frac{k_1^2}{p_1} - \frac{k_2^2}{p_2}\Bigr)$$

and

$$\hat{\mathcal{T}}(S(-2, p_1, p_2); \hat{\rho}_{k_1,k_2}) := \frac{8\sin(\frac{k_1\pi}{p_1})\sin(\frac{\pi}{p_2})}{\sqrt{2p_1p_2}}.$$

Then Ohtsuki and Takata proved the following theorem.

Theorem 1.5 ([51, Theorem 1.3]). *For odd integers n, we have the following asymptotic expansion:*

$$\begin{aligned}&\hat{\tau}_n(S(-2,p_1,p_2);e^{4\pi\sqrt{-1}/n})\\&=\frac{(-1)^{(n-1)/2}e^{\pi\sqrt{-1}/n}e^{-n(p_1+p_2)\pi\sqrt{-1}/4}n^{3/2}}{8\pi}\\&\quad\times\Bigl(\sum_{\mathrm{SU}(2)}+2\sum_{\substack{\mathrm{SL}(2;\mathbb{R})\\ \frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}}}\Bigr)(-1)^{(k_1+k_2)/2}\hat{\mathcal{T}}(S(-2,p_1,p_2);\hat{\rho}_{k_1,k_2})\\&\quad\times e^{n\widehat{\mathcal{CS}}(S(-2,p_1,p_2);\hat{\rho}_{k_1,k_2})\pi\sqrt{-1}}+O(n).\end{aligned}$$

Here

- $\sum_{\mathrm{SU}(2)}$ *means that the summation is over all* SU(2) *representations. In this case the corresponding range is*

$$\Bigl\{(k_1,k_2)\in\mathbb{Z}^2 \Bigm| \Bigl|\frac{k_1}{p_1}-\frac{k_2}{p_2}\Bigr|<\frac{1}{2}<\frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{3}{2}, k_1\underset{(2)}{\equiv}k_2\underset{(2)}{\equiv}1\Bigr\}, \tag{4}$$

- $\sum_{\mathrm{SL}(2;\mathbb{R})}$ *means that the summation is over all* $\mathrm{SL}(2;\mathbb{R})$ *representations with* $\frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}$. *In this case the corresponding range is*

$$\Bigl\{(k_1,k_2)\in\mathbb{Z}^2 \Bigm| \frac{k_1}{p_1}+\frac{k_2}{p_2}<\frac{1}{2}, k_1\underset{(2)}{\equiv}k_2\underset{(2)}{\equiv}1\Bigr\},$$

- $\widehat{\mathcal{CS}}(S(-2,p_1,p_2);\hat{\rho}_{k_1,k_2})$ *is the Chern–Simons invariant of* $S(-2,p_1,p_2)$ *associated with* $\hat{\rho}_{k_1,k_2}$,
- $\hat{\mathcal{T}}(S(-2,p_1,p_2);\hat{\rho}_{k_1,k_2})^{-2}$ *is the homological Reidemeister torsion of* $S(-2,p_1,p_2)$ *twisted by* $\hat{\rho}_{k_1,k_2}$.

Compare this formula with ours when $a=2$. If $a=2$, then our formula can be simplified as follows (see Example 9.11):

$$\begin{aligned}&\hat{\tau}_n(X_p;e^{4\pi\sqrt{-1}/n})\\&=\frac{e^{(n+1)\pi\sqrt{-1}/4}n^{3/2}}{8\pi}\\&\quad\times\Bigl(2\sum_{\substack{(h,1,l)\in\mathcal{H}_{-}^{\triangle}\\ b-l\equiv 2\pmod 4}}+\sum_{(h,1,l)\in\mathcal{H}_{+}^{\triangle}\setminus\mathcal{H}_{-}^{\triangle}}\Bigr)\mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l)e^{n\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}}\\&\quad-\frac{\sqrt{-1}e^{n(1-p)\pi\sqrt{-1}/4}n}{4\pi}\sum_{0<l<(p-1)/2}\mathcal{T}^{\mathrm{Abel}}(l)e^{n\mathcal{CS}^{\mathrm{Abel}}(l)\pi\sqrt{-1}}+O(n^{1/2})\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{(n+1)\pi\sqrt{-1}/4} n^{3/2}}{8\pi} \\
&\quad \times \Bigl(2 \sum_{\substack{\tilde{\rho}^{\mathrm{Irr}}_{h,1,l}:\mathrm{SU}(2)\text{-representation} \\ \frac{l}{2b}+\frac{h}{p-2b}<1,\, b-l\equiv 2 \pmod 4}} + \sum_{\tilde{\rho}^{\mathrm{Irr}}_{h,1,l}:\mathrm{SU}(1,1)\text{-representation}} \Bigr) \mathcal{T}^{\mathrm{Irr}}_{+}(h,1,l) \\
&\quad \times e^{n\mathcal{CS}^{\mathrm{Irr}}_{+}(h,1,l)\pi\sqrt{-1}} \\
&\quad - \frac{\sqrt{-1}e^{n(1-p)\pi\sqrt{-1}/4} n}{4\pi} \sum_{0<l<(p-1)/2} \mathcal{T}^{\mathrm{Abel}}(l) e^{n\mathcal{CS}^{\mathrm{Abel}}(l)\pi\sqrt{-1}} + O(n^{1/2}).
\end{aligned}
$$

2. Preliminaries

In this section, we describe some basic facts that are necessary for this paper.

2.1. Quantum three-manifold invariants

We will explain how to compute the SU(2) Witten–Reshetikhin–Turaev invariant [54, 67] following [36]. See also [7] and [37, Chapter 13].

Let M be a closed, oriented three-manifold. Suppose that M is obtained from the three-sphere S^3 by Dehn surgery on a framed link presented by a link diagram $D := D_1 \cup D_2 \cup \cdots \cup D_m$. Note that the surgery coefficients are integers because they are given by the framings.

If C_i $(i = 1, 2, \dots, m)$ is a link diagram in an annulus, $\langle C_1, C_2, \dots, C_m \rangle_D$ means the Kauffman bracket [29] of the link diagram obtained by inserting C_i in the regular neighborhood of D_i in the plane, respecting the under/over crossing information. We extend the definition to linear combinations of link diagrams multilinearly.

For a non-negative integer k, let $S_k(x)$ be the k-th Chebyshev polynomial defined by $S_0(x) = 1$, $S_1(x) = x$, and

$$S_{k+1}(x) = xS_k(x) - S_{k-1}(x).$$

Denoting by α the core of an annulus, let $S_k(\alpha)$ be the linear combination of α^j determined by $S_k(x)$, where α^j means the j parallels of α. So, $S_k(\alpha)$ is a linear combination of link diagrams in the annulus. It is known that $S_k(x)$ is obtained from the k-th Jones–Wenzl idempotent [66] by closing it along the annulus. Put

$$\omega := \sum_{k=0}^{n-2} \Delta_k \times S_k(\alpha),$$

with $\Delta_k := (-1)^k \frac{A^{2(k+1)} - A^{-2(k+1)}}{A^2 - A^{-2}}$.

Let $r \geq 3$ be an integer. Suppose that either A is either a primitive $4r$-th root of unity, or A is a primitive $2r$-th root of unity and r is odd. Then

$$\frac{\langle \omega, \omega, \ldots, \omega \rangle_D}{\langle \omega \rangle_{U_+}^{b_+} \langle \omega \rangle_{U_-}^{b_-}}$$

is an invariant of M, where

- b_+ (b_-, respectively) is the number of positive (negative, respectively) eigenvalues of the linking matrix of D, where the diagonal entries are the framings and the off diagonal entries are the linking numbers [36, Theorem 5].
- $U_\pm$ is a knot diagram of the unknot with framing ± 1.

Now, we define

$$\tau_r(M; \exp(2\pi\sqrt{-1}/r)) := \left.\frac{\langle \omega, \omega, \ldots, \omega \rangle_D}{\langle \omega \rangle_{U_+}^{b_+} \langle \omega \rangle_{U_-}^{b_-}}\right|_{A=\exp\left(\frac{\pi\sqrt{-1}}{2r}\right)}$$

for any integer $r \geq 3$, and

$$\hat{\tau}_n(M; \exp(4\pi\sqrt{-1}/n)) := \left.\frac{\langle \omega, \omega, \ldots, \omega \rangle_D}{\langle \omega \rangle_{U_+}^{b_+} \langle \omega \rangle_{U_-}^{b_-}}\right|_{A=\exp\left(\frac{\pi\sqrt{-1}}{n}\right)} \tag{5}$$

Here we use $\exp(2\pi\sqrt{-1}/r)$ and $\exp(4\pi\sqrt{-1}/n)$ because we want to mention the value of A^4 that is usually used for the parameter of the Jones polynomial (see Section 3).

2.2. Reidemeister torsion

Here we explain the Reidemeister torsion twisted by the adjoint action of a representation from the fundamental group of a closed three-manifold to the Lie group $\mathrm{SL}(2;\mathbb{C})$.

For a closed, oriented, connected three-manifold M, let $\rho\colon \pi_1(M) \to \mathrm{SL}(2;\mathbb{C})$ be a representation, where we take a basepoint of M appropriately. Denoting by $\widetilde{M}$ be universal cover of M, the i-th chain group $C_i(\widetilde{M};\mathbb{C})$ and the Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ can be regarded as a $\mathbb{Z}(\pi_1(M))$. Here an element in $\pi_1(M)$ acts on $C_i(\widetilde{M};\mathbb{Z})$ by a deck transformation, and acts on $\mathfrak{sl}(2;\mathbb{C})$ by $x \cdot g := \mathrm{ad}_{\rho(x)}(x)$, where $x \in \pi_1(M)$, $g \in \mathfrak{sl}(2;\mathbb{C})$ and $\mathrm{ad}_{\rho(x)}(g) := \rho(x)^{-1} g \rho(x)$ is the adjoint action. Then the tensor product $C_i(M;\rho) := C_i(\widetilde{M};\mathbb{Z}) \otimes_{\mathbb{Z}(\pi_1(M))} \mathfrak{sl}(2;\mathbb{C})$ $(i = 0, 1, 2)$ forms a chain complex

$$C_\bullet\colon \{0\} \to C_2(M;\rho) \xrightarrow{\partial_2} C_1(M;\rho) \xrightarrow{\partial_1} C_0(M;\rho) \to \{0\}.$$

Let $H_i(M;\rho)$ be the homology group of the chain complex $C_\bullet$. Let $\mathbf{c}_i$ be a basis of $C_i(M;\rho)$, $\mathbf{h}_j$ be a basis of $H_i(M;\rho)$, and $\mathbf{b}_j$ be a set of vectors in $C_i(M;\rho)$ such that the set $\partial_i(\mathbf{b}_i)$ forms a basis of $\mathrm{Im}(\partial_i)$. Then we define

$$\mathrm{Tor}(M;\rho) := \frac{[\partial_2(\mathbf{b}_2) \cup \tilde{\mathbf{h}}_1 \cup \mathbf{b} \mid \mathbf{c}_1]}{[\partial_1(\mathbf{b}_1) \cup \tilde{\mathbf{h}} \mid \mathbf{c}_0][\tilde{\mathbf{h}_2} \cup \mathbf{b}_2 \mid \mathbf{c}_2]}$$

and call it the *(homological) Reidemeister torsion twisted by the adjoint action of* ρ. Here, if $\mathbf{u}$ and $\mathbf{v}$ are bases of a vector space, then $\mathbf{x} \mid \mathbf{y}$ is the determinant of the base-change-matrix from $\mathbf{x}$ to $\mathbf{y}$.

If X is a three-manifold with torus boundary, then we can also define the Reidemeister torsion $\mathrm{Tor}_\gamma(M;\rho)$ if one fixes a simple closed curve γ in ∂X. The following facts are known to calculate the Reidemeister torsions.

Let μ be the meridian and λ be the preferred longitude.

- If ρ is a representation in the irreducible component indexed by (k,l), then $\mathrm{Tor}_\lambda(X;\rho)$ is given by

$$\mathrm{Tor}_\lambda(X;\rho) = \pm\frac{a^2b^2}{16\sin^2(\frac{k\pi}{a})\sin^2(\frac{l\pi}{b})}. \tag{6}$$

  Note that one can also determine the sign. See [15, Section 6.2].

- Suppose that $\rho(\mu) = \begin{pmatrix} e^{u/2} & * \\ 0 & e^{-u/2}\end{pmatrix}$ and $\rho(\gamma) = \begin{pmatrix} e^{w(u)/2} & * \\ 0 & e^{-w(u)/2}\end{pmatrix}$ after a certain conjugation. Then we have

$$\mathrm{Tor}_\gamma(X;\rho) = \pm\frac{d\,w(u)}{d\,u}\,\mathrm{Tor}_\mu(X,\rho) \tag{7}$$

  from [53, Théorème 4.7].

- Let M a closed three-manifold obtained from X by Dehn surgery. We assume that $M = X \cup_i D$, where $i\colon \partial D \to \partial X$ is a homeomorphism. From [53, Proposition 4.10], $\mathrm{Tor}(M;\tilde{\rho})$ is given as

$$\mathrm{Tor}(M;\tilde{\rho}) = \pm\frac{\mathrm{Tor}_{i(\mu_D)}(X;\rho)}{(\mathrm{tr}\,\rho(i(\lambda_D)))^2 - 4}, \tag{8}$$

  where μ_D and λ_D are the meridian and the longitude of D, respectively.

2.3. Chern–Simons invariant

For a representation $\rho\colon \pi_1(M) \to \mathrm{SL}(2;\mathbb{C})$, let A be a flat connection on $M \times \mathrm{SL}(2;\mathbb{C})$ that induces ρ as the holonomy representation. Then the following integral is called the $\mathrm{SL}(2;\mathbb{C})$ *Chern–Simons invariant* $\mathrm{CS}(M;\rho)$ *of* M *associated with* ρ [11]*:*

$$\mathrm{CS}(M;\rho) := \frac{1}{8\pi^2}\int_M \mathrm{tr}\Big(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\Big) \in \mathbb{C}/\mathbb{Z}.$$

Suppose that a closed three-manifold M is obtained by Dehn surgery along a knot $K \subset S^3$. Put $D := D^2 \times S^1$ and assume that M is obtained from $E := S^3 \setminus \operatorname{Int} N(K)$ and D by identifying $\mu_D \subset \partial E$ with the meridian of D. Here the meridian of D is $\partial D^2 \times \{\text{point}\}$. We also assume that the longitude $\{\text{point in } \partial D^2\} \times S^1$ of D is identified with $\lambda_D \subset \partial E$. Then the following theorem is known [32, Theorem 4.2].

Theorem 2.1 (Kirk–Klassen). *Let $\tilde{\rho}_0$ and $\tilde{\rho}_1$ be representations of $\pi_1(M) \to \mathrm{SL}(2;\mathbb{C})$. Assume that there exists a path of representations $\rho_t\colon \pi_1(E) \to \mathrm{SL}(2;\mathbb{C})$ $(0 \le t \le 1)$ avoiding parabolic representations such that $\rho_i = \tilde{\rho}_i|_E$ for $i = 0, 1$. We assume that after conjugation, the images of μ_D and λ_D are as follows:*

$$\rho_t(\mu_D) = \begin{pmatrix} e^{2\pi\sqrt{-1}\alpha(t)} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\alpha(t)} \end{pmatrix},$$

$$\rho_t(\lambda_D) = \begin{pmatrix} e^{2\pi\sqrt{-1}\beta(t)} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\beta(t)} \end{pmatrix}.$$

Then we have

$$\mathrm{CS}(M;\tilde{\rho}_1) - \mathrm{CS}(M;\tilde{\rho}_0) = 2\int_0^1 \beta(t)\frac{d\,\alpha(t)}{d\,t}\,dt$$

as an element in $\mathbb{C}/\mathbb{Z}$.

Remark 2.2. Our sign convention is different from that in [32]. See [16, p. 98, footnote].

2.4. Surgery description

In this section we show that X_p is homeomorphic to the Seifert fibered space $S(O, 0, -a/c, b/d, p - ab)$ by using techniques described in [57]. See also [56]. Here we assume that $b > a > 0$ and $(a, b) = 1$, but we do not assume that b is odd for simplicity.

Let $S \cong D^2 \times S^1$ be an unknotted solid torus, where D^2 is a 2-disk and S^1 is a circle. Let $(T(q,r) \cup L \cup M)_{(s,t,u)}$ be the 3-component link with rational surgery coefficients, where $T(q,r)$ is the torus knot with surgery coefficient s that lies on ∂S, L is the core $\{0\} \times S^1$ of S with coefficient t, and M is an unknotted circle in $S^3 \setminus S$ that is parallel to the meridian $\partial D^2 \times$ point of S with surgery coefficient u. Here we assume that $s, t, u \in \mathbb{Q} \cup \{\infty\}$, and that the knot $T(q,r)$ presents the homology class $[q \times \text{longitude} + r \times \text{meridian}] \in H_1(\partial S; \mathbb{Z})$. See Figure 2.

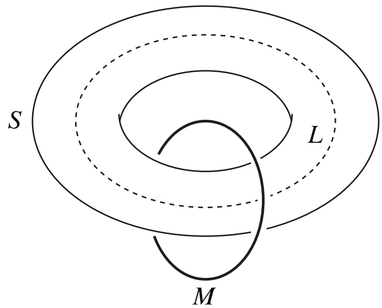

Figure 2. A solid torus S, L, and M. The torus knot $T(q,r)$ is on ∂S. It passes through M q times and goes around L r times.

Putting $r_0 =: b$ and $r_1 := a$, we have the following k equalities from the Euclidean algorithm:

$$\begin{aligned} r_0 &= q_1 r_1 + r_2, \\ r_1 &= q_2 r_2 + r_3, \\ &\vdots \\ r_{k-3} &= q_{k-2} r_{k-2} + r_{k-1}, \\ r_{k-2} &= q_{k-1} r_{k-1} + 1, \\ r_{k-1} &= q_k \end{aligned} \tag{9}$$

since $(a,b)=1$, where r_i $(i=0,1,\dots,k-1)$ and q_j $(j=1,w,\dots,k)$ are positive integers with $r_{i+1} < r_i$.

We start with $(T(r_0,r_1)\cup L\cup M)_{(p,\infty,\infty)}$. If we apply $-q_1$ times twists about L, then

$$(T(r_0,r_1)\cup L\cup M)_{(p,\infty,\infty)}$$

is changed into

$$(T(r_2,r_1)\cup L\cup M)_{(p-r_1^2 q_1,-1/q_1,\infty)}$$

without changing the associated three-manifold. Similarly, we see that $-q_2$ times twists about M changes

$$(T(r_2,r_1)\cup M\cup L)_{(p-r_1^2 q_1,\infty,-1/q_1)}$$

into

$$(T(r_2,r_3)\cup L\cup M)_{(p-r_1^2 q_1-r_2^2 q_2,-1/q_2,-1/q_1-q_2)}$$

without changing the associated three-manifold.

Continuing these twists, we get the following sequence of links with rational surgery coefficient without changing the associated three-manifold:

$$
\begin{aligned}
&(T(r_0,r_1)\cup L\cup M)_{(p,\infty,\infty)} \xrightarrow{-q_1 \text{ twists about } L} \\
&(T(r_2,r_1)\cup L\cup M)_{(p-r_1^2q_1,-1/q_1,\infty)} \xrightarrow{-q_2 \text{ twists about } M} \\
&(T(r_2,r_3)\cup L\cup M)_{(p-r_1^2q_1-r_2^2q_2,-[q_2,q_1],-1/q_2)} \xrightarrow{-q_3 \text{ twists about } L} \\
&(T(r_4,r_3)\cup L\cup M)_{(p-r_1^2q_1-r_2^2q_2-r_3^2q_3,-1/[q_3,q_2,q_1],-[q_3,q_2])} \xrightarrow{-q_4 \text{ twists about } M} \\
&\quad\vdots
\end{aligned}
$$

$$
\begin{cases}
(T(1,r_{k-1})\cup L\cup M)_{(p-\sum_{j=1}^{k-1} r_j^2q_j,-1/[q_{k-1},q_{k-2},\dots,q_1],-[q_{k-1},q_{k-2},\dots,q_2])} & \text{(if } k \text{ is even)},\\
(T(r_{k-1},1)\cup L\cup M)_{(p-\sum_{j=1}^{k-1} r_j^2q_j,-[q_{k-1},q_{k-2},\dots,q_1],-1/[q_{k-1},q_{k-2},\dots,q_2])} & \text{(if } k \text{ is odd)},
\end{cases}
$$

$$
\xrightarrow[\text{if } k \text{ is even (odd, respectively)}]{-q_r=-r_{k-1} \text{ twists about } M\ (L,\text{ respectively})}
$$

$$
\begin{cases}
(T(1,0)\cup L\cup M)_{(p-\sum_{j=1}^{k-1} r_j^2q_j-r_{k-1},-[q_k,q_{k-1},\dots,q_1],-1/[q_k,q_{k-1},\dots,q_2])} & \text{(if } k \text{ is even)},\\
(T(0,1)\cup L\cup M)_{(p-\sum_{j=1}^{k-1} r_j^2q_j-r_{k-1},-1/[q_k,q_{k-1},\dots,q_1],-[q_k,q_{k-1},\dots,q_2])} & \text{(if } k \text{ is odd)},
\end{cases}
$$

from (9), where $[m_0,m_1,m_2,\dots,m_k]$ means the continued fraction

$$
m_0+\cfrac{1}{m_1+\cfrac{1}{m_2+\dots+\frac{1}{m_k}}}.
$$

Ignoring the surgery coefficients, the links $(T(1,0)\cup L\cup M)$ and $(T(0,1)\cup L\cup M)$ are depicted in Figures 3.

Now, we study the complicated coefficients. First of all, we have

$$
\begin{aligned}
p-r_1^2q_1-\dots-r_{k-1}^2q_{k-1}-r_{k-1} &= p-r_1^2q_1-\dots-r_{k-1}r_{k-2}\\
&= p-r_1^2q_1-\dots-r_{k-2}^2q_{k-2}-r_{k-1}r_{k-2}\\
&= p-r_1^2q_1-\dots-r_{k-2}r_{k-3}\\
&\ \ \vdots\\
&= p-r_1r_0=p-ab.
\end{aligned}
$$

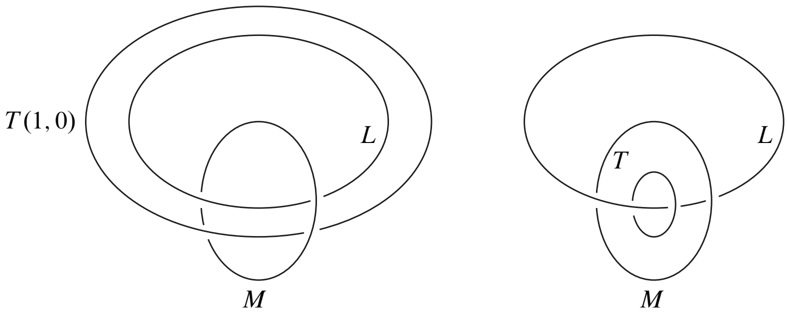

Figure 3. The links $(T(1,0) \cup L \cup M)$ (left) and $(T(0,1) \cup L \cup M)$ (right).

Next, we consider the continued fractions. Dividing j-th equation of (9) by r_j $(j = 1, 2, \dots, k-1)$, we obtain

$$\begin{aligned} r_0/r_1 &= q_1 + r_2/r_1, \\ r_1/r_2 &= q_2 + r_3/r_2, \\ &\ \vdots \\ r_{k-2}/r_{k-1} &= q_{k-1} + 1/r_{k-1}, \\ r_{k-1} &= q_k. \end{aligned}$$

From these equations, we can see that $b/a = r_0/r_1$ is expressed as the continued fraction $[q_1, q_2, \dots, q_k]$. We also see that $r_1/r_2 = [q_2, q_3, \dots, q_k]$.

Put

$$M(q) := \begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}.$$

If we write $M(q_k)M(q_{k-1})\dots M(q_j) = \left(\begin{smallmatrix} s_j & t_j \\ u_j & v_j \end{smallmatrix}\right)$, then it is easy to prove

$$\frac{s_j}{t_j} = [q_j, q_{j+1}, \dots, q_k], \quad \frac{s_j}{u_j} = [q_k, q_{k-1}, \dots, q_j], \quad \frac{t_j}{v_j} = [q_k, q_{k-1}, \dots, q_{j+1}].$$

In particular, we have

$$\frac{s_1}{t_1} = [q_1, q_2, \dots, q_k], \quad \frac{s_1}{u_1} = [q_k, q_{k-1}, \dots, q_1], \quad \frac{t_1}{v_1} = [q_k, q_{k-1}, \dots, q_2].$$

Since $b/a = [q_1, q_2, \dots, q_k] = s_1/t_1$, $s_1 > 0$, $t_1 > 0$, and $(s_1, t_1) = 1$, we have $b = s_1$ and $a = t_1$. Therefore, we have $[q_k, q_{k-2}, \dots, q_1] = b/u_1$ and $[q_k, q_{k-1}, \dots, q_2] = a/v_1$. Now, since $\det\left(\begin{smallmatrix} s_1 & t_1 \\ u_1 & v_1 \end{smallmatrix}\right) = (-1)^k$, we have $bv_1 - au_1 = (-1)^k$. Therefore, if k is even, putting $c := -v_1$ and $d := -u_1$, we have $[q_k, q_{k-1}, \dots, q_1] = -b/d$ and $[q_k, q_{k-1}, \dots, q_2] = -a/c$ with $ad - bc = 1$. If k is odd, putting $c := v_1$ and $d := u_1$, we have $[q_k, q_{k-1}, \dots, q_1] = b/d$, $[q_k, q_{k-1}, \dots, q_2] = a/c$ with $ad - bc = 1$.

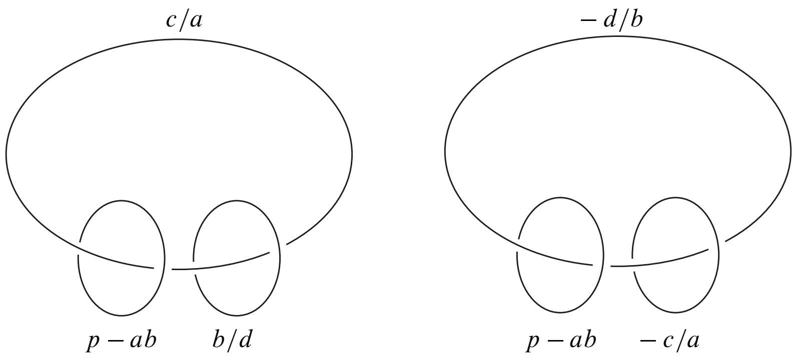

Figure 4. $(T(b,a) \cup L \cup M)_{(p,\infty,\infty)}$ is equivalent to the link to the left (right, respectively) if k is even (odd, respectively).

Therefore, $(T(b,a) \cup L \cup M)_{(p,\infty,\infty)}$ and the link in Figure 4 give the same three-manifold. Here numbers beside link components are surgery coefficients.

By the "slam-dunk" move [12, Figure 6], both links in Figure 4 and the link depicted in Figure 1 give the same three-manifold. So, we conclude that X_p is nothing but the Seifert fibered space $S(O,0,-a/c,b/d,p-ab)$.

3. Calculation of $\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n))$

We show that the summation in $\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n))$ can be expressed as the difference of two double integrals.

In this paper we are only interested in three-manifolds obtained by Dehn surgery along a knot. Let K be a knot in the three-sphere S^3 and put $X := S^3 \setminus \operatorname{Int} N(K)$, where $N(K)$ is the tubular neighborhood of K in S^3 and Int means the interior. For an integer p, denote by X_p the closed, oriented three-manifold obtained from S^3 by Dehn surgery along K with coefficient p. More precisely, we obtain X_p from the disjoint union $X \sqcup (D^2 \times S^1)$ by identifying ∂X and $\partial(D^2 \times S^1)$ so that $\partial D^2 \times \{*\} \subset \partial(D^2 \times S^1)$ is identified with the simple closed curve in ∂X that goes once along K and p times around K, where D^2 is a disk, S^1 is a circle, and $*$ is a point in S^1. So, the first homology group $H_1(X_p;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

We denote by $\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n))$ the invariant given in (5). Let E be a knot diagram presenting K with framing p. Then we have

$$
\begin{aligned}
&\langle S_k(\alpha)\rangle_E \\
&= ((-1)^k A^{(k+1)^2-1})^p \Delta_k J_{k+1}(K;A^4) \\
&= (-e^{\pi\sqrt{-1}/n})^{p((k+1)^2-1)}(-1)^{k+1}\frac{\sin(2(k+1)\pi/n)}{\sin(2\pi/n)} J_{k+1}(K;e^{4\pi\sqrt{-1}/n}),
\end{aligned}
$$

where $J_{k+1}(K;q)$ is the $(k+1)$-dimensional colored Jones polynomial of K, normalised so that $J_{k+1}(\text{unknot};q)=1$.

Remark 3.1. We need to multiply by Δ_k since $\langle S_k(\alpha)\rangle_U=\Delta_k$ with U a diagram of the unknot with no crossing.

Remark 3.2. If we start with the Kauffman bracket defined as in [29] and replace A with $t^{-1/4}$, we obtain the original Jones polynomial $V(L;t)$ [26]. In the formula above, we replace q in the colored Jones polynomial $J_{k+1}(K;q)$ with A^4, and so our 2-dimensional colored Jones polynomial $J_2(L;q)$ equals $V(L;q^{-1})$.

We have the following lemma.

Lemma 3.3. *The Witten–Reshetikhin-Turaev invariant* $\hat{\tau}_n(X_p;\exp(4\pi\sqrt{-1}/n))$ *is given by the following formula:*

$$\begin{aligned}\hat{\tau}_n(X_p;\exp(4\pi\sqrt{-1}/n)) = {} & \frac{1}{\sqrt{n}\sin(2\pi/n)} e^{\operatorname{sign}(p)(\frac{3}{n}+\frac{n+1}{4})\pi\sqrt{-1}} \\ & \times \sum_{k=1}^{n-1}\sin^2\Big(\frac{2k\pi}{n}\Big)(-e^{\frac{\pi\sqrt{-1}}{n}})^{p(k^2-1)} J_k(K;e^{4\pi\sqrt{-1}/n}),\end{aligned}$$

where $\operatorname{sign}(p)$ *is the sign of* p.

Proof. When $A=e^{\pi\sqrt{-1}/n}$, we have

$$\Delta_{k-1}=(-1)^{k-1}\frac{\sin(2k\pi/n)}{\sin(2\pi/n)}.$$

So, we have

$$\begin{aligned}\langle\omega\rangle_E &= \sum_{k=1}^{n-1}\Delta_{k-1}\times\langle S_{k-1}(\alpha)\rangle_E \\ &= \frac{1}{\sin^2(2\pi/n)}\sum_{k=1}^{n-1}(-e^{\pi\sqrt{-1}/n})^{p(k^2-1)}\sin^2(2k\pi/n)J_k(K;e^{4\pi\sqrt{-1}/n}).\end{aligned}$$

Next we calculate $\langle\omega\rangle_{U_\pm}$. Since U_- is the mirror image of U_+, $\langle\omega\rangle_{U_-}$ can be obtained from $\langle\omega\rangle_{U_+}$ by replacing A with A^{-1}, that is, $\langle\omega\rangle_{U_-}=\overline{\langle\omega\rangle_{U_+}}$ (complex conjugate). So, we will only calculate $\langle\omega\rangle_{U_+}$. Since $J_k(\text{unknot};q)=1$, we have

$$\begin{aligned}\langle\omega\rangle_{U_+} &= \frac{1}{(A^2-A^{-2})^2}\sum_{k=1}^{n-1}(-A)^{k^2-1}(A^{2k}-A^{-2k})^2 \\ &= \frac{1}{(A^2-A^{-2})^2}\sum_{k=0}^{n-1}(-A^{-5}(-A)^{(k+2)^2}+2A^{-1}(-A)^{k^2}-A^{-5}(-A)^{(k-2)^2})\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(-A^{-5}+A^{-1})}{(A^2-A^{-2})^2}\sum_{l=0}^{n-1}(-A)^{l^2} \\
&= \frac{2A^{-3}}{A^2-A^{-2}}G_n(-A),
\end{aligned}
$$

where the third equality follows since $(-A)^{(l+n)^2} = (-A)^{l^2}$, and

$$
G_n(\zeta) := \sum_{l=0}^{n-1}\zeta^{l^2}
$$

is the quadratic Gaussian sum.

Denoting by $(\frac{c}{n}) \in \{\pm 1, 0\}$ the Jacobi symbol, a generalization of the Legendre symbol, it is well known that

$$
\begin{aligned}
G_n(e^{2c\pi\sqrt{-1}/n}) &= \left(\frac{c}{n}\right)G_n(e^{2\pi\sqrt{-1}/n}), \\
\left(\frac{2}{n}\right) &= (-1)^{(n^2-1)/8}, \\
\left(\frac{cd}{n}\right) &= \left(\frac{c}{n}\right)\left(\frac{d}{n}\right).
\end{aligned}
$$

See for example [22]. So, we have

$$
\begin{aligned}
G_n(-A) &= G_n(-e^{\pi\sqrt{-1}/n}) \\
&= G_n\left(\exp\left(\frac{\frac{n+1}{2}\times 2\pi\sqrt{-1}}{n}\right)\right) \\
&= \left(\frac{(n+1)/2}{n}\right)G_n(e^{2\pi\sqrt{-1}/n}) \\
&= \frac{(\frac{n+1}{n})}{(\frac{2}{n})}G_n(e^{2\pi\sqrt{-1}/n}) \\
&= (-1)^{(n^2-1)/8}G_n(e^{2\pi\sqrt{-1}/n}),
\end{aligned}
$$

where the last equality holds since $(\frac{c}{n}) = (\frac{c'}{n})$ if $c \equiv c' \pmod{n}$. Since it is also well known that

$$
G_n(e^{2\pi\sqrt{-1}}) = \begin{cases} \sqrt{n} & \text{if } n \equiv 1 \pmod 4, \\ \sqrt{-1}\sqrt{n} & \text{if } n \equiv 3 \pmod 4, \end{cases}
$$

we have

$$
\begin{aligned}
G_n(-A) &= \begin{cases} (-1)^{(n^2-1)/8}\sqrt{n} & \text{if } n \equiv 1 \pmod 4, \\ (-1)^{(n^2-1)/8}\sqrt{-1}\sqrt{n} & \text{if } n \equiv 3 \pmod 4 \end{cases} \\
&= (-\sqrt{-1})^{(n-1)/2}\sqrt{n}.
\end{aligned}
$$

Hence, we have

$$\begin{aligned}\langle\omega\rangle_{U_+} &= \frac{e^{-3\pi\sqrt{-1}/n}}{\sqrt{-1}\sin(2\pi/n)}(-\sqrt{-1})^{(n-1)/2}\sqrt{n}\\ &= \frac{\sqrt{n}}{\sin(2\pi/n)} \times e^{-(\frac{3}{n}+\frac{n+1}{4})\pi\sqrt{-1}}.\end{aligned}$$

So, we conclude that

$$\langle\omega\rangle_{U_+}^{-b_+}\langle\omega\rangle_{U_-}^{-b_-} = \frac{\sin(2\pi/n)}{\sqrt{n}} \times e^{\operatorname{sign}(p)(\frac{3}{n}+\frac{n+1}{4})\pi\sqrt{-1}}$$

and the required formula follows. ∎

Remark 3.4. Equation (4.1) in [10] is not correct; they should have used $\langle\omega\rangle_{U_-}$ and $\langle\omega\rangle_U$ instead of $\langle\mu\omega\rangle_{U_-}$ and $\langle\mu\omega\rangle_U$.

Remark 3.5. This is the so-called *quantum* SU(2)*-invariant.* See [17, p. 930] and also [30, 37].

Let $K := T(a,b)$ be the torus knot of type (a,b). Our convention is that $T(2,3)$ is the right-handed trefoil. Note that the knot 3_1 in Rolfsen's table [57] (see also [37]) is $T(2,-3)$. Then we have

$$\begin{aligned}J_k(K;q) &= \frac{q^{1/2}-q^{-1/2}}{q^{k/2}-q^{-k/2}}q^{-ab(k^2-1)/4}\sum_{j=-(k-1)/2}^{(k-1)/2} q^{bj(aj+1)}\frac{q^{aj+1/2}-q^{-(aj+1/2)}}{q^{1/2}-q^{-1/2}}\\ &= q^{-ab(k^2-1)/4}\sum_{j=-(k-1)/2}^{(k-1)/2} q^{bj(aj+1)}\frac{q^{aj+1/2}-q^{-(aj+1/2)}}{q^{k/2}-q^{-k/2}}\end{aligned}$$

from [41, 58]. Hence, we have

$$J_k(K;e^{\frac{4\pi\sqrt{-1}}{n}}) = e^{-ab(k^2-1)\frac{\pi\sqrt{-1}}{n}}\sum_{j=-(k-1)/2}^{(k-1)/2} e^{4bj(aj+1)\frac{\pi\sqrt{-1}}{n}}\frac{\sin(\frac{(4aj+2)\pi}{n})}{\sin(\frac{2k\pi}{n})}.$$

With $l = 2j$ we have

$$\begin{aligned}\hat{\tau}_n(X_p;\exp(4\pi\sqrt{-1}/n)) = \frac{1}{\sqrt{n}\sin(2\pi/n)}&e^{\operatorname{sign}(p)(\frac{3}{n}+\frac{n+1}{4})\pi\sqrt{-1}}\\ &\times\sum_{k=1}^{n-1}\sin\Big(\frac{2k\pi}{n}\Big)(-1)^{p(k^2-1)}e^{(p-ab)(k^2-1)\frac{\pi\sqrt{-1}}{n}}\\ &\times\sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} e^{bl(al+2)\frac{\pi\sqrt{-1}}{n}}\sin\Big(\frac{2(al+1)\pi}{n}\Big).\end{aligned}$$

Putting

$$
\begin{aligned}
S &:= \sum_{k=1}^{n-1} (-1)^{pk} e^{(p-ab)k^2 h} \sinh(2kh) \sum_{\substack{-k+1 \le l \le k-1 \\ l \equiv k+1 \pmod 2}} e^{bl(al+2)h} \sinh(2(al+1)h) \\
&= e^{-\frac{b}{a}h} \sum_{k=1}^{n-1} (-1)^{pk} e^{(p-ab)k^2 h} \sinh(2kh) \sum_{\substack{-k+1 \le l \le k-1 \\ l \equiv k+1 \pmod 2}} e^{\frac{b}{a}(al+1)^2 h} \sinh(2(al+1)h)
\end{aligned}
$$

with $h := \frac{\pi\sqrt{-1}}{n}$, we have

$$
\begin{aligned}
&\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n)) \\
&\quad = -\frac{1}{\sqrt{n}\sin(2\pi/n)} e^{\operatorname{sign}(p)(\frac{3}{n}+\frac{n+1}{4})\pi\sqrt{-1}} (-1)^p e^{-(p-ab)\frac{\pi\sqrt{-1}}{n}} \times S.
\end{aligned}
$$

From now on, we consider the case where $p > ab > 0$.

We use the following formula:

$$
e^{\lambda w^2} = \frac{1}{\sqrt{\pi\lambda}} \int_{C_\theta} e^{-\frac{z^2}{\lambda} - 2zw}\, dz, \tag{10}
$$

where C_θ is the path $\{te^{\theta\sqrt{-1}} \mid t \in \mathbb{R}\}$. Here θ satisfies $-\pi/2 + \arg\lambda < 2\theta < \pi/2 + \arg\lambda$ so that the integral converges. Applying (10) with $\lambda = bh/a$ and $w = al + 1$, we have

$$
\begin{aligned}
&\sum_{\substack{-k+1 \le l \le k-1 \\ l \equiv k+1 \pmod 2}} e^{\frac{b}{a}(al+1)^2 h} \sinh(2(al+1)h) \\
&\quad = \sqrt{\frac{a}{bh\pi}} \int_{C_\theta} e^{-\frac{a}{bh}z^2} \Bigl(\sum_{\substack{-k+1 \le l \le k-1 \\ l \equiv k+1 \pmod 2}} e^{-2(al+1)z} \sinh(2(al+1)h) \Bigr)\, dz.
\end{aligned}
$$

Since $\arg(\frac{b}{a}h) = \pi/2$, θ should satisfy $0 < \theta < \pi$.

We first calculate the summation.

Lemma 3.6. *We have*

$$
\begin{aligned}
&\sum_{\substack{-k+1 \le l \le k-1 \\ l \equiv k+1 \pmod 2}} e^{-2(al+1)z} \sinh(2(al+1)h) \\
&\quad = \frac{1}{2} e^{2(h-z)} \frac{\sinh(2ka(z-h))}{\sinh(2a(z-h))} - \frac{1}{2} e^{-2(h+z)} \frac{\sinh(2ka(z+h))}{\sinh(2a(z+h))}.
\end{aligned}
$$

Proof. Since the range of summation

$$\{l \in \mathbb{Z} \mid -k+1 \le l \le k-1, l \equiv k+1 \pmod 2\}$$

is invariant under $l \leftrightarrow -l$, for any discrete function f we have

$$\sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} f(l) \quad = \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} \frac{f(l)+f(-l)}{2}.$$

Put

$$R(z) := \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} e^{-2(al+1)z} \sinh(2(al+1)h).$$

We have

$$\begin{aligned}
2R(z) &= \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} e^{-2(al+1)z} \sinh(2(al+1)h) - \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} e^{2(al-1)z} \sinh(2(al-1)h) \\
&= \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} e^{-2(al+1)z} (\sinh(2alh)\cosh(2h) + \cosh(2alh)\sinh(2h)) \\
&\quad - \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} e^{2(al-1)z} (\sinh(2alh)\cosh(2h) - \cosh(2alh)\sinh(2h)) \\
&= -(e^{2h}+e^{-2h})e^{-2z} \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} \sinh(2alz)\sinh(2alh) \\
&\quad + (e^{2h}-e^{-2h})e^{-2z} \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} \cosh(2alz)\cosh(2alh) \\
&= e^{2(h-z)} \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} \cosh(2al(z-h)) - e^{-2(h+z)} \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} \cosh(2al(z+h)).
\end{aligned}$$

Noting that

$$\sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} \cosh(xl) = \sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} \frac{\sinh((l+1)x) - \sinh((l-1)x)}{2\sinh(x)} = \frac{\sinh(kx)}{\sinh x},$$

the lemma then follows. ∎

We calculate the integral.

Proposition 3.7. *We have*

$$\int_{C_\theta} e^{-\frac{a}{bh}z^2}\Big(\sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} e^{-2(al+1)z}\sinh(2(al+1)h)\Big)\,dz$$
$$=e^{-\frac{ah}{b}}\int_{C_\theta} e^{-\frac{a}{bh}z^2}\frac{\sinh(\frac{2a}{b}z)\sinh(2z)\sinh(2akz)}{\sinh(2az)}\,dz.$$

Proof. Since $C_\theta \leftrightarrow -C_\theta$ under $z\leftrightarrow -z$, for any function f we have

$$\int_{C_\theta} f(z)dz=\int_{C_\theta} f(-z)dz.$$

.

By Lemma 3.6 the left-hand side becomes

$$\frac{1}{2}\int_{C_\theta} e^{-\frac{a}{bh}z^2}e^{2(h-z)}\frac{\sinh(2ka(z-h))}{\sinh(2a(z-h))}\,dz$$
$$-\frac{1}{2}\int_{C_\theta} e^{-\frac{a}{bh}z^2}e^{-2(h+z)}\frac{\sinh(2ka(z+h))}{\sinh(2a(z+h))}\,dz$$
$$=\frac{1}{2}\int_{C_\theta} e^{-\frac{a}{bh}z^2}e^{2(h-z)}\frac{\sinh(2ka(z-h))}{\sinh(2a(z-h))}\,dz$$
$$-\frac{1}{2}\int_{C_\theta} e^{-\frac{a}{bh}z^2}e^{-2(h-z)}\frac{\sinh(2ka(z-h))}{\sinh(2a(z-h))}\,dz$$
$$=\int_{C_\theta} e^{-\frac{a}{bh}z^2}\sinh(2(h-z))\frac{\sinh(2ka(z-h))}{\sinh(2a(z-h))}\,dz.$$

By shifting the path of integral from C_θ to $C_\theta+h$, since $\frac{\sinh(2kaz)}{\sinh(2az)}$ is analytic, and applying change of variable $z\mapsto z+h$ we get

$$-\int_{C_\theta} e^{-\frac{a}{bh}(z+h)^2}\sinh(2z)\frac{\sinh(2kaz)}{\sinh(2az)}\,dz$$
$$=-\frac{1}{2}\int_{C_\theta} e^{-\frac{a}{bh}(z+h)^2}\sinh(2z)\frac{\sinh(2kaz)}{\sinh(2az)}\,dz$$
$$+\frac{1}{2}\int_{C_\theta} e^{-\frac{a}{bh}(z-h)^2}\sinh(2z)\frac{\sinh(2kaz)}{\sinh(2az)}\,dz$$

$$= e^{-\frac{ah}{b}} \int_{C_\theta} e^{-\frac{a}{bh}z^2} \frac{\sinh(\frac{2a}{b}z)\sinh(2z)\sinh(2akz)}{\sinh(2az)}\, dz$$

as required. ∎

By Proposition 3.7 we have

$$\sum_{\substack{-k+1\le l\le k-1\\ l\equiv k+1 \pmod 2}} e^{\frac{b}{a}(al+1)^2h}\sinh(2(al+1)h)$$
$$= \sqrt{\frac{a}{bh\pi}} e^{-\frac{ah}{b}} \int_{C_\theta} e^{-\frac{a}{bh}z^2} \frac{\sinh(\frac{2a}{b}z)\sinh(2z)\sinh(2akz)}{\sinh(2az)}\, dz$$

$(az \mapsto z)$

$$= \frac{1}{\sqrt{abh\pi}} e^{-\frac{ah}{b}} \int_{C_\theta} e^{-\frac{z^2}{abh}} \frac{\sinh(\frac{2z}{b})\sinh(\frac{2z}{a})\sinh(2kz)}{\sinh(2z)}\, dz$$
$$= \frac{1}{\sqrt{abh\pi}} e^{-\frac{ah}{b}} \int_{C_\theta} e^{-\frac{z^2}{abh}} \varphi(z)\sinh(2kz)\, dz,$$

where we put

$$\varphi(z) := \frac{\sinh(\frac{2z}{b})\sinh(\frac{2z}{a})}{\sinh(2z)}.$$

Then we have

$$S = \frac{1}{\sqrt{abh\pi}} e^{-(a/b+b/a)h}$$
$$\times \sum_{k=1}^{n-1} (-1)^{pk} e^{(p-ab)k^2h}\sinh(2kh) \int_{C_\theta} e^{-\frac{z^2}{abh}} \varphi(z)\sinh(2kz)\, dz.$$

Hence, we have

$$\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n))$$
$$= -\frac{1}{\sqrt{n}\sin(2\pi/n)} e^{(\frac{3}{n}+\frac{n+1}{4})\pi\sqrt{-1}} (-1)^p e^{-(p-ab)\frac{\pi\sqrt{-1}}{n}} \frac{1}{\sqrt{abh\pi}} e^{-(a/b+b/a)h}\, T,$$

where

$$T := \sum_{k=1}^{n-1} (-1)^{pk} e^{(p-ab)k^2h}\sinh(2kh) \int_{C_\theta} e^{-\frac{z^2}{abh}} \varphi(z)\sinh(2kz)\, dz.$$

Applying (10) with $\lambda = (p-ab)h$ and $w = k$, we have

$$T = \frac{1}{\sqrt{(p-ab)h\pi}} \times \sum_{k=1}^{n-1} (-1)^{pk} \left(\int_{C_\theta} e^{-\frac{z_1^2}{(p-ab)h} - 2kz_1} \, dz_1 \right) \sinh(2kh) \times \int_{C_\theta} e^{-\frac{z_2^2}{abh}} \varphi(z_2) \sinh(2kz_2) \, dz_2.$$

Note that $\arg((p-ab)h) = \pi/2$ and θ should satisfy $0 < \theta < \pi$ as before. Applying

$$\int_{C_\theta} f(z)dz = \int_{C_\theta} \frac{f(z) + f(-z)}{2} \, dz,$$

we have

$$T = \frac{1}{\sqrt{(p-ab)h\pi}} \iint_{C_\theta \times C_\theta} e^{-\frac{z_1^2}{(p-ab)h}} e^{-\frac{z_2^2}{abh}} \varphi(z_2) \sum_{k=1}^{n-1} (-1)^{pk} \cosh(2kz_1) \sinh(2kh) \sinh(2kz_2) \, dz_1 \, dz_2.$$

Hence, we have

$$\begin{aligned} &\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n)) \\ &= -\frac{1}{\sqrt{n}\sin(2\pi/n)} e^{(\frac{3}{n} + \frac{n+1}{4})\pi\sqrt{-1}} (-1)^p e^{-(p-ab)\frac{\pi\sqrt{-1}}{n}} \\ &\quad \times \frac{1}{\sqrt{abh\pi}} e^{-(a/b+b/a)h} \frac{1}{\sqrt{(p-ab)h\pi}} U \\ &= \frac{\sqrt{n}}{\sin(2\pi/n)} \frac{\sqrt{-1}}{\pi^2} e^{(\frac{n+1}{4})\pi\sqrt{-1}} (-1)^p e^{-(p-ab+a/b+b/a-3)\frac{\pi\sqrt{-1}}{n}} \frac{1}{\sqrt{ab(p-ab)}} U, \end{aligned}$$

where

$$\begin{aligned} U &:= \iint_{C_\theta \times C_\theta} e^{-\frac{z_1^2}{(p-ab)h}} e^{-\frac{z_2^2}{abh}} \varphi(z_2) \\ &\quad \times \sum_{k=1}^{n-1} (-1)^{pk} \cosh(2kz_1) \sinh(2kh) \sinh(2kz_2) \, dz_1 \, dz_2 \\ &= \iint_{C_\theta \times C_\theta} e^{-\frac{z_1^2}{(p-ab)h}} e^{-\frac{z_2^2}{abh}} \varphi(z_2) \sum_{k=1}^{n-1} \cosh(2kz_1) \sinh(2k\tilde{h}) \sinh(2kz_2) \, dz_1 \, dz_2, \end{aligned}$$

where

$$\tilde{h} := h - p\pi\sqrt{-1}/2 = (1/n - p/2)\pi\sqrt{-1}.$$

Note that the double integral

$$\iint_{C_\theta \times C_\theta} g(z_1, z_2)\, dz_1\, dz_2$$

is unchanged if the integrand $g(z_1, z_2)$ is replaced with $g(-z_1, z_2)$, $g(z_1, -z_2)$, or $g(-z_1, -z_2)$. Note also that $\varphi(z_2)$ is an odd function.

Since

$$\begin{aligned}
&\cosh(2kz_1)\sinh(2kz_2) \\
&\quad = \frac{\sinh(2k(z_1+z_2)) + \sinh(2k(-z_1+z_2))}{2}, \\
&\sinh(2k(z_1+z_2))\sinh(2k\tilde{h}) \\
&\quad = \frac{\cosh(2k(z_1+z_2+\tilde{h})) - \cosh(2k(-z_1-z_2+\tilde{h}))}{2},
\end{aligned}$$

we have

$$\begin{aligned}
U &= \iint_{C_\theta \times C_\theta} e^{-\frac{z_1^2}{(p-ab)h}} e^{-\frac{z_2^2}{abh}} \varphi(z_2) \sum_{k=1}^{n-1} \cosh(2k(z_1+z_2+\tilde{h}))\, dz_1\, dz_2 \\
&= \frac{1}{2} \iint_{C_\theta \times C_\theta} e^{-\frac{z_1^2}{(p-ab)h}} e^{-\frac{z_2^2}{abh}} \varphi(z_2) \Big\{ \frac{\sinh((2n-1)(z_1+z_2+\tilde{h}))}{\sinh(z_1+z_2+\tilde{h})} - 1 \Big\}\, dz_1\, dz_2 \\
&= \frac{1}{2} \iint_{C_\theta \times C_\theta} e^{-\frac{z_1^2}{(p-ab)h}} e^{-\frac{z_2^2}{abh}} \varphi(z_2) \frac{\sinh((2n-1)(z_1+z_2+\tilde{h}))}{\sinh(z_1+z_2+\tilde{h})}\, dz_1\, dz_2.
\end{aligned}$$

Here we use the formula

$$\sum_{k=1}^{n-1} \cosh(2kx) = \frac{1}{2}\Big(\frac{\sinh((2n-1)x)}{\sinh(x)} - 1\Big)$$

in the second equality, and use the fact that $\varphi(z_2)$ is an odd function in the third.

Since

$$\begin{aligned}
\frac{\sinh((2n-1)(x+\tilde{h}))}{\sinh(x+\tilde{h})} &= \frac{\sinh(2n(x+\tilde{h}))\cosh(x+\tilde{h})}{\sinh(x+\tilde{h})} - \cosh(2n(x+\tilde{h})) \\
&= (-1)^{np}\{\sinh(2nx)\coth(x+\tilde{h}) - \cosh(2nx)\}
\end{aligned}$$

and $\varphi(z_2)$ is an odd function, we have

$$U = \frac{(-1)^{np}}{2} \iint\limits_{C_\theta \times C_\theta} e^{-\frac{z_1^2}{(p-ab)\hbar}} e^{-\frac{z_2^2}{ab\hbar}} \varphi(z_2) \sinh(2n(z_1+z_2)) \times \coth(z_1+z_2+\tilde{h})\, dz_1\, dz_2$$

$$= \frac{(-1)^{np}}{2} \iint\limits_{C_\theta \times C_\theta} e^{-\frac{(z_1-z_2)^2}{(p-ab)\hbar}} e^{-\frac{z_2^2}{ab\hbar}} \varphi(z_2) \sinh(2nz_1) \coth(z_1+\tilde{h})\, dz_1\, dz_2.$$

By using the symmetry of the integral, we have

$$\iint\limits_{C_\theta \times C_\theta} e^{-\frac{(z_1-z_2)^2}{(p-ab)\hbar}} e^{-\frac{z_2^2}{ab\hbar}} \varphi(z_2) \sinh(2nz_1) \coth(z_1+\tilde{h})\, dz_1\, dz_2$$

$$= -\iint\limits_{C_\theta \times C_\theta} e^{-\frac{(z_1-z_2)^2}{(p-ab)\hbar}} e^{-\frac{z_2^2}{ab\hbar}} \varphi(z_2) \sinh(2nz_1) \coth(-z_1+\tilde{h})\, dz_1\, dz_2$$

$$= \iint\limits_{C_\theta \times C_\theta} e^{-\frac{(z_1-z_2)^2}{(p-ab)\hbar}} e^{-\frac{z_2^2}{ab\hbar}} \varphi(z_2) \sinh(2nz_1) \coth(z_1-\tilde{h})\, dz_1\, dz_2.$$

Therefore, we have

$$U = \frac{(-1)^{np}}{4} \iint\limits_{C_\theta \times C_\theta} e^{-\frac{(z_1-z_2)^2}{(p-ab)\hbar}} e^{-\frac{z_2^2}{ab\hbar}} \varphi(z_2) \sinh(2nz_1) \times (\coth(z_1+\tilde{h}) - \coth(z_1-\tilde{h}))\, dz_1\, dz_2$$

$$= \frac{(-1)^{np}}{4} \iint\limits_{C_\theta \times C_\theta} e^{-\frac{(z_1-z_2)^2}{(p-ab)\hbar}} e^{-\frac{z_2^2}{ab\hbar}} \varphi(z_2) e^{2nz_1} \times (\coth(z_1+\tilde{h}) - \coth(z_1-\tilde{h}))\, dz_1\, dz_2,$$

where we use the symmetry

$$(z_1, z_2) \leftrightarrow (-z_1, -z_2)$$

of the double integral again. Hence, we have

$$\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n)) = \frac{\sqrt{n}}{\sin(2\pi/n)} \frac{(-1)^{np+p} e^{\frac{3}{4}\pi\sqrt{-1}}}{4\pi^2\sqrt{ab(p-ab)}} e^{-(p-ab+a/b+b/a-3)\frac{\pi\sqrt{-1}}{n}} e^{\frac{n}{4}\pi\sqrt{-1}} (V_1 - V_2), \tag{11}$$

where

$$V_1 = \iint\limits_{C_\theta \times C_\theta} \psi_1(z_1)\varphi(z_2)e^{nF(z_1,z_2)}\, dz_1\, dz_2,$$

$$V_2 = \iint\limits_{C_\theta \times C_\theta} \psi_2(z_1)\varphi(z_2)e^{nF(z_1,z_2)}\, dz_1\, dz_2$$

with

$$\psi_1(z_1) := \coth(z_1 + \tilde{h}),$$
$$\psi_2(z_1) := \coth(z_1 - \tilde{h}'),$$
$$F(z_1, z_2) := -\frac{(z_1 - z_2)^2}{(p - ab)\pi\sqrt{-1}} - \frac{z_2^2}{ab\pi\sqrt{-1}} + 2z_1.$$

Here we put $\tilde{h}' = \left(\frac{1}{n} + \frac{p}{2}\right)\pi\sqrt{-1}$, noting that $\coth(z_1 - \tilde{h}) = \coth(z_1 - \tilde{h}')$. We note the following facts.

- $F(z_1, z_2)$ has a unique critical point

$$(w_1, w_2) := (p\pi\sqrt{-1}, ab\pi\sqrt{-1}).$$

- $\psi_1(z_1)$ has poles

$$\xi_{1,l} = l\pi\sqrt{-1} - \tilde{h} = (l + p/2 - 1/n)\pi\sqrt{-1}$$

  with $l \in \mathbb{Z}$. Moreover, the residue of ψ_1 at $\xi_{1,l}$ is given by $\mathrm{Res}(\psi_1; \xi_{1,l}) = 1$.
- $\psi_2(z_1)$ has poles

$$\xi_{2,l} := l\pi\sqrt{-1} + \tilde{h}' = (l + p/2 + 1/n)\pi\sqrt{-1}$$

  with $l \in \mathbb{Z}$. Moreover, the residue of ψ_2 at $\xi_{2,l}$ is given by

$$\mathrm{Res}(\psi_2; \xi_{2,l}) = 1,$$

- $\varphi(z_2)$ has poles

$$\eta_m = \frac{m}{2}\pi\sqrt{-1},$$

  where $m \in \mathbb{Z}$ such that $a \nmid m$, and $b \nmid m$. Moreover, the residue is given by

$$\mathrm{Res}(\varphi; \eta_m) = -\frac{(-1)^m}{2}\sin(m\pi/a)\sin(m\pi/b).$$

4. Asymptotic expansion of V_1

In this section we consider the asymptotic behavior of V_1 as $n \to \infty$.

In the double integral in the definition of V_1, we shift $C_\theta \times C_\theta$ to $(C_\theta + w_1) \times (C_\theta + w_2)$ by using the residue theorem. Then we have

$$V_1 = I_{1,0} + 2\pi\sqrt{-1} I_{1,1} + 2\pi\sqrt{-1} I_{1,2} + (2\pi\sqrt{-1})^2 I_{1,3},$$

where we put

$$I_{1,0} := \iint_{(C_\theta+w_1)\times(C_\theta+w_2)} \psi_1(z_1)\varphi(z_2)e^{nF(z_1,z_2)}\,dz_1\,dz_2,$$

$$I_{1,1} := \sum_{-p/2+1/n<l<p/2+1/n} \operatorname{Res}(\psi_1;\xi_{1,l}) \int_{C_\theta+w_2} \varphi(z_2)e^{nF(\xi_{1,l},z_2)}\,dz_2,$$

$$I_{1,2} = \sum_{m=1}^{2ab-1} \operatorname{Res}(\varphi;\eta_m) \int_{C_\theta+w_1} \psi_1(z_1)e^{nF(z_1,\eta_m)}\,dz_1,$$

$$I_{1,3} = \sum_{\substack{-p/2+1/n<l<p/2+1/n\\ 1\le m\le 2ab-1}} \operatorname{Res}(\psi_1;\xi_{1,l}) \operatorname{Res}(\varphi;\eta_m)e^{nF(\xi_{1,l},\eta_m)}.$$

Here l runs over all integers such that $l + p/2 - 1/n$ is between 0 and p, i.e., $|l - 1/n| < p/2$, and m runs over all integers such that $0 < m < 2ab$.

Remark 4.1. If p is odd, then $\frac{1}{2}(1-p) \le l \le \frac{1}{2}(p-1)$. If p is even, then $-\frac{1}{2}p + 1 \le l \le \frac{1}{2}p$.

4.1. The double integral in V_1

In this section we calculate the double integral $I_{1,0}$. We have

$$\begin{aligned} I_{1,0} &= \iint_{(C_\theta+w_1)\times(C_\theta+w_2)} \psi_1(z_1)\varphi(z_2)e^{nF(z_1,z_2)}\,dz_1\,dz_2 \\ &= \iint_{C_\theta\times C_\theta} \psi_1(z_1+w_1)\varphi(z_2+w_2)e^{nF(z_1+w_1,z_2+w_2)}\,dz_1\,dz_2. \end{aligned}$$

Note that $\psi_1(z_1 + w_1) = \psi_1(z_1)$ and $\varphi(z_2 + w_2) = \varphi(z_2)$. Moreover, we have

$$F(z_1 + w_1, z_2 + w_2) = -\frac{(z_1 - z_2)^2}{(p-ab)\pi\sqrt{-1}} - \frac{z_2^2}{ab\pi\sqrt{-1}} + p\pi\sqrt{-1}.$$

Hence, we have

$$I_{1,0} = \iint_{C_\theta \times C_\theta} \psi_1(z_1)\varphi(z_2) \exp\Big[n\Big(-\frac{(z_1-z_2)^2}{(p-ab)\pi\sqrt{-1}} - \frac{z_2^2}{ab\pi\sqrt{-1}} + p\pi\sqrt{-1}\Big)\Big]\, dz_1\, dz_2.$$

Later, we will show this will cancel the counterpart of V_2, and so we leave it as it is.

4.2. The double sum in V_1

Since $\xi_{1,l} = (l + p/2 - 1/n)\pi\sqrt{-1}$ and $\eta_m = m\pi\sqrt{-1}/2$, we have

$$F(\xi_{1,l}, \eta_m) = \Big(-\frac{(l-m/2+p/2-1/n)^2}{p-ab} - \frac{m^2}{4ab} + 2(l + p/2 - 1/n)\Big)\pi\sqrt{-1}.$$

Hence, we have

$$nF(\xi_{1,l}, \eta_m) = \Big[\Big(-\frac{(l-m/2+p/2)^2}{p-ab} - \frac{m^2}{4ab} + 2l + p\Big)n + \frac{2l-m+p}{p-ab} - 2 - \frac{1}{(p-ab)n}\Big]\pi\sqrt{-1}.$$

Since $\operatorname{Res}(\psi_1; \xi_{1,l}) \operatorname{Res}(\varphi; \eta_m) = -\frac{(-1)^m}{2} \sin(m\pi/a)\sin(m\pi/b)$ we obtain

$$\begin{aligned} I_{1,3} = \sum_{\substack{-p/2+1/n<l<p/2+1/n \\ 1\le m\le 2ab-1}} &\frac{(-1)^{m+1}}{2} \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) \\ &\times e^{\frac{2l-m+p}{p-ab}\pi\sqrt{-1}} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\ &\times (-1)^{pn} \exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big]. \end{aligned} \tag{12}$$

4.3. The sum over l in V_1

Consider the integral

$$\int_{C_{\theta+w_2}} \varphi(z_2) e^{nF(\xi_{1,l}, z_2)}\, dz_2.$$

The polynomial $F(\xi_{1,l}, z_2) = -\frac{(\xi_{1,l}-z_2)^2}{(p-ab)\pi\sqrt{-1}} - \frac{z_2^2}{ab\pi\sqrt{-1}} + 2\xi_{1,l}$ has a unique critical point

$$\frac{ab}{p}\xi_{1,l} = ab\Big(\frac{l}{p} + \frac{1}{2}\Big)\pi\sqrt{-1} - \frac{ab}{pn}\pi\sqrt{-1}.$$

Put

$$\alpha := ab\Big(\frac{l}{p}+\frac{1}{2}\Big)\pi\sqrt{-1},\quad \beta := \frac{ab}{p}\pi\sqrt{-1},$$

so that the critical point becomes $\alpha-\beta/n$. Then we can write

$$F(\xi_{1,l},z_2) = F(\xi_{1,l},\alpha-\beta/n) - \frac{p}{ab(p-ab)\pi\sqrt{-1}}(z_2-\alpha+\beta/n)^2. \tag{13}$$

Note that $\alpha-\beta/n$ is not a pole of $\varphi(z_2)$.

We will shift the line of integration from $C_\theta + w_2$ to $C_\theta+\alpha-\beta/n$. Since $0 < l+p/2-1/n < p$, we see that $\alpha-\beta/n$ is below $w_2 = ab\pi\sqrt{-1}$ in the complex plane. So, by the residue theorem we have

$$\begin{aligned}&\int_{C_\theta+w_2}\varphi(z_2)e^{nF(\xi_{1,l},z_2)}\,dz_2\\ &\quad=\int_{C_\theta+\alpha-\beta/n}\varphi(z_2)e^{nF(\xi_{1,l},z_2)}\,dz_2\\ &\qquad-2\pi\sqrt{-1}\sum_{\frac{2ab}{p}(l+p/2-1/n)<m'<2ab}\operatorname{Res}(\varphi;\eta_{m'})e^{nF(\xi_{1,l},\eta_{m'})}.\end{aligned}$$

Note that from (13) we have

$$\begin{aligned}&\int_{C_\theta+\alpha-\beta/n}\varphi(z_2)e^{nF(\xi_{1,l},z_2)}\,dz_2\\ &\quad=e^{nF(\xi_{1,l},\alpha-\beta/n)}\int_{C_\theta-\beta/n}\varphi(z_2+\alpha)e^{\frac{-pn}{ab(p-ab)\pi\sqrt{-1}}(z_2+\beta/n)^2}\,dz_2\\ &\quad=e^{nF(\xi_{1,l},\alpha-\beta/n)}e^{\frac{-ab\pi\sqrt{-1}}{p(p-ab)n}}\\ &\qquad\times\int_{C_\theta-\beta/n}\varphi(z_2+\alpha)e^{-\frac{2}{p-ab}z_2}e^{\frac{-pn}{ab(p-ab)\pi\sqrt{-1}}z_2^2}\,dz_2\\ &\quad=e^{nF(\xi_{1,l},\alpha-\beta/n)}e^{\frac{-ab\pi\sqrt{-1}}{p(p-ab)n}}\\ &\qquad\times\int_{C_\theta}\varphi(z_2+\alpha)e^{-\frac{2}{p-ab}z_2}e^{\frac{-pn}{ab(p-ab)\pi\sqrt{-1}}z_2^2}\,dz_2,\end{aligned}$$

where we change the variable $z_2\mapsto z_2+\alpha$ at the first equality, and use the fact that $\varphi(z_2+\alpha)$ has no poles between C_θ and $C_\theta-\beta/n$ for sufficiently large n. This can be shown as follows: Suppose for a contradiction that there is a pole between C_θ

and $C_\theta - \beta/n$. It is of the form $k\pi\sqrt{-1}/2 - ab(l/p+1/2)\pi\sqrt{-1}$ with $k \in \mathbb{Z}$ not divisible by a or b. So, it should satisfy the inequality

$$-ab/(pn) \leq k/2 - ab(l/p+1/2) \leq 0.$$

If n is sufficiently large, then this implies that $2abl/p = k - ab$. Since p and ab are coprime, this means that $2l/p$ is an integer. However since $-p/2 \leq l \leq p/2$ we see that $l = -p/2, 0$, or $p/2$. Then k equals 0, ab, or $2ab$ respectively, which is divisible by a and b, a contradiction.

By the saddle point method, we have

$$\int_{C_\theta} \varphi(z_2+\alpha) e^{-\frac{2}{p-ab}z_2} e^{\frac{-pn}{ab(p-ab)\pi\sqrt{-1}}z_2^2}\, dz_2$$
$$= \sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\varphi(\alpha) + O(n^{-3/2}).$$

Hence, we have

$$\int_{C_{\theta+\alpha-\beta/n}} \varphi(z_2) e^{nF(\xi_{1,l},z_2)}\, dz_2$$
$$= e^{nF(\xi_{1,l},\alpha-\beta/n)} e^{\frac{-ab\pi\sqrt{-1}}{p(p-ab)n}} \sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\varphi(\alpha) + O(n^{-3/2}),$$

where

$$\varphi(\alpha) = (-1)^{ab+a+b}\sqrt{-1}\frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})}{\sin(\frac{2abl\pi}{p})}.$$

Since $\xi_{1,l} = (l+p/2-1/n)\pi\sqrt{-1}$ and $\alpha - \beta/n = \frac{ab}{p}\xi_{1,l}$, we have

$$\begin{aligned}
&F(\xi_{1,l}, \alpha-\beta/n) \\
&\quad = -\frac{(\xi_{1,l} - \frac{ab}{p}\xi_{1,l})^2}{(p-ab)\pi\sqrt{-1}} - \frac{(\frac{ab}{p}\xi_{1,l})^2}{ab\pi\sqrt{-1}} + 2\xi_{1,l} \\
&\quad = \Big(-\frac{p-ab}{p^2\pi\sqrt{-1}} - \frac{ab}{p^2\pi\sqrt{-1}}\Big)\xi_{1,l}^2 + 2\xi_{1,l} \\
&\quad = -\frac{1}{p\pi\sqrt{-1}}\xi_{1,l}^2 + 2\xi_{1,l} \\
&\quad = \Big(-\frac{(l+p/2-1/n)^2}{p} + 2(l+p/2-1/n)\Big)\pi\sqrt{-1} \\
&\quad = \Big(-\frac{(l+p/2)^2}{p} + 2(l+p/2) + (\frac{l+p/2}{p} - 1)\frac{2}{n} - \frac{1}{pn^2}\Big)\pi\sqrt{-1}.
\end{aligned}$$

Hence, we have

$$\int_{C_{\theta+\alpha-\beta/n}} \varphi(z_2) e^{nF(\xi_{1,l},z_2)}\, dz_2$$
$$= \sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\sqrt{-1}(-1)^{ab+a+b}$$
$$\times \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})}{\sin(\frac{2abl\pi}{p})} e^{-(\frac{ab}{p(p-ab)}+\frac{1}{p})\frac{\pi\sqrt{-1}}{n}}$$
$$\times (-1)^{np-1} e^{\frac{2l}{p}\pi\sqrt{-1}} \exp\left[n\left(-\frac{(l+p/2)^2}{p}\right)\pi\sqrt{-1}\right] + O(n^{-3/2}).$$

We finally have

$$\begin{aligned}
I_{1,1} &= \sum_{-p/2+1/n<l<p/2+1/n} \operatorname{Res}(\psi_1;\xi_{1,l}) \int_{C_\theta+w_2} \varphi(z_2) e^{nF(\xi_{1,l},z_2)}\, dz_2 \\
&= -2\pi\sqrt{-1} \sum_{\substack{-p/2+1/n<l<p/2+1/n \\ \frac{2ab}{p}(l+p/2-1/n)<m'<2ab}} \operatorname{Res}(\psi_1;\xi_{1,l}) \operatorname{Res}(\varphi;\eta_{m'}) e^{nF(\xi_{1,l}),\eta_{m'}} \\
&\quad + \sum_{-p/2+1/n<l<p/2+1/n} \operatorname{Res}(\psi_1;\xi_{1,l}) \int_{C_{\theta+\alpha-\beta/n}} \varphi(z_2) e^{nF(\xi_{1,l},z_2)}\, dz_2 \\
&= -2\pi\sqrt{-1} \sum_{\substack{-p/2+1/n<l<p/2+1/n \\ \frac{2ab}{p}(l+p/2-1/n)<m'<2ab}} \operatorname{Res}(\psi_1;\xi_{1,l}) \operatorname{Res}(\varphi;\eta_{m'}) e^{nF(\xi_{1,l},\eta_{m'})} \\
&\quad + \sum_{-p/2+1/n<l<p/2+1/n} \sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\sqrt{-1}(-1)^{ab+a+b} \\
&\quad \times \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})}{\sin(\frac{2abl\pi}{p})} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\quad \times (-1)^{np-1} e^{\frac{2l}{p}\pi\sqrt{-1}} \exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\Big] + O(n^{-3/2}).
\end{aligned} \tag{14}$$

4.4. The sum over *m* in V_1

In this section we study the asymptotic behavior of $I_{1,2}$.

Consider the integral

$$\int_{C_\theta+w_1} \psi_1(z_1) e^{nF(z_1,\eta_m)}\, dz_1.$$

The polynomial

$$F(z_1, \eta_m) = -\frac{(z_1 - \eta_m)^2}{(p-ab)\pi\sqrt{-1}} - \frac{\eta_m^2}{ab\pi\sqrt{-1}} + 2z_1$$

has a unique critical point

$$\gamma := (m/2 + p - ab)\pi\sqrt{-1}$$

and we can write

$$F(z_1, \eta_m) = F(\gamma, \eta_m) - \frac{1}{(p-ab)\pi\sqrt{-1}}(z_1 - \gamma)^2.$$

Note that γ is not a pole of $\psi_1(z_1)$. Since γ is below $w_1 = p\pi\sqrt{-1}$ in the complex plane, we have

$$\begin{aligned}
&\int_{C_{\theta+w_1}} \psi_1(z_1)e^{nF(z_1,\eta_m)}\,dz_1 \\
&\quad= \int_{C_\theta+\gamma} \psi_1(z_1)e^{nF(z_1,\eta_m)}\,dz_1 - 2\pi\sqrt{-1} \\
&\quad=\times \sum_{m/2+p/2-ab+1/n<l'<p/2+1/n} \mathrm{Res}(\psi_1;\xi_{1,l'})e^{nF(\xi_{1,l'},\eta_m)}.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{C_\theta+\gamma} \psi_1(z_1)e^{nF(z_1,\eta_m)}\,dz_1 &= e^{nF(\gamma,\eta_m)}\int_{C_\theta+\gamma} \psi_1(z_1)e^{-\frac{n(z_1-\gamma)^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1 \\
&= e^{nF(\gamma,\eta_m)}\int_{C_\theta} \psi_1(z_1+\gamma)e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1.
\end{aligned}$$

Since $\gamma = (m/2 + p - ab)\pi\sqrt{-1}$ and $\eta_m = m\pi\sqrt{-1}/2$ we have

$$F(\gamma, \eta_m) = \Big(p - ab - \frac{m^2}{4ab} + m\Big)\pi\sqrt{-1}.$$

Note that $\psi_1(z_1 + \gamma) = \coth(z_1 + \gamma + \tilde{h})$. Put

$$\delta := (m/2 + p/2 - ab)\pi\sqrt{-1}.$$

Then $\psi_1(z_1 + \gamma) = \coth(z_1 + \delta + h)$.

The integral above becomes

$$\int_{C_\theta} \coth(z_1 + \delta + h) e^{-\frac{n z_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$
$$= \int_{C_\theta + h} \coth(z_1 + \delta) e^{-\frac{n (z_1 - h)^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$
$$= e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \int_{C_\theta + h} e^{\frac{2z_1}{p-ab}} \coth(z_1 + \delta) e^{-\frac{n z_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1.$$

There are two cases to consider.

Case 1. If $m \not\equiv p \pmod 2$, then $\delta \in (\mathbb{Z} + 1/2)\pi\sqrt{-1}$ and so

$$\coth(z_1 + \delta) = \coth(z_1 + \pi\sqrt{-1}/2).$$

Hence, we have

$$\int_{C_\theta + h} e^{\frac{2z_1}{p-ab}} \coth(z_1 + \delta) e^{-\frac{n z_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$
$$= \int_{C_\theta} e^{\frac{2z_1}{p-ab}} \coth(z_1 + \pi\sqrt{-1}/2) e^{-\frac{n z_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$

since there are no poles of $\coth(z_1 + \pi\sqrt{-1}/2)$ between C_θ and $C_\theta + h$. Now, we want to apply the saddle point method. To do that, we split $\coth(z_1 + \pi\sqrt{-1}/2)$ into 1 and $\coth(z_1 + \pi\sqrt{-1}/2) - 1$. Then the integral above becomes

$$\int_{C_\theta} e^{\frac{2z_1}{p-ab}} e^{-\frac{n z_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$
$$+ \int_{C_\theta} e^{\frac{2z_1}{p-ab}} (\coth(z_1 + \pi\sqrt{-1}/2) - 1) e^{-\frac{n z_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$
$$= \sqrt{\frac{(p-ab)\pi^2\sqrt{-1}}{n}} + O(n^{-3/2}) - \left(\sqrt{\frac{(p-ab)\pi^2\sqrt{-1}}{n}} + O(n^{-3/2})\right)$$
$$= O(n^{-3/2}).$$

Case 2. If $m = p \pmod 2$, then $\delta \in \mathbb{Z}\pi\sqrt{-1}$ and

$$\coth(z_1 + \delta) = \coth(z_1).$$

Hence, we have

$$\begin{aligned}
&\int\limits_{C_\theta+h} e^{\frac{2z_1}{p-ab}}\coth(z_1+\delta)e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1\\
&\quad=\int\limits_{C_\theta+h}\Big(e^{\frac{2z_1}{p-ab}}\coth(z_1)-\frac{1}{z_1}\Big)e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1+\int\limits_{C_\theta+h}\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1\\
&\quad=\int\limits_{C_\theta}\Big(e^{\frac{2z_1}{p-ab}}\coth(z_1)-\frac{1}{z_1}\Big)e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1+\int\limits_{C_\theta+h}\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1\\
&\quad=\sqrt{\frac{(p-ab)\pi^2\sqrt{-1}}{n}}\frac{2}{p-ab}+O(n^{-3/2})+\int\limits_{C_\theta+h}\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1
\end{aligned}$$

since there are no poles of $e^{\frac{2z_1}{p-ab}}\coth(z_1)-\frac{1}{z_1}$ between C_θ and $C_\theta+h$. Note that

$$\Big(e^{\frac{2z_1}{p-ab}}\coth(z_1)-\frac{1}{z_1}\Big)\Big|_{z_1=0}=\frac{2}{p-ab}.$$

Replacing z_1+h with $-z_1+h$, we have

$$\int\limits_{C_\theta+h}\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1=-\int\limits_{C_\theta-h}\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1.$$

Hence, we have

$$\begin{aligned}
&\int\limits_{C_\theta+h}\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1\\
&\quad=\frac{1}{2}\Big(\int\limits_{C_\theta+h}\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1-\int\limits_{C_\theta-h}\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1\Big)\\
&\quad=-\pi\sqrt{-1}\,\mathrm{Res}\Big(\frac{1}{z_1}e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}};0\Big)=-\pi\sqrt{-1}.
\end{aligned}$$

So, in this case we have

$$\begin{aligned}
&\int\limits_{C_\theta+h} e^{\frac{2z_1}{p-ab}}\coth(z_1+\delta)e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1\\
&\qquad=-\pi\sqrt{-1}+\frac{2}{p-ab}\sqrt{\frac{(p-ab)\pi^2\sqrt{-1}}{n}}+O(n^{-3/2})
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{C_\theta+\gamma} \psi_1(z_1) e^{nF(z_1,\eta_m)}\,dz_1 \\
&\quad = e^{nF(\gamma,\eta_m)} e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \int_{C_\theta+h} e^{\frac{2z_1}{p-ab}} \coth(z_1+\delta) e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1 \\
&\quad = \left(-\pi\sqrt{-1} + 2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\right) e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \\
&\qquad \times \exp\left[n\left(p-ab-\frac{m^2}{4ab}+m\right)\pi\sqrt{-1}\right] + O(n^{-3/2}).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
I_{1,2} =& \sum_{m=1}^{2ab-1} \operatorname{Res}(\varphi;\eta_m) \int_{C_\theta+w_1} \psi_1(z_1) e^{nF(z_1,\eta_m)}\,dz_1 \\
=& -2\pi\sqrt{-1} \sum_{\substack{m/2+p/2-ab+1/n<l'<p/2+1/n \\ 1\le m\le 2ab-1}} \operatorname{Res}(\varphi;\eta_m) \operatorname{Res}(\psi_1;\xi_{1,l'}) e^{nF(\xi_{1,l'})\eta_m} \\
&\quad + \sum_{m=1}^{2ab-1} \operatorname{Res}(\varphi;\eta_m) \int_{C_\theta+\gamma} \psi_1(z_1) e^{nF(z_1,\eta_m)}\,dz_1 \\
=& -2\pi\sqrt{-1} \sum_{\substack{m/2+p/2-ab+1/n<l'<p/2+1/n \\ 1\le m\le 2ab-1}} \operatorname{Res}(\varphi;\eta_m) \operatorname{Res}(\psi_1;\xi_{1,l'}) e^{nF(\xi_{1,l'})\eta_m} \\
&+ \sum_{\substack{1\le m\le 2ab-1 \\ m-p\text{: even}}} \frac{(-1)^{m+1}}{2} \sin\left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi}{b}\right) \\
&\times \left(-\pi\sqrt{-1} + 2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\right) e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \exp\left[n\left(-ab-\frac{m^2}{4ab}\right)\pi\sqrt{-1}\right] \\
&+ O(n^{-3/2}),
\end{aligned}
\tag{15}
$$

where l' is an integer such that

$$
m/2 + p - ab < l' + p/2 - 1/n < p.
$$

4.5. Summary

Now, we combine the results so far.

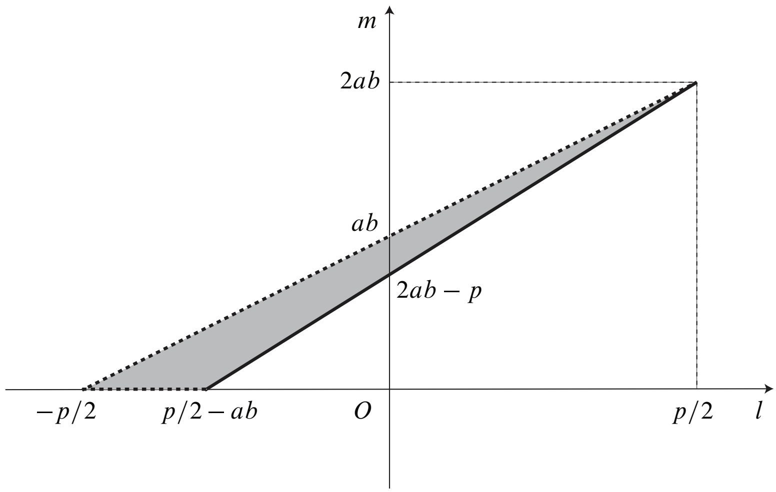

Figure 5. The gray area indicates $\mathcal{S}_1$. The dotted lines are not included.

Note that $I_{1,3}$ is the sum of $\mathrm{Res}(\psi_1; \xi_{1,l}) \, \mathrm{Res}(\varphi; \eta_m) e^{nF(\xi_{1,l}, \eta_m)}$ over (l, m). The same summand appears both in (14) and in (15). Taking cancellation into account, the range of summation turns out to be

$$
\begin{aligned}
\mathcal{S}_1 := &\left\{(l,m) \in \mathbb{Z}^2 \;\middle|\; -\frac{p}{2} < l - \frac{1}{n} < \frac{p}{2}, 0 < m < 2ab, a \nmid m, b \nmid m\right\} \\
&\quad \setminus \left\{(l,m) \in \mathbb{Z}^2 \;\middle|\; -\frac{p}{2} < l - \frac{1}{n} < \frac{p}{2}, \frac{2ab}{p}(l + \frac{p}{2} - \frac{1}{n}) < m < 2ab\right\} \\
&\quad \setminus \left\{(l,m) \in \mathbb{Z}^2 \;\middle|\; \frac{m}{2} + p - ab < l + \frac{p}{2} - \frac{1}{n} < p, 0 < m < 2ab,\right\} \\
= &\left\{(l,m) \in \mathbb{Z}^2 \;\middle|\; -\frac{p}{2} < l - \frac{1}{n} < \frac{p}{2}, m \le \frac{2ab}{p}\Big(l - \frac{1}{n}\Big) + ab,\right. \\
&\quad \left. l - \frac{1}{n} \le \frac{m}{2} + \frac{p}{2} - ab, 0 < m < 2ab, a \nmid m, b \nmid m\right\} \\
= &\left\{(l,m) \in \mathbb{Z}^2 \;\middle|\; -\frac{p}{2} < l \le \frac{p}{2}, m < \frac{2abl}{p} + ab,\right. \\
&\quad \left. l \le \frac{m}{2} + \frac{p}{2} - ab, 0 < m < 2ab, a \nmid m, b \nmid m\right\}.
\end{aligned}
$$

for large n. Note that the range $\mathcal{S}_1$ is the integer points in the interior of the triangle with vertices

$$(p/2, 2ab), \quad (-p/2, 0), \quad (p/2 - ab, 0)$$

on the lm-plane. See Figure 5.

So, we have

$$
\begin{aligned}
V_1 &= I_{1,0} \\
&\quad + \sum_{(l,m)\in\mathcal{S}_1} 2\pi^2(-1)^m \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) e^{\frac{2l-m+p}{p-ab}\pi\sqrt{-1}} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\quad \times (-1)^{pn}\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab}-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&\quad + \sum_{-p/2+1/n<l<p/2+1/n} (2\pi\sqrt{-1})\sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\sqrt{-1}(-1)^{ab+a+b} \\
&\quad \times \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})}{\sin(\frac{2abl\pi}{p})} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\quad \times (-1)^{np-1} e^{\frac{2l}{p}\pi\sqrt{-1}} \exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\Big] \\
&\quad + \sum_{\substack{1\le m\le 2ab-1 \\ m-p:\ \text{even}}} (-1)(2\pi\sqrt{-1})\frac{(-1)^m}{2}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) \\
&\quad \times \Big(-\pi\sqrt{-1}+2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\Big) e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \exp\Big[n\Big(-ab-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&\quad + O(n^{-3/2}).
\end{aligned}
$$

5. Asymptotic expansion of V_2

In this section we will study the asymptotic behavior of V_2 as $n \to \infty$. Recall that

$$V_2 = \iint_{C_\theta\times C_\theta} \psi_2(z_1)\varphi(z_2)e^{nF(z_1,z_2)}\,dz_1\,dz_2,$$

where $\psi_2(z_1) = \coth(z_1 - \tilde{h})$ with $\tilde{h} = (\frac{1}{n} - \frac{p}{2})\pi\sqrt{-1}$.

As in the case of V_1, we have

$$V_2 = I_{2,0} + 2\pi\sqrt{-1}I_{2,1} + 2\pi\sqrt{-1}I_{2,2} + (2\pi\sqrt{-1})^2 I_{2,3},$$

where

$$I_{2,0} := \iint_{(C_\theta+w_1)\times(C_\theta+w_2)} \psi_2(z_1)\varphi(z_2)e^{nF(z_1,z_2)}\,dz_1\,dz_2,$$

$$I_{2,1} := \sum_{-p/2-1/n<l<p/2-1/n} \operatorname{Res}(\psi_2; \xi_{2,l}) \int_{C_\theta + w_2} \varphi(z_2) e^{nF(\xi_{2,l}, z_2)}\, dz_2,$$

$$I_{2,2} := \sum_{m=1}^{2ab-1} \operatorname{Res}(\varphi; \eta_m) \int_{C_\theta + w_1} \psi_1(z_1) e^{nF(z_1, \eta_m)}\, dz_1,$$

$$I_{2,3} := \sum_{\substack{-p/2-1/n<l<p/2-1/n \\ 1\le m\le 2ab-1}} \operatorname{Res}(\psi_2; \xi_{2,l}) \operatorname{Res}(\varphi; \eta_m) e^{nF(\xi_{2,l}, \eta_m)}.$$

5.1. The double integral in V_2

In this section we calculate the double integral $I_{2,0}$.

By changing variables, we have

$$I_{2,0} = \iint_{C_\theta \times C_\theta} \psi_2(z_1 + w_1)\varphi(z_2 + w_2) e^{nF(z_1+w_1, z_2+w_2)}\, dz_1\, dz_2.$$

Note that $\psi_2(z_1 + w_1) = \psi_2(z_1)$ and $\varphi(z_2 + w_2) = \varphi(z_2)$. Moreover, we have

$$F(z_1 + w_1, z_2 + w_2) = -\frac{(z_1 - z_2)^2}{(p - ab)\pi\sqrt{-1}} - \frac{z_2^2}{ab\pi\sqrt{-1}} + p\pi\sqrt{-1}.$$

Hence, we have

$$I_{2,0} = \iint_{C_\theta \times C_\theta} \psi_2(z_1)\varphi(z_2) \exp\Big[n\Big(-\frac{(z_1 - z_2)^2}{(p - ab)\pi\sqrt{-1}} - \frac{z_2^2}{ab\pi\sqrt{-1}} + p\pi\sqrt{-1}\Big)\Big]\, dz_1\, dz_2.$$

Now, we change the variable $(z_1, z_2) \mapsto (-z_1, -z_2)$. Since $\varphi(-z_2) = -\varphi(z_2)$ and $\psi_2(-z_1) = -\psi_1(z_1)$, we conclude that $I_{1,0} = I_{2,0}$.

So, we see that

$$V_1 - V_2 = 2\pi\sqrt{-1}(I_{1,1} - I_{2,1} + I_{1,2} - I_{2,2}) + (2\pi\sqrt{-1})^2(I_{1,3} - I_{2,3}).$$

5.2. The double sum in V_2

In this section we calculate the double summation in V_2.

Since $\xi_{2,l} = (l + p/2 + 1/n)\pi\sqrt{-1}$ and $\eta_m = m\pi\sqrt{-1}/2$, we have

$$F(\xi_{2,l}, \eta_m) = \Big(-\frac{(l - m/2 + p/2 + 1/n)^2}{p - ab} - \frac{m^2}{4ab} + 2(l + p/2 + 1/n)\Big)\pi\sqrt{-1}.$$

Hence, we have

$$nF(\xi_{2,l},\eta_m) = \Big[\Big(-\frac{(l-m/2+p/2)^2}{p-ab} - \frac{m^2}{4ab} + 2l + p\Big)n - \frac{2l-m+p}{p-ab} + 2 - \frac{1}{(p-ab)n}\Big]\pi\sqrt{-1}.$$

Since

$$\operatorname{Res}(\psi_2;\xi_{2,l})\operatorname{Res}(\varphi;\eta_m) = -\frac{(-1)^m}{2}\sin(m\pi/a)\sin(m\pi/b),$$

we obtain

$$\begin{aligned} I_{2,3} &= \sum_{\substack{-p/2-1/n<l<p/2-1/n \\ 1\le m\le 2ab-1}} \operatorname{Res}(\psi_2;\xi_{2,l})\operatorname{Res}(\varphi;\eta_m)e^{nF(\xi_{2,l},\eta_m)} \\ &= \sum_{\substack{-p/2-1/n<l<p/2-1/n \\ 1\le m\le 2ab-1}} \frac{(-1)^{m+1}}{2}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) \\ &\quad\times e^{-\frac{2l-m+p}{p-ab}\pi\sqrt{-1}}e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\ &\quad\times(-1)^{pn}\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big]. \end{aligned}$$

5.3. The sum over l in V_2

In this section we calculate $I_{2,1}$.

Consider the integral

$$\int_{C_\theta+w_2}\varphi(z_2)e^{nF(\xi_{2,l},z_2)}\,dz_2.$$

The polynomial $F(\xi_{2,l},z_2) = -\frac{(\xi_{2,l}-z_2)^2}{(p-ab)\pi\sqrt{-1}} - \frac{z_2^2}{ab\pi\sqrt{-1}} + 2\xi_{2,l}$ has a unique critical point

$$\frac{ab}{p}\xi_{2,l} = \alpha + \beta/n$$

and

$$F(\xi_{2,l},z_2) = F(\xi_{2,l},\alpha+\beta/n) - \frac{p}{ab(p-ab)\pi\sqrt{-1}}(z_2-\alpha-\beta/n)^2. \qquad (16)$$

Recall that $\alpha = ab(\frac{l}{p}+\frac{1}{2})\pi\sqrt{-1}$ and $\beta = \frac{ab}{p}\pi\sqrt{-1}$. Note that $\alpha+\beta/n$ is not a pole of $\varphi(z_2)$.

We will shift the line of integration from $C_\theta + w_2$ to $C_\theta + \alpha + \beta/n$. Since $0 < l + p/2 + 1/n < p$, we see that $\alpha + \beta/n$ is below $w_2 = ab\pi\sqrt{-1}$. So, by the residue theorem we have

$$\begin{aligned}\int_{C_\theta + w_2} \varphi(z_2) e^{nF(\xi_{2,l}, z_2)}\, dz_2 &= \int_{C_\theta+\alpha+\beta/n} \varphi(z_2) e^{nF(\xi_{2,l}, z_2)}\, dz_2 \\ &\quad - 2\pi\sqrt{-1} \sum_{\frac{2ab}{p}(l+p/2+1/n)<m'<2ab} \mathrm{Res}(\varphi; \eta_{m'}) e^{nF(\xi_{2,l}, \eta_{m'})}\, dz_2. \end{aligned} \tag{17}$$

Note that from (16) we have

$$\begin{aligned}\int_{C_\theta+\alpha+\beta/n} \varphi(z_2) e^{nF(\xi_{2,l}, z_2)}\, dz_2 &= e^{nF(\xi_{2,l}, \alpha+\beta/n)} \int_{C_\theta+\alpha+\beta/n} \varphi(z_2) e^{\frac{-pn(z_2-\alpha-\beta/n)^2}{ab(p-ab)\pi\sqrt{-1}}}\, dz_2 \\ &= e^{nF(\xi_{2,l}, \alpha+\beta/n)} \int_{C_\theta} \varphi(z_2+\alpha+\beta/n) e^{\frac{-pnz_2^2}{ab(p-ab)\pi\sqrt{-1}}}\, dz_2 \\ &= e^{nF(\xi_{2,l}, \alpha+\beta/n)} \int_{C_\theta+\beta/n} \varphi(z_2+\alpha) e^{-\frac{pn(z_2-\beta/n)^2}{ab(p-ab)\pi\sqrt{-1}}}\, dz_2 \\ &= e^{nF(\xi_{2,l}, \alpha+\beta/n)} e^{-\frac{ab\pi\sqrt{-1}}{p(p-ab)n}} \int_{C_\theta+\beta/n} e^{\frac{2z_2}{p-ab}} \varphi(z_2+\alpha) e^{-\frac{pnz_2^2}{ab(p-ab)\pi\sqrt{-1}}}\, dz_2 \\ &= e^{nF(\xi_{2,l}, \alpha+\beta/n)} e^{-\frac{ab\pi\sqrt{-1}}{p(p-ab)n}} \int_{C_\theta} e^{\frac{2z_2}{p-ab}} \varphi(z_2+\alpha) e^{-\frac{pnz_2^2}{ab(p-ab)\pi\sqrt{-1}}}\, dz_2, \end{aligned}$$

since there are no poles of $\varphi(z_2+\alpha)$ between $C_\theta + \beta/n$ and C_θ.

By the saddle point method, we have

$$\int_{C_\theta} e^{\frac{2z_2}{p-ab}} \varphi(z_2+\alpha) e^{-\frac{pnz_2^2}{ab(p-ab)\pi\sqrt{-1}}}\, dz_2 = \sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\varphi(\alpha) + O(n^{-3/2}).$$

Hence, we have

$$\int_{C_\theta+\alpha+\beta/n} \varphi(z_2) e^{nF(\xi_{2,l},z_2)}\,dz_2$$
$$= e^{nF(\xi_{2,l},\alpha+\beta/n)} e^{-\frac{ab\pi\sqrt{-1}}{p(p-ab)n}} \sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\varphi(\alpha) + O(n^{-3/2}),$$

where

$$\varphi(\alpha) = (-1)^{ab+a+b}\sqrt{-1}\frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})}{\sin(\frac{2abl\pi}{p})}.$$

Since

$$\xi_{2,l} = (l + p/2 + 1/n)\pi\sqrt{-1}$$

and

$$\alpha + \beta/n = \frac{ab}{p}\xi_{2,l},$$

we have

$$\begin{aligned} &F(\xi_{2,l}, \alpha + \beta/n) \\ &\quad = -\frac{(\xi_{2,l} - \alpha - \beta/n)^2}{(p-ab)\pi\sqrt{-1}} - \frac{(\alpha + \beta/n)^2}{ab\pi\sqrt{-1}} + 2\xi_{2,l} \\ &\quad = -\frac{\xi_{2,l}^2}{p\pi\sqrt{-1}} + 2\xi_{2,l} \\ &\quad = \Big(-\frac{(l + p/2 + 1/n)^2}{p} + 2(l + p/2 + 1/n)\Big)\pi\sqrt{-1} \\ &\quad = \Big(-\frac{(l + p/2)^2}{p} + 2(l + p/2) - (\frac{l + p/2}{p} - 1)\frac{2}{n} - \frac{1}{pn^2}\Big)\pi\sqrt{-1}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\int_{C_\theta+\alpha+\beta/n} \varphi(z_2) e^{nF(\xi_{2,l},z_2)}\,dz_2 \\ &\quad = \sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}(-1)^{ab+a+b}\sqrt{-1} \\ &\qquad \times \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})}{\sin(\frac{2abl\pi}{p})} e^{-(\frac{ab}{p(p-ab)}+\frac{1}{p})\frac{\pi\sqrt{-1}}{n}} \\ &\qquad \times (-1)^{np-1} e^{-\frac{2l}{p}\pi\sqrt{-1}} \exp\Big[n\Big(-\frac{(l + p/2)^2}{p}\Big)\pi\sqrt{-1}\Big] + O(n^{-3/2}). \end{aligned}$$

We finally have

$$
\begin{aligned}
I_{2,1} = & \sum_{-p/2-1/n<l<p/2-1/n} \operatorname{Res}(\psi_2;\xi_{2,l}) \int_{C_\theta+w_2} \varphi(z_2) e^{nF(\xi_{2,l},z_2)}\, dz_2 \\
= & -2\pi\sqrt{-1} \sum_{\substack{-p/2-1/n<l<p/2-1/n \\ \frac{2ab}{p}(l+p/2+1/n)<m'<2ab}} \operatorname{Res}(\psi_2;\xi_{2,l}) \operatorname{Res}(\varphi;\eta_{m'}) e^{nF(\xi_{2,l},\eta_{m'})} \\
& + \sum_{-p/2-1/n<l<p/2-1/n} \operatorname{Res}(\psi_2;\xi_{2,l}) \int_{C_\theta+\alpha+\beta/n} \varphi(z_2) e^{nF(\xi_{2,l},z_2)}\, dz_2 \\
= & -2\pi\sqrt{-1} \sum_{\substack{-p/2-1/n<l<p/2-1/n \\ \frac{2ab}{p}(l+p/2+1/n)<m'<2ab}} \operatorname{Res}(\psi_2;\xi_{2,l}) \operatorname{Res}(\varphi;\eta_{m'}) e^{nF(\xi_{2,l},\eta_{m'})} \\
& + \sum_{-p/2-1/n<l<p/2-1/n} \sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}} (-1)^{ab+a+b}\sqrt{-1} \\
& \times \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})}{\sin(\frac{2abl\pi}{p})} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
& \times (-1)^{np-1} e^{-\frac{2l}{p}\pi\sqrt{-1}} \exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\Big] + O(n^{-3/2}),
\end{aligned}
$$

where

$$
\frac{2ab}{p}(l+p/2+1/n) < m' < 2ab.
$$

5.4. The sum over m in V_2

In this section we study the asymptotic behavior of $I_{2,2}$.

Consider

$$
\int_{C_\theta+w_1} \psi_2(z_1) e^{nF(z_1,\eta_m)}\, dz_1.
$$

The polynomial $F(z_1,\eta_m) = -\frac{(z_1-\eta_m)^2}{(p-ab)\pi\sqrt{-1}} - \frac{\eta_m^2}{ab\pi\sqrt{-1}} + 2z_1$ has a unique critical point

$$
\gamma := (m/2+p-ab)\pi\sqrt{-1}
$$

and

$$
F(z_1,\eta_m) = F(\gamma,\eta_m) - \frac{1}{(p-ab)\pi\sqrt{-1}}(z_1-\gamma)^2.
$$

Note that γ is not a pole of $\psi_2(z_1)$. We have

$$\int\limits_{C_\theta+w_1} \psi_2(z_1)e^{nF(z_1,\eta_m)}\,dz_1$$
$$= \int\limits_{C_\theta+\gamma} \psi_2(z_1)e^{nF(z_1,\eta_m)}$$
$$-2\pi\sqrt{-1} \sum_{m/2+p/2-ab-1/n<l'<p/2-1/n} \operatorname{Res}(\psi_2;\xi_{2,l'})e^{nF(\xi_{2,l'},\eta_m)}\,dz_1. \quad (18)$$

Note that

$$\int\limits_{C_\theta+\gamma} \psi_2(z_1)e^{nF(z_1,\eta_m)}\,dz_1 = e^{nF(\gamma,\eta_m)} \int\limits_{C_\theta+\gamma} \psi_2(z_1)e^{-\frac{n(z_1-\gamma)^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$
$$= e^{nF(\gamma,\eta_m)} \int\limits_{C_\theta} \psi_2(z_1+\gamma)e^{-\frac{nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1.$$

Since $\gamma = (m/2 + p - ab)\pi\sqrt{-1}$ and $\eta_m = m\pi\sqrt{-1}/2$, we have

$$F(\gamma,\eta_m) = \Big(p - ab - \frac{m^2}{4ab} + m\Big)\pi\sqrt{-1}.$$

Note that $\psi_2(z_1+\gamma) = \coth(z_1+\gamma-\tilde{h}')$. Put

$$\delta := (m/2 + p/2 - ab)\pi\sqrt{-1}.$$

Then

$$\psi_2(z_1+\gamma) = \coth(z_1+\delta-h).$$

The integral above becomes

$$\int\limits_{C_\theta} \cot(z_1+\delta-h)e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$
$$= \int\limits_{C_\theta-h} \coth(z_1+\delta)e^{\frac{-n(z_1+h)^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1$$
$$= e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \int\limits_{C_\theta-h} e^{\frac{-2z_1}{p-ab}} \coth(z_1+\delta)e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\,dz_1.$$

There are two cases to consider.

Case 1. If $m \neq p \pmod 2$, then $\delta \in (\mathbb{Z} + 1/2)\pi\sqrt{-1}$ and so

$$\coth(z_1 + \delta) = \coth(z_1 + \pi\sqrt{-1}/2).$$

Hence, we have

$$\begin{aligned}
&\int\limits_{C_\theta - h} e^{\frac{-2z_1}{p-ab}} \coth(z_1 + \delta) e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1 \\
&\quad = \int\limits_{C_\theta} e^{\frac{-2z_1}{p-ab}} \coth(z_1 + \pi\sqrt{-1}/2) e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1 \\
&\quad = \sqrt{\frac{(p-ab)\pi^2\sqrt{-1}}{n}} \coth(\pi\sqrt{-1}/2) + O(n^{-3/2}) = O(n^{-3/2})
\end{aligned}$$

since there are no poles of $\coth(z_1 + \pi\sqrt{-1}/2)$ between C_θ and $C_\theta - h$.

Case 2. If $m = p \pmod 2$, then $\delta \in \mathbb{Z}\pi\sqrt{-1}$ and

$$\coth(z_1 + \delta) = \coth(z_1).$$

Hence, we have

$$\begin{aligned}
&\int\limits_{C_\theta - h} e^{\frac{-2z_1}{p-ab}} \coth(z_1 + \delta) e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1 \\
&= \int\limits_{C_\theta - h} \Big(e^{\frac{-2z_1}{p-ab}} \coth(z_1) - \frac{1}{z_1}\Big) e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1 + \int\limits_{C_\theta - h} \frac{1}{z_1} e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1 \\
&= \int\limits_{C_\theta} \Big(e^{\frac{-2z_1}{p-ab}} \coth(z_1) - \frac{1}{z_1}\Big) e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1 + \int\limits_{C_\theta - h} \frac{1}{z_1} e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1 \\
&= \sqrt{\frac{(p-ab)\pi^2\sqrt{-1}}{n}} \Big(-\frac{2}{p-ab}\Big) + O(n^{-3/2}) + \int\limits_{C_\theta - h} \frac{1}{z_1} e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1
\end{aligned}$$

since there are no poles of $e^{\frac{-2z_1}{p-ab}} \coth(z_1) - \frac{1}{z_1}$ between C_θ and $C_\theta - h$. Note that

$$\Big(e^{\frac{-2z_1}{p-ab}} \coth(z_1) - \frac{1}{z_1}\Big)\Big|_{z_1=0} = \frac{-2}{p-ab}.$$

Moreover, we have

$$\int\limits_{C_\theta - h} \frac{1}{z} e^{\frac{-nz^2}{(p-ab)\pi\sqrt{-1}}}\, dz = -\int\limits_{C_\theta + h} \frac{1}{z} e^{\frac{-nz^2}{(p-ab)\pi\sqrt{-1}}}\, dz = \pi\sqrt{-1}.$$

Therefore, in this case we have

$$\int_{C_\theta - h} e^{\frac{-2z_1}{p-ab}} \cosh(z_1+\delta) e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1$$

$$= \pi\sqrt{-1} - \frac{2}{p-ab}\sqrt{\frac{(p-ab)\pi^2\sqrt{-1}}{n}} + O(n^{-3/2})$$

and

$$\int_{C_{\theta+\gamma}} \psi_2(z_1) e^{nF(z_1,\eta_m)}\, dz_1$$

$$= e^{nF(\gamma,\eta_m)} e^{\frac{-\pi\sqrt{-1}}{(p-ab)n}} \int_{C_\theta - h} e^{\frac{-2z_1}{p-ab}} \coth(z_1+\delta) e^{\frac{-nz_1^2}{(p-ab)\pi\sqrt{-1}}}\, dz_1$$

$$= \left(\pi\sqrt{-1} - 2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\right) e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}$$

$$\times \exp\left[n\left(p-ab-\frac{m^2}{4ab}+m\right)\pi\sqrt{-1}\right] + O(n^{-3/2}).$$

Hence, we have

$$I_{2,2} = \sum_{m=1}^{2ab-1} \operatorname{Res}(\varphi;\eta_m) \int_{C_\theta + w_1} \psi_2(z_1) e^{nF(z_1,\eta_m)}\, dz_1$$

$$= -2\pi\sqrt{-1} \sum_{\substack{m/2+p/2-ab-1/n<l'<p/2-1/n \\ 1\le m\le 2ab-1}} \operatorname{Res}(\varphi;\eta_m) \operatorname{Res}(\psi_2;\xi_{2,l'}) e^{nF(\xi_{2,l'},\eta_m)}$$

$$+ \sum_{m=1}^{2ab-1} \operatorname{Res}(\varphi;\eta_m) \int_{C_{\theta+\gamma}} \psi_2(z_1) e^{nF(z_1,\eta_m)}\, dz_1$$

$$= -2\pi\sqrt{-1} \sum_{\substack{m/2+p/2-ab-1/n<l'<p/2-1/n \\ 1\le m\le 2ab-1}} \operatorname{Res}(\varphi;\eta_m) \operatorname{Res}(\psi_2;\xi_{2,l'}) e^{nF(\xi_{2,l'},\eta_m)}$$

$$+ \sum_{\substack{1\le m\le 2ab-1 \\ m-p:\ \text{even}}} (-1)(2\pi\sqrt{-1}) \frac{(-1)^m}{2} \sin\left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi}{b}\right)$$

$$\times \left(\pi\sqrt{-1} - 2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\right) e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \exp\left[n\left(-ab-\frac{m^2}{4ab}\right)\pi\sqrt{-1}\right]$$

$$+ O(n^{-3/2}).$$

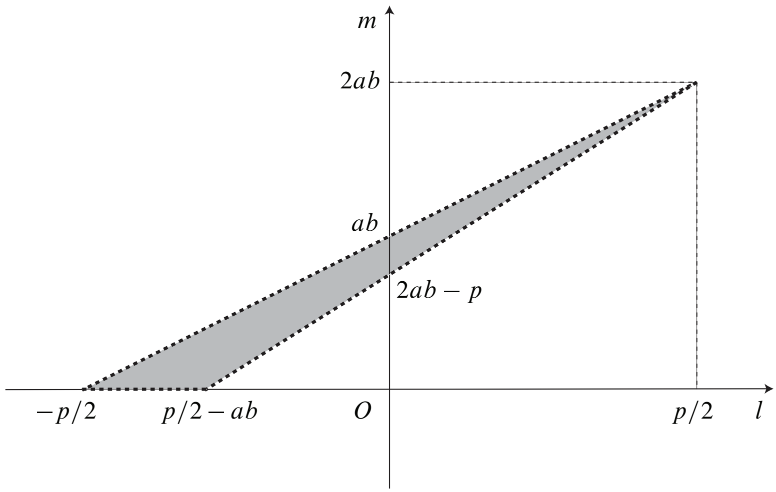

Figure 6. The gray area indicates $\mathcal{S}_2$. The edges are not included.

5.5. Summary

Now, we combine the results so far.

Note that $I_{2,3}$ is the sum of $\mathrm{Res}(\psi_2; \xi_{2,l}) \, \mathrm{Res}(\varphi; \eta_m) e^{nF(\xi_{2,l}, \eta_m)}$ over (l, m). The same summand appears both in (17) and in (18). Taking cancellation into account, the range of summation turns out to be

$$\begin{aligned}
\mathcal{S}_2 := & \left\{(l, m) \in \mathbb{Z}^2 \;\middle|\; -\frac{p}{2} < l + \frac{1}{n} < \frac{p}{2}, 0 < m < 2ab, a \nmid m, b \nmid m\right\} \\
& \setminus \left\{(l, m) \in \mathbb{Z}^2 \;\middle|\; -\frac{p}{2} < l + \frac{1}{n} < \frac{p}{2}, \frac{2ab}{p}\left(l + \frac{p}{2} + \frac{1}{n}\right) < m < 2ab\right\} \\
& \setminus \left\{(l, m) \in \mathbb{Z}^2 \;\middle|\; \frac{m}{2} + p - ab < l + \frac{p}{2} + \frac{1}{n} < p, 0 < m < 2ab\right\} \\
= & \left\{(l, m) \in \mathbb{Z}^2 \;\middle|\; -\frac{p}{2} < l + \frac{1}{n} < \frac{p}{2}, m \le \frac{2ab}{p}\left(l + \frac{1}{n}\right) + ab,\right. \\
& \left. l + \frac{1}{n} \le \frac{m}{2} + \frac{p}{2} - ab, 0 < m < 2ab, a \nmid m, b \nmid m\right\} \\
= & \left\{(l, m) \in \mathbb{Z}^2 \;\middle|\; -\frac{p}{2} \le l < \frac{p}{2}, m \le \frac{2abl}{p} + ab, l < \frac{m}{2} + \frac{p}{2} - ab,\right. \\
& \left. 0 < m < 2ab, a \nmid m, b \nmid m\right\}.
\end{aligned}$$

Observe that the line

$$m = 2abl/p + ab$$

contains no integer points. This can be shown as follows. Suppose that $(l,m) \in \mathbb{Z}^2$ satisfies $m = 2abl/p + ab$. Then we have $pm = ab(2l + p)$ and so p divides $2l$ since $(p, ab) = 1$. So, m is a multiple of ab, contradicting the assumption that $a \nmid m$ and $b \nmid m$. Therefore, $\mathcal{S}_2$ is indeed of the form

$$\mathcal{S}_2 = \Big\{(l,m) \in \mathbb{Z}^2 \;\Big|\; -\frac{p}{2} < l < \frac{p}{2}, m < \frac{2abl}{p} + ab, l < \frac{m}{2} + \frac{p}{2} - ab, \\ 0 < m < 2ab, a \nmid m, b \nmid m\Big\}.$$

See Figure 6.

Note that

$$\mathcal{S}_1 = \mathcal{S}_2 \cup \partial S_1,$$

where ∂S_1 denotes the edge connecting $(p/2, 2ab)$ and $(p/2 - ab, 0)$, i.e,

$$\partial S_1 = \{(l,m) \in \mathbb{Z}^2 \mid 0 < m < 2ab, a \nmid m, b \nmid m, l = m/2 + p/2 - ab\}.$$

We have

$$\begin{aligned}
V_2 = I_{2,0} &+ \sum_{(l,m)\in\mathcal{S}_2} 2\pi^2(-1)^m \sin\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{b}\right) e^{-\frac{2l-m+p}{p-ab}\pi\sqrt{-1}} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\times (-1)^{pn}\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&+ \sum_{-p/2-1/n<l<p/n-1/n} (2\pi\sqrt{-1})\sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\sqrt{-1}(-1)^{a+b-ab} \\
&\times \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})}{\sin(\frac{2abl\pi}{p})} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\times (-1)^{np-1} e^{-\frac{2l}{p}\pi\sqrt{-1}} \exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\Big] \\
&+ \sum_{\substack{1\le m\le 2ab-1 \\ m-p\text{: even}}} (-1)(2\pi\sqrt{-1})\frac{(-1)^m}{2}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) \\
&\times \left(\pi\sqrt{-1} - 2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\right) e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \exp\Big[n\Big(-ab - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&+ O(n^{-3/2}).
\end{aligned}$$

6. V_1 and V_2

First of all note that $I_{1,0} = I_{2,0}$. Recall that $\mathcal{S}_1 = \mathcal{S}_2 \cup \partial\mathcal{S}_1$. Hence, we have

$$\begin{aligned}
V_1 - V_2 = &\sum_{(l,m)\in\mathcal{S}_1} 2\pi^2(-1)^m \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) e^{\frac{2l-m+p}{p-ab}\pi\sqrt{-1}} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\times (-1)^{pn}\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&-\sum_{(l,m)\in\mathcal{S}_2} 2\pi^2(-1)^m \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) e^{-\frac{2l-m+p}{p-ab}\pi\sqrt{-1}} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\times (-1)^{pn}\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&+ 2\pi\sqrt{-1}\sum_{l}\sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\sqrt{-1}(-1)^{ab+a+b} \\
&\times \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})\sin(\frac{2l\pi}{p})}{\sin(\frac{2abl\pi}{p})} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\quad\times (-1)^{np-1}\exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\Big] \\
&+ 2\sum_{m-p:\text{ even}}(-1)(2\pi\sqrt{-1})\frac{(-1)^m}{2}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) \\
&\times\Bigg(-\pi\sqrt{-1} + 2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\Bigg) e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}\exp\Big[n\Big(-ab - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&+ O(n^{-3/2}).
\end{aligned}$$

Since

$$e^{\frac{2l-m+p}{p-ab}\pi\sqrt{-1}} - e^{-\frac{2l-m+p}{p-ab}\pi\sqrt{-1}} = 2\sqrt{-1}\sin\Big(\frac{(2l-m+p)\pi}{p-ab}\Big),$$

we have

$$\begin{aligned}
V_1 - V_2 = &\sum_{(l,m)\in\partial\mathcal{S}_1} 2\pi^2(-1)^m \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) e^{\frac{2l-m+p}{p-ab}\pi\sqrt{-1}} e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\times (-1)^{pn}\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&+\sum_{(l,m)\in\mathcal{S}_2} 2\pi^2(-1)^m \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times 2\sqrt{-1}\sin\Big(\frac{(2l-m+p)\pi}{p-ab}\Big)e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\times (-1)^{pn}\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab}-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&+2\pi\sqrt{-1}\sum_{l}\sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}\sqrt{-1}(-1)^{ab+a+b} \\
&\times \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})\sin(\frac{2l\pi}{p})}{\sin(\frac{2abl\pi}{p})}e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\times (-1)^{np-1}\exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\Big] \\
&+2\sum_{m-p\text{: even}}(-1)(2\pi\sqrt{-1})\frac{(-1)^m}{2}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) \\
&\times\left(-\pi\sqrt{-1}+2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\right)e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}\exp\Big[n\Big(-ab-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&+O(n^{-3/2}).
\end{aligned}
$$

When $l = m/2 + p/2 - ab$, the summand in the first term becomes

$$
\begin{aligned}
&2\pi^2(-1)^m\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\quad\times(-1)^{pn}\exp\Big[n\Big(-(p-ab)-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big]
\end{aligned}
$$

and the summation is over all m such that $0 < m < 2ab$, $a \nmid m$, $b \nmid m$, and that $p-m$ is even. So, this term cancels with a part of the last term involving $-\pi\sqrt{-1}$. Therefore, we have

$$
\begin{aligned}
V_1 - V_2 = &\sum_{(l,m)\in\mathcal{S}_2}2\pi^2(-1)^m\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big) \\
&\times 2\sqrt{-1}\sin\Big(\frac{(2l-m+p)\pi}{p-ab}\Big)e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}} \\
&\times (-1)^{pn}\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab}-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big] \\
&+2\pi\sqrt{-1}\sum_{l}\sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}} \\
&\times\sqrt{-1}(-1)^{ab+a+b}\frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})\sin(\frac{2l\pi}{p})}{\sin(\frac{2abl\pi}{p})}e^{-\frac{1}{p-ab}\frac{\pi\sqrt{-1}}{n}}
\end{aligned}
$$

$$\times (-1)^{np-1} \exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\Big]$$

$$+ 2\sum_{m-p:\ \text{even}} (-1)(2\pi\sqrt{-1})\frac{(-1)^m}{2}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)$$

$$\times \left(2\sqrt{\frac{\pi^2\sqrt{-1}}{(p-ab)n}}\right)e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}\exp\Big[n\Big(-ab-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big]$$

$$+ O(n^{-3/2})$$

$$= (-1)^{pn}4\pi^2\sqrt{-1}e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}\sum_{(l,m)\in\mathcal{S}_2}(-1)^m\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)$$

$$\times \sin\Big(\frac{(2l-m+p)\pi}{p-ab}\Big)\exp\Big[n\Big(-\frac{(l-m/2+p/2)^2}{p-ab}-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big]$$

$$+ 2\pi(-1)^{ab+a+b+pn}\sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}$$

$$\times \sum_l \frac{\sin\left(\frac{2al\pi}{p}\right)\sin\left(\frac{2bl\pi}{p}\right)\sin\left(\frac{2l\pi}{p}\right)}{\sin\left(\frac{2abl\pi}{p}\right)}\exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\Big]$$

$$- 4\pi^2 e^{3\pi\sqrt{-1}/4}e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}\frac{(-1)^{abn}}{\sqrt{(p-ab)n}}$$

$$\times \sum_{m-p:\ \text{even}}(-1)^m\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-\frac{m^2 n}{4ab}\pi\sqrt{-1}\Big)$$

$$+ O(n^{-3/2}).$$

By Lemma 10.1 we have

$$\sum_{\substack{0\le m\le 2ab\\ m-p:\ \text{even}}}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big) = 0.$$

This implies that

$$V_1 - V_2 = (-1)^{pn}4\pi^2\sqrt{-1}e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}\sum_{(l,m)\in\mathcal{S}_2}(-1)^m\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)$$

$$\times \sin\Big(\frac{(2l-m+p)\pi}{p-ab}\Big)\exp\Big[n\Big(-\frac{(2l-m+p)^2}{4(p-ab)}-\frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big]$$

$$+ (-1)^{a+b+ab+pn}4\pi\sqrt{-1}\sqrt{\frac{ab(p-ab)\pi^2\sqrt{-1}}{pn}}e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}$$

$$\times \sum_{|l|\le (p-1)/2} \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})\sin(\frac{2l\pi}{p})}{\sin(\frac{2abl\pi}{p})} \exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\,\Big]$$
$$+ O(n^{-3/2}).$$

From (11), we have

$$\begin{aligned}
&\hat{\tau}_n(X_p;\exp(4\pi\sqrt{-1}/n))\\
&= \frac{e^{3\pi\sqrt{-1}/4}\sqrt{n}}{4\pi^2\sin(2\pi/n)}(-1)^{np+p}\\
&\quad\times e^{-(p-ab+a/b+b/a-3)\frac{\pi\sqrt{-1}}{n}} e^{\frac{n}{4}\pi\sqrt{-1}} \frac{1}{\sqrt{ab(p-ab)}}(V_1-V_2)\\
&= (-1)^{p+1} e^{-(p-ab+a/b+b/a-3)\frac{\pi\sqrt{-1}}{n}} e^{\frac{n}{4}\pi\sqrt{-1}} \frac{\sqrt{n}}{\sqrt{ab(p-ab)}\sin(2\pi/n)}\\
&\quad\times \Bigg\{ e^{\pi\sqrt{-1}/4} e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}} \sum_{(l,m)\in\mathcal{S}_2} (-1)^m \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\sin\Big(\frac{(2l-m+p)\pi}{p-ab}\Big)\\
&\qquad\times \exp\Big[n\Big(-\frac{(2l-m+p)^2}{4(p-ab)} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\,\Big]\\
&\qquad+ \sqrt{-1}\sqrt{\frac{ab(p-ab)}{pn}}(-1)^{a+b+ab} e^{-\frac{\pi\sqrt{-1}}{(p-ab)n}}\\
&\qquad\times \sum_{|l|\le (p-1)/2} \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})\sin(\frac{2l\pi}{p})}{\sin(\frac{2abl\pi}{p})}\\
&\qquad\times \exp\Big[n\Big(-\frac{(l+p/2)^2}{p}\Big)\pi\sqrt{-1}\,\Big] + O(n^{-3/2}) \Bigg\}.
\end{aligned}$$

Hence, we have

$$\hat{\tau}_n(X_p;\exp(4\pi\sqrt{-1}/n)) = \frac{(-1)^{p+1}n^{3/2}}{2\pi}(A(n)+B(n)n^{-1/2}+O(n^{-1})), \tag{19}$$

where

$$\begin{aligned}
A(n) := e^{\frac{n+1}{4}\pi\sqrt{-1}} \sum_{(l,m)\in\mathcal{S}_2} (-1)^m \frac{\sin(\frac{m\pi}{a})\sin(\frac{m\pi}{b})\sin(\frac{(2l-m+p)\pi}{p-ab})}{\sqrt{ab(p-ab)}}\\
\times \exp\Big[n\Big(-\frac{(2l-m+p)^2}{4(p-ab)} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\,\Big],
\end{aligned}$$

and

$$B(n) := 2\sqrt{-1}(-1)^{a+b+ab} e^{\frac{(1-p)n}{4}\pi\sqrt{-1}} \sum_{1\le l\le (p-1)/2} (-1)^l \frac{\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})\sin(\frac{2l\pi}{p})}{\sqrt{p}\sin(\frac{2abl\pi}{p})} \times \exp\Big[n\Big(-\frac{l^2}{p}\Big)\pi\sqrt{-1}\Big].$$

If we put $g := 2l - m + p$, then the summation range $\mathcal{S}_2$ becomes

$$\widetilde{\mathcal{S}} := \Big\{(g,m) \in \mathbb{Z}^2 \;\Big|\; 0 < m < 2ab, \frac{p-ab}{ab}m < g < 2(p-ab), g \equiv p - m \pmod 2, a \nmid m, b \nmid m\Big\},$$

which is the interior of the right-angled triangle with vertices $(0,0)$, $(2(p-ab),0)$, and $(2(p-ab),2ab)$. Using parameters (g,m), we have

$$A(n) = \frac{e^{\frac{n+1}{4}\pi\sqrt{-1}}}{\sqrt{ab(p-ab)}} \sum_{(g,m)\in\widetilde{\mathcal{S}}} G(g,m) \tag{20}$$

with

$$G(g,m) := (-1)^m \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\sin\Big(\frac{g\pi}{p-ab}\Big) \times \exp\Big[n\Big(-\frac{g^2}{4(p-ab)} - \frac{m^2}{4ab}\Big)\pi\sqrt{-1}\Big]. \tag{21}$$

We have the following symmetries of $G(g,m)$:

$$G(2(p-ab)-g,m) = (-1)^{ab+m+1}G(g,m),$$
$$G(g,2ab-m) = (-1)^{ab+m}G(g,m).$$

We divide $\widetilde{\mathcal{S}}$ into three parts $\mathcal{R}^{\Delta}$, $\mathcal{S}'$ and $\widetilde{\mathcal{R}}$:

$$\mathcal{R}^{\Delta} := \widetilde{\mathcal{S}} \cap \{(g,m)\in\mathbb{Z}^2 \mid g < p-ab\},$$
$$\mathcal{S}' := \widetilde{\mathcal{S}} \cap \{(g,m)\in\mathbb{Z}^2 \mid m > ab\},$$
$$\widetilde{\mathcal{R}} := \widetilde{\mathcal{S}} \cap \{(g,m)\in\mathbb{Z}^2 \mid p-ab < g, m < ab\}.$$

Note that we can exclude the line $g = p - ab$, since $A(n)$ vanishes when $g = p - ab$. Then we have

$$\sum_{(g,m)\in\widetilde{\mathcal{S}}} G(g,m) = \sum_{(g,m)\in\mathcal{R}^{\Delta}\cup\mathcal{S}'\cup\widetilde{\mathcal{R}}} G(g,m) = \sum_{(g,m)\in\mathcal{R}^{\Delta}} G(g,m) - \sum_{(g,m)\in\mathcal{R}^{\nabla}} G(g,m) + \sum_{(g,m)\in\mathcal{R}} (-1)^{ab+m+1} G(g,m), \tag{22}$$

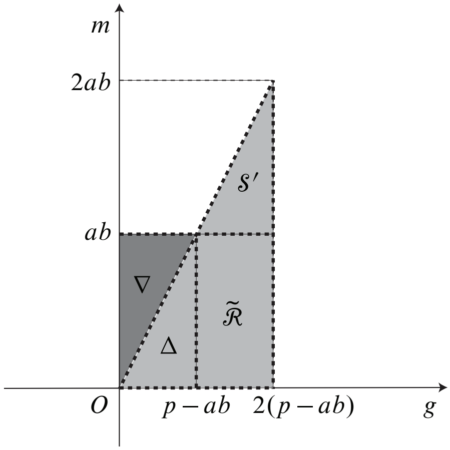

Figure 7. The light gray area indicates $\widetilde{\mathcal{S}} = \mathcal{R}^{\Delta} \cup \mathcal{S}' \cup \widetilde{\mathcal{R}}$. The dark gray area is $\mathcal{R}^{\nabla}$. We put $\mathcal{R} := \mathcal{R}^{\Delta} \cup \mathcal{R}^{\nabla}$.

where

$$\begin{aligned}\mathcal{R}^{\nabla} &:= \Big\{(g,m) \,\Big|\, 0 < m < ab, 0 < g < p - ab, \\ &\qquad g < \frac{p-ab}{ab} m, g \underset{(2)}{\equiv} p - m, a \nmid m, b \nmid m\Big\}, \\ \mathcal{R} &:= \{(g,m) \mid 0 < m < ab, 0 < g < p - ab, g \underset{(2)}{\equiv} p - m, a \nmid m, b \nmid m\}.\end{aligned}$$

Since

$$\begin{aligned}\mathcal{R}^{\Delta} = \Big\{(g,m) \,\Big|\, 0 < m < 2ab, \frac{p-ab}{ab} m < g < p - ab, \\ g \underset{(2)}{\equiv} p - m, a \nmid m, b \nmid m\Big\},\end{aligned}$$

$\mathcal{R}$ splits into $\mathcal{R}^{\Delta}$ and $\mathcal{R}^{\nabla}$. So, we can regard $A(n)$ as a summation over $\mathcal{R}$.

7. The Reidemeister torsion and the Chern–Simons invariant

7.1. Fundamental group

We assume that b is odd. For the torus knot $T(a,b)$ we have

$$\pi_1(S^3 \setminus T(a,b)) = \langle x, y \mid x^a = y^b \rangle.$$

Note that

- $z := x^a = y^b$ is central,
- $\mu = x^{-c}y^d$ is a meridian, where c and d are integers such that $ad - bc = 1$, and
- $\lambda = z\mu^{-ab}$ is the preferred longitude.

Put $X := S^3 \setminus \operatorname{Int} N(T(a,b))$, and $D := D^2 \times S^1$. Let X_p be the closed three-manifold obtained from X by p-surgery. Then X_p is $X \cup_{i_p} D$, where $i_p \colon \partial D \to \partial X$ sending $\partial D^2 \times \{\text{point}\}$ to $\mu_D := \lambda\mu^p$, and $\{\text{point in } \partial D^2\} \times S^1$ to μ^{-1}, where we identify a simple closed curve on the torus ∂X with an element in $\pi_1(\partial X)$.

Since in $\pi_1(X_p)$, $\mu_D = 1$, which is equivalent to $\mu^{p-ab} = z^{-1}$, we have the following presentation of $\pi_1(X_p)$:

$$\pi_1(X_p) = \langle x, y, \mu \mid x^a = y^b, \mu = x^{-c}y^d, \mu^{p-ab} = x^{-a}\rangle. \tag{G}$$

Note that we do not need the generator μ.

Regarding X_p as the Seifert fibered space $S(-a/c, b/d, p-ab)$, we have another presentation:

$$\begin{aligned}\pi_1(S(-a/c, b/d, p-ab)) = \langle \alpha, \beta, \gamma, f \mid [\alpha, f] = [\beta, f] = [\gamma, f] = \alpha^a f^{-c} \\ = \beta^b f^d = \gamma^{p-ab} f = \alpha\beta\gamma = 1\rangle,\end{aligned} \tag{23}$$

where α, β, and γ go around the singular fibers with indices $-a/b$, b/d, and $p-ab$ respectively, and f is a regular fiber. We use $\pi_1(X_p)$ for the presentation (G) and $\pi_1(S(-a/c, b/d, p-ab))$ for that described in (23).

We will construct a concrete isomorphism between $\pi_1(X_p)$ and $\pi_1(S(-a/c, b/d, p-ab))$.

Define a homomorphism $\Phi \colon \pi_1(S(-a/c, b/d, p-ab)) \to \pi_1(X_p)$ by

$$\begin{aligned}\Phi(\alpha) &:= x^{-c}\\ \Phi(\beta) &:= y^d,\\ \Phi(\gamma) &:= \mu^{-1},\\ \Phi(f) &:= \mu^{p-ab}.\end{aligned}$$

Since we have

$$\begin{aligned}\Phi(\alpha^a f^{-c}) &= x^{-ac}\mu^{-c(p-ab)} = x^{-ac}x^{ac} = 1,\\ \Phi(\beta^b f^d) &= y^{bd}\mu^{d(p-ab)} = y^{bd}x^{-ad} = 1,\\ \Phi(\gamma^{p-ab} f) &= \mu^{ab-p}\mu^{p-ab} = 1,\\ \Phi([\alpha, f]) &= [x^{-c}, \mu^{p-ab}] = [x^{-c}, x^{-a}] = 1,\end{aligned}$$

$$\begin{aligned}\Phi([\beta, f]) &= [y^d, \mu^{p-ab}] = [y^d, y^{-b}] = 1,\\ \Phi([\gamma, f]) &= [\mu^{-1}, \mu^{p-ab}] = 1,\\ \Phi(\alpha\beta\gamma) &= x^{-c} y^d \mu^{-1} = 1,\end{aligned}$$

Φ is well defined.

We also define $\Psi\colon \pi_1(X_p) \to \pi_1(S(-a/c, b/d, p-ab))$ by

$$\begin{aligned}\Psi(x) &:= \alpha^b f^{-d},\\ \Psi(y) &:= \beta^a f^c,\\ \Phi(\mu) &:= \gamma^{-1}.\end{aligned}$$

Since we have

$$\begin{aligned}\Psi(x^a y^{-b}) &= \alpha^{ab} f^{-ad} \beta^{-ab} f^{-bc} = f^{bc} f^{-ad} f^{ad} f^{-bc} = 1,\\ \Psi(\mu y^{-d} x^c) &= \gamma^{-1} f^{-cd} \beta^{-ad} \alpha^{bc} f^{-cd} = \gamma^{-1} \beta^{bc} \beta^{-ad} \alpha^{bc} \alpha^{-ad}\\ &= \gamma^{-1} \beta^{-1} \alpha^{-1} = 1,\\ \Psi(\mu^{p-ab} x^a) &= \gamma^{ab-p} (\alpha^b f^{-d})^a = f \alpha^{ab} f^{-ad} = f \cdot f^{bc} f^{-ad} = 1,\end{aligned}$$

Ψ is also well defined.

Since $\Phi \circ \Psi = \mathrm{Id}_{\pi_1(X_p)}$ and $\Psi \circ \Phi = \mathrm{Id}_{\pi_1(S(-a/c, b/d, p-ab))}$, both Φ and Ψ are isomorphisms, as desired.

7.2. Representations

It is known that the $\mathrm{SL}(2;\mathbb{C})$ character variety of $T(a,b)$ has $(a-1)(b-1)/2$ irreducible components and an Abelian component. Put

$$\mathcal{P} := \{(k,l) \mid 0 < k < a, 0 < l < b, k \equiv l \pmod 2\}.$$

Then the irreducible components are indexed by $\mathcal{P}$ as follows. Let $\rho^{\mathrm{Irr}}_{k,l}\colon \pi_1(X) \to \mathrm{SL}(2;\mathbb{C})$ be an irreducible representation that belongs to the component indexed by $(k,l) \in \mathcal{P}$. Then it satisfies

$$\operatorname{tr} \rho^{\mathrm{Irr}}_{k,l}(x) = 2\cos\left(\frac{k\pi}{a}\right), \tag{24}$$

$$\operatorname{tr} \rho^{\mathrm{Irr}}_{k,l}(y) = 2\cos\left(\frac{l\pi}{b}\right). \tag{25}$$

See [33,43].

The irreducible component indexed by $(k,l) \in \mathcal{P}$ intersects with the Abelian component in two representations whose traces of the meridian μ are $2\cos(\frac{(adl-bck)\pi}{ab})$ and $2\cos(\frac{(adl+bck)\pi}{ab})$, where c and d are integers satisfying $ad - bc = 1$.

We want to extend the representation $\rho^{\mathrm{Irr}}_{k,l}$ to a representation of $\pi_1(X_p)$. Since $\operatorname{tr}\rho^{\mathrm{Irr}}_{k,l}(x) \neq \pm 2$, we can find $Q \in \mathrm{SL}(2;\mathbb{C})$ such that

$$Q^{-1}\rho^{\mathrm{Irr}}_{k,l}(x)Q = \begin{pmatrix} e^{k\pi\sqrt{-1}/a} & 0 \\ 0 & e^{-k\pi\sqrt{-1}/a} \end{pmatrix}.$$

Then since $\mu_D = \lambda\mu^p = z\mu^{p-ab} = x^a\mu^{p-ab}$, we have

$$Q^{-1}\rho^{\mathrm{Irr}}_{k,l}(\mu_D)Q = (-1)^k Q^{-1}(\rho^{\mathrm{Irr}}_{k,l}(\mu))^{p-ab}Q.$$

Therefore, if $\operatorname{tr}\rho^{\mathrm{Irr}}_{k,l}(\mu) = 2\cos(\frac{h\pi}{p-ab})$ with $0 < h < p-ab$ and $h \equiv k \equiv l \pmod 2$, then $\rho^{\mathrm{Irr}}_{k,l}(\mu_D)$ becomes the identity and so we can extend $\rho^{\mathrm{Irr}}_{k,l}\colon \pi_1(X) \to \mathrm{SL}(2;\mathbb{C})$ to an irreducible representation $\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}\colon \pi_1(X_p) \to \mathrm{SL}(2;\mathbb{C})$.

Therefore, we have the following proposition.

Proposition 7.1. *Put*

$$\mathcal{H} := \{(h,k,l) \in \mathbb{Z}^3 \mid 0 < h < p-ab, 0 < k < a, 0 < l < b, h \equiv k \equiv l \pmod 2\}. \tag{26}$$

Then for any triple (h,k,l) there exists an irreducible representation $\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}\colon \pi_1(X_p) \to \mathrm{SL}(2;\mathbb{C})$ such that

$$\begin{aligned}
\operatorname{tr}\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}(x) &= 2\cos\Bigl(\frac{k\pi}{a}\Bigr),\\
\operatorname{tr}\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}(y) &= 2\cos\Bigl(\frac{l\pi}{b}\Bigr),\\
\operatorname{tr}\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}(\mu) &= 2\cos\Bigl(\frac{h\pi}{p-ab}\Bigr).
\end{aligned}$$

Moreover, the representation is unique up to conjugation.

We can describe these representations in terms of $\pi_1(S(-a/c, b/d, p-ab))$ by using the isomorphism $\Phi\colon \pi_1(S(-a/c, b/d, p-ab)) \to \pi_1(X_p)$. The generators of the presentation (23) is mapped as follows:

$$\operatorname{tr}\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}(\alpha) = 2\cos\Bigl(\frac{ck\pi}{a}\Bigr), \tag{27}$$

$$\operatorname{tr}\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}(\beta) = 2\cos\Bigl(\frac{dl\pi}{b}\Bigr), \tag{28}$$

$$\operatorname{tr}\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}(\gamma) = 2\cos\Bigl(\frac{h\pi}{p-ab}\Bigr), \tag{29}$$

$$\operatorname{tr}\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}(f) = 2(-1)^h. \tag{30}$$

Remark 7.2. Since X_p is the Seifert fibered space $S(-a/c, b/d, p-ab)$, the result above is well known.

We also introduce reducible Abelian representations indexed by

$$\{l \in \mathbb{Z} \mid 0 < l < p\}.$$

Definition 7.3. For an integer with $0 < l < p$, let $\tilde{\rho}_l^{\mathrm{Abel}}\colon \pi_1(X_p) \to \mathrm{SL}(2;\mathbb{C})$ be the Abelian representation sending

$$\mu \to \begin{pmatrix} e^{2l\pi\sqrt{-1}/p} & 0 \\ 0 & e^{-2l\pi\sqrt{-1}/p} \end{pmatrix}.$$

Since $H_1(X_p) = \mathbb{Z}/p\mathbb{Z}$, this is well defined.

7.3. Reidemeister torsion

In this section we describe the *homological* Reidemeister torsion of X_p twisted by the adjoint action of the irreducible representation $\tilde{\rho}_{h,k,l}^{\mathrm{Irr}}$ with $(h,k,l) \in \mathcal{H}$.

For a representation $\rho\colon \pi_1(X) \to \mathrm{SL}(2;\mathbb{C})$, let $\mathrm{Tor}_\gamma(X;\rho)$ be the twisted Reidemeister torsion associated with a simple closed curve $\gamma \subset \partial X$. Here we assume that ρ is γ-regular [53, Définition 3.21].

If M is an oriented, closed three-manifold and $\tilde{\rho}\colon \pi_1(M) \to \mathrm{SL}(2;\mathbb{C})$ is a representation. We denote by $\mathrm{Tor}(M;\tilde{\rho})$ the Reidemeister torsion twisted by the adjoint action of ρ.

Now, we can calculate $\mathrm{Tor}(X_p;\tilde{\rho}_{h,k,l}^{\mathrm{Irr}})$ as follows.

Lemma 7.4. *The Reidemeister torsion twisted by the irreducible representation $\tilde{\rho}_{h,k,l}^{\mathrm{Irr}}$ is given by*

$$\mathrm{Tor}(X_p;\tilde{\rho}_{h,k,l}^{\mathrm{Irr}}) = \pm\frac{ab(p-ab)}{64\sin^2(\frac{k\pi}{a})\sin^2(\frac{l\pi}{b})\sin^2(\frac{h\pi}{p-ab})}.$$

Proof. Since $\lambda = z\mu^{-ab}$ and z is central, by a conjugation we may assume that

$$\rho_{k,l}^{\mathrm{Irr}}(\mu) = \begin{pmatrix} e^{\frac{h\pi\sqrt{-1}}{p-ab}} & * \\ 0 & e^{-\frac{h\pi\sqrt{-1}}{p-ab}} \end{pmatrix}$$

and that

$$\rho_{k,l}^{\mathrm{Irr}}(\lambda) = \pm\begin{pmatrix} e^{\frac{-abh\pi\sqrt{-1}}{p-ab}} & * \\ 0 & e^{\frac{abh\pi\sqrt{-1}}{p-ab}} \end{pmatrix}.$$

Therefore, we have

$$\mathrm{Tor}_\mu(X;\rho_{k,l}^{\mathrm{Irr}}) = \pm\frac{1}{ab}\frac{a^2b^2}{16\sin^2(\frac{k\pi}{a})\sin^2(\frac{l\pi}{b})} = \pm\frac{ab}{16\sin^2(\frac{k\pi}{a})\sin^2(\frac{l\pi}{b})}$$

from (7). Note that this holds for any irreducible representation in the component indexed by (k, l).

Since

$$\rho^{\mathrm{Irr}}_{k,l}(\lambda\mu^p) = \pm\begin{pmatrix} e^{h\pi\sqrt{-1}} & * \\ 0 & e^{-h\pi\sqrt{-1}} \end{pmatrix} = \pm\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}(\mu)^{p-ab},$$

we have

$$\mathrm{Tor}_{i_p(\mu_D)}(X;\rho^{\mathrm{Irr}}_{k,l}) = \pm(p-ab)\,\mathrm{Tor}_\mu(X;\rho^{\mathrm{Irr}}_{k,l}) = \pm\frac{ab(p-ab)}{16\sin^2(\frac{k\pi}{a})\sin^2(\frac{l\pi}{b})}$$

from (7).

Since $\rho^{\mathrm{Irr}}_{k,l}(i_p(\lambda_D)) = \rho^{\mathrm{Irr}}_{k,l}(\mu)^{-1}$, we have $\mathrm{tr}\,\rho^{\mathrm{Irr}}_{k,l}(i_p(\lambda_D)) = 2\cos(\frac{h\pi}{p-ab})$. So, we finally have

$$\begin{aligned}\mathrm{Tor}(X_p;\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}) &= \pm\frac{ab(p-ab)}{16\sin^2(\frac{k\pi}{a})\sin^2(\frac{l\pi}{b})}\times\frac{1}{4\cos^2(\frac{h\pi}{p-ab})-4} \\ &= \pm\frac{ab(p-ab)}{64\sin^2(\frac{k\pi}{a})\sin^2(\frac{l\pi}{b})\sin^2(\frac{h\pi}{p-ab})}\end{aligned}$$

from (8). ∎

Let ρ^{Abel}_l be the reducible Abelian representation

$$\pi_1(X)\to \mathrm{SL}(2;\mathbb{C}),\quad \mu\to\begin{pmatrix} e^{2l\pi\sqrt{-1}/p} & 0 \\ 0 & e^{-2l\pi\sqrt{-1}/p}\end{pmatrix}.$$

Note that ρ^{Abel}_l can be extended to

$$\tilde{\rho}^{\mathrm{Abel}}_l\colon \pi_1(X_p)\to\mathrm{SL}(2;\mathbb{C}),\quad \rho^{\mathrm{Abel}}_l = \tilde{\rho}^{\mathrm{Abel}}_l|_{\pi_1(X)}.$$

The Reidemeister torsion $\mathrm{Tor}_\mu(X;\rho^{\mathrm{Abel}}_l)$ of X twisted by ρ^{Abel}_l associated with μ is given by $\pm(\frac{\Delta(T(a,b);e^{4l\pi\sqrt{-1}/p})}{2\sinh(2l\pi\sqrt{-1}/p)})^2$ ([40, Theorem 4], [64, Theorem 1.1.2]; see also [47, Proposition 5.1]), where $\Delta(K;t)$ is the normalised Alexander polynomial of a knot K.

Then we show

Lemma 7.5. *The Reidemeister torsion twisted by the Abelian representation ρ^{Abel}_l is given by*

$$\mathrm{Tor}(X_p;\tilde{\rho}^{\mathrm{Abel}}_l) = \pm\frac{p\sin^2(\frac{2lab\pi}{p})}{16\sin^2(\frac{2la\pi}{p})\sin^2(\frac{2lb\pi}{p})\sin^2(\frac{2l\pi}{p})}.$$

Proof. Since it is well known (see for example [37, Chapter 11]) that

$$\Delta(T(a,b);t) = \frac{(t^{ab/2} - t^{-ab/2})(t^{1/2} - t^{-1/2})}{(t^{a/2} - t^{-a/2})(t^{b/2} - t^{-b/2})},$$

we have

$$\mathrm{Tor}_{\mu}(X;\rho_l^{\mathrm{Abel}}) = \pm\frac{\sin^2(\frac{2abl\pi}{p})}{4\sin^2(\frac{2al\pi}{p})\sin^2(\frac{2bl\pi}{p})}.$$

Since $\rho_l^{\mathrm{Abel}}(i_p(\mu_D)) = \rho_l^{\mathrm{Abel}}(\mu)^p$, we have

$$\mathrm{Tor}_{i_p(\mu_D)}(X;\rho_l^{\mathrm{Abel}}) = \pm\frac{p\sin^2(\frac{2ab\pi}{p})}{4\sin^2(\frac{2al\pi}{p})\sin^2(\frac{2bl\pi}{p})}$$

from (7). Since we also know that $\rho_l^{\mathrm{Abel}}(i_p(\lambda_D)) = \rho_l^{\mathrm{Abel}}(\mu)^{-1}$, from (8) we have

$$\begin{aligned}\mathrm{Tor}(X_p;\tilde{\rho}_l^{\mathrm{Abel}}) &= \pm\frac{\mathrm{Tor}_{i_p(\mu_D)}(X;\rho_l^{\mathrm{Abel}})}{(\mathrm{tr}\,\rho_l^{\mathrm{Abel}}(i_p(\lambda_D)))^2 - 4}\\ &= \pm\frac{p\sin^2(\frac{2abl\pi}{p})}{4\sin^2(\frac{2al\pi}{p})\sin^2(\frac{2bl\pi}{p})} \times \frac{1}{4\cos^2(\frac{2l\pi}{p}) - 4}\\ &= \pm\frac{p\sin^2(\frac{2abl\pi}{p})}{16\sin^2(\frac{2al\pi}{p})\sin^2(\frac{2bl\pi}{p})\sin^2(\frac{2l\pi}{p})},\end{aligned}$$

completing the proof ■

7.4. Chern–Simons invariant

In this section we calculate the Chern–Simons invariants of X_p associated with representations described in Section 7.2. We denote by $\mathrm{CS}(M;\rho) \in \mathbb{C}/\mathbb{Z}$ the Chern–Simons invariant of a closed three-manifold associated with a representation ρ. We use a formula by Kirk and Klassen [32, Theorem 4.2; see also p. 354, first paragraph]. Note that in our case, $K = T(a,b)$, $E = X$, $M = X_p$, $\mu_D = \mu^p\lambda \in \pi_1(X)$ and $\lambda_D = \mu^{-1} \in \pi_1(X)$.

First we consider the Abelian representations. See [31, Theorem 4.5], noting our sign convention (Remark 2.2).

Lemma 7.6. *Let $\tilde{\rho}_l^{\mathrm{Abel}}$ be the Abelian representation defined in Definition 7.3. Then the Chern–Simons invariant of $\tilde{\rho}_l^{\mathrm{Abel}}$ is given as*

$$\mathrm{CS}(X_p;\tilde{\rho}_l^{\mathrm{Abel}}) = -\frac{l^2}{p} \in \mathbb{C}/\mathbb{Z}.$$

Proof. Now, let us consider a path of representations $\rho_t\colon \pi_1(X) \to \mathrm{SL}(2;\mathbb{C})$ sending μ to $\begin{pmatrix} e^{2tl\pi\sqrt{-1}/p} & 0 \\ 0 & e^{-2tl\pi\sqrt{-1}/p} \end{pmatrix}$ and λ to the identity matrix. Then ρ_0 is trivial and $\rho_1(\mu) = \tilde{\rho}_l^{\mathrm{Abel}}|_X(\mu)$. Therefore, both ρ_0 and ρ_1 can be extended to representations $\tilde{\rho}_0$, which is trivial, and $\tilde{\rho}_1 = \tilde{\rho}_l^{\mathrm{Abel}}$ of $\pi_1(X_p)$. We also see that if we put $\alpha(t) := lt$ and $\beta(t) := -lt/p$, then we have

$$\rho_t(\mu_D) = \begin{pmatrix} e^{2\pi\sqrt{-1}\alpha(t)} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\alpha(t)} \end{pmatrix},$$
$$\rho_t(\lambda_D) = \begin{pmatrix} e^{2\pi\sqrt{-1}\beta(t)} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\beta(t)} \end{pmatrix}.$$

Therefore, from Theorem 2.1 we have

$$\begin{aligned}\mathrm{CS}(X_p; \tilde{\rho}_l^{\mathrm{Abel}}) &= \mathrm{CS}(X_p; \tilde{\rho}_1) - \mathrm{CS}(X_p; \tilde{\rho}_0) \\ &= 2\int_0^1 \frac{-l^2 t}{p}\, dt = -\frac{l^2}{p},\end{aligned}$$

since $\mathrm{CS}(X_p; \tilde{\rho}_0) = 0$. See [32, Theorem 5.1]. ∎

Next we consider the irreducible representations. The following lemma is also well known. See [32, Theorem 5.2] and [5, Proposition 2.3]. Notice again our sign convention; ours agrees with that of [5].

Lemma 7.7. *Let $\tilde{\rho}_{h,k,l}^{\mathrm{Irr}}$ be the irreducible representation described in Proposition 7.1. Then we have*

$$\mathrm{CS}(X_p; \tilde{\rho}_{h,k,l}^{\mathrm{Irr}}) = -\frac{h^2}{4(p-ab)} - \frac{(adl-bck)^2}{4ab} \in \mathbb{C}/\mathbb{Z}. \tag{31}$$

Proof. Fix integers c and d such that $ad - bc = 1$ as usual.

We will define a path of representations

$$\varphi_t\colon \pi_1(X) \to \mathrm{SL}(2;\mathbb{C}) \quad (0 \le t \le 1)$$

as follows.

For $0 \le t \le 1/2$, define

$$\begin{cases} \varphi_t(x) := \begin{pmatrix} e^{2(adl-bck)t\pi\sqrt{-1}/a} & 0 \\ 0 & e^{-2(adl-bck)t\pi\sqrt{-1}/a} \end{pmatrix}, \\ \varphi_t(y) := \begin{pmatrix} e^{2(adl-bck)t\pi\sqrt{-1}/b} & 0 \\ 0 & e^{-2(adl-bck)t\pi\sqrt{-1}/b} \end{pmatrix}. \end{cases} \tag{32}$$

Note that φ_0 is trivial and so it can be extended to the trivial representation

$$\tilde{\varphi}_0 \colon \pi_1(X_p) \to \mathrm{SL}(2;\mathbb{C}).$$

Note also that

$$\begin{cases} \varphi_{1/2}(x) = \begin{pmatrix} e^{k\pi\sqrt{-1}/a} & 0 \\ 0 & e^{-k\pi\sqrt{-1}/a} \end{pmatrix}, \\ \varphi_{1/2}(y) = \begin{pmatrix} e^{l\pi\sqrt{-1}/b} & 0 \\ 0 & e^{-l\pi\sqrt{-1}/b} \end{pmatrix}, \end{cases}$$

since $k \equiv l \pmod 2$.

• For $1/2 \le t \le 1$, we will construct a path from $\varphi_{1/2}$ to $\rho^{\mathrm{Irr}}_{k,l} \colon \pi_1(X) \to \mathrm{SL}(2;\mathbb{C})$. Recall that the representation $\rho^{\mathrm{Irr}}_{k,l}$ satisfies $\operatorname{tr}\rho^{\mathrm{Irr}}_{k,l}(x) = 2\cos(k\pi/a)$ and $\operatorname{tr}\rho^{\mathrm{Irr}}_{k,l}(y) = 2\cos(l\pi/b)$ (see Proposition 7.1). Since the Chern–Simons invariant does not depend on conjugacy classes, we may assume that $\rho^{\mathrm{Irr}}_{k,l}(x) = \begin{pmatrix} e^{k\pi\sqrt{-1}/a} & 0 \\ 0 & e^{-k\pi\sqrt{-1}/a} \end{pmatrix}$. Let $Q \in \mathrm{SL}(2;\mathbb{C})$ be a matrix such that $Q^{-1}\rho^{\mathrm{Irr}}_{k,l}(y)Q = \begin{pmatrix} e^{l\pi\sqrt{-1}/b} & 0 \\ 0 & e^{-l\pi\sqrt{-1}/b} \end{pmatrix}$. Since $\mathrm{SL}(2;\mathbb{C})$ is connected, there exists a path $P(t) \colon [1/2, 1] \to \mathrm{SL}(2;\mathbb{C})$ such that one has $P(1/2) = I_2$ and $P(1) = Q$, where I_2 is the 2×2 identity matrix. Now, we put

$$\begin{cases} \varphi_t(x) := \begin{pmatrix} e^{k\pi\sqrt{-1}/a} & 0 \\ 0 & e^{-k\pi\sqrt{-1}/a} \end{pmatrix}, \\ \varphi_t(y) := P(t)\begin{pmatrix} e^{l\pi\sqrt{-1}/b} & 0 \\ 0 & e^{-l\pi\sqrt{-1}/b} \end{pmatrix} P(t)^{-1}, \end{cases} \tag{33}$$

for $1/2 \le t \le 1$. Then, since $\varphi_t(y)^b = (-1)^l I_2$ and $k \equiv l \pmod 2$, we see that φ_t is well defined. Note that $\varphi_{1/2}$ in (33) coincides with that in (32), and that φ_1 equals $\rho^{\mathrm{Irr}}_{k,l}$, which can be extended to $\tilde{\rho}^{\mathrm{Irr}}_{h,k,l} \colon \pi_1(X_p) \to \mathrm{SL}(2;\mathbb{C})$ for $(h,k,l) \in \mathcal{H}$.

Next, we calculate $\varphi_t(\mu_D)$ and $\varphi_t(\lambda_D)$. Recall that $\mu = x^{-c}y^d$, $\lambda = x^a\mu^{-ab}$, $\mu_D = \lambda\mu^p$, and $\lambda_D = \mu^{-1}$.

• For $0 \le t \le 1/2$, we see that $\varphi_t(\mu)$ is a diagonal matrix with $(1,1)$-entry

$$e^{2(adl-bck)t\pi\sqrt{-1}/(ab)}$$

and that $\varphi_t(\lambda) = I_2$. Therefore, we have

$$\begin{cases} \varphi_t(\mu_D) = \begin{pmatrix} e^{2p(adl-bck)t\pi\sqrt{-1}/(ab)} & 0 \\ 0 & e^{-2p(adl-bck)t\pi\sqrt{-1}/(ab)} \end{pmatrix}, \\ \varphi_t(\lambda_D) = \begin{pmatrix} e^{-2(adl-bck)t\pi\sqrt{-1}/(ab)} & 0 \\ 0 & e^{2(adl-bck)t\pi\sqrt{-1}/(ab)} \end{pmatrix}, \end{cases}$$

and so we can put

$$\alpha(t) := \frac{p(adl - bck)}{ab} t \quad \text{and} \quad \beta(t) := \frac{-(adl - bck)}{ab} t$$

to use Theorem 2.1. Then we have

$$2 \int_0^{1/2} \beta(t) \frac{d\,\alpha(t)}{d\,t}\, dt = -\frac{p(adl - bck)^2}{4a^2b^2}. \tag{34}$$

- For $1/2 \leq t \leq 1$, we see that $\varphi_t(x^a) = (-1)^k I_2$ for any $1/2 \leq t \leq 1$. So, we have $\varphi_t(\lambda) = (-1)^k \varphi_t(\mu)^{-ab}$, $\varphi_t(\mu_D) = (-1)^k \varphi_t(\mu)^{p-ab}$ and $\varphi_t(\lambda_D) = \varphi_t(\mu)^{-1}$. Therefore, if we assume that $\varphi_t(\mu) = \begin{pmatrix} e^{2\pi\sqrt{-1} f(t)} & 0 \\ 0 & e^{-2\pi\sqrt{-1} f(t)} \end{pmatrix}$ after conjugation, then we have

$$\begin{cases} \varphi_t(\mu_D) = \begin{pmatrix} e^{2\pi\sqrt{-1}((p-ab)f(t)+k/2)} & 0 \\ 0 & e^{-2\pi\sqrt{-1}((p-ab)f(t)+k/2)} \end{pmatrix}, \\ \varphi_t(\lambda_D) = \begin{pmatrix} e^{-2\pi\sqrt{-1} f(t)} & 0 \\ 0 & e^{2\pi\sqrt{-1} f(t)} \end{pmatrix}. \end{cases}$$

Therefore, we can put $\alpha(t) := (p - ab) f(t) + k/2$ and $\beta(t) := -f(t)$ to use Theorem 2.1. Then we have

$$\begin{aligned} 2 \int_{1/2}^{1} \beta(t) \frac{d\,\alpha(t)}{d\,t}\, dt &= -2 \int_{1/2}^{1} (p - ab) f'(t) f(t)\, dt \\ &= -(p - ab)[f(t)^2]_{1/2}^{1} \\ &= -\frac{h^2}{4(p - ab)} + \frac{(p - ab)(adl - bck)^2}{4a^2b^2} \end{aligned} \tag{35}$$

since $f(1) = \frac{h}{2(p-ab)}$ and $f(\frac{1}{2}) = \frac{adl-bck}{2ab}$.

Therefore, from (34), (35), and Theorem 2.1, we conclude that

$$\mathrm{CS}(X_p; \tilde{\rho}^{\mathrm{Irr}}_{h,k,l}) = 2 \int_0^1 \beta(t) \frac{d\,\alpha(t)}{d\,t} dt = -\frac{h^2}{4(p - ab)} - \frac{(adl - bck)^2}{4ab},$$

completing the proof. ∎

Remark 7.8. Let us confirm that the right-hand side of (31) does not depend on the choice of (c, d) as an element in $\mathbb{C}/\mathbb{Z}$.

Suppose that c' and d' are integers such that $ad' - bc' = 1$. Then we have

$$\begin{aligned}
&(adl - bck)^2 - (ad'l - bc'k)^2 \\
&\quad = a^2l^2(d^2 - d'^2) + b^2k^2(c^2 - c'^2) - 2abkl(cd - c'd') \\
&\quad = al^2(d + d')b(c - c') + bk^2(c + c')a(d - d') - 2abkl(cd - c'd') \\
&\quad = ab(l^2(d + d')(c - c') + k^2(c + c')(d - d') - 2kl(cd - c'd'))
\end{aligned}$$

since $a(d - d') = b(c - c')$. So, $\frac{(adl-bck)^2}{4ab} - \frac{(ad'l-bc'k)}{4ab}$ is an integer because $k \equiv l \pmod 2$ and the right-hand side of (31) does not depend on the choice of c and d.

8. $\mathcal{H}$ and $\mathcal{R}$

In Section 7.4 we have shown that the irreducible representations of $\pi_1(X_p)$ to $\mathrm{SL}(2;\mathbb{C})$ are indexed by $\mathcal{H}$. In this section we construct two injections $\widetilde{\Gamma}_+$ and $\widetilde{\Gamma}_-$ from $\mathcal{H}$ to $\mathcal{R}$ so that $\mathcal{R} = \widetilde{\Gamma}_+(\mathcal{H}) \sqcup \widetilde{\Gamma}_-(\mathcal{H})$, where $\sqcup$ means a disjoint union.

Let a and b positive coprime integers. We assume that b is odd.

We first define two finite sets $\mathcal{P}$ and $\mathcal{Q}$ as follows:

$$\mathcal{P} := \{(k,l) \mid 1 \le k \le a-1, 1 \le l \le b-1, k \underset{(2)}{\equiv} l\},$$

$$\mathcal{Q} := \{m \in \mathbb{Z} \mid 1 \le m \le ab-1, a \nmid m, b \nmid m\},$$

where $k \underset{(2)}{\equiv} l$ means $k \equiv l \pmod 2$.

Define maps Γ_+ and Γ_- from $\mathcal{P}$ to $\mathcal{Q}$ by

$$\Gamma_+(k,l) := [adl - bck]_{ab}, \quad \Gamma_-(k,l) := [-adl - bck]_{ab}.$$

Here we choose integers c and d such that $ad - bc = 1$, and $[x]_p$ is the integer satisfying $0 \le [x]_p < p$ and $x \equiv [x]_p \pmod p$.

Remark 8.1. By Sunzi's theorem [34], $\Gamma_\pm(k,l)$ is characterised as follows:

$$\begin{aligned}
([\Gamma_+(k,l)]_a, [\Gamma_+(k,l)]_b) &= (k,l), \\
([\Gamma_-(k,l)]_a, [\Gamma_-(k,l)]_b) &= (k, b-l)
\end{aligned}$$

since $\Gamma_+(k,l) \equiv k \pmod a$ and $\Gamma_+(k,l) \equiv l \pmod b$, and $\Gamma_-(k,l) \equiv k \pmod a$ and $\Gamma_-(k,l) \equiv b - l \pmod b$.

We also have

$$\Gamma_+([m]_a, [m]_b) = m, \tag{36}$$

$$\Gamma_-([m]_a, [-m]_b) = m \tag{37}$$

for any integer m with $0 < m < ab - 1$.

Note that both Γ_+ and Γ_- are injective, and that $\Gamma_+(\mathcal{P}) \cap \Gamma_-(\mathcal{P}) = \emptyset$. The former follows from Sunzi's theorem. The latter is because

$$[\Gamma_+(k,l)]_a \underset{(2)}{\equiv} [\Gamma_+(k,l)]_b$$

but

$$[\Gamma_-(k,l)]_a \underset{(2)}{\not\equiv} [\Gamma_-(k,l)]_b$$

since b is odd.

Since $\#\mathcal{P} = (a-1)(b-1)/2$ and $\#\mathcal{Q} = (a-1)(b-1)$, we conclude that

$$\mathcal{Q} = \mathcal{Q}_+ \sqcup \mathcal{Q}_-,$$

where $\sqcup$ denotes the disjoint union and $\mathcal{Q}_\pm := \Gamma_\pm(\mathcal{P})$. Note that

$$\mathcal{Q}_+ = \{m \in \mathcal{Q} \mid [m]_a \underset{(2)}{\equiv} [m]_b\}.$$
$$\mathcal{Q}_- = \{m \in \mathcal{Q} \mid [m]_a \underset{(2)}{\not\equiv} [m]_b\}.$$

We also define maps $\Theta_+ \colon \mathcal{Q}_+ \to \mathcal{P}$ and $\Theta_- \colon \mathcal{Q}_- \to \mathcal{P}$ by

$$\Theta_\pm(m) := ([m]_a, [\pm m]_b)$$

Then we have

$$\begin{aligned}(\Theta_\pm \circ \Gamma_\pm)(k,l) &= \Theta_\pm([\pm adl - bck]_{ab}) \\ &= ([\pm adl - bck]_a, [adl \mp bck)]_b = (k,l).\end{aligned}$$

We also have

$$\begin{aligned}(\Gamma_+ \circ \Theta_+)(m) &= \Gamma_+([m]_a, [m]_b) = m \quad && \text{if } m \in \mathcal{Q}_+, \\ (\Gamma_- \circ \Theta_-)(m) &= \Gamma_-([m]_a, [-m]_b) = m \quad && \text{if } m \in \mathcal{Q}_-\end{aligned}$$

from (36) and (37). So, $\Theta_\pm$ is the inverse of Γ_+.

Example 8.2. When $a = 2$, we have

$$\mathcal{P} = \{(1,l) \mid 1 \le l \le b-1, l\colon \text{odd}\}.$$

and

$$\begin{aligned}\mathcal{Q}_+ &= \{1, 3, \dots, b-2\}, \\ \mathcal{Q}_- &= \{b+2, b+4, \dots, 2b-1\}.\end{aligned}$$

We also see that $\Gamma_+(1,l) = l$ and $\Gamma_-(1,l) = 2b - l$, and that $\Theta_+(m) = (1,m)$ if $m \in \mathcal{Q}_+$ and $\Theta_-(m) = (1, 2b-m)$ if $m \in \mathcal{Q}_-$.

Example 8.3. When $(a,b) = (4,3)$, we have

$$\begin{aligned}
\mathcal{P} &= \{(1,1),(2,2),(3,1)\},\\
\mathcal{Q}_+ &= \{1,2,7\} \quad \mathcal{Q}_- = \{5,10,11\},\\
\Gamma_+&: (1,1) \to 1, (2,2) \to 2, (3,1) \to 7,\\
\Gamma_-&: (1,1) \to 5, (2,2) \to 10, (3,1) \to 11,
\end{aligned}$$

and

$$\begin{array}{llll}
\Theta_+: & 1 \to (1,1), & 2 \to (2,2), & 7 \to (3,1),\\
\Theta_-: & 5 \to (1,1), & 10 \to (2,2), & 11 \to (3,1).
\end{array}$$

Example 8.4. When $(a,b) = (3,5)$, we have

$$\begin{aligned}
\mathcal{P} &= \{(1,1),(1,3),(2,2),(2,4)\},\\
\mathcal{Q}_+ &= \{1,2,13,14\}, \quad \mathcal{Q}_- = \{4,7,8,11\},\\
\Gamma_+&: (1,1) \to 1, (1,3) \to 13, (2,2) \to 2, (2,4) \to 14,\\
\Gamma_-&: (1,1) \to 4, (1,3) \to 7, (2,2) \to 8, (2,4) \to 11,
\end{aligned}$$

and

$$\begin{array}{lllll}
\Theta_+: & 1 \to (1,1), & 2 \to (2,2), & 13 \to (1,3), & 14 \to (2,4),\\
\Theta_-: & 4 \to (1,1), & 7 \to (1,3), & 8 \to (2,2), & 11 \to (2,4).
\end{array}$$

Next we construct maps $\widetilde{\Gamma}_\pm$ from $\mathcal{H}$ to $\mathcal{R}$.

Recall the following:

$$\begin{aligned}
\mathcal{H} &:= \{(h,k,l) \in \mathbb{Z}^3 \mid 0 < h < p-ab, 0 < k < a, 0 < l < b, h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l\},\\
\mathcal{R} &:= \{(g,m) \in \mathbb{Z}^2 \mid 0 < m < ab, 0 < g < p-ab,\\
&\qquad\qquad g \underset{(2)}{\equiv} p-m, a \nmid m, b \nmid m\},
\end{aligned}$$

$$\Gamma_\pm(k,l) := [\pm adl - bck]_{ab}.$$

We put

$$\widetilde{\Gamma}_\pm(h,k,l) := \begin{cases} (p-ab-h, \Gamma_\pm(k,l)) & \text{if } \Gamma_\pm(k,l) + ab + h \underset{(2)}{\equiv} 0,\\ (p-ab-h, ab-\Gamma_\pm(k,l)) & \text{if } \Gamma_\pm(k,l) + ab + h \underset{(2)}{\not\equiv} 0. \end{cases}$$

Note that if a is even, we have

$$\begin{aligned}
\Gamma_\pm(k,l) + ab + h &= [\pm adl - bck]_{ab} + ab + h\\
&= [bc(-k \pm l) \pm l]_{ab} + ab + h \underset{(2)}{\equiv} 0
\end{aligned}$$

since $k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} h$. So, in this case $\widetilde{\Gamma}_\pm(h,k,l) = (p-ab-h, \Gamma_\pm(k,l))$.

Let us check whether the image of $\widetilde{\Gamma}_\pm$ is contained in $\mathcal{R}$ or not. First assume that

$$\Gamma_\pm(k,l) + ab + h \underset{(2)}{\equiv} 0.$$

Then we have

$$(p - ab - h) - \widetilde{\Gamma}_\pm(h,k,l) \underset{(2)}{\equiv} p$$

and so the image is in $\mathcal{R}$. Next assume that

$$\Gamma_\pm(k,l) + ab + h \underset{(2)}{\not\equiv} 0.$$

Noting that a should be odd, we have

$$(p - ab - h) - (ab - \widetilde{\Gamma}_\pm(h,k,l)) \underset{(2)}{\equiv} (p - ab - h) + (h + 1) \underset{(2)}{\equiv} p.$$

Therefore, the image of $\widetilde{\Gamma}_\pm$ is also contained in $\mathcal{R}$.

Since

$$\begin{aligned}\Gamma_\pm(a-k, b-l) &= [\pm ad(b-l) - bc(a-k)]_{ab} = [-(\pm adl - bck)]_{ab} \\ &= ab - \Gamma_\pm(k,l)\end{aligned}$$

and $\Gamma_+(\mathcal{P}) \cap \Gamma_-(\mathcal{P}) = \emptyset$, we see that $\widetilde{\Gamma}_+(\mathcal{H}) \cap \widetilde{\Gamma}_-(\mathcal{H}) = \emptyset$.

We show that $\widetilde{\Gamma}_\pm$ is injective. Suppose that

$$\widetilde{\Gamma}_\pm(h,k,l) = \widetilde{\Gamma}_\pm(k',k',h') \quad \text{for } (h,k,l), (k',l',h') \in \mathcal{H}.$$

Then from the definition we have $h = h'$.

We also have either

i. $\Gamma_\pm(k,l) \underset{(2)}{\equiv} ab + h \underset{(2)}{\equiv} \Gamma_\pm(k',l')$ and $\Gamma_\pm(k,l) = \Gamma_\pm(k',l')$,

ii. $\Gamma_\pm(k,l) \underset{(2)}{\equiv} ab + h \underset{(2)}{\not\equiv} \Gamma_\pm(k',l')$ and $\Gamma_\pm(k,l) = ab - \Gamma_\pm(k',l')$,

iii. $\Gamma_\pm(k,l) \underset{(2)}{\not\equiv} ab + h \underset{(2)}{\equiv} \Gamma_\pm(k',l')$ and $ab - \Gamma_\pm(k,l) = \Gamma_\pm(k',l')$, or

iv. $\Gamma_\pm(k,l) \underset{(2)}{\not\equiv} ab + h \underset{(2)}{\not\equiv} \Gamma_\pm(k',l')$ and $\Gamma_\pm(k,l) = \Gamma_\pm(k',l')$.

For cases (i) and (iv), we have $(k,l) = (k',l')$ from the injectivity of $\Gamma_\pm$. For case (ii), since $ab - \Gamma_\pm(k',l') = \Gamma_\pm(a-k', b-l')$, we have $b = l + l'$, which contradicts the condition

$$l \underset{(2)}{\equiv} h \underset{(2)}{\equiv} l'.$$

So, case (ii) does not happen. Similarly, case (iii) does not happen, either. Therefore, we have $(h,k,l) = (k',l',h')$ and $\widetilde{\Gamma}_\pm$ are injective.

Noting that

$$\#\mathcal{H} = \begin{cases} \dfrac{1}{4}(a-1)(b-1)(p-ab-1) & \text{when } p \text{ is odd,} \\ \dfrac{(a-1)(b-1)}{2}\left\lfloor \dfrac{p-ab-1}{2} \right\rfloor & \text{when } p \text{ is even.} \end{cases}$$

$$\#\mathcal{R} = \begin{cases} \dfrac{1}{2}(a-1)(b-1)(p-ab-1) & \text{when } p \text{ is odd,} \\ (a-1)(b-1)\left\lfloor \dfrac{p-ab-1}{2} \right\rfloor & \text{when } p \text{ is even.} \end{cases}$$

we see that $\widetilde{\Gamma}_+(\mathcal{H}) \sqcup \widetilde{\Gamma}_-(\mathcal{H}) = \mathcal{R}$.

We denote by $\mathcal{R}_\pm$ the image of $\widetilde{\Gamma}_\pm$.

Lemma 8.5. *We can characterise $\mathcal{R}_\pm$ as follows.*

$$\mathcal{R}_+ = \{(g,m) \in \mathcal{R} \mid [m]_a \underset{(2)}{\equiv} [m]_b\},$$
$$\mathcal{R}_- = \{g,m) \in \mathcal{R} \mid [m]_a \underset{(2)}{\not\equiv} [m]_b\}.$$

Proof. It is sufficient to show that

$$[\Gamma_+(k,l)]_a \underset{(2)}{\equiv} [\Gamma_+(k,l)]_b \quad \text{and} \quad [\Gamma_-(k,l)]_a \underset{(2)}{\not\equiv} [\Gamma_-(k,l)]_b.$$

Since $\Gamma_\pm(k,l) = [\pm adl - bck]_{ab}$ and $ad - bc = 1$, we have

$$[\Gamma_\pm(k,l)]_a = [k]_a = k,$$
$$[\Gamma_+(k,l)]_b = [l]_b = l,$$
$$[\Gamma_-(k,l)]_b = [-l]_b = b - l.$$

The conclusion follows since b is odd and $k \underset{(2)}{\equiv} l$. ∎

We define maps $\widetilde{\Theta}_+\colon \mathcal{R}_+ \to \mathcal{H}$ and $\widetilde{\Theta}_-\colon \mathcal{R}_- \to \mathcal{H}$ as follows:

$$\widetilde{\Theta}_\pm(g,m) := (p-ab-g, [m]_a, [\pm m]_b) \qquad \text{if } ab + m + [m]_a \underset{(2)}{\equiv} 0,$$
$$\widetilde{\Theta}_\pm(g,m) := (p-ab-g, [-m]_a, [\mp m]_b) \quad \text{if } ab + m + [m]_a \underset{(2)}{\not\equiv} 0.$$

Note that if a is even, then $[m]_a \underset{(2)}{\equiv} m$ and so $ab + m + [m]_a \underset{(2)}{\equiv} 0$. As a result, if $b + m + [m]_a \underset{(2)}{\not\equiv} 0$, then a is odd. We need to check that $\widetilde{\Theta}_\pm$ is a map to $\mathcal{H}$.

Assume that $ab + m + [m]_a \underset{(2)}{\equiv} 0$. Then, since $(g, m) \in \mathcal{R}$, we have

$$(p - ab - g) - [m]_a \underset{(2)}{\equiv} p + g + m \underset{(2)}{\equiv} 0.$$

If $(g, m) \in \mathcal{R}_+$, then $[m]_a \underset{(2)}{\equiv} [m]_b$ and so $\widetilde{\Theta}_+(g, m) \in \mathcal{H}$. If $(g, m) \in \mathcal{R}_-$, then $[m]_a \underset{(2)}{\not\equiv} [m]_b$. Since $[-m]_b = b - [m]_b \underset{(2)}{\not\equiv} [m]_b$, $[m]_a \underset{(2)}{\equiv} [-m]_b$ and so $\widetilde{\Theta}_+(g, m) \in \mathcal{H}$.

Assume that $ab + m + [m]_a \underset{(2)}{\not\equiv} 0$. Since a is odd as mentioned before, we have

$$(p - ab - g) - [-m]_a = (p - ab - g) - (a - [m]_a) \underset{(2)}{\equiv} p + m + g \underset{(2)}{\equiv} 0.$$

If $(g, m) \in \mathcal{R}_+$, then $[-m]_a = a - [m]_a \underset{(2)}{\equiv} b - [m]_b \underset{(2)}{\equiv} [-m]_b$ and so $\widetilde{\Theta}_+(g, m) \in \mathcal{H}$. If $(g, m) \in \mathcal{R}_-$, then $[-m]_a = a - [m]_a \underset{(2)}{\not\equiv} b - [m]_b \underset{(2)}{\not\equiv} [m]_b$ and so $\widetilde{\Theta}_-(g, m) \in \mathcal{H}$. Therefore, the image of $\widetilde{\Theta}_\pm$ is in $\mathcal{H}$.

Next we show that $\widetilde{\Theta}_\pm$ is the inverse of $\widetilde{\Gamma}_\pm$.

When $\Gamma_\pm(k, l) + ab + h \underset{(2)}{\equiv} 0$, we have

$$\begin{aligned}(\widetilde{\Theta}_\pm \circ \widetilde{\Gamma}_\pm)(h, k, l) &= \widetilde{\Theta}_\pm(p - ab - h, \Gamma_\pm(k, l)) \\ &= (h, [\Gamma_\pm(k, l)]_a, [\pm\Gamma_\pm(k, l)]_b) \\ &= (h, k, l)\end{aligned}$$

since

$$ab + \Gamma_\pm(k, l) + [\Gamma_\pm(k, l)]_a = ab + \Gamma_\pm(k, l) + k \underset{(2)}{\equiv} 0.$$

When $\Gamma_\pm(k, l) + ab + h \underset{(2)}{\not\equiv} 0$, we have

$$\begin{aligned}(\widetilde{\Theta}_\pm \circ \widetilde{\Gamma}_\pm)(h, k, l) &= \widetilde{\Theta}_\pm(p - ab - h, \Gamma_\pm(a - k, b - l)) \\ &= (h, [-\Gamma_\pm(a - k, b - l)]_a, [\mp\Gamma_\pm(a - k, b - l)]_b) \\ &= (h, [\mp ad(b - l) + bc(a - k)]_a, [-ad(b - l) \pm bc(a - k)]_b) \\ &= (h, k, l).\end{aligned}$$

Here the second equality follows since

$$\begin{aligned}ab + \Gamma_\pm(a - k, b - l) + [\Gamma_\pm(a - k, b - l)]_a &\underset{(2)}{\equiv} \Gamma_\pm(k, l) + b - l \\ &\underset{(2)}{\not\equiv} ab + h + b + l \underset{(2)}{\equiv} 0.\end{aligned}$$

Therefore, $\widetilde{\Theta}_\pm \circ \widetilde{\Gamma}_\pm$ is the identity on $\mathcal{H}$.

When $(g,m) \in \mathcal{R}_+$ and $ab+m+[m]_a \underset{(2)}{\equiv} 0$, we have

$$(\widetilde{\Gamma}_+ \circ \widetilde{\Theta}_+)(g,m) = \widetilde{\Gamma}_+(p-ab-g,[m]_a,[m]_b).$$

Since $\Gamma_+([m]_a,[m]_b) = m$ from (36), and $m+ab+(p-ab-g) \underset{(2)}{\equiv} 0$, we have

$$\widetilde{\Gamma}_+(p-ab-g,[m]_a,[m]_b) = (g,\Gamma_+([m]_a,[m]_b)) = (g,m).$$

When $(g,m) \in \mathcal{R}_+$ and $ab+m+[m]_a \underset{(2)}{\not\equiv} 0$, we have

$$(\widetilde{\Gamma}_+ \circ \widetilde{\Theta}_+)(g,m) = \widetilde{\Gamma}_+(p-ab-g,[-m]_a,[-m]_b).$$

Since

$$\Gamma_+([-m]_a,[-m]_b) = \Gamma_+([ab-m]_a,[ab-m]_b) = ab-m$$

and

$$(ab-m)+ab+(p-ab-g) \underset{(2)}{\equiv} ab \underset{(2)}{\not\equiv} 0,$$

we have

$$\begin{aligned}\widetilde{\Gamma}_+(p-ab-g,[-m]_a,[-m]_b) &= (g,ab-\Gamma_+([-m]_a,[-m]_b)) \\ &= (g,ab-\Gamma_+([ab-m]_a,[ab-m]_b)) \\ &= (g,m).\end{aligned}$$

Therefore, $\widetilde{\Gamma}_+ \circ \widetilde{\Theta}_+$ is the identity on $\mathcal{R}_+$.

When $(g,m) \in \mathcal{R}_-$ and $ab+m+[m]_a \underset{(2)}{\equiv} 0$, we have

$$(\widetilde{\Gamma}_- \circ \widetilde{\Theta}_-)(g,m) = \widetilde{\Gamma}_-(p-ab-g,[m]_a,[-m]_b).$$

Since $\Gamma_-([m]_a,[-m]_b) = m$ from (37), and $m+ab+(p-ab-g) \underset{(2)}{\equiv} 0$, we have

$$\widetilde{\Gamma}_-(p-ab-g,[m]_a,[-m]_b) = (g,\Gamma_-([m]_a,[-m]_b)) = (g,m).$$

When $(g,m) \in \mathcal{R}_-$ and $ab+m+[m]_a \underset{(2)}{\not\equiv} 0$, we have

$$(\widetilde{\Gamma}_- \circ \widetilde{\Theta}_-)(g,m) = \widetilde{\Gamma}_-(p-ab-g,[-m]_a,[m]_b).$$

Since

$$\Gamma_-([-m]_a,[m]_b) = \Gamma_-([ab-m]_a,[m-ab]_b) = ab-m,$$

and

$$(ab-m)+ab+(p-ab-g) \underset{(2)}{\equiv} ab \underset{(2)}{\not\equiv} 0,$$

we have

$$\widetilde{\Gamma}_-(p-ab-g,[-m]_a,[m]_b) = (g, ab - \widetilde{\Gamma}_-([-m]_a,[m]_b)) = (g,m).$$

Therefore, $\widetilde{\Gamma}_\pm \circ \widetilde{\Theta}_-$ is the identity on $\mathcal{R}_-$.

Example 8.6. . When $a = 2$, b and p are odd from the assumption. So, we have

$$\begin{aligned}
\mathcal{H} &= \{(h,1,l) \mid h = 1,3,\ldots,p-2b-2, l = 1,3,\ldots,b-2\},\\
\mathcal{R} &= \{(g,m) \mid g = 2,4,\ldots,p-2b-1, m = 1,3,\ldots,b-2,b+2,\ldots,2b-1\},\\
\mathcal{R}_+ &= \{(g,m) \mid g = 2,4,\ldots,p-2b-1, m = 1,3,\ldots,b-2\},\\
\mathcal{R}_- &= \{(g,m) \mid g = 2,4,\ldots,p-2b-1, m = b+2,b+4,\ldots,2b-1\}.
\end{aligned}$$

We can put $d := (1-b)/2$ and $c := -1$. So, we have

$$\begin{aligned}
\widetilde{\Gamma}_+(h,1,l) &= (p-2b-h,[(1-b)l+b]_{2b}) = (p-2b-h,l),\\
\widetilde{\Gamma}_-(h,1,l) &= (p-2b-h,[-(1-b)l+b]_{2b}) = (p-2b-h,2b-l),
\end{aligned}$$

since l is odd.

We also have

$$\widetilde{\Theta}_+(g,m) = (p-2b-g,1,m)$$

when $(g,m) \in \mathcal{R}_+$ and

$$\widetilde{\Theta}_-(g,m) = (p-2b-g,1,2b-m)$$

when $(g,m) \in \mathcal{R}_-$.

Example 8.7. When $(a,b) = (3,5)$ and $p = 19$, we have

$$\begin{aligned}
\mathcal{H} &= \{(1,1,1),(1,1,3),(2,2,2),(2,2,4),(3,1,1),(3,1,3)\},\\
\mathcal{R} &= \{(1,2),(1,4),(1,8),(1,14),(2,1),(2,7),(2,11),(2,13),(3,2),(3,4),(3,8),\\
&\qquad (3,14)\},\\
\mathcal{R}_+ &= \{(1,2),(1,14),(2,1),(2,13),(3,2),(3,14)\},\\
\mathcal{R}_- &= \{(1,4),(1,8),(2,7),(2,11),(3,4),(3,8)\}.
\end{aligned}$$

From Example 8.4, we have

$$\begin{aligned}
\widetilde{\Gamma}_+&\colon (1,1,1)\to(3,14),(1,1,3)\to(3,2),(2,2,2)\to(2,13),(2,2,4)\to(2,1),\\
&\quad (3,1,1)\to(1,14),(3,1,3)\to(1,2),\\
\widetilde{\Gamma}_-&\colon (1,1,1)\to(3,4),(1,1,3)\to(3,8),(2,2,2)\to(2,7),(2,2,4)\to(2,11),\\
&\quad (3,1,1)\to(1,4),(3,1,3)\to(1,8),
\end{aligned}$$

$$\widetilde{\Theta}_+ \colon (1,2) \to (3,1,3), (1,14) \to (3,1,1), (2,1) \to (2,2,4), (2,13) \to (2,2,2), \\ (3,2) \to (1,1,3), (3,14) \to (1,1,1),$$
$$\widetilde{\Theta}_- \colon (1,4) \to (3,1,1), (1,8) \to (3,1,3), (2,7) \to (2,2,2), (2,11) \to (2,2,4), \\ (3,4) \to (1,1,1), (3,8) \to (1,1,3).$$

9. Topological interpretations of $A(n)$ and $B(n)$

As shown in (26), the irreducible representations of $\pi_1(X_p)$ to $\mathrm{SL}(2;\mathbb{C})$ are indexed by $\mathcal{H}$.

9.1. Topological interpretation of $A(n)$

In this section, we give a topological interpretation of $A(n)$.

Let $\widetilde{\Gamma}_\pm \colon \mathcal{H} \to \mathcal{R}_\pm$ be the bijections described in Section 8. Recall the following subsets of $\mathcal{R}$ (see Figure 8):

$$\mathcal{R} = \{(g,m) \in \mathbb{Z}^2 \mid 0 < m < ab, 0 < g < p-ab, g \underset{(2)}{\equiv} p-m, a \nmid m, b \nmid m\},$$
$$\mathcal{R}^\Delta = \left\{(g,m) \in \mathcal{R} \;\middle|\; \frac{m}{ab} < \frac{g}{p-ab}\right\},$$
$$\mathcal{R}^\nabla = \left\{(g,m) \in \mathcal{R} \;\middle|\; \frac{m}{ab} > \frac{g}{p-ab}\right\},$$
$$\mathcal{R}_+ = \left\{(g,m) \in \mathcal{R} \;\middle|\; [m]_a \underset{(2)}{\equiv} [m]_b\right\},$$
$$\mathcal{R}_- = \{(g,m) \in \mathcal{R} \mid [m]_a \underset{(2)}{\not\equiv} [m]_b\}.$$

Put $\mathcal{R}^\Delta_\pm := \mathcal{R}_\pm \cap \mathcal{R}^\Delta$, $\mathcal{R}^\nabla_\pm := \mathcal{R}_\pm \cap \mathcal{R}^\nabla$, $\mathcal{H}^\Delta_\pm := \widetilde{\Theta}_\pm(\mathcal{R}^\Delta_\pm)$ and $\mathcal{H}^\nabla_\pm := \widetilde{\Theta}_\pm(\mathcal{R}^\nabla_\pm)$. Then we have two decompositions of $\mathcal{H}$ and one decompositon of $\mathcal{R}$:

$$\mathcal{H} = \mathcal{H}^\Delta_+ \sqcup \mathcal{H}^\nabla_+,$$
$$\mathcal{H} = \mathcal{H}^\Delta_- \sqcup \mathcal{H}^\nabla_-,$$
$$\mathcal{R} = \mathcal{R}^\Delta_+ \sqcup \mathcal{R}^\nabla_+ \sqcup \mathcal{R}^\Delta_- \sqcup \mathcal{R}^\nabla_-.$$

Remark 9.1. Note that $\mathcal{H}^\Delta_\pm \cap \mathcal{H}^\nabla_\pm = \emptyset$ but that $\mathcal{H}^\Delta_+ \cap \mathcal{H}^\Delta_- \neq \emptyset$ and $\mathcal{H}^\Delta_- \cap \mathcal{H}^\Delta_+ \neq \emptyset$ in general.

Remark 9.2. Since an element (h,k,l) in $\mathcal{H}$ belongs to $\mathcal{H}^\Delta_\pm$ ($\mathcal{H}^\nabla_\pm$, respectively) if and only if $\widetilde{\Gamma}_\pm(h,k,l) \in \mathcal{R}^\Delta$ ($\widetilde{\Gamma}_\pm(h,k,l) \in \mathcal{R}^\nabla$, respectively). Therefore, concretely

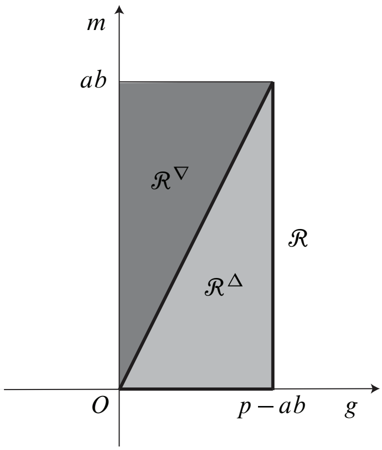

Figure 8. The dark gray area is $\mathcal{R}^{\nabla}$, the light gray area is $\mathcal{R}^{\Delta}$, and $\mathcal{R} = \mathcal{R}^{\Delta} \cup \mathcal{R}^{\nabla}$.

speaking, the sets $\mathcal{H}_{\pm}^{\Delta}$ and $\mathcal{H}_{\pm}^{\nabla}$ are given as follows.

$$\mathcal{H}_{\pm}^{\Delta} = \Big\{(h,k,l) \in \mathcal{H} \;\Big|\; \frac{\Gamma_{\pm}(k,l)}{ab} + \frac{h}{p-ab} < 1 \text{ (if } \Gamma_{\pm}(k,l) + ab + h \underset{(2)}{\equiv} 0),$$
$$\frac{\Gamma_{\pm}(k,l)}{ab} > \frac{h}{p-ab} \text{ (if } \Gamma_{\pm}(k,l) + ab + h \underset{(2)}{\equiv} 1)\Big\},$$
$$\mathcal{H}_{\pm}^{\nabla} = \Big\{(h,k,l) \in \mathcal{H} \;\Big|\; \frac{\Gamma_{\pm}(k,l)}{ab} + \frac{h}{p-ab} > 1 \text{ (if } \Gamma_{\pm}(k,l) + ab + h \underset{(2)}{\equiv} 0),$$
$$\frac{\Gamma_{\pm}(k,l)}{ab} < \frac{h}{p-ab} \text{ (if } \Gamma_{\pm}(k,l) + ab + h \underset{(2)}{\equiv} 1)\Big\}.$$

Example 9.3. Suppose that $a = 2$. Then we have

$$\mathcal{R}_{+} = \{(g,m) \mid h = 2,4,\dots,p-2b-1, m = 1,3,\dots,b-2\},$$
$$\mathcal{R}_{-} = \{(g,m) \mid h = 2,4,\dots,p-2b-1, m = b+2,b+4,\dots,2b-1\}$$

and so

$$\mathcal{R}_{+}^{\Delta} = \Big\{(g,m) \;\Big|\; 2 \le h \le p-2b-1, 1 \le m \le b-2, \frac{h}{p-2b} > \frac{m}{2b},$$
$$h \underset{(2)}{\equiv} 0, m \underset{(2)}{\equiv} 1\Big\},$$
$$\mathcal{R}_{-}^{\Delta} = \Big\{(g,m) \;\Big|\; 2 \le h \le p-2b-1, b+2 \le m \le 2b-1, \frac{h}{p-2b} > \frac{m}{2b},$$
$$h \underset{(2)}{\equiv} 0, m \underset{(2)}{\equiv} 1\Big\},$$

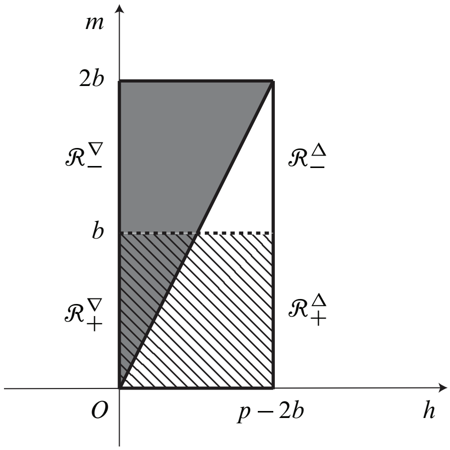

Figure 9. The decomposition of $\mathcal{R}$ when $a = 2$ ($k = 1$).

$$\mathcal{R}_+^\nabla = \left\{(g, m) \,\middle|\, 2 \le h \le p - 2b - 1, m \le l \le b - 2, \frac{h}{p-2b} < \frac{m}{2b}, \right.$$
$$\left. h \underset{(2)}{\equiv} 0, m \underset{(2)}{\equiv} 1\right\},$$
$$\mathcal{R}_-^\nabla = \left\{(g, m) \,\middle|\, 2 \le h \le p - 2b - 1, b + 2 \le m \le 2b - 1, \frac{h}{p-2b} > \frac{m}{2b}, \right.$$
$$\left. h \underset{(2)}{\equiv} 0, m \underset{(2)}{\equiv} 1\right\}.$$

See Figure 9. Therefore, from Example 8.6, we have

$$\mathcal{H}_+^\Delta = \left\{(h, 1, l) \,\middle|\, 1 \le h \le p - 2b - 2, 1 \le l \le b - 2, \frac{h}{p-2b} + \frac{l}{2b} < 1, \right.$$
$$\left. h \underset{(2)}{\equiv} 1, l \underset{(2)}{\equiv} 1\right\},$$
$$\mathcal{H}_+^\nabla = \left\{(h, 1, l) \,\middle|\, 1 \le h \le p - 2b - 2, 1 \le l \le b - 2, \frac{h}{p-2b} + \frac{l}{2b} > 1, \right.$$
$$\left. h \underset{(2)}{\equiv} 1, l \underset{(2)}{\equiv} 1\right\},$$
$$\mathcal{H}_-^\Delta = \left\{(h, 1, l) \,\middle|\, 1 \le h \le p - 2b - 2, 1 \le l \le b - 2, \frac{h}{p-2b} < \frac{l}{2b}, \right.$$
$$\left. h \underset{(2)}{\equiv} 1, l \underset{(2)}{\equiv} 1\right\},$$

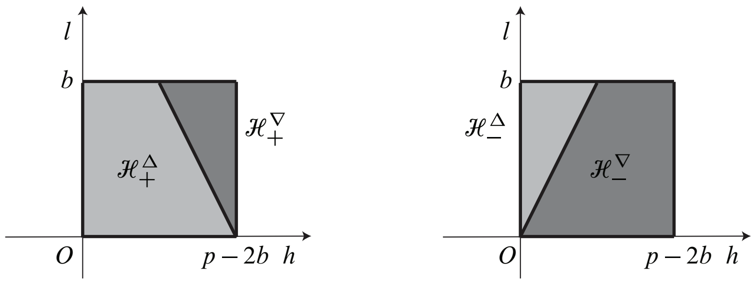

Figure 10. Two decompositions of $\mathcal{H}$.

$$\mathcal{H}_-^\nabla = \left\{(h,1,l) \,\middle|\, 1 \le h \le p-2b-2, 1 \le l \le b-2, \frac{h}{p-2b} > \frac{l}{2b}, \right.$$
$$\left. h \underset{(2)}{\equiv} 1, l \underset{(2)}{\equiv} 1\right\}$$

as indicated in Figures 10. Note that $\mathcal{H}_-^\Delta \subset \mathcal{H}_+^\Delta$ and that $\mathcal{H}_+^\nabla \subset \mathcal{H}_-^\nabla$.

Example 9.4. If $(a,b) = (3,5)$ and $p = 19$, we have

$$\mathcal{R}^\Delta = \{(1,2),(2,1),(3,2),(3,4),(3,8)\}$$
$$\mathcal{R}^\nabla = \{(1,4),(1,8),(1,14),(2,7),(2,11),(2,13),(3,14)\},$$

$$\mathcal{R}_+^\Delta = \{(1,2),(2,1),(3,2)\},$$
$$\mathcal{R}_+^\nabla = \{(1,14),(2,13),(3,14)\},$$
$$\mathcal{R}_-^\Delta = \{(3,4),(3,8)\},$$
$$\mathcal{R}_-^\nabla = \{(1,4),(1,8),(2,7),(2,11)\},$$

and

$$\mathcal{H}_+^\Delta = \{(3,1,3),(2,2,4),(1,1,3)\},$$
$$\mathcal{H}_+^\nabla = \{(3,1,1),(2,2,2),(1,1,1)\},$$
$$\mathcal{H}_-^\Delta = \{(1,1,1),(1,1,3)\},$$
$$\mathcal{H}_-^\nabla = \{(3,1,1),(3,1,3),(2,2,2),(2,2,4)\},$$

from 8.7.

Note that

$$\mathcal{H}_+^\Delta \cap \mathcal{H}_-^\Delta = \{(1,1,3)\} \quad \text{and} \quad \mathcal{H}_+^\nabla \cap \mathcal{H}_-^\nabla = \{(3,1,1),(2,2,2)\}.$$

Since

$$\mathcal{R} = \mathcal{R}^{\Delta} \sqcup \mathcal{R}^{\nabla}, \quad \mathcal{R}^{\Delta} = \mathcal{R}^{\Delta}_{+} \sqcup \mathcal{R}^{\Delta}_{-}, \quad \mathcal{R}^{\nabla} = \mathcal{R}^{\nabla}_{+} \sqcup \mathcal{R}^{\nabla}_{-},$$

from (22) and the condition that $g \underset{(2)}{\equiv} p - m$ when $(g, m) \in \mathcal{R}$, we have

$$\begin{aligned}
&e^{-\frac{(n+1)\pi\sqrt{-1}}{4}} \sqrt{ab(p-ab)} A(n) \\
&= \sum_{(g,m)\in\mathcal{R}^{\Delta}} G(g,m) - \sum_{(g,m)\in\mathcal{R}^{\nabla}} G(g,m) - \sum_{(g,m)\in\mathcal{R}} (-1)^{ab+p+g} G(g,m) \\
&= \sum_{(g,m)\in\mathcal{R}^{\Delta}_{+}} G(g,m) + \sum_{(g,m)\in\mathcal{R}^{\Delta}_{-}} G(g,m) - \sum_{(g,m)\in\mathcal{R}^{\nabla}_{+}} G(g,m) - \sum_{(g,m)\in\mathcal{R}^{\nabla}_{-}} G(g,m) \\
&\quad - \sum_{(g,m)\in\mathcal{R}^{\Delta}_{+}} (-1)^{ab+p+g} G(g,m) - \sum_{(g,m)\in\mathcal{R}^{\Delta}_{-}} (-1)^{ab+p+g} G(g,m) \\
&\quad - \sum_{(g,m)\in\mathcal{R}^{\nabla}_{+}} (-1)^{ab+p+g} G(g,m) - \sum_{(g,m)\in\mathcal{R}^{\nabla}_{-}} (-1)^{ab+p+g} G(g,m) \\
&= \sum_{(g,m)\in\mathcal{R}^{\Delta}_{+}} (1-(-1)^{ab+p+g}) G(g,m) + \sum_{(g,m)\in\mathcal{R}^{\Delta}_{-}} (1-(-1)^{ab+p+g}) G(g,m) \\
&\quad - \sum_{(g,m)\in\mathcal{R}^{\nabla}_{+}} (1+(-1)^{ab+p+g}) G(g,m) - \sum_{(g,m)\in\mathcal{R}^{\nabla}_{-}} (1+(-1)^{ab+p+g}) G(g,m) \\
&= \sum_{(h,k,l)\in\mathcal{H}^{\Delta}_{+}} (1-(-1)^{h}) G(\widetilde{\Gamma}_{+}(h,k,l)) + \sum_{(h,k,l)\in\mathcal{H}^{\Delta}_{-}} (1-(-1)^{h}) G(\widetilde{\Gamma}_{-}(h,k,l)) \\
&\quad - \sum_{(h,k,l)\in\mathcal{H}^{\nabla}_{+}} (1+(-1)^{h}) G(\widetilde{\Gamma}_{+}(h,k,l)) - \sum_{(h,k,l)\in\mathcal{H}^{\nabla}_{-}} (1+(-1)^{h}) G(\widetilde{\Gamma}_{-}(h,k,l)),
\end{aligned}$$

since $ab + p + g \underset{(2)}{\equiv} h$ if $(g, m) = \widetilde{\Gamma}_{\pm}(h, k, l)$.

Therefore, we have

$$\begin{aligned}
&e^{-\frac{(n+1)\pi\sqrt{-1}}{4}} \sqrt{ab(p-ab)} A(n) \\
&= 2 \sum_{\substack{(h,k,l)\in\mathcal{H}^{\Delta}_{+} \\ h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 1}} G(\widetilde{\Gamma}_{+}(h,k,l)) + 2 \sum_{\substack{(h,k,l)\in\mathcal{H}^{\Delta}_{-} \\ h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 1}} G(\widetilde{\Gamma}_{-}(h,k,l)) \\
&\quad - 2 \sum_{\substack{(h,k,l)\in\mathcal{H}^{\nabla}_{+} \\ h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 0}} G(\widetilde{\Gamma}_{+}(h,k,l)) - 2 \sum_{\substack{(h,k,l)\in\mathcal{H}^{\nabla}_{-} \\ h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 0}} G(\widetilde{\Gamma}_{-}(h,k,l)).
\end{aligned}$$

Now, we define $\mathcal{CS}_{\pm}(h,k,l)$ and $\mathcal{T}_{\pm}(h,k,l)$ as follows:

$$\mathcal{CS}^{\mathrm{Irr}}_{\pm}(h,k,l)$$
$$:= \begin{cases} -\dfrac{(p-ab-h)^2}{4(p-ab)} - \dfrac{\Gamma_{\pm}(k,l)^2}{4ab} & \text{if } \Gamma_{\pm}(k,l)+ab+h \underset{(2)}{\equiv} 0, \\ -\dfrac{(p-ab-h)^2}{4(p-ab)} - \dfrac{(ab-\Gamma_{\pm}(k,l))^2}{4ab} & \text{if } \Gamma_{\pm}(k,l)+ab+h \underset{(2)}{\equiv} 1, \end{cases}$$

$$\mathcal{T}^{\mathrm{Irr}}_{\pm}(h,k,l)$$
$$:= \begin{cases} (-1)^{\Gamma_{\pm}(k,l)} \dfrac{8\sin(\frac{\Gamma_{\pm}(k,l)\pi}{a})\sin(\frac{\Gamma_{\pm}(k,l)\pi}{b})\sin(\frac{(p-ab-h)\pi}{p-ab})}{\sqrt{ab(p-ab)}} \\ \qquad\qquad \text{if } \Gamma_{\pm}(k,l)+ab+h \underset{(2)}{\equiv} 0, \\ (-1)^{ab-\Gamma_{\pm}(k,l)} \dfrac{8\sin(\frac{(ab-\Gamma_{\pm}(k,l))\pi}{a})\sin(\frac{(ab-\Gamma_{\pm}(k,l))\pi}{b})\sin(\frac{(p-ab-h)\pi}{p-ab})}{\sqrt{ab(p-ab)}} \\ \qquad\qquad \text{if } \Gamma_{\pm}(k,l)+ab+h \underset{(2)}{\equiv} 1. \end{cases}$$

Then from (21) we have

$$\begin{aligned} e^{-\frac{(n+1)\pi\sqrt{-1}}{4}} A(n) = \frac{1}{4} & \sum_{\substack{(h,k,l)\in\mathcal{H}^{\Delta}_{+} \\ h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 1}} \mathcal{T}^{\mathrm{Irr}}_{+}(h,k,l) e^{n\mathcal{CS}^{\mathrm{Irr}}_{+}(h,k,l)\pi\sqrt{-1}} \\ &+ \frac{1}{4} \sum_{\substack{(h,k,l)\in\mathcal{H}^{\Delta}_{-} \\ h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 1}} \mathcal{T}^{\mathrm{Irr}}_{-}(h,k,l) e^{n\mathcal{CS}^{\mathrm{Irr}}_{-}(h,k,l)\pi\sqrt{-1}} \\ &- \frac{1}{4} \sum_{\substack{(h,k,l)\in\mathcal{H}^{\nabla}_{+} \\ h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 0}} \mathcal{T}^{\mathrm{Irr}}_{+}(h,k,l) e^{n\mathcal{CS}^{\mathrm{Irr}}_{+}(h,k,l)\pi\sqrt{-1}} \\ &- \frac{1}{4} \sum_{\substack{(h,k,l)\in\mathcal{H}^{\nabla}_{-} \\ h \underset{(2)}{\equiv} k \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 0}} \mathcal{T}^{\mathrm{Irr}}_{-}(h,k,l) e^{n\mathcal{CS}^{\mathrm{Irr}}_{-}(h,k,l)\pi\sqrt{-1}}. \end{aligned} \tag{38}$$

Lemma 9.5. *We have*

$$\mathcal{T}^{\mathrm{Irr}}_{\pm}(h,k,l)^{-2} = |\operatorname{Tor}(X_p; \tilde{\rho}^{\mathrm{Irr}}_{p-ab-h,a-k,b-l})|.$$

Proof. We will show

$$\mathcal{T}^{\mathrm{Irr}}_{\pm}(h,k,l) = \pm\sin\Big(\frac{k\pi}{a}\Big)\sin\Big(\frac{l\pi}{b}\Big)\sin\Big(\frac{h\pi}{p-ab}\Big).$$

It is sufficient to prove

$$\sin\Big(\frac{\Gamma_{\pm}(k,l)\pi}{a}\Big)\sin\Big(\frac{\Gamma_{\pm}(k,l)\pi}{b}\Big)=\pm\sin\Big(\frac{k\pi}{a}\Big)\sin\Big(\frac{l\pi}{b}\Big).$$

Since $\Gamma_{\pm}(k,l)=\pm adl-bck\equiv -bck\equiv -k \pmod{a}$ and $\Gamma_{\pm}(k,l)\equiv adl\equiv l \pmod{b}$, the equality above holds. ■

Lemma 9.6. *We have*

$$\mathcal{CS}(h,k,l)\equiv \mathrm{CS}(X_p;\tilde{\rho}^{\mathrm{Irr}}_{p-ab-h,a-k,b-l})\in\mathbb{C}/(2\mathbb{Z}).$$

Proof. We will show that

$$\begin{aligned}&(ad(b-l)-bc(a-k))^2\\ &\equiv\begin{cases}\Gamma_{\pm}(k,l)^2 \mod 2ab & \text{if } \Gamma_{\pm}(k,l)+ab+h \underset{(2)}{\equiv} 0,\\ (ab-\Gamma_{\pm}(k,l))^2 \pmod{2ab} & \text{if } \Gamma_{\pm}(k,l)+ab+h \underset{(2)}{\equiv} 1.\end{cases}\end{aligned}$$

- $\Gamma_{\pm}(k,l)+ab+h \underset{(2)}{\equiv} 0$. Put $u_{\pm}:=\Gamma_{\pm}(k,l)$. Then there exists $v_{\pm}\in\mathbb{Z}$ such that

$$\pm adl-bck=abv_{\pm}+u_{\pm}.$$

So, we have

$$\begin{aligned}&\Gamma_{+}(k,l)^2-(ad(b-l)-bc(a-k))^2\\ &=(adl-bck-abv_{+})^2-(ad(b-l)-bc(a-k))^2\\ &=ab(d-c-v_{+})(2adl-abd+abc-2bck-abv_{+}).\\ &\equiv ab(d-c-v_{+})(abd+abc+abv_{+}) \pmod{2ab}.\end{aligned}$$

However since

$$\begin{aligned}abv_{+}&=adl-bck-u_{+}\\ &\underset{(2)}{\equiv}(ad-bc)l+\Gamma_{+}(k,l)\\ &\underset{(2)}{\equiv} l+ab+h\underset{(2)}{\equiv} ab,\end{aligned}$$

this is congruent to

$$ab(d-c-v_{+})(abd+abc+ab)=a^2b^2(d-c-v_{+})(d+c+1).$$

If a is even, this is congruent to 0 modulo $2ab$. If a is odd, then $d+c+1$ is congruent to 0 modulo 2 since $ad-bc=1$ and so this is also congruent to 0 modulo $2ab$.

We also have

$$\begin{aligned}
&\Gamma_-(k,l)^2 - (ad(b-l) - bc(a-k))^2 \\
&\quad = (-adl - bck - abv_-)^2 - (ad(b-l) - bc(a-k))^2 \\
&\quad = (abd - 2adl - abc - abv_-)(-abd + abc - 2bck - abv_-) \\
&\quad = ab(bd - 2dl - bc - bv_-)(-ad + ac - 2ck - av_-) \\
&\quad \equiv ab(bd - bc - bv_-)(-ad + ac - av_-) \\
&\quad \equiv (abd - abc - abv_-)(-abd + abc - abv_-) \\
&\quad \equiv 0 \pmod{2ab},
\end{aligned}$$

where the last congruence follows by the same reason as above.

Therefore, we conclude that $\Gamma_\pm(k,l)^2 \equiv (ad(b-l) - bc(a-k))^2 \pmod{2ab}$.

- $\Gamma_\pm(k,l) + ab + h \underset{(2)}{\equiv} 1$. Let $u_\pm$ and $v_\pm$ be as above.

We have

$$\begin{aligned}
&(ab - \Gamma_+(k,l))^2 - (ad(b-l) - bc(a-k))^2 \\
&\quad = (ab - adl + bck + abv_+)^2 - (ad(b-l) - bc(a-k))^2 \\
&\quad = (ab + abd - 2adl - abc + 2bck + abv_+)(ab - abd + abc + abv_+) \\
&\quad = ab(ab + abd - 2adl - abc + 2bck + abv_+)(1 - d + c + v_+) \\
&\quad = ab(ab + abd - abc + abv_+)(1 - d + c + v_+) \pmod{2ab}.
\end{aligned}$$

Now, since $abv_+ = adl - bck - u_+ \underset{(2)}{\equiv} ab + 1$, this is congruent to

$$\begin{aligned}
&ab(ab + abd - abc + ab + 1)(1 - d + c + v_+) \\
&\quad \equiv a^2b^2(c + d + 1)(1 - d + c + v_+) \pmod{2ab},
\end{aligned}$$

which is congruent by the same reason as above.

We also have

$$\begin{aligned}
&\Gamma_-(k,l)^2 - (ad(b-l) - bc(a-k))^2 \\
&\quad = (ab + adl + bck + abv_-)^2 - (ad(b-l) - bc(a-k))^2 \\
&\quad = (ab + abd + 2bck - abc + abv_-)(ab - abd + 2adl + abc + abv_-) \\
&\quad = ab(a + ad + 2ck - ac + av_-)(b - bd + 2dl + bc + bv_-) \\
&\quad \equiv ab(a + ad - ac + av_-)(b - bd + bc + bv_-) \\
&\quad = ab(1 + d - c + v_-)(ab - abd + abc + abv_-) \\
&\quad \equiv 0 \pmod{2ab}
\end{aligned}$$

by the same reason as above.

Therefore, we conclude that $\Gamma_\pm(k,l)^2 \equiv (ad(b-l) - bc(a-k))^2 \pmod{2ab}$.

The proof is complete. ■

From Lemmas 9.5 and 9.6, we have topological interpretations of the terms in the right-hand side of (38).

Example 9.7. Suppose that $a = 2$. If $(h,k,l) \in \mathcal{H}$, then $k = 1$, and l and h are odd. So, the last two terms in (38) vanish.

Moreover, from Example 9.3, we see that $\mathcal{H}_{-}^{\Delta} \subset \mathcal{H}_{+}^{\Delta}$ and that $\mathcal{H}_{+}^{\nabla} \subset \mathcal{H}_{-}^{\nabla}$. So, (38) becomes

$$e^{-\frac{(n+1)\pi\sqrt{-1}}{4}} A(n) = \frac{1}{4} \sum_{(h,1,l)\in\mathcal{H}_{+}^{\Delta}} \mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l) e^{n\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}} + \frac{1}{4} \sum_{(h,1,l)\in\mathcal{H}_{-}^{\Delta}} \mathcal{T}_{-}^{\mathrm{Irr}}(h,1,l) e^{n\mathcal{CS}_{-}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}}$$

From Example 8.2, $\Gamma_{+}(1,l) = l$ and $\Gamma_{-}(1,l) = 2b - l$. Note that $\Gamma_{\pm}(k,l) + ab + h \equiv 0$ in this case. So, we have

$$\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l) = -\frac{(p-2b-h)^2}{4(p-2b)} - \frac{l^2}{8b},$$
$$\mathcal{CS}_{-}^{\mathrm{Irr}}(h,1,l) = -\frac{(p-2b-h)^2}{4(p-2b)} - \frac{(2b-l)^2}{8b} = \mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l) + \frac{b-l}{2}$$

from Example 8.6. Note that $(b-l)/2 \in \mathbb{Z}$. We also have

$$\mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l) = -\frac{8\sin(\frac{l}{2}\pi)\sin(\frac{l}{b}\pi)\sin(\frac{p-2b-h}{p-2b}\pi)}{\sqrt{2b(p-2b)}},$$
$$\mathcal{T}_{-}^{\mathrm{Irr}}(h,1,l) = -\frac{8\sin(\frac{2b-l}{2}\pi)\sin(\frac{2b-l}{b}\pi)\sin(\frac{p-2b-h}{p-2b}\pi)}{\sqrt{2b(p-2b)}} = -\mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l).$$

So, we have

$$\begin{aligned} e^{-\frac{(n+1)\pi\sqrt{-1}}{4}} A(n) &= \frac{1}{4} \sum_{(h,1,l)\in\mathcal{H}_{+}^{\Delta}} \mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l) e^{n\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}} \\ &\quad - \frac{1}{4} \sum_{(h,1,l)\in\mathcal{H}_{-}^{\Delta}} (-1)^{(b-l)/2} \mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l) e^{n\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}} \\ &= \frac{1}{2} \sum_{\substack{(h,1,l)\in\mathcal{H}_{-}^{\Delta} \\ b-l\equiv 2 \pmod 4}} \mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l) e^{n\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}} \\ &\quad + \frac{1}{4} \sum_{(h,1,l)\in\mathcal{H}_{+}^{\Delta}\setminus\mathcal{H}_{-}^{\Delta}} \mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l) e^{n\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}} \end{aligned}$$

since $\mathcal{H}_{-}^{\Delta} \subset \mathcal{H}_{+}^{\Delta}$.

9.2. Topological interpretation of $B(n)$

In this section we give a topological interpretation of $B(n)$.

First note that $H_1(M_p;\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ and so a reducible Abelian representation σ_l of $\pi_1(M_p)$ is characterised as $\operatorname{tr}\sigma_l = \cos(2l\pi/p)$ with $1 < l < (p-1)/2$.

Put

$$\mathcal{T}^{\mathrm{Abel}}(l) := (-1)^l \frac{4\sin(\frac{2al\pi}{p})\sin(\frac{2bl\pi}{p})\sin(\frac{2l\pi}{p})}{\sqrt{p}\sin(\frac{2abl\pi}{p})}, \quad \mathcal{CS}^{\mathrm{Abel}}(l) := -\frac{l^2}{p}.$$

Then we can write $B(n)$ as follows:

$$B(n) = \frac{1}{2}\sqrt{-1}(-1)^{a+b+ab} e^{n(1-p)\pi\sqrt{-1}/4} \sum_{0<l<(p-1)/2} \mathcal{T}^{\mathrm{Abel}}(l) e^{n\mathcal{CS}^{\mathrm{Abel}}(l)\pi\sqrt{-1}}. \tag{39}$$

Note that

$$(\mathcal{T}^{\mathrm{Abel}}(l))^{-2} = |\operatorname{Tor}(X_p;\tilde{\rho}_l^{\mathrm{Abel}})|, \quad \mathcal{CS}^{\mathrm{Abel}}(l) = \mathrm{CS}(X_p;\tilde{\rho}_l^{\mathrm{Abel}}) \pmod{\mathbb{Z}}. \tag{40}$$

From (19), (38), (39), (40), and Lemmas 9.5 and 9.6 we have the following theorem.

Theorem 9.8. *The Witten–Reshetikhin-Turaev invariant of X_p evaluated at $e^{4\pi\sqrt{-1}/n}$ has the following asymptotic expansion.*

$$\hat{\tau}_n(X_p;\exp(4\pi\sqrt{-1}/n)) = \frac{(-1)^{p+1}n^{3/2}}{2\pi}\Big(A(n) + B(n)n^{-1/2} + O(n^{-1})\Big)$$

with

$$A(n) = 2e^{\frac{n+1}{4}\pi\sqrt{-1}}\Bigg(\sum_{\substack{(h,k,l)\in\mathcal{H}_+^{\triangle}\\ h\underset{(2)}{\equiv}k\underset{(2)}{\equiv}l\underset{(2)}{\equiv}1}} - \sum_{\substack{(h,k,l)\in\mathcal{H}_+^{\triangledown}\\ h\underset{(2)}{\equiv}k\underset{(2)}{\equiv}l\underset{(2)}{\equiv}0}}\Bigg)\mathcal{T}_+^{\mathrm{Irr}}(h,k,l)e^{n\mathcal{CS}_+^{\mathrm{Irr}}(h,k,l)\pi\sqrt{-1}}$$
$$+ 2e^{\frac{n+1}{4}\pi\sqrt{-1}}\Bigg(\sum_{\substack{(h,k,l)\in\mathcal{H}_-^{\triangle}\\ h\underset{(2)}{\equiv}k\underset{(2)}{\equiv}l\underset{(2)}{\equiv}1}} - \sum_{\substack{(h,k,l)\in\mathcal{H}_-^{\triangledown}\\ h\underset{(2)}{\equiv}k\underset{(2)}{\equiv}l\underset{(2)}{\equiv}0}}\Bigg)\mathcal{T}_-^{\mathrm{Irr}}(h,k,l)e^{n\mathcal{CS}_-^{\mathrm{Irr}}(h,k,l)\pi\sqrt{-1}}$$

and

$$B(n) = \frac{1}{2}\sqrt{-1}(-1)^{a+b+ab} e^{n(1-p)\pi\sqrt{-1}/4} \sum_{0<l<(p-1)/2} \mathcal{T}^{\mathrm{Abel}}(l) e^{n\mathcal{CS}^{\mathrm{Abel}}(l)\pi\sqrt{-1}},$$

where $\mathcal{T}_{\pm}^{\mathrm{Irr}}(h,k,l)$ and $\mathcal{T}^{\mathrm{Abel}}(l)$ are related to the twisted Reidemeister torsions, and $\mathcal{CS}_{\pm}^{\mathrm{Irr}}(h,k,l)$ and $\mathcal{CS}^{\mathrm{Abel}}(l)$ are related to the Chern–Simons invariant as described above.

9.3. SU(2) representations and SU(1, 1) representations

Motivated by [51, Theorem 1.3] (see Theorem 1.5), we will study when a given irreducible representation $\rho\colon \pi_1(X_p) \to \mathrm{SL}(2;\mathbb{C})$ is an SU(2) or an SU(1, 1) representation. As a result, at least in the case where $a = 2$, we show that $A(n)$ in Theorem 9.8 can be written in terms of these representations.

A 2×2 complex matrix L is in SU(2) if and only if $L^* L = I_2$ and $\det L = 1$, where L^* is the conjugate transpose of L. Note that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \,\middle|\, u, v \in \mathbb{C}, |u|^2 + |v|^2 = 1 \right\},$$

where $\bar{u}$ is the complex conjugate of u. A 2×2 complex matrix M is in SU(1, 1) if and only if $M^* \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\det M = 1$. Note that

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \,\middle|\, u, v \in \mathbb{C}, |u|^2 - |v|^2 = 1 \right\}.$$

The Cayley map defined by $M \mapsto \mathcal{C} M \mathcal{C}^{-1}$ gives an isomorphism between SU(1, 1) and $\mathrm{SL}(2;\mathbb{R})$, where $\mathcal{C} := \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{smallmatrix}\right) \in \mathrm{SL}(2;\mathbb{C})$.

Proposition 9.9. *An irreducible representation $\tilde{\rho}^{\mathrm{Irr}}_{h,k,l} \to \mathrm{SL}(2;\mathbb{C})$ is either an* SU(2) *representation or an* SU(1, 1) *representation. Moreover, it is an* SU(2) *representation if and only if*

$$\begin{aligned}&\left(\cos\Big(\frac{h\pi}{p-ab}\Big) - \cos\Big(\frac{(adl+bck)\pi}{ab}\Big)\right) \\ &\quad \times \left(\cos\Big(\frac{h\pi}{p-ab}\Big) - \cos\Big(\frac{(adl-bck)\pi}{ab}\Big)\right) > 0, \end{aligned} \tag{41}$$

and it is an SU(1, 1) *representation if and only if*

$$\begin{aligned}&\left(\cos\Big(\frac{h\pi}{p-ab}\Big) - \cos\Big(\frac{(adl+bck)\pi}{ab}\Big)\right) \\ &\quad \times \left(\cos\Big(\frac{h\pi}{p-ab}\Big) - \cos\Big(\frac{(adl-bck)\pi}{ab}\Big)\right) < 0. \end{aligned} \tag{42}$$

To prove the proposition, we prepare a lemma.

Lemma 9.10. *For a matrix $M \in \mathrm{SL}(2;\mathbb{C})$ and an integer m, we have*

$$M^n = S_{n-1}(\operatorname{tr} M) M - S_{n-2}(\operatorname{tr} M) I_2,$$

where $S_n(z)$ is the n-th Chebyshev polynomial defined by $S_0(z) = 1$, $S_1(z) = z$, and $S_n(z) = z S_{n-1} - S_{n-2}(z)$ and tr *is the trace.*

Proof. By the Cayley–Hamilton theorem, $M^2 = (\operatorname{tr} M)M - I_2$. The lemma follows easily by induction. ■

Note that

$$S_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$$

for $\theta \in \mathbb{R}$.

Now, we prove Proposition 9.9.

Proof of Proposition 9.9. In this proof we use ρ instead of $\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}$ for short. Recall the following from Proposition 7.1:

$$\operatorname{tr}\rho(x) = 2\cos\Big(\frac{k\pi}{a}\Big), \quad \operatorname{tr}\rho(y) = 2\cos\Big(\frac{l\pi}{b}\Big), \quad \operatorname{tr}\rho(\mu) = 2\cos\Big(\frac{h\pi}{p-ab}\Big).$$

First, we show that ρ is a representation to SU(2) if and only if (41) holds. Suppose that the image of ρ is in $\mathrm{SU}(2) \subset \mathrm{SL}(2;\mathbb{C})$. Since a unitary matrix is diagonalizable, we may assume that $\rho(x) = \begin{pmatrix} e^{k\pi\sqrt{-1}/a} & 0 \\ 0 & e^{-k\pi\sqrt{-1}/a} \end{pmatrix}$ up to conjugation. Note that $\rho(y)$ is conjugate to $\begin{pmatrix} e^{l\pi\sqrt{-1}/b} & 0 \\ 0 & e^{-l\pi\sqrt{-1}/b} \end{pmatrix}$. Write $\rho(y^d) = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$ with $|u|^2 + |v|^2 = 1$ and $v \neq 0$ since ρ is irreducible. Since $u + \bar{u} = \operatorname{tr}\rho(y^d) = 2\cos(dl\pi/b)$, the real part of u equals $\cos(dl\pi/b)$. So, we can put $u = \cos(dl\pi/b) + r\sqrt{-1}$ for some $r \in \mathbb{R}$. Now, we have

$$\begin{aligned}
\operatorname{tr}\rho(\mu) &= \operatorname{tr}\rho(x^{-c}y^d) \\
&= \operatorname{tr}\left(\begin{pmatrix} e^{-ck\pi\sqrt{-1}/a} & 0 \\ 0 & e^{ck\pi\sqrt{-1}/a} \end{pmatrix}\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}\right) \\
&= \operatorname{tr}\begin{pmatrix} ue^{-ck\pi\sqrt{-1}/a} & ve^{-ck\pi\sqrt{-1}/a} \\ -\bar{v}e^{ck\pi\sqrt{-1}/a} & \bar{u}e^{ck\pi\sqrt{-1}/a} \end{pmatrix} \\
&= (\cos(dl\pi/b) + r\sqrt{-1})(\cos(ck\pi/a) - \sin(ck\pi/a)\sqrt{-1}) \\
&\quad + (\cos(dl\pi/b) - r\sqrt{-1})(\cos(ck\pi/a) + \sin(ck\pi/a)\sqrt{-1}) \\
&= 2\cos(dl\pi/b)\cos(ck\pi/a) + 2r\sin(ck\pi/a).
\end{aligned}$$

Since $\operatorname{tr}\rho(u) = 2\cos\big(h\pi/(p-ab)\big)$, we obtain

$$r = \frac{\cos\big(h\pi/(p-ab)\big) - \cos(dl\pi/b)\cos(ck\pi/a)}{\sin(ck\pi/a)}. \tag{43}$$

Since $|u|^2 + |v|^2 = 1$ and $v \neq 0$, we have $|u|^2 < 1$ and so $\cos^2(dl\pi/b) + r^2 < 1$. Hence, we have

$$\cos^2(dl\pi/b) + \frac{(\cos(h\pi/(p-ab)) - \cos(dl\pi/b)\cos(ck\pi/a))^2}{\sin^2(ck\pi/a)} < 1,$$

which means that

$$\begin{aligned} 0 > & -\sin^2(ck\pi/a) + \sin^2(ck\pi/a)\cos^2(dl\pi/b) \\ & + (\cos(h\pi/(p-ab)) - \cos(dl\pi/b)\cos(ck\pi/a))^2 \\ = & (\cos(h\pi/(p-ab)) - \cos(dl\pi/b)\cos(ck\pi/a))^2 \\ & + \sin^2(ck\pi/a)\sin^2(dl\pi/b) \\ = & (\cos(h\pi/(p-ab)) - \cos(dl\pi/b + ck\pi/a)) \\ & \times (\cos(h\pi/(p-ab)) - \cos(dl\pi/b - ck\pi/a)). \end{aligned}$$

Therefore, if ρ is a representation to $\mathrm{SU}(2)$, then (41) holds true.

Conversely, suppose that (41) is satisfied. With r given by (43) and

$$u := \cos(dl\pi/b) + r\sqrt{-1},$$

we have $|u|^2 < 1$. So, there exists $v \in \mathbb{C}$ such that $|u|^2 + |v|^2 = 1$. We now show that there exists $\rho(y) \in \mathrm{SU}(2)$ such that $\rho(y^d) = \left(\begin{smallmatrix} u & v \\ -\bar{v} & \bar{u} \end{smallmatrix}\right)$.

By Lemma 9.10, we have

$$\rho(y^d) = S_{d-1}(2\cos(l\pi/b))\rho(y) - S_{d-2}(2\cos(l\pi/b))I_2. \tag{44}$$

Since

$$S_{d-1}(2\cos(l\pi/b)) = \frac{\sin(dl\pi/b)}{\sin(l\pi/b)} \neq 0,$$

we put

$$\rho(y) := \frac{1}{S_{d-1}(2\cos(l\pi/b))}\begin{pmatrix} u + S_{d-2}(2\cos(l\pi/b)) & v \\ -\bar{v} & \bar{u} + S_{d-2}(2\cos(l\pi/b)) \end{pmatrix}. \tag{45}$$

Then from (44)

$$\begin{aligned} \rho(y^d) &= \begin{pmatrix} u + S_{d-2}(2\cos(l\pi/b)) & v \\ -\bar{v} & \bar{u} + S_{d-2}(2\cos(l\pi/b)) \end{pmatrix} \\ &\quad - S_{d-2}(2\cos(l\pi/b))I_2 \\ &= \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}. \end{aligned}$$

We need to show that the right-hand side of (45) is an element in $\mathrm{SU}(2)$. It will be sufficient if we can show that

$$|v|^2 + |u + S_{d-2}(2\cos(l\pi/b))|^2 = (S_{d-1}(2\cos(l\pi/b)))^2.$$

Put $\theta := l\pi/b$. Since $|u|^2 + |v|^2 = 1$ and $u + \bar{u} = 2\cos(d\theta)$, we have

$$\begin{aligned}
&|v|^2 + |u + S_{d-2}(2\cos(l\pi/b))|^2 - (S_{d-1}(2\cos(l\pi/b)))^2 \\
&\quad = |v|^2 + (u + S_{d-2}(2\cos\theta))(\bar{u} + S_{d-2}(2\cos\theta)) - (S_{d-1}(2\cos(\theta)))^2 \\
&\quad = |v|^2 + |u|^2 + (u+\bar{u})S_{d-2}(2\cos\theta) + (S_{d-2}(2\cos\theta))^2 - (S_{d-1}(2\cos(\theta)))^2 \\
&\quad = 1 + 2\cos(d\theta)\frac{\sin((d-1)\theta)}{\sin\theta} + \Big(\frac{\sin((d-1)\theta)}{\sin\theta}\Big)^2 - \Big(\frac{\sin(d\theta)}{\sin\theta}\Big)^2 \\
&\quad = 1 + 2\cos(d\theta)\Big(\frac{\sin(d\theta)}{\sin\theta}\cos\theta - \cos(d\theta)\Big) + \Big(\frac{\sin(d\theta)}{\sin\theta}\cos\theta - \cos(d\theta)\Big)^2 \\
&\qquad - \Big(\frac{\sin(d\theta)}{\sin\theta}\Big)^2 \\
&\quad = 1 - 2\cos^2(d\theta) + \Big(\frac{\sin(d\theta)}{\sin\theta}\cos(d\theta)\Big)^2 + \cos^2(d\theta) - \Big(\frac{\sin(d\theta)}{\sin\theta}\Big)^2 \\
&\quad = 1 - \cos^2(d\theta) - \Big(\frac{\sin(d\theta)}{\sin\theta}\Big)^2\Big(1 - \cos^2\theta\Big) = 0.
\end{aligned}$$

Next, we show that ρ is a representation to SU(1, 1) if and only if (42) holds.

Suppose that the image of ρ is in $\mathrm{SU}(1,1) \subset \mathrm{SL}(2;\mathbb{C})$. Since the eigenvalues of $\rho(x)$ are $e^{\pm k\pi\sqrt{-1}/a}$, we may assume that

$$\rho(x) = \begin{pmatrix} e^{k\pi\sqrt{-1}/a} & 0 \\ 0 & e^{-k\pi\sqrt{-1}/a} \end{pmatrix}$$

up to conjugation. By the same reason, $\rho(y)$ is conjugate to $\begin{pmatrix} e^{l\pi\sqrt{-1}/b} & 0 \\ 0 & e^{-l\pi\sqrt{-1}/b} \end{pmatrix}$. Write $\rho(y^d) = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$ with $|u|^2 - |v|^2 = 1$ and $v \neq 0$. Since $u + \bar{u} = \operatorname{tr}\rho(y^d) = 2\cos(dl\pi/b)$, the real part of u equals $\cos(dl\pi/b)$. So, we can put $u = \cos(dl\pi/b) + r\sqrt{-1}$ for some $r \in \mathbb{R}$. Now, we have

$$\begin{aligned}
\operatorname{tr}\rho(u) &= \operatorname{tr}\rho(x^{-c}y^d) \\
&= \operatorname{tr}\left(\begin{pmatrix} e^{-ck\pi\sqrt{-1}/a} & 0 \\ 0 & e^{ck\pi\sqrt{-1}/a} \end{pmatrix}\begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}\right) \\
&= \operatorname{tr}\begin{pmatrix} ue^{-ck\pi\sqrt{-1}/a} & ve^{-ck\pi\sqrt{-1}/a} \\ \bar{v}e^{ck\pi\sqrt{-1}/a} & \bar{u}e^{ck\pi\sqrt{-1}/a} \end{pmatrix} \\
&= (\cos(dl\pi/b) + r\sqrt{-1})(\cos(ck\pi/a) - \sin(ck\pi/a)\sqrt{-1}) \\
&\quad + (\cos(dl\pi/b) - r\sqrt{-1})(\cos(ck\pi/a) + \sin(ck\pi/a)\sqrt{-1}) \\
&= 2\cos(dl\pi/b)\cos(ck\pi/a) + 2r\sin(ck\pi/a).
\end{aligned}$$

Since $\operatorname{tr}\rho(u) = 2\cos(h\pi/(p-ab))$, we obtain

$$r = \frac{\cos(h\pi/(p-ab)) - \cos(dl\pi/b)\cos(ck\pi/a)}{\sin(ck\pi/a)}.$$

Since $|u|^2 - |v|^2 = 1$ and $v \neq 0$, we have $|u|^2 > 1$ and so $\cos^2(dl\pi/b) + r^2 > 1$. In the same way as above, we can prove (42).

Conversely, suppose that (42) is satisfied. With r given by (43) and

$$u := \cos(dl\pi/b) + r\sqrt{-1},$$

we have $|u|^2 > 1$. So, there exists $v \in \mathbb{C}$ such that $|u|^2 - |v|^2 = 1$. We now show that there exists $\rho(y) \in \mathrm{SU}(1,1)$ such that $\rho(y^d) = \left(\begin{smallmatrix} u & v \\ \bar{v} & \bar{u} \end{smallmatrix}\right)$.

By Lemma 9.10, we have

$$\rho(y^d) = S_{d-1}(2\cos(l\pi/b))\rho(y) - S_{d-2}(2\cos(l\pi/b))I_2. \tag{46}$$

Since $S_{d-1}(2\cos(l\pi/b)) = \frac{\sin(dl\pi/b)}{\sin(l\pi/b)} \neq 0$, we put

$$\rho(y) := \frac{1}{S_{d-1}(2\cos(l\pi/b))}\begin{pmatrix} u + S_{d-2}(2\cos(l\pi/b)) & v \\ \bar{v} & \bar{u} + S_{d-2}(2\cos(l\pi/b)) \end{pmatrix}. \tag{47}$$

Then from (46)

$$\begin{aligned}\rho(y^d) &= \begin{pmatrix} u + S_{d-2}(2\cos(l\pi/b)) & v \\ \bar{v} & \bar{u} + S_{d-2}(2\cos(l\pi/b)) \end{pmatrix} \\ &\quad - S_{d-2}(2\cos(l\pi/b))I_2 \\ &= \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}.\end{aligned}$$

We need to show that the right-hand side of (47) is an element in $\mathrm{SU}(1,1)$. It will be sufficient if we can show that

$$-|v|^2 + |u + S_{d-2}(2\cos(l\pi/b))|^2 = (S_{d-1}(2\cos(l\pi/b)))^2.$$

Put $\theta := l\pi/b$. Since $|u|^2 - |v|^2 = 1$ and $u + \bar{u} = 2\cos(d\theta)$, we have

$$\begin{aligned}&-|v|^2 + |u + S_{d-2}(2\cos(l\pi/b))|^2 - (S_{d-1}(2\cos(l\pi/b)))^2 \\ &\quad = -|v|^2 + (u + S_{d-2}(2\cos\theta))(\bar{u} + S_{d-2}(2\cos\theta)) - (S_{d-1}(2\cos(\theta)))^2 \\ &\quad = -|v|^2 + |u|^2 + (u + \bar{u})S_{d-2}(2\cos\theta) + (S_{d-2}(2\cos\theta))^2 \\ &\qquad - (S_{d-1}(2\cos(\theta)))^2 \\ &\quad = 1 + 2\cos(d\theta)\frac{\sin((d-1)\theta)}{\sin\theta} + \left(\frac{\sin((d-1)\theta)}{\sin\theta}\right)^2 - \left(\frac{\sin(d\theta)}{\sin\theta}\right)^2 \\ &\quad = 1 + 2\cos(d\theta)\left(\frac{\sin(d\theta)}{\sin\theta}\cos\theta - \cos(d\theta)\right) + \left(\frac{\sin(d\theta)}{\sin\theta}\cos\theta - \cos(d\theta)\right)^2 \\ &\qquad - \left(\frac{\sin(d\theta)}{\sin\theta}\right)^2\end{aligned}$$

$$= 1 - 2\cos^2(d\theta) + \Big(\frac{\sin(d\theta)}{\sin\theta}\cos(d\theta)\Big)^2 + \cos^2(d\theta) - \Big(\frac{\sin(d\theta)}{\sin\theta}\Big)^2$$
$$= 1 - \cos^2(d\theta) - \Big(\frac{\sin(d\theta)}{\sin\theta}\Big)^2(1-\cos^2\theta) = 0. \qquad \blacksquare$$

Example 9.11. When $a = 2$, we can put $d := (b+1)/2$ and $c := 1$. Since $k = 1$, the representation $\tilde{\rho}^{\mathrm{Irr}}_{h,1,l}$ is an SU(2) representation if and only if

$$\begin{aligned}
&\Big(\cos\Big(\frac{h\pi}{p-2b}\Big) - \cos\Big(\frac{\big((b+1)l+b\big)\pi}{2b}\Big)\Big)\\
&\times\Big(\cos\Big(\frac{h\pi}{p-2b}\Big) - \cos\Big(\frac{\big((b+1)l-b\big)\pi}{2b}\Big)\Big)\\
&\quad = \Big(\cos\Big(\frac{h\pi}{p-2b}\Big) - (-1)^{(l+1)/2}\cos\Big(\frac{l\pi}{2b}\Big)\Big)\\
&\qquad \times\Big(\cos\Big(\frac{h\pi}{p-2b}\Big) - (-1)^{(l-1)/2}\cos\Big(\frac{l\pi}{2b}\Big)\Big)\\
&\quad = \Big(\cos\Big(\frac{h\pi}{p-2b}\Big) - \cos\Big(\frac{l\pi}{2b}\Big)\Big)\Big(\cos\Big(\frac{h\pi}{p-2b}\Big) + \cos\Big(\frac{l\pi}{2b}\Big)\Big) > 0.
\end{aligned}$$

The second equality follows since $(l+1)/2$ and $(l-1)/2$ have different parities. Since $0 < \frac{h}{p-2b} < 1$ and $0 < \frac{l}{2b} < \frac{1}{2}$, we have

$$\frac{h}{p-2b} < \frac{l}{2b} \quad \text{or} \quad \frac{l}{2b} + \frac{h}{p-2b} > 1.$$

Therefore, ρ is an irreducible SU(2) representation if and only if the pair (h,l) is in the following set:

$$\begin{aligned}
&\Big\{(h,l)\in\mathbb{Z}^2 \;\Big|\; 1 \le h \le p-2b-2,\, 1 \le l \le b-2,\, h \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 1,\, \frac{h}{p-2b} < \frac{l}{2b}\Big\}\\
&\quad\cup \Big\{(h,l)\in\mathbb{Z}^2 \;\Big|\; 1 \le h \le p-2b-2,\, 1 \le l \le b-2,\, h \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 1,\\
&\quad \frac{l}{2b} + \frac{h}{p-2b} > 1\Big\}.
\end{aligned}$$

Since the first set equals $\mathcal{H}^{\Delta}_{-}$ from Example 9.3, we can write $\mathcal{H}^{\Delta}_{-}$ as

$$\mathcal{H}^{\Delta}_{-} = \Big\{(h,1,l) \;\Big|\; \tilde{\rho}^{\mathrm{Irr}}_{(h,k,l)}\colon \text{SU(2)-representation},\, \frac{l}{2b} + \frac{h}{p-2b} < 1\Big\}$$

It is an $SL(2;\mathbb{R})$ representation if and only if

$$\frac{h}{p-2b} > \frac{l}{2b} \quad \text{and} \quad \frac{l}{2b} + \frac{h}{p-2b} < 1.$$

Therefore, ρ is an irreducible SU(1, 1) representation if and only if the pair (h, l) is in the following set:

$$\Big\{(h,l) \in \mathbb{Z}^2 \;\Big|\; 1 \le h \le p-2b-2, 1 \le l \le b-2, h \underset{(2)}{\equiv} l \underset{(2)}{\equiv} 1, \frac{h}{p-2b} > \frac{l}{2b}, \frac{l}{2b} + \frac{h}{p-2b} < 1\Big\},$$

which equals $\mathcal{H}_{+}^{\Delta} \setminus \mathcal{H}_{-}^{\Delta}$.

Now, from Example 9.7, we have

$$\begin{aligned}
&\hat{\tau}_n(X_p; \exp(4\pi\sqrt{-1}/n)) \\
&= \frac{e^{(n+1)\pi\sqrt{-1}/4} n^{3/2}}{8\pi} \Big(2\sum_{\substack{(h,1,l)\in\mathcal{H}_{-}^{\Delta} \\ b-l\equiv 2 \pmod 4}} + \sum_{(h,1,l)\in\mathcal{H}_{+}^{\Delta}\setminus\mathcal{H}_{-}^{\Delta}} \Big) \mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l) \\
&\quad \times e^{n\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}} \\
&\quad - \frac{\sqrt{-1}e^{n(1-p)\pi\sqrt{-1}/4} n}{4\pi} \sum_{0<l<(p-1)/2} \mathcal{T}^{\mathrm{Abel}}(l) e^{n\mathcal{CS}^{\mathrm{Abel}}(l)\pi\sqrt{-1}} + O(n^{1/2}) \\
&= \frac{e^{(n+1)\pi\sqrt{-1}/4} n^{3/2}}{8\pi} \Big(2\sum_{\substack{\tilde{\rho}_{h,1,l}^{\mathrm{Irr}}:\,\mathrm{SU}(2)\text{-representation} \\ \frac{l}{2b}+\frac{h}{p-2b}<1,\, b-l\equiv 2 \pmod 4}} + \sum_{\tilde{\rho}_{h,1,l}^{\mathrm{Irr}}:\,\mathrm{SU}(1,1)\text{-representation}} \Big) \\
&\quad \times \mathcal{T}_{+}^{\mathrm{Irr}}(h,1,l) e^{n\mathcal{CS}_{+}^{\mathrm{Irr}}(h,1,l)\pi\sqrt{-1}} \\
&\quad - \frac{\sqrt{-1}e^{n(1-p)\pi\sqrt{-1}/4} n}{4\pi} \sum_{0<l<(p-1)/2} \mathcal{T}^{\mathrm{Abel}}(l) e^{n\mathcal{CS}^{\mathrm{Abel}}(l)\pi\sqrt{-1}} + O(n^{1/2}).
\end{aligned}$$

Example 9.12. When $a = 4$, $b = 3$, and $p = 17$, we have

$$\begin{aligned}
\mathcal{H} &= \{(1,1,1), (1,1,3), (2,2,2), (3,1,1), (3,1,3), (4,2,2)\}, \\
\mathcal{H}_{+}^{\Delta} &= \{(1,1,1), (1,1,3), (2,2,2), (3,1,1), (4,2,2)\}, \\
\mathcal{H}_{-}^{\Delta} &= \{(1,1,1)\}, \\
\mathcal{H}_{+}^{\nabla} &= \{(3,1,3)\}, \\
\mathcal{H}_{-}^{\nabla} &= \{(1,1,3), (2,2,2), (3,1,1), (3,1,3), (4,2,2)\}.
\end{aligned}$$

We also see that $\tilde{\rho}_{h,k,l}^{\mathrm{Irr}}$ is an SU(2) representation if and only if (h, k, l) is in

$$\{(1,1,3), (3,1,1)\},$$

and is an SU(1, 1) representation if and only if (h, k, l) is in

$$\{(1,1,1), (2,2,2), (3,1,3), (4,2,2)\}.$$

So, there seems to be no good interpretation as in the case $a = 2$.

Example 9.13. When $a = 3$, $b = 5$, and $p = 19$, we have

$$\begin{aligned}
\mathcal{H} &= \{(1,1,1),(1,1,3),(2,2,2),(2,2,4),(3,1,1),(3,1,3)\},\\
\mathcal{H}_+^{\Delta} &= \{(1,1,3),(2,2,4),(3,1,3)\},\\
\mathcal{H}_-^{\Delta} &= \{(1,1,1),(1,1,3),(2,2,2)\},\\
\mathcal{H}_+^{\nabla} &= \{(1,1,1),(2,2,2),(3,1,1)\},\\
\mathcal{H}_-^{\nabla} &= \{(2,2,4),(3,1,1),(3,1,3)\}.
\end{aligned}$$

We also see that $\tilde{\rho}^{\mathrm{Irr}}_{h,k,l}$ is an SU(2) representation if and only if (h,k,l) is in

$$\{(1,1,3),(3,1,1)\},$$

and is an SU(1, 1) representation if and only if (h,k,l) is in

$$\{(1,1,1),(2,2,2),(2,2,4),(3,1,3)\}.$$

So, there seems to be no good interpretation as in the case $a = 2$, either.

10. Lemma

In this section we prove the following lemma that we use in this paper.

Lemma 10.1. *Suppose that a and b are coprime positive integers. Then for any odd integer $n \geq 3$ we have*

$$\sum_{\substack{0\le m\le 2ab\\ m:\text{ even}}} \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big) = 0, \tag{48}$$

$$\sum_{\substack{0\le m\le 2ab\\ m:\text{ odd}}} \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big) = 0. \tag{49}$$

Proof. We consider the following four cases:

i. (48) for ab odd,

ii. (48) for ab even,

iii. (49) for ab odd,

iv. (49) for ab even.

First of all, replacing m with $2ab - m$ in the summation, we have

$$\begin{aligned}
&\sum_{0\le m\le 2ab} \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)\\
&\quad= \sum_{0\le m\le 2ab} \sin\Big(\frac{(2ab-m)\pi}{a}\Big)\sin\Big(\frac{(2ab-m)\pi}{b}\Big)\exp\Big(-n\frac{(2ab-m)^2}{4ab}\pi\sqrt{-1}\Big).
\end{aligned}$$

However, since

$$\begin{aligned}
\sin\Big(\frac{(2ab-m)\pi}{a}\Big) &= -\sin\Big(\frac{m\pi}{a}\Big),\\
\sin\Big(\frac{(2ab-m)\pi}{b}\Big) &= -\sin\Big(\frac{m\pi}{b}\Big),\\
\exp\Big(-n\frac{(2ab-m)^2}{4ab}\pi\sqrt{-1}\Big) &= (-1)^{n(m-ab)}\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big),
\end{aligned}$$

and n is odd, we have

$$\begin{aligned}
&\sum_{0\le m\le 2ab}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)\\
&\quad=\sum_{0\le m\le 2ab}(-1)^{m-ab}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big). \qquad (50)
\end{aligned}$$

Therefore, if ab is even, then we have

$$\begin{aligned}
&\sum_{0\le m\le 2ab}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)\\
&\quad=\sum_{0\le m\le 2ab}(-1)^{m}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)\\
&\quad=\sum_{\substack{0\le m\le 2ab\\ m:\text{ even}}}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)\\
&\qquad-\sum_{\substack{0\le m\le 2ab\\ m:\text{ odd}}}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)
\end{aligned}$$

So, we have

$$\sum_{\substack{0\le m\le 2ab\\ m:\text{ odd}}}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)=0,$$

proving (iv). Similarly, if ab is odd, we have

$$\sum_{\substack{0\le m\le 2ab\\ m:\text{ even}}}\sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)=0$$

from (50), proving (i).

Next we consider the case (ii). We assume that ab is even. Putting $m = 2k$, we have

$$\sum_{\substack{0\le m\le 2ab\\ m:\,\text{even}}} \sin\Big(\frac{m\pi}{a}\Big)\sin\Big(\frac{m\pi}{b}\Big)\exp\Big(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Big)$$
$$= \sum_{k=0}^{ab} \sin\Big(\frac{2k\pi}{a}\Big)\sin\Big(\frac{2k\pi}{b}\Big)\exp\Big(-n\frac{k^2}{ab}\pi\sqrt{-1}\Big).$$

We denote the right-hand side by W_{ii}. Note that

$$\sum_{k=1}^{ab}(e^{2k\pi\sqrt{-1}/a} - e^{-2k\pi\sqrt{-1}/a})(e^{2k\pi\sqrt{-1}/b} - e^{-2k\pi\sqrt{-1}/b})e^{-n\frac{k^2}{ab}\pi\sqrt{-1}} = -4W_{\text{ii}}. \tag{51}$$

We will use the following Gauss sum reciprocity formula (see [8, Chapter IX], for example):

Theorem 10.2 (Cauchy–Kronecker). *Suppose that c and d are positive integers. Let w be a rational number such that $cd + 2cw \equiv 0 \pmod 2$. Then we have*

$$\frac{1}{\sqrt{d}}\sum_{k=1}^{d} e^{\frac{c}{d}(k+w)^2\pi\sqrt{-1}} = \frac{e^{\pi\sqrt{-1}/4}}{\sqrt{c}}\sum_{l=1}^{c} e^{-\frac{d}{c}l^2\pi\sqrt{-1}+2lw\pi\sqrt{-1}}.$$

From (51), we have

$$-4W_{\text{ii}} = \sum_{k=1}^{ab}\Big(e^{(2k\frac{a+b}{ab}-n\frac{k^2}{ab})\pi\sqrt{-1}} - e^{(2k\frac{-a+b}{ab}-n\frac{k^2}{ab})\pi\sqrt{-1}}$$
$$- e^{(2k\frac{a-b}{ab}-n\frac{k^2}{ab})\pi\sqrt{-1}} + e^{(2k\frac{-a-b}{ab}-n\frac{k^2}{ab})\pi\sqrt{-1}}\Big)$$

Putting $c := ab$ and $d := n$, and choose $w := \frac{\pm a\pm b}{ab}$, we can apply Theorem 10.2 because ab is even. We have

$$-4W_{\text{ii}} = \frac{\sqrt{ab}}{e^{\pi\sqrt{-1}/4}\sqrt{n}}\sum_{k=1}^{n}\Big(e^{\frac{ab}{n}(k+\frac{a+b}{ab})^2\pi\sqrt{-1}} - e^{\frac{ab}{n}(k+\frac{-a+b}{ab})^2\pi\sqrt{-1}}$$
$$- e^{\frac{ab}{n}(k+\frac{a-b}{ab})^2\pi\sqrt{-1}} + e^{\frac{ab}{n}(k+\frac{-a-b}{ab})^2\pi\sqrt{-1}}\Big)$$
$$= \sqrt{\frac{ab}{n}}e^{-\pi\sqrt{-1}/4}\sum_{k=1}^{n}\Big(e^{\frac{\pi\sqrt{-1}}{abn}(abk+a+b)^2} - e^{\frac{\pi\sqrt{-1}}{abn}(abk-a+b)^2}$$
$$- e^{\frac{\pi\sqrt{-1}}{abn}(abk+a-b)^2} + e^{\frac{\pi\sqrt{-1}}{abn}(abk-a-b)^2}\Big).$$

Replacing k with $n-k$, we see that

$$\sum_{k=1}^{n} e^{\frac{\pi\sqrt{-1}}{abn}(abk-a-b)^2} = \sum_{k=1}^{n} e^{\frac{\pi\sqrt{-1}}{abn}(abk+a+b)^2}$$

$$\sum_{k=1}^{n} e^{\frac{\pi\sqrt{-1}}{abn}(abk+a-b)^2} = \sum_{k=1}^{n} e^{\frac{\pi\sqrt{-1}}{abn}(abk-a+b)^2}.$$

Therefore, we have

$$\begin{aligned}
-4W_{\mathrm{ii}} &= 2\sqrt{\frac{ab}{n}}e^{-\pi\sqrt{-1}/4}\sum_{k=1}^{n}\left(e^{\frac{\pi\sqrt{-1}}{abn}(abk+a+b)^2} - e^{\frac{\pi\sqrt{-1}}{abn}(abk-a+b)^2}\right) \\
&= 2\sqrt{\frac{ab}{n}}e^{-\pi\sqrt{-1}/4}e^{\frac{a\pi\sqrt{-1}}{bn}+\frac{b\pi\sqrt{-1}}{an}}\sum_{k=1}^{n}e^{\frac{bk(ak+2)\pi\sqrt{-1}}{n}}\Big(e^{\frac{2(ak+1)\pi\sqrt{-1}}{n}} \\
&\qquad - e^{\frac{-2(ak+1)\pi\sqrt{-1}}{n}}\Big)
\end{aligned}$$

Now, let us consider the colored Jones polynomial of the torus knot $T(a,b)$. We put

$$\begin{aligned}
\widetilde{J}_k(q) &:= q^{ab(k^2-1)/4}(q^{k/2}-q^{-k/2})J_k(T(a,b);q) \\
&= \sum_{j=-(k-1)/2}^{(k-1)/2} q^{bj(aj+1)}(q^{aj+1/2}-q^{-aj-1/2}).
\end{aligned}$$

Note that this is nothing but the Kauffman bracket of $T(a,b)$ with $n-1$-th Jones–Wenzl idempotent inserted, replacing A with $q^{-1/4}$. Then we have

$$\begin{aligned}
&\widetilde{J}_n(e^{4\pi\sqrt{-1}/n}) \\
&= \sum_{j=-(n-1)/2}^{(n-1)/2} e^{4bj(aj+1)\pi\sqrt{-1}/n}\left(e^{4(aj+1/2)\pi\sqrt{-1}/n} - e^{-4(aj+1/2)\pi\sqrt{-1}/n}\right)
\end{aligned}$$

(put $l=2j$, noting that n is odd)

$$\begin{aligned}
&= \sum_{\substack{1-n\le l\le n-1\\ l:\text{ even}}} e^{bl(al+2)\pi\sqrt{-1}/n}\left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right) \\
&= \sum_{\substack{0\le l\le n-1\\ l:\text{ even}}} e^{bl(al+2)\pi\sqrt{-1}/n}\left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right) \\
&\quad + \sum_{\substack{1\le l\le n-2\\ l\text{ odd}}} e^{b(l-n)(a(l-n)+2)\pi\sqrt{-1}/n}\left(e^{2(a(l-n)+1)\pi\sqrt{-1}/n} - e^{-2(a(l-n)+1)\pi\sqrt{-1}/n}\right)
\end{aligned}$$

(since ab is even and n is odd)

$$= \sum_{l=0}^{n-1} e^{\frac{\pi\sqrt{-1}}{n}bl(al+2)}\left(e^{\frac{2\pi\sqrt{-1}}{n}(al+1)} - e^{-\frac{2\pi\sqrt{-1}}{n}(al+1)}\right).$$

Hence, we have

$$
\begin{aligned}
-4W_{\mathrm{ii}} &= 2\sqrt{\frac{ab}{n}}e^{-\pi\sqrt{-1}/4}e^{\frac{a\pi\sqrt{-1}}{bn}+\frac{b\pi\sqrt{-1}}{an}}\widetilde{J}_n(K;e^{4\pi\sqrt{-1}/n}) \\
&= 2\sqrt{\frac{ab}{n}}e^{-\pi\sqrt{-1}/4}e^{\frac{a\pi\sqrt{-1}}{bn}+\frac{b\pi\sqrt{-1}}{an}} \\
&\quad \times e^{ab(n^2-1)\pi\sqrt{-1}/n}(e^{2\pi\sqrt{-1}}-e^{-2\pi\sqrt{-1}})\widetilde{J}_n(e^{4\pi\sqrt{-1}/n}) = 0
\end{aligned}
$$

proving (ii).

We consider (iii). Note that we are assuming that both a and b are odd. We have

$$
\begin{aligned}
&\sum_{\substack{0\le m\le 2ab\\ m:\text{ odd}}}\sin\Bigl(\frac{m\pi}{a}\Bigr)\sin\Bigl(\frac{m\pi}{b}\Bigr)\exp\Bigl(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Bigr) \\
&\quad= \sum_{\substack{0\le m\le ab\\ m:\text{ odd}}}\sin\Bigl(\frac{m\pi}{a}\Bigr)\sin\Bigl(\frac{m\pi}{b}\Bigr)\exp\Bigl(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Bigr) \\
&\qquad+ \sum_{\substack{0\le m\le ab\\ m:\text{ odd}}}\sin\Bigl(\frac{(2ab-m)\pi}{a}\Bigr)\sin\Bigl(\frac{(2ab-m)\pi}{b}\Bigr)\exp\Bigl(-n\frac{(2ab-m)^2}{4ab}\pi\sqrt{-1}\Bigr) \\
&\quad= 2\sum_{\substack{0\le m\le ab\\ m:\text{ odd}}}\sin\Bigl(\frac{m\pi}{a}\Bigr)\sin\Bigl(\frac{m\pi}{b}\Bigr)\exp\Bigl(-n\frac{m^2}{4ab}\pi\sqrt{-1}\Bigr). \\
&\quad= 2\sum_{\substack{0\le k\le ab\\ k:\text{ even}}}\sin\Bigl(\frac{(ab-k)\pi}{a}\Bigr)\sin\Bigl(\frac{(ab-k)\pi}{b}\Bigr)\exp\Bigl(-n\frac{(ab-k)^2}{4ab}\pi\sqrt{-1}\Bigr) \\
&\quad= 2\exp\Bigl(\frac{-abn\pi\sqrt{-1}}{4}\Bigr)\sum_{\substack{0\le k\le ab\\ k:\text{ even}}}\sqrt{-1}^k\sin\Bigl(\frac{k\pi}{a}\Bigr)\sin\Bigl(\frac{k\pi}{b}\Bigr)\exp\Bigl(-n\frac{k^2}{4ab}\pi\sqrt{-1}\Bigr).
\end{aligned}
$$

On the other hand, since we have

$$
\begin{aligned}
&\sum_{\substack{0\le k\le 2ab\\ k:\text{ even}}}\sqrt{-1}^k\sin\Bigl(\frac{k\pi}{a}\Bigr)\sin\Bigl(\frac{k\pi}{b}\Bigr)\exp\Bigl(-n\frac{k^2}{4ab}\pi\sqrt{-1}\Bigr) \\
&\quad= \sum_{\substack{0\le k\le ab\\ k:\text{ even}}}\sqrt{-1}^k\sin\Bigl(\frac{k\pi}{a}\Bigr)\sin\Bigl(\frac{k\pi}{b}\Bigr)\exp\Bigl(-n\frac{k^2}{4ab}\pi\sqrt{-1}\Bigr) \\
&\qquad+ \sum_{\substack{0\le k\le ab\\ k:\text{ even}}}\sqrt{-1}^{2ab-k}\sin\Bigl(\frac{(2ab-k)\pi}{a}\Bigr)\sin\Bigl(\frac{(2ab-k)\pi}{b}\Bigr) \\
&\qquad\times\exp\Bigl(-n\frac{(2ab-k)^2}{4ab}\pi\sqrt{-1}\Bigr)
\end{aligned}
$$

$$= 2\sum_{\substack{0\le k\le ab\\ k:\text{ even}}} \sqrt{-1}^{k} \sin\Bigl(\frac{k\pi}{a}\Bigr)\sin\Bigl(\frac{k\pi}{b}\Bigr)\exp\Bigl(-n\frac{k^2}{4ab}\pi\sqrt{-1}\Bigr)$$

$$= 2\sum_{l=0}^{ab}(-1)^l \sin\Bigl(\frac{2l\pi}{a}\Bigr)\sin\Bigl(\frac{2l\pi}{b}\Bigr)\exp\Bigl(-n\frac{l^2}{ab}\pi\sqrt{-1}\Bigr),$$

equation (49) becomes

$$\exp\Bigl(\frac{-abn\pi\sqrt{-1}}{4}\Bigr)\sum_{l=0}^{ab}(-1)^l \sin\Bigl(\frac{2l\pi}{a}\Bigr)\sin\Bigl(\frac{2l\pi}{b}\Bigr)\exp\Bigl(-n\frac{l^2}{ab}\pi\sqrt{-1}\Bigr).$$

In a similar way, the summation above becomes

$$\sum_{l=0}^{ab}(-1)^l \sin\Bigl(\frac{2l\pi}{a}\Bigr)\sin\Bigl(\frac{2l\pi}{b}\Bigr)\exp\Bigl(-n\frac{l^2}{ab}\pi\sqrt{-1}\Bigr)$$

$$= \sum_{\substack{0\le l\le ab\\ l:\text{ even}}} \sin\Bigl(\frac{2l\pi}{a}\Bigr)\sin\Bigl(\frac{2l\pi}{b}\Bigr)\exp\Bigl(-n\frac{l^2}{ab}\pi\sqrt{-1}\Bigr)$$

$$- \sum_{\substack{0\le l\le ab\\ l:\text{ even}}} \sin\Bigl(\frac{2(ab-l)\pi}{a}\Bigr)\sin\Bigl(\frac{2(ab-l)\pi}{b}\Bigr)\exp\Bigl(-n\frac{(ab-l)^2}{ab}\pi\sqrt{-1}\Bigr)$$

(since ab and n are odd)

$$= 2\sum_{\substack{0\le l\le ab\\ l:\text{ even}}} \sin\Bigl(\frac{2l\pi}{a}\Bigr)\sin\Bigl(\frac{2l\pi}{b}\Bigr)\exp\Bigl(-n\frac{l^2}{ab}\pi\sqrt{-1}\Bigr)$$

$$= \sum_{\substack{0\le l\le ab\\ l:\text{ even}}} \sin\Bigl(\frac{2l\pi}{a}\Bigr)\sin\Bigl(\frac{2l\pi}{b}\Bigr)\exp\Bigl(-n\frac{l^2}{ab}\pi\sqrt{-1}\Bigr)$$

$$+ \sum_{\substack{0\le l\le ab\\ l:\text{ even}}} \sin\Bigl(\frac{2(2ab-l)\pi}{a}\Bigr)\sin\Bigl(\frac{2(2ab-l)\pi}{b}\Bigr)\exp\Bigl(-n\frac{(2ab-l)^2}{ab}\pi\sqrt{-1}\Bigr)$$

$$\sum_{\substack{0\le l\le 2ab\\ l:\text{ even}}} \sin\Bigl(\frac{2l\pi}{a}\Bigr)\sin\Bigl(\frac{2l\pi}{b}\Bigr)\exp\Bigl(-n\frac{l^2}{ab}\pi\sqrt{-1}\Bigr)$$

$$= \sum_{h=0}^{ab} \sin\Bigl(\frac{4h\pi}{a}\Bigr)\sin\Bigl(\frac{4h\pi}{b}\Bigr)\exp\Bigl(-4n\frac{h^2}{ab}\pi\sqrt{-1}\Bigr).$$

We denote the right-hand side by W_{iii}, and show that it vanishes.

Using the equality

$$\sum_{k=1}^{ab}(e^{4k\pi\sqrt{-1}/a}-e^{4k\pi\sqrt{-1}/a})(e^{4k\pi\sqrt{-1}/b}-e^{4k\pi\sqrt{-1}/b})e^{-4n\frac{k^2}{ab}\pi\sqrt{-1}}=-4W_{\mathrm{iii}},$$

we apply the Gauss sum reciprocity formula. Putting

$$c:=ab,\quad d:=4n,\quad w:=\frac{2(\pm a\pm b)}{ab}$$

in Theorem 10.2, we have

$$\begin{aligned}
-4W_{\mathrm{iii}}&=\sum_{k=1}^{ab}(e^{(4k\frac{a+b}{ab}-4n\frac{k^2}{ab})\pi\sqrt{-1}}-e^{(4k\frac{-a+b}{ab}-4n\frac{k^2}{ab})\pi\sqrt{-1}}\\
&\qquad -e^{(4k\frac{a-b}{ab}-4n\frac{k^2}{ab})\pi\sqrt{-1}}+e^{(4k\frac{-a-b}{ab}-4n\frac{k^2}{ab})\pi\sqrt{-1}})\\
&=\frac{\sqrt{ab}}{e^{\pi\sqrt{-1}/4}\sqrt{4n}}\sum_{l=1}^{4n}(e^{\frac{ab}{4n}(l+\frac{2(a+b)}{ab})^2\pi\sqrt{-1}}-e^{\frac{ab}{4n}(l+\frac{2(-a+b)}{ab})^2\pi\sqrt{-1}}\\
&\qquad -e^{\frac{ab}{4n}(l+\frac{2(a-b)}{ab})^2\pi\sqrt{-1}}+e^{\frac{ab}{4n}(l+\frac{2(-a-b)}{ab})^2\pi\sqrt{-1}})\\
&=\sqrt{\frac{ab}{4n}}e^{-\pi\sqrt{-1}/4}\sum_{l=1}^{4n}(e^{\frac{\pi\sqrt{-1}}{4abn}(abl+2a+2b)^2\pi\sqrt{-1}}-e^{\frac{\pi\sqrt{-1}}{4abn}(abl-2a+2b)^2\pi\sqrt{-1}}\\
&\qquad -e^{\frac{\pi\sqrt{-1}}{4abn}(abl+2a-2b)^2\pi\sqrt{-1}}\\
&\qquad +e^{\frac{\pi\sqrt{-1}}{4abn}(abl-2a-2b)^2\pi\sqrt{-1}}).
\end{aligned}$$

Replacing l with $4n-l$, we have

$$\sum_{l=1}^{4n}e^{\frac{\pi\sqrt{-1}}{4abn}(abl-2a-2b)^2\pi\sqrt{-1}}=\sum_{l=1}^{4n}e^{\frac{\pi\sqrt{-1}}{4abn}(abl+2a+2b)^2\pi\sqrt{-1}},$$

$$\sum_{l=1}^{4n}e^{\frac{\pi\sqrt{-1}}{4abn}(abl+2a-2b)^2\pi\sqrt{-1}}=\sum_{l=1}^{4n}e^{\frac{\pi\sqrt{-1}}{4abn}(abl+2a-2b)^2\pi\sqrt{-1}}.$$

Therefore, we have

$$\begin{aligned}
-4W_{\mathrm{iii}}&=\sqrt{\frac{ab}{n}}e^{-\pi\sqrt{-1}/4}\sum_{l=1}^{4n}(e^{\frac{\pi\sqrt{-1}}{4abn}(abl+2a+2b)^2\pi\sqrt{-1}}\\
&\qquad -e^{\frac{\pi\sqrt{-1}}{4abn}(abl-2a+2b)^2\pi\sqrt{-1}}).\\
&=\sqrt{\frac{ab}{n}}e^{-\pi\sqrt{-1}/4}e^{\frac{a\pi\sqrt{-1}}{bn}+\frac{b\pi\sqrt{-1}}{an}}\sum_{l=1}^{4n}e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}\\
&\qquad -e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}).
\end{aligned}$$

We denote the summation in the right-hand side by $\widetilde{W}_{\text{iii}}$. Note that

$$
\begin{aligned}
&\sum_{\substack{1\le l\le 4n\\ l:\text{ even}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad=\sum_{\substack{1\le l\le 2n\\ l:\text{ even}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right)\\
&\qquad+\sum_{\substack{1\le l\le 2n\\ l:\text{ even}}} e^{\frac{b(l+2n)(a(l+2n)+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(a(l+2n)+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(a(l+2n)+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad=\sum_{\substack{1\le l\le 2n\\ l:\text{ even}}}(1+(-1)^{b(2+al+an)})e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right)=0.
\end{aligned}
$$

In a similar way we can prove

$$
\begin{aligned}
&\sum_{\substack{1\le l\le 4n\\ l:\text{ odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad=\sum_{\substack{1\le l\le 2n\\ l:\text{ odd}}}(1+(-1)^{b(2+al+an)})e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad=2\sum_{\substack{1\le l\le 2n\\ l:\text{ odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right).
\end{aligned}
$$

Hence, we have the following four equalities:

$$
\widetilde{W}_{\text{iii}}=2\sum_{\substack{1\le l\le 2n\\ l:\text{ odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right), \tag{52}
$$

$$
\begin{aligned}
\widetilde{W}_{\text{iii}}&=2\sum_{\substack{1\le l\le 2n\\ l:\text{ odd}}} e^{\frac{b(2n-l)(a(2n-l)+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(a(2n-l)+2)\pi\sqrt{-1}}{n}}-e^{\frac{-(a(2n-l)+2)\pi\sqrt{-1}}{n}}\right)\\
&=2\sum_{\substack{1\le l\le 2n\\ l:\text{ odd}}} e^{\frac{bl(al-4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{-(al-2)\pi\sqrt{-1}}{n}}-e^{\frac{(al-2)\pi\sqrt{-1}}{n}}\right),
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
&\sum_{\substack{n+1\le l\le 2n-1\\ l:\text{ odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad=\sum_{\substack{1\le l\le n-1\\ l:\text{ odd}}} e^{\frac{b(2n-l)(a(2n-l)+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(a(2n-l)+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(a(2n-l)+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad=\sum_{\substack{1\le l\le n-1\\ l:\text{ odd}}} e^{\frac{bl(al-4)\pi\sqrt{-1}}{4n}}\left(e^{-\frac{(al-2)\pi\sqrt{-1}}{n}}-e^{\frac{(al-2)\pi\sqrt{-1}}{n}}\right),
\end{aligned} \tag{54}
$$

$$\begin{aligned}
&\sum_{\substack{n+1\leq l\leq 2n-1\\ l:\text{ odd}}} e^{\frac{bl(al-4)\pi\sqrt{-1}}{4n}}(e^{-\frac{(al-2)\pi\sqrt{-1}}{n}}-e^{\frac{(al-2)\pi\sqrt{-1}}{n}})\\
&\quad=\sum_{\substack{1\leq l\leq n-1\\ l:\text{ odd}}} e^{\frac{b(2n-l)(a(2n-l)-4)\pi\sqrt{-1}}{4n}}(e^{-\frac{(a(2n-l)-2)\pi\sqrt{-1}}{n}}-e^{\frac{(a(2n-l)-2)\pi\sqrt{-1}}{n}})\\
&\quad=\sum_{\substack{1\leq l\leq n-1\\ l:\text{ odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}})
\end{aligned}\tag{55}$$

since n is odd.

Add (52) and (53), and divide it by two, we have

$$\begin{aligned}
\widetilde{W}_{\text{iii}}=&\sum_{\substack{1\leq l\leq 2n\\ l:\text{ odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}})\\
&+\sum_{\substack{1\leq l\leq 2n\\ l:\text{ odd}}} e^{\frac{bl(al-4)\pi\sqrt{-1}}{4n}}(e^{-\frac{(al-2)\pi\sqrt{-1}}{n}}-e^{\frac{(al-2)\pi\sqrt{-1}}{n}}).
\end{aligned}\tag{56}$$

From (54), the first summation in the right-hand side of (56) becomes

$$\begin{aligned}
&\sum_{\substack{1\leq l\leq n-1\\ l:\text{ odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}(e^{\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al+2)\pi\sqrt{-1}}{n}})\\
&+e^{\frac{bn(an-4)\pi\sqrt{-1}}{4n}}(e^{\frac{(an-2)\pi\sqrt{-1}}{n}}-e^{-\frac{(an-2)\pi\sqrt{-1}}{n}})\\
&+\sum_{\substack{1\leq l\leq n-1\\ l:\text{ odd}}} e^{\frac{bl(al-4)\pi\sqrt{-1}}{4n}}(e^{\frac{(al-2)\pi\sqrt{-1}}{n}}-e^{-\frac{(al-2)\pi\sqrt{-1}}{n}}).
\end{aligned}$$

From (55), the second summation in the right-hand side of (56) becomes

$$\begin{aligned}
&\sum_{\substack{1\leq l\leq n-1\\ l:\text{ odd}}} e^{\frac{bl(al-4)\pi\sqrt{-1}}{4n}}(e^{-\frac{(al-2)\pi\sqrt{-1}}{n}}-e^{\frac{(al-2)\pi\sqrt{-1}}{n}})\\
&+e^{\frac{bn(an-4)\pi\sqrt{-1}}{4n}}(e^{-\frac{(an-2)\pi\sqrt{-1}}{n}}-e^{\frac{(an-2)\pi\sqrt{-1}}{n}})\\
&+\sum_{\substack{1\leq l\leq n-1\\ l:\text{ odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}(e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}-e^{\frac{(al+2)\pi\sqrt{-1}}{n}}).
\end{aligned}$$

Therefore, (56) turns out to be

$$
\begin{aligned}
\widetilde{W}_{\mathrm{iii}} &= 2\sum_{\substack{1\le l\le n-1\\ l:\,\text{odd}}} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}} - e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad + 2\sum_{\substack{1\le l\le n-1\\ l:\,\text{odd}}} e^{\frac{bl(al-4)\pi\sqrt{-1}}{4n}}\left(e^{-\frac{(al-2)\pi\sqrt{-1}}{n}} - e^{\frac{(al-2)\pi\sqrt{-1}}{n}}\right)\\
&\quad - 2e^{abn\pi\sqrt{-1}/4}\left(e^{\frac{2\pi\sqrt{-1}}{n}} - e^{-\frac{2\pi\sqrt{-1}}{n}}\right)\\
&= 2\sum_{\substack{1\le l\le n-1\\ l:\,\text{even}}} e^{\frac{b(n-l)(a(n-l)+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(a(n-l)+2)\pi\sqrt{-1}}{n}} - e^{-\frac{(a(n-l)+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad + 2\sum_{\substack{1\le l\le n-1\\ l:\,\text{even}}} e^{\frac{b(n-l)(a(n-l)-4)\pi\sqrt{-1}}{4n}}\left(e^{-\frac{(a(n-l)-2)\pi\sqrt{-1}}{n}} - e^{\frac{(a(n-l)-2)\pi\sqrt{-1}}{n}}\right)\\
&\quad - 2e^{abn\pi\sqrt{-1}/4}\left(e^{\frac{2\pi\sqrt{-1}}{n}} - e^{-\frac{2\pi\sqrt{-1}}{n}}\right)\\
&= 2\sum_{\substack{1\le l\le n-1\\ l:\,\text{even}}} (-1)^{l/2} e^{abn\pi\sqrt{-1}/4} e^{\frac{bl(al-4)\pi\sqrt{-1}}{4n}}\left(e^{-\frac{(al-2)\pi\sqrt{-1}}{n}} - e^{\frac{(al-2)\pi\sqrt{-1}}{n}}\right)\\
&\quad + 2\sum_{\substack{1\le l\le n-1\\ l:\,\text{even}}} (-1)^{l/2} e^{abn\pi\sqrt{-1}/4} e^{\frac{bl(al+4)\pi\sqrt{-1}}{4n}}\left(e^{\frac{(al+2)\pi\sqrt{-1}}{n}} - e^{-\frac{(al+2)\pi\sqrt{-1}}{n}}\right)\\
&\quad - 2e^{abn\pi\sqrt{-1}/4}\left(e^{\frac{2\pi\sqrt{-1}}{n}} - e^{-\frac{2\pi\sqrt{-1}}{n}}\right)\\
&= 2e^{abn\pi\sqrt{-1}/4}\sum_{l=1}^{(n-1)/2} (-1)^l e^{\frac{bl(al-2)\pi\sqrt{-1}}{n}}\left(e^{-\frac{2(al-1)\pi\sqrt{-1}}{n}} - e^{\frac{2(al-1)\pi\sqrt{-1}}{n}}\right)\\
&\quad + 2e^{abn\pi\sqrt{-1}/4}\sum_{l=1}^{(n-1)/2} (-1)^l e^{\frac{bl(al+2)\pi\sqrt{-1}}{n}}\left(e^{\frac{2(al+1)\pi\sqrt{-1}}{n}} - e^{-2\frac{(al+1)\pi\sqrt{-1}}{n}}\right)\\
&\quad - 2e^{abn\pi\sqrt{-1}/4}\left(e^{\frac{2\pi\sqrt{-1}}{n}} - e^{-\frac{2\pi\sqrt{-1}}{n}}\right).
\end{aligned}
$$

Now, we will calculate $\widetilde{J}_n(e^{4\pi\sqrt{-1}/4})$. Since ab is odd, we have

$$
\begin{aligned}
&\widetilde{J}_n(e^{4\pi\sqrt{-1}/n})\\
&\quad = \sum_{j=-(n-1)/2}^{(n-1)/2} e^{4bj(aj+1)\pi\sqrt{-1}/n}\left(e^{4(aj+1/2)\pi\sqrt{-1}/n} - e^{-4(aj+1/2)\pi\sqrt{-1}/n}\right)
\end{aligned}
$$

(put $l = 2j$, noting that n is odd)

$$
\quad = \sum_{\substack{1-n\le l\le n-1\\ l:\,\text{even}}} e^{bl(al+2)\pi\sqrt{-1}/n}\left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right)
$$

$$= \sum_{\substack{0 \le l \le n-1 \\ l:\text{ even}}} e^{bl(al+2)\pi\sqrt{-1}/n} \left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right)$$
$$+ \sum_{\substack{1 \le l \le n-2 \\ l:\text{ odd}}} e^{b(l-n)(a(l-n)+2)\pi\sqrt{-1}/n} \left(e^{2(a(l-n)+1)\pi\sqrt{-1}/n} - e^{-2(a(l-n)+1)\pi\sqrt{-1}/n}\right)$$

(since ab and n are odd)

$$= \sum_{\substack{0 \le l \le n-1 \\ l:\text{ even}}} e^{bl(al+2)\pi\sqrt{-1}/n} \left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right)$$
$$- \sum_{\substack{1 \le l \le n-2 \\ l:\text{ odd}}} e^{bl(al+2)\pi\sqrt{-1}/n} \left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right)$$
$$= \sum_{l=0}^{n-1} (-1)^l e^{bl(al+2)\pi\sqrt{-1}/n} \left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right).$$

Since we have

$$\sum_{l=0}^{n-1} (-1)^l e^{bl(al+2)\pi\sqrt{-1}/n} \left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right)$$
$$= \sum_{l=1}^{n} (-1)^{n-l} e^{b(n-l)(a(n-l)+2)\pi\sqrt{-1}/n} \left(e^{2(a(n-l)+1)\pi\sqrt{-1}/n} - e^{-2(a(n-l)+1)\pi\sqrt{-1}/n}\right)$$
$$= \sum_{l=0}^{n-1} (-1)^l e^{bl(al-2)\pi\sqrt{-1}/n} \left(e^{-2(al-1)\pi\sqrt{-1}/n} - e^{2(al-1)\pi\sqrt{-1}/n}\right),$$

we conclude that

$$\widetilde{J}_n(e^{4\pi\sqrt{-1}/n})$$
$$= \frac{1}{2} \sum_{l=0}^{n-1} (-1)^l e^{bl(al+2)\pi\sqrt{-1}/n} \left(e^{2(al+1)\pi\sqrt{-1}/n} - e^{-2(al+1)\pi\sqrt{-1}/n}\right)$$
$$+ \frac{1}{2} \sum_{l=0}^{n-1} (-1)^l e^{bl(al-2)\pi\sqrt{-1}/n} \left(e^{-2(al-1)\pi\sqrt{-1}/n} - e^{2(al-1)\pi\sqrt{-1}/n}\right). \quad (57)$$

Now, we have

$$
\begin{aligned}
&\sum_{l=0}^{n-1}(-1)^l e^{bl(al+2)\pi\sqrt{-1}/n}\left(e^{2(al+1)\pi\sqrt{-1}/n}-e^{-2(al+1)\pi\sqrt{-1}/n}\right) \\
&\quad=\sum_{l=0}^{(n-1)/2}(-1)^l e^{bl(al+2)\pi\sqrt{-1}/n}\left(e^{2(al+1)\pi\sqrt{-1}/n}-e^{-2(al+1)\pi\sqrt{-1}/n}\right) \\
&\qquad+\sum_{l=1}^{(n-1)/2}(-1)^{n-l} e^{b(n-l)(a(n-l)+2)\pi\sqrt{-1}/n}\left(e^{2(a(n-l)+1)\pi\sqrt{-1}/n}\right. \\
&\qquad\qquad\left.-e^{-2(a(n-l)+1)\pi\sqrt{-1}/n}\right) \\
&\quad=\sum_{l=0}^{(n-1)/2}(-1)^l e^{bl(al+2)\pi\sqrt{-1}/n}\left(e^{2(al+1)\pi\sqrt{-1}/n}-e^{-2(al+1)\pi\sqrt{-1}/n}\right) \\
&\qquad+\sum_{l=1}^{(n-1)/2}(-1)^l e^{bl(al-2)\pi\sqrt{-1}/n}\left(e^{-2(al-1)\pi\sqrt{-1}/n}-e^{2(al-1)\pi\sqrt{-1}/n}\right)
\end{aligned}
\tag{58}
$$

and

$$
\begin{aligned}
&\sum_{l=0}^{n-1}(-1)^l e^{bl(al-2)\pi\sqrt{-1}/n}\left(e^{-2(al-1)\pi\sqrt{-1}/n}-e^{2(al-1)\pi\sqrt{-1}/n}\right) \\
&=\sum_{l=0}^{(n-1)/2}(-1)^l e^{bl(al-2)\pi\sqrt{-1}/n}\left(e^{-2(al-1)\pi\sqrt{-1}/n}-e^{2(al-1)\pi\sqrt{-1}/n}\right) \\
&\quad+\sum_{l=1}^{(n-1)/2}(-1)^l e^{bl(al+2)\pi\sqrt{-1}/n}\left(e^{2(al+1)\pi\sqrt{-1}/n}-e^{-2(al+1)\pi\sqrt{-1}/n}\right).
\end{aligned}
\tag{59}
$$

Adding (58) and (59), and dividing it by two, we obtain from (57)

$$
\begin{aligned}
&\widetilde{J}_n(e^{4\pi\sqrt{-1}/n}) \\
&\quad=\sum_{l=0}^{(n-1)/2}(-1)^l e^{bl(al+2)\pi\sqrt{-1}/n}\left(e^{2(al+1)\pi\sqrt{-1}/n}-e^{-2(al+1)\pi\sqrt{-1}/n}\right) \\
&\qquad+\sum_{l=0}^{(n-1)/2}(-1)^l e^{bl(al-2)\pi\sqrt{-1}/n}\left(e^{-2(al-1)\pi\sqrt{-1}/n}-e^{2(al-1)\pi\sqrt{-1}/n}\right) \\
&\quad-\left(e^{2\pi\sqrt{-1}/n}-e^{-2\pi\sqrt{-1}/n}\right).
\end{aligned}
$$

Therefore, we finally have

$$
\widetilde{W}_{\mathrm{iii}}=2e^{abn\pi\sqrt{-1}/4}\,\widetilde{J}_n(e^{4\pi\sqrt{-1}/n})=0
$$

and

$$-4W_{\mathrm{iii}} = \sqrt{\frac{ab}{n}} e^{-\pi\sqrt{-1}/4} e^{\frac{a\pi\sqrt{-1}}{bn} + \frac{b\pi\sqrt{-1}}{an}} \widetilde{W}_{\mathrm{iii}} = 0.$$

This completes the proof. ■

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